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L^p AND HOLDER ESTIMATES FOR PSEUDODIFFERENTIAL OPERATORS : SUFFICIENT CONDITIONS (1)

by Richard BEALS

1. Introduction.

Boundedness results for classical pseudodifferential operators of order zero in L^p and Hölder spaces are well known. They provide one means of obtaining the interior regularity theory of elliptic operators. Such results have been extended to the quasi-homogeneous case and applied to operators of parabolic or semi-elliptic type.

Much more general classes of pseudodifferential operators, defined by conditions on the symbols which may not be translation invariant nor quasi-homogeneous in the dual variables, have been introduced in the L^2 theory of partial differential operators. In particular, Hörmander [5] has recently developed a very general theory under essentially minimal conditions ; see also Unterberger [10]. In this generality the operators of order zero are bounded in L^2, but usually not in L^p for other p. However, Nagel and Stein [9] have recently obtained L^p and Hölder results for important classes of pseudodifferential operators associated with various hypoelliptic differential operators. These classes (essentially) fit into those introduced in the L^2 framework by Hörmander, so it seems natural to seek within that framework for necessary conditions and for sufficient conditions in order that L^p or Hölder boundedness hold. We have considered necessity in [1].

In Section 2 we introduce a number of conditions on the "metric" defining a class of pseudodifferential operators of order zero. One of these conditions is (one form of) the necessary condi-

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tion from [1] and the others are more or less natural for classes arising from partial differential operators. These conditions are shown to be sufficient for L^p boundedness and for boundedness in suitable Hölder spaces in Sections 3 and 4, respectively. Once certain preliminary consequences of our conditions are established, the arguments are close to those of Nagel and Stein. In particular, the L^p boundedness is obtained via the Coifman-Weiss version of the Calderón-Zygmund theory [4].

In Section 5 we give two examples associated with hypoelliptic operators, and indicate what the corresponding a priori estimates are for these cases.

2. Sufficient conditions.

Let V be a real vector space of dimension $n > 1$, let V' be the dual space, and let $\langle\ ,\ \rangle$ be the canonical pairing between V' and V. Let σ denote the associated symplectic form on the space $W = V \times V'$:

$$\sigma(w, w') = \langle \xi, x' \rangle - \langle \xi', x \rangle, \quad w = (x, \xi), \quad w' = (x', \xi').$$

Let dx be a translation invariant measure on V and $d\xi$ the dual measure on V', normalized so that for the Fourier transform we have

$$\hat{u}(\xi) = \int e^{-i\langle \xi, x\rangle} u(x)\, dx, \quad u(x) = \int e^{i\langle \xi, x\rangle} \hat{u}(\xi)\, d\xi.$$

If $a \in C^\infty(W)$, we define an associated pseudodifferential operator according to one of the formulas

$$A_1 u(x) = \int\int e^{i\langle \xi, x-y\rangle} a(x, \xi) u(y)\, dy\, d\xi \tag{2.1}$$

or

$$A_2 u(x) = \iint e^{i\langle \xi, x-y\rangle} a\left(\frac{1}{2}(x+y), \xi\right) u(y)\, dy\, d\xi. \tag{2.2}$$

Consider a Riemannian metric on W, i.e. a continuous function g from W to the space of positive definite quadratic forms on W. The corresponding class of symbols of order 0 is denote $S(g)$; it consists of those $a \in C^\infty(W)$ such that for each $k \in Z_+$ the corresponding Frechet differential satisfies

$$|a_w^{(k)}(w_1, w_2, \ldots, w_k)| \leqslant C_k \Pi g_w(w_j)^{1/2}, \quad \text{all } w, w_j, \tag{2.3}$$

where g_w is the quadratic form assigned to $w \in W$. Hörmander [5] has given very general conditions on g which guarantee that the operators with symbols in $S(g)$ are an algebra. The conditions also imply that these operators are bounded in $L^2(V)$. Here we give conditions under which the operators are bounded in $L^p(V)$, $1 < p < \infty$, and in certain suitable spaces of Hölder type.

It will be slightly more convenient to work on

$$W_0 = V \times (V' \backslash (0)),$$

which will not affect the local smoothness properties of the operators. For $w \in W_0$, g_w is assumed to be a positive definite quadratic form on W.

