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INTERPOLATION BY BOUNDED FUNCTIONS

par W. HAYMAN.

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1. Let D be a domain in the plane or more generally a Riemann surface, which admits bounded analytic functions. In a recent lecture R. C. Buck raised the following problem. Do there exist infinite sequences z_n in D, such that an arbitrary bounded sequence w_n can be interpolated at z_n by a function $f(z)$ regular and bounded in D, and if so does every sequence z_n, which approaches the boundary of D sufficiently rapidly have this property? Although the existence and uniqueness problem for fixed sequences w_n and z_n has been extensively treated by Pick, Schur, Grunsky, Carathéodory, Denjoy, Nevanlinna and others (1), Buck's questions does not seem answerable by the classical methods.

We shall in this paper supply an affirmative answer to both problems in case D is the unit circle. A sequence z_n, $n = 1, 2, \ldots$ will be called a *universal interpolation sequence*, (u.i.s.) if

$$|z_n| < 1, \qquad n = 1, 2, \ldots$$

and given any complex sequence w_n satisfying

$$|w_n| \leqslant 1, \qquad n = 1, 2, \ldots$$

we can find $f(z)$ regular and bounded in $|z| < 1$ and such that

$$f(z_n) = w_n. \tag{1. 1}$$

(1) See e. g. R. Nevanlinna, Uber beschränkte analytische Funktionen, *Annales Acad. Sci. Fenn.* 32, nr. 7 (1929), for a good account of the problem.

The conditions evidently imply that the z_n are distinct and have no limit point in $|z| < 1$. We write

$$r_{m,n} = \left|\frac{z_m - z_n}{1 - \bar{z}_m z_n}\right|.$$

We shall denote by $C, C_1, C_2, \ldots$ positive constants independent of m, n not necessarily the same each time. The letter A will denote positive absolute constants and $A(\varepsilon)$ constants depending only on ε. Our main result can now be stated as follows.

THEOREM I. — *A necessary condition for a sequence z_n to be a u.i.s. is that*

$$\Pi_n = \prod_{\substack{m=1 \\ m \neq n}}^{\infty} r_{m,n} \geqslant C_1, \quad \text{all} \quad n. \tag{1. 2}$$

A sufficient condition is that there exists $\lambda < 1$ and $C_2 > 0$ so that

$$\Pi_n(\lambda) = \prod_{\substack{m=1 \\ m \neq n}}^{\infty} [1 - (1 - r_{m,n})^{\lambda}] \geqslant C_2, \quad \text{all} \quad n. \tag{1. 3}$$

We note that (1. 3) reduces to (1. 2) if we put $\lambda = 1$. Thus the necessary and sufficient conditions are not too far apart. It seems quite possible that (1. 2) is in fact sufficient as well as necessary, but I have been unable to prove this.

From Theorem 1 we shall be able to deduce

THEOREM 2. — *A sufficient condition for a sequence of distinct numbers z_n in $|z| < 1$ to be a u.i.s. is that*

$$\overline{\lim_{n \to \infty}} \frac{1 - |z_{n+1}|}{1 - |z_n|} < 1. \tag{1. 4}$$

If z_n is positive increasing, the condition is also necessary.

2. PROOF OF THEOREM 1, NECESSITY. — Suppose that z_n is a u.i.s. and that (1. 2) is false. Then we can find an increasing sequence of integers n_p, $p = 1, 2, \ldots$, such that

$$\Pi_{n_p} \to 0, \quad \text{as} \quad p \to \infty. \tag{2. 1}$$

Since $\{z_n\}$ is a u.i.s. $\{z_n\}$ has no limit point in $|z| < 1$ and so

$$r_{m,n} \to 1, \quad \text{as} \quad m \to \infty \quad \text{for fixed} \quad n.$$

