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TRANSVERSELY HOMOGENEOUS FOLIATIONS

by Robert A. BLUMENTHAL

1. Introduction and statement of main results.

One way of defining a smooth codimension q foliation $\mathfrak{F}$ of a manifold M is by a smooth N^q-cocyle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta\in A}$ where N^q is a smooth q-dimensional manifold and

(i) $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M.

(ii) $f_\alpha : U_\alpha \to N^q$ is a smooth submersion whose level sets are the leaves of $\mathfrak{F}/U_\alpha$.

(iii) $g_{\alpha\beta} : f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism satisfying

$$f_\alpha = g_{\alpha\beta} \circ f_\beta \qquad \text{on} \qquad U_\alpha \cap U_\beta .$$

If N^q is a homogeneous space G/K (here G is a Lie group and $K \subset G$ is a closed subgroup) and each $g_{\alpha\beta}$ is (the restriction of) a G-translation of G/K, then $\mathfrak{F}$ is called a (transversely) homogeneous G/K-foliation.

Let us consider an important example due to Roussarie. Let $G = SL(2,\mathbf{R})$, $K = \left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : ac=1, a>0\right\}$, and let Γ be a uniform discrete subgroup of $SL(2,\mathbf{R})$. The foliation of $SL(2,\mathbf{R})$ whose leaves are the left cosets of K induces on $M = \Gamma\backslash SL(2,\mathbf{R})$ a homogeneous $SL(2,\mathbf{R})/K \cong S^1$-foliation $\mathfrak{F}$. Moreover, $\mathfrak{F}$ is defined by a smooth nowhere zero one-form ω on M satisfying $d\omega = \omega \wedge \omega_1$, $d\omega_1 = \frac{1}{2}\omega_2 \wedge \omega$, $d\omega_2 = \omega_1 \wedge \omega_2$. Later in this paper we shall show that this set of equations completely characterizes the homogeneous $SL(2,\mathbf{R})/K \cong S^1$-foliations.

Let G be a Lie group acting effectively on the connected homogeneous space G/K and let $\mathfrak{F}$ be a homogeneous G/K-foliation of a connected manifold M.

THEOREM 1. — *To a homogeneous* G/K*-foliation* $\mathfrak{F}$ *on* M *is associated a homomorphism* $\Phi : \pi_1(M) \to G$ *well-defined up to conjugation. Let* Γ *be its image. The induced foliation* $\tilde{\mathfrak{F}}$ *on the cover* $\tilde{M}$ *of* M *associated to the kernel of* Φ *is given by a* Γ*-equivariant submersion* $f : \tilde{M} \to G/K$ (Γ *acting on* $\tilde{M}$ *by covering transformations). The hononomy group of a leaf* L *of* $\mathfrak{F}$ *is isomorphic to the isotropy subgroup* $\Gamma_{\tilde{L}}$ *of* Γ *at* $\tilde{L}$, *where* $\tilde{L}$ *is a leaf of* $\tilde{\mathfrak{F}}$ *projecting to* L. *If* M *is compact (whence each leaf of* $\mathfrak{F}$ *has a well-defined growth type), the growth of* L *is dominated by the growth of the orbit* $\Gamma(x)$, $x = f(\tilde{L})$. *Thus, if* $\pi_1(M)$ *has non-exponential growth (respectively, polynomial growth of degree d), then all the leaves of* $\mathfrak{F}$ *have non-exponential growth (respectively, polynomial growth of degree d).*

See [3] for a more general statement of the first part of the theorem.

In Section 3 we provide a differential forms characterization of a large class of homogeneous foliations. Let $\{\theta_1, \ldots, \theta_{m-k}, \theta_{m-k+1}, \ldots, \theta_m\}$ be a basis of the space of left-invariant one-forms on G such that $\{\theta_{m-k+1}, \ldots, \theta_m\}$ is a basis of the left-invariant one-forms on K. We then have the structure equations of G relative to this basis :

$$d\theta_i = \sum_{1 \leq j < l \leq m} c^i_{jl}\theta_j \wedge \theta_l, \qquad i = 1, \ldots, m \text{ where } c^i_{jl} \in \mathbf{R}.$$

THEOREM 2. — *If* $H^1(M;K)$ *is trivial, then the normal bundle of* $\mathfrak{F}$ *is trivial and there exist* $m - k$ *independent one-forms* $\omega_1, \ldots, \omega_{m-k} \in A^1(M)$ *defining* $\mathfrak{F}$ *and one-forms* $\omega_{m-k+1}, \ldots, \omega_m \in A^1(M)$ *satisfying*

$$d\omega_i = \sum_{1 \leq j < l \leq m} c^i_{jl}\omega_j \wedge \omega_l, \quad i = 1, \ldots, m.$$

COROLLARY 1. — *A codimension one foliation of* M *is a homogeneous* $SL(2,\mathbf{R})/K \cong S^1$ *-foliation if and only if it is defined by a smooth nowhere zero one-form* $\omega \in A^1(M)$ *satisfying* $d\omega = \omega \wedge \omega_1$, $d\omega_1 = \frac{1}{2}\omega_2 \wedge \omega$, $d\omega_2 = \omega_1 \wedge \omega_2$ *where* $\omega_1, \omega_2 \in A^1(M)$.

In Section 4 we shall establish the following :

THEOREM 3. — *If* M *and* K *are compact, then*

i) The universal cover of M *fibers over the universal cover of* G/K, *the fibers being the leaves of the lifted foliation.*

ii) *The closure of each leaf* $L \in \mathfrak{F}$ *is a submanifold of* M (thus $\bar{\mathfrak{F}} = \{\bar{L} : L \in \mathfrak{F}\}$ *is a foliation, possibly with singularities) foliated by the leaves of* $\mathfrak{F}$, *this foliation being a homogeneous* $G'/K_{\bar{L}}$*-foliation where* G' *is a Lie group and* $K_{\bar{L}}$ *is a compact subgroup of* G'.

iii) *If* G/K *is compact with finite fundamental group (e.g., if* G *is a compact semi-simple Lie group), then there exists a connected, open, dense,* $\bar{\mathfrak{F}}$*-saturated submanifold of* M *which fibers over a connected Hausdorff manifold, the fibers being the leaves of* $\bar{\mathfrak{F}}$. *Moreover if* $L \in \mathfrak{F}$ *is a compact leaf whose holonomy group has non-exponential growth, then all the leaves of* $\mathfrak{F}$ *have polynomial growth.*

COROLLARY 2. — *If* M *and* K *are compact and if the universal cover* $(\widetilde{G/K})$ *of* G/K *is contractible, then the universal cover of* M *is a product* $\tilde{L} \times (\widetilde{G/K})$ *where* $\tilde{L}$ *is the (common) universal cover of the leaves of* $\mathfrak{F}$ *and the leaves of the lifted foliation become identified with the sets* $\tilde{L} \times \{\text{pt.}\}$. *Furthermore, the inclusion of a leaf* $L \hookrightarrow M$ *induces a monomorphism* $\pi_1(L) \to \pi_1(M)$ *between fundamental groups.*

In Section 5 we study several particular types of homogeneous foliations and the case where the leaves are one-dimensional. For instance, it is proved that if $\mathfrak{F}$ is a one-dimensional homogeneous $SO(2q+1)/SO(2q) \cong S^{2q}$-foliation of a compact manifold M, then $\pi_1(M)$ has polynomial growth of degree $\leqslant 1$ and $\mathfrak{F}$ has a compact leaf. If M^3 is compact and $\pi_1(M^3)$ is not solvable, then M^3 does not support a codimension 2 Euclidean (homogeneous $SO(2)\cdot\mathbf{R}^2/SO(2)$ -) foliation.

