ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 30, nº 2 (1980), p. 109-169

<http://www.numdam.org/item?id=AIF_1980__30_2_109_0>

Ann. Inst. Fourier, Grenoble
30, 2 (1980), 109-169.

ON THE EIGENVALUES OF A CLASS OF HYPO-ELLIPTIC OPERATORS. IV

by **Johannes SJÖSTRAND**

0. Introduction.

Let X be a compact smooth manifold of dimension n and let P be a classical pseudodifferential operator of order $m > 1$, formally selfadjoint with respect to some fixed smooth positive density, and with non-negative principal symbol $p(x,\xi) \in C^\infty(T^*X\backslash 0)$. We shall always assume that P is hypoelliptic with loss of 1 derivative and it is then easy to prove that P is a selfadjoint unbounded operator $H^0(X) \to H^0(X)$ with domain $\mathscr{D}_P = \{u \in H^0(X)\,;\ Pu \in H^0(X)\} \subset H^{m-1}(X)$, and that P has discrete spectrum with locally finite multiplicity. In [6], [8] A. Menikoff and the author determined the eigenvalue asymptotics in the case when $\Sigma = p^{-1}(0)$ is a symplectic submanifold and p vanishes on Σ to precisely the second order. In [7] they eliminated the assumption that Σ should be symplectic. Using the results of A. Melin [3] and L. Hörmander [1] it was proved in [8] that microlocally; either $\Sigma = p^{-1}(0)$ is a symplectic manifold on which p vanishes to precisely the second order, or

$$(0.1) \qquad S_P + \frac{1}{2}\widetilde{\mathrm{tr}} > 0 \quad \text{on} \quad p^{-1}(0),$$

where S_P is the subprincipal symbol of P and $\widetilde{\mathrm{tr}}$ is the sum of the positive eigenvalues of $\frac{1}{i}F$, where F is the Hamilton matrix associated to the Hessian of p. When (0.1) holds, we also know from [3] that P is bounded from below.

In the present paper we shall generalize the results of [6], [7] by assuming (0.1) but not necessarily that $p^{-1}(0)$ is a manifold. A part from a trivial microlocalization, this will then together with the results of [8], cover general self-adjoint operators of order > 1 with non-negative principal symbol, hypoelliptic with loss of 1-derivative. The result on the eigenvalue distribution (theorem 8.9) is somewhat technical to formulate and will be given in the end of the paper. Here we only give a weaker statement :

THEOREM 0.1. — *Let* P *be a formally selfadjoint classical pseudo-differential operator of order* $m > 1$ *with non-negative principal symbol* p *and assume* (0.1). *Then the number of eigenvalues of* P *smaller than or equal to* λ *is of the same order of magnitude as*

$$\iint_{p(x,\xi)+|\xi|^{m-1}\leqslant\lambda} dx\, d\xi$$

as $\lambda \to +\infty$. *Here* $|\xi|$ *is the norm with respect to some Riemannian metric and* $dx\, d\xi$ *is the invariant symplectic volume on* T^*X.

The more precise formula for $N(\lambda)$ (the number of eigenvalues $\leqslant \lambda$) in section 8 is given by a similar integral, where $p + |\xi|^{m-1}$ is replaced by $p + S_P + \frac{1}{2}\widetilde{\mathrm{tr}}$ and $dx\, d\xi$ by a measure depending on p which can be thought of as a discretization of $dx\, d\xi$. We thank L. Hörmander for having suggested the use of such measures already in the formulation of the results of [6].

As in [6], [7], [8] the proof is based on the construction of $\exp(-tP)$ as a complex Fourier integral operator with quasi homogeneous phase, but instead of using Taylor expansions in the study of the characteristic and transport equations, we shall make estimates of the type already used by A. Melin and the author in [4]. As in [6], [7], [8] we have a global problem for the phase and the amplitude with respect to the time-variable, so the estimates have to be pushed considerably further than in [4].

The treatment of the characteristic equation is carried out in sections 1-3, the transport equation is treated in section 4 and in section 5 we conclude the construction of $\exp(-tP)$. In section 6 we introduce the discrete measures, the trace of $\exp(-tP)$ is studied in section 8, and the eigenvalue distribution is derived by using Karamata's Tauberian theorem, that we recall in section 7.

Using $\exp(-tP)$ one can also write down a parametrix of P (namely $\int_0^\infty \exp(-tP)\, dt$) and prove the semiboundness result of A. Melin [3]. We hope that the methods of sections 1-5 will also apply to other problems for operators with double characteristics so that the eigenvalue distribution will not be the only justification of these constructions (*).

The main result, theorem 8.9, was announced in a slightly different form in [9]. The full proof of the result in [9] also requires some rather long and dull estimates on the measures introduced in section 6 below, that we have chosen to omit, since the presentation in theorem 8.9 is simpler and more natural anyway.

1. Estimates along the Hamilton flow.

Let $X \subset \mathbf{R}^n$ be open and $p \in C^\infty(T^*X\backslash 0)$ be homogeneous of degree 1 and such that $p(x,\xi) \geqslant 0$. (In section 5 we shall show how the treatment of the transport- and characteristic equations can be reduced to the case when the principal symbol is homogeneous of degree 1). We fix once and for all an almost analytic extension of p to some complex neighborhood $\widetilde{T^*X\backslash 0}$ of $T^*X\backslash 0$. Let

$$\tilde{\varepsilon}(T) = \delta(1+T)^{-N} \tag{1.1}$$

where $\delta > 0$, $N > 0$ will be fixed later on. We shall study various estimates for integral curves $[0,T] \ni t \longmapsto \rho_t$ of $H_{\frac{1}{i}p}$ (cf. [4]) under the assumption that ρ_0 belongs to some small complex neighborhood of a real point where p vanishes and that

$$\sup_{0 \leqslant t \leqslant T} |\operatorname{Im} \rho_t| + \int_0^T |p'(\operatorname{Re} \rho_t)|\, dt < \tilde{\varepsilon}(T). \tag{1.2}$$

If $N \geqslant 1$ and $\delta > 0$ is sufficiently small, (1.2) implies that ρ_t stays in some fixed compact set. The new feature compared with [4] is precisely that the length of the time interval may tend to infinity when the initial point tends to a real point, where p vanishes, while our estimates remain uniform.

PROPOSITION 1.1. — *Assume that* $N \geqslant 2$ *in* (1.1) *and that* δ *is sufficiently small. Let* $S(x,\xi) = -\langle \operatorname{Im} x, \operatorname{Re} \xi \rangle$. *Then if* $0 \leqslant s \leqslant \sigma \leqslant t \leqslant T$ *and*

(*) *Added in Prof:* C. and N. Iwasaki have announced a slightly different construction of $\exp(-tP)$ in Proc. Japan Acad., Vol. 55, Ser. A, No 7 (1979), 237-240.

(1.2) *holds, we have the estimates*

$$(1.3)\qquad S(\rho_t) - S(\rho_s) \geqslant \frac{1}{3}\int_s^t p(\mathrm{Re}\ \rho_\tau)\, d\tau - C(t-s)|\mathrm{Im}\ \rho_t|^3$$

$$(1.4)\qquad |\mathrm{Im}\ \rho_\sigma| \leqslant C(1+(t-s)^{1/2})(|\mathrm{Im}\ \rho_t|^{1/2} + |\mathrm{Im}\ \rho_s|^{1/2}).$$

Here $C > 0$ *is independent of* T.

Proof. – From

$$\frac{d\rho_t}{dt} = \frac{1}{i} H_p(\rho_t)$$

we get by Taylor expanding $\frac{1}{i} H_p(\rho_t)$ at $\mathrm{Re}\ \rho_t$,

$$(1.5)\qquad \frac{d\rho_t}{dt} = \frac{1}{i} H_p(\mathrm{Re}\ \rho_t) + F_{\mathrm{Re}\,\rho_t}(\mathrm{Im}\ \rho_t) + \mathcal{O}(|\mathrm{Im}\ \rho_t|^2).$$

Here F is real (it is the Hamilton matrix). Taking imaginary parts of (1.5), we get

$$(1.6)\qquad \frac{d}{dt}\mathrm{Im}\ \rho_t + A(\rho_t)\,\mathrm{Im}\ \rho_t = -\ H_p(\mathrm{Re}\ \rho_t),$$

where the matrix $A(\rho_t)$ satisfies :

$$(1.7)\qquad A(\rho_t) = \mathcal{O}(|\mathrm{Im}\ \rho_t|) \leqslant \mathcal{O}(\varepsilon(T)).$$

Let $B_{t,s}$, $0 \leqslant s, t \leqslant T$ be the family of matrices which satisfy

$$(1.8)\qquad \frac{\partial B_{t,s}}{\partial t} + A(\rho_t)B_{t,s} = 0, \qquad B_{t,t} = I.$$

Then (1.6) gives

$$(1.9)\qquad \mathrm{Im}\ \rho_t = B_{t,s}\,\mathrm{Im}\ \rho_s - \int_s^t B_{t,\tau} H_p(\mathrm{Re}\ \rho_\tau)\, d\tau.$$

If $N \geqslant 1$ we have $B_{t,s} = \mathcal{O}(1)$ and (1.9) gives

$$(1.10)\qquad |\mathrm{Im}\ \rho_t| \leqslant C(|\mathrm{Im}\ \rho_s| + |\int_s^t |p'(\mathrm{Re}\ \rho_\tau)\,|d\tau|).$$

Now recall from [4] that

$$\frac{dS(\rho_t)}{dt} \geqslant \frac{1}{2} p(\mathrm{Re}\ \rho_t) - C\,|\mathrm{Im}\ \rho_t|^3,$$

or rather

$$S(\rho_t) - S(\rho_s) \geqslant \frac{1}{2}\int_s^t p(\operatorname{Re}\rho_\tau)\,d\tau - C\int_s^t |\operatorname{Im}\rho_\tau|^3\,d\tau,$$
$$0 \leqslant s \leqslant t \leqslant T.$$

(The constants « C » are different each time). By (1.10)

$$(1.11)\quad S(\rho_t) - S(\rho_s) \geqslant \frac{1}{2}\int_s^t p(\operatorname{Re}\rho_\tau)\,d\tau - C(t-s)\,|\operatorname{Im}\rho_t|^3 - C\int_s^t\left(\int_\tau^t |p'(\operatorname{Re}\rho_\mu)|\,d\mu\right)^3 d\tau.$$

Using the Cauchy-Schwartz inequality

$$\int_s^t |p'|\,d\mu \leqslant (t-s)^{1/2}\left(\int_s^t |p'|^2\,d\mu\right)^{1/2},$$

and the fact that $|p'| \leqslant Cp^{1/2}$ on the real domain, the last remainder in (1.11) can be estimated by

$$C(t-s)\left(\int_s^t |p'|\,d\mu\right)^3 \leqslant C(t-s)^2\int_s^t |p'|\,d\mu\int_s^t p\,d\mu.$$

Now

$$C(1 + T^2)\int_0^T |p'|\,d\mu \leqslant \frac{1}{6}$$

if $N \geqslant 2$ and $\delta > 0$ is small enough, so the last term in (1.11) can be absorbed, and we get (1.3).

For $0 \leqslant s \leqslant \sigma \leqslant t \leqslant T$ we get using (1.10), (1.3) :

$$\begin{aligned}
|\operatorname{Im}\rho_\sigma| &\leqslant C\left(|\operatorname{Im}\rho_t| + \int_\sigma^t |p'|\,d\tau\right)\\
&\leqslant C_1\left(|\operatorname{Im}\rho_t| + (t-s)^{1/2}\left(\int_s^t p\,d\tau\right)^{1/2}\right)\\
&\leqslant C_2\left(|\operatorname{Im}\rho_t| + (t-s)^{1/2}(S(\rho_t) - S(\rho_s) + C(t-s)|\operatorname{Im}\rho_t|^3)^{1/2}\right)\\
&\leqslant C_3\left(|\operatorname{Im}\rho_t| + (t-s)^{1/2}(|\operatorname{Im}\rho_t|^{1/2} + |\operatorname{Im}\rho_s|^{1/2} + (t-s)^{1/2}|\operatorname{Im}\rho_t|^{3/2})\right)\\
&\leqslant C_4(1+(t-s)^{1/2})(|\operatorname{Im}\rho_t|^{1/2}+|\operatorname{Im}\rho_s|^{1/2})
\end{aligned}$$

and this completes the proof.

Let C_t be the union of all (ρ_t,ρ_0) for all integral curves satisfying (1.2). We write $\rho_t = (x_t,\xi_t)$, $\rho_0 = (x_0,\xi_0)$ and define the generating function $\varphi(t,.)$ on C_t by

$$\varphi = \langle x_0,\xi_0\rangle .$$

Somewhat incorrectly we shall write $\varphi = \varphi(t,x_t,\xi_0)$ although φ may possibly be a multivalued function of (t,x_t,ξ_0). We shall estimate $\mathrm{Im}\,\varphi$ *when* (x_t,ξ_0) *is real.* Then $\mathrm{Im}\,\varphi(t,x_t,\xi_0) = S(\rho_t) - S(\rho_0)$ and (1.3) gives

$$\mathrm{Im}\,\varphi(t,x_t,\xi_0) \geqslant \frac{1}{3}\int_0^t p(\mathrm{Re}\,\rho_\tau)\,d\tau - Ct\,|\mathrm{Im}\,\rho_t|^3 . \tag{1.12}$$

LEMMA 1.2. – *When* (x_t,ξ_0) *is real,* $N \geqslant 1$, $\delta > 0$ *sufficiently small, we have*

$$|\mathrm{Im}\,\rho_\tau| \leqslant C\int_0^t |p'(\mathrm{Re}\,\rho_\tau)|\,d\tau, \quad 0 \leqslant \tau \leqslant t . \tag{1.13}$$

Proof. – From (1.5) we get, since ξ_0 is real :

$$|\mathrm{Im}\,\xi_\tau| \leqslant C\left(\int_0^t |p'(\mathrm{Re}\,\rho_\tau)|\,d\tau + \int_0^t |\mathrm{Im}\,\rho_\tau|^2\,d\tau\right)$$

and the same estimate holds for $\mathrm{Im}\,x_\tau$ (using that x_t is real) and hence also for $\mathrm{Im}\,\rho_\tau$. So if $a = \max\limits_{0\leqslant\tau\leqslant t} |\mathrm{Im}\,\rho_\tau|$, we have

$$a \leqslant C\int_0^t |p'(\mathrm{Re}\,\rho_\tau)|\,d\tau + ta^2 . \tag{1.14}$$

Using (1.2) we can absorb the last term and (1.13) follows.

The last term in (1.12) can now be estimated by

$$Ct\Big(\int_0^t |p'(\mathrm{Re}\,\rho_\tau)|\,d\tau\Big)^3 \leqslant C_1 \int_0^t |p'(\mathrm{Re}\,\rho_\tau)|\,d\tau\; t^2 \int_0^t p(\mathrm{Re}\,\rho_\tau)\,d\tau .$$

Thus by (1.2) :

LEMMA 1.3. – *If* $N \geqslant 2$ *and* δ *sufficiently small we have*

$$\mathrm{Im}\,\varphi(t,x_t,\xi_0) \geqslant \frac{1}{4}\int_0^t p(\mathrm{Re}\,\rho_\tau)\,d\tau , \tag{1.15}$$

when (x_t,ξ_0) *is real.*

We finally compare $\operatorname{Im} \varphi(t,x_t,\xi_0)$ with $p(x_t,\xi_0)$ assuming still that (x_t,ξ_0) is real. Let $\ell(\operatorname{Re} \rho_t, \operatorname{Re} \rho_0)$ denote the length of the curve $[0,t] \ni s \longmapsto \operatorname{Re} \rho_s$. Then by (1.5), (1.13) we get

$$\ell(\operatorname{Re} \rho_t, \operatorname{Re} \rho_0) \leqslant Ct \int_0^t |p'(\operatorname{Re} \rho_\tau)| \, d\tau . \tag{1.16}$$

In particular $|x_t - \operatorname{Re} x_\tau|$, $|\xi_0 - \operatorname{Re} \xi_\tau|$ satisfy the same estimate and Taylor's formula gives

$$|p(x_t,\xi_0) - p(\operatorname{Re} \rho_\tau)| \leqslant Ct \int_0^t |p'(\operatorname{Re} \rho_\sigma)| \, d\sigma |p'(\operatorname{Re} \rho_\tau)| + Ct^2 \left(\int_0^t |p'(\operatorname{Re} \rho_\sigma)| \, d\sigma \right)^2 .$$

Integrating this inequality from 0 to t we get

$$\left|tp(x_t,\xi_0) - \int_0^t p(\operatorname{Re} \rho_\tau) \, d\tau\right| \leqslant Ct(1+t^2)\left(\int_0^t |p'(\operatorname{Re} \rho_\tau)| \, d\tau \right)^2 , \tag{1.17}$$

which implies the somewhat weaker estimate

$$\left|tp(x_t,\xi_0) - \int_0^t p(\operatorname{Re} \rho_\tau) \, d\tau\right| \leqslant Ct^2(1+t^2) \int_0^t p(\operatorname{Re} \rho_\tau) \, d\tau . \tag{1.18}$$

In particular

$$\int_0^t p(\operatorname{Re} \rho_\tau) \, d\tau \geqslant \frac{tp(x_t,\xi_0)}{(1+C(t^4+t^2))}$$

which together with (1.15) gives :

LEMMA 1.4. — *For real* x_t, ξ_0 *we have, when* $N \geqslant 2$ *and* $\delta > 0$ *sufficiently small :*

$$\operatorname{Im} \varphi(t,x_t,\xi_0) \geqslant \frac{t}{4(1+C(t^4+t^2))} p(x_t,\xi_0) . \tag{1.19}$$

2. Estimates on the tangents and the curvatures.

To start with we shall study the linearized situation. Let M be a real symplectic vector space of finite dimension, let $\tilde{M}$ be its complexification. Let

$[0,T] \ni t \longmapsto a_t$ be a continuous family of real quadratic forms on M satisfying

$$(2.1) \qquad 0 \leqslant a_t(u,u) \leqslant C_0\|u\|^2, \qquad u \in M$$

for some fixed norm $\| \ \|$ and some fixed constant C_0. Let $A_t : M \to M$ be the corresponding Hamilton matrix defined by

$$(2.2) \qquad \sigma(u,A_t v) = 2a_t(u,v).$$

Here σ is the symplectic form and we extend A_t and σ, a_t to be complex linear on $\tilde{M}$. Let $B_t : \tilde{M} \to \tilde{M}$ be a continuous family of real-linear maps satisfying

$$(2.3) \qquad \|B_t\| \leqslant \varepsilon(T)$$

where

$$(2.4) \qquad \varepsilon(T) = \delta(1+T)^{-N}$$

and $\delta > 0$, $1/N > 0$ will be chosen sufficiently small. We shall first estimate the solutions of the homogeneous equation

$$(2.5) \qquad \frac{du_t}{dt} = \left(\frac{1}{i}A_t + B_t\right)u_t, \qquad 0 \leqslant t \leqslant T,$$

where in place of the function $S(x,\xi)$, we shall use the function

$$(2.6) \qquad [u,u] = \frac{1}{2i}\sigma(u,\bar{u}).$$

Our estimates will only depend on C_0 and the choice of norm, (that we extend to $\tilde{M}$ by putting $\|u_2 + iu_2\|^2 = \|u_1\|^2 + \|u_2\|^2$), but not on the size of T.

PROPOSITION 2.1. — *There is a constant* $C > 0$ *such that*

$$(2.7) \qquad [u_t,u_t] - [u_0,u_0] \geqslant \frac{\ell(u_0,u_t)^2}{Ct} - C(1+t)\varepsilon(T)\|u_{s_0}\|^2, \qquad 0 \leqslant s_0 \leqslant t \leqslant T,$$

$$(2.8) \qquad \|u_s\| \leqslant C(1+t)^{1/2}(\|u_0\| + \|u_t\|), \qquad 0 \leqslant s \leqslant t \leqslant T$$

for all solution curves of (2.5), *provided that* $\delta > 0$ *is sufficiently small, and* $N \geqslant 2$. *Here* $\ell(u_t,u_0)$ *is the length of the curve* $[0,t] \ni s \longmapsto u_s$.