By choosing a subsequence of our sequence n_p if necessary, we may therefore suppose in addition to (2. 1) that, given $n_1, n_2, \ldots, n_{p-1}$; n_p is chosen so large that

$$r_{n_p, n_k} > \exp[-2^{-(p-k)}], \qquad k = 1, 2, \ldots, p-1.$$

We deduce that

$$\mathrm{Q}_k = \prod_{\substack{p=1 \\ p \neq k}}^{\infty} r_{n_p, n_k} > \exp\left[-\left(\sum_{\substack{p=1 \\ p\neq k}}^{\infty} 2^{-|p-k|}\right)\right]$$

$$> \exp\left[-2\sum_{t=1}^{\infty} 2^{-t}\right] = e^{-2}. \qquad (2.\ 2)$$

Suppose then that our sequence n_k satisfies (2. 1) and (2. 2). We choose w_n so that

$$w_{n_p} = 1, \qquad p = 1, 2, \ldots,$$
$$w_n = 0, \qquad \text{if} \quad n \neq n_p \quad \text{for any} \quad p,$$

and suppose that there exists $f(z)$ regular in $|z| < 1$ and satisfying (1. 1) and $|f(z)| < \mathrm{M}$ there. Let N be a positive integer and set

$$\varphi(z) = f(z) \prod_{n=1}^{\mathrm{N}}{}' \left|\frac{1-\bar{z}_n z}{z_n - z}\right|$$

where the prime denotes a product over integers not belonging to the sequence n_p. Then $\varphi(z)$ is regular in $|z| < 1$ and

$$\varlimsup_{|z| \to 1} |\varphi(z)| \leqslant \mathrm{M}.$$

Thus the maximum modulus principle gives $|\varphi(z)| \leqslant \mathrm{M}$ in $|z| < 1$, and so

$$|f(z)| \leqslant \mathrm{M} \prod_{n=1}^{\mathrm{N}}{}' \left|\frac{z - z_n}{1-\bar{z}_n z}\right|$$

Setting $z = z_{n_k}$ for a fixed k and making $\mathrm{N} \to \infty$ we deduce

$$1 \leqslant \mathrm{M} \prod_{n=1}^{\infty}{}' r_{n, n_k} = \mathrm{M}\frac{\Pi_{n_k}}{\mathrm{Q}_k} \leqslant \mathrm{M}e^2 \Pi_{n_k}.$$

This contradicts (2. 1) and so proves the necessity part of Theorem 1.

3. Proof. of Theorem 1, sufficiency. — Let z_n be a sequence of points in $|z| < 1$ satisfying (1. 3), or more gene-

rally (1. 2) and suppose that we can find a sequence of functions $f_n(z)$ regular in $|z| < 1$ and satisfying

$$|f_n(z_n)| \geqslant C', \qquad \text{all} \quad n \tag{3. 1}$$

and

$$\sum_{n=1}^{\infty} |f_n(z)| \leqslant C'', \qquad |z| < 1. \tag{3. 2}$$

We write

$$g_n(z) = f_n(z) \prod_{\substack{m=1 \\ m \neq n}}^{\infty} \left\{ \frac{z_m - z}{1 - \bar{z}_m z} \cdot \frac{\bar{z}_m}{|z_m|} \right\}. \tag{3. 3}$$

Then the condition (1. 2) implies that $g_n(z)$ is regular in $|z| < 1$,

$$g_n(z_m) = 0, \qquad m \neq n,$$

and

$$|g_n(z_n)| = |f_n(z_n)| \, \Pi_n \geqslant C.$$

We now put

$$h_n(z) = \frac{g_n(z)}{g_n(z_n)}. \tag{3. 4}$$

Then we have for $|z| < 1$

$$|h_n(z)| \leqslant \frac{|g_n(z)|}{C} \leqslant \frac{|f_n(z)|}{C},$$

and so by (3. 2)

$$\sum_{n=1}^{\infty} |h_n(z)| \leqslant \frac{C''}{C'}, \qquad |z| < 1. \tag{3. 5}$$

Also by (3. 3) and (3. 4) we have

$$h_n(z_n) = 1, \qquad h_n(z_m) = 0, \qquad n \neq m. \tag{3. 6}$$

Thus if w_n is any bounded sequence we set

$$f(z) = \sum_{n=1}^{\infty} w_n h_n(z).$$

It now follows from (3. 6) that $f(z)$ satisfies (1. 1) and from (3. 5) that $f(z)$ is bounded in $|z| < 1$.

In order to complete the proof or Theorem 1 it therefore remains only to construct the sequence $f_n(z)$ satisfying (3. 1) and (3. 2), given a sequence z_n satisfying (1. 3) and this we proceed to do.