Let $S(g)$ denote the set of symbols in $C^\infty(W)$ which satisfy (2.3). We assume that g splits: $g_w(y, \eta) = g_w(y, 0) + g_w(0, \eta)$, $y \in V$, $\eta \in V'$, and set

$$\Phi_w(\eta) = g_w(0, \eta), \quad \eta \in V', \tag{2.4}$$

$$\varphi_w(y) = \sup \{\langle \eta, y \rangle^2 \Phi_w(\eta)^{-1} : \eta \in V', \eta \neq 0.$$

Let $\delta(x, \xi)$ be the volume of the unit ellipsoid for $\varphi_{x,\xi}$:

$$\delta(x, \xi) = |\{y : \varphi_{x,\xi}(y) < 1\}| = c_n |\{\eta : \Phi_{x,\xi}(\eta) < 1\}|^{-1}. \tag{2.5}$$

Our conditions are that there are positive constants c, C, α, β, N such that

$$g_w(y, 0) \leqslant \varphi_w(y), \quad w \in W, y \in V; \tag{2.6}$$

$$c\Phi_{x,\xi} \leqslant \Phi_{x,\eta} \leqslant C\Phi_{x,\xi} \quad \text{if} \quad \Phi_{x,\xi}(\eta - \xi) < 1; \tag{2.7}$$

$$\varphi_{y,\xi} \leqslant C\varphi_{x,\xi}(1 + \varphi_{x,\xi}(y - x))^N; \tag{2.8}$$

$$c\lambda^{-\beta}\Phi_{x,\xi} \leqslant \Phi_{x,\lambda\xi} \leqslant \lambda^{-\alpha}\Phi_{x,\xi}, \quad \text{if} \quad \lambda \geqslant 1; \tag{2.9}$$

$$\Phi_{x,\xi}(\xi) \leqslant C; \tag{2.10}$$

$$c\Phi_{x,\xi} \leqslant \Phi_{x,\eta} \leqslant C\Phi_{x,\xi} \quad \text{if} \quad \delta(x, \xi) = \delta(x, \eta); \tag{2.11}$$

$$a \in S(g) \Longrightarrow A \in \mathcal{L}(L^2(V)). \tag{2.12}$$

Remarks on the conditions (2.6) - (2.12). —The conditions (2.6), (2.7), (2.8), and (2.12) are consequences of Hörmander's conditions [5]. In fact (2.6) follows from the "uncertainty principle" $g \leqslant g^\sigma$;

(2.7) follows from the "slowly varying" condition on g ; (2.8) follows from the "σ-temperate" condition ; (2.12) is the basic boundedness theorem in this case. As indicated in the examples in § 5, it is useful to work with these assumptions in the present weaker form. Only condition (2.6) appears to involve the metric g_w restricted to V ; in practice, as the later examples show, there may be a further condition implicit in (2.12).

Condition (2.9), which implies that Φ decreases algebraically on each ray, is satisfied in most examples of classes of pseudodifferential operators introduced in connection with partial differential equations (at least for $|\xi| \geqslant 1$, $\lambda \geqslant 1$; the extension to small ξ is technically convenient and does not affect the local theory). Condition (2.11) is also satisfied in many examples of such classes.

The remaining condition, (2.10), was shown in [1] to be necessary in the translation invariant case. Here it is the key assumption for sufficiency, through its use in Lemma 2.1 below and in estimates for kernels in Proposition 2.6.

The simplest example to consider here is the class $S_{\rho,\delta}$, for which g is essentially

$$g_{x,\xi}(y, \eta) = |\xi|^{2\delta} |y|^2 + |\xi|^{-2\rho} |\eta|^2 ,$$

where $0 \leqslant \delta$, $\rho \leqslant 1$. Then (2.6) implies $\delta \leqslant \rho$; (2.7), (2.8), (2.9), (2.11) are automatic. For (2.10) we require $\rho = 1$, and for (2.12) we require $\delta < 1$ [3].

Throughout the remainder of this section we assume (2.6)-(2.11) and derive the consequences for coverings and kernels needed to prove L^p and Hölder space estimates.