Proof. – Since $A_t u$ is of the same size as $\text{grad}_u a_t(u,u)$; we have

$$\|A_t u\| \leqslant C(a_t(u,u))^{1/2}, \qquad u \in M.$$

For $u = u_1 + iu_2$ in the complexification we then have

$$\|A_t u\|^2 \leqslant (\|A_t u_1\|^2 + \|A_t u_2\|^2) \leqslant C^2(a_t(u_1,u_1) + a_t(u_2,u_2)) = C^2 a_t(u,\bar{u}),$$

$$\text{(2.9)} \qquad \|A_t u\| \leqslant C(a_t(u,\bar{u}))^{1/2}, \qquad u \in \tilde{M}.$$

From (2.3), (2.5) it follows that

$$\text{(2.10)} \qquad \ell(u_t,u_0) \leqslant \int_0^t \|A_s u_s\|\, ds + \varepsilon(T) \int_0^t \|u_s\|\, ds$$

and hence by (2.9) and the Cauchy-Schwartz inequality :

$$\ell(u_t,u_0) \leqslant C(t^{1/2}\left(\int_0^t a_s(u_s,\overline{u_s})\, ds\right)^{1/2} + (\varepsilon(T)T)^{1/2}(\varepsilon(T)\int_0^t \|u_s\|^2\, ds)^{1/2}).$$

If $\delta > 0$ is small enough and $N \geqslant 2$, we have $\varepsilon(T)T \leqslant 1$ and hence

$$\text{(2.11)} \qquad \ell(u_t,u_0)^2 \leqslant Ct \int_0^t a_s(u_s,\overline{u_s})\, ds + \varepsilon(T) \int_0^t \|u_s\|^2\, ds.$$

Next, we notice that since A_t is antisymetric for σ :

$$\left[\frac{1}{i} A_t u, u\right] + \left[u, \frac{1}{i} A_t u\right] = \sigma(u, A_t \bar{u}) = 2a_t(u,\bar{u}),$$

so by (2.3), (2.5)

$$\frac{d}{dt}[u_t,u_t] = 2a_t(u_t,\overline{u_t}) + \mathcal{O}(\varepsilon(T)\|u_t\|^2).$$

Hence

$$\text{(2.12)} \qquad [u_t,u_t] - [u_0,u_0] \geqslant 2 \int_0^t a_s(u_s,\overline{u_s})\, ds - C\varepsilon(T) \int_0^t \|u_s\|^2\, ds.$$

Combining (2.11), (2.12) we get

(2.13)

$$[u_t,u_t] - [u_0,u_0] \leqslant \frac{\ell(u_t,u_0)^2}{Ct} - C\varepsilon(T)\left(1 + \frac{1}{t}\right) \int_0^t \|u_s\|^2\, ds.$$

Now we use that $||u_s|| \leqslant ||u_{s_0}|| + \ell(u_0,u_t)$ for $s, s_0 \in [0,t]$ and get

$$(2.14)\quad [u_t,u_t] - [u_0,u_0] \geqslant \frac{\ell(u_t,u_0)^2}{Ct} - C\varepsilon(T)(1+t)(||u_{s_0}||^2 + \ell(u_0,u_t)^2).$$

Now if $N \geqslant 2$ and δ is sufficiently small we have $C\varepsilon(T)(1+t) \ll \frac{1}{Ct}$ and (2.7) follows.

For $0 \leqslant s \leqslant t$ we then have

$$||u_s||^2 \leqslant C(||u_0||^2 + \ell(u_0,u_t)^2) \leqslant C(Ct([u_t,u_t]-[u_0,u_0]) + 2||u_0||^2),$$

which implies (2.8) and the proof is complete.

We next study the transversality properties of the real linear subspace $C_t = \{(u_t,u_0)\,;\, u_0 \in \tilde{M}\} \subset \tilde{M} \times \tilde{M}$, where u_t are solution curves of (2.5). Let $M = \mathbf{R}^n \times \mathbf{R}^n$ with the standard symplectic form, so that $\tilde{M} = \mathbf{C}^n \times \mathbf{C}^n$.

PROPOSITION 2.2. – *If* $N \geqslant 2$ *and* $\delta > 0$ *sufficiently small, then there is a constant* $C > 0$ *such that*

$$(2.15)\qquad ||\xi|| + ||y|| \leqslant C(1+t)(||x|| + ||\eta||)$$

for all $(x,\xi,y,\eta) \in \mathscr{C}_t$.

Proof. – We shall actually prove a little more : Let $L \subset \tilde{M} \times \tilde{M}$ be a complex linear canonical relation which is negative. More precisely we assume that $\dim L = \dim \tilde{M}$, $\sigma' - \sigma''|_L = 0$, where $\sigma'(\sigma'')$ is the symplectic form on the first (second) copy of $\tilde{M}$ in $\tilde{M} \times \tilde{M}$ and that $[u,u] - [v,v] \leqslant 0$ for all $(u,v) \in L$. (We shall take $L = \{(0,\xi,y,0)\,;\, y, \xi \in \mathbf{C}^n\}$ below). We also assume that L and $\mathscr{C}_0$ = graph (Identity) are transversal.

To measure the degree of transversality between L and $\mathscr{C}_t$, choose $(u_t,u_0) \in \mathscr{C}_t$ and distinguish two cases :

Case 1 : $||u_t - u_0|| < \alpha||u_0||$, where $\alpha > 0$ is small. Then if d denotes the distance :

$$\begin{aligned} d((u_t,u_0),L) &\geqslant d((u_0,u_0),L) - ||u_t - u_0|| \\ &\geqslant C||u_0|| - \alpha||u_0|| \geqslant \frac{C}{2}||(u_t,u_0)||. \end{aligned}$$

Here C is the constant measuring the degree of transversality between $\mathscr{C}_0$ and L.

Case 2 : With $\alpha > 0$ small but fixed, assume that $\|u_t - u_0\| \geqslant \alpha\|u_0\|$. Let $(v,w) \in L$ be the point which satisfies :

$$d((u_t,u_0), (v,w)) = d((u_t,u_0),L).$$

Now $\ell(u_t,u_0)$ is at least of the same order of magnitude as $\|u_0\|$, so if we choose $s_0 = 0$ in (2.7), the remainder term there can be absorbed, and we get

$$[u_t,u_t] - [u_0,u_0] \geqslant \frac{\ell(u_0,u_t)^2}{Ct}. \tag{2.16}$$

We estimate $[u_t,u_t] - [u_0,u_0]$ from above :

$$\begin{aligned} [u_t,u_t] &= [u_t-v,u_t-v] + [u_t-v,v] + [v,u_t-v] + [v,v] \\ -[u_0,u_0] &= -[u_0-w,u_0-w] - [u_0-w,w] - [w,u_0-w] - [w,w]. \end{aligned}$$

Since $[v,v] - [w,w] \leqslant 0$ by the negativity of L, we get

$$[u_t,u_t] - [u_0,u_0] \leqslant C(\|u_t - v\|^2 + \|v\|\,\|u_t - v\| + \|w\|\,\|u_0 - w\| + \|u_0 - w\|^2),$$

so

$$[u_t,u_t] - [u_0,u_0] \leqslant C(d((u_t,u_0), (v,w))^2 + \|(v,w)\|\, d((u_t,u_0), (v,w))).$$

Since (v,w) minimizes the distance to (u_t,u_0) we have

$$\|(v,w)\| \leqslant \|(u_t,u_0)\|, \qquad d((u_t,u_0), (v,w)) \leqslant \|(u_t,u_0)\|, \tag{2.17}$$

(at least after choosing $\|\ \ \|$ to be a Hilbert space norm) so

$$[u_t,u_t] - [u_0,u_0] \leqslant C\|(u_t,u_0)\|\, d((u_t,u_0),L). \tag{2.18}$$

On the other hand,

$$\begin{aligned} \|(u_t,u_0)\| \leqslant C(\|u_t - u_0\| + \|u_0\|) &\leqslant C\left(1 + \frac{1}{\alpha}\right)\|u_t - u_0\| \\ &\leqslant C\left(1 + \frac{1}{\alpha}\right)\ell(u_t,u_0) \end{aligned}$$

so (2.16), (2.18) give

$$d((u_t,u_0),L) \geqslant \frac{C}{t}\,\|(u_t,u_0)\|. \tag{2.19}$$

Combining the two cases we get in general

$$d((u_t,u_0),\mathrm{L}) \geqslant \frac{\mathrm{C}}{(1+t)}\,\|(u_t,u_0)\|$$

and with the choice of L indicated above, (2.15) follows.

PROPOSITION 2.3. — *Let* $a_t = a$ *be independent of* t *and assume that* $\mathbf{B}_t = 0$. *Then* $\mathscr{C}_t$ *is of the form*

$$\xi = \Phi'_x(t,x,\eta), \qquad y = \Phi'_\eta(t,x,\eta) \tag{2.20}$$

where Φ *is a quadratic form in* (x,η) *satisfying*

$$\operatorname{Im}\Phi(t,x,\eta) \geqslant \frac{t}{\mathrm{C}(1+t^2)}\,a(x,\eta,x,\eta),\ (x,\eta)\in\mathbf{R}^{2n}. \tag{2.21}$$

Proof. — When $\mathbf{B}_t = 0$, $\mathscr{C}_t$ will be a canonical relation and proposition 2.2 implies (2.20) with $\Phi(t,x,\eta) = \frac{1}{2}(\langle x,\xi\rangle + \langle y,\eta\rangle)|_{\mathscr{C}_t}$. An easy computation shows that for $(x,\xi,y,\eta)\in\mathscr{C}_t$

$$[(x,\xi),(x,\xi)] - [(y,\eta),(y,\eta)] = 2\operatorname{Im}\Phi(t,x,\eta)$$

when (x,η) is real.

On the other hand, with $u_t = (x,\xi)$, $u_0 = (y,\eta)$, (2.7) and (2.12) simplify to

$$[u_t,u_t] - [u_0,u_0] \geqslant \frac{\ell(u_0,u_t)^2}{\mathrm{C}t}, \tag{2.22}$$

$$[u_t,u_t] - [u_0,u_0] \geqslant 2\int_0^t a(u_s,\overline{u_s})\,ds. \tag{2.23}$$

Since $\sup\limits_{0\leqslant s\leqslant t}\|u_s - v\| \leqslant \mathrm{C}\ell(u_0,u_t)$ if $v = (x,\eta)$,

$$\begin{aligned} ta(v,\bar v) = \int_0^t a(v,\bar v)\,ds &\leqslant 2\int_0^t a(v-u_s,\overline{v-u_s})\,ds + 2\int_0^t a(u_s,\bar u_s)\,ds \\ &\leqslant \mathrm{C}(t\ell(u_0,u_t)^2 + [u_t,u_s] - [u_0,u_0]) \\ &\leqslant \mathrm{C}_1(1+t^2)([u_t,u_t] - [u_0,u_0]). \end{aligned}$$

Thus

$$[u_t,u_t] - [u_0,u_0] \geqslant \frac{t}{\mathrm{C}(1+t^2)}\,a(v,\bar v)$$

and (2.21) follows.

Assuming still (2.1), (2.3) we next study the inhomogeneous equation

$$\frac{\partial u_s}{\partial s} = \left(\frac{1}{i}\,\mathrm{A}_s + \mathrm{B}_s\right)u_s + v_s, \tag{2.24}$$

where v_s depends continuously on $s \in [0,\mathrm{T}]$. We write $(u_t)_x, (u_t)_\xi$ for the x, ξ-components of x, ξ (identifying M with $\mathbf{R}^n \times \mathbf{R}^n$).

PROPOSITION 2.4. — *If* $\delta > 0$ *is sufficiently small and* $\mathrm{N} \geqslant 2$ *in* (2.4), *then there is a constant* $\mathrm{C} > 0$ *such that if* $u, v : [0,\mathrm{T}] \to \tilde{\mathrm{M}}$ *satisfy* (2.24) *and* $(u_t)_x = (u_0)_\xi = 0$ *for some* $t \in [0,\mathrm{T}]$, *then*

$$\sup_{0 \leqslant s \leqslant t} \|u_s\| \leqslant \mathrm{C}(1+t)t \sup_{0 \leqslant s \leqslant t} \|v_s\|.$$

Proof. — The assumptions imply that

$$[u_t,u_t] - [u_0,u_0] = 0. \tag{2.25}$$

Put $\mathrm{U} = \sup_{0 \leqslant s \leqslant t} \|u_s\|$, $\mathrm{V} = \sup_{0 \leqslant s \leqslant t} \|v_s\|$. Integrating (2.24) gives

$$\ell(u_0,u_t) \leqslant \mathrm{C}t^{1/2}\left(\int_0^t a_s(u_s,\bar{u}_s)\,ds\right)^{1/2} + \mathrm{C}t\varepsilon(\mathrm{T})\mathrm{U} + \mathrm{C}t\mathrm{V}. \tag{2.26}$$

Also for $0 \leqslant s \leqslant t$:

$$\frac{d}{ds}[u_s,u_s] \geqslant 2a_s(u_s,\bar{u}_s) - \mathrm{C}\varepsilon(\mathrm{T})\|u_s\|^2 - \mathrm{C}\|u_s\|\mathrm{V}, \tag{2.27}$$

so an integration gives :

$$[u_t,u_t] - [u_0,u_0] \geqslant 2\int_0^t a_s(\bar{u}_s,u_s)\,ds - \mathrm{C}\varepsilon(\mathrm{T})t\mathrm{U}^2 - \mathrm{C}t\mathrm{U}\mathrm{V}, \tag{2.28}$$

and (2.25) then implies :

$$\int_0^t a_s(u_s,\bar{u}_s)\,ds \leqslant \mathrm{C}t\varepsilon(\mathrm{T})\mathrm{U}^2 + \mathrm{C}t\mathrm{U}\mathrm{V}. \tag{2.29}$$

Combining this with (2.26) shows that

$$\ell(u_0,u_t)^2 \leqslant \mathrm{C}t(t\varepsilon(\mathrm{T})\mathrm{U}^2 + t\mathrm{U}\mathrm{V}) + \mathrm{C}(t^2\varepsilon(\mathrm{T})^2\mathrm{U}^2 + t^2\mathrm{V}^2). \tag{2.30}$$

Hence

$$\ell(u_0,u_t)^2 \leqslant C(t^2\varepsilon(T)U^2 + t^2UV + t^2V^2). \tag{2.31}$$

Since $(u_t)_x = 0$, $(u_0)_\xi = 0$ we have $U \leqslant C\ell(u_0,u_t)$ and we get

$$U^2 \leqslant C(t^2\varepsilon(T)U^2 + t^2UV + t^2V^2). \tag{2.32}$$

For every $\alpha > 0$,

$$t^2UV \leqslant \frac{\alpha}{2}U^2 + \frac{1}{2\alpha}t^4V^2 ;$$

so (2.32) implies

$$U^2 \leqslant C\left(\left(t^2\varepsilon(T) + \frac{\alpha}{2}\right)U^2 + \left(t^2 + \frac{1}{2\alpha}t^4\right)V^2\right). \tag{2.33}$$

Choosing δ, α sufficiently small, we can absorb the first term to the right and the proposition follows.

We shall now apply our linearized estimates to study the tangents and curvatures of C_t, introduced in section 1. Let $[0,T] \ni t \longmapsto \rho_t$ be an integral curve of $\frac{1}{i}H_p$ satisfying (1.2). We also assume that $\rho_t = \rho_t(r)$ is a smooth function of a real parameter r. Differentiating the equation

$$\frac{\partial \rho_t}{\partial t} = \frac{1}{i}H_p(\rho_t) \tag{2.34}$$

with respect to r, we get with $\rho'_t = \frac{\partial \rho_t}{\partial r}$:

$$\frac{\partial \rho'_t}{\partial t} = \frac{1}{i}\frac{\partial H_p}{\partial \rho}\rho'_t + \frac{1}{i}\frac{\partial H_p}{\partial \bar\rho}\overline{\rho'_t}.$$

Hence

$$\frac{\partial \rho'_t}{\partial t} = \frac{1}{i}F_t\rho'_t + B_t^{(1)}\ \rho'_t \tag{2.35}$$

where F_t is the Hamilton matrix of p at ρ_t and $B_t^{(1)}$ is a real-linear matrix satisfying $\|B_t^{(1)}\| = \mathcal{O}(|\mathrm{Im}\ \rho_t|^N)$ for all N.

In general if $f(x) \geqslant 0$ is a smooth function then (locally) from the estimate $f(x+t) + f(x-t) \geqslant 0$, we get $f(x) + \frac{1}{2}\langle f''(x)t, t\rangle \geqslant -\mathcal{O}(t^4)$, and it follows that locally :

$$f''(x) \geqslant -Cf(x)^{1/2}I.$$

Thus if $C > 0$ is sufficiently large the form

$$a_t(u,u) = \frac{1}{2}\langle p''(\operatorname{Re}\rho_t)u, u\rangle + Cp(\operatorname{Re}\rho_t)^{1/2}\|u\|^2$$

is positive semi-definite and we define A_t to be the corresponding Hamilton matrix. Then from (2.35) :

$$\frac{\partial \rho_t'}{\partial t} = \frac{1}{i}A_t\rho_t' + B_t\rho_t' \tag{2.36}$$

where $\|B_t\| \leqslant C(|\operatorname{Im}\rho_t| + p(\operatorname{Re}\rho_t)^{1/2})$. Assume now that $T \geqslant 1$. Then for $t \in [0,T]$, let I be an interval of length 1 such that $t \in I \subset [0,T]$. Then

$$\begin{aligned} p(\operatorname{Re}\rho_t) &= \int_I p(\operatorname{Re}\rho_t)\,ds \leqslant \int_0^T p(\operatorname{Re}\rho_s)\,ds + \int_I p(\operatorname{Re}\rho_t) - p(\operatorname{Re}\rho_s)\,ds \\ &\leqslant \int_0^T p(\operatorname{Re}\rho_s)\,ds + C\int_I |p'(\operatorname{Re}\rho_s)|^2\,ds + C\int_I |\rho_t - \rho_s|^2\,ds. \end{aligned}$$

From (1.5), (1.10) it follows that for $s \in I$:

$$|\rho_t - \rho_s| \leqslant C\left(\int_I |p'(\operatorname{Re}\rho_\sigma)|d\sigma + |\operatorname{Im}\rho_t|\right)$$

so we get

$$p(\operatorname{Re}\rho_t) \leqslant C\,(|\operatorname{Im}\rho_t|^2 + \int_0^T |p'(\operatorname{Re}\rho_s)|ds) \leqslant C\tilde{\varepsilon}(T).$$

Hence

$$\|B_t\| \leqslant C\tilde{\varepsilon}(T)^{1/2}, \tag{2.37}$$

so to take $N \geqslant 4$ in (1.1) corresponds to take $N \geqslant 2$ in (2.4), and all our estimates so far are valid.

PROPOSITION 2.5. — *Let* $N \geqslant 4$ *in* (1.1) *and* $\delta > 0$ *sufficiently small. Then there is a constant* $C > 0$ *such that for all* $(\rho_t, \rho_0) \in C_t$, $(\delta\rho_t, \delta\rho_0) = (\delta x, \delta\xi, \delta y, \delta\eta) \in T_{(\rho_t,\rho_0)}(C_t) = \mathscr{C}_t$:

$$\|\delta\xi\| + \|\delta y\| \leqslant C(1+t)(\|\delta x\| + \|\delta\eta\|), \tag{2.38}$$

$$|\sigma(u_t,v_t) - \sigma(u_0,v_0)| \leqslant C_N t(1+t)^{N+\frac{1}{2}} |\mathrm{Im}\,(\rho_t,\rho_0)|^N \|(u_t,u_0)\| \cdot \|(v_t,v_0)\| \tag{2.39}$$

for all $N \geqslant 0$, (u_t,u_0), $(v_t,v_0) \in \mathscr{C}_t$,

$$d(i\mathscr{C}_t,\mathscr{C}_t) \leqslant C_N t(1+t)^{N+\frac{3}{2}} |\mathrm{Im}\,(\rho_t,\rho_0)|^N. \tag{2.40}$$

Here d *denotes a distance in the appropriate Grassmannian manifold.*

Proof. — (2.38) follows from proposition 2.2. (The case $T < 1$ can easily be treated along the same lines and is in fact already treated in [5].) To prove (2.39) we replace ρ_t' by u_t and v_t in (2.35). Then we obtain for all $N \geqslant 0$:

$$\frac{d}{ds}\sigma(u_s,v_s) = \mathcal{O}(1)\,|\mathrm{Im}\,\rho_s|^{2N}\|u_s\| \cdot \|v_s\|.$$

Integrating and using (1.4), (2.8) we get (2.39).

Let $\mathscr{I}$ denote multiplication by i in $\tilde{M}$, the complexification of $T^*\mathbf{R}^n$. Let $(u_t,u_0) \in \mathscr{C}_t$ and let α be its projection in the (x,η)-space. Let $(v_t,v_0) \in \mathscr{C}_t$ be the vector which projects to $\mathscr{I}\alpha$. From (2.35) we get

$$\frac{\partial u_s}{\partial s} = G_s u_s, \qquad \frac{\partial v_s}{\partial s} = G_s v_s, \qquad 0 \leqslant s \leqslant t \tag{2.41}$$

where $[G_s,\mathscr{I}] = \mathcal{O}(|\mathrm{Im}\,\rho_s|^{2N})$ for all N. Then

$$\frac{\partial}{\partial s}(\mathscr{I}u_s - v_s) = G_s(\mathscr{I}u_s - v_s) - [G_s,\mathscr{I}]u_s. \tag{2.42}$$

Since $(\mathscr{I}u_t - v_t)_x = (\mathscr{I}u_0 - v_0)_\eta = 0$, we can apply proposition 2.4 and (2.8), (1.4)

$$\sup_{0\leqslant s\leqslant t} \|\mathscr{I}u_s - v_s\| \leqslant C_N t(1+t)^{N+\frac{3}{2}} |\mathrm{Im}\,(\rho_t,\rho_0)|^N \|(u_t,u_0)\|. \tag{2.43}$$

In particular, the same estimate holds for $\|\mathscr{I}(u_t,u_0) - (v_t,v_0)\|$ and (2.40) follows.

