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an ideal in its von Neumann algebra**

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## ON GROUP REPRESENTATIONS WHOSE C\*-ALGEBRA IS AN IDEAL IN ITS VON NEUMANN ALGEBRA

by Edmond E. GRANIRER

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*Introduction.* — Let  $\tau$  be a unitary continuous representation of the locally compact group  $G$  on the Hilbert space  $H_\tau$  and denote by  $L(H_\tau)[LC(H_\tau)]$  the algebra of all bounded [compact] linear operators on  $H_\tau$ .  $\tau$  can be lifted in the usual way to a  $*$ -representation of  $L^1(G)$ . Denote by  $C_\tau^*(G) = C_\tau^*$  the norm closure of  $\tau[L^1(G)]$  in  $L(H_\tau)$  (with operator norm) and by  $VN_\tau(G) = VN_\tau$  the  $W^*$ -algebra generated by  $\tau[L^1(G)]$  in  $L(H_\tau)$ . Let

$$M_\tau(C_\tau^*) = \{\varphi \in VN_\tau; \varphi C_\tau^* + C_\tau^* \varphi \subset C_\tau^*\}$$

i.e. the two sided multipliers of  $C_\tau^*$  in  $VN_\tau$  (not in the bidual  $(C_\tau^*)''$  of  $C_\tau^*$ ).

The representation  $\tau$  is said to be CCR if  $C_\tau^* \subset LC(H_\tau)$ . Furthermore,  $\text{supp } \tau$  will denote the closed subset of  $\hat{G}$  of all  $\pi$  in  $\hat{G}$  which are weakly contained (à la Fell) in  $\tau$  (see the notations that follow).

One of the main results in this paper (in slightly shortened fashion) is :

**THEOREM 1.** — *Let  $G$  be  $\sigma$ -compact and  $\tau$  a unitary continuous representation of  $G$  such that  $M_\tau(C_\tau^*) = VN_\tau$ . Then  $\text{supp } \tau$  is a (closed) discrete subset of  $\hat{G}$  and each  $\pi$  in  $\text{supp } \tau$  is CCR (i.e.  $C_\tau^* = LC(H_\tau)$ ).*

A result of I. Kaplanski will hence imply that moreover  $C_\tau^*$  is a dual  $C^*$ -algebra (see [6] (10.10.6)).

Our main application of this theorem is to induced representations and in particular to the quasiregular representation  $\pi_H$  on  $L^2(G/H)$ , for some

closed subgroup  $H$ , as detailed in what follows :

**THEOREM 2** <sup>(1)</sup>. — *Let  $H$  be a subgroup of the  $\sigma$ -compact group  $G$  and  $\nu = \nu^\chi$  the representation of  $G$  induced by the representation  $\chi$  of  $H$ . If  $I_G$  is weakly contained in  $\nu$  and  $M_\nu(C_\nu^*) = VN_\nu$  then  $H$  has finite covolume in  $G$ ,  $\text{supp } \nu$  is discrete and any  $\pi$  in  $\text{supp } \nu$  is CCR.*

Note that  $I_G$  is the unit representation of  $G$  on  $C$ .

We improve somewhat theorem 2 for the case that  $\nu = \pi_H$  is the quasiregular representation of  $G$  on  $L^2(G/H)$  in

**THEOREM 6**. — *Let  $G$  be  $\sigma$ -compact,  $H$  a closed subgroup and  $\nu = \pi_H$ . If  $\pi_H$  weakly contains a nonzero finite dimensional representation and  $M_{\pi_H}(C_{\pi_H}^*) = VN_{\pi_H}$  then  $H$  has finite covolume in  $G$ ,  $\text{supp } \pi_H$  is discrete and any  $\pi$  in  $\text{supp } \pi_H$  is CCR.*

It seems to be in the folklore that if  $G$  is arbitrary and  $H$  is a closed subgroup such that  $G/H$  is a compact coset space then  $M_{\pi_H}(C_{\pi_H}^*) = VN_{\pi_H}$  (see proposition 4).

It seems to us that the fact that  $H$  has finite covolume in  $G$  and  $\text{supp } \pi_H$  is discrete should imply, at least for  $\sigma$ -compact  $G$  that  $G/H$  is compact.

This would generalize a result of L. Baggett [20], A. H. Shtern and S. P. Wang [17] who have proved it for the regular representation (i.e.  $H = \{e\}$ ) of a  $\sigma$ -compact group  $G$ . We pose the above as an open question.

The assumptions of theorem 6 still imply that  $G/H$  is compact at least in the following cases (1)  $H$  is compact. (2)  $H$  is a semidirect summand. (3)  $H$  is open in  $G$ . (4)  $G$  is connected and  $H = Z_G(A)$  for some set  $A$  of automorphisms of  $G$ . (5)  $G$  is a connected Lie group and  $H$  is a connected subgroup. (6)  $G$  is a solvable Lie group and  $H$  is any subgroup (both (5) and (6) using some deep theorems of G.D. Mostow). (7)  $G$  is any Lie group and  $H = Z_G(A)$  ( $A$  as above) etc... See corollaries 1, 2, 3 after theorem 6.

In case  $\pi_H$  does not weakly contain any finite dimensional nonzero representation we still have the

**THEOREM 5**. — *Let  $G$  be  $\sigma$ -compact  $H$  a closed subgroup. If  $M_{\pi_H}(C_{\pi_H}^*) = VN_{\pi_H}$  then  $\text{supp } \pi_H$  is discrete and any  $\pi \in \text{supp } \pi_H$  is CCR.*

<sup>(1)</sup> For second countable  $G$  thm. 8.2 of Mackey in [22] p. 120 implies that  $I_G$  can be replaced by any finite dimensional representation. Thanks are due to L. Baggett for an inspiring conversation connected with this fact.

It seems to us that the fact that  $\text{supp } \pi_H$  is discrete, each  $\pi$  in  $\text{supp } \pi_H$  is CCR and  $G$  is  $\sigma$ -compact should imply that  $G/H$  is compact.