2. Proof of Theorem 1.

Let $\{(U_\alpha, f_\alpha, \lambda_{g_{\alpha\beta}})\}_{\alpha,\beta\in A}$ be a G/K-cocycle defining $\mathfrak{F}$. Here $g_{\alpha\beta} \in G$ and $\lambda_{g_{\alpha\beta}}$ denotes the diffeomorphism of G/K sending aK to $g_{\alpha\beta}aK$. Let $P = \{[\lambda_g \circ f_\alpha]_x : x \in U_\alpha,\ \alpha\in A,\ g\in G\}$, where $[\lambda_g \circ f_\alpha]_x$ denotes the germ of $\lambda_g \circ f_\alpha$ at x. By analyticity and the connectivity of G/K, P admits a differentiable structure such that the natural projection $\pi : P \to M$ is a smooth regular covering with G as the group of covering transformations. Let $\tilde{M}$ be a connected component of P. The group of covering transformations of the covering $\pi : \tilde{M} \to M$ is a subgroup Γ of G and is the image of a homomorphism $\Phi : \pi_1(M) \to G$. The evaluation map $f : \tilde{M} \to G/K$ is a smooth Γ-equivariant submersion constant along the leaves of $\pi^{-1}(\mathfrak{F})$. Let L be a leaf of $\mathfrak{F}$ and choose a leaf $\tilde{L}$ of $\pi^{-1}(\mathfrak{F})$ which projects to L. Then

$\pi/\tilde{L} : \tilde{L} \to L$ is a regular covering whose group of covering transformations, namely $\Gamma_{\tilde{L}}$, is isomorphic to the holonomy group of L. Note that the holonomy group of L can be realized as a subgroup of K and that if σ is a loop in L which is homotopically trivial in M, then the element of holonomy determined by σ is trivial.

We now assume that M is compact. Let $\{U_i, f_i, \lambda_{\gamma_{ij}}\}_{i,j=1}^{m}$ be a finite G/K-cocycle defining $\mathfrak{F}$ such that $\{U_i\}_{i=1}^{m}$ is a regular covering of M in the sense of [6, pp. 336-337] and such that $\gamma_{ij} \in \Gamma$ for $i, j = 1, \ldots, m$. By a plaque $\rho \subset U_i$ of the leaf $L \in \mathfrak{F}$ is meant a connected component of $L \cap U_i$. Fix $L \in \mathfrak{F}$ and let ρ be a plaque of L. Without loss of generality, we may assume that $\rho \subset U_1$. For each $i = 1, \ldots, m$, let $\nu_i(n)$ be the number of distinct plaques in U_i which can be reached from the plaque ρ by a chain of plaques of length $\leqslant n$. If g_ρ denotes the growth function of L at ρ with respect to the regular covering $\{U_i\}_{i=1}^{m}$, then $g_\rho(n) = \nu_1(n) + \ldots + \nu_m(n)$, $n \in \mathbf{Z}^+$ [6]. Let $\Gamma^1 \subset \Gamma$ be a finite symmetric generating set for Γ such that $\gamma_{ij} \in \Gamma^1$ for all $i, j = 1, \ldots, m$. Set $z = f_1(\rho) \in G/K$. We may assume that $\Gamma(z)$ is the orbit $\Gamma(f(\tilde{L}))$ where $\tilde{L} \in \pi^{-1}(\mathfrak{F})$ is a leaf projecting to L. The growth function of Γ at z is $g_z(n) = |\Gamma^n(z)|$, $n \in \mathbf{Z}^+$ where $|\ \ |$ denotes cardinality and $\Gamma^n(z) = \{z' \in G/K : z' = \lambda_{\gamma_j} \circ \ldots \circ \lambda_{\gamma_1}(z)$ for some $\gamma_1, \ldots, \gamma_j \in \Gamma^1$ where $j \leqslant n\}$. Let τ be a plaque of L in U_i which can be reached from the plaque ρ by a chain of plaques of length $\leqslant n$. Let $(\rho = \rho_1, \rho_2, \ldots, \rho_l = \tau)$ be such a chain, $l \leqslant n$. For each $k = 1, \ldots, l$ choose U_{i_k} such that $\rho_k \subset U_{i_k}$, $U_{i_1} = U_1$, $U_{i_l} = U_i$. Then $\lambda_{\gamma_{i_l i_{l-1}}} \circ \ldots \circ \lambda_{\gamma_{i_2 i_1}}(z) = f_i(\tau)$. Thus $f_i(\tau) \in \Gamma^{n-1}(z)$. If $\tau' \neq \tau$ is another such plaque, then $f_i(\tau') \neq f_i(\tau)$. Hence $\nu_i(n) \leqslant g_z(n-1)$ and so $g_\rho(n) \leqslant m g_z(n-1)$ which completes the proof of the theorem.

2.1. Corollary. — *If* G *is nilpotent or if* $\pi_1(M)$ *is nilpotent then all the leaves of* $\mathfrak{F}$ *have polynomial growth. Moreover, the holonomy group of every leaf is finitely generated and has polynomial growth.*

Proof. — Since Γ is a finitely generated nilpotent group, it has polynomial growth [1], [11] and hence all the leaves of $\mathfrak{F}$ have polynomial growth. Let L be a leaf of $\mathfrak{F}$ and $\tilde{L} \in \pi^{-1}(\mathfrak{F})$ a leaf projecting to L. Since $\Gamma_{\tilde{L}} \subset \Gamma$ is a subgroup of a finitely generated nilpotent group, we have that $\Gamma_{\tilde{L}}$ is finitely generated [7] and hence has polynomial growth.

2.2. Corollary. — *If* G *is solvable and* $\pi_1(M)$ *has non-exponential growth, then all the leaves of* $\mathfrak{F}$ *have polynomial growth.*

Proof. — Since Γ is a finitely generated solvable group with non-exponential growth, it follows that Γ has polynomial growth [11].

2.3. COROLLARY. — *If* $L \in \mathfrak{F}$ *is a leaf satisfying*

i) $[\pi_1(M) : i_*\pi_1(L)] < \infty$,

ii) *The holonomy group of* L *has non-exponential growth (respectively, polynomial growth of degree* d),

then all the leaves of $\mathfrak{F}$ *have non-exponential growth (respectively, polynomial growth of degree* d).