Next we study the curvatures of C_t. Differentiating (2.34) N times with respect to r we get

$$\frac{\partial \rho_t^{(N)}}{\partial t} = \frac{1}{i}\frac{\partial H_p}{\partial \rho}\rho_t^{(N)} + \frac{1}{i}\frac{\partial H_p}{\partial \bar{\rho}}\overline{\rho_t^{(N)}} \tag{2.44}$$
$$+ \sum_{|k_1|+\cdots+(N-1)|k_{N-1}|+|\ell_1|+\cdots+(N-1)|\ell_{N-1}|=N} a_{k,\ell}(\rho_t)(\rho_t^{(1)})^{k_1}\dots (\rho_t^{(N-1)})^{k_{N-1}}(\overline{\rho_t^{(1)}})^{\ell_1}\dots(\overline{\rho^{(N-1)}})^{\ell_{N-1}}.$$

Here $\rho_t^{(N)} = \dfrac{\partial^N \rho_t}{\partial r^N}$, k_j, ℓ_j are multiindices and $a_{k,\ell}$ are smooth functions (and in particular bounded, since ρ_t stays in a compact set). (2.44) gives

$$\frac{\partial \rho_t^{(N)}}{\partial t} = \left(\frac{1}{i}A_t + B_t\right)\rho_t^{(N)} \tag{2.45}$$
$$+ \mathcal{O}(1)\sum_{k_1+\cdots+(N-1)k_{N-1}=N} \|\rho_t^{(1)}\|^{k_1}\dots\|\rho_t^{(N-1)}\|^{k_{N-1}}.$$

Here we recall the inequality

$$a_1^{\alpha_1}\dots a_N^{\alpha_N} \leqslant \sum_1^N \alpha_j a_j, \tag{2.46}$$

when $a_j \geqslant 0$, $\alpha_j \geqslant 0$, $\Sigma\alpha_j = 1$.

Writing $\|\rho_t^{(j)}\|^{k_j} = (\|\rho_t^{(j)}\|^{N/j})^{jk_j/N}$, we get

$$\|\rho_t^{(1)}\|^{k_1}\dots\|\rho_t^{(N-1)}\|^{k_{N-1}} \leqslant \sum_1^{N-1}\frac{jk_j}{N}\|\rho_t^{(j)}\|^{N/j},$$

since $\Sigma jk_j/N = 1$. Then (2.45) implies

$$\frac{\partial \rho_t^{(N)}}{\partial t} = \left(\frac{1}{i}A_t + B_t\right)\rho_t^{(N)} + \mathcal{O}(1).\sum_1^{N-1}\|\rho_t^{(j)}\|^{N/j}. \tag{2.47}$$

PROPOSITION 2.6. – *Let* $N \geqslant 4$ *in* (1.4) *and* $\delta > 0$ *sufficiently small. For a fixed* $t > 0$, *let* $I \ni r \longmapsto (\rho_t(r), \rho_0(r)) \in C_t$ *be a curve such that* $((\rho_t^{(1)}(r))_x, (\rho_0^{(1)}(r))_\xi)$ *is constant and of length* 1. (Here I is some interval). *Then there exist constants* $C_k > 0$ *independent of* t *and the choice of the curve above, such that*

$$\sup_{0\leqslant s\leqslant t}\|\rho_s^{(k)}\| \leqslant C_k(1+t)^{k\left(\frac{3}{2}+2\log k\right)} \tag{2.48}$$

Proof. – From (2.38), (2.8) we first obtain (2.48) for $k = 1$. Put

$$\mu_k = \sup_{0\leqslant s\leqslant t} \|\rho_s^{(k)}\|.$$

We shall prove by induction that

$$(2.49)_k \qquad \mu_k \leqslant C_k(1+t)^{a_k}.$$

This clearly holds for $k = 1$, with $a_k = 3/2$. Assume $(2.49)_k$ for $1 \leqslant k \leqslant N - 1$. By (2.47) :

$$(2.50) \qquad \frac{\partial\rho_s^{(N)}}{\partial s} = \left(\frac{1}{i}A_s + B_s\right)\rho_s^{(N)} + \mathcal{O}(1)\sum_{1}^{N-1}(1+t)^{Na_j/j}.$$

Since $(\rho_t^{(N)})_x = (\rho_0^{(N)})_\xi = 0$, proposition 2.4 shows that $(2.49)_N$ holds with

$$(2.51) \qquad a_N \geqslant 2 + \max_{1\leqslant k\leqslant N-1} N\frac{a_k}{k},\ a_1 \geqslant \frac{3}{2}.$$

With $b_N = \max_{1\leqslant k\leqslant N} a_k/k$, (2.51) is equivalent to

$$(2.52) \qquad b_1 \geqslant \frac{3}{2}, \qquad b_N \geqslant \frac{2}{N} + b_{N-1}.$$

The optimal choice is therefore

$$b_N = \frac{3}{2} + \sum_{2}^{N}\frac{2}{k} \leqslant \frac{3}{2} + 2\log N,$$

so we can take $a_N = N\left(\frac{3}{2} + 2\log N\right)$, and the proposition follows.

3. The phase function.

We still consider the situation described in the beginning of section 1. In this section we keep $N = N_0$ in (1.4) fixed, $\geqslant 4$, and choose $\delta > 0$ sufficiently small so that all the estimates of section 1 and 2 are valid. As above let C_t be the set of points (ρ_t,ρ_0), $\rho_t = \exp tH_{\frac{1}{i}p}(\rho_0)$ with the restriction (1.2) and the restriction that ρ_0 belongs to a small open neighborhood W of a real point where p vanishes. Put

$$\tilde{C} = \{(t,\rho_t,\rho_0)\,;\, t \geqslant 0,\, (\rho_t,\rho_0) \in C_t\}.$$

An element in $T_{(t,\rho_t,\rho_0)}(\tilde{C})$ can be written $h = (k,u_t + kH_{\frac{1}{i}p},u_0)$ with $k \in \mathbf{R}$, $(u_t,u_0) \in T_{(\rho_t,\rho_0)}(C_t)$, so

$$((u_t)_x,(u_0)_\eta) = \mathcal{O}(|h_{(x,\eta)}| + |k|) = \mathcal{O}(|h_{(t,x,\eta)}|).$$

Thus by (2.15); $(u_t,u_0) = \mathcal{O}((1+t)|h_{(t,x,\eta)}|)$ and we conclude that the projection

$$\text{(3.1)} \qquad T_{(t,\rho_t,\rho_0)}(\tilde{C}) \ni (t,v_t,v_0) \to (t,(v_t)_x,(v_0)_\xi) \in \mathbf{R} \times \mathbf{C}^{2n}$$

is bijective, with inverse $= \mathcal{O}((1+t))$. The projection

$$\text{(3.2)} \qquad \tilde{C} \ni (t,x,\xi,y,\eta) \longmapsto (t,x,\eta) \in \mathbf{R} \times \mathbf{C}^{2n}$$

is therefore *locally* a diffeomorphism, and by restricting it suitably we shall achieve injectivity.

Fix a point $(x,\eta) \in W$. Then for $t \geqslant 0$ sufficiently small, there is a unique point $(x,\xi_t,y_t,\eta) \in C_t$ depending smoothly on t. Let $T_\delta(x,\eta) \in]0,+\infty]$ be the largest number such that there exists $(x,\xi_t,y_t,\eta) \in C_t$, *depending smoothly on* t, for $0 \leqslant t < T_\delta(x,\eta)$. Clearly ξ_t, y_t are unique and we put

$$\Omega_\delta = \{(t,x,\eta)\,;\, 0 \leqslant t < T_\delta(x,\eta)\}.$$

Then Ω_δ is open and over Ω_δ we have a unique « branch » of $\tilde{C}$ given by smooth equations

$$\text{(3.3)} \qquad \xi = H(t,x,\eta), \qquad y = G(t,x,\eta).$$

For $(t,x,\eta) \in \Omega_\delta$, let $[0,t] \ni s \longmapsto \rho_s(t,x,\eta)$ be the corresponding integral curve of $H_{\frac{1}{i}p}$ and put

$$\text{(3.4)} \qquad \gamma(t,x,\eta) = \sup_{0 \leqslant s \leqslant t} |\operatorname{Im} \rho_s| + \int_0^t |p'(\operatorname{Re} \rho_s)|\, ds.$$

LEMMA 3.1. — *There is a constant* $C > 0$ *such that for all* (t,x,η) :

$$\text{(3.5)} \qquad \varlimsup_{(t',x',\eta') \to (t,x,\eta)} \frac{|\gamma(t',x',\eta') - \gamma(t,x,\eta)|}{\|(t-t',x-x',\eta-\eta')\|} \leqslant C(1+t)^{5/2}.$$

Proof. – Let $r \to (t(r),x(r),\eta(r)) \in \Omega_\delta$ be a C^1-curve with derivative bounded by 1 in norm, and let

$$[0,t(r)] \ni s \longmapsto \rho_s(r) = (x_s(r),\xi_s(r))$$

be the corresponding integral curves of $H_{\frac{1}{i}p}$ so that $x_{t(r)}(r) = x(r)$, $\xi_0(r) = \eta(r)$. Since the inverse of (3.1) is $\mathcal{O}((1+t))$ we have

$$\left\|\frac{d\rho_{t(r)}(r)}{dr}\right\| + \left\|\frac{d\rho_0(r)}{dr}\right\| = \mathcal{O}((1+t(r))).$$

Then (2.8) shows that

$$\frac{\partial \rho_s(r)}{\partial r} = \mathcal{O}((1+t(r))^{3/2}), \qquad s \in [0,t(r)].$$

It follows easily that for r' close to r,

$$|\gamma(t(r'),x(r'),\eta(r')) - \gamma(t(r),x(r),\eta(r))| \\ = \mathcal{O}((1+t(r))^{5/2}|r - r'| + (1+t(r))^4|r - r'|^2),$$

and we deduce (3.5).

Now assume that for some (x,η) and some $\delta' > 0$, we have $T_\delta(x,\eta) < \infty$ and

$$\gamma(t,x,\eta) \leqslant \frac{\delta}{(1+t)^{N_0}} - \delta', \qquad 0 \leqslant t < T_\delta(x,\eta).$$

Then it is clear that $\lim_{t\to T_\delta} (x,\xi_t,y_t,\eta)$ exists in C_{T_δ} and this contradicts the maximality of T_δ. Thus, in view of the local Lipschitz property of γ :

$$\text{(3.6)} \qquad \lim_{t\to T_\delta} \gamma(t,x,\eta) = \frac{\delta}{(1+T_\delta(x,\eta))^{N_0}}, \qquad \text{when} \qquad T_\delta < \infty.$$

From now on we denote also by $\tilde{C}$, C_t the restricted parts given by (3.3).

To estimate the derivatives of H and G we let ν be a constant vectorfield in the (x,η)-space of length $\leqslant 1$. Proposition 2.6 then shows that

$$\text{(3.7)} \qquad \left(\frac{\partial}{\partial \nu}\right)^N H, \left(\frac{\partial}{\partial \nu}\right)^N G = \mathcal{O}((1+t)^{aN}), \qquad (t,x,\eta) \in \Omega_\delta,$$

for all N, where, here and below a_N, $a_{k,\alpha,\beta}$, etc. will denote some positive constants, that we shall not try to estimate closer. Moreover (3.7) is uniform with respect to ν. Since the vectorfield $\frac{\partial}{\partial t} + H_{\frac{1}{i}p}$ is tangential to $\tilde{C}$ we have

$$\left(\frac{\partial}{\partial t} + (H_{\frac{1}{i}p})_x\right)(H_{\frac{1}{i}p})_\xi = 0$$

$$\left(\frac{\partial}{\partial t} + (H_{\frac{1}{i}p})_x\right)G(t,x,\eta) = 0$$

where $(H_{\frac{1}{i}p})_x$, $(H_{\frac{1}{i}p})_\xi$ are the x and ξ-components of the Hamilton field, computed at the point $\xi = H(t,x,\eta)$. Differentiating these equations successively with respect to t, we get

$$\text{(3.8)} \qquad D_t^k D_{x,\eta}^\alpha H, \qquad D_t^k D_{x,\eta}^\alpha G = \mathcal{O}((1+t)^{a_{\alpha,k}}),$$

if we first notice that (3.8) for $k = 0$ follows from (3.7). (By D_x, D_η we denote derivatives with respect to the real variables $\operatorname{Re} x$, $\operatorname{Im} x$, $\operatorname{Re} \eta$, $\operatorname{Im} \eta$.)

From proposition 2.5 if follows that

$$\text{(3.9)} \qquad \|\bar{\partial}_{(x,\eta)}H\| + \|\bar{\partial}_{(x,\eta)}G\| = \mathcal{O}(1)(1+t)^{a_N}\gamma(t,x,\eta)^N.$$

To estimate the derivatives here we put $u = \bar{\partial}H$, $(u = \bar{\partial}G)$ and let $(t,x,\eta) \in \Omega_\delta$, $v = (t,x,\eta) \in \mathbf{R} \times \mathbf{C}^{2n}$, $\|v\| \leqslant \frac{\delta}{C(1+t)^{5/2+N_0}}$ so that $(t,x,\eta) + v \in \Omega_{2\delta}$ in view of lemma 3.1. (3.8) and Taylor's formula imply that for a certain $k_2 > 0$:

$$\langle u'(t,x,\eta),v)\rangle = u'((t,x,\eta)+v) - u(t,x,\eta) + \mathcal{O}((1+t)^{k_2}\|v\|^2).$$

Here u' is the gradient in the real sense. Choosing $v = C_1^{-1}(1+t)^{-k_1}u'$, where $k_1 > k_2$ and C_1 are so large that $\|v\| \leqslant \frac{\delta}{C}(1+t)^{-5/2-N_0}$, we get

$$\frac{\|u'\|^2}{C_1(1+t)^{k_1}} = \mathcal{O}\left((1+t)^{a_N}\gamma^N + \frac{\|u'\|^2}{C_1^2(1+t)^{2k_1-k_2}}\right)$$

so when $k + |\alpha| + |\beta| = 1$, we get for all $N \geqslant 0$,

$$\text{(3.10)} \quad \|D_t^k D_x^\alpha D_\eta^\beta \bar{\partial}H\| + \|D_t^k D_x^\alpha D_\eta^\beta \bar{\partial}G\| = \mathcal{O}((1+t)^{a_{k,\alpha,\beta,N}}\gamma^N).$$

Repeating the same argument we get (3.10) for all k, α, β, N.

Since C_t is conic, (2.39) implies with $\xi\, dx = \xi_1\, dx_1 + \cdots + \xi_n\, dx_n$ etc.,

$$(3.11)\qquad \xi\, dx - \eta\, dy|_{C_t} = \mathcal{O}(1)(1+t)^{a_N}\gamma^N, \qquad \forall N \geqslant 0,$$

and we claim that

$$(3.12)\qquad ip(x,\xi)\, dt + \xi\, dx - \eta\, dy|_{\tilde{C}} = \mathcal{O}(1)(1+t)^{a_N}\gamma^N, \qquad \forall N \geqslant 0.$$

By (3.11) this certainly holds when applied to tangent vectors with vanishing t-component, so it suffices to apply the form in (3.12) to the tangent-vector $\frac{\partial}{\partial t} + H_{\frac{1}{i}p}$. By the almost analyticity and the homogeneity we obtain

$$ip + \frac{1}{i} < \frac{\partial p}{\partial \xi}, \ \xi > + \mathcal{O}(\gamma^N) = \mathcal{O}(\gamma^N), \qquad \forall N,$$

so (3.12) follows.

Now recall that the generating function, φ is defined as

$$(3.13)\qquad \varphi(t,x,\eta) = \langle y,\eta\rangle|_{\tilde{C}}.$$

Mostly, we consider φ as a function on Ω_δ, but sometimes as a function on $\tilde{C}$. From (3.12), (3.13), we get

$$(3.14)\qquad d\varphi = \eta\, dy + y\, d\eta|_{\tilde{C}} = ip(x,\xi)\, dt + \xi\, dx + y\, d\eta + \mathcal{O}(1)(1+t)^{a_N}\gamma^N.$$

On the other hand

$$(3.15)\qquad d\varphi = \frac{d\varphi}{\partial t}\, dt + \frac{\partial \varphi}{\partial x}\, dx + \frac{\partial \varphi}{\partial \eta}\, d\eta + \frac{\partial \varphi}{\partial \bar{x}}\, \overline{dx} + \frac{\partial \varphi}{\partial \bar{\eta}}\, \overline{d\eta},$$

so comparing (3.14), (3.15), we get

$$(3.16)\qquad \frac{\partial \varphi}{\partial \bar{x}}, \quad \frac{\partial \varphi}{\partial \bar{\eta}} = \mathcal{O}(1)(1+t)^{a_N}\gamma^N, \qquad \forall N,$$

$$(3.17)\qquad \xi - \frac{\partial \varphi}{\partial x}, \quad y - \frac{\partial \varphi}{\partial \eta} = \mathcal{O}(1)(1+t)^{a_N}\gamma^N, \qquad \forall N, \ \text{on } \tilde{C},$$

$$(3.18)\qquad \frac{\partial \varphi}{\partial t} + \frac{1}{i}\, p\left(x, \frac{\partial \varphi}{\partial x}\right) = \mathcal{O}(1)(1+t)^{a_N}\gamma^N, \qquad \forall N.$$

It follows from (3.8), that

$$\text{(3.19)}\quad D_t^k D_x^\alpha D_\eta^\beta \varphi(t,x,\eta) = \mathcal{O}(1)(1+t)^{a_{N,k,\alpha,\beta}}, \qquad \forall N,\ k,\ \alpha,\ \beta.$$

Using (3.19) we can now sharpen (3.16), (3.17), (3.18) as in the proof of (3.10) :

PROPOSITION 3.2. – *If* φ *is defined by* (3.13) *we have for all* $N \geqslant 0$:

$$\text{(3.16')}\quad \left|D_t^k D_x^\alpha D_\eta^\beta \frac{\partial \varphi}{\partial \bar{x}}\right| + \left|D_t^k D_x^\alpha D_\eta^\beta \frac{\partial \varphi}{\partial \bar{\eta}}\right| = \mathcal{O}(1)(1+t)^{a_{k,\alpha,\beta,N}} \gamma^N,$$

(3.17′) For all $(t,x,\xi,y,\eta) \in \tilde{C}$:

$$\left|D_t^k D_x^\alpha D_\eta^\beta \left(\xi - \frac{\partial \varphi}{\partial x}, y - \frac{\partial \varphi}{\partial \eta}\right)\right| = \mathcal{O}(1)(1+t)^{a_{k,\alpha,\beta,N}} \gamma^N,$$

$$\text{(3.18')}\quad \left|D_t^k D_x^\alpha D_\eta^\beta \left(\frac{\partial \varphi}{\partial t} + \frac{1}{i} p\left(x, \frac{\partial \varphi}{\partial x}\right)\right)\right| = \mathcal{O}(1)(1+t)^{a_{k,\alpha,\beta,N}} \gamma^N.$$

We now restrict the attention to the real domain ; $\Omega_{\delta,\mathbf{R}}$ and notice first, by (1.13), (1.15) that

$$\gamma^2 \leqslant C\left(\int_0^t |p'|\, d\tau\right)^2 \leqslant Ct \int_0^t p\, d\tau \leqslant C't \operatorname{Im} \varphi(t,x,\eta),$$

so that

$$\text{(3.20)}\qquad \operatorname{Im} \varphi(t,x,\eta) \geqslant \frac{\gamma^2}{Ct}, \qquad (t,x,\eta) \in \Omega_{\delta,\mathbf{R}}.$$

PROPOSITION 3.3. – *There exists a function*

$$\chi(t,x,\eta) \in C^\infty(\mathbf{R}_+ \times \mathbf{W}_\mathbf{R};\, [0,1])$$

with support in $\Omega_{\delta,\mathbf{R}}$ *such that :*

$$\text{(3.21)}\qquad \partial_t^k \partial_x^\alpha \partial_\eta^\beta \chi = \mathcal{O}((1+t)^{a_{k,\alpha,\beta}})$$

$$\text{(3.22)}\quad \operatorname{Im} \varphi(t,x,\eta) \geqslant \frac{1}{C(1+t)^{1+2N_0}}, \qquad \text{when} \quad \chi(t,x,\eta) \in\,]0,1[,$$

where $N_0 \geqslant 2$ is given in (1.4).

Proof. – We may assume that W is contained in some larger neighborhood $\tilde{W}$, such that all our constructions and estimates are valid with W replaced by $\tilde{W}$. If $(t,x,\eta) \in \partial\Omega_{\frac{\delta}{2},R}$, we notice from (3.20) that

$$\operatorname{Im} \varphi(t,x,\eta) \geqslant \frac{1}{C(1+t)^{1+2N_0}}.$$

Using the estimate on the Lipschitz continuity of γ and the temperate growth of the derivatives of φ, it is easy to see that the function

$$\chi_\delta(t,x,\eta) = \int \Phi_{\frac{\delta}{2}}(t,x',\eta')(1+t)^{N(2n+1)}\Phi((1+t)^N(t-t',x-x',\eta-\eta'))\, dt'\, dx'\, d\eta',$$

will have the required properties, if $\Phi_{\frac{\delta}{2}}$ is the characteristic function of $\Omega_{\delta/2}$ and $0 \leqslant \Phi \in C_0^\infty(\mathbf{R}^{2n+1})$ has integral 1 and its support in a sufficiently small neighborhood of 0, and $N \geqslant 0$ is sufficiently large.