A proof for this statement would provide an answer to the above open question. We commission hereby a proof of this statement from people in the know. It would imply, together with our theorem 5 the following statement :

« Let  $G$  be  $\sigma$ -compact  $H$  a closed subgroup. Then  $M_{\pi_H}(C_{\pi_H}^*) = VN_{\pi_H}$  if and only if  $G/H$  is a compact coset space. »

In case  $H = e$  we have a slightly better result than the above, namely :

**THEOREM 3.** — *Let  $G$  be any locally compact group and  $\rho$  the left regular representation. If  $M_\rho(C_\rho^*) = VN_\rho$  then  $G$  is compact (and conversely).*

This result improves a result of ours in [10] where it was assumed in addition that  $G$  is amenable. M. A. Barnes has informed us that he has also obtained this improvement of our result in [10] using different methods.

*Notations.* — Most of the notations in this paper are consistent with Dixmier [6] and Eymard [7] and [8].

Let  $A$  be a  $C^*$ -algebra. Let  $A''$  be the bidual (or the enveloping)  $C^*$ -algebra of  $A$  as in [6] 12.1.4. Denote by  $M(A) = \{\varphi \in A''; \varphi A + A\varphi \subset A\}$  the multipliers of  $A$  in  $A''$  (or the idealizer of  $A$  in  $A''$ ). If  $H$  is a Hilbert space,  $L(H)$  [ $LC(H)$ ] will denote the algebra of all [compact] bounded linear operators on  $H$ .

$G$  will always stand for a locally compact group with a given left Haar measure. We say that  $\pi$  is a representation of  $G$  if  $\pi$  is a unitary continuous representation of  $G$  as in [6] 13.1.1.  $H_\pi$  will denote a Hilbert space on which the operators  $\{\pi(x); x \in G\}$  act.  $\pi$  can be lifted as usual to a  $*$ -representation of  $L^1(G)$  [in fact of  $M(G)$ ] in the usual way. We denote by  $C_\pi^*(G)$  or  $C_\pi^*$ , when  $G$  is obvious, the  $C^*$ -algebra which is the operator norm closure of  $\pi[L^1(G)]$  in  $L(H_\pi)$ .  $VN_\pi(G) = VN_\pi$  will denote the  $W^*$ -algebra generated by  $\pi[L^1(G)]$  in  $L(H_\pi)$ .

The following notation is important : if  $\pi$  is a representation of  $G$  on  $H_\pi$  then

$$M_\pi(C_\pi^*) = M_\pi(C_\pi^*(G)) = \{\varphi \in VN_\pi; \varphi C_\pi^* + C_\pi^* \varphi \subset C_\pi^*\}$$

i.e. the idealizer of  $C_\pi^*$  in  $VN_\pi$ . Note that  $M_\pi(C_\pi^*)$  does not usually coincide with  $M(C_\pi^*)$  which is the idealizer of  $C_\pi^*$  in its bidual  $(C_\pi^*)''$ . Proposition

12.1.15 of [6] is important in this respect and it shows that the bidual of  $C_\pi^*$  is universal in a certain sense not shared usually by  $VN_\pi$ .

$\rho$  will denote the left regular representation of  $G$  on  $L^2(G)$  and  $VN$  will denote  $VN_\rho$ .  $C^*(G)$  will denote the full  $C^*$ -algebra of  $G$  as in [6] 13.9. The set  $P(G)$  of all positive definite continuous functions on  $G$  is identified with the positive linear functionals on  $C^*(G)$ .  $P_\pi(G) = P_\pi$  will be the set of those  $u$  in  $P(G)$  which are weakly associated with  $\pi$  (this set is canonically identified with the positive linear functionals on  $C_\pi^*$ ).  $B_\pi(G) = B_\pi[B(G)]$  will be the complex linear span of  $P_\pi[P(G)]$  and is identified canonically with the dual of  $C_\pi^*[C^*(G)]$  see Eymard [7], pp. 189-191.

$B(G)$  is equipped with the dual norm on  $C^*(G)$  and  $B_\pi(G) \subset B(G)$  with the subspace norm.  $A_\pi(G) = A_\pi$  will denote the norm closure in  $B_\pi$  of the set of all coefficient functions of the representation  $\pi$  i.e. of  $\{\langle \pi(x)\zeta, \eta \rangle, \zeta, \eta \in H_\pi\}$ .  $A_\pi(G)$  is canonically identified with the predual of the  $W^*$ -algebra  $VN_\pi$ . Many results on  $A_\pi(G)$  are contained in G. Arzac's elegant thesis [3] which unfortunately was never published. We were unable to find them in such an elegant and suitable form anywhere else and will quote hence reference [3].

If  $\pi$  is a representation of  $G$ , it will be identified with its unitary equivalence class.  $\hat{G}$  will as usual denote the set (of equivalence classes) of irreducible representations of  $G$  with the usual weak containment topology of Fell, i.e. that one transported from the hull-kernel topology of  $C^*(G)^\wedge$  as in Dixmier [6] 18.1, p. 314. Let  $\pi_1, \pi_2$  be representations of  $G$ . We say that  $\pi_1$  is weakly contained (à la Fell) in  $\pi_2$  if  $B_{\pi_1} \subset B_{\pi_2}$  (see [7], p. 189).

We denote the support of  $\pi$ ,  $\text{supp } \pi$  as the set of  $\tau$  in  $\hat{G}$  which are weakly contained in  $\pi$  (à la Fell) see [6] 18.1.  $\text{supp } \pi$  is a closed subset of  $\hat{G}$ .  $I_G$  will denote the unit representation of  $G$  on  $\mathbb{C}$  (the complex numbers) i.e.  $\langle I_G(x)\alpha, \beta \rangle = \alpha\beta$  for each  $x$  in  $G$  and  $\alpha, \beta$  in  $\mathbb{C}$ .

A representation  $\pi$  of  $G$  is CCR if  $C_\pi^* \subset LC(H_\pi)$ .

Let  $H$  be a closed subgroup of  $G$ .  $G/H = \{\dot{x} = xH; x \in G\}$  is the space of left cosets of  $G$  with the quotient topology.  $G/H$  admits a quasi invariant measure (see S. Gaal [9], V.3). If  $\Delta_G(\Delta_H)$  are the modular functions of  $G(H)$  and if  $\Delta_G = \Delta_H$  on  $H$  then  $G/H$  admits a (not necessarily finite) invariant measure [9] V. 3, p. 266.  $G/H$  may be compact and  $G/H$  need not admit an invariant measure. It happens in numerous important cases that  $G/H$  admits a *finite* invariant measure (in other terminology that  $H$  has finite

covolume in  $G$ ) and still  $G/H$  is not compact (see for ex G. D. Mostow [11], [12]).