Proof. — If $\tilde{L} \in \pi^{-1}(\mathfrak{F})$ is a leaf projecting to L, then we have $[\Gamma : \Gamma_{\tilde{L}}] < \infty$. Hence $\Gamma_{\tilde{L}}$ is finitely generated [1] and, by (ii), has non-exponential growth (respectively, polynomial growth of degree d). Hence Γ has non-exponential growth (respectively, polynomial growth of degree d) [1].

3. Structure equations and the normal bundle.

The following is established using arguments similar to those found, e.g., in Chapter 10 of [9].

3.1. PROPOSITION. — *Let* $\mathfrak{F}$ *be a codimension* $m-k$ *foliation of* M *defined by* $m-k$ *linearly independent one-forms* $\omega_1, \dots, \omega_{m-k} \in A^1(M)$ *and suppose that there are also one-forms* $\omega_{m-k+1}, \dots, \omega_m \in A^1(M)$ *such that*

$$d\omega_i = \sum_{1 \leqslant j < l \leqslant m} c^i_{jl}\omega_j \wedge \omega_l, \quad i = 1, \dots, m.$$

Then $\mathfrak{F}$ *is a homogeneous* G/K*-foliation.*

We now prove Theorem 2. The canonical projection $p : G \to G/K$ makes G a smooth principal K-bundle over G/K. We may pull back this bundle via f to obtain a smooth principal K-bundle $\rho : f^*(G) \to \tilde{M}$ where $f^*(G) = \{(y,g) \in \tilde{M} \times G : f(y) = p(g)\}$ and $\rho(y,g) = y$. We also have a map $\bar{f} : f^*(G) \to G$, defined by $\bar{f}(y,g) = g$, such that $p \circ \bar{f} = f \circ \rho$. Define a left action of Γ on $\tilde{M} \times G$ by $\gamma(y,g) = (\gamma y, \gamma g)$ for $\gamma \in \Gamma$, $(y,g) \in \tilde{M} \times G$. This action of Γ preserves $f^*(G)$ and thus defines a smooth left action of Γ on $f^*(G)$ such that $\gamma \circ \rho = \rho \circ \gamma$ for each $\gamma \in \Gamma$. Let $\Gamma \backslash f^*(G)$ denote the space of orbits and let $\tau : f^*(G) \to \Gamma \backslash f^*(G)$ be the natural projection. The map $\rho : f^*(G) \to \tilde{M}$ induces a continuous surjection $\bar{\rho} : \Gamma \backslash f^*(G) \to M$ satisfying $\bar{\rho} \circ \tau = \pi \circ \rho$. Since the right action of K on $f^*(G)$ commutes with the left action of Γ, there is a free right action of K on $\Gamma \backslash f^*(G)$ such that $\bar{\rho} : \Gamma \backslash f^*(G) \to M$ is a smooth principal K-bundle. We remark that Γ acts freely and properly discontinuously on $f^*(G)$ and hence $\tau : f^*(G) \to \Gamma \backslash f^*(G)$ is a smooth regular covering with Γ as the group of covering transformations.

Now $p \circ \bar{f}$ is a submersion, whence $\bar{f}$ is transverse to $\mathfrak{F}_0$ (where $\mathfrak{F}_0$ is the foliation of G by the left cosets of K) and so $\bar{f}^{-1}(\mathfrak{F}_0)$ is a well-defined foliation of $f^*(G)$. The foliation $\mathfrak{F}_0$ of G is defined by $\theta_1, \ldots, \theta_{m-k}$ and hence $\rho^{-1}(\pi^{-1}(\mathfrak{F})) = \bar{f}^{-1}(\mathfrak{F}_0)$ is defined by the $m-k$ linearly independent smooth one-forms $\bar{f}^*\theta_1, \ldots, \bar{f}^*\theta_{m-k} \in A^1(f^*(G))$ which satisfy

$$d(\bar{f}^*\theta_i) = \sum_{1 \leqslant j < l \leqslant m} c^i_{jl} (\bar{f}^*\theta_j) \wedge (f^*\theta_l), i = 1, \ldots, m.$$

Note that $L_\gamma \circ \bar{f} \circ \gamma$ for each $\gamma \in \Gamma$ where $L_\gamma : G \to G$ denotes left translation by γ. Now $\mathfrak{F}_0$ and $\theta_1, \ldots, \theta_m$ are invariant under left translation by elements of Γ and hence $\bar{f}^{-1}(\mathfrak{F}_0)$ and $\bar{f}^*\theta_1, \ldots, \bar{f}^*\theta_m$ are invariant under the action of Γ on $f^*(G)$. Thus $\bar{f}^{-1}(\mathfrak{F}_0)$ projects to a well-defined foliation $\bar{\mathfrak{F}}$ of $\Gamma \backslash f^*(G)$ and $\bar{f}^*\theta_1, \ldots, \bar{f}^*\theta_m$ project to well-defined smooth one-forms $\alpha_1, \ldots, \alpha_m$ respectively such that $\bar{\mathfrak{F}}$ is defined by the $m - k$ linearly independent smooth one-forms

$$\alpha_1, \ldots, \alpha_{m-k} \in A^1(\Gamma \backslash f^*(G))$$

which satisfy

$$d\alpha_i = \sum_{1 \leqslant j < l \leqslant m} c^i_{jl} \alpha_j \wedge \alpha_l,$$

$i = 1, \ldots, m$. Note that $\bar{\mathfrak{F}} = \bar{\rho}^{-1}(\mathfrak{F})$. Since $H^1(M;K)$ is trivial, there exists a smooth section $s : M \to \Gamma \backslash f^*(G)$. Now s is transverse to $\bar{\rho}^{-1}(\mathfrak{F})$ and $s^{-1}(\bar{\rho}^{-1}(\mathfrak{F})) = \mathfrak{F}$; hence, setting $\omega_i = s^*\alpha_i$ for $i = 1, \ldots, m$, we see that $\mathfrak{F}$ is defined by the $m-k$ independent one-forms $\omega_1, \ldots, \omega_{m-k} \in A^1(M)$ which satisfy $d\omega_i = \sum_{1 \leqslant j < l \leqslant m} c^i_{jl}\omega_j \wedge \omega_l$, $i = 1, \ldots, m$. In particular, the normal bundle of $\mathfrak{F}$ is trivial.

Noting that the two-dimensional affine group

$$K = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : ac = 1, a > 0 \right\}$$

is contractible, we see that Corollary 1 follows from Proposition (3.1) and Theorem 2.