Note that by (2.8) and the duality between φ and Φ we obtain (with new constants) the analogous inequalities

$$\varphi_{x,\xi} \leqslant C\varphi_{y,\xi}(1 + \varphi_{x,\xi}(y - x))^N ; \tag{2.8$'$}$$

$$(\Phi_{y,\xi}/\Phi_{x,\xi})^{\pm 1} \leqslant C(1 + \varphi_{x,\xi}(y - x))^N . \tag{2.8$''$}$$

It is convenient to introduce notions of equivalence and similarity of positive functions f and g with the same domain. Set

$$f \approx g \qquad \text{if} \qquad cf \leqslant g \leqslant Cf$$

and

$$f \sim g \qquad \text{if} \qquad c \min \{f^\alpha, f^\beta\} \leqslant g \leqslant C \max \{f^\alpha, f^\beta\} ,$$

where c, C, α, β are positive constants.

LEMMA 2.1. – *There is a constant* C *such that for any* $x \in V$ *and any* $\delta > 0$, *the set* $\{\xi : \delta(x, \xi) > \delta\}$ *has measure* $\leqslant C\delta^{-1}$.

Proof. – Let $\Sigma = \{\xi : \delta(x, \xi) = \delta\}$; this is the boundary of the set defined by the inequality. Conditions (2.10) and (2.11) imply that Σ is contained in a fixed multiple of the unit ball as determined by $\Phi_{x,\xi}$ for any given $\xi \in \Sigma$. The result follows from this and (2.5).

We say that a positive function f defined on R_+ is *slowly varying* if

$$c\, f(t) \leqslant f(2t) \leqslant C\, f(t), \qquad t > 0. \tag{2.13}$$

COROLLARY 2.2. – *If* f *is a slowly varying function, then*

$$\int f(\delta(x, \xi))\, d\xi \approx \int_0^\infty f(t)\, t^{-2}\, dt = \int_0^\infty f(t^{-1})\, dt.$$

Proof. – Let $\Omega_j = \{\xi : \delta(x, \xi) \in [2^j, 2^{j+1})\}$. By Lemma 2, Ω_j has measure $\approx 2^{-j}$. The desired equivalence follows.

Let Γ be a fixed ray from the origin in V'. Given $x \in V$ and $\delta > 0$, let

$$B_\delta(x) = \{y : \varphi_{x,\xi}(y - x) < 1\}, \tag{2.14}$$

where $\xi \in \Gamma$ is chosen so that $\delta(x, \xi) = \delta$. The basic properties of the open ellipsoids $B_\delta(x)$ are given in the following.

PROPOSITION 2.3. –

a) $B_\delta(x)$ *has measure* δ.

b) $\{B_\delta(x) : \delta > 0\}$ *is a base of neighborhoods of* x.

c) *There is a positive constant* K *such that if* $B_\delta(x) \cap B_\delta(y) \neq \emptyset$, *then* $B_\delta(y) \subset B_{K\delta}(x)$.

Proof. – Part a) is true by definition of $B_\delta(x)$ and $\delta(x, \xi)$. Part b) follows easily from (2.9). Suppose $z \in B_\delta(x) \cap B_\delta(y)$, and suppose ξ, $\eta \in \Gamma$ are chosen so that $\delta(x, \xi) = \delta = \delta(y, \eta)$. Then $\varphi_{x,\xi}(z - x) < 1$ and $\varphi_{y,\eta}(z - y) < 1$. By (2.8) and (2.8)', this means $\delta(z, \xi) \approx \delta(x, \xi) = \delta(y, \eta)$. By (2.9) we have $|\xi| \approx |\eta|$. Using (2.9) and (2.8)'', we get $\Phi_{x,\xi} \approx \Phi_{z,\xi} \approx \Phi_{z,\eta} \approx \Phi_{y,\eta}$. Part c) follows easily.

We now define a distance function

$$d_x(y) = \inf\{\delta > 0 : y \in B_\delta(x)\}\,, \tag{2.15}$$

so that

$$B_\delta(x) = \{y : d_x(y) < \delta\}\,. \tag{2.16}$$

Let

$$d(x, y) = d_x(y) + d_y(x)\,, \tag{2.17}$$

so that $d(x, y) = d(y, x)$.

