

ANNALES DE L'INSTITUT FOURIER

HIROHIKO SHIMA

Homogeneous hessian manifolds

Annales de l'institut Fourier, tome 30, n° 3 (1980), p. 91-128

http://www.numdam.org/item?id=AIF_1980__30_3_91_0

© Annales de l'institut Fourier, 1980, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

HOMOGENEOUS HESSIAN MANIFOLDS

by Hirohiko SHIMA

Introduction.

In [8] [9] [10] we introduced the notion of Hessian manifolds and studied the geometry of such manifolds. We first recall the definition of Hessian manifolds(*). Let M be a flat affine manifold, i.e., M admits open charts $(U_\alpha, \{x_\alpha^1, \dots, x_\alpha^n\})$ such that $M = \cup U_\alpha$ and whose coordinate changes are all affine functions. Such local coordinate systems $\{x_\alpha^1, \dots, x_\alpha^n\}$ will be called affine local coordinate systems. Throughout this paper the local expressions for geometric concepts on M will be given in terms of affine local coordinate systems.

A Riemannian metric g on M is said to be *Hessian* if for each point $p \in M$ there exists a C^∞ -function ϕ defined on a neighbourhood of p such that $g_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}$. Let D denote the covariant differential with respect to the flat affine structure on M . Using D we may define the exterior differentiation for cotangent bundle valued forms. We know that a Riemannian metric g is Hessian if and only if the cotangent bundle valued 1-form g^0 corresponding to g has an exterior differential zero [8];

$$D_X g^0(Y) - D_Y g^0(X) - g^0([X, Y]) = 0$$

for all vector fields X, Y on M . A flat affine manifold provided

(*) In this paper for the sake of brevity we adopt the term of *Hessian* instead of *locally Hessian* used in [8] [9] [10].

with a Hessian metric is called a *Hessian manifold*. As we see (Proposition 0.1), the tangent bundle over a Hessian manifold admits in a natural way a Kählerian structure. Thus the geometry of Hessian manifolds is related with that of certain Kählerian manifolds.

Let M be a Hessian manifold. A diffeomorphism of M onto itself is called an automorphism of M if it preserves both the flat affine structure and the Hessian metric. The set of all automorphisms of M , denoted by $\text{Aut}(M)$, forms a Lie group. A Hessian manifold M is said to be homogeneous if the group $\text{Aut}(M)$ acts transitively on M .

For homogeneous Kählerian manifolds Vinberg and Gindikin proposed the following conjecture and settled the related problems [1] [14].

Every homogeneous Kählerian manifold admits a holomorphic fibering, whose base space is holomorphically isomorphic with a homogeneous bounded domain, and whose fiber is, with the induced Kählerian structure, isomorphic with the direct product of a locally flat homogeneous Kählerian manifold and a simply connected compact homogeneous Kählerian manifold.

In this paper we consider analogous problems for homogeneous Hessian manifolds and obtain the following results.

MAIN THEOREM. — *Let M be a connected homogeneous Hessian manifold. Then we have*

1) *The domain of definition E_x for the exponential mapping \exp_x at $x \in M$ given by the flat affine structure is a convex domain. Moreover E_x is the universal covering manifold of M with affine projection $\exp_x: E_x \rightarrow M$.*

2) *The universal covering manifold E_x of M has a decomposition $E_x = E_x^0 + E_x^+$ where E_x^0 is a uniquely determined vector subspace of the tangent space $T_x M$ of M at x and E_x^+ is an affine homogeneous convex domain not containing any full straight line. Thus E_x admits a unique fibering with the following properties:*

(i) *The base space is E_x^+ .*

(ii) *The projection $p: E_x \rightarrow E_x^+$ is given by the canonical projection from $E_x = E_x^0 + E_x^+$ onto E_x^+ .*

(iii) *The fiber $E_x^0 + v$ through $v \in E_x$ is characterized as the set of all points which can be joined with v by full straight lines contained in E_x . Moreover each fiber is an affine subspace of $T_x M$ and is a Euclidean space with respect to the induced metric.*

(iv) *Every automorphism of E_x is fiber preserving.*

(v) *The group of automorphisms of E_x which preserve every fiber, acts transitively on the fibers.*

COROLLARY 1. — *Let β denote the canonical bilinear form on a connected homogeneous Hessian manifold M ; $\beta_{ij} = \frac{\partial^2 \log F}{\partial x^i \partial x^j}$ where $F = \sqrt{\det [g_{ij}]}$. Then we have*

(i) *β is positive semi-definite.*

(ii) *The null space of β at $x \in M$ coincides with E_x^0 . In particular*

(iii) *$\beta = 0$ if and only if $E_x = T_x M$ and it is a Euclidean space with respect to the induced metric.*

(iv) *β is positive definite if and only if E_x is an affine homogeneous convex domain not containing any full straight line.*

In [5] Kobayashi considered pseudo-distances c_M^a , c_M , d_M^a and d_M on a flat affine (more generally flat projective) manifold M (see also [11]).

COROLLARY 2. — *Let M be a connected homogeneous Hessian manifold and let d be one of the pseudo-distances on E_x listed above. Then the fiber through a point $v \in E_x$ is characterized by the set of all points $w \in E_x$ such that $d(v, w) = 0$. In particular we have:*

(i) *$d = 0$ if and only if $E_x = T_x M$ and it is a Euclidean space with respect to the induced metric.*

(ii) *d is a distance on E_x if and only if E_x is an affine homogeneous convex domain not containing any full straight line.*

COROLLARY 3. — *Let M be a connected homogeneous Hessian manifold. If there is no affine map of \mathbf{R} into M except for constant*

maps, then the universal covering manifold of M is an affine homogeneous convex domain not containing any full straight line.

COROLLARY 4. — *If a connected Lie subgroup G of $\text{Aut}(M)$ acts transitively on a Hessian manifold M and if the isotropy subgroup of G at a point in M is discrete, then G is a solvable Lie group.*

COROLLARY 5. — *If a connected homogeneous Hessian manifold M admits a transitive reductive Lie subgroup of $\text{Aut}(M)$, then the universal covering manifold of M is a direct product of a Euclidean space and an affine homogeneous convex self-dual cone not containing any full straight line.*

COROLLARY 6. — *A compact connected homogeneous Hessian manifold is a Euclidean torus.*

At the conclusion of this introduction we show the relation between Hessian manifolds and Kählerian manifolds. Let M be a flat affine manifold and let $\pi: TM \rightarrow M$ be the tangent bundle over M with projection π . Then the space TM admits in a natural way a complex structure induced by the flat affine structure on M . Indeed, for an affine local coordinate system $\{x^1, \dots, x^n\}$ we put $z^i = y^i + \sqrt{-1} y^{n+i}$ where $y^i = x^i \circ \pi$, $y^{n+i} = dx^i$, $i = 1, \dots, n$. The systems $\{z^1, \dots, z^n\}$ defined as above give a complex structure on TM (cf. [2]).

Let g be a Riemannian metric on M . If we set

$$g^T = \sum_{i,j=1}^n (g_{ij} \circ \pi) dz^i d\bar{z}^j,$$

then g^T is a Hermitian metric on TM (the definition of g^T is independent of the choice of affine local coordinate systems).

PROPOSITION 0.1. — *A Riemannian metric g on M is Hessian if and only if the corresponding Hermitian metric g^T on TM is Kählerian.*

Proof. — Since the fundamental 2-form ρ of the Hermitian metric g^T is expressed locally as

$$\rho = 2 \sum_{i,j=1}^n (g_{ij} \circ \pi) dy^i \wedge dy^{n+j},$$

we know that $d\rho = 0$ if and only if $\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{kj}}{\partial x^i}$, which is equivalent to g being Hessian (cf. [8]). q.e.d.

1. Proof of Main Theorem 1).

In this section we prove the first part of Main Theorem along the same line as Koszul [6] [7]. Let M be a Hessian manifold with Hessian metric g . A C^∞ -function ϕ defined on an open set U in M is called a *primitive* of g on U if it satisfies the condition

$$g_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j} \text{ on a neighbourhood of each point in } U.$$

From now on we always assume that M is a connected homogeneous Hessian manifold.

LEMMA 1.1. — *Let $\{x^1, \dots, x^n\}$ be an affine local coordinate system in U . If ϕ is a primitive of g on U , then $\frac{\partial \phi}{\partial x^j}$ ($j = 1, \dots, n$) are regular rational functions in x^1, \dots, x^n (*).*

Proof. — Let \mathfrak{g} be the Lie algebra of the automorphism group $\text{Aut}(M)$. For $X \in \mathfrak{g}$ we denote by X^* the vector field on M induced by $\exp(-tX)$. For fixed $p \in U$ there exist a neighbourhood W of p in U and elements X_1, \dots, X_n in \mathfrak{g} such that the values of the vector fields X_1^*, \dots, X_n^* at each point $q \in W$ form a basis of the tangent space of M at q . So we have $\frac{\partial}{\partial x^j} = \sum_i \eta_j^i X_i^*$ on W , where each η_j^i is a C^∞ -function on W . Since X_i^* is an infinitesimal affine transformation, the components ξ_i^j of $X_i^* = \sum_j \xi_i^j \frac{\partial}{\partial x^j}$ are affine functions in x^1, \dots, x^n . Therefore η_j^i are rational functions in x^1, \dots, x^n . Since $X^* = \sum_j \xi^j \frac{\partial}{\partial x^j}$ ($X \in \mathfrak{g}$) is an infi-

(*) The author learned this result from Professor Koszul.

tesimal isometry and its components are affine functions, we get

$$\frac{\partial^2 X^* \phi}{\partial x^i \partial x^j} = \sum_p \frac{\partial \xi^p}{\partial x^i} g_{pj} + \sum_p \frac{\partial \xi^p}{\partial x^j} g_{pi} + \sum_p \xi^p \frac{\partial g_{ij}}{\partial x^p} = 0,$$

and so $X^* \phi$ is an affine function in x^1, \dots, x^n . Thus $\frac{\partial \phi}{\partial x^j} = \sum_i \eta_j^i X_i^* \phi$ is a regular rational function in x^1, \dots, x^n on W , and also on U because p is an arbitrary point in U . q.e.d.

We now need the following lemma due to Koszul [7].

LEMMA 1.2. — *Let M be a connected flat affine manifold and let E_x be the domain of definition for the exponential mapping \exp_x at $x \in M$ given by the flat affine structure. Then \exp_x is an affine mapping from E_x to M and its rank is maximum at every point in E_x and equal to $\dim M$. Moreover if E_x is convex it is the universal covering manifold of M with covering projection \exp_x .*

It follows from this lemma that the induced metric $\tilde{g} = \exp_x^* g$ on E_x is Hessian.

LEMMA 1.3. — *There exists a primitive ψ of \tilde{g} on E_x .*

Proof. — Let $\{y^1, \dots, y^n\}$ be an affine coordinate system on $T_x M$. Define a 1-form γ_i on E_x by $\gamma_i = \sum_j \tilde{g}_{ij} dy^j$. We have then $d\gamma_i = \sum_{k < j} \left(\frac{\partial \tilde{g}_{ij}}{\partial y^k} - \frac{\partial \tilde{g}_{ik}}{\partial y^j} \right) dy^k \wedge dy^j = 0$. Since E_x is star-shaped with respect to the origin 0 , by Poincaré Lemma there exists a C^∞ -function h_i on E_x such that $\gamma_i = dh_i$. If we define a 1-form γ on E_x by $\gamma = \sum_i h_i dy^i$, we get $d\gamma = \sum_{j < i} \left(\frac{\partial h_i}{\partial y^j} - \frac{\partial h_j}{\partial y^i} \right) dy^j \wedge dy^i = 0$. Again by Poincaré Lemma there exists a C^∞ -function ψ such that $\gamma = d\psi$. Thus we have $\tilde{g}_{ij} = \frac{\partial^2 \psi}{\partial y^i \partial y^j}$. q.e.d.

LEMMA 1.4 (Koszul [6]). — *Let a be an element in $T_x M$ such that $ta \in E_x$ for $0 \leq t < 1$ and $a \notin E_x$. Then we have*

$$\lim_{t \rightarrow 1} \psi(ta) = \infty,$$

where ψ is a primitive of \tilde{g} on E_x .

Proof. — The length of the curve $\exp_x(ta)$ ($0 \leq t < \theta$) with respect to g is given by

$$l(\theta) = \int_0^\theta g(\exp_x(ta), \exp_x(ta))^{1/2} dt = \int_0^\theta \left(\frac{dF}{dt}\right)^{1/2} dt,$$

where $F(t) = \frac{d}{dt} \psi(ta)$. Since the Riemannian metric g on M is complete because M is homogeneous, we have

$$\lim_{\theta \rightarrow 1} l(\theta) = \lim_{\theta \rightarrow 1} \int_0^\theta \left(\frac{dF}{dt}\right)^{1/2} dt = \infty.$$

For each $0 \leq t_0 < 1$ there exists a primitive ϕ_{t_0} defined on a neighbourhood of $\exp_x(t_0a)$ such that $\psi = \phi_{t_0} \circ \exp_x$ and so by Lemma 1.1 and 1.2 $F(t)$ is a regular rational function in t ($0 \leq t < 1$).