3. 1. In order to construct our sequence $f_n(z)$ we shall construct functions $U_n(z)$ positive and harmonic in $|z| < 1$ and such that for some positive ε

$$U_n(z_n) \leqslant C_1, \tag{3. 7}$$

$$\max\{U_n(z), U_m(z)\} \geqslant (1 - r_{m,n})^{-\varepsilon}, \quad m \neq n, \quad |z| < 1. \tag{3. 8}$$

We then define $f_n(z)$ by the equation

$$|f_n(z)| = e^{-U_n(z)}.$$

Then (3. 7) shows that (3. 1) holds. Also (3. 8) shows that

$$\min\{|f_m(z)|, |f_n(z)|\} \leqslant \exp(1 - r_{m,n})^{-\varepsilon}, \quad |z| < 1, \quad m \neq n.$$

For any z in $|z| < 1$ let

$$t(z) = \sup_m |f_m(z)| = f_M(z),$$

say. Then if

$$\exp[-(1 - r_{M,n})^{-\varepsilon}] < t(z), \tag{3. 9}$$

we have

$$|f_n(z)| \leqslant \exp[-(1 - r_{M,n})^{-\varepsilon}]. \tag{3. 10}$$

Now if $N = N(r)$ is the total number of indices n for which $r_{M,n} \leqslant r$ it follows from (1. 3) that

$$[1 - (1 - r)^{\lambda}]^{N} \geqslant C,$$

and hence

$$N(r) \leqslant C(1 - r)^{-\lambda}.$$

We choose r so that

$$\exp[-(1 - r)^{-\varepsilon}] = t(z), \qquad (1 - r)^{-\varepsilon} = \log[1/t(z)].$$

Thus in this case

$$N \leqslant C\{\log[1/t(z)]\}^{\lambda/\varepsilon}. \tag{3. 11}$$

We see that the number N of indices n for which (3. 9) is false satisfies (3. 11) for any z in $|z| < 1$. For all other values of n we have (3. 10). Thus

$$\sum_{n=1}^{\infty} |f_n(z)| \leqslant Nt(z) + \sum_{\substack{n=1 \\ n \neq M}}^{\infty} \exp[-(1 - r_{M,n})^{-\varepsilon}]$$

$$\leqslant Ct(z)\{\log[1/t(z)]\}^{\lambda/\varepsilon} + A(\varepsilon) \sum_{\substack{n=1 \\ n \neq M}}^{\infty} (1 - r_{M,n}) \leqslant C,$$

in view of (1. 3). This yields (3. 2). Thus our problem of constructing the regular functions $f_n(z)$ is reduced to the construction of the positive harmonic functions $U_n(z)$ satisfying (3. 7) and (3. 8).

4. Construction of the functions $U_n(z)$. — For any pair of points z, z' in the unit circle we set

$$r(z, z') = \left|\frac{z - z'}{1 - \bar{z}z'}\right|.$$

We shall need a number of lemmas.

Lemma 1. — *Given* $\varepsilon > 0$ *and* ρ *such that* $0 < \rho < 1$, *there exists* $u(z)$ *harmonic and positive in* $|z| < 1$ *and such that* $u(\rho) = 1$,

$$u(z) > \sin\left(\frac{\pi}{2}\varepsilon\right)\left\{\frac{1+\rho}{1-\rho}\cdot\left|\frac{1-z}{1+z}\right|\right\}^{1-\varepsilon}, \quad |z| < 1.$$

Choose

$$u = \Re\left\{\frac{1+\rho}{1-\rho}\cdot\frac{1-z}{1+z}\right\}^{1-\varepsilon},$$

and write

$$\frac{1-z}{1+z} = Te^{i\varphi}, \qquad u = \left\{\frac{1+\rho}{1-\rho}T\right\}^{1-\varepsilon}\cos\left[(1-\varepsilon)\varphi\right].$$

Then $|\varphi| < \frac{\pi}{2}$ and so

$$\cos\left[(1-\varepsilon)\varphi\right] \geqslant \cos\left[(1-\varepsilon)\frac{\pi}{2}\right] = \sin\left(\frac{\pi}{2}\varepsilon\right).$$

and this proves the Lemma.