LEMMA 0. — Let  $\sigma$  be a continuous unitary representation of the locally compact group  $G$  such that  $M_\sigma(C_\sigma^*) = VN_\sigma$ . If  $v_n \in A_\sigma \cap P_\sigma(G)$  is a sequence such that  $v_n \rightarrow v_0$  uniformly on compacta then  $v_0 \in A_\sigma \cap P_\sigma(G)$  and  $v_n \rightarrow v_0$  weakly in the Banach space  $A_\sigma$  (i.e. in  $w(A_\sigma, VN_\sigma)$ ).

*Proof.* — Clearly  $\|v_n\| = v_n(e)$  is bounded and by the  $w^*$  compactness of closed bounded balls in the dual  $B_\sigma$  of  $C_\sigma^*$  a subnet of  $v_n$  will converge  $w^*$  (hence in  $w(B_\sigma, L^1)$ ) and also uniformly on compacta to some  $u_0 \in B_\sigma$ . Hence  $u_0(x) = v_0(x)$  for each  $x \in G$  i.e.  $v_0 \in P_\sigma$ . By Akeman and Walter's proposition 2 of [2], p. 458 it follows that  $\langle \Phi, v_n \rangle'' \rightarrow \langle \Phi, u_0 \rangle''$  for each  $\Phi \in M(C_\sigma^*)$  where  $\langle \rangle''$  ( $\langle \rangle$ ) will stand for the  $\langle (C_\sigma^*)'', B_\sigma \rangle$  ( $\langle VN_\sigma, A_\sigma \rangle$ ) duality. Let  $i : C_\sigma^* \rightarrow VN_\sigma$  be the identity embedding then  $i$  can be extended uniquely to a faithful representation also denoted by  $i$  to all of  $M(C_\sigma^*)$  and  $i(M(C_\sigma^*))$  is the idealiser of  $i(C_\sigma^*)$  in its weak closure i.e. in  $VN_\sigma$ , by a result of Akeman Pedersen and Tomiyama [1], p. 280, Prop. 2.4 (and independently obtained by M.C. Flanders in his thesis at Tulane Univ. 1968). If  $\kappa : C_\sigma^* \rightarrow (C_\sigma^*)''$  is the canonical embedding then by Dixmier [6] (12.1.5) there exist a unique ultraweakly continuous representation  $\tilde{i}$  of  $(C_\sigma^*)''$  to  $VN_\sigma$  such that  $\tilde{i}\kappa(\varphi) = i(\varphi)$  for each  $\varphi \in C_\sigma^*$ . As remarked in the proof of [1], p. 280, prop. 2.4  $\tilde{i}$  restricted to  $M(C_\sigma^*)$  is just  $i$ . Since we assume that  $M_\sigma(C_\sigma^*) = VN_\sigma$  it follows that  $\tilde{i}(M(C_\sigma^*)) = VN_\sigma$ .

Claim 1. — If  $\Phi \in (C_\sigma^*)''$  and  $v \in A_\sigma \subset B_\sigma$  the predual of  $VN_\sigma$  then  $\langle \Phi v \rangle'' = \langle \tilde{i}\Phi, v \rangle$ . In fact let  $f_\alpha \in L^1(G)$  be such that

$$\langle \kappa\sigma(f_\alpha), u \rangle'' \rightarrow \langle \Phi u \rangle''$$

for each  $u \in B_\sigma$  (i.e. ultraweakly in  $(C_\sigma^*)''$ , see Dixmier [6] 12.1). Then  $\tilde{i}\kappa\sigma(f_\alpha) \rightarrow \tilde{i}\Phi$  ultraweakly in  $VN_\sigma$  by the u.w. continuity of  $\tilde{i}$ . But  $\tilde{i}\kappa\sigma(f_\alpha) = i\sigma(f_\alpha)$  and since  $v \in A_\sigma$  one has

$$\langle \tilde{i}\kappa\sigma(f_\alpha), v \rangle = \langle i\sigma(f_\alpha), v \rangle = \int f_\alpha(x)v(x) dx = \langle \kappa\sigma(f_\alpha), v \rangle'' \rightarrow \langle \Phi, v \rangle''$$

by Dixmier [6] 12.13 (ii). But the left hand side converges to  $\langle \tilde{i}\Phi, v \rangle$  which proves the present claim.

Claim 2. —  $v_n$  is a weak (i.e.  $w(A_\sigma, VN_\sigma)$ ) Cauchy sequence in  $A_\sigma$ . In fact by the Akeman Pedersen Tomiyama result quoted above each element of

$VN_\sigma$  can be expressed as  $\tilde{v}\Phi$  for some  $\Phi \in M(C_\sigma^*)$ . Hence

$$\langle \tilde{v}\Phi, v_n \rangle = \langle \Phi, v_n \rangle'' \rightarrow \langle \Phi, u_0 \rangle''$$

by the Akeman Walter prop. 2.4 of [2], p. 458, which proves claim 2.

Now  $A_\sigma$  is weakly sequentially complete as any predual of any  $W^*$  algebra by Sakai [13]. Hence there is some  $v' \in A_\sigma$  such that  $v_n \rightarrow v'$  in  $w(A_\sigma, VN_\sigma)$  and in particular  $\int f(x)v_n(x) dx \rightarrow \int f(x)v'(x) dx$  for each  $f \in L^1(G)$ . But the left hand side converges to  $\int f(x)u_0(x) dx$ . Thus  $u_0 = v'$  almost everywhere and since  $u_0$  and  $v'$  are continuous  $u_0(x) = v'(x)$  for each  $x$ . Thus  $u_0 \in A_\sigma \cap P_\sigma(G)$  and  $v_n \rightarrow u_0$  weakly as claimed.

**COROLLARY.** — *If  $G$  is  $\sigma$ -compact and  $\sigma$  is a representation of  $G$  such that  $M_\sigma(C_\sigma^*) = VN_\sigma$  then  $A_\sigma = B_\sigma$ .*

*Proof.* — For any  $v_0 \in P_\sigma$  there exists a sequence  $v_n \in A_\sigma \cap P_\sigma$  such that  $v_n \rightarrow v_0$  uniformly on compacta. Thus  $v_0 \in A_\sigma$ . Hence  $P_\sigma \subset A_\sigma$  and  $A_\sigma = B_\sigma$ .