This together with $\lim_{\theta \rightarrow 1} \int_0^\theta \left(\frac{dF}{dt}\right)^{1/2} dt = \infty$ means that $F(t)$ has a pole of order ≥ 1 at $t = 1$. Thus we get

$$\lim_{t \rightarrow 1} \psi(ta) = \lim_{\theta \rightarrow 1} \int_0^\theta F(t) dt + \psi(0) = \infty. \quad \text{q.e.d.}$$

According to Lemma 1.4, Lemma 4.2 in [6] and the fact that E_x is star-shaped with respect to the origin 0 , E_x is a convex domain in T_xM . Moreover by Lemma 1.2 E_x is the universal covering manifold of M with projection $\exp_x: E_x \rightarrow M$. Thus Main Theorem 1) is completely proved.

2. Normal Hessian algebras.

Let Ω be an affine homogeneous domain in \mathbb{R}^n with an invariant Hessian metric g . In this section we first show that Ω admits a simply transitive triangular subgroup of $\text{Aut}(\Omega)$ and using this we construct a normal Hessian algebra (Definition 2.3). According to Theorem 2.1 the study of affine homogeneous domains with invariant Hessian metric is reduced to that of normal Hessian algebras.

Let $A(n)$ denote the group of all affine transformations of \mathbb{R}^n and $\text{Aff}(\Omega)$ the set of all elements in $A(n)$ leaving Ω invariant. Then it is easy to see that $\text{Aff}(\Omega)$ is a closed subgroup of $A(n)$. Denoting by $I(\Omega)$ the group of all isometries of Ω with respect

to the Hessian metric g it follows $\text{Aut}(\Omega) = \text{Aff}(\Omega) \cap \text{I}(\Omega)$. A subgroup of $\text{A}(n)$ is said to be algebraic if it is selected from $\text{A}(n)$ by polynomial equations connecting the coefficients of an affine transformation in an affine coordinate system.

LEMMA 2.1. — *Let N be the normalizer of the identity component of $\text{Aff}(\Omega)$ in $\text{A}(n)$. Then N is algebraic and N , $\text{Aff}(\Omega)$ have the same identity component.*

For the proof see Vinberg [13].

PROPOSITION 2.1. — *The identity component $\text{Aut}_0(\Omega)$ of $\text{Aut}(\Omega)$ coincides with that of an algebraic group in $\text{A}(n)$.*

Proof. — Let $\{x^1, \dots, x^n\}$ be an affine coordinate system on \mathbf{R}^n . For $a \in \text{A}(n)$ we denote by $\mathbf{f}(a) = [\mathbf{f}(a)_j^i]$ and $\mathbf{q}(a) = [\mathbf{q}(a)^i]$ the linear part and the translation part of a respectively, where $x^i \circ a = \sum_j \mathbf{f}(a)_j^i x^j + \mathbf{q}(a)^i$. An element $a \in \text{Aff}(\Omega)$ is contained in $\text{I}(\Omega)$ if and only if $\sum_{r,s} \mathbf{f}(a)_i^r \mathbf{f}(a)_j^s g_{rs}(ap) = g_{ij}(p)$ holds for all $p \in \Omega$. Let ϕ be a primitive of g on Ω . Then by Lemma 1.1 the functions $g_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}$ defined on Ω are rational functions in x^1, \dots, x^n . Therefore we may regard g_{ij} as rational functions on \mathbf{R}^n with respect to x^1, \dots, x^n . Put

$$\text{H} = \left\{ a \in \text{A}(n) \mid \sum_{r,s} \mathbf{f}(a)_i^r \mathbf{f}(a)_j^s g_{rs}(ax) = g_{ij}(x) \text{ for all } x \in \mathbf{R}^n, \right. \\ \left. i, j = 1, \dots, n \right\}.$$

Then H is an algebraic group in $\text{A}(n)$ and $\text{Aut}(\Omega) = \text{Aff}(\Omega) \cap \text{H}$. Therefore by Lemma 2.1 $\text{Aut}_0(\Omega)$ coincides with the identity component of the algebraic group $\text{N} \cap \text{H}$. q.e.d.

PROPOSITION 2.2. — *The isotropy subgroup of $\text{Aut}_0(\Omega)$ at a point in Ω is a maximal compact subgroup of $\text{Aut}_0(\Omega)$.*

Proof. — Let K be the isotropy subgroup of $\text{Aut}_0(\Omega)$ at $p \in \Omega$. Since $\text{Aff}(\Omega)$ and H are closed in $\text{A}(n)$, $\text{Aut}_0(\Omega)$ is closed in $\text{A}(n)$ and so K is closed in $\text{A}(n)$. Let $\{x^1, \dots, x^n\}$ be an affine coordinate system such that $x^i(p) = 0$ and $g_{ij}(p) = \delta_{ij}$

where δ_{ij} is Kronecker's delta. Representing affine transformations in terms of x^1, \dots, x^n it follows $K \subset O(n)$ where $O(n)$ is the orthogonal matrix group. Therefore K is a compact subgroup of $\text{Aut}_0(\Omega)$. Let K' be a maximal compact subgroup of $\text{Aut}_0(\Omega)$ containing K . Then there exists a fixed point $p' \in \Omega$ for K' because Ω is a convex domain. Taking $a \in \text{Aut}_0(\Omega)$ such that $ap' = p$ we get $aK'a^{-1} \subset K$. Since $aK'a^{-1}$ is a maximal compact subgroup of $\text{Aut}_0(\Omega)$ we obtain $K = aK'a^{-1}$ and so K is a maximal compact subgroup of $\text{Aut}_0(\Omega)$. q.e.d.

A subgroup T of $A(n)$ is said to be *triangular* if the linear parts of the transformation in T can be written as upper triangular matrices with respect to some affine coordinate system.

By Proposition 2.1 and by a theorem of Vinberg [12] we get a decomposition $\text{Aut}_0(\Omega) = TK$, where T and K are a maximal connected triangular subgroup and a maximal compact subgroup of $\text{Aut}_0(\Omega)$ respectively, and $T \cap K$ consists of the unit element only. Using this together with Proposition 2.2 we have

PROPOSITION 2.3. — *Let Ω be an affine homogeneous domain in \mathbb{R}^n with an invariant Hessian metric. Then Ω admits a simply transitive triangular subgroup of $\text{Aut}(\Omega)$.*

Choose a point $o \in \Omega$ and an affine coordinate system $\{x^1, \dots, x^n\}$ such that $x^i(o) = 0$ ($i = 1, \dots, n$). Let T be a connected triangular subgroup of $\text{Aut}(\Omega)$ acting simply transitively on Ω and \mathfrak{t} the Lie algebra of T . For $X \in \mathfrak{t}$ we denote by X^* the vector field on Ω induced by a one parameter subgroup of $\exp(-tX)$. We have then $X^* = - \sum_i \left(\sum_j f(X)_j^i x^j + q(X)^i \right) \frac{\partial}{\partial x^i}$, where $f(X)_j^i$ and $q(X)^i$ are constants determined by X . Let V be the tangent space of Ω at o . Define mappings $q: \mathfrak{t} \rightarrow V$ and $f: \mathfrak{t} \rightarrow \mathfrak{gl}(V)$ by

$$q(X) = \sum_i q(X)^i \left(\frac{\partial}{\partial x^i} \right)_o,$$

$$f(X)q(Y) = \sum_{i,j} f(X)_j^i q(X)^j \left(\frac{\partial}{\partial x^i} \right)_o.$$

Then we have

- (1) f is a representation of \mathfrak{t} in V .
 (2) q is a linear isomorphism from \mathfrak{t} onto V satisfying

$$q([X, Y]) = f(X)q(Y) - f(Y)q(X) \quad \text{for } X, Y \in \mathfrak{t}.$$

We now define an operation of multiplication in V by the formula

$$x \cdot y = f(q^{-1}(x))y \quad \text{for } x, y \in V. \quad (3)$$

The algebra V with this multiplication is called *the algebra of the affine homogeneous domain Ω with respect to the point $o \in \Omega$ and the simply transitive connected triangular group T* . Using the notation

$$\begin{aligned} x \cdot y &= L_x y = R_y x, \\ [x \cdot y \cdot z] &= x \cdot (y \cdot z) - (x \cdot y) \cdot z, \end{aligned}$$

from (1) (2) we get

$$[L_x, L_y] = L_{x \cdot y} - L_{y \cdot x}, \quad (4)$$

$$[x \cdot y \cdot z] = [y \cdot x \cdot z], \quad (5)$$

$$[L_x, R_y] = R_{x \cdot y} - R_y R_x, \quad (6)$$

for $x, y, z \in V$. The conditions (4), (5) and (6) are mutually equivalent.

DEFINITION 2.1 – *An algebra satisfying one of the conditions (4) (5) (6) is said to be left symmetric (cf. Vinberg [13]).*

DEFINITION 2.2. – *A left symmetric algebra is said to be normal if all operators L_x have only real eigenvalues (cf. [13]).*

Let \langle, \rangle denote the inner product on V given by the Hessian metric. Then we have

$$\langle x \cdot y, z \rangle + \langle y, x \cdot z \rangle = \langle y \cdot x, z \rangle + \langle x, y \cdot z \rangle \quad (7)$$

for all $x, y, z \in V$ (cf. [8]).

DEFINITION 2.3. – *A left symmetric algebra endowed with an inner product satisfying (7) is called a Hessian algebra.*

Summing up the obtained results, we have

PROPOSITION 2.4. — *Let Ω be an affine homogeneous domain with an invariant Hessian metric. Then the algebra of Ω with respect to a point in Ω and a simply transitive connected triangular group is a normal Hessian algebra.*

Conversely we shall prove that a normal Hessian algebra determines an affine homogeneous domain with an invariant Hessian metric.

Let V be a normal Hessian algebra endowed with an inner product \langle, \rangle . Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of V with respect to \langle, \rangle and $\{x^1, \dots, x^n\}$ the affine coordinate system on V given by $v = \sum_i x^i(v) e_i$ for all $v \in V$. We denote by $f(a) \in GL(V)$ and $q(a) \in V$ the linear part and the translation part of $a \in A(n)$ respectively; $av = f(a)v + q(a)$. For $v \in V$ we define an infinitesimal affine transformation X_v^* by

$$X_v^* = - \sum_{i,j} (L_{v_j}^i x^j + v^i) \frac{\partial}{\partial x^i}, \tag{8}$$

where $L_{v_j}^i, v^i$ are the components of L_v, v with respect to $\{e_1, \dots, e_n\}$; $L_v e_j = \sum_i L_{v_j}^i e_i, v = \sum_i v^i e_i$. From (4) it follows

$$[X_v^*, X_w^*] = X_{v.w - w.v}^* \text{ for } v, w \in V, \tag{9}$$

and so $\mathfrak{t}(V) = \{X_v^* \mid v \in V\}$ forms a Lie algebra. Let $T(V)$ denote the connected Lie subgroup of $A(n)$ generated by $\mathfrak{t}(V)$. We denote by $\Omega(V)$ the open orbit of $T(V)$ through the origin 0 ; $\Omega(V) = T(V)0$, which we call *the affine homogeneous domain corresponding to V* .

We first show that $T(V)$ acts simply transitively on $\Omega(V)$. By (8) the isotropy subgroup B of $T(V)$ at 0 is discrete. Suppose $b \in B$. Since the exponential mapping $\exp: \mathfrak{t}(V) \rightarrow T(V)$ is surjective because $T(V)$ is triangular, there exists $X_w^* \in \mathfrak{t}(V)$ such that $b = \exp X_w^*$. If we put $b' = \exp 1/2 X_w^*$, then we have $0 = b0 = b'^2 0 = f(b')q(b') + q(b')$ and so $f(b')q(b') = -q(b')$. Since $f(b') = \exp(-1/2 L_w)$ and since L_w is triangular, the eigenvalues of $f(b')$ are all positive. This means $b'0 = q(b') = 0$ and so $b' = \exp 1/2 X_w^* \in B$. By the same argument we have $\exp 1/2^n X_w^* \in B$ for all non-negative integer n . Thus $X_w^* = 0$ because B is discrete. Therefore B consists of the unit element only and $T(V)$ acts simply transitively on $\Omega(V)$.