We have next

Lemma 2. — *Let* D *be a subdomain of* $|z| < 1$ *bounded by an arc of a circle orthogonal to* $|z| = 1$ *and an arc of* $|z| = 1$. *Let* z_0 *be a point of* $|z| < 1$ *outside* D *and such that for every* z *in* D *we have* $r(z, z_0) \geqslant r_0$.

Then, given $\varepsilon > 0$, *we can find* $v(z)$ *harmonic and positive in* $|z| < 1$ *and such that* $v(z_0) = 1$ *and*

$$v(z) \geqslant \sin\left(\frac{\pi}{2}\varepsilon\right)\left(\frac{1+r_0}{1-r_0}\right)^{1-\varepsilon} \textit{ in } D.$$

We may suppose without loss in generality that z_0 is the origin and that D is bisected by the positive real axis, since these results may be achieved by a conformal map of $|z| < 1$ onto itself, which leaves $r(z, z_0)$ invariant. It now follows that D is the domain given by

$$\left|\frac{1+z}{1-z}\right| \geqslant R, \qquad \text{where} \qquad R \geqslant \frac{1+r_0}{1-r_0}.$$

We now set

$$\varphi(z) = \Re\left(\frac{1+z}{1-z}\right)^{1-\varepsilon},$$

and note as in Lemma 1, that

$$\varphi(z) \geqslant \sin\left(\frac{\pi}{2}\varepsilon\right)\left|\frac{1+z}{1-z}\right|^{1-\varepsilon} \geqslant \sin\left(\frac{\pi}{2}\varepsilon\right)\left(\frac{1+r_0}{1-r_0}\right)^{1-\varepsilon}$$

for z in D, and this proves the Lemma.

4. 1. In order to make use of Lemmas 1 and 2 in our construction we need some inequalities for $r(z, z')$.

LEMMA 3. — *Suppose that z_1, z_2, z_3, z_4 are points in $|z| < 1$ and that $0 < z_2 \leqslant z_4 < 1$. Suppose further that*

$$2\left|\frac{1+z_1}{1-z_1}\right| \leqslant \frac{1+z_2}{1-z_2} \leqslant \left|\frac{1+z_3}{1-z_3}\right|.$$

Then we have

$$1 - r(z_1, z_3) \leqslant A\,\frac{1-r(z_1, z_4)}{1-r(z_2, z_4)}.$$

Write

$$Z_1 = \frac{1+z_1}{1-z_1} = R_1 e^{i\varphi_1}, \qquad \frac{1+z_2}{1-z_2} = R_2,$$

$$Z_3 = \frac{1+z_3}{1-z_3} = R_3 e^{i\varphi_3}, \qquad \frac{1+z_4}{1-z_4} = R_4,$$

where by hypothesis $2R_1 \leqslant R_2 \leqslant R_3$, $R_2 \leqslant R_4$ and also $|\varphi_1| < \frac{\pi}{2}$, $|\varphi_3| < \frac{\pi}{2}$. Then

$$z_1 = \frac{Z_1 - 1}{Z_1 + 1}, \qquad z_3 = \frac{Z_3 - 1}{Z_3 + 1},$$

and

$$r(z_1,\ z_3) = \left|\frac{z_1 - z_3}{1 - \bar{z}_1 z_3}\right| = \left|\frac{Z_1 - Z_3}{\bar{Z}_1 + Z_3}\right|.$$

Also

$$1 - r(z_1,\ z_3)^2 = \frac{|\bar{Z}_1 + Z_3|^2 - |Z_1 - Z_3|^2}{|\bar{Z}_1 + Z_3|^2} = \frac{4R_1R_3 \cos\varphi_1 \cos\varphi_3}{|\bar{Z}_1 + Z_3|^2}.$$