To end this section we shall give some approximations for φ, that will serve, later for the study of the trace of the heat-kernel.

PROPOSITION 3.4. — *For* $(t,x,\eta) \in \Omega_{\delta,R}$ *we have*

(3.23) $\|x - \varphi'_\eta(t,x,\eta)\| + \|\eta - \varphi'_x(t,x,\eta)\| \leqslant Ce^{C(1+t)^2}t\|p'(x,\eta)\|$

(3.24) $\varphi(t,x,\eta) = \langle x,\eta\rangle + itp(x,\eta) + \mathcal{O}(1)t^2e^{C(1+t)^2}\|p'(x,\eta)\|^2$.

Proof. — For $(t,x,\eta) \in \Omega_{\delta,R}$, let $[0,t] \ni s \to \rho_s(t,x,\eta)$ be the corresponding integral curve of $H_{\frac{1}{i}p}$, so that $(\rho_0)_\xi = \eta$, $(\rho_t)_x = \xi$. We fix (x,η) and shall first estimate how $(\rho_t(t,x,\eta), \rho_0(t,x,\eta))$ varies with t. Passing from t to $t + \delta t$, a first candidate for $(\rho_{t+\delta t}(t+\delta t,x,\eta),\ \rho_0(t+\delta t,x,\eta))$ can be chosen to be $(\rho_t(t)+\delta t\, H_{\frac{1}{i}p}(\rho_t),\rho_0(t))$. However, in the x-projection, we then have an error, which is $\mathcal{O}(|\delta t||H_p(\rho_t)|)$ and since the differential of the projection $\tilde{C} \to \Omega_\delta$ is invertible with inverse $= \mathcal{O}((1+t))$, we get the correct point, after adding a correction, which is $\mathcal{O}((1+t)|\delta t||H_p(\rho_t)|)$. In other words :

$$\frac{\partial}{\partial t}(\rho_t(t,x,\eta),\ \rho_0(t,x,\eta)) = \mathcal{O}(1)|p'(\rho_t(t,x,\eta))|(1+t). \tag{3.25}$$

If we put $A_t = \|\rho_t(t,x,\eta) - (x,\eta)\| + \|\rho_0(t,x,\eta) - (x,\eta)\|$, then by (3.25)

$$\partial_t^\pm A_t = \mathcal{O}(1)(|p'(x,\eta)|+A_t)(1+t), \qquad A_0 = 0, \tag{3.26}$$

where $\partial_t^+ (\partial_t^-)$ denotes the right (left) derivative. Then

$$\partial_t^\pm A_t - C(1+t)A_t \leqslant C|p'(x,\eta)|(1+t)$$

if C is sufficiently large. Writing this as

$$\partial_t^{\pm} e^{-\frac{C(1+t)^2}{2}} A_t \leqslant C|p'(x,\eta)|(1+t)\, e^{-\frac{C(1+t)^2}{2}}$$

and integrating, we get with a new constant C :

$$(3.27)\quad \|\rho_t(t,x,\eta) - (x,\eta)\| + \|\rho_0(t,x,\eta) - (x,\eta)\| \leqslant Ct|p'(x,\eta)|\,.\,e^{C(1+t)^2}.$$

One can also estimate intermediate points on the integral curve. (3.27), (3.25) imply

$$(3.28)\qquad \frac{\partial}{\partial t}(\rho_t(t,x,\eta), \rho_0(t,x,\eta)) = \mathcal{O}(1)e^{C(1+t)^2}\|p'(x,\eta)\|.$$

Using proposition 2.1 it follows easily that

$$(3.29)\qquad \frac{\partial}{\partial t}\rho_s(t,x,\eta) = \mathcal{O}(1)e^{C(1+t)^2}\|p'(x,\eta)\|,$$

for $0 \leqslant s \leqslant t$. With (3.27) this shows that

$$(3.30)\ \|\rho_s(t,x,\eta) - (x,\eta)\| \leqslant Ct\, e^{C(1+t)^2}\|p'(x,\eta)\|, \qquad 0 \leqslant s \leqslant t.$$

When (x,η) is real (3.30) shows that

$$(3.31)\qquad \gamma(t,x,\eta) \leqslant Ct\, e^{C(1+t)^2}|p'(x,\eta)|.$$

If $\rho_t(t,x,\eta) = (x,\xi)$, $\rho_0(t,x,\eta) = (y,\eta)$, we see from (3.27), (3.17) that (3.23) follows. By (3.18)

$$(3.32)\qquad \frac{\partial\varphi}{\partial t} = ip(x,\varphi'_x) + \mathcal{O}(1)(1+t)^{a_2}\,\gamma(t,x,\eta)^2.$$

Then (3.24) follows if we use (3.23), (3.31) and integrate.

We also need an approximation result for the second order derivatives.

PROPOSITION 3.5. — *Let $p_0 \geqslant 0$ be homogeneous of degree 1 and vanish at the (real) point (x_0,η_0). Let $\varphi_0(t,x,\eta)$ be the corresponding phase function. Then for $(t,x_0,\eta_0) \in \Omega_\delta$:*

$$\varphi''(t,x_0,\eta_0) - \varphi''_0(t,x_0,\eta_0) = \mathcal{O}(1)(e^{C(1+t)^2}t\|p'(x_0,\eta_0)\| + t(1+t)^3\,\|p''(x_0,\eta_0) - p''_0(x_0,\eta_0)\|).$$

Proof. — Let $C_{0,t}$ be the canonical relations generated by φ_0, so that $T_{(x_0,\eta_0,x_0,\eta_0)}(C_{0,t})$ is the set of points (u_t,u_0) obtained by integrating the system

$$\frac{\partial u_s}{\partial s} = \frac{1}{i} A u_s, \qquad 0 \leqslant s \leqslant t,$$

where A is the Hamilton matrix of p_0 at (x_0,ξ_0).

Similarly $T_{(\rho_t(t,x,\eta),\,\rho_0(t,x,\eta))}(C_t)$ is obtained by integrating a system

$$\frac{\partial v_s}{\partial s} = \left(\frac{1}{i} A + B_{t,s}\right) v_s, \qquad 0 \leqslant s \leqslant t,$$

where

$$\|B_{t,s}\| \leqslant C(te^{C(1+t)^2}\|p'(x_0,\eta_0)\| + \|p''(x_0,\eta_0) - p_0''(x_0,\eta_0)\|).$$

Now assume that $(v_t)_x = (u_t)_x = X$, $(v_0)_\xi = (u_0)_\xi = \Xi$, $\|(X,\Xi)\| \leqslant 1$. Then $v_s = \mathcal{O}((1+t)^{3/2})$ by (2.38), (2.8), and since

$$\frac{\partial u_s - v_s}{\partial s} = \frac{1}{i} A(u_s - v_s) - B_{t,s} v_s = \frac{1}{i} A(u_s - v_s) + \mathcal{O}(1)(e^{C(1+t)^2}\|p'(x_0,\eta_0)\| + (1+t)^{3/2}\|p'' - p_0''\|),$$

we get from proposition 2.4

$$(3.33)\quad \|u_s - v_s\| \leqslant C(te^{C(1+t)^2}\|p'(x_0,\eta_0)\| + t(1+t)^{5/2}\|p'' - p_0''\|).$$

On the other hand the two tangent spaces are given by

$$(\delta\xi,\delta y) = \varphi_0''(\delta x,\delta\eta), \qquad (\delta\xi,\delta y) = \varphi''(\delta x,\delta\eta)$$

so the proposition follows from (3.33) by choosing $s = t$ and $s = 0$.

4. The transport-equation.

Let p be as before and $p_{m-1}(x,\xi)$ another smooth function. We shall study the transport operator :

$$(4.1) \qquad L = \frac{1}{i}\left(\frac{\partial}{\partial t} + \frac{1}{i}\sum_{j=1}^{n} p^{(j)}(x,\varphi_x')\frac{\partial}{\partial x_j} + q_{m-1}\right),$$

where

$$(4.2)\qquad q_{m-1}(t,x,\eta) = p_{m-1}(x,\varphi'_x) + \frac{1}{2i}\sum_j\sum_k \frac{\partial^2 p(x,\varphi'_x)}{\partial\xi_j\partial\xi_k}\frac{\partial^2\varphi}{\partial x_j\partial x_k}.$$

Here φ is the generating function studied in the previous section. Since φ'_x is approximately $= \xi$ when $(t,x,\xi,y,\eta) \in \tilde{C}$, we shall replace φ'_x by ξ in (4.1) and consider iL as an ordinary differential operator of the form $\frac{\partial}{\partial t} + H_{\frac{1}{i}p} + q_{m-1}$ along the integral curves of $\frac{\partial}{\partial t} + H_{\frac{1}{i}p}$ in $\tilde{C}$. These approximations will be justified a posteriori when we have proved that the solutions to the modified transport equations are small for large t and sufficiently almost analytic. Along each integral curve iL will take the form.

$$(4.3)\qquad \frac{d}{dt} + q_{m-1}(t).$$

We want to estimate solutions of such an equation when $t \to +\infty$. In order to do this we first do some approximations. Let $T \geqslant 1$ and let $\gamma : [0,T] \ni s \longmapsto \rho_s$ be an integral curve of $H_{\frac{1}{i}p}$ satisfying (1.1), (1.2), with $N \geqslant 4$. Let ρ^0 be a real point such that

$$(4.4)\qquad \|\rho_s - \rho^0\| \leqslant CT\tilde{\varepsilon}(T), \qquad 0 \leqslant s \leqslant T.$$

In view of (1.5), for any $t \in [0,T]$, we can take $\rho^0 = \operatorname{Re}(x,\eta)$ if $\rho_t = (x,\xi)$, $\rho_0 = (y,\eta)$. As noticed in section 2 (see (2.37)) we know that

$$p''(\rho^0) \geqslant -C\varepsilon(T),$$

where $\varepsilon(T) = \tilde{\varepsilon}(T)^{1/2}$. We can therefore construct a smooth real-valued function $\tilde{p}$ such that

$$(4.5)\qquad \begin{aligned} &\tilde{p}(\rho^0) = 0, \quad \tilde{p}'(\rho^0) = 0, \quad \tilde{p}''(\rho^0) \geqslant 0, \\ &\tilde{p}''(\rho^0) - p''(\rho^0) = \mathcal{O}(\varepsilon(T)). \end{aligned}$$

Let $\tilde{\varphi}$ solve

$$(4.6)\qquad \begin{aligned} &\frac{\partial\tilde{\varphi}}{\partial t} + \frac{1}{i}\tilde{p}(x,\tilde{\varphi}'_x) = 0 \\ &\tilde{\varphi}|_{t=0} = \langle x,\eta\rangle \end{aligned}$$

to infinite order at ρ^0 and define $\tilde{L}$ as above, by replacing (p,φ) by $(\tilde{p},\tilde{\varphi})$.

At ρ^0, $\tilde{L}$ is given by

$$i\tilde{L} = \frac{\partial}{\partial t} + \tilde{q}_{m-1}(t), \tag{4.7}$$

where

$$\tilde{q}_{m-1} = p_{m-1}(\rho^0) + \frac{1}{2i}\Sigma\Sigma\frac{\partial^2\tilde{p}}{\partial\xi_j\,\partial\xi_k}(\rho^0)\frac{\partial^2\tilde{\varphi}}{\partial x_j\,\partial x_k}.$$

Corresponding to $\tilde{\varphi}$, there is a family of canonical relations $\tilde{C}_t$, and the tangent space $T_{(\rho^0,\rho^0)}(\tilde{C}_t)$ is the set of (u_t, u_0) obtained by integrating

$$\frac{\partial u_s}{\partial s} = \frac{1}{i}Fu_s, \qquad 0 \leqslant s \leqslant t. \tag{4.8}$$

Here F is the Hamilton matrix of $\tilde{p}$ at ρ^0. We can compare this with $T_{\rho_t,\rho_0}(C_t)$, which is the space of all (v_t, v_0) obtained by integrating

$$\frac{\partial v_s}{\partial s} = \left(\frac{1}{i}F + B_s\right)v_s, \qquad 0 \leqslant s \leqslant t, \tag{4.9}$$

where now

$$B_s = \mathcal{O}(\varepsilon(T)). \tag{4.10}$$

Using proposition 2.4 as in the proof of proposition 2.5, we obtain

$$u_s - v_s = \mathcal{O}(T^{7/2}\varepsilon(T)), \qquad 0 \leqslant s \leqslant t$$

if $(u_0)_\xi = (v_0)_\xi$, $(u_t)_x = (v_t)_x$ have norm $\leqslant 1$. Hence, if $T_{\rho_t,\rho_0}(C_t)$, $T_{\rho^0,\rho^0}(\tilde{C}_t)$ are given by $(\xi, y) = H_t(x,\eta)$, $(\xi,y) = \tilde{H}_t(x,\eta)$ respectively, then

$$\|H_t - \tilde{H}_t\| = \mathcal{O}(T^{7/2}\varepsilon(T)). \tag{4.11}$$

It follows that

$$|q(t) - \tilde{q}(t)| = \mathcal{O}(T^{7/2}\varepsilon(T)). \tag{4.12}$$

We require from now on that $N \geqslant \frac{9}{2}$ in (2.4) (i.e. $N \geqslant 9$ in (1.1)).

To study $\tilde{L}$ of ρ^0 we may first make a (symplectic) translation so that ρ^0 becomes the point (0,0) and without any loss of generality we may then assume that p is a quadratic form. We are then reduced to the linearized situation. As in [7], we notice that on $\tilde{C} = \{(t,x,\xi,y,\eta)\,;\ (x,\xi,y,\eta) \in \tilde{C}_t\}$:

$$(4.13)\qquad (i\tilde{L}a)\sqrt{dt\,dx\,d\eta} = (\mathscr{L}_\nu + \tilde{S})(a\sqrt{dt\,dx\,d\eta}),$$

where $\nu = \dfrac{\partial}{\partial t} + H_{\frac{1}{i}\tilde{p}}$ and $\mathscr{L}_\nu$ is the Lie derivative, and

$$\tilde{S} = p_{m-1} - \frac{1}{2i}\Sigma\frac{\partial^2\tilde{p}}{\partial x_j\,\partial\xi_j}$$

is the subprincipal symbol.

PROPOSITION 4.1. – *If* $\tilde{L}a(t) = 0,\ a(0) = 1,$ *then*

$$(4.14)\qquad a(t) = e^{-t(\kappa+o(1))}, \qquad t \to \infty.$$

Here $\kappa = \tilde{S} + \frac{1}{2}\tilde{tr}(\tilde{p})$ *and the estimate « $o(1)$ » is uniform when p varies in a compact set of non-negative quadratic forms.*

Proof. – Taking (t,y,η) as local coordinates on $\tilde{C}$, we simply have $\mathscr{L}_\nu = \dfrac{\partial}{\partial t}$, so if $\tilde{L}a = 0$ and we define b by

$$a\sqrt{dt\,dx\,d\eta} = b\sqrt{dt\,dy\,d\eta},$$

we have $b = e^{-\tilde{S}t}$. Our problem is therefore to prove that on $\tilde{C}_t$:

$$(4.15)\qquad |dy\,d\eta| = e^{-t(\tilde{tr}(\tilde{p})+o(1))}|dx\,d\eta|.$$

On $\tilde{M} = T_0(\tilde{T}^*X)$ we introduce new (linear) symplectic coordinates $(\tilde{x},\tilde{\xi})$ so that the Lagrangian planes $\Lambda_+ = \{\tilde{\xi}=0\}$ and $\Lambda_- = \{\tilde{x}=0\}$ are positive and negative respectively $(\Lambda_+)_{\mathbf{R}} = (\Lambda_-)_{\mathbf{R}} = 0$ and $\Lambda_+ = \bar{\Lambda}_-$. Then $\{(x,\xi,y,\eta)\,;\ \tilde{x} = 0,\ \tilde{\eta}=0\}$ is a negative canonical relation, hence transversal to $\tilde{C}_t$, and it follows easily that $\tilde{C}_t$ is given by

$$(4.16)\qquad (\tilde{\xi},\tilde{y}) = G_t(\tilde{x},\tilde{\eta})$$

where the matrix G_t is bounded as $t \to \infty$. In particular

$$(4.17)\qquad |dx\,d\eta| \leqslant C|d\tilde{x}\,d\tilde{\eta}| \qquad \text{on} \qquad \tilde{C}_t.$$

Also in view of (2.38) we have

$$|d\tilde{x}\, d\tilde{\eta}| \leqslant C(1+t)^{2n}|dx\, d\eta| \quad \text{on} \quad \tilde{C}_t . \tag{4.18}$$

It is therefore enough to prove (4.15) with (x,ξ,y,η) replaced by $(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta})$. (Notice that $|d\tilde{y}\, d\tilde{\eta}| = |dy\, d\eta|$.)

To do this we shall first consider $\tilde{p}$ as the polarization of a bilinear form :

$$a(u,u) = \tilde{p}(u).$$

Then we have adopted the notation of section 2, and we shall first show that for $(u_t,u_0) \in \tilde{C}_t$:

$$|a(u_t,u_t)| \leqslant \frac{[u_t,u_t] - [u_0,u_0]}{t}. \tag{4.19}$$

Indeed, by (2.23) we have

$$2t \inf_{0 \leqslant s \leqslant t} a(u_s,\bar{u}_s) \leqslant [u_t,u_t] - [u_0,u_0]. \tag{4.20}$$

On the other hand

$$\begin{aligned} |a(u,u)| &\leqslant a(\text{Re}\, u, \text{Re}\, u) + 2|a(\text{Re}\, u, \text{Im}\, u)| + a(\text{Im}\, u, \text{Im}\, u) \\ &\leqslant 2(a(\text{Re}\, u, \text{Re}\, u) + a(\text{Im}\, u, \text{Im}\, u)) = 2a(u,\bar{u}), \end{aligned}$$

and since $a(u_s,u_s)$ is independent of s, we get (4.19).

We now consider $a(u,u)$ as a quadratic form on $\tilde{M} \times \tilde{M}$, constant along the second factor, and write $(u_t,u_0) = (\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta})$. Let Δa denote the restriction of a to $\tilde{C}_t$, extended to be constant in $(\tilde{\xi},\tilde{y})$, and put $b_t = a - \Delta a$. Then by (4.16), (4.19) :

$$b_t|_{\tilde{C}_t} = 0, \qquad \|b_t - a\| = \mathcal{O}\left(\frac{1}{(1+t)}\right). \tag{4.21}$$

Let $A, B_t : \tilde{M} \times \tilde{M} \to \tilde{M} \times \tilde{M}$ be the Hamilton matrices associated to a and b_t. By (4.21) we know that B_t maps $\tilde{C}_t$ into itself, and that

$$\|A - B_t\| = \mathcal{O}\left(\frac{1}{(1+t)}\right).$$

We can therefore find eigenvalues $\mu_1, \ldots, \mu_{2n}$ of $\frac{1}{i}A$ such that $\frac{1}{i}B_t|_{\tilde{C}_t}$ has the eigenvalues $\mu_1(t), \ldots, \mu_{2n}(t)$, with

$$|\mu_j - \mu_j(t)| = \mathcal{O}((1+t)^{-1/4n}).$$

We know from [7], that $\tilde{C}_t$ will converge to a limiting canonical relation $\tilde{C}_\infty$ (although the convergence may be very slow, if $\frac{1}{i}A$ has small non-vanishing eigenvalues), so clearly $\mu_1, \ldots, \mu_{2n}$ have to be the eigenvalues of $\frac{1}{i}A|_{\tilde{C}_\infty}$. It also follows from [7] that

$$\operatorname{tr}\frac{1}{i}A|_{\tilde{C}_\infty} = \mu_1 + \cdots + \mu_{2n} = \tilde{\operatorname{tr}}\, a,$$

so we obtain

$$\operatorname{tr}\frac{1}{i}B_t|_{\tilde{C}_t} = \tilde{\operatorname{tr}}\, a + \mathcal{O}((1+t)^{-1/4n}). \tag{4.22}$$

On the other hand $\operatorname{tr}\frac{1}{i}B_t|_{\tilde{C}_t}$ is the divergence of ν (on $\tilde{C}$) computed with respect to the coordinates $(t,\tilde{x},\tilde{\eta})$ and since

$$\mathcal{L}_\nu(dt\, d\tilde{x}\, d\tilde{\eta}) = \operatorname{div}(\nu)\,(dt\, d\tilde{x}\, d\tilde{\eta}),$$

while $\mathcal{L}_\nu(dt\, d\tilde{y}\, d\tilde{\eta}) = 0$, we conclude that

$$\frac{|dt\, d\tilde{x}\, d\tilde{\eta}|}{|dt\, d\tilde{y}\, d\tilde{\eta}|} = e^{t(\tilde{\operatorname{tr}} + o(1))}, \qquad t \to \infty \tag{4.23}$$

at a point $(t,x,\xi,y,\eta) \in \tilde{C}$. The proof is complete in view of (4.17), (4.18).