Now we denote by g the $T(V)$ -invariant Riemannian metric on $\Omega(V)$ satisfying $g_{ij}(0) = \delta_{ij}$ (Kronecker's delta). It follows then

$$g_{ij}(a0) = \sum_p \mathbf{f}(a^{-1})_i^p \mathbf{f}(a^{-1})_j^p \quad \text{for } a \in T(V) \quad (10)$$

where $\mathbf{f}(a)_i^j$ are the components of $\mathbf{f}(a)$ with respect to $\{e_1, \dots, e_n\}$. Denoting by $\exp tX_v^*$ the one parameter group generated by X_v^* we get $\left. \frac{d}{dt} \right|_{t=0} \mathbf{f}(\exp tX_v^*) = -L_v$ and $\left. \frac{d}{dt} \right|_{t=0} \mathbf{q}(\exp tX_v^*) = -v$. Choose an element $a \in T(V)$ and define an isomorphism $v \rightarrow v'$ of V by $a^{-1} \exp tX_v^* a = \exp tX_{v'}^*$. Then we have

$$\begin{aligned} v' &= \mathbf{f}(a)^{-1} L_v \mathbf{q}(a) + \mathbf{f}(a)^{-1} v = L_{v'} \mathbf{f}(a)^{-1} \mathbf{q}(a) + \mathbf{f}(a)^{-1} v, \\ L_{v'} &= \mathbf{f}(a)^{-1} L_v \mathbf{f}(a). \end{aligned} \quad (11)$$

Let D denote the natural flat linear connection on $\Omega(V)$ given by $Ddx^i = 0$. Put $A_{X^*} = L_{X^*} - D_{X^*}$ where L_{X^*} and D_{X^*} are the Lie differentiation and the covariant differentiation by a vector field X^* respectively. We have

$$(A_{X_u^*} X_v^*)_x = - \sum_i (L_u L_v x + L_u v)^i \left(\frac{\partial}{\partial x^i} \right)_x, \quad (12)$$

for all $x \in \Omega(V)$. Since $A_{X_u^*}$ is a derivation of the algebra of tensor fields and maps every function into zero and since $L_{X^*} g = 0$, it follows

$$(D_{X_u^*} g)(X_v^*, X_w^*) = g(A_{X_u^*} X_v^*, X_w^*) + g(X_v^*, A_{X_u^*} X_w^*). \quad (13)$$

Using (10) (11) (12) we obtain

$$\begin{aligned} g(a0) ((A_{X_u^*} X_v^*)_{a0}, (X_w^*)_{a0}) &= \sum_{i,j,p} \mathbf{f}(a^{-1})_i^p \mathbf{f}(a^{-1})_j^p (L_u L_v a0 + L_u v)^i (L_w a0 + w)^j \\ &= \sum_p (\mathbf{f}(a^{-1}) (L_u L_v \mathbf{q}(a) + L_u v))^p (\mathbf{f}(a^{-1}) (L_w \mathbf{q}(a) + w))^p \\ &= \sum_p (L_{u'} L_{v'} \mathbf{f}(a)^{-1} \mathbf{q}(a) + L_{u'} \mathbf{f}(a)^{-1} v)^p (L_{w'} \mathbf{f}(a)^{-1} \mathbf{q}(a) \\ &\quad + \mathbf{f}(a)^{-1} w)^p \\ &= \sum_p (u' \cdot v')^p w'^p \\ &= \langle u' \cdot v', w' \rangle. \end{aligned}$$

This together with (7) (13) implies

$$(D_{X_u^*} g)(X_v^*, X_w^*) = (D_{X_v^*} g)(X_u^*, X_w^*),$$

and so g is a Hessian metric (cf. [8]).

Let Ω be an affine homogeneous domain in \mathbf{R}^n with an invariant Hessian metric and V the normal Hessian algebra of Ω with respect to $0 \in \Omega$ and a simply transitive triangular group. Identifying the tangent space V of Ω at 0 with \mathbf{R}^n the domain $\Omega(V)$ corresponding to V coincides with Ω . Therefore we have

THEOREM 2.1. — *Let V be a normal Hessian algebra. Then the domain $\Omega(V)$ constructed as above is an affine homogeneous domain with invariant Hessian metric. All affine homogeneous domains with invariant Hessian metric are obtained in this way.*

DEFINITION 2.4 (cf. [3]). — *A normal left symmetric algebra U is called a clan if it admits a linear function ω satisfying the condition*

- (i) $\omega(x \cdot y) = \omega(y \cdot x)$ for all $x, y \in U$,
- (ii) $\omega(x \cdot x) > 0$ for all $x \neq 0 \in U$.

Remark. — Let U be a clan with ω . If we put $\langle x, y \rangle = \omega(x \cdot y)$, then \langle, \rangle is an inner product on U satisfying the condition (7) and so U is a normal Hessian algebra.

The following theorem is due to Vinberg [13].

THEOREM 2.2. — *Let V be a clan. Then the domain $\Omega(V)$ is an affine homogeneous convex domain not containing any full straight line. All affine homogeneous convex domains not containing any full straight line are obtained in this way.*

3. Structure of normal Hessian algebras.

In this section we state a fundamental theorem for normal Hessian algebras. Let V be a normal Hessian algebra.

DEFINITION 3.1. — *Let W be a vector subspace of V .*

- (a) W is called a commutative subalgebra of V if $W \cdot W = \{0\}$.

(b) W is said to be an ideal of V if $W \cdot V \subset W$ and $V \cdot W \subset W$.

THEOREM 3.1. — Let V be a normal Hessian algebra. Then V is decomposed into the semi-direct sum $V = I + U$, where I is a commutative ideal of V and U is a subalgebra with an element s satisfying the following properties:

(i) $s \cdot s = s$,

(ii) the restriction of L_s on U is diagonalizable and has eigenvalues $1, 1/2$,

(iii) $R_s = 2L_s - 1$ on U ,

where 1 is the identity transformation of U . (An element s in U satisfying the above conditions is called a principal idempotent of U .)

The proof of this theorem is carried out by induction on the dimension of normal Hessian algebras in an analogous way as Gindikin and Vinberg [1] [14].

For later use we prepare some lemmas.

LEMMA 3.1. — Let W be an ideal of V . Then the orthogonal complement W^\perp of W in V is a subalgebra.

Proof. — Let $x, y \in W^\perp$ and $a \in W$. We have then

$$\langle a, x \cdot y \rangle = -\langle x \cdot a, y \rangle + \langle a \cdot x, y \rangle + \langle x, a \cdot y \rangle = 0.$$

This implies $x \cdot y \in W^\perp$.

q.e.d.

LEMMA 3.2. — Let u be a non-zero element in V and let $P = \{p \in V \mid p \cdot u = 0\}$. Suppose P is invariant by L_u . Then for $p \in P, x \in V$ we have

(i) $L_u(p \cdot x) = (L_u p) \cdot x + p \cdot (L_u x)$,

(ii) $\exp tL_u(p \cdot x) = (\exp tL_u p) \cdot (\exp tL_u x)$,

(iii) $\frac{d}{dt} \langle \exp tL_u p, \exp tL_u x \rangle = \langle u, \exp tL_u(p \cdot x) \rangle$.

Proof. — (i) follows from

$$u \cdot (p \cdot x) = (u \cdot p) \cdot x + p \cdot (u \cdot x) - (p \cdot u) \cdot x.$$

(ii) is a consequence of (i). Using (7) in 2 and (ii) we obtain

$$\begin{aligned} \frac{d}{dt} \langle \exp tL_u p, \exp tL_u x \rangle &= \langle L_u \exp tL_u p, \exp tL_u x \rangle + \langle \exp tL_u p, L_u \exp tL_u x \rangle \\ &= \langle (\exp tL_u p) \cdot u, \exp tL_u x \rangle + \langle u, (\exp tL_u p) \cdot (\exp tL_u x) \rangle \\ &= \langle u, \exp tL_u(p \cdot x) \rangle. \end{aligned} \quad \text{q.e.d.}$$

LEMMA 3.3. — *Let W be a subspace of V . Suppose that an element $a \neq 0 \in V$ satisfies the following conditions :*

- (a) $a \cdot a = \epsilon a$, where $\epsilon = 0, 1$,
- (b) L_a and R_a leave W invariant,
- (c) a is orthogonal to $W \cdot W$.

Then we have:

(i) If $\epsilon = 0$, $L_a = R_a = 0$ on W .

(ii) If $\epsilon = 1$, the restriction of L_a on W is symmetric and its eigenvalues are $0, 1/2$. Moreover $R_a = 2L_a$ on W .

Proof. — From (6) in 2, (a) and (b) it follows

$$[L_a, R_a] = \epsilon R_a - R_a^2 \text{ on } W. \quad (1)$$

By (c) we have

$$\langle a \cdot x, y \rangle + \langle x, a \cdot y \rangle = \langle x \cdot a, y \rangle + \langle a, x \cdot y \rangle = \langle x \cdot a, y \rangle$$

for all $x, y \in W$. This implies

$$L_a + {}^tL_a = R_a \text{ on } W. \quad (2)$$

Put $S = \epsilon R_a - R_a^2$. S being commutative with R_a we have $\text{Tr}_W S^2 = \text{Tr}_W [L_a, R_a] S = \text{Tr}_W [L_a S, R_a] = 0$. This means $S = 0$ on W because S is symmetric on W by (2) and so

$$R_a^2 = \epsilon R_a \text{ on } W, [L_a, R_a] = 0 \text{ on } W. \quad (3)$$

Suppose $\epsilon = 0$. The facts that R_a is symmetric on W and that $R_a^2 = 0$ on W imply $R_a = 0$ on W . Using this and (2), L_a is skew symmetric on W and its eigenvalues are purely imaginary. Therefore we must have $L_a = 0$ on W . Suppose $\epsilon = 1$. Since $R_a^2 = R_a$ on W the eigenvalues of R_a on W are $0, 1$. From (2) it follows $L_a - {}^tL_a = 2L_a - R_a$ on W . Since $[L_a, R_a] = 0$

on W and since the eigenvalues of L_a, R_a on W are real, the eigenvalues of $2L_a - R_a$ on W are real. On the other hand $L_a - {}^tL_a$ is skew symmetric and its eigenvalues are purely imaginary. Therefore we have $L_a - {}^tL_a = 2L_a - R_a = 0$ on W and so ${}^tL_a = L_a$ on W , $R_a = 2L_a$ on W . This means (ii). q.e.d.

The following lemmas 3.4*-3.7* are immediate consequences of Theorem 3.1.

LEMMA 3.4.* - Let U_λ denote the eigenspaces of L_s on U corresponding to λ . Then we have:

- (i) $U = U_1 + U_{1/2}$,
 $U_\lambda \cdot U_\mu \subset U_{\mu-\lambda+1}$.
(ii) U is a clan.

Proof. - For $x \in U_\lambda$, $y \in U_\mu$ we have

$$\begin{aligned} s \cdot (x \cdot y) &= (s \cdot x) \cdot y + x \cdot (s \cdot y) - (x \cdot s) \cdot y \\ &= \lambda x \cdot y + \mu x \cdot y - (2\lambda - 1)x \cdot y = (\mu - \lambda + 1)x \cdot y \end{aligned}$$

and so $x \cdot y \in U_{\mu-\lambda+1}$. Define a linear function ω on U by

$$\omega(x) = \frac{1}{\lambda} \langle s, x \rangle \text{ for } x \in U_\lambda.$$

Let $x \in U_\lambda$, $y \in U_\mu$. Using

$$\langle s \cdot x, y \rangle + \langle x, s \cdot y \rangle = \langle x \cdot s, y \rangle + \langle s, x \cdot y \rangle,$$

$\mu - \lambda + 1 \neq 0$ and $x \cdot y \in U_{\mu-\lambda+1}$ we get

$$\langle x, y \rangle = \frac{1}{\mu - \lambda + 1} \langle s, x \cdot y \rangle = \omega(x \cdot y).$$

Thus we have $\langle x, y \rangle = \omega(x \cdot y)$ for all $x, y \in U$. Therefore U is a clan. q.e.d.

LEMMA 3.5.* - (i) The restriction of L_s on I is symmetric and its eigenvalues are $0, 1/2$.

(ii) Let I_λ denote the eigenspace of L_s on I corresponding to λ . Then we have $I = I_0 + I_{1/2}$,

$$U_\lambda \cdot I_\mu \subset I_{\mu-\lambda+1}, \quad I_\lambda \cdot U_\mu \subset I_{\mu-\lambda}.$$

(iii) $R_s = 2L_s$ on I .

Proof. — Since I is a commutative ideal of V and since $s \cdot s = s$, applying Lemma 3.3 it follows that the restriction of L_s on I is symmetric and its eigenvalues are $0, 1/2$ and moreover $R_s = 2L_s$ on I . Let $x \in U_\lambda, a \in I_\mu$. By Theorem 3.1 (iii) we obtain

$$\begin{aligned} s \cdot (x \cdot a) &= (s \cdot x) \cdot a + x \cdot (s \cdot a) - (x \cdot s) \cdot a \\ &= \lambda x \cdot a + \mu x \cdot a - (2\lambda - 1) x \cdot a = (\mu - \lambda + 1) x \cdot a \end{aligned}$$

and $x \cdot a \in I_{\mu-\lambda+1}$. Let $a \in I_\lambda, x \in U_\mu$. By (iii) we have

$$\begin{aligned} s \cdot (a \cdot x) &= (s \cdot a) \cdot x + a \cdot (s \cdot x) - (a \cdot s) \cdot x \\ &= \lambda a \cdot x + \mu a \cdot x - 2\lambda a \cdot x = (\mu - \lambda) a \cdot x \end{aligned}$$

and so $a \cdot x \in I_{\mu-\lambda}$. q.e.d.

LEMMA 3.6*. — *The commutative ideal I of V is characterized by the set of all points $x \in V$ such that $x \cdot x = 0$.*

Proof. — Suppose $x \cdot x = 0$. If $x = a + y$ where $a \in I$ and $y \in U$, we have $0 = x \cdot x = a \cdot y + y \cdot a + y \cdot y$ and so $y \cdot y = 0$. By Lemma 3.4* (ii) there exists a linear function ω on U satisfying the conditions in Definition 2.4. Since $\omega(y \cdot y) = 0$, we have $y = 0$ and $x = a \in I$. q.e.d.

LEMMA 3.7*. — *The subspaces $I_0, I_{1/2}$ and U are mutually orthogonal with respect to \langle, \rangle .*

Proof. — By Lemma 3.5* (i) I_0 and $I_{1/2}$ are orthogonal. For $a \in I_\lambda$ we have

$$0 = \langle s \cdot a, s \rangle + \langle a, s \cdot s \rangle - \langle a \cdot s, s \rangle - \langle s, a \cdot s \rangle = (-3\lambda + 1) \langle a, s \rangle$$

and so $\langle a, s \rangle = 0$ because $\lambda = 0, 1/2$. This implies s and I are orthogonal. Applying this, for $a \in I_\lambda, x \in U_\mu$ we obtain

$$0 = \langle s \cdot a, x \rangle + \langle a, s \cdot x \rangle - \langle a \cdot s, x \rangle - \langle s, a \cdot x \rangle = (\mu - \lambda) \langle a, x \rangle$$

$$\begin{aligned} \text{and } 0 &= \langle s \cdot x, a \rangle + \langle x, s \cdot a \rangle - \langle x \cdot s, a \rangle - \langle s, x \cdot a \rangle \\ &= (\lambda - \mu + 1) \langle a, x \rangle. \end{aligned}$$

This shows $\langle a, x \rangle = 0$. Therefore I and U are orthogonal. q.e.d.