Similarly

$$1 - r(z_1,\ z_2)^2 = \frac{4R_1R_2 \cos\varphi_1}{|\bar{Z}_1 + R_2|^2},$$

$$1 - r(z_2,\ z_4)^2 = \frac{4R_2R_4}{(R_2 + R_4)^2},$$

$$1 - r(z_1,\ z_4)^2 = \frac{4R_1R_4 \cos\varphi_1}{|\bar{z}_1 + R_4|^2}.$$

Now we have by hypothesis $2R_1 \leqslant R_2 \leqslant R_3,\ R_2 \leqslant R_4$, and so

$$\frac{1}{4}R_2^2 \leqslant \frac{1}{4}R_3^2 \leqslant |\bar{Z}_1 + Z_3|^2,$$

$$R_4^2 \leqslant (R_2 + R_4)^2,$$

$$|\bar{Z}_1 + R_4|^2 \leqslant \frac{9}{4}R_4^2.$$

Thus

$$\frac{1 - r(z_1,\ z_4)}{1 - r(z_2,\ z_4)} \geqslant \frac{1}{2}\,\frac{1 - r(z_1,\ z_4)^2}{1 - r(z_2,\ z_4)^2} \geqslant \frac{4R_1R_4\cos\varphi_1}{2\cdot\frac{9}{4}R_4^2}\cdot\frac{R_4^2}{4R_2R_4}$$

$$= \frac{2}{9}\,\frac{R_1\cos\varphi_1}{R_2} \geqslant \frac{2}{9}\,\frac{R_1R_3\cos\varphi_1}{R_3^2} \geqslant \frac{1}{18}\,\frac{R_1R_3\cos\varphi_1}{|\bar{Z}_1 + Z_3|^2}$$

$$\geqslant \frac{1}{72}\left[1 - r(z_1,\ z_3)^2\right] \geqslant \frac{1}{72}\left[1 - r(z_1,\ z_3)\right].$$

This proves the Lemma.

4. 2. The key result in our construction is

LEMMA 4. — *Suppose that* ρ, $u(z)$ *are defined as in Lemma* 1, *that* $0 < \lambda < 1$, *and* $\varepsilon = \frac{1}{4}(1 - \lambda)$, *further that* $z' = \rho' e^{i\varphi}$ *where* $0 \leqslant \rho' \leqslant \rho$. *Let*

$$r = r(z',\ \rho) = \left|\frac{z' - \rho}{1 - \rho z'}\right|.$$

Then there exists $v(z)$, positive and harmonic in $|z| < 1$, and such that $v(z') = (1 - r)^{\lambda}$ and

$$u(z) + v(z) \geqslant A(\varepsilon)(1 - r)^{-\varepsilon}, \qquad |z| < 1.$$

We distinguish two cases. Suppose first that

$$2\left|\frac{1+z'}{1-z'}\right| \leqslant \frac{1+t}{1-t}, \tag{4. 1}$$

where

$$(1-t) = (1-\rho)(1-r)^{-2\varepsilon}. \tag{4. 2}$$

Let D be the set given by

$$\left|\frac{1+z}{1-z}\right| \geqslant \frac{1+t}{1-t}.$$

Then if z lies in D we have by Lemma 3, with z_1, z_2, z_3, z_4 replaced by z', t, z, ρ

$$\begin{aligned} 1 - r(z, z') &\leqslant A\frac{1 - r(z', \rho)}{1 - r(t, \rho)} \\ &= \frac{A(1-r)(1-t\rho)}{(1-\rho)(1+t)} \leqslant \frac{A(1-r)(1-t^2)}{(1-\rho)(1+t)} = A(1-r)^{1-2\varepsilon}. \end{aligned}$$

Hence by Lemma 2 we can construct a positive harmonic function $v_1(z)$ such that $v_1(z') = 1$, and for all z in D

$$v_1(z) \geqslant A(\varepsilon)(1-r)^{-(1-\varepsilon)(1-2\varepsilon)} \geqslant A(\varepsilon)(1-r)^{-1+3\varepsilon}.$$

Also outside D we have by Lemma 1 and (4. 2)

$$\begin{aligned} u(z) &\geqslant A(\varepsilon)\left\{\frac{1-t}{1+t}\cdot\frac{1+\rho}{1-\rho}\right\}^{1-\varepsilon} \geqslant A(\varepsilon)(1-r)^{-2\varepsilon(1-\varepsilon)} \\ &\geqslant A(\varepsilon)(1-r)^{-\varepsilon}, \end{aligned} \tag{4, 3}$$

since $\varepsilon \leqslant \frac{1}{2}$.