**THEOREM 1.** — *Let  $G$  be any  $\sigma$ -compact loc. cpt. group and  $\sigma$  a unitary continuous representation of  $G$  such that  $M_\sigma(C_\sigma^*) = VN_\sigma$ .*

*Then  $\text{supp } \sigma$  is a discrete (closed) subset of  $\hat{G}$ , and  $A_\nu = B_\nu$  for each representation  $\nu$  weakly contained  $\sigma$ . In addition :*

*a) for some cardinal  $c$ ,  $c\sigma \simeq c\{\Sigma \oplus \pi; \pi \in \text{supp } \sigma\}$ , ( $\simeq$  stands for equivalence of representations and  $\Sigma \oplus$  for direct Hilbert sum),*

*(b)  $B_\sigma = A_\sigma = \{\Sigma \oplus A_\pi; \pi \in \text{supp } \sigma\}$  (the  $l^1$  direct sum, see Arzac [3], p. 27 and 39) and*

*(c)  $C_\pi^*(G) = LC(H_\pi)$  for each  $\pi \in \text{supp } \sigma$ .*

By a theorem of Kaplanski, as stated in Dixmier [6] (10.10.6)  $C_\sigma^*$  is moreover, a dual  $C^*$ -algebra (see also [6](4.7.20) and [1] prop. 2.4 and thm. 2.8).

*Remark.* — If for some  $\pi \in \hat{G}$ ,  $C_\pi^* = LC(H_\pi)$  then clearly  $M_\pi(C_\pi^*) = VN_\pi = L(H_\pi)$ .

*Proof.* — Let  $\nu$  be a representation weakly contained in  $\sigma$ . Let  $v_0 \in P_\nu \subset P_\sigma$  and  $v_n \in P_\nu \cap A_\nu \subset P_\sigma = P_\sigma \cap A_\sigma$  (by the above corollary) be such that  $v_n \rightarrow v_0$  uniformly on compacta. We apply lemma 0 to the

sequence  $v_n \in A_\sigma$ ,  $v_0 \in B_\sigma$ . It implies that  $v_0 \in A_\sigma$  and  $v_n \rightarrow v_0$  in  $w(A_\sigma, VN_\sigma)$ . Thus a net of convex combinations of the  $v_n$ 's will converge in the norm of  $A_\sigma$  to  $v_0$ . Since  $v_n \in A_\nu$  and  $A_\nu$  is norm closed it follows that  $v_0 \in A_\nu$  i.e.  $A_\nu = B_\nu$ .

Let  $\pi_0 \in \text{supp } \sigma$ . We show that  $\{\pi_0\}$  is open in  $\text{supp } \sigma$ . Let

$$v = \{\Sigma \oplus \pi; \pi \in \text{supp } \sigma, \pi \neq \pi_0\}.$$

If  $\pi_0$  is not open in  $\text{supp } \sigma$  then  $\pi_0 \in \text{supp } v$ . Thus  $A_{\pi_0} = B_{\pi_0} \subset B_\nu = A_\nu$  by above. Thus  $\pi_0$  is quasi-equivalent to a subrepresentation  $v_0$  of  $v$  by Arzac [3] Cor. 3.14, p. 40. Thus  $v_0$  is equivalent to  $k\pi_0$  for some cardinal  $k$  (see Dixmier [6], p. 105 (iii)  $\Rightarrow$  (iv)). But then  $\pi_0$  which is irreducible, is a subrepresentation of  $v_0$  and hence of  $v$ , which cannot be, by the definition of  $v$ . Thus  $\{\pi_0\}$  is open in  $\text{supp } \sigma$  which has hence, to be discrete.

Let now  $\tau = \{\Sigma \oplus \pi; \pi \in \text{supp } \sigma\}$ . Then, by Arzac [3], p. 39, Cor. 1,  $A_\tau$  is the  $l^1$  direct sum of all  $A_\pi = B_\pi \subset B_\sigma = A_\sigma$  with  $\pi \in \text{supp } \sigma$ . Hence  $A_\tau \subset A_\sigma$  and by Arzac [3], p. 43, thm. 3.18  $A_\sigma = A_\tau \oplus A_{\tau'}$  where  $\tau'$  is the linear span of the coefficients of the representations  $v$  of  $G$  contained in  $\sigma$  and disjoint from  $\tau$  (see also Dixmier [6], 5.2. p. 101).

Let  $\pi_1 \in \text{supp } \tau' \subset \text{supp } \sigma$ . Then  $\pi_1 \leq \tau$ . Thus  $A_{\pi_1} \subset A_\tau \cap A_{\tau'} = \{0\}$  by Arzac [3], p. 37, (3.12). This shows that  $A_\sigma = A_\tau$  which implies that for some cardinal  $c$ ,  $c\sigma \simeq c\tau$  (see Dixmier [6], p. 105).

Now  $A_\sigma = A_{c\sigma} = A_{c\tau} = A_\tau$  by Arzac [3], p. 29 and  $A_\tau = \{\Sigma \oplus A_\pi; \pi \in \text{supp } \sigma\}$  (the  $l_1$  direct sum) by [3], p. 39, Cor. 1. To complete the proof one still has to show that  $C_\pi^*(G) = LC(H_\pi)$  for each  $\pi \in \text{supp } \sigma$ . Fix now such a  $\pi$ . It is enough to show that  $C_\pi^*(G)$  is a norm separable C\*-algebra, since then using the fact that  $A_\pi = B_\pi$  we get by Arzac [3], p. 47 that  $C_\pi^*(G) = LC(H_\pi)$ . (This result is stated in [3] only for separable groups  $G$ . However, only the separability of  $C_\pi^*(G)$  is used in the proof). Let  $a \in H_\pi$ ,  $a \neq 0$ . Then  $\{\pi(x)a; x \in G\}$  spans a dense linear subspace of  $H_\pi$ . If  $K \subset G$  is compact then  $\{\pi(x)a; x \in K\}$  is a norm compact hence separable subset of  $H_\pi$ . Since  $G$  is  $\sigma$ -compact, it follows that  $\{\pi(x)a; x \in G\}$  is separable hence so is  $H_\pi$ . Thus  $C_\pi^*(G)$  is a C\*-algebra acting on the separable  $H_\pi$ , whose dual is the singleton  $\{\pi\}$  (if  $\pi'$  is irreducible and weakly contained in  $\pi$  then, since  $\pi$  is closed in  $\hat{G}$ ,  $\pi'$  is equivalent to  $\pi$ ). We apply now lemma 1.5 of S. P. Wang [17], p. 21 and get that  $C_\pi^*(G)$  is separable. This finishes the proof.