4. The case $u \cdot u = u$.

Since V is a normal left symmetric algebra, by Lie's Theorem there exists an element $u \neq 0 \in V$ such that $x \cdot u = \kappa(x)u$ for all $x \in V$, where κ is a linear function on V . Multiplying u by non-zero scalar (if necessary) the following two cases are possible ;

$$\begin{aligned} u \cdot u &= u, \\ u \cdot u &= 0. \end{aligned}$$

In this section we consider the case $u \cdot u = u$ and prove the following.

PROPOSITION 4.1. — *Suppose $u \cdot u = u$. Then the operator L_u is diagonalizable and has eigenvalues $0, 1/2, 1$. Denoting by V_λ the eigenspace of L_u corresponding to λ we have:*

- (i) $V = V_1 + V_{1/2} + V_0$ (orthogonal decomposition).
- (ii) $V_1 = \{u\}$.
- (iii) $u \cdot p = \frac{1}{2} p$, $p \cdot u = 0$ for $p \in V_{1/2}$.
- (iv) $u \cdot q = 0$, $q \cdot u = 0$ for $q \in V_0$.
- (v) $V_0 \cdot V_{1/2} \subset V_{1/2}$, $V_{1/2} \cdot V_0 \subset V_{1/2}$,
 $V_0 \cdot V_0 \subset V_0$, $V_{1/2} \cdot V_{1/2} \subset V_1$.

In particular $V_1 + V_{1/2}$ is an ideal of V with principal idempotent u and V_0 is a subalgebra.

Let P denote the kernel of R_u ;

$$P = \{p \in V \mid p \cdot u = 0\}. \quad (1)$$

Then we have

$$L_u P \subset P, \quad (2)$$

$$V = \{u\} + P. \quad (3)$$

Indeed for $p \in P$ we have

$$(u \cdot p) \cdot u = u \cdot (p \cdot u) + (p \cdot u) \cdot u - p \cdot (u \cdot u) = 0,$$

which implies (2). (3) follows from $x - \kappa(x)u \in P$ for all $x \in V$.

LEMMA 4.1. — *The restriction of L_u on P is diagonalizable and has eigenvalues $0, 1/2$.*

Proof. – By Lemma 3.2 for $p \in P$ we have

$$\frac{d}{dt} \langle \exp tL_u p, \exp tL_u u \rangle = \langle u, \exp tL_u(p \cdot u) \rangle = 0,$$

and so

$$\langle \exp tL_u p, u \rangle = ae^{-t}, \tag{4}$$

where a is a constant determined by p not depending on t . Using this for $x = cu + p \in V$ ($c \in \mathbf{R}$, $p \in P$) we obtain

$$\begin{aligned} \langle u, \exp tL_u x \rangle &= \langle u, ce^t u + \exp tL_u p \rangle \\ &= \langle u, \exp tL_u p \rangle + c \langle u, u \rangle e^t = ae^{-t} + be^t, \end{aligned} \tag{5}$$

where a, b are constants determined by x not depending on t . Applying Lemma 3.2 and (5) we have for $p, q \in P$

$$\frac{d}{dt} \langle \exp tL_u p, \exp tL_u q \rangle = \langle u, \exp tL_u(p \cdot q) \rangle = ae^{-t} + be^t,$$

and consequently

$$\langle \exp tL_u p, \exp tL_u q \rangle = -ae^{-t} + be^t + c, \tag{6}$$

where a, b and c are constants determined by p, q not depending on t . From (6) it follows that L_u is diagonalizable on P . Indeed, if L_u is not diagonalizable on P there exist non-zero elements $p, q \in P$ such that $L_u p = \lambda p$, $L_u q = \lambda q + p$. We have then

$$\begin{aligned} \langle \exp tL_u p, \exp tL_u q \rangle &= \langle e^{\lambda t} p, e^{\lambda t} q + te^{\lambda t} p \rangle \\ &= te^{2\lambda t} \langle p, p \rangle + e^{2\lambda t} \langle p, q \rangle, \end{aligned}$$

which contradicts to (6). Let λ be an eigenvalue of L_u on P and $p \neq 0 \in P$ an eigenvector corresponding to λ . It follows then

$$\frac{d}{dt} \langle \exp tL_u p, \exp tL_u p \rangle = 2\lambda \langle p, p \rangle e^{2\lambda t}.$$

On the other hand (6) implies

$$\frac{d}{dt} \langle \exp tL_u p, \exp tL_u p \rangle = ae^{-t} + be^t.$$

Therefore we obtain

$$2\lambda \langle p, p \rangle e^{2\lambda t} = ae^{-t} + be^t, \tag{7}$$

consequently $\lambda = 0, 1/2, -1/2$. By (4) we get $\langle p, u \rangle e^{(\lambda+1)t} = a$, so $\langle p, u \rangle = 0$ and $a = 0$ because $\lambda + 1 \neq 0$. Thus we have

$$\langle p, u \rangle = 0 \quad \text{for all } p \in P, \quad (4')$$

$$\langle u, \exp tL_u x \rangle = be^t \quad \text{for } x \in V, \quad (5')$$

$$2\lambda \langle p, p \rangle e^{2\lambda t} = be^t. \quad (7')$$

(7') shows $\lambda = 0, 1/2$. q.e.d.

Let P_λ denote the eigenspace of L_u in P corresponding to λ . From Lemma 4.1 and (3) it follows

$$V = V_1 + V_{1/2} + V_0, \quad (8)$$

where $V_1 = \{u\}$, $V_{1/2} = P_{1/2}$ and $V_0 = P_0$.

LEMMA 4.2. — *The decomposition (8) is orthogonal and we have $P_\lambda \cdot P_\mu \subset V_{\lambda+\mu}$.*

Proof. — For $p \in P_\lambda$ and $q \in P_\mu$ we have

$$u \cdot (p \cdot q) = (u \cdot p) \cdot q + p \cdot (u \cdot q) - (p \cdot u) \cdot q = (\lambda + \mu) p \cdot q.$$

This implies $P_\lambda \cdot P_\mu \subset V_{\lambda+\mu}$. The orthogonality of $\{u\}$ and P follows from (4'). Applying this for $p \in P_{1/2}$ and $q \in P_0$ we obtain $1/2 \langle p, q \rangle = \langle u \cdot p, q \rangle = -\langle p, u \cdot q \rangle + \langle p \cdot u, q \rangle + \langle u, p \cdot q \rangle = 0$ because $p \cdot q \in P_{1/2}$. Thus $P_{1/2}$ and P_0 are orthogonal. q.e.d.

The assertion of Proposition 4.1 follows from Lemma 4.2 and (8).

5. The case $u \cdot u = 0$.

The purpose of this section is to prove the following.

PROPOSITION 5.1. — *Suppose $u \cdot u = 0$. Then there exists a commutative ideal of V containing u .*

LEMMA 5.1. — $L_u^2 = 0$.

Proof. — Let P denote the kernel of R_u ; $P = \{p \in V \mid p \cdot u = 0\}$. Then we have

$$L_u V \subset P, \quad (1)$$

because $(u \cdot x) \cdot u = u \cdot (x \cdot u) + (x \cdot u) \cdot u - x \cdot (u \cdot u) = 0$ for all $x \in V$. For $p \in P$, $x \in V$ it follows from (1) and Lemma 3.2

$$\begin{aligned} \frac{d^3}{dt^3} \langle \exp tL_u p, \exp tL_u x \rangle &= \frac{d^2}{dt^2} \langle u, \exp tL_u(p \cdot x) \rangle = \langle u, u \cdot p' \rangle \\ &= -\langle p', u \cdot u \rangle + \langle p' \cdot u, u \rangle + \langle u, p' \cdot u \rangle = 0, \end{aligned}$$

where $p' = L_u \exp tL_u(p \cdot x) \in P$, and consequently

$$\langle \exp tL_u p, \exp tL_u x \rangle = at^2 + bt + c, \tag{2}$$

where a, b, c are constants independent of t . Let λ be an eigenvalue of L_u on P and $p \neq 0 \in P$ an eigenvector corresponding to λ . By (2) we get $e^{2\lambda t} \langle p, p \rangle = at^2 + bt + c$, and so $\lambda = 0$. This together with (1) implies that the eigenvalues of L_u are equal to 0. Assume $L_u^2 \neq 0$. Then there exist non-zero elements $x, y, z \in V$ such that $u \cdot x = 0, u \cdot y = x, u \cdot z = y$. From this we have $\exp tL_u y = y + tx, \exp tL_u z = z + ty + \frac{t^2}{2}x$. Since $y = u \cdot z \in P$, applying (2) we obtain $\langle y + tx, z + ty + \frac{t^2}{2}x \rangle = at^2 + bt + c$. This is a contradiction because $\langle x, x \rangle \neq 0$. Thus we have $L_u^2 = 0$.
q.e.d.

Using $L_u^2 = 0$ we define a filtration of V . Consider the subspaces of V

$$\begin{aligned} V^{(-1)} &= V, \\ V^{(0)} &= \{x \in V \mid L_u x \in \{u\}\}, \\ V^{(1)} &= L_u V + \{u\}, \\ V^{(2)} &= \{u\}. \end{aligned}$$

Then we have

LEMMA 5.2. — *The subspaces $V^{(i)}$ form a filtration of the algebra V ;*

- (i) $V^{(-1)} \supset V^{(0)} \supset V^{(1)} \supset V^{(2)}$,
- (ii) $V^{(i)} \cdot V^{(j)} \subset V^{(i+j)}$.

Moreover we have

$$(iii) \quad V^{(1)} \cdot V^{(1)} = \{0\}.$$

Proof. – (i) follows from $u \cdot u = 0$ and $L_u^2 = 0$. Note that

$$(u \cdot x) \cdot (u \cdot y) = 0 \quad \text{for all } x, y \in V. \quad (3)$$

In fact for $x, y \in V$ we have

$$\begin{aligned} 0 &= u \cdot (u \cdot (x \cdot y)) = u \cdot ((u \cdot x) \cdot y + x \cdot (u \cdot y) - (x \cdot u) \cdot y) \\ &= u \cdot ((u \cdot x) \cdot y) + u \cdot (x \cdot (u \cdot y)) - \kappa(x) u \cdot (u \cdot y) = (u \cdot (u \cdot x)) \cdot y \\ &\quad + (u \cdot x) \cdot (u \cdot y) - ((u \cdot x) \cdot u) \cdot y + (u \cdot x) \cdot (u \cdot y) + x \cdot (u \cdot (u \cdot y)) \\ &\quad - (x \cdot u) \cdot (u \cdot y) = 2(u \cdot x) \cdot (u \cdot x) \end{aligned}$$

because $L_u^2 = 0$, $V \cdot u \subset \{u\}$ and $L_u V \subset P$. Let

$$u \cdot x + \lambda u, u \cdot y + \mu u \in V^{(1)} \quad (x, y \in V, \lambda, \mu \in \mathbf{R}).$$

Using (1) and (3) we get

$$\begin{aligned} (u \cdot x + \lambda u) \cdot (u \cdot y + \mu u) &= (u \cdot x) \cdot (u \cdot y) + \mu(u \cdot x) \cdot u \\ &\quad + \lambda u \cdot (u \cdot y) + \lambda \mu u \cdot u \\ &= 0. \end{aligned}$$

This implies (iii). Let $x \in V^{(0)}$, $u \cdot y + \mu u \in V^{(1)}$ ($y \in V$, $\mu \in \mathbf{R}$). We have then $u \cdot x = \nu u$ ($\nu \in \mathbf{R}$) and

$$\begin{aligned} x \cdot (u \cdot y + \mu u) &= x \cdot (u \cdot y) + \mu x \cdot u = (x \cdot u) \cdot y + u \cdot (x \cdot y) \\ &\quad - (u \cdot x) \cdot y + \mu x \cdot u \\ &= \kappa(x) u \cdot y + u \cdot (x \cdot y) - \nu u \cdot y + \mu \kappa(x) u \in V^{(1)}. \end{aligned}$$

In the same way $(u \cdot y + \mu u) \cdot x \in V^{(1)}$. Therefore we have

$$V^{(0)} \cdot V^{(1)} \subset V^{(1)}, \quad V^{(1)} \cdot V^{(0)} \subset V^{(1)}. \quad (4)$$

Let $u \cdot x + \mu u \in V^{(1)}$ ($x \in V$, $\mu \in \mathbf{R}$) and $y \in V^{(-1)}$. By (iii) we have

$$\begin{aligned} u \cdot ((u \cdot x + \mu u) \cdot y) &= u \cdot ((u \cdot x) \cdot y) + \mu u \cdot (u \cdot y) = (u \cdot (u \cdot x)) \cdot y \\ &\quad + (u \cdot x) \cdot (u \cdot y) - ((u \cdot x) \cdot u) \cdot y + \mu u \cdot (u \cdot y) = 0 \end{aligned}$$

$$\begin{aligned} \text{and } u \cdot (y \cdot (u \cdot x + \mu u)) &= u \cdot (y \cdot (u \cdot x)) + \mu u \cdot (y \cdot u) \\ &= (u \cdot y) \cdot (u \cdot x) + y \cdot (u \cdot (u \cdot x)) - (y \cdot u) \cdot (u \cdot x) \\ &\quad + \mu u \cdot (y \cdot u) \\ &= 0. \end{aligned}$$

This implies

$$V^{(1)} \cdot V^{(-1)} \subset V^{(0)}, \quad V^{(-1)} \cdot V^{(1)} \subset V^{(0)}. \quad (5)$$

Let $x, y \in V^{(0)}$. We have then $u \cdot x = \mu u$, $u \cdot y = \nu u$ and so

$$\begin{aligned} u \cdot (x \cdot y) &= (u \cdot x) \cdot y + x \cdot (u \cdot y) - (x \cdot u) \cdot y = \mu \nu u \\ &\quad + \nu \kappa(x) u - \kappa(x) \nu u = \mu \nu u. \end{aligned}$$

This means

$$V^{(0)} \cdot V^{(0)} \subset V^{(0)}. \tag{6}$$

The other relations $V^{(i)} \cdot V^{(j)} \subset V^{(i+j)}$ are trivial. q.e.d.