Choose now

$$v(z) = (1-r)^{\lambda} v_1(z) = (1-r)^{1-4\varepsilon} v_1(z).$$

Then

$$v(z') = (1-r)^{\lambda},$$

and in D we have

$$v(z) \geqslant A(\varepsilon)(1-r)^{\lambda-1+3\varepsilon} = A(\varepsilon)(1-r)^{-\varepsilon},$$

while outside D (4. 3) holds. Thus Lemma 4 is proved in this case.

We next consider the case in which (4. 1) is false. Suppose first that

$$1-\rho' \leqslant |\varphi| \leqslant \pi.$$

In this case

$$1-r^2 = 1-r(z',\rho)^2 = \frac{|1-\rho\rho' e^{i\varphi}|^2 - |\rho' e^{i\varphi}-\rho|^2}{|1-\rho\rho' e^{i\varphi}|^2}$$
$$= \frac{(1-\rho^2)(1-\rho'^2)}{(1-\rho\rho')^2+2\rho\rho'(1-\cos\varphi)} \geqslant \frac{A(1-\rho)(1-\rho')}{\varphi^2},$$

while

$$\left|\frac{1+z'}{1-z'}\right|^2 = \frac{1+2\rho'\cos\varphi+\rho'^2}{(1-\rho')^2+2\rho'(1-\cos\varphi)} \leqslant \frac{A}{\varphi^2}.$$

Since (4. 1) is false, we deduce

$$\frac{A}{\varphi^2} \geqslant \frac{A}{(1-t)^2} = \frac{A(1-r)^{4\varepsilon}}{(1-\rho)^2} \geqslant \frac{A}{(1-\rho)^2}\left[\frac{(1-\rho)(1-\rho')}{\varphi^2}\right]^{4\varepsilon},$$
$$|\varphi|^{2(1-4\varepsilon)} \leqslant A(1-\rho)^{2-4\varepsilon}(1-\rho')^{-4\varepsilon} \leqslant A(1-\rho')^{2(1-4\varepsilon)},$$

and so, since $\lambda = 1-4\varepsilon > 0$,

$$|\varphi| \leqslant A(\varepsilon)(1-\rho').$$

This inequality thus holds in any case if (4. 1) is false. Thus in this case

$$1-r^2 = \frac{(1-\rho^2)(1-\rho'^2)}{(1-\rho\rho')^2+2\rho\rho'(1-\cos\varphi)} > \frac{A(\varepsilon)(1-\rho)}{(1-\rho')}. \tag{4. 4}$$

We now put

$$v(z) = c\Re\left(\frac{1+z}{1-z}\right)^{1-\varepsilon} \geqslant c\sin\left(\frac{\pi\varepsilon}{2}\right)\left|\frac{1+z}{1-z}\right|^{1-\varepsilon},$$

where c is so shosen that

$$v(z') = (1-r)^{\lambda}.$$

Then we have for $|z| < 1$

$$u(z)+v(z) \geqslant \sin\left(\frac{\pi}{2}\varepsilon\right)\left\{\left(\frac{1+\rho}{1-\rho}\right)^{1-\varepsilon}\left|\frac{1-z}{1+z}\right|^{1-\varepsilon} + c\left|\frac{1+z}{1-z}\right|^{1-\varepsilon}\right\}$$
$$\geqslant \sin\left(\frac{\pi}{2}\varepsilon\right)\left[c\left(\frac{1+\rho}{1-\rho}\right)^{1-\varepsilon}\right]^{\frac{1}{2}}.$$

We have

$$(1-r)^\lambda = v(z') \leqslant c\left|\frac{1+z'}{1-z'}\right|^{1-\varepsilon} \leqslant c\left(\frac{1+|z'|}{1-|z'|}\right)^{1-\varepsilon},$$

so that

$$c \geqslant \frac{1}{2}(1-\rho')^{1-\varepsilon}(1-r)^\lambda \geqslant A(1-\rho')^{1-\varepsilon}\left(\frac{1-\rho}{1-\rho'}\right)^{1-4\varepsilon},$$

by (4. 4). Thus

$$u(z)+v(z) \geqslant A(\varepsilon)\left[(1-\rho')^{3\varepsilon}(1-\rho)^{1-4\varepsilon}(1-\rho)^{\varepsilon-1}\right]^{\frac{1}{2}},$$

$$\geqslant A(\varepsilon)\left(\frac{1-\rho'}{1-\rho}\right)^{\frac{3}{2}\varepsilon} \geqslant A(\varepsilon)(1-r)^{-\frac{3}{2}\varepsilon},$$

again by (4. 4), so that Lemma 4 follows also in this case.