*Remark.* — If  $A_\sigma = B_\sigma$  for some representation  $\sigma$ , it does not follow that  $M_\sigma(C_\sigma^*) = VN_\sigma$ , even if  $G$  is abelian. In fact let  $G$  be locally compact



abelian and  $\{\gamma_n\}_0^\infty = S$  be such that  $\gamma_n \rightarrow \gamma_0$  uniformly on compact subsets of  $G$ . (Each  $\gamma_n$  is a continuous character on  $G$ ). Then  $S$  is a compact subset of  $\hat{G}$ .

Let  $\kappa$  be the counting measure on  $S$  and  $\sigma$  the representation of  $G$  on  $L^2(S, \kappa) = l^2(S)$  given by  $(\sigma(x)f)(\gamma) = \langle \gamma, x \rangle f(\gamma)$  for  $\gamma \in S$ ,  $x \in G$  and  $f \in l^2(S)$ . If  $\mu \in M(G)$  then for  $f_1, f_2 \in l^2(S)$  one has

$$\begin{aligned} \langle \sigma(\mu)f_1, f_2 \rangle &= \int_S \int_G \gamma(x) d\mu(x) f_1(\gamma) \overline{f_2(\gamma)} d\kappa \\ &= \int_S \hat{\mu}(\gamma) f_1(\gamma) \overline{f_2(\gamma)} d\kappa = \langle \hat{\mu}f_1, f_2 \rangle \end{aligned}$$

i.e.  $\sigma(\mu)f_1(\gamma) = \hat{\mu}(\gamma)f_1(\gamma)$  (pointwise multiplication of functions) and the operator norm of  $\sigma(\mu)$  on  $l^2(S)$  is just  $\sup \{|\mu(\gamma)|; \gamma \in S\}$ . Thus  $C_\sigma^* = \{ \text{sup norm closure of } L^1(G)^\wedge \text{ restricted to } S \} = C(S)$  the continuous complex functions on  $S$ . Furthermore  $VN_\sigma = l^\infty(S)$  hence

$$M_\sigma(C_\sigma^*) = \{f \in l^\infty(S); fg \in C(S) \text{ for each } g \in C(S)\} = C(S) \neq l^\infty(S) = VN_\sigma$$

since

$$C(S) = \{f \in l^\infty(S); \lim_{n \rightarrow \infty} f(\gamma_n) = f(\gamma_0)\}.$$

(Note that multiplication of operators in  $VN_\sigma$  is just pointwise multiplication of functions.) However  $A_\sigma = B_\sigma$  since  $B_\sigma$  (the dual of  $C(S)$ ) is just  $M(S) = l^1(S)$ , since  $S$  is countable and the predual  $A_\sigma$  of  $l^\infty(S)$  is just  $l^1(S)$ . In fact if  $\gamma_0 \in S$  and  $h \in l^1(G)$  then

$$\langle \delta_{\gamma_0}, \hat{h} \rangle = \langle \delta_{\gamma_0}^\vee, h \rangle = \int \delta_{\gamma_0}^\vee(x) h(x) dx = \hat{h}(\gamma_0).$$

Thus  $A_\sigma = B_\sigma$  but  $M_\sigma(C_\sigma^*) \neq VN_\sigma$ .

Denote by  $I_G$  the trivial unit representation of  $G$  on  $H = C$ . For definitions related to induced representations we follow S. Gaal [9]. Applying the previous theorem to induced representations we get.

**THEOREM 2.** — *Let  $H$  be a closed subgroup of the  $\sigma$ -compact group  $G$  and  $\nu = \nu^x$  the representation induced on  $G$  by the representation  $\chi$  of  $H$ . If  $I_G$  is weakly contained in  $\nu$  and  $M_\nu(C_\nu^*) = VN_\nu$  then  $H$  has finite covolume in  $G$  (i.e.  $G/H$  admits a finite invariant measure) and  $\text{supp } \nu$  is discrete. Thus ([19] Gaal, p. 407),  $n(1_G, \nu^x) = n(1_H, \chi)$ . Any  $\pi$  in  $\text{supp } \nu$  is CCR.*

*Remark.* — In particular, if  $H$  is open in  $G$  i.e.  $G/H$  is discrete, then  $G/H$  is finite. See also footnote (1) in introduction.

*Proof.* —  $I_G$  is weakly contained in  $v$  which by our theorem 1 necessarily contains  $I_G$  as a direct summand. Hence, by proposition 3 in S. Gaal [9], p. 406,  $G/H$  admits a finite invariant measure. The result on the multiplicities  $n$  of  $1_G(1_H)$  in  $v^\lambda(\chi)$  follows from Gaal [9], p. 407.

We use next our main theorem to the regular representation  $\rho$  of any locally compact group.  $VN_\rho(G)$  is denoted by  $VN(G)$  in Eymard [7].

**THEOREM 3.** — *Let  $G$  be any locally group such that  $M_\rho(C_\rho^*(G)) = VN_\rho(G)$ , where  $\rho$  denotes the left regular representation. Then  $G$  is compact.*

*Proof.* — Let  $G_1$  be any open  $\sigma$ -compact subgroup of  $G$ . If  $u \in B_{\rho_1}(G_1)$  ( $\rho_1$  is the left regular representation of  $G_1$ ) let  $\hat{u}$  extend  $u$  to all of  $G$  by  $\hat{u}(x) = 0$  if  $x \in G \sim G_1$ . Then  $\hat{u} \in B_\rho(G)$  (see Eymard [7], 2.31, p. 205). Also,  $P_\rho(G)|_{G_1} = \{u|_{G_1}; u \in P_\rho(G)\} = P_{\rho_1}(G_1)$ . ([7], 2.31, p. 205).