In view of (4.12) and proposition 4.1 we get :

PROPOSITION 4.2. — *If* $\left(\frac{d}{dt} + q_{m-1}(t)\right)a_0(t) = 0$, $a_0(0) = 1$, *then*

$$a_0(t) = e^{-t(\kappa + o(1))}, \qquad t \to \infty. \tag{4.24}$$

From now on we assume that the condition (0.1) holds, so that $\kappa > 0$. It is then easy to consider inhomogeneous transport equations.

PROPOSITION 4.3. — *Suppose that* $b(t) = \mathcal{O}(1)e^{-t(\kappa+o(1))}$, $t \to \infty$ *and that*

$$\left(\frac{d}{dt} + q_{m-1}(t)\right)a(t) = b(t). \tag{4.25}$$

Then

$$a(t) = \mathcal{O}(1)e^{-t(\kappa+o(1))}, \qquad t \to \infty.$$

Proof. — (4.25) can be written :

$$\frac{d}{dt} a_0^{-1}a = a_0^{-1}b,$$

so

$$a(t) = \int_0^t \frac{a_0(t)b(s)}{a_0(s)}\, ds + Ca_0(t).$$

For $0 \leqslant s \leqslant t$:

$$\frac{a_0(t)b(s)}{a_0(s)} = \mathcal{O}(1)e^{-t\kappa+o(1)t+o(1)s},$$

and the proposition follows.

We shall now estimate the derivatives of a solution to the equation $La = 0$. Let $\vec{\alpha}$ be a constant vectorfield on Ω_δ with vanishing t-component and modulus $\leqslant 1$. For a given $t > 0$, $\vec{\alpha}$ gives rise to a vectorfield (ν_t,ν_0) on C_t, where $\nu_t = \exp(tH_{\frac{1}{i}p})_*(\nu_0)$. Put $\nu_s = \exp(sH_{\frac{1}{i}p})_*(\nu_0)$, $0 \leqslant s \leqslant t$. We know from section 2 that $(\nu_t,\nu_0) = \mathcal{O}((1+t))$, $\nu_s = \mathcal{O}((1+t)^{3/2})$.

Let $r \longmapsto (\rho_t(r), \rho_0(r))$ be an integral curve of (ν_t,ν_0), so that $r \longmapsto (\rho_s(r),\rho_0(r))$ is an integral curve of $\mu_s = (\nu_s,\nu_0)$. Then considering q_{m-1} as a function on $\tilde{C}$ we obtain

$$\mu_s^N(q) = \left(\frac{\partial}{\partial r}\right)^N (q_{m-1}(s,\rho_s(r), \rho_0(r)) = \mathcal{O}(1)(1+t)^{a_N} \tag{4.26}$$

in view of proposition 2.6 and the polynomial bounds on the derivatives of φ.

PROPOSITION 4.4. — *Let* $a \in C^\infty(\Omega_\delta)$ *be a solution of* $La = 0$ (*where* L *is identified with the ordinary differential operator* $\frac{\partial}{\partial t} + H_{\frac{1}{i}p} + q_{m-1}$ *on* $\tilde{C}$). *Then for all multiindices* α *and all* $k \in \mathbf{N}$:

$$D_t^k D_{x,\eta}^\alpha a(t,x,\eta) = \mathcal{O}(1)e^{-t(\kappa+o(1))}, \qquad t \to \infty. \tag{4.27}$$

Proof. – We already know that (4.27) holds for $t = 0$. We fix a $t \geqslant 0$, and choose $\vec{\alpha}$, (ν_t, ν_0), ν_s, μ_s as above.

By construction, $\frac{\partial}{\partial s} + H_{\frac{1}{i}p}$ and μ_s commute so we obtain

$$(4.28) \qquad \left(\frac{\partial}{\partial s} + \frac{1}{i} H_p\right)(\mu_s a) + q_{m-1}(\mu_s a) = \mu_s(q_{m-1})a, \qquad 0 \leqslant s \leqslant t.$$

Applying (4.26) and proposition 4.3 we obtain

$$(4.29) \qquad \mu_s a = \mathcal{O}(1)(1+t)^{a_1} e^{-s(\kappa+o(1))}, \qquad 0 \leqslant s \leqslant t.$$

(Here $o(1)$ denotes a term tending to zero when $s \to \infty$ uniformly with respect to $t \in [s, +\infty[$.) Continuing to apply μ_s to (4.28) we get

$$(4.30) \qquad \mu_s^{N}(a) = \mathcal{O}(1)(1+t)^{a_N} e^{-s(\kappa+o(1))}, \qquad 0 \leqslant s \leqslant t$$

and hence by taking $s = t$ we obtain (4.27) for $k = 0$.

To estimate also the t-derivatives of a we can write the transport equation explicitely and differentiate successively in t, x, η.

Remark 4.5. – If we consider the inhomogeneous transport equation $La = b$, where b satisfies the estimates (4.27) then a will satisfy the same estimates. Indeed, the same proof works.

We shall finally estimate $\bar{\partial}_{(x,\eta)}a$. If ν is a real vector field we denote by $i\nu$ the differential operator defined by $(i\nu)(f) = i(\nu(f))$ for all functions f, and by $J\nu$ the unique real vectorfield such that $i\nu(f) = J\nu(f)$ for all *holomorphic* functions f. Let $\vec{\alpha}$, $\mu_t = (\nu_t, \nu_0)$ be the same as above and let $\hat{\mu}_t = (\hat{\nu}_t, \hat{\nu}_0)$ be the vectorfield on C_t which projects on $J\alpha$.

Proposition 2.5 then shows that

$$(4.31) \qquad J\mu_t - \tilde{\mu}_t = \mathcal{O}(1)(1+t)^{a_N}\gamma(t,x,\eta)^{N}, \qquad \forall N.$$

If $\hat{\nu}_s = \exp(s H_{\frac{1}{i}p})_*(\hat{\nu}_0)$, $\hat{\mu}_s = (\hat{\nu}_s, \hat{\nu}_0)$, then by proposition 2.4 :

$$(4.32) \qquad J\mu_s - \hat{\mu}_s = \mathcal{O}(1)(1+t)^{a_N}\gamma(t,x,\eta)^{N}, \qquad 0 \leqslant s \leqslant t.$$

Applying $\hat{\mu}_s - i\mu_s$ to the transport equation $La = 0$, we get

$$\left(\frac{\partial}{\partial s} + H_{\frac{1}{i}p}\right)(\hat{\mu}_s - i\mu_s)a + q_{m-1}(\hat{\mu}_s - i\mu_s)a = (\hat{\mu}_s - i\mu_s)(q_{m-1})a.$$

Estimating the almost analyticity of q_{m-1} by means of (3.16′) and applying (4.31) we see that

$$(4.33) \qquad (\hat{\mu}_s - i\mu_s)(q_{m-1}) = \mathcal{O}(1)(1+t)^{a_N}\gamma^N, \qquad \forall N \geqslant 0.$$

Applying proposition 4.4 and remark 4.5 we get

$$(4.34) \qquad (\hat{\mu}_t - i\mu_t)a = \mathcal{O}(1)e^{-t(\kappa+o(1))}\gamma^N, \qquad \forall N,$$

provided that a is almost analytic in the usual sense for $t = 0$. Projecting (4.34) into the (t,x,η)-space we get :

$$(4.35) \qquad \bar{\partial}_{(x,\eta)}a(t,x,\eta) = \mathcal{O}(1)e^{-t(\kappa+o(1))}\gamma^N, \qquad \forall N,$$

for $(t,x,\eta) \in \Omega_\delta$. As in section 3 we can use the Lipschitz continuity of γ and the estimates (4.27), to strengthen (4.35) to

$$(4.36) \qquad D_t^k D_{(x,\eta)}^\alpha \bar{\partial}_{(x,\eta)}a(t,x,\eta) = \mathcal{O}(1)e^{-t(\kappa+o(1))}\gamma^N, \qquad \forall N, k, \alpha.$$

The same arguments apply to the inhomogeneous transport equations and we obtain

THEOREM 4.6. — *Let* $b \in C^\infty(\Omega_\delta)$ *satisfy the estimates*

$$(4.37) \qquad D_t^k D_{(x,\eta)}^\alpha b = \mathcal{O}(1)e^{-t(\kappa+o(1))}$$

$$(4.38) \qquad D_t^k D_{(x,\eta)}^\alpha \bar{\partial}_{(x,\eta)} b = \mathcal{O}(1)e^{-t(\kappa+o(1))}\gamma^N$$

for all k, α, N. *Let a be a solution of*

$$(4.39) \qquad \left(\frac{\partial}{\partial t} + H_{\frac{1}{i}p} + q_{m-1}\right)a = b$$

such that $a(0,x,\eta)$ *is almost analytic. Then a also satisfies* (4.37), (4.38) *and*

$$(4.40) \qquad D_t^k D_x^\alpha D_\eta^\beta\left(b - \left(\frac{\partial}{\partial t} + \frac{1}{i}\sum_{j=1}^n p^{(j)}(x,\varphi'_x)\frac{\partial}{\partial x_j} + q_{m-1}\right)a\right) = \mathcal{O}(1)e^{-t(\kappa+o(1))}\gamma^N$$

for all k, α, β.

In fact, the almost analyticity of a permits us to pass from the ordinary differential equation (4.39), to the estimates (4.40).

Remark 4.7. — In the estimates (4.37), (4.38), (4.40) for a, we may replace γ by $\operatorname{Im}\varphi$, when (x,η) is real. Also, if χ is the cut off function given in proposition 3.3, and we extend φ to be defined for real (x,η) and all $t \geqslant 0$ by replacing φ by $\chi\varphi + i(1-\chi)$, then we get from (3.18′), (3.21), (3.22) that for real (x,η) and $t \geqslant 0$:

$$(4.41) \quad |D_t^k D_{(x,\eta)}^{\alpha}\left(\frac{\partial \varphi}{\partial t} + \frac{1}{i} p\left(x, \frac{\partial \varphi}{\partial x}\right)\right) = \mathcal{O}(1)(1+t)^{a_{k,\alpha,N}} \mathrm{Im}\ \varphi^N, \quad \forall N, k, \alpha.$$

Similarly in theorem 4.6 we can replace a by χa so that for real (x,η) and all $t \geqslant 0$, (4.40) is valid with γ replaced by $\mathrm{Im}\,\varphi$. (Provided that b is extended similarly).

5. Parametrics for the heat equation.

As in [6], [7] we apply the Fourier integral operator approach, and we may rely on [7] for the general properties of our operators, and asymptotic expansions.

Let $X \subset \mathbf{R}^n$ be open and let $P \in L_c^m(X)$ be a property supported classical pseudo diferential operator satisfying the assumptions of theorem 0.1. The first step in our construction of e^{-tP} will be to find a function $\varphi \in C^\infty(\bar{\mathbf{R}}_+ \times X \times \dot{\mathbf{R}}^n)$ with $\mathrm{Im}\,\varphi \geqslant 0$ such that

$$(5.1) \qquad \varphi(0,x,\eta) = \langle x,\eta\rangle$$

$$(5.2) \qquad \frac{\partial \varphi}{\partial t} + \frac{1}{i} p(x,\varphi'_x) = 0 + \text{« small error »}.$$

The difference with section 2 is that p is now homogeneous of degree $m > 1$, and as noted in [6], [7], φ will no more be homogeneous of degree 1 but quasi-homogeneous (see [6], [7] for a definition) of degree 1 :

$$(5.3) \qquad \varphi(t,x,\lambda\eta) = \lambda\varphi(t\lambda^{m-1},x,\eta), \qquad \lambda > 0.$$

To get a reduction to the situation studied in section 1-4, we introduce an extra variable x_0 and put

$$\tilde{x} = (x_0,x), \qquad \tilde{\xi} = (\xi_0,\xi), \qquad \tilde{p}(\tilde{x},\tilde{\xi}) = \xi_0 p(x,\xi/\xi_0) = \xi_0^{1-m} p(x,\xi).$$

Then $\tilde{p}$ is homogeneous of degree 1 and according to proposition 3.2 and remark 4.7 we can find a smooth function $\tilde{\varphi}(t,\tilde{x},\tilde{\eta})$ defined for $t \in \bar{\mathbf{R}}_+$,

$x_0 \in \mathbf{R}$, $x \in X$, $\eta \in \dot{\mathbf{R}}^n$, $\eta_0 > 0$ such that uniformly for $(\tilde{x},\tilde{\eta})$ in any compact set :

$$\text{(5.4)} \quad D_t^k D_{\tilde{x}}^\alpha D_{\tilde{\eta}}^\beta\left(\frac{\partial\tilde{\varphi}}{\partial t} + \frac{1}{i}\tilde{p}(\tilde{x},\tilde{\varphi}'_{\tilde{x}})\right) = \mathcal{O}(1)(1+t)^{a_{k,\alpha,\beta,N}}(\operatorname{Im}\tilde{\varphi})^N$$

for all k, α, β, N, and such that

$$\text{(5.5)} \quad \tilde{\varphi}|_{t=0} = \langle\tilde{x},\tilde{\eta}\rangle = x_0\eta_0 + \langle x,\eta\rangle .$$

We also have a polynomial control on the growth of the derivatives of $\tilde{\varphi}$.

Now p is independant of the variable x_0, so the corresponding Hamiltonfield has a vanishing ξ_0-component. Therefore $\tilde{\varphi}$ can be constructed with $\dfrac{\partial\varphi}{\partial x_0} = \eta_0$. Now put

$$\varphi(t,x,\eta) = \tilde{\varphi}(t,(0,x),(1,\eta)) \in C^\infty(\bar{\mathbf{R}}_+ \times X \times \dot{\mathbf{R}}^n).$$

Then we get

Proposition 5.1. — *We have* $\operatorname{Im}\varphi \geqslant 0$, *and uniformly for* (x,η) *in any compact set :*

$$\text{(5.6)} \quad D_t^k D_x^\alpha D_\eta^\beta \varphi(t,x,\eta) = \mathcal{O}(1)(1+t)^{a_{k,\alpha,\beta}}$$

$$\text{(5.7)} \quad D_t^k D_x^\alpha D_\eta^\beta\left(\frac{\partial\varphi}{\partial t} + \frac{1}{i}p(x,\varphi'_x)\right) = \mathcal{O}(1)(1+t)^{a_{k,\alpha,\beta,N}}|\operatorname{Im}\varphi|^N$$

$$\text{(5.8)} \quad \varphi(0,x,\eta) = \langle x,\eta\rangle .$$

Replacing φ by its quasi-homogeneous extension from $|\eta| = 1$, we may also assume that φ is quasi homogeneous of degree 1.

To make the same reduction for the transport equations we define microlocally in the domain $\xi_0 > 0$:

$$\tilde{P}(\tilde{x},D_{\tilde{x}}) = D_{x_0}^{1-m}P(x,D_x).$$

Then the full symbol of P is

$$\tilde{P}(\tilde{x},\tilde{\xi}) = \tilde{p}(\tilde{x},\tilde{\xi}) + \tilde{p}_0(\tilde{x},\tilde{\xi}) + \tilde{p}_{-1} + \cdots$$

where

$$\tilde{p} = \xi_0^{1-m}p(x,\xi), \qquad \tilde{p}_0 = \xi_0^{1-m}p_{m-1}(x,\xi).$$

To P, $\tilde{\mathrm{P}}$ we then associate the transport operators :

$$\mathrm{L_P} = \frac{1}{i}\left(\frac{\partial}{\partial t} + \frac{1}{i}\sum_1^n p^{(j)}(x,\varphi'_x)\frac{\partial}{\partial x_j} + \left(p_{m-1}(x,\varphi'_x) + \frac{1}{2i}\sum_{j,k} p^{(jk)}(x,\varphi'_x)\frac{\partial^2\varphi}{\partial x_j \partial x_k}\right)\right)$$

$$\mathrm{L}_{\tilde{\mathrm{P}}} = \frac{1}{i}\left(\frac{\partial}{\partial t} + \frac{1}{i}\sum_0^n \tilde{p}^{(j)}(\tilde{x},\tilde{\varphi}'_{\tilde{x}})\frac{\partial}{\partial x_j} + \left(\tilde{p}_0(\tilde{x},\tilde{\varphi}'_{\tilde{x}}) + \frac{1}{2i}\sum_{j,k} \tilde{p}^{(jk)}(\tilde{x},\tilde{\varphi}'_{\tilde{x}})\frac{\partial^2\tilde{\varphi}}{\partial \tilde{x}_j \partial \tilde{x}_k}\right)\right).$$

We note that

$$\tilde{p}^{(j)}(\tilde{x},\tilde{\varphi}'_{\tilde{x}})|_{\eta_0=1} = p^{(j)}(x,\varphi'_x), \qquad 1 \leqslant j \leqslant n, \tag{5.9}$$

$$\tilde{p}^{(jk)}(\tilde{x},\tilde{\varphi}'_{\tilde{x}})\left.\frac{\partial^2\tilde{\varphi}}{\partial x_j\,\partial x_k}\right|_{\eta_0=1} = \begin{cases} p^{(jk)}(x,\varphi'_x)\dfrac{\partial^2\varphi}{\partial x_j\,\partial x_k} & \text{when} \quad j,k \geqslant 1 \\ 0 \quad \text{when} \quad j=0 \quad \text{or} \quad k=0 \end{cases} \tag{5.10}$$

$$\tilde{p}_0(\tilde{x},\tilde{\varphi}'_{\tilde{x}})|_{\eta_0=1} = p_{m-1}(x,\varphi'_x). \tag{5.11}$$

Thus, if $\mathrm{L}_{\tilde{\mathrm{P}}}\tilde{a} = \tilde{b}$ where $\tilde{a}$, $\tilde{b}$ are independent of x_0, we get $\mathrm{L_P}a = b$ with $a = \tilde{a}|_{\eta_0=1}$, $b = \tilde{b}|_{\eta_0=1}$. Also, $\tilde{\mathrm{P}}$ satisfies (0.1) since $\mathrm{S}_{\tilde{\mathrm{P}}}|_{\eta_0=1} = \mathrm{S_P}$ and since the nonvanishing eigenvalues of the Hamiltonian matrix of $\tilde{p}$ at a point where $\xi_0 = 1$, $\tilde{p} = 0$, are the same as those of the Hamilton matrix of p. The results of Section 4 (see Remark 4.7) then imply :

Proposition 5.2.

(A) There is a function $a \in \mathrm{C}^\infty(\bar{\mathbf{R}}_+ \times \mathrm{X} \times \dot{\mathbf{R}}^n)$ *quasihomogeneous of degree* 0 *such that for all* k, α *and all* $\mathrm{K} \subset\subset \mathrm{X} \times \dot{\mathbf{R}}^n$,

$$\partial_t^k \partial_{(x,\eta)}^\alpha a = 0(1)e^{-t(\kappa+o(1))}, \qquad t \geqslant 0, \qquad (x,\eta) \in \mathrm{K}$$
$$\mathrm{L_P}a = b_{-\infty}, \qquad a|_{t=0} = 1,$$

where

$$\partial_t^k \partial_{(x,\eta)}^\alpha b_{-\infty} = 0(1)e^{-t(\kappa+o(1))}(\mathrm{Im}\,\varphi)^{\mathrm{N}} \qquad \text{for all} \qquad \mathrm{N}, k, \alpha,$$

when $(x,\eta) \in \mathrm{K} \subset\subset \mathrm{X} \times \dot{\mathbf{R}}^n$.

(B) Let $b \in \mathrm{C}^\infty(\bar{\mathbf{R}}_+ \times \mathrm{X} \times \dot{\mathbf{R}}^n)$ be q.h. of degree $k + m - 1$, such that

$$\partial_t^k \partial_{(x,\eta)}^\alpha a = 0(1)e^{-t(\kappa+o(1))}, \qquad t > 0, \qquad (x,\eta) \in \mathrm{K}$$

for all $k, \alpha, \mathrm{K} \subset\subset \mathrm{X} \times \dot{\mathbf{R}}^n$. Then there exists $a \in \mathrm{C}^\infty(\bar{\mathbf{R}}_+ \times \mathrm{X} \times \dot{\mathbf{R}}^n)$, q.h. of degree k, satisfying the same kind of estimates such that

$$\mathrm{L_P}a = b + b_{-\infty}, \qquad a|_{t=0} = 0.$$

Here $b_{-\infty}$ is of the same type as in (A).

The function $\kappa = \kappa(x,\eta)$ is continuous, > 0, homogeneous of degree $m-1$ and $\kappa = S_P + \frac{1}{2}\widetilde{\mathrm{tr}}$ on $p^{-1}(0)$.