5. Completion of proof of theorem 1. — We can now construct our harmonic functions $U_n(z)$ to satisfy (3. 7) and (3. 8). Let $z_n = \rho_n e^{i\theta_n}$ be the members of our sequence and suppose that

$$\rho_n \leqslant \rho_{n+1}, \quad n = 1, 2, \ldots$$

Set

$$V_n(z) = \Re\left\{\frac{1+\rho_n}{1-\rho_n}\cdot\frac{1-ze^{-i\theta_n}}{1+ze^{-i\theta_n}}\right\}^{1-\varepsilon}.$$

Then after a rotation of the unit circle we can deduce from Lemma 4 that we can, for $m < n$, construct a function $u_{m,n}(z)$, positive and harmonic in $|z| < 1$ and such that

$$u_{m,n}(z_m) = (1-r_{m,n})^\lambda,$$

and

$$u_{m,n}(z) + V_n(z) \geqslant A(\varepsilon)(1-r_{m,n})^{-\varepsilon}, \qquad |z| < 1.$$

Set now

$$U_m(z) = V_m(z) + \sum_{n=m+1}^{\infty} u_{m,n}(z).$$

Then

$$U_m(z_m) = 1 + \sum_{n=m+1}^{\infty}(1-r_{m,n})^\lambda$$

$$\leqslant 1 - \sum_{n=m+1}^{\infty}\log\left[1-(1-r_{m,n})^\lambda\right] \leqslant C \quad (5.\ 1)$$

by (1. 3). On the other hand if $m < n$ and $|z| < 1$

$$\max\{U_m(z), U_n(z)\} \geqslant \frac{1}{2}[U_m(z) + U_n(z)]$$

$$\geqslant \frac{1}{2}[u_{m,n}(z) + V_n(z)] \geqslant A(\varepsilon)(1 - r_{m,n})^{-\varepsilon}. \quad (5.\ 2)$$

If we write $A(\varepsilon)U_n(z)$ instead of $U_n(z)$ in (5. 1), (5. 2) we obtain (3. 7), (3. 8) as required. This completes the proof of Theorem 1.

6. Proof of Theorem 2. — We proceed to deduce Theorem 2 from Theorem 1. We prove first the sufficiency part of Theorem 2. Suppose that $z = \rho e^{i\theta}$, $z' = \rho' e^{i\theta'}$, where $\rho \leqslant \rho'$. Then

$$1 - r(z, z')^2 = \frac{(1-\rho^2)(1-\rho'^2)}{(1-\rho\rho')^2 + 2\rho\rho'[1-\cos(\theta-\theta')]}$$

$$\leqslant \frac{(1-\rho^2)(1-\rho'^2)}{(1-\rho\rho')^2} = 1 - r(\rho, \rho')^2.$$

Thus also

$$1 - r(z, z') \leqslant 1 - r(\rho, \rho') = 1 - \frac{\rho'-\rho}{1-\rho\rho'} = \frac{(1-\rho')(1+\rho)}{1-\rho\rho'} \leqslant \frac{2(1-\rho')}{1-\rho}.$$

Suppose now that $z_n = \rho_n e^{i\theta_n}$ is the sequence of Theorem 2 and that we have for $n \geqslant n_0$,

$$1 - |z_{n+1}| < K(1 - |z_n|)$$

where $K < 1$. Then for $n > m \geqslant n_0$ we have

$$1 - |z_n| \leqslant K^{n-m}(1 - |z_m|)$$

and hence for $n > n_0$, $m > n_0$

$$1 - r_{m,n} < 2K^{|n-m|}. \quad (6.\ 1)$$

Similarly if $n > n_0$, $m \leqslant n_0$

$$1 - r_{m,n} \leqslant 2K^{n-n_0}. \quad (6.\ 2)$$

Finally since $r_{m,n} \neq 0$, for $m < n \leqslant n_0$, we have for $m < n \leqslant n_0$