We claim that  $(G_1)_{\rho_1}^\wedge$  the reduced dual of  $G_1$  is discrete. It is enough to prove that  $M_{\rho_1}(C_{\rho_1}^*(G_1)) = VN(G_1)$ . By [7], (3.21) 2°, p. 215, the map  $v \rightarrow v|_{G_1}$  from  $A(G)$  onto  $A(G_1)$  has as its transpose a map  $T \rightarrow \hat{T}$  from  $VN(G_1)$  to  $VN(G)$  which is an isomorphism of  $W^*$ -algebras onto  $VN_{G_1}$  (the  $w^*$ -subalgebra of  $VN(G)$  generated by  $\{\rho(y); y \in G_1\}$ ) which maps  $C_{\rho_1}(G_1)$  into  $C_\rho^*(G)$ . Let  $T \in VN(G_1)$ ,  $\Phi \in C_{\rho_1}^*(G_1)$ . Then  $(T\Phi)^\circ = \hat{T}\hat{\Phi} \in C_\rho^*(G)$  by the assumption of our theorem. Note that  $T \rightarrow \hat{T}$  is one to one since  $u \rightarrow u|_{G_1}$  is onto. We show that  $T\Phi \in C_{\rho_1}^*(G_1)$ . If  $\Lambda \in VN(G)$  let  $\Lambda|_{G_1} \in VN(G_1)$  be defined for  $v \in A(G_1)$  by  $\langle \Lambda|_{G_1}, v \rangle = \langle \Lambda, v \rangle$ , as in [7] 3.21, 1°. If  $\psi \in VN(G_1)$  then  $\hat{\psi}|_{G_1} = \psi$ . Since if  $u \in A(G_1)$  then

$$\langle \hat{\psi}|_{G_1}, u \rangle = \langle \hat{\psi}, \hat{u} \rangle = \langle \psi, \hat{u}|_{G_1} \rangle = \langle \psi, u \rangle.$$

It follows that  $T\Phi = [(T\Phi)^\circ]|_{G_1}$  where  $(T\Phi)^\circ \in C_\rho(G)$ . But by [7], 3.21, 1°  $(T\Phi)^\circ|_{G_1} \in C_{\rho_1}^*(G_1)$ . Thus  $T\Phi \in C_{\rho_1}^*(G_1)$  for each  $T \in VN(G_1)$ .

Hence  $VN(G_1) = M_{\rho_1}(C_{\rho_1}^*(G_1))$ . Our previous theorem implies that  $\text{supp } \rho_1$  is discrete. We use now thm. 7.6, p. 33 of S. P. Wang [17] and get that  $G_1$  is compact.

Since any  $\sigma$ -compact subgroup  $G_1$  of  $G$  is compact it follows that  $G$  is compact.

### Applications to the quasiregular representation.

*Notations.* —  $H$  will always be a closed subgroup of the locally compact group  $G$ . We denote by  $\pi_H$  the quasiregular representation of  $G$  on  $L^2(G/H, \lambda)$  where  $\lambda$  is a quasi invariant measure on  $G/H$  (for definition see Eymard [8], p. 27) and  $G/H = \{\dot{x} = xH; x \in G\}$ .

**PROPOSITION 4.** — *Let  $H$  be a closed subgroup of  $G$  such that the coset space  $G/H$  is compact. Then  $M_{\pi_H}(C_{\pi_H}^*) = VN_{\pi_H}$  (In fact  $C_{\pi_H}^*$  is even an ideal in its second dual).*

*Proof.* — As known [8], p. 27,  $\pi_H$  is the representation of  $G$  induced by  $I_H$  the trivial one dimensional representation of  $H$  on  $\mathbb{C}$  (which is certainly CCR). Hence, if  $G$  is second countable, then  $\pi_H$  is CCR by I Schochetman [14] thm. 4.1, p. 482. i.e.  $\pi_H(f)$  is compact for all  $f$  in  $L^1(G)$ . Schochetman's result has been generalized to arbitrary  $G$  by J.M.G. Fell in [21], p. 57. Consequently  $C_{\pi_H}^*$  contains only compact operators for arbitrary  $G$ .

Let  $A = C_{\pi_H}^*$ . Then by Berglund thm. 5.5, p. 25,  $A$  is an ideal in its second dual  $A''$ . Let  $\pi_0$  be the identity representation of  $A$ ,  $\pi_0(a) = a$  for all  $a \in A$  where  $a$  is an operator on  $L^2(G/H)$ . Then  $\pi_0$  is faithful and hence lifts to a faithful representation again denoted by  $\pi_0$  of

$$A'' = M(A) = \{b \in A''; bA + Ab \subset A\}$$

into  $VN_{\pi_H}$  (see Ackeman Pedersen Tomiyama [1] prop. 2.4, attributed there to M. C. Flanders in his Tulane Univ. 1968 thesis) and furthermore  $\pi_0 M(A)$  is the idealiser of  $A$  in  $VN_{\pi_H}$  i.e. exactly our  $M_{\pi_H}(C_{\pi_H}^*)$ . Note also that by the same prop. 2.4,  $\pi_0$  lifted to  $M(A)$  is just the restriction of the normal extension from  $A$  to  $A''$  as in Dixmier [6] 12.1.5. Hence  $\pi_0 A'' = VN_{\pi_H}$  i.e.

$$M_{\pi_H}(C_{\pi_H}^*) = \pi_0 M(A) = \pi_0(A'') = VN_{\pi_H}$$

which finishes the proof.

*Remark.* — The following known fact has been shown above : If  $\tau$  is a representation of the dual  $C^*$ -algebra  $A$  (i.e.  $A$  is an ideal in  $A''$ ) on the Hilbert space  $H_\tau$  and  $VN_\tau$  is the  $W^*$ -algebra generated by  $\tau(A)$  then  $M_\tau[\tau(A)] = VN_\tau$ .

Our main goal in what follows is to get some converse to the above proposition.

Our main theorem yields immediately the following.

**THEOREM 5.** — *Let  $G$  be  $\sigma$ -compact,  $H \subset G$  a closed subgroup and  $\pi_H$  the quasiregular representation. If  $M_{\pi_H}(C_{\pi_H}^*) = VN_{\pi_H}$  then (1)  $\text{supp } \pi_H$  is a discrete closed subset of  $\hat{G}$  (and  $A_v = B_v$  for any  $v$  weakly contained in  $\pi_H$ ) (2) for some cardinal  $c$ ,  $c\pi_H$  is equivalent to  $c\{\Sigma \oplus \pi; \pi \in \text{supp } \sigma\}$  and (3) any  $v \in \text{supp } \pi_H$  is CCR.*

*Remarks.* — We hoped to find in the literature a result stating the following :

(\*) Let  $G$  be  $\sigma$ -compact  $H \subset G$  a closed subgroup. If  $\text{supp } \pi_H$  is discrete then the coset space  $G/H$  is compact.