If $V^{(0)} = V$, then $V^{(2)} = \{u\}$ is a commutative ideal of V and consequently Proposition 5.1 is proved. From now on we assume $V^{(0)} \neq V$. Since $V^{(0)}$ is a subalgebra of dimension less than $\dim V$, by the inductive hypothesis we have $V^{(0)} = I + U$, where I is a commutative ideal of $V^{(0)}$ and U is a subalgebra with a principal idempotent s .

LEMMA 5.3. — $V^{(1)} \subset I$.

Proof. — According to Lemma 3.6* it follows

$$I = \{x \in V^{(0)} \mid x \cdot x = 0\}.$$

This and $V^{(1)} \cdot V^{(1)} = \{0\}$ imply $V^{(1)} \subset I$. q.e.d.

LEMMA 5.4. — $V \cdot I \subset V^{(0)}$, $I \cdot V \subset V^{(0)}$.

Proof. — Let $x \in V$ and $a \in I$. Since I is commutative and since $u, u \cdot x, x \cdot u \in I$ by Lemma 5.2 and 5.3, we have

$$u \cdot (x \cdot a) = (u \cdot x) \cdot a + x \cdot (u \cdot a) - (x \cdot u) \cdot a = 0$$

and $u \cdot (a \cdot x) = (u \cdot a) \cdot x + a \cdot (u \cdot x) - (a \cdot u) \cdot x = 0$.

This means $x \cdot a, a \cdot x \in V^{(0)}$. q.e.d.

If $I = V^{(0)}$, Lemma 5.4 implies that I is a commutative ideal of V containing u and Proposition 5.1 is proved. Henceforth we assume $I \neq V^{(0)}$, i.e., $U \neq \{0\}$.

Let s be a principal idempotent of U . Since $V^{(1)} \subset I$ and since $V^{(1)}$ is invariant by L_s and R_s , by Lemma 3.3 we have:

The restriction of L_s on $V^{(1)}$ is symmetric and its eigenvalues are $0, 1/2$. Therefore denoting by $V_\lambda^{(1)}$ the eigenspace of L_s corresponding to λ we obtain the orthogonal decomposition

$$V^{(1)} = V_0^{(1)} + V_{1/2}^{(1)}. \tag{7}$$

$$R_s = 2L_s \text{ on } V^{(1)}. \tag{8}$$

We set $s \cdot u = \alpha u$. From (8) it follows $u \cdot s = 2s \cdot u = 2\alpha u$. Thus

$$\begin{aligned} L_s u &= \alpha u, \\ R_s u &= 2\alpha u, \text{ where } \alpha = 0, 1/2. \end{aligned} \tag{9}$$

Consider the graded algebra \bar{V} associated to the filtered algebra V : $\bar{V} = \bar{V}^{(-1)} + \bar{V}^{(0)} + \bar{V}^{(1)} + \bar{V}^{(2)}$, where $\bar{V}^{(i)} = V^{(i)} / V^{(i+1)}$ ($-1 \leq i \leq 1$) and $\bar{V}^{(2)} = V^{(2)}$. For $x \in V^{(i)}$ we denote by \bar{x} the element in $\bar{V}^{(i)}$ corresponding to x and by $L_{\bar{x}}$ (resp. $R_{\bar{x}}$) the left (resp. right) multiplication by \bar{x} .

LEMMA 5.5. – (i) *The mapping $L_{\bar{u}}: \bar{V}^{(-1)} \longrightarrow \bar{V}^{(1)}$ is an isomorphism.*

(ii) $L_{\bar{s}} L_{\bar{u}} = L_{\bar{u}} (L_{\bar{s}} - \alpha)$ on $\bar{V}^{(-1)}$. *In particular the restriction of $L_{\bar{s}}$ on $\bar{V}^{(-1)}$ is diagonalizable and its eigenvalues are α , $\alpha + 1/2$.*

(iii) $R_{\bar{s}} L_{\bar{u}} = L_{\bar{u}} R_{\bar{s}}$ on $\bar{V}^{(-1)}$.

Proof. – The mapping $L_{\bar{u}}: \bar{V}^{(-1)} \longrightarrow \bar{V}^{(1)}$ is surjective because $\bar{V}^{(1)} = L_{\bar{u}} V + \{u\} / \{u\}$. Suppose $L_{\bar{u}} \bar{x} = 0$ ($x \in V^{(-1)}$). Then it follows $u \cdot x \in \{u\}$, consequently $x \in V^{(0)}$ and $\bar{x} = 0$. Thus (i) is proved. By (9) we have

$$\begin{aligned} L_{\bar{s}} L_{\bar{u}} \bar{x} &= \overline{s \cdot (u \cdot x)} = \overline{(s \cdot u) \cdot x} + \overline{u \cdot (s \cdot x)} - \overline{(u \cdot s) \cdot x} \\ &= \overline{u \cdot (s \cdot x)} - \alpha u \cdot x = L_{\bar{u}} (L_{\bar{s}} - \alpha) \bar{x} \end{aligned}$$

for all $x \in V^{(-1)}$, which implies $L_{\bar{s}} L_{\bar{u}} = L_{\bar{u}} (L_{\bar{s}} - \alpha)$ on $\bar{V}^{(-1)}$. Using this together with (7) the restriction of $L_{\bar{s}}$ on $\bar{V}^{(-1)}$ is diagonalizable and has eigenvalues α , $\alpha + 1/2$. This shows (ii). By (9) we obtain

$$\begin{aligned} R_{\bar{s}} L_{\bar{u}} \bar{x} &= \overline{(u \cdot x) \cdot s} = \overline{u \cdot (x \cdot s)} + \overline{(x \cdot u) \cdot s} - \overline{x \cdot (u \cdot s)} = \overline{u \cdot (x \cdot s)} \\ &+ \kappa(x) \overline{u \cdot s} - 2\alpha x \cdot u = \overline{u \cdot (x \cdot s)} = L_{\bar{u}} R_{\bar{s}} \bar{x} \text{ for all } x \in V^{(-1)}, \end{aligned}$$

which means (iii). q.e.d.

According to Lemma 3.5*, (7) and Lemma 5.5 the operator $L_{\bar{s}}$ leaves each subspace $\bar{V}^{(i)}$ invariant and is diagonalizable on $\bar{V}^{(i)}$. We denote by $\bar{V}_{\lambda}^{(i)}$ the eigenspace of $L_{\bar{s}}$ in $\bar{V}^{(i)}$ corresponding to $\lambda \in \mathbb{R}$.

LEMMA 5.6. – *Let $\bar{a} \in \bar{V}_{\lambda}^{(-1)}$. Then we have*

(i) $L_{\bar{s}} \bar{a} = \lambda \bar{a}$,

(ii) $R_{\bar{s}} \bar{a} = 2(\lambda - \alpha) \bar{a}$.

Proof. – Using Lemma 5.4 and (8) we obtain

$$\begin{aligned} L_{\bar{u}} R_{\bar{s}} \bar{a} &= R_{\bar{s}} L_{\bar{u}} \bar{a} = \overline{R_s L_u a} = \overline{2L_s L_u a} = 2L_{\bar{s}} L_{\bar{u}} \bar{a} \\ &= 2L_{\bar{u}} (L_{\bar{s}} - \alpha) \bar{a} = L_{\bar{u}} (2(\lambda - \alpha) \bar{a}). \end{aligned}$$

This implies $R_{\bar{s}} \bar{a} = 2(\lambda - \alpha) \bar{a}$ because $L_{\bar{u}} : \bar{V}^{(-1)} \rightarrow \bar{V}^{(1)}$ is an isomorphism. q.e.d.

For simplicity we denote by $a' \in V^{(1)}$ the element $u \cdot a$ where $a \in V^{(-1)}$.

LEMMA 5.7. –

(i) If $\bar{a} \in V_{\lambda}^{(-1)}$, then $\bar{a}' \in \bar{V}_{\lambda-\alpha}^{(1)}$.

(ii) Let $\bar{a} \in V_{\lambda}^{(-1)}$, $\bar{b} \in \bar{V}_{\mu}^{(-1)}$. Then we have

$$\bar{a}' \cdot \bar{b}, \bar{a} \cdot \bar{b}' \in \bar{V}_{-\lambda+\mu+\alpha}^{(0)}.$$

Proof. – From Lemma 5.5 (ii) it follows

$$L_{\bar{s}} \bar{a}' = L_{\bar{s}} L_{\bar{u}} \bar{a} = L_{\bar{u}} (L_{\bar{s}} - \alpha) \bar{a} = (\lambda - \alpha) L_{\bar{u}} \bar{a} = (\lambda - \alpha) \bar{a}',$$

which implies (i). Using (i), (8) and Lemma 5.6 (ii) we obtain

$$\begin{aligned} \bar{s} \cdot (\bar{a}' \cdot \bar{b}) &= (\bar{s} \cdot \bar{a}') \cdot \bar{b} + \bar{a}' \cdot (\bar{s} \cdot \bar{b}) - (\bar{a}' \cdot \bar{s}) \cdot \bar{b} = (\lambda - \alpha) \bar{a}' \cdot \bar{b} \\ &+ \mu \bar{a}' \cdot \bar{b} - 2(\lambda - \alpha) \bar{a}' \cdot \bar{b} = (-\lambda + \mu + \alpha) \bar{a}' \cdot \bar{b} \end{aligned}$$

and

$$\begin{aligned} \bar{s} \cdot (\bar{a} \cdot \bar{b}') &= (\bar{s} \cdot \bar{a}) \cdot \bar{b}' + \bar{a} \cdot (\bar{s} \cdot \bar{b}') - (\bar{a} \cdot \bar{s}) \cdot \bar{b}' = \lambda \bar{a} \cdot \bar{b}' \\ &+ (\mu - \alpha) \bar{a} \cdot \bar{b}' - 2(\lambda - \alpha) \bar{a} \cdot \bar{b}' = (-\lambda + \mu + \alpha) \bar{a} \cdot \bar{b}'. \end{aligned}$$

This shows (ii). q.e.d.

According to Lemma 3.5* and Lemma 5.3 we get

$$\begin{aligned} V_{\lambda}^{(0)} \cdot V_{\mu'}^{(1)} &\subset V^{(1)} \cap (I_{\lambda} + U_{\lambda}) \cdot I_{\mu'} = V^{(1)} \cap U_{\lambda} \cdot I_{\mu'} \subset V^{(1)} \cap I_{\mu'-\lambda+1} \\ &= V_{\mu'-\lambda+1}^{(1)} \end{aligned}$$

and

$$\begin{aligned} V_{\lambda'}^{(1)} \cdot V_{\mu}^{(0)} &\subset V^{(1)} \cap I_{\lambda'} \cdot (I_{\mu} + U_{\mu}) = V^{(1)} \cap I_{\lambda'} \cdot U_{\mu} \subset V^{(1)} \cap I_{\mu-\lambda'} \\ &= V_{\mu-\lambda'}^{(1)}. \end{aligned}$$

Thus we have

$$\begin{aligned} V_{\lambda}^{(0)} \cdot V_{\mu'}^{(1)} &\subset V_{-\lambda+\mu'+1}^{(1)}, \\ V_{\lambda'}^{(1)} \cdot V_{\mu}^{(0)} &\subset V_{-\lambda'+\mu}^{(1)}. \end{aligned} \tag{10}$$

Consider the subspace $W^{(1)}$ of $V^{(1)}$ defined by

$$W^{(1)} = \{a \in V^{(1)} \mid \langle a, u \rangle = 0\}.$$

The subspace $W^{(1)}$ is invariant by L_s . In fact for $a \in W^{(1)}$ using (8), (9) and $V^{(1)}$. $V^{(1)} = \{0\}$ we have

$$\begin{aligned} \langle s \cdot a, u \rangle &= -\langle a, s \cdot u \rangle + \langle a \cdot s, u \rangle + \langle s, a \cdot u \rangle = -\alpha \langle a, u \rangle \\ &\quad + 2 \langle s \cdot a, u \rangle = 2 \langle s \cdot a, u \rangle \end{aligned}$$

and $\langle s \cdot a, u \rangle = 0$, consequently $s \cdot a \in W^{(1)}$. We denote by $W_\lambda^{(1)}$ the eigenspace of L_s in $W^{(1)}$.