3.2. COROLLARY. — *Suppose* G/K *is a simply connected Riemannian homogeneous space of constant curvature. Let* $\mathfrak{F}$ *be a codimension* $m-k$ *foliation of* M *with trivial normal bundle. Then* $\mathfrak{F}$ *is a homogeneous* G/K*-foliation if and only if there exist* $m-k$ *linearly independent one-forms* $\omega_1, \ldots, \omega_{m-k} \in A^1(M)$ *defining* $\mathfrak{F}$ *and one-forms* $\omega_{m-k+1}, \ldots, \omega_m \in A^1(M)$ *satisfying* $d\omega_i = \sum_{1 \leqslant j < l \leqslant m} c^i_{jl}\omega_j \wedge \omega_l$, $i = 1, \ldots, m$.

Proof. — If $\mathfrak{F}$ is defined by such one-forms, then $\mathfrak{F}$ is a homogeneous G/K-foliation by Proposition (3.1). If $\mathfrak{F}$ is a homogeneous G/K-foliation, then the metric on G/K induces a smooth Riemannian metric on the normal bundle of $\mathfrak{F}$. The hypothesis on G/K implies that K is the full orthogonal group $0(m-k)$ and hence the principal K-bundle $\bar{\rho} : \Gamma\backslash f^*(G) \to M$ constructed in the proof of the theorem is just the bundle of orthonormal frames of the normal bundle of $\mathfrak{F}$. The conclusion now follows from the triviality of this principal K-bundle.

3.3. COROLLARY. — *Let* $\mathfrak{F}$ *be a codimension two foliation of* M *with trivial normal bundle. Then*

a) $\mathfrak{F}$ *is transversely Euclidean* (*homogeneous* $SO(2)\cdot\mathbf{R}^2/SO(2)$) *if and only if it is defined by independent one-forms* ω_1, ω_2 *satisfying* $d\omega_1 = \frac{1}{2}\omega_2 \wedge \omega_3$, $d\omega_2 = -\frac{1}{2}\omega_1 \wedge \omega_3$, $d\omega_3 = 0$.

b) $\mathfrak{F}$ *is transversely hyperbolic* (*homogeneous* $SL(2,\mathbf{R})/SO(2)$) *if and only if it is defined by independent one-forms* ω_1, ω_2 *satisfying* $d\omega_1 = \frac{1}{2}\omega_2 \wedge \omega_3$, $d\omega_2 = \omega_1 \wedge \omega_2 - 2\omega_1 \wedge \omega_3$, $d\omega_3 = -\omega_1 \wedge \omega_3$.

c) $\mathfrak{F}$ *is transversely elliptic* (*homogeneous* $SO(3)/SO(2)$) *if and only if it is defined by independent one-forms* ω_1, ω_2 *satisfying* $d\omega_1 = \frac{1}{2}\omega_2 \wedge \omega_3$, $d\omega_2 = -\frac{1}{2}\omega_1 \wedge \omega_3$, $d\omega_3 = \frac{1}{2}\omega_1 \wedge \omega_2$.

3.4. COROLLARY. — *Let* $\mathfrak{F}$ *be a codimension* q *foliation of* M *with trivial normal bundle. Then* $\mathfrak{F}$ *is transversely affine* (*homogeneous* $GL(q,\mathbf{R})\cdot\mathbf{R}^q/GL(q,\mathbf{R})$) *if and only if there exist* $q = m-k$ *independent one-*

forms $\omega_1, \ldots, \omega_{m-k} \in A^1(M)$ *defining* $\mathfrak{F}$ (*where* $k = q^2$) *and one-forms* $\omega_{m-k+1}, \ldots, \omega_m \in A^1(M)$ *satisfying*

$$d\omega_i = \sum_{1 \leqslant j < l \leqslant m} c^i_{jl} \omega_j \wedge \omega_l, \quad i = 1, \ldots, m.$$

Proof. – If $\mathfrak{F}$ is transversely affine, then the principal K-bundle $\bar{\rho} : \Gamma \backslash f^*(G) \to M$ is just the bundle of frames of the normal bundle of $\mathfrak{F}$.

4. Riemannian homogeneous foliations.

Throughout this section we assume that M and K are compact. Pick and fix a G-invariant Riemannian metric on G/K.

4.1. THEOREM. – *The universal cover of* M *fibers over the universal cover of* G/K, *the fibers being the leaves of the lifted foliation.*

Proof. – Let $\pi : \tilde{M} \to M$ and $f : \tilde{M} \to G/K$ be as in Theorem 1. Let $\hat{M}$ be the universal cover of M and choose a covering map $\pi' : \hat{M} \to \tilde{M}$. Let $\hat{\mathfrak{F}}$ be the foliation of $\hat{M}$ obtained by lifting $\mathfrak{F}$ via $\pi \circ \pi'$ and let $\hat{f} = f \circ \pi'$. Then $\hat{f}$ is a submersion defining $\hat{\mathfrak{F}}$. Let $E \subset T(M)$ be the bundle tangent to the leaves of $\mathfrak{F}$. Choose a subbundle $Q \subset T(M)$ such that $T(M) = E \oplus Q$. If $\{(U_\alpha, f_\alpha, \lambda_{g_{\alpha\beta}})\}_{\alpha,\beta \in A}$ is a G/K-cocycle defining $\mathfrak{F}$, then Q inherits a smooth Riemannian metric by the requirement that $f_{\alpha *_x} : Q_x \to T_{f_\alpha(x)}(G/K)$, $x \in U_\alpha$ be a vector space isometry. Choose a Riemannian metric on E and define a Riemannian metric on M by the requirement that E_x is orthogonal to Q_x for all $x \in M$. Then this metric on the foliated manifold $(M,\mathfrak{F})$ is bundle-like in the sense of [8]. Moreover, since M is compact, this bundle-like metric is complete.

The complete bundle-like metric on M lifts via $\pi \circ \pi'$ to a complete bundle-like metric on the foliated manifold $(\hat{M},\hat{\mathfrak{F}})$. But $\hat{\mathfrak{F}}$ is regular since it is defined by a global submersion. Hence, by Corollary 3 in [8], the space of leaves $\hat{M}/\hat{\mathfrak{F}}$ of $\hat{\mathfrak{F}}$ is a complete, Riemannian, Hausdorff manifold and the natural projection $\hat{M} \to \hat{M}/\hat{\mathfrak{F}}$ is a fibration. Since $\hat{M}$ is simply connected, it follows that $\hat{M}/\hat{\mathfrak{F}}$ is simply connected. Now $\hat{f}$ induces a local isometry $\hat{M}/\hat{\mathfrak{F}} \to G/K$ which lifts to a local isometry $\hat{M}/\hat{\mathfrak{F}} \to \widetilde{(G/K)}$ where $\widetilde{(G/K)}$ denotes the simply connected Riemannian cover of G/K. Hence $\hat{M}/\hat{\mathfrak{F}}$ and

$\widetilde{(G/K)}$ are isometric [4] since each is a connected, simply connected, complete, analytic, Riemannian manifold.