(\*) Would generalize the well known result of L. Baggett [20], A. I. Shtern and S. P. Wang [17], p. 33, that if  $G$  is  $\sigma$ -compact  $H = \{e\}$  (thus  $\pi_H = \rho$  is the regular representation) and  $\text{supp } \rho$  is discrete then  $G$  is compact. We could neither find such a result in the literature, nor prove it ourselves. Clearly, our main theorem together with (\*) would imply that if  $H$  is a closed subgroup of the  $\sigma$ -compact  $G$  and  $M_{\pi_H}(C_{\pi_H}^*) = VN_{\pi_H}$  then  $G/H$  is compact.

We still will be able to prove this assertion in many cases.

$G/H$  is said to be an *amenable* coset space, if the identity representation  $I_G$  is weakly contained in the quasiregular representation  $\pi_H$ . We follow Eymard [8] in notations and results regarding  $\pi_H$ . Many equivalent conditions as well as many examples of amenable coset spaces  $G/H$  are given in Eymard [8]. In particular, if  $G$  is an amenable group then  $G/H$  is an amenable coset space for any  $H$ . In quite a few interesting cases,  $G$  and  $H \subset G$  are not amenable while  $G/H$  is an amenable coset space. For example if  $G = \text{SL}(2\mathbb{R})$ ,  $H = \text{SL}(2, \mathbb{Z})$  then  $G/H$  is an amenable coset space even though none of  $G$  or  $H$  are amenable groups. Furthermore, if  $G/H$  is compact then it is not necessarily an amenable coset space.

**THEOREM 6.** — *Let  $G$  be  $\sigma$ -compact and  $H \subset G$  a closed subgroup such that the quasiregular representation weakly contains a finite dimensional nonzero representation.*

*If  $M_{\pi_H}(C_{\pi_H}^*) = VN_{\pi_H}$  then  $G/H$  admits a finite invariant measure and the support of  $\pi_H$  is discrete. In addition any  $\pi$  in  $\text{supp } \pi_H$  is CCR.*

*Proof.* — Assume that  $\alpha$  is a non zero representation of  $G$  on the Hilbert space  $H_\alpha$  with  $\dim H_\alpha = n$ ,  $0 < n < \infty$ . Let  $v(x) = \langle \alpha(x)\xi, \eta \rangle$ ,  $\eta \in H_\alpha$  be such that  $v(x_0) \neq 0$  at some  $x_0$ . Then  $v(x)$  is a bounded continuous almost periodic function on  $G$  ( $AP(G)$ ) and  $M(|v|) > 0$  where  $M$  is the unique invariant mean on the continuous weakly almost periodic functions on  $G$  (denoted  $WAP(G)$ ). Note that  $|v|$  is defined by  $|v|(x) = |v(x)|$  for all  $x$ . This is due to the direct sum decomposition  $WAP(G) = AP(G) \oplus W_0(G)$  where  $W_0(G) = \{f \in WAP(G); M(|f|) = 0\}$  see Burckel [5], pp. 29-30. Clearly  $v \in B_\alpha \subset B_{\pi_H} = A_{\pi_H}$  (by our theorem 1). Apply now the majorisation principle of Hertz as given in Arzac [3], p. 54 (take  $\omega = I_H$ ). Then there is some  $u \in A_{\pi_H}$  such that  $\|u\| \leq \|v\|$  (in  $B(G)$  norm) but  $|v(x)| \leq u(x)$  for each  $x$  in  $G$ . Hence  $M(u) = d \neq 0$  and by Dixmier [6], 13.11.6, p. 274 the constant  $d = M(u)$  function belongs to the uniform closure of  $Co\{l_x u; x \in G\}$  where  $(l_x u)(y) = u(xy)$  and  $Co$  denotes convex hull. Clearly, by M. Walter [15], p. 33,  $l_x u \in B_{\pi_H}$ . Also, for any  $w \in Co\{l_x u; x \in G\}$   $\|w\| \leq \|v\|$  see [7], p. 186. Let  $w_\beta \in Co\{l_x u; x \in G\}$  be a net such that  $w_\beta(x) \rightarrow d > 0$  uniformly in  $x \in G$ . Then a subnet  $w_{\beta_\gamma}$  will converge  $w^*$  and a fortiori  $\sigma(B_{\pi_H}, L^1(G))$  to some  $w_0 \in B_{\pi_H}$  (which is the dual of  $C_{\pi_H}^*$ ). If  $f \in C_{00}(G)$  then  $\int w_{\beta_\gamma} f dx \rightarrow \int w_0 f dx$ . Thus  $\int (w_0 - d)f dx = 0$  for each  $f \in C_{00}(G)$ . This shows that  $w_0 = d$  a.e. ( $G$  is  $\sigma$ -compact). Since  $w_0$  is continuous  $w_0 = d > 0$  everywhere. Thus  $1 \in B_{\pi_H}$  and  $I_G$  is weakly contained in  $\pi_H$ . Apply now theorem 2 and the fact that  $\pi_H$  is in fact  $I_H$  induced to  $G$ . Thus  $H$  has finite covolume in  $G$  and  $\text{supp } \pi_H$  is discrete.

**COROLLARY 1.** — *Let  $G$  be  $\sigma$ -compact  $H$  a closed subgroup be such that  $\pi_H$  weakly contains a finite dimensional  $\neq 0$  representation. If  $M_{\pi_H}(C_{\pi_H}^*) = VN_{\pi_H}$ . Then  $G/H$  is compact and admits a finite invariant measure at least in the following cases :*

1.  $H$  is compact.
2.  $H$  is a semidirect summand.
3.  $H$  is open in  $G$  (i.e.  $G/H$  is discrete)<sup>(2)</sup>. In particular if  $G$  is discrete.
4.  $G$  is connected and  $H = Z_G(A) = \{x \in G; \alpha x = x \text{ for all } \alpha \in A\}$  where  $A$  is any set of continuous automorphisms of  $G$ .