LEMMA 5.8. — *Suppose $\rho' = \nu' - \beta + 1$. If $W_{\rho'}^{(1)} \cdot V_\beta^{(0)} \subset \{u\}$, then $V_\beta^{(0)} \cdot W_{\nu'}^{(1)} \subset \{u\}$.*

Proof. — Let $a_1 \in W_{\nu'}^{(1)}$, $b_1 \in W_{\rho'}^{(1)}$ and $x \in V_\beta^{(0)}$. By (10) we have $x \cdot a_1 \in V_{\rho'}^{(1)}$ and $x \cdot b_1 \in V_{\rho' - \beta + 1}^{(1)}$. Since $b_1 \cdot x \in \{u\}$ and $W^{(1)} \cdot W^{(1)} = \{0\}$, we obtain

$$\langle x \cdot b_1, a_1 \rangle + \langle b_1, x \cdot a_1 \rangle = \langle b_1 \cdot x, a_1 \rangle + \langle x, b_1 \cdot a_1 \rangle = 0.$$

If $\rho' - \beta + 1 \neq \nu'$, the orthogonality of the decomposition $V^{(1)} = V_0^{(1)} + V_{1/2}^{(1)}$ implies $\langle b_1, x \cdot a_1 \rangle = 0$ and consequently $x \cdot a_1 \in \{u\}$. If $\rho' - \beta + 1 = \nu'$, then $\beta = 1$ and $\rho' = \nu'$. From this it follows $L_x W_{\nu'}^{(1)} \subset V_{\nu'}^{(1)}$. Define the mapping

$$A_x = pr \circ L_x : W^{(1)} \longrightarrow W^{(1)}$$

where pr is the projection from $V^{(1)} = W^{(1)} + \{u\}$ onto $W^{(1)}$. Then we have $\langle A_x b_1, a_1 \rangle + \langle b_1, A_x a_1 \rangle = 0$ for all $a_1, b_1 \in W_{\nu'}^{(1)}$ and so A_x is skew symmetric on $W_{\nu'}^{(1)}$. On the other hand A_x has only real eigenvalues because the eigenvalues of L_x are real. This means $A_x = 0$ on $W_{\nu'}^{(1)}$ and $L_x W_{\nu'}^{(1)} \subset \{u\}$. Thus the proof of this lemma is completed. q.e.d.

LEMMA 5.9. — *Let $a, b, c \in V^{(-1)}$. Then the products of \bar{a}, \bar{b}' and \bar{c}' are equal to 0 where $b' = u \cdot b$ and $c' = u \cdot c$.*

Proof. — For each $b \in V^{(-1)}$ we denote by b_1 the element in $W^{(1)}$ such that $\bar{b}_1 = \bar{b}'$. Let $\bar{a} \in \bar{V}_\lambda^{(-1)}$, $\bar{b} \in \bar{V}_\mu^{(-1)}$ and $\bar{c} \in \bar{V}_\nu^{(-1)}$. By Lemma 5.7 we see $\bar{b}' = \bar{b}_1 \in \bar{V}_{\mu - \alpha}^{(1)}$, $\bar{c}' = \bar{c}_1 \in \bar{V}_{\nu - \alpha}^{(-1)}$ and $\bar{a} \cdot \bar{b}' \in \bar{V}_{-\lambda + \mu + \alpha}^{(0)}$. We first prove

$$(i) \quad (\bar{a} \cdot \bar{b}') \cdot \bar{c}' = 0.$$

According to Lemma 5.8, for the proof of (i) it suffices to show

$$(i)' \quad \bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) = 0 \quad \text{for all } \bar{d} \in \bar{V}_\rho^{(-1)},$$

where $\rho = \lambda - \mu + \nu - \alpha + 1$. From $W^{(1)} \cdot W^{(1)} = \{0\}$ it follows

$$\bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) = (\bar{d}_1 \cdot \bar{a}) \cdot \bar{b}_1 - (\bar{a} \cdot \bar{d}_1) \cdot \bar{b}_1.$$

Using Lemma 5.7 and (10) we have

$$\begin{aligned} \bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) &\in \bar{V}_{-2\lambda+2\mu-\nu+3\alpha-1}^{(1)}, \\ (\bar{d}_1 \cdot \bar{a}) \cdot \bar{b}_1 &\in \bar{V}_{\nu-3\alpha+2}^{(1)}, \\ (\bar{a} \cdot \bar{d}_1) \cdot \bar{b}_1 &\in \bar{V}_{2\mu-\nu-\alpha}^{(1)}. \end{aligned}$$

(A) the case $\alpha = 0$. By Lemma 5.5 we know $\lambda, \mu, \nu = 0, 1/2$. This implies $\nu - 3\alpha + 2 = \nu + 2 = 2, 5/2$. Consequently by (7) we have $(\bar{d}_1 \cdot \bar{a}) \cdot \bar{b}_1 = 0$ and $\bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) = -(\bar{a} \cdot \bar{d}_1) \cdot \bar{b}_1$. If $\bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) \neq 0$, then we obtain $-2\lambda + 2\mu - \nu + 3\alpha - 1 = 2\mu - \nu - \alpha$ and so $\lambda = -\frac{1}{2}$, which is a contradiction. Thus (i)' holds.

(B) the case $\alpha = \frac{1}{2}$. By Lemma 5.5 we have $\lambda, \mu, \nu = \frac{1}{2}, 1$. Therefore we obtain $\nu - 3\alpha + 2 = \nu + \frac{1}{2} = 1, \frac{3}{2}$, so by (7) $(\bar{d}_1 \cdot \bar{a}) \cdot \bar{b}_1 = 0$ and

$$(a) \quad \bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) = -(\bar{a} \cdot \bar{d}_1) \cdot \bar{b}_1.$$

This shows $\bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) = 0$ if $-2\lambda + 2\mu - \nu + 3\alpha - 1 \neq 2\mu - \nu - \alpha$. Thus we may assume $-2\lambda + 2\mu - \nu + 3\alpha - 1 = 2\mu - \nu - \alpha$. Then it follows

$$(b) \quad \alpha = \frac{1}{2}, \quad \lambda = \frac{1}{2}, \quad \rho = -\mu + \nu + 1.$$

Let $h_1 \in W_{2\mu-\nu-\frac{1}{2}}^{(1)}$. Since $W^{(1)} \cdot W^{(1)} = \{0\}$, we have

$$(c) \quad \langle (a \cdot d_1) \cdot b_1, h_1 \rangle = -\langle b_1, (a \cdot d_1) \cdot h_1 \rangle + \langle b_1 \cdot (a \cdot d_1), h_1 \rangle.$$

Applying Lemma 5.7 and (10) we obtain

$$(\bar{a} \cdot \bar{d}_1) \cdot \bar{h}_1 \in \bar{V}_{3\mu-2\nu-\frac{1}{2}}^{(1)}, \quad \bar{b}_1 \cdot (\bar{a} \cdot \bar{d}_1) \in \bar{V}_{-2\mu+\nu+\frac{3}{2}}^{(1)}.$$

Therefore we have $\langle b_1, (a \cdot d_1) \cdot h_1 \rangle = 0$ if $\mu - \alpha \neq 3\mu - 2\nu - \frac{1}{2}$, i.e.,

$$(d) \quad \langle b_1, (a \cdot d_1) \cdot h_1 \rangle = 0 \quad \text{if } \mu \neq \nu.$$

If $-2\mu + \nu + \frac{3}{2} \neq 2\mu - \nu - \frac{1}{2}$, then $\langle b_1 \cdot (a \cdot d_1), h_1 \rangle = 0$. Suppose $-2\mu + \nu + \frac{3}{2} = 2\mu - \nu - \frac{1}{2}$. Then we get $\nu = 2\mu - 1$ and so $\mu = 1, \nu = 1$ or $\mu = \frac{1}{2}, \nu = 0$. The case $\mu = \frac{1}{2}, \nu = 0$ is impossible because $\nu = \frac{1}{2}, 1$. Consequently we have

(e) $\langle b_1 \cdot (a \cdot d_1), h_1 \rangle = 0$ except for $\mu = \nu = 1$.

(B') The case $\mu \neq \nu$. By (c) (d) (e) we have $(a \cdot d_1) \cdot b_1 \in \{u\}$ and so by (a) $\bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) = -(\bar{a} \cdot \bar{d}_1) \cdot \bar{b}_1 = 0$.

(B'') The case $\mu = \nu = \frac{1}{2}$. It follows then $b_1, h_1, (a \cdot d_1) \cdot b_1, (a \cdot d_1) \cdot h_1 \in V_0^{(1)}$ and $L_{a \cdot d_1} W_0^{(1)} \subset V_0^{(1)}$. Define the mapping $A_{a \cdot d_1} = pr \circ L_{a \cdot d_1} : W_0^{(1)} \rightarrow W_0^{(1)}$ where pr is the projection from $V_0^{(1)} = W_0^{(1)} + \{u\}$ onto $W_0^{(1)}$. (c) and (e) imply

$$\langle A_{a \cdot d_1} b, h_1 \rangle = -\langle b_1, A_{a \cdot d_1} h_1 \rangle$$

and so $A_{a \cdot d_1}$ is skew symmetric. Since the eigenvalues of

$$A_{a \cdot d_1} = pr \circ L_{a \cdot d_1}$$

are all real, we obtain $A_{a \cdot d_1} = 0$, $(a \cdot d_1) \cdot b_1 \in \{u\}$, and so by (a) $\bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) = -(\bar{a} \cdot \bar{d}_1) \cdot \bar{b}_1 = 0$.

Summing up the results mentioned above (A), (B') and (B'') we have

$$(f) \quad \begin{aligned} \bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) &= 0, \\ (\bar{a} \cdot \bar{b}_1) \cdot \bar{c}_1 &= 0 \end{aligned}$$

except for the case $\alpha = \frac{1}{2}, \lambda = \frac{1}{2}, \mu = \nu = 1, \rho = 1$.

(B''') The case $\mu = \nu = 1$. Then it follows $\alpha = \frac{1}{2}, \lambda = \frac{1}{2}, \mu = \nu = 1, \rho = 1$. Using $a \cdot b' + a' \cdot b, d \cdot b' + d' \cdot b \in V^{(1)}$ and $V^{(1)} \cdot V^{(1)} = \{0\}$, we get

$$\begin{aligned} \bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) &= \bar{d}' \cdot (\bar{a} \cdot \bar{b}') = -\bar{d}' \cdot (\bar{a}' \cdot \bar{b}) = -(\bar{d}' \cdot \bar{a}') \cdot \bar{b} \\ -\bar{a}' \cdot (\bar{d}' \cdot \bar{b}) + (\bar{a}' \cdot \bar{d}') \cdot \bar{b} &= -\bar{a}' \cdot (\bar{d}' \cdot \bar{b}) = \bar{a}' \cdot (\bar{d} \cdot \bar{b}') = \bar{a}_1 \cdot (\bar{d} \cdot \bar{b}_1). \end{aligned}$$

For $h_1 \in W_{1/2}^{(1)}$ we obtain

$$\begin{aligned} \langle a_1 \cdot (d \cdot b_1), h_1 \rangle &= -\langle d \cdot b_1, a_1 \cdot h_1 \rangle + \langle (d \cdot b_1) \cdot a_1, h_1 \rangle \\ &\quad + \langle a_1, (d \cdot b_1) \cdot h_1 \rangle = \langle (d \cdot b_1) \cdot a_1, h_1 \rangle \end{aligned}$$

because $a_1 \cdot h_1 = 0$ and $(\bar{d} \cdot \bar{b}_1) \cdot \bar{h}_1 \in \bar{V}_1^{(1)} = \{0\}$. Since $\bar{a}_1 \cdot (\bar{d} \cdot \bar{b}_1)$, $(\bar{d} \cdot \bar{b}_1) \cdot \bar{a}_1 \in \bar{V}_{1/2}^{(1)}$, we have $a_1 \cdot (d \cdot b_1) - (d \cdot b_1) \cdot a_1 \in \{u\}$ and $\bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) = \bar{a}_1 \cdot (\bar{d} \cdot \bar{b}_1) = (\bar{d} \cdot \bar{b}_1) \cdot \bar{a}_1$. (f) implies $(\bar{d} \cdot \bar{b}_1) \cdot \bar{a}_1 = 0$. Thus (i)' holds.

Therefore the proof of (i)' is completed.

Finally we show

- (ii) $\bar{c}' \cdot (\bar{a} \cdot \bar{b}') = 0$,
- (iii) $(\bar{b}' \cdot \bar{a}) \cdot \bar{c}' = 0$,
- (iv) $\bar{c}' \cdot (\bar{b}' \cdot \bar{a}) = 0$.

Using (i) and $V^{(1)}$. $V^{(1)} = \{0\}$, for $d_1 \in W^{(1)}$ we get

$$\begin{aligned} \langle c_1 \cdot (a \cdot b_1), d_1 \rangle &= -\langle a \cdot b_1, c_1 \cdot d_1 \rangle + \langle (a \cdot b_1) \cdot c_1, d_1 \rangle \\ &\quad + \langle c_1, (a \cdot b_1) \cdot d_1 \rangle \\ &= 0. \end{aligned}$$

This implies (ii). From (i), $b' \cdot a + b \cdot a' \in V^{(1)}$ and $V^{(1)}$. $V^{(1)} = \{0\}$ we obtain $(\bar{b}' \cdot \bar{a}) \cdot \bar{c}' = -(\bar{b} \cdot \bar{a}') \cdot \bar{c}' = 0$. In the same way (iv) follows from (ii).