Remark. – Corollary 2 is an immediate consequence of Theorem (4.1). If we do not assume that $\widetilde{(G/K)}$ is contractible, but only that $\pi_2\widetilde{(G/K)} = 0$, then the leaves of $\hat{\mathfrak{F}}$ are simply connected and we still have that $i_* : \pi_1(L) \to \pi_1(M)$ is injective for all $L \in \mathfrak{F}$.

4.2. COROLLARY. – *Suppose the universal cover of* G/K *is contractible.*

i) *If* $\dim \mathfrak{F} = 1$, *then* $\hat{M}$ *is contractible.*

ii) *If* $\dim \mathfrak{F} = 2$, *then either* $\hat{M}$ *is contractible or has the homotopy type of* S^2. *In the latter case, all the leaves of* $\mathfrak{F}$ *are compact with universal cover* S^2.

We adopt the following notation for the rest of this section and the next. Let N denote the connected Riemannian manifold G/K. Thus G is a transitive group of isometries of N. Let $\tilde{N}$ be the universal cover of N endowed with the Riemannian metric lifted from N and let $\tilde{G}$ be the full group of isometries of $\tilde{N}$. Then $\tilde{G}$ acts transitively on $\tilde{N}$. Theorem (4.1) tells us that we have a fibration $\rho : \hat{M} \to \hat{M}/\hat{\mathfrak{F}} \cong \tilde{N}$. Let $\pi_1(M)$ denote the group of covering transformations of $\hat{M}$. Each element $\tau \in \pi_1(M)$ induces an isometry $\psi(\tau) : \hat{M}/\hat{\mathfrak{F}} \to \hat{M}/\hat{\mathfrak{F}}$. We regard $\psi(\tau)$ as an isometry of $\tilde{N}$. Hence $\psi(\tau) \in \tilde{G}$ and we have a homomorphism $\psi : \pi_1(M) \to \tilde{G}$. Let $\Sigma = \text{image}\,\psi \subset \tilde{G}$. For $x \in \tilde{N}$, let $\Sigma_x = \{\sigma \in \Sigma : \sigma(x) = x\}$ and $\Sigma(x) = \{\sigma(x) : \sigma \in \Sigma\}$. Let $L \in \mathfrak{F}$ and choose a leaf $L' \in \hat{\mathfrak{F}}$ which projects to L. Then the orbit $\Sigma(x)$ of $x = \rho(L')$ under Σ depends only on L and we denote this orbit by Σ^L. The following lemma is elementary.

4.3. LEMMA. – *Let* L *be a leaf of* $\mathfrak{F}$. *Then*

i) L *is proper* (*i.e.*, L *is an imbedded submanifold of* M) *if and only if* Σ^L *is a discrete subset of* $\tilde{N}$.

ii) L *is compact if and only if* Σ^L *is discrete and closed.*

iii) L *is dense if and only if* Σ^L *is dense.*

iv) *The space of leaves of* $\mathfrak{F}$ *is homeomorphic to the orbit space* $\Sigma\backslash\tilde{N}$.

v) *Let* $\bar{\Sigma}$ *denote the closure of* Σ *in* $\tilde{G}$. *Then for* $x \in \tilde{N}$, *we have* $\overline{\Sigma(x)} = \bar{\Sigma}(x)$; *that is, the orbit of* x *under* $\bar{\Sigma}$ *is the closure of the orbit of* x *under* Σ.

4.4. THEOREM. – *Let* $L \in \mathfrak{F}$. *Then* $\bar{L}$ *is a submanifold of* M *and the foliation of* $\bar{L}$ *by the leaves of* $\mathfrak{F}$ *is a homogeneous* $G'/K_{\bar{L}}$-*foliation where* G' *is a closed subgroup of* $\tilde{G}$ *and* $K_{\bar{L}}$ *is a compact subgroup of* G'.

Proof. – Let $\pi : \hat{M} \to M$ be the universal cover of M. We have a fibration $\rho : \hat{M} \to \tilde{N}$ whose fibers are the leaves of $\hat{\mathfrak{F}} = \pi^{-1}(\mathfrak{F})$. Choose a leaf $L' \subset \pi^{-1}(L)$ and let $x = \rho(L') \in \tilde{N}$. Then

$$\pi^{-1}(\bar{L}) = \overline{\pi^{-1}(L)} = \overline{\rho^{-1}(\Sigma(x))} = \rho^{-1}(\overline{\Sigma(x)}) = \rho^{-1}(\bar{\Sigma}(x))$$

and so we have a fibration $\rho/\pi^{-1}(\bar{L}) : \pi^{-1}(\bar{L}) \to \bar{\Sigma}(x)$. Now $\bar{\Sigma}(x)$ is the orbit of x under the action of the Lie group $\bar{\Sigma}$ and hence is a submanifold of $\tilde{N}$ from which it follows that $\pi^{-1}(\bar{L})$ is a submanifold of $\hat{M}$. Hence $\bar{L}$ is a submanifold of M.

It remains to demonstrate that the foliation of $\bar{L}$ by the leaves of $\mathfrak{F}$ is a homogeneous foliation. Now $\pi/\pi^{-1}(\bar{L}) : \pi^{-1}(\bar{L}) \to \bar{L}$ is a regular covering and $\rho/\pi^{-1}(\bar{L}) : \pi^{-1}(\bar{L}) \to \bar{\Sigma}(x)$ is a submersion defining $\hat{\mathfrak{F}}/\pi^{-1}(\bar{L}) = \pi^{-1}(\mathfrak{F}/\bar{L})$. Moreover, for each covering transformation τ of $\pi^{-1}(\bar{L})$ we have $(\psi(\tau)/\bar{\Sigma}(x)) \circ (\rho/\pi^{-1}(\bar{L})) = (\rho/\pi^{-1}(\bar{L})) \circ (\tau/\pi^{-1}(\bar{L}))$. Hence there exists a $\bar{\Sigma}(x)$-cocycle $\{(U_\alpha, f_\alpha, \sigma_{\alpha\beta})\}_{\alpha,\beta\in A}$ defining $\mathfrak{F}/\bar{L}$ such that $\sigma_{\alpha\beta} \in \Sigma \subset \bar{\Sigma}$ for all $\alpha, \beta \in A$. But $\bar{\Sigma}(x)$ inherits a Riemannian metric from $\tilde{N}$ and $\psi(\tau)/\bar{\Sigma}(x) : \bar{\Sigma}(x) \to \bar{\Sigma}(x)$ is an isometry. Moreover, $\bar{\Sigma}$ acts transitively on $\bar{\Sigma}(x)$ and so $\bar{\Sigma}(x) \cong \bar{\Sigma}/K_{\bar{L}}$ where $K_{\bar{L}}$ is a compact subgroup of $\bar{\Sigma}$. Taking $G' = \bar{\Sigma}$, we have that $\mathfrak{F}/\bar{L}$ is a homogeneous $G'/K_{\bar{L}}$-foliation.