We can now follow [7] closely : First we look for a formal object $a(t,x,\eta) \sim \sum_0^\infty a_j(t,x,\eta)$ where a_j is q.h. of degree $-j$ such that the formal asymptotic expansion of $\left(D_t + \frac{1}{i}P\right)(e^{i\varphi}a)$ vanishes. Collecting the terms according to their degree of quasihomogeneity, we obtain the transport equations :

$$(T_0) \quad L(t,x,\eta,D_t,D_x)a_0 = 0$$
$$\vdots$$
$$(T_j) \quad L(t,x,\eta,D_t,D_x)a_j + \ell_j(t,x,\eta,a_0,\ldots,a_{j-1}) = 0$$
$$\vdots$$

where $L = L_P$ is the transport operator above and ℓ_j are linear differential operators acting on $a_0, \ldots, a_{j-1}$, whose coefficients have the right degree of quasi-homogeneity, and are of temperate growth as $t \to +\infty$. We also impose the initial conditions :

$$(5.12) \qquad a_0(0,x,\eta) = 1, \qquad a_j(0,x,\eta) = 0, \qquad j \geqslant 1.$$

Proposition 5.2 tells us that this system can be solved with certain errors : Formally we get

$$(5.13) \qquad \left(D_t + \frac{1}{i}P\right)(ae^{i\varphi}) \sim be^{i\varphi}$$

where $b \sim \sum_0^\infty b_j(t,x,\eta)$, b_j q.h. of degree $m-j$ and

$$(5.14) \qquad \partial_t^k \partial_{(x,\eta)}^\alpha a_j = 0(1)\, e^{-t(\kappa + o(1))},$$
$$\partial_t^k \partial_{(x,\eta)}^\alpha b_j = 0(1)\, e^{-t(\kappa + o(1))}\, \mathrm{Im}\, \varphi^N.$$

Following [7] we introduce $\tilde{S}^k(\bar{\mathbf{R}}_+ \times X \times \mathbf{R}^n) \subset C^\infty(\bar{\mathbf{R}}_+ \times X \times \mathbf{R}^n)$ as the space of symbols a, such that for all ν, α, β, N, $K \subset\subset X$, we have

$$(5.15) \quad |(|\eta|^{-(m-1)}D_t)^\nu D_x^\alpha D_\eta^\beta a(t,x,\eta)| \leqslant C_{K,\alpha,\beta,N,\nu}(1+t|\eta|^{m-1})^{-N}(1+|\eta|)^{k-|\beta|}, \qquad x \in K, \qquad |\eta| \geqslant 1.$$

We put $\tilde{S}^{-\infty} = \bigcap_{k \in \mathbf{R}} \tilde{S}^k$ and define asymptotic sums as usual. By $\tilde{S}^k_c$ we denote the elements in $\tilde{S}^k$ which have asymptotic expressions $\sim \sum_0^\infty a_j$, where $a_j \in \tilde{S}^{k-j}$ is q.h. of degree $k-j$ (in the region where $|\eta| \geqslant 1$). The formal symbols a, b constructed above can now be defined in $\tilde{S}^0_c$ and $\tilde{S}^m_c$ respectively and from (5.14) it follows that

$$be^{i\varphi} \in \tilde{S}^{-\infty} \tag{5.16}$$

Applying theorem 3.1 of [7] we now get

$$\left(D_t + \frac{1}{i} P\right)(e^{i\varphi}a) \in \tilde{S}^{-\infty}, \tag{5.17}$$

and if we define

$$A_t u(x) = \int e^{i\varphi(t,x,\eta)} a(t,x,\eta) \hat{u}(\eta) \frac{d\eta}{(2\pi)^n}, \quad u \in C_0^\infty(X) \tag{5.18}$$

we get as in [7], section 4 :

THEOREM 5.3. — *Let* P, φ *and* a *be as above. Then* A_t *defined by* (5.18) *is a continuous map* $C_0^\infty(X) \to C^\infty(\bar{\mathbf{R}}_+ ; C^\infty(X))$ *and has a continuous extension to* $\mathscr{E}'(X) \to C^\infty(\bar{\mathbf{R}}_+ ; \mathscr{D}'(X))$. *The distribution kernel* $A_t(x,y)$ *restricted to* $x \neq y$ *belongs to* $C^\infty(\bar{\mathbf{R}}_+ \times (X \times X \backslash \Delta))$ *and the distribution kernel of* $\left(D_t + \frac{1}{i} P\right) A_t$ *belongs to* $C^\infty(\bar{\mathbf{R}}_+ \times X \times C)$. *Moreover* $A_0 = I$.

Naturally, in the situation of the introduction, where X is a compact manifold, we get a parametrix for the heat equation by a partition of unity. This parametrix differs from $\exp(-tP)$ by an operator with kernel in $C^\infty(\bar{\mathbf{R}}_+ \times X \times X)$.

6. Some special measures on $T^*X\backslash 0$.

We now make the assumptions of theorem 0.1, and fix some Riemannian metric on X, so that S^*X is embedded naturally in $T^*X\backslash 0$ and also carries a Riemannian metric. We write $p(\theta) = p|_{S^*X}$ so that at a point $(x,\xi) = r\theta$ in

$T^*X\backslash 0 : p = r^m p(\theta)$. We fix an $\varepsilon_0 \in \left]0, \frac{1}{4}\right[$. For $r_0 \geqslant 1$, we may cover the set $\Gamma = \{\theta \in S^*X ; p(\theta) \leqslant r_0^{-1+\varepsilon_0}\}$ by a finite number of balls of radius $\frac{1}{2} r_0^{-\varepsilon_0}$ centered in Γ, such that

(6.1) no point in S^*X is contained in more than N_0 of the concentric balls with radius $3r_0^{-\varepsilon_0}$.

Here N_0 is independent of $r_0 \geqslant 1$ and of $\varepsilon_0 \in]0,1/4[$. We shall later take $r_0 = 2^k$, $k \in \mathbf{N}$. If $\theta_0 \in \Gamma$ is the center of one of these balls, let $B(\theta_0, r_0^{-\varepsilon_0})$ be the ball with center at θ_0 and radius $r_0^{-\varepsilon_0}$. Since S^*X is a Riemannian manifold we can define the Hessian matrix p'' invariantly at points of S^*X where $p' = 0$, and more generally if we only make changes of coordinates which are bounded in C^∞ (when the parameter r_0 tends to infinity) we can define the Hessian matrix invariantly up to $0(\|p'\|) = 0(p^{1/2})$. In particular in the ball $B = B(\theta_0, r_0^{-\varepsilon_0})$ we can define p'' invariantly up to an error $0(1)(r_0^{-(1-\varepsilon_0)} + r_0^{-2\varepsilon_0})^{1/2} = 0(r_0^{-\varepsilon_0})$. We can now find $\varepsilon_1 = \frac{1}{k}\varepsilon_0$, $4 \leqslant k \leqslant 2^{2n+2}$ (depending on $B(\theta_0, r_0^{-\varepsilon_0})$) such that

(6.2) $p''(\theta_0)$ has no eigenvalues in the interval $[r_0^{-2\varepsilon_1}, r_0^{-\varepsilon_1}]$.

After a (bounded) change of coordinates we may assume that $\theta = (\theta', \theta'')$, $dx\, d\xi = r^n (dr/r)\, d\theta$ and that

(6.3) $p''_{\theta''\theta''}(\theta_0) \geqslant r_0^{-\varepsilon_1}, \qquad p''_{\theta''\theta'}(\theta_0), \qquad p''_{\theta',\theta'}(\theta_0) = 0(r^{-2\varepsilon_1})$.

Then (at least for r_0 sufficiently large),

$$(6.4) \qquad p''_{\theta'',\theta''} \geqslant \frac{1}{2} r_0^{-\varepsilon_1}, \qquad p''_{\theta''\theta'}, \qquad p''_{\theta'\theta'} = 0(r_0^{-2\varepsilon_1})$$

in the ball $B(\theta_0, r_0^{-\varepsilon_0})$. Let Σ_B be the points in B where $\theta'' \to p(\theta',\theta'')$ takes its minimum. By the implicit function theorem, Σ_B is of the form :

$$(6.5) \qquad \theta'' = h(\theta').$$

At the point θ_0, we have $\|p'_{\theta''}\| \geqslant (r_0^{-\varepsilon_1}/C)\|\theta''_0 - h(\theta'_0)\|$, and on the other hand $\|p'_{\theta''}(\theta_0)\| \leqslant C r_0^{-(1-\varepsilon_0)/2}$, so

$$(6.6) \qquad \|\theta''_0 - h(\theta'_0)\| \leqslant C r_0^{-(1-\varepsilon_0-2\varepsilon_1)/2} << r_0^{-\varepsilon_0}$$

when r_0 is large. To fix the ideas we may assume therefore, that $\theta_0 \in \Sigma_B$. Set for a while $y = \theta''$, $x = \theta'$. Differentiating the equation $p'_y(x,h(x)) = 0$, shows that

$$(6.7) \qquad p''_{yx} + p''_{yy}\, h'(x) = 0,$$

so that

$$(6.8) \qquad h' = 0(r_0^{-2\varepsilon_1+\varepsilon_1}) = 0(r_0^{-\varepsilon_1}).$$

Put $p = p_B + q_B$, where $q_B = p(x,h(x))$. Then

$$(6.9) \qquad q_B|_{\Sigma_B} = 0, \qquad p_B \geqslant \frac{1}{C} r_0^{-\varepsilon_1}(d_{\Sigma_B})^2,$$

where d_{Σ_B} denotes the distance to Σ_B. We may differentiate q_B :

$$(6.10) \qquad (q_B)'_x = p'_x(x,h(x)) + p'_y(x,h(x))h'(x) = p'_x(x,h(x)) = 0(r_0^{-\varepsilon_0}),$$

$$(6.11) \qquad (q_B)''_{xx} = p''_{xx} + p''_{xy}h' = 0(r_0^{-2\varepsilon_1}).$$

From (6.11) and the positivity of q_B we also get

$$(6.12) \qquad (q_B)'_x = 0(r_0^{-\varepsilon_1}(q_B)^{1/2}).$$

LEMMA 6.1. — *In* $B(\theta_0, r_0^{-\varepsilon_0})$, *there are coordinates*

$$(\tilde{x},\tilde{y}) = (x,y+F(x,y))$$

with $F \in C^1$, $dF = 0(r_0^{-\varepsilon_1})$ *such that*

$$p = q_B(\tilde{x}) + \frac{1}{2}\langle p''_{\theta''\theta''}(\theta_0)\tilde{y},\tilde{y}\rangle .$$

Proof. — We may first replace y by $y - h(x)$ so that

$$f(x,y) = q_B(x) + \frac{1}{2}\langle p''_{\theta''\theta''}(x,0)y,y\rangle + 0(|y|^3).$$

Then put $\hat{y} = t(x,y)y$ in order to absorb the error-term. We get $t = 1 + s(x,y)$, where

$$s = 0(r_0^{-\varepsilon_0/2}), \qquad s' = 0(r_0^{2\varepsilon_1}),$$

and hence $(sy)' = 0(r_0^{-\varepsilon_0/2})$. Finally to replace $p''_{\theta''\theta''}(x,0)$ by $p''_{\theta''\theta''}(\theta_0)$ we put $\tilde{y} = S(x)\hat{y}$ for a suitable matrix S, close to the identity. We omit the details.

After another linear change of coordinates in $\tilde{y}$ we can find local coordinates ω in $B(\theta_0, r_0^{-\varepsilon_0})$ such that $\omega' = x$, Σ_B is given by $\omega'' = 0$ and

$$p = q_B(\omega') + \omega''^2.$$

Moreover the Jacobian is

$$\frac{d\omega}{d\theta} = \frac{1}{f(\omega')}(1+0(r_0^{-\varepsilon_1})) \tag{6.13}$$

where $1/f$ is the square root of the product of the eigenvalues of $p''_{\theta''\theta''}(\theta',h(\theta'))$, so that

$$\frac{1}{C} \leqslant f \leqslant C r_0^{-n\varepsilon_1}. \tag{6.14}$$

Passing to the cone $B(\theta_0, r_0^{-\varepsilon_0}) \times \mathbf{R}_+$ we can express the symplectic volume as

$$dx\, d\xi = f(1+0(r_0^{-\varepsilon_1}))r^{n-1}\, dr\, d\omega. \tag{6.15}$$

Let $\hat{B} = B(\theta_0, r_0^{-\varepsilon_0}) \times \left[-\frac{r_0}{2}, 2r_0\right]$. Let the codimension of Σ_B be $2d$ and let $0 \leqslant \mu_1(\theta') \leqslant \ldots \leqslant \mu_{[d]}(\theta')$ be the positive eigenvalues of i times the fundamental matrix of p_B on $S^*X \cap \Sigma_B$ (by $\hat{\Sigma}_B$ we shall denote the corresponding cone in $T^*X\backslash 0$). Write $\mu_j(r,\theta) = r^{m-1}\mu_j(\theta)$. On $\hat{B}$ we use the coordinates (r,ω) to define a projection

$$\pi_{\hat{B}} : (r,\omega) \to (r,\omega',\tilde{\omega}''),$$

where, if $\omega'' = (\omega''_1,\ldots,\omega''_{[d]})$, $\tilde{\omega}'' = (\tilde{\omega}''_1,\ldots,\tilde{\omega}''_{[d]})$, $\tilde{\omega}''$ is given by

$$(\tilde{\omega}''_{2j-1},\tilde{\omega}''_{2j}) = c_j(r,\omega)(\omega''_{2j-1},\omega''_{2j})$$

and $\tilde{\omega}''_{2d} = \omega''_{2d}$ when $2d$ is odd. Here $c_j = 1$ if $\mu_j = 0$, otherwise it is the nonnegative number such that $\tilde{\omega}''^2_{2j-1} + \tilde{\omega}''^2_{2j} = r^{-1}k_j\mu_j(\omega')$, where $k_j \in \mathbf{N}$ is determined by

$$k_j\mu_j(\omega')/r \leqslant \omega''^2_{2j-1} + \omega''^2_{2j} < (k_j+1)\mu_j(\omega')/r.$$

On $\hat{B}$ we define the measure $\Omega_{\hat{B}}$ as the direct image of $fr^{n-1}\,dr\,d\omega$ under the projection $\pi_{\hat{B}}$ (naturally $\Omega_{\hat{B}}$ also depends on how we choose the coordinates ω).

Let now r_0 take the values 1, 2, 2^2, 2^3, ... and for each such value choose a covering of Γ by balls $B\left(\theta_0, \frac{1}{2} r_0^{-\varepsilon_0}\right)$ as above. We then get a covering of the set

$$\hat{\Gamma} = \left\{(r,\theta) \in T^*X\backslash 0 \,;\, r \geqslant 1,\, p(r,\theta) \leqslant \frac{1}{C} r^{m-1+\varepsilon_0}\right\}$$

with sets $\hat{B}_\nu$ of the type $\hat{B}$ above and the analogue of (6.1) holds. We can then find a partition of unity

$$1 = \chi_0 + \Sigma\chi_\nu$$

where $\chi_\nu \in C_0^\infty(B_\nu)$, $0 \leqslant \chi_\nu \leqslant 1$, $0 \leqslant \chi_0 \leqslant 1$, $\operatorname{supp}\chi_0 \subset \complement\hat{\Gamma}$ and

$$\partial_x^\alpha \partial_\xi^\beta \chi_\nu = 0(|\xi|^{\varepsilon_0(|\alpha|+|\beta|)-|\beta|}) \tag{6.16}$$

uniformly with respect to ν. As a discretization of the symplectic volume, we now put

$$\Omega = \chi_0\, dx\, d\xi + \Sigma\chi_\nu\Omega_{\hat{B}_\nu}. \tag{6.17}$$

Let $\kappa > 0$ be a continuous function on $T^*X\backslash 0$, homogeneous of degree $m-1$, and equal to $S_P + \frac{1}{2}\tilde{\operatorname{tr}}$ on $p^{-1}(0)$. We shall study

$$W(\lambda) = \int_{p+\kappa \leqslant \lambda} \Omega(dx\, d\xi) \tag{6.18}$$

and as a rough estimate of $W(\lambda)$ we will use

$$V(\lambda) = \int_{p+|\xi|^{m-1} \leqslant \lambda} dx\, d\xi. \tag{6.19}$$

LEMMA 6.2. – *For $a \geqslant 1$, $\lambda \geqslant 1$ we have*

$$V(a\lambda) \leqslant a^{n/(m-1)} V(\lambda). \tag{6.20}$$

Proof. – If $p(x,\xi) + |\xi|^{m-1} \leqslant a\lambda$, then

$$p(x, a^{-1/(m-1)}\xi) + |a^{-1/(m-1)}\xi|^{m-1} = a^{-\frac{m}{m-1}}p(x,\xi) + a^{-1}|\xi|^{m-1} \leqslant \lambda$$

and the lemma follows.

Notice that the order of magnitude of $V(\lambda)$ or the validity of Lemma 6.2 is not affected if we replace $|\xi|^{m-1}$ by a positive continuous function, homogeneous of degree $m-1$.

For every $\hat{B}_\nu$ let $\kappa_\nu = \kappa_\nu(r,\omega')$ be defined as $S_{P_\nu} + \frac{1}{2}\tilde{\mathrm{tr}}(P_\nu)$ where P_ν is obtained from P, by replacing p by p_{B_ν}. Put

$$\tilde{W}(\lambda) = \int_{p+\kappa\leqslant\lambda} \chi_0 \, dx \, d\xi + \Sigma \int_{p+\kappa_\nu\leqslant\lambda} \chi_\nu \Omega_{\hat{B}_\nu}(dx \, d\xi). \tag{6.21}$$

PROPOSITION 6.3. – *$W(\lambda)/\tilde{W}(\lambda) \to 1$ when $\lambda \to +\infty$ and $\tilde{W}(\lambda)$ and $V(\lambda)$ are of the same order of magnitude. Moreover there is a constant $C > 0$ such that*

$$\tilde{W}(a\lambda) - \tilde{W}(\lambda) \leqslant C(a^{\frac{n}{m-1}} - 1)\tilde{W}(\lambda) \tag{6.22}$$

for $\lambda \geqslant 1$, $a \geqslant 1$.

Proof. – We first compare the order of magnitude of $\tilde{W}(\lambda)$ and $V(\lambda)$. First we write

$$\tilde{W}(\lambda) \leqslant \int_{p+\kappa\leqslant\lambda} \chi_0 \, dx \, d\xi + \Sigma \int_{\substack{p+\kappa_\nu\leqslant\lambda \\ (x,\xi)\in\hat{B}_\nu}} \Omega_{\hat{B}_\nu}(dx \, d\xi). \tag{6.23}$$

We fix a ν and assume for simplicity that codim $\Sigma_{B_\nu} = 2d$ is even and that $\mu_1, \ldots, \mu_d \neq 0$. To compute the integral over $\hat{B}_\nu$ in (6.23) we then have to compute the volume with respect to $f(\omega')r^n \frac{dr}{r} d\omega$ of the sets

$$\hat{B}_{\nu,k_1,\ldots k_d} = \Big\{(x,\xi) \in \hat{B}_\nu ; \frac{k_j\mu_j}{r} \leqslant \omega_{2j-1}''^2 + \omega_{2j}''^2 \leqslant \frac{(k_j+1)\mu_j}{r},$$
$$r^m q_B(\omega') + r^{m-1}(\kappa_\nu(\omega') + \langle k,\mu\rangle) \leqslant \lambda\Big\}$$

and the sum over all $k = (k_1, \ldots, k_d) \in \mathbf{N}^d$. Let

$$2\hat{B}_\nu = 2B_\nu \times \left[\frac{r_\nu}{4}, 4r_\nu\right],$$

where $\hat{B}_\nu = B_\nu \times \left[\frac{r_\nu}{2}, 2r_\nu\right]$ and $2B_\nu$ is the concentric ball of double radius. Then it is easy to see that the volume of $\hat{B}_{\nu,k_1,\ldots k_d}$ is smaller than a constant times the volume of

$$\left\{(x,\xi) \in 2\hat{B}_\nu ; \frac{k_j\mu_j}{r} \leqslant \omega''^2_{2j-1} + \omega''^2_{2j} \leqslant \frac{(k_j+1)\mu_j}{r}, p + \kappa_\nu \leqslant \lambda\right\},$$

so finally if $C > 0$ is large enough (using also (6.15)) :

(6.24) $$\tilde{W}(\lambda) \leqslant C\left(\int_{p+(1/C)|\xi|^{m-1} \leqslant \lambda} \chi_0 \, dx \, d\xi + \Sigma \int_{2\hat{B}_\nu \cap \{p+(1/C)|\xi|^{m-1} \leqslant \lambda\}} dx \, d\xi\right).$$

No more than a fixed finite number of sets $2\hat{B}_\nu$ intersect at any given point, so with a new constant C we obtain

(6.25) $$\tilde{W}(\lambda) \leqslant CV(\lambda).$$

Essentially the same argument shows that

(6.26) $$V(\lambda) \leqslant C\tilde{W}(\lambda),$$

so $V(\lambda)$ and $\tilde{W}(\lambda)$ are of the same order of magnitude.