$$1 - (1 - r_{m,n})^{\frac{1}{2}} \geqslant C. \quad (6.\ 3)$$

This inequality remains true for general distinct m, n. In fact if $m \leqslant n_0 < n$ we have

$$r_{m,n} = \left|\frac{z_m - z_n}{1 - \bar{z}_m z_n}\right| \geqslant \frac{\rho_n - \rho_m}{1 - \rho_n\rho_m} \geqslant \frac{\rho_{n_0+1} - \rho_{n_0}}{1 - \rho_{n_0}\rho_{n_0+1}} = C,$$

and if $n > m \geqslant n_0$, we have

$$r_{m,n} \geqslant \frac{\rho_{m+1} - \rho_m}{1 - \rho_m \rho_{m+1}} = \frac{(1-\rho_m) - (1-\rho_{m+1})}{(1-\rho_m) + \rho_m(1-\rho_{m+1})}$$
$$\geqslant \frac{(1-\mathrm{K})(1-\rho_m)}{2(1-\rho_m)} = \frac{1-\mathrm{K}}{2}.$$

Thus (6. 3) holds in all cases.

Let now t_0 be the smallest positive integer, such that $2\mathrm{K}^{t_0} < \frac{1}{2}$. Suppose first $n \leqslant n_0 + t_0$. Then

$$\Pi_n\left(\frac{1}{2}\right) = \prod_{\substack{m=1 \\ m \neq n}}^{\infty} \left[1 - (1 - r_{m,})^{\frac{1}{2}}\right] = \prod_{\substack{m \leqslant n_0 + 2t_0 \\ m \neq n}} \prod_{m > n_0 + 2t_0} = \Pi' \Pi'',$$

say. Here $\Pi' \geqslant \mathrm{C}$ by (6. 3) and by (6. 1), (6. 2)

$$\Pi'' \geqslant \prod_{t=t_0+1}^{\infty} \left[1 - (2\mathrm{K}^t)^{\frac{1}{2}}\right] = \mathrm{C} > \frac{1}{2}.$$

Thus in this case $\Pi_n\left(\frac{1}{2}\right) \geqslant \mathrm{C}$ in (1. 3).

Similarly if $n > n_0 + t_0$

$$\Pi_n\left(\frac{1}{2}\right) \geqslant \prod_{m \leqslant n_1} \prod_{1 \leqslant |m-n| \leqslant t_0} \prod_{\substack{|n-m| > t_0 \\ m > n_0}} \left[1 - (1 - r_{m,n})^{\frac{1}{2}}\right]$$
$$\geqslant \mathrm{C}^{n_0} \mathrm{C}^{2t_0} \left\{ \prod_{t=t_0+1}^{\infty} \left[1 - (2\mathrm{K}^t)^{\frac{1}{2}}\right] \right\}^2 \geqslant \mathrm{C},$$

and so (1. 3) holds again with $\lambda = \frac{1}{2}$. This completes the sufficiency part of Theorem 2.

To prove necessity if the z_n are all positive, suppose that they are arranged in order of magnitude. Then (1. 2) must be satisfied and it follows that

$$r_{m,m+1} = \frac{z_{m+1} - z_m}{1 - z_m z_{m+1}} \geqslant \mathrm{C} > 0, \qquad m = 1 \quad \text{to} \quad \infty,$$
$$z_{m+1} \geqslant \frac{\mathrm{C} + z_m}{1 + \mathrm{C} z_m},$$
$$(1 - z_{m+1}) \leqslant \frac{(1-\mathrm{C})(1-z_m)}{1 + \mathrm{C} z_m} \leqslant (1-\mathrm{C})(1-z_m).$$

Since this holds for all m, we have (1. 4). This completes the proof of Theorem 2.

Since receiving the proofs of this paper, Prof. L. Carleson has kindly shown me the proofs of a very elegant paper of his, to be published in the American Journal of Mathematics, in which he proves that the condition (1.2) is sufficient as well as necessary for z_n to be a u.i.s. However his proof is nonconstructive, so that the present paper, in which an interpolations series is actually constructed, may still have some interest.