Remark. – Since $\{\overline{\Sigma(x)} : x \in \tilde{N}\} : \{\bar{\Sigma}(x) : x \in \tilde{N}\}$ partitions $\tilde{N}$, we see that $\{\bar{L} : L \in \mathfrak{F}\}$ partitions M and hence $\bar{\mathfrak{F}} = \{\bar{L} : L \in \mathfrak{F}\}$ is a foliation of M, possibly with singularities.

4.5. THEOREM. – *If $\tilde{N}$ is compact (i.e., if $N = G/K$ is compact with finite fundamental group), there exists a connected, open, dense, $\bar{\mathfrak{F}}$-saturated submanifold V of M which fibers over a connected Hausdorff manifold, the fibers being the leaves of $\bar{\mathfrak{F}}$. Thus, in particular, the leaves of $\bar{\mathfrak{F}}$ contained in V are mutually diffeomorphic.*

Proof. – Since $\tilde{N}$ is compact, it follows that $\tilde{G}$ is compact. Thus $\bar{\Sigma}$ is a compact Lie group acting smoothly on $\tilde{N}$. Let $W \subset \tilde{N}$ be the union of the principal orbits [2]. Then W is an open, dense, saturated subset of $\tilde{N}$ and $\bar{\Sigma}\backslash W$ is a connected Hausdorff manifold [2]. Let $V = \pi(\rho^{-1}(W)) \subset M$. Then V is an open, dense, $\bar{\mathfrak{F}}$-saturated submanifold of M. We have a smooth submersion $V \to \bar{\Sigma}\backslash W$ defining $\bar{\mathfrak{F}}/V$ and hence $\bar{\mathfrak{F}}/V$ is a regular foliation of V with all leaves compact. Thus $V \to \bar{\Sigma}\backslash W = V/\bar{\mathfrak{F}}$ is a fibration [5], the fibers being the leaves of $\bar{\mathfrak{F}}/V$. In particular, V is connected.

4.6. THEOREM. — *Suppose* $(\widetilde{G/K})$ *is compact and that* $L \in \mathfrak{F}$ *is a compact leaf. If the holonomy group of* L *has non-exponential growth (respectively, polynomial growth of degree* d), *then all the leaves of* $\mathfrak{F}$ *have polynomial growth (respectively, polynomial growth of degree* d). *If the fundamental group of* L *has non-exponential growth (respectively, polynomial growth of degree* d), *then* $\pi_1(M)$ *has non-exponential growth (respectively, polynomial growth of degree* d) *and all the leaves of* $\mathfrak{F}$ *have polynomial growth (respectively, polynomial growth of degree* d).

Proof. — Since L is compact we have that Σ^L is discrete and closed, hence finite. Thus $\pi^{-1}(L)$ is a finite union of leaves. Let L' be a leaf of $\hat{\mathfrak{F}}$ which projects to L and let $\pi_1(M)_{L'} = \{\tau \in \pi_1(M) : \tau(L') = L'\}$. Then the index of $\pi_1(M)_{L'}$ in $\pi_1(M)$ is finite and hence $[\pi_1(M) : i_*\pi_1(L)] < \infty$. The holonomy group of L is isomorphic to the linear holonomy group of L and hence is a finitely generated linear group with non-exponential growth, hence polynomial growth [10] (respectively, polynomial growth of degree d). Hence, by Corollary (2.3), all the leaves of $\mathfrak{F}$ have polynomial growth (respectively, polynomial growth of degree d). If $\pi_1(L)$ has non-exponential growth (respectively, polynomial growth of degree d), then so does $i_*\pi_1(L)$. Since $[\pi_1(M) : i_*\pi_1(L)] < \infty$, it follows from [1] that $\pi_1(M)$ has non-exponential growth (respectively, polynomial growth of degree d).

Combining Theorems (4.1), (4.4), (4.5) and (4.6), we obtain Theorem 3.

4.7. PROPOSITION. — *If* $\pi_1(\bar{L})$ *is finite for some leaf* $L \in \mathfrak{F}$, *then all the leaves of* $\mathfrak{F}$ *are compact. If an addition* $(\widetilde{G/K})$ *is compact, then* $\pi_1(M)$ *is finite.*

Proof. — Let $\pi : \hat{M} \to M$ be the universal cover of M and let $\pi^{-1}(\bar{L})_0$ be a connected component of $\pi^{-1}(\bar{L})$. Then $\pi/\pi^{-1}(\bar{L})_0 : \pi^{-1}(\bar{L})_0 \to \bar{L}$ is a covering of $\bar{L}$. Since $\bar{L}$ is compact with finite fundamental group, it follows that $\pi^{-1}(\bar{L})_0$ is compact. Let L' be a leaf of $\hat{\mathfrak{F}}$ contained in $\pi^{-1}(\bar{L})_0$. Then L' is a closed subset of the compact space $\pi^{-1}(\bar{L})_0$ and hence L' is compact. Thus all the leaves of $\hat{\mathfrak{F}}$ are compact and so all the leaves of $\mathfrak{F}$ are compact. If in addition $\tilde{N}$ is compact, then $\pi : \hat{M} \to \tilde{N}$ is a fibration with compact base and compact fiber. Thus $\hat{M}$ is compact and so $\pi_1(M)$ is finite.

5. Elliptic, Euclidean, and hyperbolic foliations.

Throughout this section M denotes a closed manifold.

5.1. PROPOSITION. — *Let* $\mathfrak{F}$ *be a one-dimensional homogeneous* $SO(2q+1)/SO(2q) \cong S^{2q}$*-foliation of* M. *Then* $\pi_1(M)$ *has polynomial growth of degree* $d \leqslant 1$ *and* $\mathfrak{F}$ *has a compact leaf.*

Proof. — Suppose that no leaf of $\mathfrak{F}$ is compact. Then all the leaves of $\tilde{\mathfrak{F}}$ are simply connected and hence $i_*\pi_1(\mathrm{L})$ is trivial for all $\mathrm{L} \in \mathfrak{F}$. Let $\tau \in \pi_1(\mathrm{M})$. Then, since $\psi(\tau) : \mathrm{S}^{2q} \to \mathrm{S}^{2q}$ has a fixed point, there exists a leaf $\mathrm{L}' \in \tilde{\mathfrak{F}}$ such that $\tau(\mathrm{L}') = \mathrm{L}'$. If $\mathrm{L} = \pi(\mathrm{L}') \in \mathfrak{F}$, we have $\tau \in \pi_1(\mathrm{M})_{\mathrm{L}'} = \{\mu \in \pi_1(\mathrm{M}) : \mu(\mathrm{L}') = \mathrm{L}'\} \cong i_*\pi_1(\mathrm{L})$ and hence τ is the identity covering transformation and so M is simply connected. Thus M fibers over S^{2q}, the fibers being the leaves of $\mathfrak{F}$. But this implies that all the leaves of $\mathfrak{F}$ are circles which is impossible. Hence there exists a compact leaf $\mathrm{L} \in \mathfrak{F}$. Since the fundamental group of L has polynomial growth of degree 1, it follows from Theorem (4.6) that $\pi_1(\mathrm{M})$ has polynomial growth of degree $d \leqslant 1$.