<sup>(2)</sup> Any  $\pi$  in  $\text{supp } \pi_H$  is CCR and since  $G$  is discrete  $\pi$  is finite dimensional. The requirement that  $\pi_H$  weakly contains a finite dimensional rep. is superfluous.

4'. It is enough in 4 that  $G$  is only  $\sigma$ -compact  $H = Z_G(A)$  and  $F(G/G_0)$  is open in  $G/G_0$  (where  $G_0$  is the connected component of  $e$  and  $F(K) = \{x \in K; x \text{ has relatively compact conjugacy class in } K\}$ ).

*Proof.* — 1.  $A_{\pi_H} = B_{\pi_H}$  by our theorem 1 and  $1 \in B_{\pi_H}$  by theorem 6. Thus  $1 \in A_{\pi_H}$ . However  $A_{\pi_H} \subset A(G)$  by Arsac [3], p. 83, proposition. Since  $1 \in A(G) \subset C_0(G)$  it follows that  $G$  is compact.

2.  $G = HN$  and  $H \times N \rightarrow HN$  defined by  $(h, n) \rightarrow hn$  is a homeomorphism. Also  $H \cap N = \{e\}$  and  $N$  is normal in  $G$ . Thus  $N = N/N \cap H \rightarrow G/H$  and  $H = H/N \cap N \rightarrow G/N$  (canonical maps) are homeomorphisms. By prop. 4.4 of S. P. Wang [18], p. 413,  $H$  has finite invariant covolume in  $G$  (which holds by our theorem 6 above) iff  $\{e\} = N \cap H$  has finite covolume in  $N$ . This just means that the *Haar* measure  $\lambda$  on the group  $N$  satisfies  $\lambda(N) < \infty$ . Thus  $N$  has to be a compact group (as well known) and since  $N$  and  $G/H$  are homeomorphic,  $G/H$  is compact.

3.  $G/H$  is discrete and admits a finite invariant measure by our theorem. Thus  $G/H$  is finite.

4 and 4'. Our theorem 6 implies that  $H = Z_G(A)$  has finite invariant covolume in  $G$ . We apply now Corollary 5.7, p. 416 of S. P. Wang [18] and get that  $G/H$  is compact. Note that a slightly better result than 4 can be obtained, by using thm. 5.6., p. 416 of S. P. Wang [18].

**COROLLARY 2.** — *Let  $G$  be  $\sigma$ -compact  $H$  a closed subgroup be such that  $\pi_H$  weakly contains a  $\neq 0$  finite dimensional representation. If  $M_{\pi_H}(C_{\pi_H}^*) = VN_{\pi_H}$ , then  $G/H$  is compact and admits a finite invariant measure at least in the following cases :*

1.  $G$  is a connected Lie group and  $H$  is a connected subgroup.
2.  $G$  is a solvable Lie group and  $H$  any closed subgroup.
3.  $G$  is any Lie group and  $H = Z_G(A)$  where  $A$  is any set of continuous automorphisms of  $G$ .
4.  $G$  is a Lie group all whose connected semisimple subgroups are compact and  $H$  any closed subgroup.

*Proof.* — In any case  $H$  has finite covolume in  $G$  by our theorem 6. We use now some deep results of G. D. Mostow which state that if  $G$  is a connected [or solvable] Lie group and  $H$  is any connected [arbitrary] closed

subgroup such that  $H$  has finite covolume in  $G$  then  $G/H$  is compact (see [19], p. 317 and [16], p. 392 or Mostow [11]). To prove (3) use S. P. Wang's thm. 3.6 on p. 319 of [19] which states that  $H = Z_G(A)$  has finite covolume in  $G$  if and only if  $G/H$  is compact.

To prove 4 note that by theorem 2.1 of S. P. Wang [19], p. 315 any closed subgroup  $H$  of such  $G$  for which  $G/H$  has finite invariant volume necessarily satisfies that  $G/H$  is compact.

**COROLLARY 3.** — *Let  $G$  be  $\sigma$ -compact  $H$  a closed subgroup be such that  $\pi_H$  weakly contains a  $\neq 0$  finite dimensional representation and  $M_{\pi_H}(C_{\pi_H}^*) = VN_{\pi_H}$ .*

*If  $G$  is a Lie group  $H$  any closed subgroup then  $G/H_1$  is compact and has a finite invariant measure, where  $H_1 = Z_G(Z_G(H))$ .*

*Note.* —  $Z_G(K) = \{x \in G; xk = kx \text{ for all } k \text{ in } K\}$ .

*Proof.* — Our theorem 6 implies that  $G/H$  has a finite invariant measure Thus by Wang [19] Cor. 3.7  $G/H_1$  is compact. But  $H \subset H_1$ . Thus by Mostow [11] lemma 2.5  $G/H_1$  admits a finite invariant measure.

The reader will notice that in all above corollaries 1, 2 and 3,  $G$  and  $H$  were so chosen that the assertion  $G/H$  admits a finite invariant measure (i.e.  $I_G$  is an isolated point of  $\text{supp } \pi_H$ ) implied that  $G/H$  is in fact compact.

Our theorem 6 implies however more that this fact, namely it implies that  $\text{supp } \pi_H$  is discrete and  $G/H$  admits a finite invariant measure, and any  $\pi$  in  $\text{supp } \pi_H$  is CCR.

We expect in fact that the following statement (weaker than the statement (\*) after theorem 5) to be true :

(\*\*) Let  $G$  be  $\sigma$ -compact,  $H$  a closed subgroup be such that  $G/H$  admits a finite invariant measure. If  $\text{supp } \pi_H$  is discrete then  $G/H$  is compact. The truth of (\*\*) would imply an improved version of theorem 6 namely.

*Assertion 6'.* — Let  $G$  be  $\sigma$ -compact  $H$  a closed subgroup such that  $\pi_H$  weakly contains a finite dimensional nonzero representation.

If  $M_{\pi_H}(C_{\pi_H}^*) = VN_{\pi_H}$  then  $G/H$  is compact and admits a finite invariant measure. (Theorem 6 only asserts that  $\text{supp } \pi_H$  is discrete and  $G/H$  admits a

finite invariant measure). One notes that Assertion 6' would make corollaries 1, 2, 3 superfluous.

We pose hereby, statements (\*) or at least (\*\*) as open questions.

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