Now (6.22) follows for $a \geqslant 2$ and in order to prove the estimate for $a \in [1,2]$ we first write

(6.27)
$$\tilde{W}(a\lambda) - \tilde{W}(\lambda) \leqslant \int_{\lambda \leqslant p+\kappa \leqslant a\lambda} \chi_0 \, dx \, d\xi + \sum_{\substack{\lambda \leqslant p+\kappa_\nu \leqslant a\lambda \\ (x,\xi) \in \hat{B}_\nu}} \int \Omega_{\hat{B}_\nu}(dx \, d\xi).$$

The same licing argument as above, shows that

$$\int_{\substack{\lambda \leqslant p+K \leqslant a\lambda \\ (x,\xi) \in \hat{B}_\nu}} \Omega_{\hat{B}_\nu}(dx \, d\xi) \leqslant C \int_{\substack{\lambda \leqslant p+K \leqslant a\lambda \\ (x,\xi) \in \hat{B}_\nu}} dx \, d\xi.$$

Since $\kappa_\nu/\kappa \to 1$ when $r_\nu \to \infty$ we can estimate the last integral by a constant times

$$\int_{\substack{\lambda \leqslant p+\mathrm{K} \leqslant a\lambda \\ (x,\xi)\in 3\hat{\mathrm{B}}_\nu}} dx\, d\xi$$

where $3\hat{\mathrm{B}}_\nu = 3\mathrm{B}_\nu \times \left[\frac{r_\nu}{6}, 6r_\nu\right]$. Thus as above we get

$$\tilde{\mathrm{W}}(a\lambda) - \tilde{\mathrm{W}}(\lambda) \leqslant \mathrm{C} \int_{\lambda \leqslant p+\kappa \leqslant a\lambda} dx\, d\xi. \tag{6.28}$$

The proof of Lemma 6.2 shows that

$$\int_{\lambda \leqslant p+\kappa \leqslant a\lambda} dx\, d\xi \leqslant (a^{n/(m-1)}-1) \int_{p+\kappa \leqslant \lambda} dx\, d\xi \tag{6.29}$$

and (6.22) follows from the equivalence of $\tilde{\mathrm{W}}(\lambda)$ and $\mathrm{V}(\lambda)$ and the equivalence of $\mathrm{V}(\lambda)$ and $\int_{p+\kappa\leqslant\lambda} dx\, d\xi$. The statement about $\mathrm{W}(\lambda)/\tilde{\mathrm{W}}(\lambda)$ is finally proved along the same lines.

7. Karamata's Tauberian Theorem.

It will be useful to recall also a proof of Karamata's theorem, since similar techniques will be used in the next section.

THEOREM 7.1 (Karamata [2]). — *Let* $a(\tau)$, $b(\tau)$ *be increasing functions on* $[0,+\infty[$, *with* $a(0) = b(0) = 0$, $b(\tau) \to +\infty$, $\tau \to +\infty$, *such that*

(7.1) *There is a constant* $\mathrm{C} > 0$ *such that*

$$b(k\tau) \leqslant \mathrm{C}k^{\mathrm{C}}b(\tau), \qquad k \geqslant 1, \qquad \tau \geqslant 1.$$

(7.2) *There is a positive function* $h(\delta)$, $\delta \in]0,1]$ *with* $h(\delta) \to 0$, $\delta \to 0$, *such that for every* $\delta \in]0,1]$, *there is a* $\tau_\delta \geqslant 1$ *such that* $b((1+\delta)\tau) - b(\tau) \leqslant h(\delta)b(\tau)$, *when* $\tau \geqslant \tau_\delta$.

(7.3) $a(\tau)$ *is of temperate growth at infinity.*

Let $\mathrm{A}(t) = \int_0^\infty e^{-t\tau}\, da(\tau)$, $\qquad \mathrm{B}(t) = \int_0^\infty e^{-t\tau}\, db(\tau)$, *for* $t > 0$. *Then if* $A(t)/B(t) \to 1$, *when* $t \to 0$, *we have* $a(\tau)/b(\tau) \to 1$, *when* $\tau \to +\infty$.

Proof. – The space L of all finite linear combinations of functions $t \to e^{-t\tau}$ with $\tau > 0$, is dense in $S(\bar{\mathbf{R}}_+)$. In fact, otherwise there would be an element $0 \neq u \in S'(\mathbf{R})$ with support in $[0, +\infty[$ such that $\langle u, e^{-t\tau}\rangle = 0$, $\forall \tau > 0$. Thus the Laplace transform $\hat{u}(\tau)$ vanishes identically for $\operatorname{Re}\tau \geqslant 0$, since it is holomorphic for $\operatorname{Re}\tau > 0$, and we conclude that $u = 0$, which is a contradiction.

Let $\hat{\chi}(\tau) \in C_0^\infty(\bar{\mathbf{R}}_+)$ satisfy, $0 \leqslant \hat{\chi} \leqslant 1$, $\hat{\chi}(\tau) = 1$ for $0 \leqslant \tau \leqslant 1$, $\hat{\chi}(\tau) = 0$, $\tau \geqslant 1 + \delta$. $\hat{\chi}$ is not a Laplace transform, but for every $N > 0$, there exists $\chi_1 \in C_0^\infty(\mathbf{R}_+)$ (depending on δ, N) such that

$$|\hat{\chi}_1(\tau) - \hat{\chi}(\tau)| \leqslant \delta(1+\tau)^{-N}, \qquad \tau \geqslant 0, \tag{7.4}$$

if $\hat{\chi}_1$ denotes the Laplace transform.

Notice that the Laplace transform of $\lambda\chi_1(\lambda t)$ is $\hat{\chi}_1(\tau/\lambda)$. We have the identities

$$\int_0^\infty \lambda\chi_1(\lambda t)A(t)\,dt = \int_0^\infty \hat{\chi}_1\left(\frac{\tau}{\lambda}\right) da(\tau) \tag{7.5}$$

$$\int_0^\infty \lambda\chi_1(\lambda t)B(t)\,dt = \int_0^\infty \hat{\chi}_1\left(\frac{\tau}{\lambda}\right) db(\tau). \tag{7.6}$$

The difference of the two integrals to the left can be written

$$\int_0^\infty \chi_1(t)\left(A\left(\frac{t}{\lambda}\right) - B\left(\frac{t}{\lambda}\right)\right) dt.$$

Since $A(t)/B(t) \to 1$, $t \to 0$ and χ_1 has compact support, we have

$$\left|\int_0^\infty \chi_1(t)\left(A\left(\frac{t}{\lambda}\right) - B\left(\frac{t}{\lambda}\right)\right) dt\right| = o(1)\int_0^\infty |\chi_1(t)| B\left(\frac{t}{\lambda}\right) dt, \quad \lambda \to +\infty. \tag{7.7}$$

LEMMA 7.2. – *Let* $f(\tau)$ *be a positive decreasing continuous function on* $[0, \infty[$ *and let* $0 \leqslant a_1(\tau) \leqslant a_2(\tau)$ *be functions of locally bounded total variation, such that for some* $N_0 > 0$, *the total variation of* a_j *on each interval* $[\tau, \tau+1]$ *is* $0((1+\tau)^{N_0})$ *and* $f(\tau) = 0(1)(1+\tau)^{-N_0-2}$. *Then*

$$\int_0^\infty f(\tau)\,da_1(\tau) \leqslant f(0)(a_2(0) - a_1(0)) + \int_0^\infty f(\tau)\,da_2(\tau). \tag{7.8}$$

Proof. — Put $c(\tau) = a_2(\tau) - a_1(\tau)$. By a density argument it is enough to prove (7.8) when $f \in S(\bar{\mathbf{R}}_+)$. Then

$$\int_0^\infty f(\tau)\, dc(\tau) + f(0)c(0) = -\int_0^\infty f'(\tau)c(\tau)\, d\tau \geqslant 0$$

and (7.8) follows.

Now, if $0 < t \leqslant 1$ we have

$$\begin{aligned} B(t) &= \int_0^{1/t} e^{-t\tau}\, db(\tau) + \int_{1/t}^\infty e^{-t\tau}\, db(\tau) \\ &= \int_0^1 e^{-\sigma}\, db\left(\frac{\sigma}{t}\right) + \int_1^\infty e^{-\sigma}\, db\left(\frac{\sigma}{t}\right) \\ &\leqslant b\left(\frac{1}{t}\right) + \int_1^\infty e^{-\sigma}\, db\left(\frac{\sigma}{t}\right). \end{aligned}$$

Here we apply (7.8) with $a_1(\sigma) = b(\sigma/t)$, $a_2(\sigma) = C\sigma^C b(1/t)$ and get

$$\text{(7.9)} \quad B(t) \leqslant (1+C)b\left(\frac{1}{t}\right) + C\int_1^\infty e^{-\sigma}\, d(\sigma^C)\, b\left(\frac{1}{t}\right) \leqslant C' b\left(\frac{1}{t}\right).$$

Applying this estimate to $B(t/\lambda)$ in (7.7), we get

$$\text{(7.10)} \left|\int_0^\infty \chi_1(t)\left(A\left(\frac{t}{\lambda}\right) - B\left(\frac{t}{\lambda}\right)\right) dt\right| = o(1)b(\lambda), \qquad \lambda \to +\infty,$$

so in view of (7.5), (7.6) :

(7.11)

$$\left|\int_0^\infty \hat{\chi}_1\left(\frac{\tau}{\lambda}\right) da(\tau) - \int_0^\infty \hat{\chi}_1\left(\frac{\tau}{\lambda}\right) db(\tau)\right| = o(1)b(\lambda), \qquad \lambda \to \infty.$$

Here we want to replace $\hat{\chi}_1$ by $\hat{\chi}$ so we have to estimate

$$\text{(7.12)} \qquad \int_0^\infty \left|\hat{\chi}_1\left(\frac{\tau}{\lambda}\right) - \hat{\chi}\left(\frac{\tau}{\lambda}\right)\right| da(\tau)$$

and the corresponding (less troublesome) term with a replaced by b.

From the inequality

$$A\left(\frac{1}{\tau}\right) \geqslant \int_0^\tau e^{-t/\tau}\, da(t')$$

we get

$$a(\tau) \leqslant eA\left(\frac{1}{\tau}\right).$$

Thus by (7.9) and the fact that $A(t)/B(t) \to 1$, $t \to \infty$, we get

$$(7.13) \qquad a(\tau) \leqslant Cb(\tau) \qquad \text{for } \tau \text{ sufficiently large.}$$

Now fix $N \geqslant C_0 + 2$, where C_0 is such that $a(\tau),\ b(\tau) = 0(\tau^{C_0})$, $\tau \to \infty$. Then (7.4), Lemma 7.2, (7.13) give when λ is sufficiently large :

$$\begin{aligned}\int_0^\infty \left|\hat{\chi}_1\left(\frac{\tau}{\lambda}\right) - \hat{\chi}\left(\frac{\tau}{\lambda}\right)\right| da(\tau) &\leqslant \delta \int_0^\infty \frac{1}{\left(1+\frac{\tau}{\lambda}\right)^N} da(\tau) \\ &\leqslant \delta a(\lambda) + \delta \int_\lambda^\infty \frac{1}{\left(1+\frac{\tau}{\lambda}\right)^N} da(\tau) \\ &\leqslant \delta a(\lambda) + \delta \int_1^\infty \frac{1}{(1+\tau)^N} da(\lambda\tau) \\ &\leqslant C\delta b(\lambda) + C\delta \int_1^\infty \frac{1}{(1+\tau)^N} d(\tau^C)b(\lambda) \\ &\leqslant C'\delta b(\lambda).\end{aligned}$$

Applying this to (7.11) gives :

$$(7.14) \qquad \left|\int_0^\infty \hat{\chi}\left(\frac{\tau}{\lambda}\right) da(\tau) - \int_0^\infty \hat{\chi}\left(\frac{\tau}{\lambda}\right) db(\tau)\right| \leqslant (C\delta + o(1))b(\lambda), \quad \lambda \to \infty;$$

so recalling the properties of $\hat{\chi}$:

$$(7.15) \qquad a(\lambda) \leqslant b((1+\delta)\lambda) + (C\delta + o(1))b(\lambda), \qquad \lambda \to \infty,$$

$$(7.16) \qquad a(\lambda) \geqslant b\left(\frac{\lambda}{1+\delta}\right) - (C\delta + o(1))b\left(\frac{\lambda}{1+\delta}\right), \qquad \lambda \to \infty.$$

Now $b((1+\delta)\lambda) \leqslant (1+h(\delta))b(\lambda)$, and $b\left(\frac{\lambda}{1+\delta}\right) \geqslant \frac{1}{1+h(\delta)} b(\lambda)$ for λ sufficiently large so we get

$$(7.17) \qquad \frac{1 - C\delta + o(1)}{1 + h(\delta)} b(\lambda) \leqslant a(\lambda) \leqslant (1+h(\delta))(1+C\delta+o(1))b(\lambda), \quad \lambda \to +\infty,$$

so $b(\lambda)/a(\lambda) \to 1$ and the proof is complete.

8. Study of the trace.

We shall first only study the order of magnitude of $\operatorname{tr} e^{-tP}$ when $t \to 0$, $t > 0$. After a partition of unity we are then reduced to the situation of Section 5 (i.e. where X is an open set in $\mathbf{R}^n$) and we have to study the integral

$$(8.1) \qquad I_\chi(t) = \iint e^{i(\varphi(t,x,\xi) - \langle x,\xi\rangle)} a(t,x,\xi)\chi(x) \frac{dx\, d\xi}{(2\pi)^n},$$

where $\chi \in C_0^\infty(X)$, $\chi \geqslant 0$, and φ and a are given in theorem 5.3. Put $\psi = \operatorname{Im} \varphi$.

LEMMA 8.1. — *Let* $F(\alpha,\beta) \geqslant 0$ *be a rapidly decreasing continuous function on* $\bar{\mathbf{R}}_+ \times \bar{\mathbf{R}}_+$. *Then there is a constant* $C > 0$, *such that*

$$(8.2) \qquad \iint F(\psi(t,x,\xi), t|\xi|^{m-1})\chi(x)\, dx\, d\xi \leqslant C \iint_{p(x,\xi) + |\xi|^{m-1} \leqslant \frac{1}{t}} \chi(x)\, dx\, d\xi.$$

Proof. — Put $F_1(\tau) = \sup\limits_{\alpha+\beta \geqslant \tau} F(\alpha,\beta)$ so that F_1 is decreasing, rapidly decreasing and $F(\alpha,\beta) \leqslant F_1(\alpha+\beta)$. Since by lemma 1.4 and the quasihomogeneity

$$\psi(t,x,\xi) \geqslant \frac{t}{C(1+t|\xi|^{m-1})^4}\, p(x,\xi)$$

we have

$$(8.3) \qquad \iint F(\psi, t|\xi|^{m-1})\chi\, dx\, d\xi \leqslant \iint F_1\left(\frac{tp(x,\xi)}{C(1+t|\xi|^{m-1})^4} + t|\xi|^{m-1}\right)\chi\, dx\, d\xi.$$

Now $\alpha/C(1+\beta)^4 + (1+\beta)$ is homogeneous of degree 1 in

$$(\alpha^{1/5}, \beta+1) \in \mathbf{R}_+ \times \mathbf{R}_+,$$

so

$$\frac{\alpha}{C(1+\beta)^4} + (1+\beta) \geqslant \frac{1}{C_0}(\alpha^{1/5} + \beta + 1) \geqslant \frac{1}{C_1}(\alpha+\beta+1)^{1/5}.$$

Thus

$$\frac{tp(x,\xi)}{C(1+t|\xi|^{m-1})^4} + t|\xi|^{m-1} \geqslant \frac{1}{C_1}(tp(x,\xi) + t|\xi|^{m-1})^{1/5} - 1,$$

so if

$$F_2(x) = F_1\left(\max\left(\frac{1}{C_1}x^{1/5}-1,0\right)\right),$$

then

$$F_1\left(\frac{tp}{C1+t|\xi|^{m-1}} + t|\xi|^{m-1}\right) \leqslant F_2(t(p+|\xi|^{m-1})).$$

The integral to the left in (8.2) can therefore be estimated by

$$(8.4)\quad \iint F_2(t(p+|\xi|^{m-1}))\chi(x)\,dx\,d\xi = \int_0^\infty F_2(t\lambda)\,dV_\chi(\lambda) = \int_0^\infty F_2(s)\,dV_\chi\left(\frac{s}{t}\right),$$

where

$$V_\chi(\lambda) = \iint_{p+|\xi|^{m-1}\leqslant\lambda} \chi(x)\,dx\,d\xi.$$

By lemma 6.2. and its proof, we have $V(a\lambda) \leqslant a^{n/(m-1)}V(\lambda)$, for $a \geqslant 1$, $\lambda \geqslant 1$, so applying lemma 7.2., we get

$$\begin{aligned}\int_0^\infty F_2(s)\,dV_\chi\left(\frac{s}{t}\right) &= \int_0^1 F_2(s)\,dV_\chi\left(\frac{s}{t}\right) + \int_1^\infty F_2(s)\,dV_\chi\left(\frac{s}{t}\right)\\ &\leqslant F_2(0)V_\chi\left(\frac{1}{t}\right) + \int_1^\infty F_2(s)\,d(s^{\frac{n}{m-1}})V_\chi\left(\frac{1}{t}\right)\\ &\leqslant CV_\chi\left(\frac{1}{t}\right),\end{aligned}$$

and the proof is complete.

Lemma 8.1. can be applied to estimate $I_\chi(t)$, since

$$a(t,x,\eta) = 0(1)(1+t|\eta|^{m-1})^{-N}$$

for every N. We immediately get the estimate from above in

THEOREM 8.2. — *Under the assumptions of theorem* 0.1, *there is a constant* $C > 0$, *such that for* $0 < t \leqslant 1$:

$$(8.5)\quad \frac{1}{C}\iint_{p(x,\xi)+|\xi|^{m-1}\leqslant 1/t} dx\,d\xi \leqslant \operatorname{tr} e^{-tP} \leqslant C\iint_{p(x,\xi)+|\xi|^{m-1}\leqslant 1/t} dx\,d\xi.$$

Since we shall compute the trace more precisely below, we just indicate briefly how to prove the left half of (8.5). The first observation is that $e^{-tP} = e^{-tP/2}(e^{-tP/2})^*$, so the operator e^{-tP} can be expressed locally with the « selfadjoint » phase function $\varphi(t/2,x,\eta) - \bar{\varphi}(t/2,y,\eta)$ and with amplitude $a(t/2,x,\eta)\bar{a}(t/2,y,\eta)$. Thus the trace can be studied, by looking at the integrals

$$\tilde{I}_\chi(t) = \iint e^{-2\psi(t/2,x,\xi)}\chi(x)\left|a\left(\frac{t}{2}, x,\xi\right)\right|^2 \frac{dx\,d\xi}{(2\pi)^n}. \tag{8.6}$$

For $0 \leqslant t|\xi|^{m-1} \leqslant 1$, we can estimate $|a|$ from below by $1/C$ and ψ from above by $Ctp(x,\xi)$, so

$$\begin{aligned}\tilde{I}_\chi(t) &\geqslant \frac{1}{C}\iint_{t|\xi|^{m-1}\leqslant 1} e^{-Ct(p(x,\xi)+|\xi|^{m-1})}\chi(x)\,dx\,d\xi \\ &\geqslant \frac{1}{C}\int_0^{1/t} e^{-Ct\lambda}\,dV_\chi(\lambda) \geqslant \frac{1}{C_1}V_\chi\left(\frac{1}{t}\right),\end{aligned} \tag{8.7}$$

and the left half of (8.5) follows.

In order to study $\operatorname{tr} e^{-tP}$ more closely, we take a partition of unity $1 = \chi_0 + \sum_1^\infty \chi_\nu$ on T^*X as in section 6. For each χ_ν with $\nu \neq 0$, we have some choice of local coordinates x in X near the projection of the support of χ_ν, and somewhat incorrectly, we assume that the same is true for χ_0. For $\chi = \chi_\nu$ we then have to study

$$I_\chi(t) = \iint e^{i(\varphi(t,x,\xi)-\langle x,\xi\rangle)}a(t,x,\xi)\chi(x,\xi)\frac{dx\,d\xi}{(2\pi)^n}. \tag{8.8}$$

The first case is when $\chi = \chi_0$.

PROPOSITION 8.3.

$$I_{\chi_0}(t) = \iint e^{-tp(x,\xi)}\chi_0(x,\xi)\frac{dx\,d\xi}{(2\pi)^n} + o(1)V\left(\frac{1}{t}\right),$$
$$t \to 0, \qquad t > 0.$$

Proof. – On the support of χ_0 we have $p(x,\xi) \geqslant C^{-1}|\xi|^{m-1+\varepsilon_0}$. Set $\varepsilon_3 = \varepsilon_0/2$ and split the integration in (8.8) into two regions.

First region : $t|\xi|^{m-1} \leqslant |\xi|^{-\varepsilon_3}$. Let $I'_{\chi_0}(t)$ be the corresponding integral. In this region (3.24) gives the approximation

$$(8.9) \qquad \varphi(t,x,\xi) - \langle x,\xi\rangle = (1+0(|\xi|^{-\varepsilon_3}))itp(x,\xi),$$

while the amplitude satisfies :

$$(8.10) \qquad a(t,x,\xi) = 1 + 0(|\xi|^{-\varepsilon_3}).$$

Put $\Phi = \varphi - \langle x,\xi\rangle$, $\Phi_0 = it\ p(x,\xi)$, $\Phi_s = s\Phi + (1-s)\Phi_0$. Then for $0 \leqslant s \leqslant 1$,

$$\frac{d}{ds} e^{i\Phi_s} a = i(\Phi-\Phi_0)ae^{i\Phi_s} = 0(1)|\xi|^{-\varepsilon_3}e^{-\frac{t}{2}p(x,\xi)}$$

so the corresponding integral is $o(1)V(1/t)$ as $t \to 0$. Thus

$$\begin{aligned} I'_{\chi_0}(t) &= \iint_{t|\xi|^{m-1} \leqslant |\xi|^{-\varepsilon_3}} e^{-tp(x,\xi)}a(t,x,\xi)\chi_0(x,\xi)\frac{dx\,d\xi}{(2\pi)^n} + o(1)V\left(\frac{1}{t}\right) \\ &= \iint_{t|\xi|^{m-1} \leqslant |\xi|^{-\varepsilon_3}} e^{-tp(x,\xi)}\chi_0(x,\xi)\frac{dx\,d\xi}{(2\pi)^n} + o(1)V\left(\frac{1}{t}\right). \end{aligned}$$

Second region : $t|\xi|^{m-1} \geqslant |\xi|^{-\varepsilon_3}$. Let $I''_{\chi_0}(t)$ be the corresponding integral. As in the proof of lemma 8.1, we can estimate $I''_{\chi_0}(t)$ by

$$(8.11) \qquad \iint_{t|\xi|^{m-1} \geqslant |\xi|^{-\varepsilon_3}} F_2(tp(x,\xi)+t|\xi|^{m-1})\chi_0(x,\xi)\,dx\,d\xi,$$

where $F_2 \geqslant 0$ is rapidly decreasing. Now

$$tp(x,\xi) \geqslant |\xi|^{\varepsilon_0-\varepsilon_3} = |\xi|^{\varepsilon_0/2},$$

so it follows that $I''_{\chi_0}(t)$ is bounded when $t \to 0$. The same argument shows that

$$\iint_{t|\xi|^{m-1} \geqslant |\xi|^{-\varepsilon_3}} e^{-tp(x,\xi)}\chi_0(x,\xi)\frac{dx\,d\xi}{(2\pi)^n};$$

is bounded as $t \to 0$, and the proposition follows.