According to (i) – (iv) and $V^{(1)}$. $V^{(1)} = \{0\}$, the proof of this lemma is completed. q.e.d.

LEMMA 5.10. – Let $a, b \in V^{(-1)}$. Then the products of u, a', b are equal to 0 where $a' = u \cdot a$.

Proof. – By $V^{(1)}$. $V^{(1)} = \{0\}$ we obtain

- (i) $u \cdot (a' \cdot b) = 0$,
- (ii) $u \cdot (b \cdot a') = 0$.

In fact we have $u \cdot (a' \cdot b) = (u \cdot a') \cdot b + a' \cdot (u \cdot b) - (a' \cdot u) \cdot b = 0$ and $u \cdot (b \cdot a') = (u \cdot b) \cdot a' + b \cdot (u \cdot a') - (b \cdot u) \cdot a' = 0$. From (i) it follows

$\langle (a' \cdot b) \cdot u, u \rangle + \langle u, (a' \cdot b) \cdot u \rangle = \langle u \cdot (a' \cdot b), u \rangle + \langle a' \cdot b, u \cdot u \rangle = 0$,
so $\langle (a' \cdot b) \cdot u, u \rangle = 0$. This implies

- (iii) $(a' \cdot b) \cdot u = 0$.

In the same way by (ii) we get

$$(iv) (b \cdot a') \cdot u = 0.$$

The other cases easily follow from $V^{(1)} \cdot V^{(1)} = \{0\}$. q.e.d.

From Lemma 5.10 we have

LEMMA 5.10'. — Let $a \in V^{(-1)}$ and $b^1 \in V^{(1)}$. Then the products of a, b^1, u are equal to 0.

LEMMA 5.11. — Let $a \in V^{(-1)}$ and $b^1, c^1 \in V^{(1)}$. Then the products of a, b^1, c^1 are equal to 0.

Proof. — By Lemma 5.9 we have $(a \cdot b^1) \cdot c^1 \in \{u\}$. Using Lemma 5.10' and $V^{(1)} \cdot V^{(1)} = \{0\}$ we get

$$\begin{aligned} \langle u, (a \cdot b^1) \cdot c^1 \rangle &= -\langle (a \cdot b^1) \cdot u, c^1 \rangle + \langle u \cdot (a \cdot b^1), c^1 \rangle \\ &\quad + \langle a \cdot b^1, u \cdot c^1 \rangle = 0. \end{aligned}$$

Thus we have $(a \cdot b^1) \cdot c^1 = 0$. By the same way we obtain

$$c^1 \cdot (a \cdot b^1) = 0, \quad (b^1 \cdot a) \cdot c^1 = 0, \quad c^1 \cdot (b^1 \cdot a) = 0.$$

The other cases follow from $V^{(1)} \cdot V^{(1)} = \{0\}$. q.e.d.

Consider the centralizer Z of $V^{(1)}$ in V ;

$$Z = \{z \in V \mid z \cdot a^1 = a^1 \cdot z = 0 \text{ for all } a^1 \in V^{(1)}\}.$$

Then we have

LEMMA 5.12. — Z is an ideal of V .

Proof. — Let $z \in Z, a \in V$. We have

$$u \cdot (z \cdot a) = (u \cdot z) \cdot a + z \cdot (u \cdot a) - (z \cdot u) \cdot a = 0,$$

$$u \cdot (a \cdot z) = (u \cdot a) \cdot z + a \cdot (u \cdot z) - (a \cdot u) \cdot z = 0$$

and so $z \cdot a, a \cdot z \in V^{(0)}$. From this $V^{(1)}$ is invariant by $L_{z \cdot a}, R_{z \cdot a}, L_{a \cdot z}$ and $R_{a \cdot z}$. Using Lemma 5.11 and $V^{(1)} \cdot V^{(1)} = \{0\}$, for $b^1, c^1 \in V^{(1)}$ we get

$$\begin{aligned} \langle L_{z \cdot a} b^1, c^1 \rangle + \langle b^1, L_{z \cdot a} c^1 \rangle &= \langle b^1 \cdot (z \cdot a), c^1 \rangle + \langle z \cdot a, b^1 \cdot c^1 \rangle \\ &= \langle b^1 \cdot (z \cdot a), c^1 \rangle = \langle (b^1 \cdot z) \cdot a + z \cdot (b^1 \cdot a) - (z \cdot b^1) \cdot a, c^1 \rangle \\ &= \langle z \cdot (b^1 \cdot a), c^1 \rangle = -\langle z \cdot c^1, b^1 \cdot a \rangle + \langle c^1 \cdot z, b^1 \cdot a \rangle \\ &\quad + \langle z, c^1 \cdot (b^1 \cdot a) \rangle \\ &= 0. \end{aligned}$$

This means that $L_{z \cdot a}$ is skew symmetric. On the other hand the eigenvalues of $L_{z \cdot a}$ are all real. Therefore it must be $L_{z \cdot a} = 0$ on $V^{(1)}$, i.e., $(z \cdot a) \cdot b^1 = 0$ for all $b^1 \in V^{(1)}$. From this it follows

$$\begin{aligned} \langle b^1 \cdot (z \cdot a), c^1 \rangle &= -\langle z \cdot a, b^1 \cdot c^1 \rangle + \langle (z \cdot a) \cdot b^1, c^1 \rangle \\ &+ \langle b^1, (z \cdot a) \cdot c^1 \rangle = 0 \text{ for all } b^1, c^1 \in V^{(1)} \end{aligned}$$

and so $b^1 \cdot (z \cdot a) = 0$ for all $b^1 \in V^{(1)}$. Thus we get

(a)
$$z \cdot a \in Z.$$

Applying Lemma 5.11 and $V^{(1)} \cdot V^{(1)} = \{0\}$, we obtain

$$\begin{aligned} \langle L_{a \cdot z} b^1, c^1 \rangle + \langle b^1, L_{a \cdot z} c^1 \rangle &= \langle b^1 \cdot (a \cdot z), c^1 \rangle + \langle a \cdot z, b^1 \cdot c^1 \rangle \\ &= \langle b^1 \cdot (a \cdot z), c^1 \rangle = \langle (b^1 \cdot a) \cdot z + a \cdot (b^1 \cdot z) - (a \cdot b^1) \cdot z, c^1 \rangle \\ &= \langle (b^1 \cdot a - a \cdot b^1) \cdot z, c^1 \rangle = -\langle (b^1 \cdot a - a \cdot b^1) \cdot c^1, z \rangle \\ &+ \langle c^1 \cdot (b^1 \cdot a - a \cdot b^1), z \rangle + \langle b^1 \cdot a - a \cdot b^1, c^1 \cdot z \rangle = 0 \end{aligned}$$

for all $b^1, c^1 \in V^{(1)}$. Consequently $L_{a \cdot z}$ is skew symmetric on $V^{(1)}$. Since the eigenvalues of $L_{a \cdot z}$ are real, we have $L_{a \cdot z} = 0$ on $V^{(1)}$, i.e., $(a \cdot z) \cdot b^1 = 0$ for all $b^1 \in V^{(1)}$. Using this and $V^{(1)} \cdot V^{(1)} = \{0\}$ we get

$$\begin{aligned} \langle b^1 \cdot (a \cdot z), c^1 \rangle &= -\langle a \cdot z, b^1 \cdot c^1 \rangle + \langle (a \cdot z) \cdot b^1, c^1 \rangle + \langle b^1, (a \cdot z) \cdot c^1 \rangle \\ &= 0 \end{aligned}$$

for all $b^1, c^1 \in V^{(1)}$ and hence

(b)
$$b^1 \cdot (a \cdot z) = 0 \text{ for all } b^1 \in V^{(1)}.$$

Therefore we have $a \cdot z \in Z$. (a) and (b) imply that Z is an ideal of V . q.e.d.

Let C denote the center of Z ;

$$C = \{c \in Z \mid c \cdot z = z \cdot c = 0 \text{ for all } z \in Z\}.$$

Then we have

LEMMA 5.13. — C is a commutative ideal of V containing u .

Proof. — From $C \supset V^{(1)}$ it follows $u \in C$. Let $c \in C, x \in V$. Since Z is an ideal of V , we have

$$z \cdot (c \cdot x) = (z \cdot c) \cdot x + c \cdot (z \cdot x) - (c \cdot z) \cdot x = 0$$

and $z \cdot (x \cdot c) = (z \cdot x) \cdot c + x \cdot (z \cdot c) - (x \cdot z) \cdot c = 0$ for all $z \in Z$.

This implies

(a) $R_a = 0$ on Z

where $a = c \cdot x$ or $x \cdot c$. Using this and Lemma 3.2, for $z, z' \in Z$ we get

$$(b) \quad \frac{d^2}{dt^2} \langle \exp tL_a z, \exp tL_a z' \rangle = \frac{d}{dt} \langle a, \exp tL_a(z \cdot z') \rangle \\ = \langle a, L_a(\exp tL_a(z \cdot z')) \rangle = -\langle w, a \cdot a \rangle + \langle w \cdot a, a \rangle \\ + \langle a, w \cdot a \rangle = 0,$$

where $w = \exp tL_a(z \cdot z') \in Z$. Let λ be an eigenvalue of L_a on Z and z an eigenvector corresponding to λ . Then we have $\frac{d^2}{dt^2} \langle \exp tL_a z, \exp tL_a z \rangle = \frac{d^2}{dt^2} \langle z, z \rangle e^{2\lambda t} = (2\lambda)^2 e^{2\lambda t} \langle z, z \rangle$ and by (b) $\lambda = 0$. Thus the eigenvalues of L_a on Z are equal to 0. We show

(c) $L_a = 0$ on Z .

Suppose $L_a \neq 0$ on Z . Then there exist elements $z, w \in V$ such that $L_a w = 0$, $w = L_a z \neq 0$. Since $\exp tL_a z = z + tw$, we have

$$\frac{d^2}{dt^2} \langle \exp tL_a z, \exp tL_a z \rangle = \frac{d^2}{dt^2} \langle z + tw, z + tw \rangle \\ = 2 \langle w, w \rangle t + 2 \langle z, w \rangle,$$

which contradicts to (b). Thus (c) holds. (a) and (c) imply $a \in C$ and consequently $c \cdot x, x \cdot c \in C$. Therefore C is an ideal of V .
q.e.d.

Proposition 5.1 follows from Lemma 5.13.

6. Proof of Theorem 3.1.

We first consider the case $u \cdot u = u$. By Proposition 4.1 we have the orthogonal decomposition $V = \{u\} + V_{1/2} + V_0$. Since V_0 is a subalgebra, by the inductive assumption we get $V_0 = I + U_0$, where I is a commutative ideal of V_0 and U_0 is a subalgebra with principal idempotent s_0 . Put $E = \{u\} + V_{1/2}$. Then E is an ideal of V . Let $a \in I$. Since E is invariant under L_a, R_a and is orthogonal to a and since $a \cdot a = 0$, by Lemma 3.3 we obtain $L_a = R_a = 0$ on E . From this we know that I is a commutative ideal of V . Put

$$U = E + U_0, \\ s = u + s_0,$$

Using Proposition 4.1 (iv), $u \cdot u = u$ and $s_0 \cdot s_0 = s_0$, we have

(i) $s \cdot s = s$.

By Proposition 4.1 (iv) we have $L_u = R_u = 0$ on U_0 . Therefore $L_s = L_{s_0}$ is diagonalizable on U_0 and its eigenvalues on U_0 are equal to $1/2$, 1 and moreover $R_s = R_{s_0} = 2L_{s_0} - 1 = 2L_s - 1$ on U_0 . Since E is invariant under L_{s_0}, R_{s_0} and is orthogonal to s_0 and since $s_0 \cdot s_0 = s_0$, applying Lemma 3.3 it follows that the restriction of L_{s_0} on E is diagonalizable and its eigenvalues are $0, 1/2$ and that $R_{s_0} = 2L_{s_0}$ on E . Therefore using $L_{s_0}u = 0, L_{s_0}V_{1/2} \subset V_{1/2}$ and $L_u = 1/2$ on $V_{1/2}$, $L_s = L_u + L_{s_0}$ is diagonalizable on E and its eigenvalues on E are equal to $1/2, 1$. Since $R_u = 2L_u - 1$ and $R_{s_0} = 2L_{s_0}$ hold on E , we have $R_s = 2L_s - 1$ on E . Thus we obtain

(ii) The restriction of L_s on U is diagonalizable and its eigenvalues on U are equal to $1/2, 1$.

(iii) $R_s = 2L_s - 1$ on U .

(i) (ii) (iii) imply that s is a principal idempotent of U . Thus in the case $u \cdot u = u$ the proof of Theorem 3.2 is completed.

Next we consider the case $u \cdot u = 0$. By Proposition 5.1 there exists a commutative ideal C of V containing u . Let V' be the orthogonal complement of C in V . By Lemma 3.1 V' is a subalgebra. From the inductive assumption we get $V' = I' + U$, where I' is a commutative ideal of V' and U is a subalgebra with principal idempotent s . Let $a' \in I'$. Since C is invariant under $L_{a'}, R_{a'}$ and $C \cdot C = \{0\}$ and since $a' \cdot a' = 0$, by Lemma 3.3 we obtain $L_{a'} = R_{a'} = 0$ on C . This shows that $I = C + I'$ is a commutative ideal of V . Thus the decomposition $V = I + U$ has the desired properties.

Therefore the proof of Theorem 3.1 is completed. q.e.d.