5.2. Proposition. — *Let $\mathfrak{F}$ be a codimension two elliptic foliation of* M. *Then either*

i) *all the leaves of $\mathfrak{F}$ are compact,*

ii) *all the leaves of $\mathfrak{F}$ are dense, or*

iii) *all the leaves of $\mathfrak{F}$ have polynomial growth and there exists a compact leaf.*

Proof. — In this case $\mathrm{N} = \tilde{\mathrm{N}} = \mathrm{S}^2$, $\tilde{\mathrm{G}} = \mathrm{SO}(3)$ and so we have $\psi : \pi_1(\mathrm{M}) \to \mathrm{SO}(3)$ with image $\Sigma \subset \mathrm{SO}(3)$. Thus $\bar{\Sigma}$ is a compact Lie group and hence has only finitely many connected components. Let $(\bar{\Sigma})_0$ be the connected component of the identity. Then $(\bar{\Sigma})_0$ is a compact connected Lie group and $[\bar{\Sigma} : (\bar{\Sigma})_0] < \infty$. Let $\Sigma_0 = \Sigma \cap (\bar{\Sigma})_0$. Then Σ_0 is a subgroup of Σ and $[\Sigma : \Sigma_0] < \infty$. Hence, by [1], Σ_0 is finitely generated and has the same growth type as Σ. We consider four cases :

a) $(\bar{\Sigma})_0$ is zero-dimensional : Then Σ is discrete and hence finite. Thus $\Sigma(x)$ is finite for all $x \in \mathrm{S}^2$ and so all the leaves of $\mathfrak{F}$ are compact by Lemma (4.3).

b) $(\bar{\Sigma})_0$ is one-dimensional : Then $(\bar{\Sigma})_0$ is isomorphic to S^1 and so Σ_0 is (finitely generated) abelian. Hence Σ has polynomial growth. Thus, by arguments identical to those used to establish Theorem 1, all the leaves of $\mathfrak{F}$ have polynomial growth. Now Σ_0 is not trivial for otherwise $(\bar{\Sigma})_0$ would be zero-dimensional. Choose a non-identity element $\mathrm{A} \in \Sigma_0$. Then A has exactly two fixed points $x, y \in \mathrm{S}^2$. If $\mathrm{B} \in \Sigma_0$ is nontrivial, then $\mathrm{AB}x = \mathrm{BA}x = \mathrm{B}x$ and $\mathrm{AB}y = \mathrm{BA}y = \mathrm{B}y$. Hence either $\mathrm{B}x = x$ and $\mathrm{B}y = y$ or else $\mathrm{B}x = y$ and $\mathrm{B}y = x$. Either way, the orbit of x under Σ_0 is finite. Thus $\Sigma(x)$ is finite and so $\mathfrak{F}$ has a compact leaf.

c) $(\bar{\Sigma})_0$ is two-dimensional : Then $(\bar{\Sigma})_0$ is isomorphic to the two-dimensional torus which is impossible.

d) $(\bar{\Sigma})_0$ is three-dimensional : Then $(\bar{\Sigma})_0 = \mathrm{SO}(3)$ and so Σ is dense in $\mathrm{SO}(3)$. Hence $\Sigma(x)$ is dense in S^2 for all $x \in S^2$ and so all the leaves of $\mathfrak{F}$ are dense.

5.3. PROPOSITION. — *Let $\mathfrak{F}$ be a codimension two Euclidean, elliptic or hyperbolic foliation of* M. *If* $L \in \mathfrak{F}$ *is a compact leaf with* $H^1(L) = 0$, *then all the leaves of* $\mathfrak{F}$ *are compact. If* $L \in \mathfrak{F}$ *is a (not necessarily compact) leaf with* $H_1(L,\mathbf{Z}) = 0$, *then* $i_*\pi_1(L)$ *is a normal subgroup of* $\pi_1(M)$.

Proof. — Let $L \in \mathfrak{F}$ and choose a leaf $L' \in \hat{\mathfrak{F}}$ which projects to L. Let $x = \rho(L') \in \tilde{N}$. Since Σ_x is abelian, the composition

$$\pi_1(L) \xrightarrow{i_*} i_*\pi_1(L) \cong \pi_1(M)_{L'} \xrightarrow{\psi} \Sigma_x$$

induces a surjection $H_1(L,\mathbf{Z}) \to \Sigma_x$. Suppose L is compact. Then $\Sigma(x) = \Sigma^L$ is discrete and closed. If $H^1(L) = 0$, then $H_1(L,\mathbf{Z})$ is finite and so Σ_x is finite. Thus Σ is a discrete subgroup of $\tilde{G}$ and hence $\Sigma(x)$ is discrete and closed for all $x \in \tilde{N}$ [4]. Thus all the leaves of $\mathfrak{F}$ are compact. If $H_1(L,\mathbf{Z}) = 0$, then Σ_x is trivial and hence $\pi_1(M)_{L'} = \text{kernel}\,\psi$. Thus $i_*\pi_1(L)$ is normal in $\pi_1(M)$.

5.4. PROPOSITION. — *Let $\mathfrak{F}$ be a codimension two Euclidean or hyperbolic foliation of* M. *If* $i_*\pi_1(L_0)$ *is trivial for some leaf* $L_0 \in \mathfrak{F}$, *then the fundamental group of every leaf is abelian.*

Proof. — Choose a leaf $L'_0 \subset \pi^{-1}(L_0)$. Then

$$\text{kernel}\,\psi \subset \pi_1(M)_{L'_0} \cong i_*\pi_1(L_0)$$

and hence ψ is injective. Let $L \in \mathfrak{F}$. Choose $L' \subset \pi^{-1}(L)$ and let $z = \rho(L') \in \tilde{N}$. Since $\pi_2(\tilde{N}) = 0$ we know that $i_* : \pi_1(L) \to \pi_1(M)$ is one-one and so $\psi \circ i_*$ maps $\pi_1(L)$ isomorphically onto Σ_z. But $\Sigma_z \subset \{\tilde{g} \in \tilde{G} : \tilde{g}(z) = z\} \cong \mathrm{SO}(2)$ and so Σ_z is abelian.

5.5. PROPOSITION. — *Suppose* $\tilde{G}$ *is solvable. If* $\mathfrak{F}$ *contains a leaf whose fundamental group is solvable, then* $\pi_1(M)$ *is solvable. Thus if* $\tilde{G}$ *is solvable and* $\dim \mathfrak{F} = 1$, *then* $\pi_1(M)$ *is solvable.*

Proof. – Suppose $L \in \mathfrak{F}$ is such that $\pi_1(L)$ is solvable. Then $i_* \pi_1(L)$ is solvable. Choose a leaf $L' \in \tilde{\mathfrak{F}}$ which projects to L. Then we have kernel $\psi \subset \pi_1(M)_{L'} \cong i_* \pi_1(L)$ and hence kernel ψ is solvable. But $\pi_1(M)/\text{kernel}\,\psi \cong \Sigma$ is solvable since $\Sigma \subset \tilde{G}$. Thus $\pi_1(M)$ is solvable.