We now study $I_{\chi_\nu}(t)$ for $\nu \neq 0$, so we put $\chi = \chi_\nu$, $B = B_\nu$, $r_0 = r_\nu = r_B$, $p = p_B + q_B$. Let $\tilde{\varphi}_B(t,x,\eta)$ be the phase corresponding to

p_B. Applying proposition 3.4 and (6.12) we get

$$(8.12) \qquad \|x - \varphi'_\xi\| + \|\xi - \varphi'_x\| \leqslant Ce^{C(1+t)^2} t r_B^{-\varepsilon_1} p(x,\xi)^{1/2}, \quad (x,\xi) \in \Sigma_B(|\xi|=1),$$

$$(8.13) \quad \varphi(t,x,\xi) - \langle x,\xi\rangle = itp(x,\xi) + 0(1)t^2 e^{C(1+t)^2} r_B^{-2\varepsilon_1} p(x,\xi), \quad (x,\xi) \in \Sigma_B.$$

Proposition 3.5 and (6.11) show that

$$(8.14) \qquad \|\varphi''(t,x,\xi) - \tilde{\varphi}''_B(t,x,\xi)\| \leqslant Ce^{C(1+t)^2} t r_B^{-2\varepsilon_1}, \quad (x,\xi) \in \Sigma_B.$$

Now put $\varphi_B = itq_B + \tilde{\varphi}_B$ and choose coordinates $\theta = (\theta',\theta'')$ on S^*X as in the beginning of section 6. Let Γ be a leaf $\theta' =$ const., intersecting Σ_B at ρ^0 : Then the above estimates give on Γ :

$$(8.15) \qquad \|(\varphi - \varphi_B)|'_\Gamma \Phi\rho^0)\| \leqslant Ce^{C(1+t)^2} t r_B^{-\varepsilon_1} p(\rho^0)^{1/2}$$

$$(8.16) \qquad \|(\varphi - \varphi_B)(\rho^0)\| \leqslant Ce^{C(1+t)^2} t^2 r_B^{-2\varepsilon_1} p(\rho^0)$$

$$(8.17) \qquad \|(\varphi - \varphi_B)|''_\Gamma(\rho^0)\| \leqslant Ce^{C(1+t)^2} t r_B^{-2\varepsilon_1}.$$

We now modify $\tilde{\varphi}_B$ without changing the derivatives up to second order on $\hat{\Sigma}_B$, so that $\tilde{\varphi}_B - \langle x,\eta\rangle$ becomes quadratic in $\theta'' - h(\theta')$. Applying Taylor's formula, (8.15)-(8.17) and the fact that φ''' is of temperate growth as $t \to +\infty$, we get for $(x,\xi) \in B(|\xi|=1)$,

$$(8.18) \quad |\varphi - \varphi_B| \leqslant Cte^{C(1+t)^2}(r_B^{-2\varepsilon_1} q_B + r_B^{-\varepsilon_1} q_B^{1/2} |\theta'' - h(\theta')| + r_B^{-2\varepsilon_1}|\theta'' - h(\theta')|^2) + Ct(1+t)^{N_0}|\theta'' - h(\theta')|^3.$$

In B we have $|\theta'' - h(\theta')| \leqslant Cr_B^{-2\varepsilon_1}$ so the last term can easily be absorbed. We also have the estimate

$$r_0^{-\varepsilon_1} q_B^{1/2} |\theta'' - h(\theta')| \leqslant r_0^{-\varepsilon_1/2} q_B + r_B^{-3\varepsilon_1/2} |\theta'' - h(\theta')|^2 \quad r_B^{-\varepsilon_1/2}(q_B + r_B^{-\varepsilon_1}|\theta'' - h(\theta')|^2).$$

By proposition 2.3,

$$\operatorname{Im} \varphi_B \geqslant tq_B + \frac{t}{C(1+t)^2} p_B \geqslant \frac{t}{C_1(1+t^2)} (q_B + r_B^{-\varepsilon_1}|\theta'' - h(\theta')|^2),$$

so by (8.18),

$$(8.19) \qquad |\varphi - \varphi_B| \leqslant Ce^{C(1+t)^2} r_B^{-\varepsilon_1/2} \operatorname{Im} \varphi_B, \qquad (x,\xi) \in B.$$

Thus there is a function $N(r_B) \to +\infty$, $r_B \to +\infty$ such that

$$(8.20) \qquad |\varphi - \varphi_B| \leqslant C r_B^{-\varepsilon_{1/4}} \operatorname{Im} \varphi_B, \qquad \text{for} \qquad (x,\xi) \in B, \quad 0 \leqslant t \leqslant N(r_B).$$

The same arguments applied to the transport equation show that if a_0 is the leading quasihomogeneous part of a, and a_B is the quasihomogeneous symbol of degree 0 obtained by solving the first transport equation on Σ_B with p replaced by p_B and then extended to be independent of θ'', then

$$(8.21) \quad |a_0 - a_B| \leqslant C r_0^{-\varepsilon_{1/4}} |a_B|, \qquad 0 \leqslant t \leqslant N(r_B), \qquad (x,\xi) \in B.$$

We say that a quantity $\gamma(t,\nu)$ is negligible if it can be estimated by an expression

$$f(r_\nu) \iint_{2\hat{B}_\nu} F(t(p + |\xi|^{m-1}))\, dx\, d\xi$$

where $F \geqslant 0$ is a rapidly decreasing function on $\bar{\mathbf{R}}_+$, independent of ν, and $f(r) \geqslant 0$, $f(r) \to 0$, $r \to \infty$. A finite sum of negligible terms is negligible. If $\gamma(t,\nu)$ is negligible, then

$$\sum_\nu \gamma(t,\nu) = o(1) V\left(\frac{1}{t}\right), \qquad t \to 0.$$

LEMMA 8.4. — *Modulo a negligible term, we have*

$$(8.22) \qquad I_{\chi_\nu}(t) \equiv \iint e^{i(\varphi_B - \langle x,\xi \rangle)} a_{B_\nu} \chi_\nu \frac{dx\, d\xi}{(2\pi)^n}.$$

Proof. — First notice that

$$\iint_{t|\xi|^{m-1} \geqslant N(r_B)} e^{i(\varphi - \langle x,\xi \rangle)} a \chi_\nu\, dx\, d\xi$$

is negligible, and similarly if φ is replaced by φ_B or a by a_0 or a_B. Also

$$\iint e^{i(\varphi - \langle x,\xi \rangle)} (a - a_0) \chi_\nu\, dx\, d\xi \quad \text{is negligible.}$$

The problem is then to show that

$$\iint_{0 \leqslant t|\xi|^{m-1} \leqslant N(r_B)} (e^{i(\varphi_B - \langle x,\xi \rangle)} - e^{i(\varphi - \langle x,\xi \rangle)}) a_0 \chi_\nu\, dx\, d\xi$$

and

$$\iint_{0 \leqslant t|\xi|^{m-1} \leqslant N(r_B)} e^{i(\varphi - \langle x,\xi\rangle)}(a_0 - a_B)\chi_\nu \, dx \, d\xi$$

are negligible. For the first integral, this is easily done by introducing $\varphi_s = s\varphi + (1-s)\varphi_B$ and noticing that in view of (8.20),

$$\frac{d}{ds} \iint_{0 \leqslant t|\xi|^{m-1} \leqslant N(r_B)} e^{i(\varphi_s - \langle x,\xi\rangle)} a_0 \chi_\nu \, dx \, d\xi$$

is (uniformly) negligible, for $0 \leqslant s \leqslant 1$. The second integral is negligible, in view of (8.21).

Let $\varepsilon_3 = \varepsilon_1/2$, where ε_1 was defined in section 6. We shall express $I_{\chi_\nu}(t)$ in intrinsic terms, in the two complementary regions $tr_B^{m-1} \leqslant r_B^{-\varepsilon_3}$ and $tr_B^{m-1} \geqslant r_B^{-\varepsilon_3}$.

In the region $tr_B^{m-1} \leqslant r_B^{-\varepsilon_3}$, the same argument as in the proof of proposition 8.3, shows that modulo a negligible term :

$$I_{\chi_\nu}(t) \equiv \iint e^{-t(p(x,\xi) + \kappa_\nu(x,\xi))} \chi_\nu(x,\xi) \frac{dx \, d\xi}{(2\pi)^n}. \tag{8.23}$$

In the region $tr_B^{m-1} \geqslant r_B^{-\varepsilon_3}$, we first apply lemma 8.4. Let r, θ be the polar coordinates introduced in section 6 so that Σ_B (and $\hat{\Sigma}_B$) takes the form $\theta'' = h(\theta')$, and $dx \, d\xi = r^n(dr/r) \, d\theta$. Passing to these coordinates, we shall first show that $\chi_\nu(r,\theta)$ can be replaced by $\chi_\nu(r,\theta',h(\theta'))$ in the integral in (8.22). (Somewhat incorrectly we write $\chi_\nu(x,\xi)$, $\chi_\nu(r,\theta)$, $\chi_\nu(r,\omega)$ etc. in order to express the same function in different coordinate systems.) Now

$$\chi_\nu(r,\theta) - \chi_\nu(r,\theta',h(\theta')) = 0(r_B^{\varepsilon_0}|\theta'' - h(\theta')|)$$

so we have to show that

$$r_B^{\varepsilon_0} \iint_{(r,\theta',h(\theta')) \in \hat{B}} e^{i(\varphi_B - \langle x,\xi\rangle)} a_B |\theta'' - h(\theta')| \, dx \, d\xi \tag{8.24}$$

is negligible. This expression can be estimated by

$$r_B^{\varepsilon_0} \iint_{(r,\theta',h(\theta')) \in B} F(t\tilde{p}, tr^{m-1}) |\theta'' - h(\theta')| r^n \frac{dr}{r} \, d\theta, \tag{8.25}$$

where $F \geqslant 0$ is rapidly decreasing on $\bar{\mathbf{R}}_+ \times \bar{\mathbf{R}}_+$ and

$$\tilde{p}(r,\theta) = q_B(r,\theta') + \frac{1}{2}\langle p''_{\theta''\theta''}(r,\theta',h(\theta'))(\theta''-h(\theta')), (\theta''-h(\theta'))\rangle = q_B + \tilde{p}_B.$$

In the region under consideration we have $tr_B^{m-1} \geqslant r_B^{-\varepsilon_3}$ and $\tilde{p} \geqslant \frac{1}{C} r_B^{m-\varepsilon_1}|\theta'' - h(\theta')|^2$, so we deduce that

$$|\theta'' - h(\theta')| \leqslant Cr_B^{-(1-\varepsilon_3-\varepsilon_1)/2}(t\tilde{p})^{1/2},$$

and hence with a new decreasing, rapidly decreasing function F, it suffices to estimate

$$\text{(8.26)} \qquad r_B^{\varepsilon_0-(1-\varepsilon_3-\varepsilon_1)2} \iint_{(r,\theta',h(\theta'))\in\hat{B}} F(t(\tilde{p}+r^{m-1}))r^n \frac{dr}{r}\, d\theta.$$

Here $\varepsilon_0 - (1-\varepsilon_2-\varepsilon_1)/2 < 0$, and the only trouble is that the integration in θ'' is over an unbounded domain. We shall therefore estimate the integral in (8.26) by a similar integral over $2B$. Put

$$v(r,\theta',\lambda) = \int_{\tilde{p}_B(r,\theta',\theta'')\leqslant\lambda} d\theta''$$

so that v is homogeneous in λ of degree d, if codim $\Sigma_B = 2d$. Then for r, θ' fixed,

$$\int_0^\infty F(t(q_B(r,\theta')+r^{m-1}+s))\, dv(r,\theta',s)$$
$$= \int_0^\infty F(tq_B+tr^{m-1}+\sigma)\, dv\left(r,\theta',\frac{\sigma}{t}\right) \leqslant$$
$$\int_0^1 F(tq_B+tr^{m-1}+\sigma)\, dv\left(r,\theta',\frac{\sigma}{t}\right) + \int_1^\infty F(tq_B+tr^{m-1}+\sigma)\, d(\sigma^d)v\left(r,\theta',\frac{1}{t}\right)$$
$$\leqslant \tilde{F}(tq_B+tr^{m-1})v\left(r,\theta',\frac{1}{t}\right)$$

where $\tilde{F} \geqslant 0$ is decreasing and rapidly decreasing. The last expression can be bounded by

$$\int_0^{t^{-1}} \hat{F}(t(q_B+r^{m-1}+s))\, dv\,(r,\theta',s),$$

for a suitable rapidly decreasing function $\hat{F}$, so the expression (8.26) can be estimated by a similar expression where the integration is restricted to : $(r,\theta',h(\theta')) \in \hat{B}$, $\tilde{p}_B \leqslant 1/t$. Since $\tilde{p}_B \geqslant (1/C) r_B^{m-\varepsilon_1} |\theta'' - h|^2$ we have $|\theta'' - h(\theta')| \leqslant C\, r_B^{(\varepsilon_3+\varepsilon_1-1)/2}$ in the new domain of integration, so this domain is contained in $2B$. This means that (8.26), (8.25), (8.24) are negligible, and summing up, we have proved :

LEMMA 8.5. — *In the region* $tr_B^{m-1} \geqslant r_B^{-\varepsilon_3}$, $\varepsilon_3 = \varepsilon_1/2$, *we have modulo a negligible term :*

$$(8.27) \qquad I_{\chi_\nu}(t) \equiv \iint e^{i(\varphi_B - \langle x,\xi\rangle)} a_B \chi_\nu(r,\theta',h(\theta')) r^n \frac{dr}{r} \frac{d\theta}{(2\pi)^n}.$$

The θ''-integration in (8.27) can be eliminated, using the stationary phase formula. This is a pure case, since a_B is independent of θ'', and $\varphi_B - \langle x,\xi\rangle = itq_B + H$, where $H = \tilde{\varphi}_B - \langle x,\xi\rangle$ is a quadratic form in $\theta'' - h(\theta')$. With the usual convention, about the choice of branch of $\left(\det \frac{1}{i} H''_{\theta''\theta''}\right)^{1/2}$, we get modulo the same negligible term as in lemma 8.5 that

$$(8.28) \quad I_{\chi_\nu}(t) \equiv (2\pi)^{-(n-d)} \iint e^{-tq_B} \frac{a_B \chi_\nu(r,\theta',h(\theta'))}{\left(\det \frac{1}{i} H''_{\theta''\theta''}\right)^{1/2}} r^n \frac{dr}{r} d\theta',$$

when $tr_B^{m-1} \geqslant r_B^{-\varepsilon_3}$.

The density $(a_B/(\det(1/i)H''_{\theta''\theta''})^{1/2}) r^n (dr/r)\, d\theta'$ on $\hat{\Sigma}_B$ was effectively computed in [7, lemma 5.3]. With $f(\theta')$ defined as in section 6, we get

$$2^{-d} e^{-t\kappa_\nu(r,\theta')} \left(\sum_1^{[d]} \frac{t\mu_j(r,\theta')}{(1-\exp\{-t\mu_j(r,\theta')\})} t^{-d} f(\theta') \right) (1 + O(r_B^{-\varepsilon_1})) r^{n-md} \frac{dr}{r} d\theta'.$$

Here $[d]$ is the integer part of d. In the case when codim Σ_B is even, we get modulo a negligible term from (8.28) :

$$(8.29) \quad I_{\chi_\nu}(t) \equiv (2\pi)^{-n} \iint \sum_{k\in \mathbf{N}^d} e^{-t(q_B+\kappa_\nu+\langle k,\mu\rangle)} \chi_\nu(r,\theta',h(\theta')) \frac{\pi^d \mu_1 \ldots \mu_d f}{r^{md}} r^n \frac{dr}{r} d\theta'$$

(when d' of the μ_j's are 0, the sum over $\mathbf{N}^d$ can be given a meaning as a sum over $\mathbf{N}^{d-d'}$ of integrals.) A similar formula holds in the odd-dimensional case, and we obtain :

LEMMA 8.6. — *In the region* $tr_{\mathrm{B}}^{m-1} \geqslant r_{\mathrm{B}}^{-\varepsilon_3}$, $\varepsilon_3 = \varepsilon_1/2$, *we have, modulo a negligible term :*

$$\text{(8.30)} \qquad \mathrm{I}_{\chi_v}(t) \equiv (2\pi)^{-n} \int e^{-t(p+\kappa_v)}\chi_v(r,\theta',h(\theta'))\Omega_{\hat{\mathrm{B}}}\,(dx\,d\xi),$$

(*where* $\mathrm{B} = \mathrm{B}_v$).

It remains to prove

PROPOSITION 8.7. — *Modulo a negligible term, we have*

$$\text{(8.31)} \qquad \mathrm{I}_{\chi_v}(t) \equiv (2\pi)^{-n} \iint e^{-t(p+\kappa_v)}\chi_v(x,\xi)\Omega_{\hat{\mathrm{B}}}\,(dx\,d\xi).$$

Proof. — We split this into the same regions as before.

First region : $tr_{\mathrm{B}}^{m-1} \leqslant r_{\mathrm{B}}^{-\varepsilon_3}$. Up to a negligible error the right-hand side of (8.31) is

$$\text{(8.32)} \qquad (2\pi)^{-n} \iint e^{-t(p^*+\kappa_v)}\chi_v^*\,dx\,d\xi$$

where p^*, χ_v^* denote the pullbacks of p, χ_v under the projection $\pi = \pi_{\hat{\mathrm{B}}}$, used in section 6, to define the measure $\Omega_{\hat{\mathrm{B}}}$. In the present region $tp - tp^* = 0(r_{\mathrm{B}}^{-\varepsilon_3})$,

$$\chi_v^* - \chi_v = 0(r_{\mathrm{B}}^{\varepsilon_0-(1-\varepsilon_1)/2}) = 0(r_{\mathrm{B}}^{-\varepsilon_3}),$$

so up to a negligible error the integral (8.32) is

$$\text{(8.33)} \qquad (2\pi)^{-n} \iint e^{-t(p+\kappa_v)}\chi_v\,dx\,d\xi,$$

and (8.31) follows in this case from (8.23).

Second region : $tr_{\mathrm{B}}^{m-1} \geqslant r_{\mathrm{B}}^{-\varepsilon_3}$. We now write the integral in (8.31) as

$$\text{(8.34)} \qquad (2\pi)^{n} \iint e^{-t(p^*+\kappa_v)}\chi_v^* f(\theta) r^n \frac{dr}{r}\,d\omega,$$

(where the coordinates $\omega = (\omega'\omega'')$ were introduced in section 6). The same argument as in the proof of lemma 8.6, shows that χ_ν^* may be replaced by $\chi_\nu(r,\theta',h(\theta'))$ and then (8.31) follows from (8.30).

Propositions 8.7 and 8.3 give

COROLLARY 8.8. — *Under the assumptions of theorem* 0.1,

$$\operatorname{tr} e^{-tP} = \frac{1+o(1)}{(2\pi)^n}\int e^{-t\tau}\, d\tilde{W}(\tau), \qquad t \to 0, \qquad t > 0,$$

where $\tilde{W}(\tau)$ *is defined in section* 6.

Applying proposition 6.4, theorem 7.1, we get the main result of this paper :

THEOREM 8.9. — *Under the assumptions of theorem* 0.1, *let* $\Omega\,(dx\,d\xi)$ *be a measure on* $T^*X\backslash 0$, *constructed as in section* 6. *Let* $\kappa(x,\xi) > 0$ *be a continuous function on* $T^*X\backslash 0$, *homogeneous of degree* $m-1$, *and equal to* $S_P + \frac{1}{2}\widetilde{\operatorname{tr}}$ *on* $p^{-1}(0)$. *Then*

$$N(\lambda) = (1+o(1))\iint_{p+\kappa\leqslant\lambda} \Omega\,(dx\,d\xi), \qquad \lambda \to +\infty.$$

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Manuscrit reçu le 29 novembre 1979.

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