7. Proof of Main Theorem 2) and Corollaries.

Let V be the tangent space of M at x . In view of Main Theorem 1), Proposition 2.4 and Theorem 2.1 V admits a structure of normal Hessian algebra and $E_x = T(V)0$.

Proof of Main Theorem 2). — According to Theorem 3.1 the normal Hessian algebra V is decomposed in $V = I + U$, where I is a commutative ideal of V and U is a clan. Denote by $T(I)$ the commutative normal subgroup of $T(V)$ generated by $\{X_a^* \mid a \in I\}$ and $T(U)$ the subgroup of $T(V)$ generated by $\{X_w^* \mid w \in U\}$. Then we get $T(V) = T(I)T(U)$. Let E_x^+ denote the orbit of $T(U)$ through the origin 0 ; $E_x^+ = T(U)0$. For $a \in I, v^+ \in E_x^+$ we have $\exp X_a^* v^+ = v^+ + \sum_{k=0}^{\infty} \frac{L_a^{k+1}}{(k+1)!} (L_a v^+ + a) = a + a \cdot v^+ + v^+$ because I is a commutative ideal of V . Thus $T(I)v^+ \subset I + v^+$. Suppose $v^+ = h0$ where $h \in T(U)$. Since

$$T(I)v^+ = T(I)h0 = hh^{-1}T(I)h0 = hT(I)0 = hI$$

and since h is an affine transformation of V , we obtain $T(I)v^+ = I + v^+$.

Therefore, putting $E_x^0 = I$ we get

$$E_x = T(V)0 = T(I)T(U)0 = T(I)E_x^+ = E_x^0 + E_x^+.$$

Let $p: E_x \rightarrow E_x^+$ denote the projection from $E_x = E_x^0 + E_x^+$ onto E_x^+ . Then E_x admits a fibering with projection p . Since U is a clan, applying Theorem 2.2 (Vinberg's result) the base space E_x^+ is an affine homogeneous convex domain not containing any full straight line. The fiber $p^{-1}(v^+) = T(I)v^+ = E_x^0 + v^+$ over $v^+ \in E_x^+$ is an affine subspace of V and a Euclidean space with respect to the induced metric because $T(I)$ is commutative. It is clear that the fiber $E_x^0 + v$ through $v \in E_x$ is characterized as the set of all points which can be joined with v by full straight lines contained in E_x . This implies that our fibering of E_x is unique and that every affine transformation of E_x is fiber preserving. q.e.d.

Proof of Corollary 1. — If we put $\alpha_x(v) = \text{Tr } L_v$ for $v \in V$, the value β_x of the canonical bilinear form β at x has an expression (cf. [8]) $\beta_x(v, w) = \alpha_x(v \cdot w)$ for $v, w \in V$. By Theorem 3.1 V is decomposed in $V = I + U$, where I is a commutative ideal of V and U is a clan. I being a commutative ideal of V we get

$$\begin{aligned} \alpha_x(a) &= 0, \\ \beta_x(a, v) &= 0, \quad \text{for } a \in I, v \in V. \end{aligned} \tag{1}$$

Because $\langle v \cdot a, b \rangle + \langle a, v \cdot b \rangle = \langle a \cdot v, b \rangle + \langle v, a \cdot b \rangle = \langle a \cdot v, b \rangle$ for $a, b \in I$ and $v \in V$, we have

$$L_v + {}^tL_v = R_v \quad \text{on } I. \tag{2}$$

Since U is a clan, it follows

$$\text{Tr}_U L_{v,v} > 0 \quad \text{for } v \neq 0 \in U. \tag{3}$$

Using $R_{v,v} = R_v R_v + [L_v, R_v]$ and (2) we obtain

$$\text{Tr}_I L_{v,v} = \frac{1}{2} \text{Tr}_I R_{v,v} = \frac{1}{2} \text{Tr}_I R_v {}^t R_v \geq 0.$$

From this and (3) we get $\beta_{x_0}(v, v) = \text{Tr}_I L_{v,v} + \text{Tr}_U L_{v,v} > 0$ for all $v \neq 0 \in U$. This together with (1) implies that β_x is positive semi-definite and that the null space of β_x coincides with $E_x^0 = I$.
 q.e.d.

Proof of Corollary 2. — Let $v \in E_x$. Since the fiber $E_x^0 + v$ through v is an affine subspace of V , it follows $d(v, w) = 0$ for all $w \in E_x^0 + v$ (cf. [5]). Conversely, suppose $d(v, w) = 0$. Then we get $0 \leq c_{E_x^+}^a(p(v), p(w)) \leq c_{E_x}^a(v, w) \leq d(v, w) = 0$ because the projection $p: E_x \rightarrow E_x^+$ is an affine mapping. By a result of Vey [11] $c_{E_x^+}^a$ is a distance on E_x^+ . This implies $p(v) = p(w)$. Therefore we get $E_x^0 + v = \{w \in E_x \mid d(v, w) = 0\}$.
 q.e.d.

Proof of Corollary 3. — Our assertion follows from the facts that the covering projection $\exp_x: E_x = E_x^0 + E_x^+ \rightarrow M$ is an affine mapping and that E_x^0 is an affine subspace of V .
 q.e.d.

Let G be a connected Lie subgroup of $\text{Aut}(M)$ which acts transitively on M and B the isotropy subgroup of G at a point x in M ; $M = G/B$. We denote by \tilde{G} the universal covering group of G and by $\pi: \tilde{G} \rightarrow G$ the covering projection. Then $\tilde{M} = \tilde{G}/\tilde{B}$ is the universal covering manifold of $M = G/B$, where \tilde{B} is the identity component of $\pi^{-1}(B)$. Let \tilde{N} be the normal subgroup of \tilde{G} consisting of all elements in \tilde{G} which induce the identity transformation of \tilde{M} . We put $G^* = \tilde{G}/\tilde{N}$, $B^* = \tilde{B}/\tilde{N}$. According to Main Theorem 1) it follows that $\tilde{M} = G^*/B^*$ is a convex domain in \mathbb{R}^n and that G^* is a subgroup of the affine transformation group $A(n)$ of \mathbb{R}^n .

Proof of Corollary 4. — Assume G is not solvable. Since G^* is not solvable, there exists a connected semi-simple Lie subgroup S^* of G^* . Let K^* be a maximal compact subgroup of S^* . Since \tilde{M} is a convex domain in \mathbb{R}^n , K^* has a fixed point \tilde{y} in \tilde{M} .

Therefore we have

$$\dim G^* = \dim \tilde{M} = \dim G^* \tilde{y} \leq \dim G^* - \dim K^* < \dim G^*,$$

which is a contradiction. Thus G must be a solvable Lie group. q.e.d.

Proof of Corollary 5. — Let G be a transitive reductive Lie subgroup of $\text{Aut}(M)$ and let \mathfrak{g} be the Lie algebra of G . Then \mathfrak{g} is decomposed into the direct sum $\mathfrak{g} = \mathfrak{c} + \mathfrak{s}$ where \mathfrak{c} is the center of \mathfrak{g} and \mathfrak{s} the semi-simple part of \mathfrak{g} . Denoting by C^* and S^* the connected Lie subgroup of G^* corresponding to \mathfrak{c} and \mathfrak{s} respectively, we have $G^* = C^*S^*$. Since S^* is a connected semi-simple Lie subgroup of $A(n)$, S^* is closed in $A(n)$ (cf. [15]). Let \overline{C}^* denote the closure of C^* in $A(n)$. Then the subgroup \overline{C}^*S^* is closed in $A(n)$ (cf. [3]) and so coincides with the closure \overline{G}^* of G^* in $A(n)$. It is easy to see that every element in \overline{G}^* preserves the domain \tilde{M} and leaves invariant the Hessian metric on \tilde{M} . Denoting by K_c^* and K_s^* maximal compact subgroups of \overline{C}^* and S^* respectively, the group $K^* = K_c^*K_s^*$ is a maximal compact subgroup of $\overline{G}^* = \overline{C}^*S^*$ because the center of S^* is finite. Since \tilde{M} is a convex domain in \mathbf{R}^n , K^* has a fixed point \tilde{o} in \tilde{M} . We may assume that \tilde{o} is the origin in \mathbf{R}^n . The isotropy subgroup $K^{*'}$ of \overline{G}^* at \tilde{o} is contained in an orthogonal group and is closed in \overline{G}^* . Thus $K^{*'}$ is a compact subgroup of \overline{G}^* containing K^* and so $K^{*'} = K^* = K_s^*$. Since \overline{G}^* acts effectively on \tilde{M} , \overline{K}_c^* is reduced to the identity and so $K^{*'} = K^* = K_s^*$. We denote by $\overline{\mathfrak{g}}^*$, $\overline{\mathfrak{c}}^*$, \mathfrak{s}^* and \mathfrak{k}_s^* the Lie algebras of \overline{G}^* , \overline{C}^* , S^* and K_s^* respectively, and by \mathfrak{p}_s^* the orthogonal complement of \mathfrak{k}_s^* in \mathfrak{s}^* with respect to the Killing form of \mathfrak{s}^* . Putting $\mathfrak{t}^* = \mathfrak{k}_s^*$ and $\mathfrak{p}^* = \overline{\mathfrak{c}}^* + \mathfrak{p}_s^*$, we have

$$\overline{\mathfrak{g}}^* = \mathfrak{t}^* + \mathfrak{p}^*, \quad [\mathfrak{t}^*, \mathfrak{t}^*] \subset \mathfrak{t}^*, \quad [\mathfrak{t}^*, \mathfrak{p}^*] \subset \mathfrak{p}^*, \quad [\mathfrak{p}^*, \mathfrak{p}^*] \subset \mathfrak{t}^*.$$

From this using the same argument as in [9] it follows that $\tilde{M} = G^*/K^*$ is the direct product of a Euclidean space and an affine homogeneous convex self-dual cone not containing any full straight line. q.e.d.

Proof of Corollary 6. — Since M is compact, the automorphism group $G = \text{Aut}(M)$ is compact. Therefore the Lie algebra \mathfrak{g} of G is decomposed into the direct sum $\mathfrak{g} = \mathfrak{c} + \mathfrak{s}$ where \mathfrak{c} is the center of \mathfrak{g} and \mathfrak{s} is a compact semi-simple subalgebra of \mathfrak{g} . Denote by

C^* and S^* the connected Lie subgroups of G^* corresponding to \mathfrak{c} and \mathfrak{s} respectively. Then $G^* = C^*S^*$, and S^* is compact by a theorem of Weyl. Since $\tilde{M} = G^*/B^*$ is a convex domain, S^* has a fixed point in M . Therefore $B^* \supset S^*$ and so S^* is a normal subgroup of G^* contained in B^* . From this and the effectiveness, S^* is reduced to the identity. Thus we have $\mathfrak{g} = \mathfrak{c}$. Consequently G is commutative and M is a Euclidean torus. q.e.d.

BIBLIOGRAPHY

- [1] S.G. GINDIKIN, I.I. PJATECKII-SAPIRO and E.B. VINBERG, Homogeneous Kähler manifolds, in "Geometry of Homogeneous Bounded Domains", Centro Int. Math. Estivo, 3 Ciclo, Urbino, Italy, 1967, 3-87.
- [2] P. DOMBROWSKI, On the geometry of the tangent bundles, *J. Reine Angew. Math.*, 210 (1962), 73-88.
- [3] M. GOTO, Faithful representations of Lie groups I, *Math. Japon.*, 1 (1948), 1-13.
- [4] J. HELMSTETTER, Doctorat de 3^e cycle "Radical et groupe formel d'une algèbre symétrique à gauche" novembre 1975, Grenoble.
- [5] S. KOBAYASHI, Intrinsic distances associated with flat affine or projective structures, *J. Fac. Sci. Univ. of Tokyo*, IA 24 (1977), 129-135.
- [6] J.L. KOSZUL, Domaines bornés homogènes et orbites de groupes de transformations affines, *Bull. Soc. Math. France*, 89 (1961), 515-533.
- [7] J.L. KOSZUL, Variétés localement plates et convexité, *Osaka J. Math.*, 2 (1965), 285-290.
- [8] H. SHIMA, On certain locally flat homogeneous manifolds of solvable Lie groups, *Osaka J. Math.*, 13 (1976), 213-229.
- [9] H. SHIMA, Symmetric spaces with invariant locally Hessian structures, *J. Math. Soc. Japan*, 29 (1977), 581-589.
- [10] H. SHIMA, Compact locally Hessian manifolds, *Osaka J. Math.*, 15 (1978) 509-513.

- [11] J. VEY, Une notion d'hyperbolicité sur les variétés localement plates, *C.R. Acad. Sci. Paris*, 266 (1968), 622-624.
- [12] E.B. VINBERG, The Morozov-Borel theorem for real Lie groups, *Soviet Math. Dokl.*, 2 (1961), 1416-1419.
- [13] E.B. VINBERG, The theory of convex homogeneous cones, *Trans. Moscow Math. Soc.*, 12 (1963), 340-403.
- [14] E.B. VINBERG and S.G. GINDIKIN, Kaehlerian manifolds admitting a transitive solvable automorphism group, *Math. Sb.*, 75 (116) (1967), 333-351.
- [15] K. YOSHIDA, A theorem concerning the semi-simple Lie groups, *Tohoku Math. J.*, 43 (Part II) (1937), 81-84.

Manuscrit reçu le 11 septembre 1979
révisé le 21 janvier 1980.

Hirohiko SHIMA,
Department of Mathematics
Yamaguchi University
Yamaguchi 753 (Japan).