5.6. Corollary. – *If* $\pi_1(M^3)$ *is not solvable, then* M^3 *does not support a codimension two Euclidean foliation.*

5.7. Proposition. – *If* $H_1(M,\mathbf{Z}) = 0$, *then* M *does not support a codimension two Euclidean foliation.*

Proof. – If $\mathfrak{F}$ is a codimension two Euclidean foliation of M, then we have $\psi : \pi_1(M) \to \tilde{G} = SO(2).\mathbf{R}^2$. Let $h : SO(2)\cdot\mathbf{R}^2 \to SO(2)$ be the projection. Since $H_1(M,\mathbf{Z}) = 0$ and $SO(2)$ is abelian, it follows that $h \circ \psi$ is the trivial homomorphism. Thus if $\tau \in \pi_1(M)$, then $\psi(\tau)$ is a translation of $\mathbf{R}^2$ and hence $\mathfrak{F}$ is a Lie $\mathbf{R}^2$-foliation. Thus $\mathfrak{F}$ is defined by two linearly independent closed one-forms and hence, since $H^1(M) = 0$, there exists a submersion $M \to \mathbf{R}^2$ defining $\mathfrak{F}$ which is impossible.

5.8. Proposition. – *Let* $\mathfrak{F}$ *be a codimension two Euclidean foliation of* M. *If all the leaves of* $\mathfrak{F}$ *are simply connected, then* $\mathfrak{F}$ *is a Lie* $\mathbf{R}^2$*-foliation and* $\pi_1(M)$ *is abelian.*

Proof. – Let $\sigma \in \Sigma$. Suppose $\sigma(x) = x$ for some $x \in \mathbf{R}^2$. Choose $\tau \in \pi_1(M)$ such that $\psi(\tau) = \sigma$ and let $L' \in \tilde{\mathfrak{F}}$ be a leaf such that $\rho(L') = x$. Then $\tau(L') = L'$. Setting $L = \pi(L') \in \mathfrak{F}$, we have that $\tau \in \pi_1(M)_{L'} \cong i_* \pi_1(L)$. Since L is simply connected, it follows that τ, and hence σ, is the identity transformation. Thus Σ acts freely on $\mathbf{R}^2$ and so Σ is a group of translations. Hence $\mathfrak{F}$ is a Lie $\mathbf{R}^2$-foliation. Finally, $\psi : \pi_1(M) \to \Sigma$ is an isomorphism and so $\pi_1(M)$ is abelian.

5.9. Corollary. – *Let* $\mathfrak{F}$ *be a codimension two Euclidean foliation of the* 3*-manifold* M.

i) *If* $\pi_1(M)$ *is not abelian, then* $\mathfrak{F}$ *has a compact leaf.*

ii) *If* $\mathfrak{F}$ *is not a Lie* $\mathbf{R}^2$*-foliation, then* $\mathfrak{F}$ *has a compact leaf.*

5.10. Proposition. – *Let* $\mathfrak{F}$ *be a codimension two Euclidean foliation of* M *and suppose that* $\pi_1(M)$ *is abelian. Then either*

i) $\mathfrak{F}$ *is a Lie* $\mathbf{R}^2$*-foliation, or*

ii) $\mathfrak{F}$ *has a compact leaf* L *such that* $i_* : \pi_1(L) \to \pi_1(M)$ *is an isomorphism.*

Proof. – Since $\pi_1(M)$ is abelian, we have that Σ is abelian. Hence all the non-identity elements of Σ have the same fixed point set Z. Either Z is empty or Z has one element. If Z is empty, then Σ is a group of translations and so $\mathfrak{F}$ is a Lie $\mathbf{R}^2$-foliation. Suppose $Z = \{x\}$, $x \in \mathbf{R}^2$. Let $L' \in \hat{\mathfrak{F}}$ be a leaf such that $\rho(L') = x$ and let $L = \pi(L') \in \mathfrak{F}$. Then $\Sigma^L = \Sigma(x) = \{x\}$ and hence L is compact. Let $\tau \in \pi_1(M)$. Then, since $\psi(\tau)(x) = x$, we must have that $\tau(L') = L'$. Thus $\tau \in \pi_1(M)_{L'}$ and so

$$i_*\pi_1(L) \cong \pi_1(M)_{L'} = \pi_1(M).$$

5.11. Corollary. – *Let* $\mathfrak{F}$ *be a codimension two Euclidean foliation of* M *where* M *is a* 3*-manifold with* $\pi_1(M)$ *abelian,* $\pi_1(M) \neq \mathbf{Z}$. *Then* $\mathfrak{F}$ *is a Lie* $\mathbf{R}^2$*-foliation.*

Proof. – If not, then $\mathfrak{F}$ has a compact leaf L such that $i_* : \pi_1(L) \to \pi_1(M)$ is an isomorphism. Since $\mathfrak{F}$ is one-dimensional, we have that $L \cong S^1$ and hence $\pi_1(M) = \mathbf{Z}$, a contradiction.

5.12. Corollary. – *Let* $\mathfrak{F}$ *be a codimension two Euclidean foliation of* M *where* M *is an orientable* 4*-manifold with* $\pi_1(M)$ *abelian,* $\pi_1(M) \neq \mathbf{Z} \times \mathbf{Z}$. *Then* $\mathfrak{F}$ *is a Lie* $\mathbf{R}^2$*-foliation.*

Proof. – If not, then $\mathfrak{F}$ has a compact leaf L such that $i_* : \pi_1(L) \to \pi_1(M)$ is an isomorphism. Since M is orientable and $\mathfrak{F}$ is transversely orientable, we have that the leaves of $\mathfrak{F}$ are orientable. Thus L is a compact orientable surface. If the genus of L is one, then $\pi_1(M) = \mathbf{Z} \times \mathbf{Z}$, contrary to assumption. If the genus of L is greater than one, then $\pi_1(M)$ is not abelian, contrary to assumption. If the genus of L is zero, then M is simply connected and hence doesn't support a codimension two Euclidean foliation.

5.13. Proposition. – *If* $\pi_1(M)$ *has non-exponential growth, then* M *does not support a codimension two hyperbolic foliation.*

Proof. – In this case $\tilde{G} = SL(2,\mathbf{R})$ and we have $\psi : \pi_1(M) \to SL(2,\mathbf{R})$ with image $\psi = \Sigma \subset SL(2,\mathbf{R})$. By Lemma (4.3), $\Sigma\backslash\tilde{N}$ is compact and hence $\Sigma\backslash\tilde{G}$ is compact. Thus $\Sigma\backslash SL(2,\mathbf{R})$ is compact which is impossible since Σ has non-exponential growth.

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