PROPOSITION 2.4. –

a) $d(x, y) \approx d_x(y) \approx d_y(x)$.

b) $d(x, z) \leqslant C(d(x, y) + d(y, z))$.

c) $d(x, y)\,\delta(x, \xi)^{-1} \sim \varphi_{x,\xi}(y - x)$.

d) $d(x, x + \lambda z)\,d(x, x + z)^{-1} \sim \lambda$ *for* $\lambda > 0$.

e) $d(y, y \pm z) \approx d(x, x + z)$ *if* $d(x, x + z) \approx d(x, y)$.

Proof. – Parts a) and b) follows easily from Proposition 2.3 c). For part c) condition (2.11) implies that we may assume $\xi \in \Gamma$. Choose $\eta \in \Gamma$ such that $\varphi_{x,\eta}(y - x) = 1$. Then $d_x(y) = \delta(x, \eta)$. By (2.7) and (2.9),

$$d(x, y)\,\delta(x, \xi)^{-1} \approx \delta(x, \eta)\,\delta(x, \xi)^{-1} \sim |\xi|\,|\eta|^{-1}$$
$$\sim \varphi_{x,\xi}(y - x)\,\varphi_{x,\eta}(y - x)^{-1} = \varphi_{x,\xi}(y - x)\,.$$

For part d), choose $\xi(\lambda) \in \Gamma$ so that $\delta(x, \xi(\lambda)) = d(x, x + \lambda z)$. Then $\varphi_{x,\xi(\lambda)}(\lambda z) = 1$, and

$$d(x, x + \lambda z)\,d(x, x + z)^{-1} = \delta(x, \xi(\lambda))\,\delta(x, \xi(1))^{-1}$$
$$\sim |\xi(1)|\,|\xi(\lambda)|^{-1} \sim \varphi_{x,\xi(1)}(\lambda z)\,\varphi_{x,\xi(\lambda)}(\lambda z) = \lambda^2\,.$$

Finally, for part e) choose ξ, $\eta \in \Gamma$ so that $\delta(x, \xi) = d(x, y) = \delta(y, \eta)$. As in the proof of Proposition 2.3 c), $\varphi_{x,\xi} \approx \varphi_{y,\eta}$, so $\varphi_{y,\eta}(z) \approx 1$, so $d(y, y \pm z) \approx \delta(y, \eta) = d(x, y)$. This completes the proof.

Suppose now that $\mathcal{K}$ is a bounded subset of $S(g)$, and that the symbol a is in $\mathcal{K} \cap \mathcal{D}(W_0)$. Then the operators A_1 and A_2 are integral operators, with kernels given by

$$k_1(x, y) = \int e^{i\langle \xi, x - y\rangle}\,a(x, \xi)\,d\xi\,, \tag{2.18}$$

$$k_2(x,y) = \int e^{i\langle\xi, x-y\rangle} a\left(\frac{1}{2}(x+y), \xi\right) d\xi . \tag{2.19}$$

In the propositions below, the constants depend on the bounded set $\mathcal{K}$, but not on the particular symbol a, and k denotes k_1 or k_2.

PROPOSITION 2.5. — $|k(x,y)| \leqslant C d(x,y)^{-1}$.

Proof. — Suppose first that $k = k_1$, and that

$$\operatorname{supp} a(x,\cdot) \subset \{\xi : \Phi_w(\xi - \xi_0) < 1\}, \quad w = (x, \xi_0) . \tag{2.20}$$

Choose $\eta \in V' \backslash (0)$ so that

$$\langle \eta, x-y\rangle^2 = \varphi_w(x-y)\,\Phi_w(\eta) . \tag{2.21}$$

Let L be the differential operator

$$L = (1 + \varphi_w(x-y))^{-1}(1 - \Phi_w(\eta)^{-1}(\eta\cdot\nabla_\xi)^2) .$$

Then

$$L^M(e^{i\langle x-y,\xi\rangle}) = e^{i\langle x-y,\xi\rangle} . \tag{2.22}$$

We use the identity (2.22) in (2.18) and integrate by parts to replace a by $({}^tL)^M a$. Note that (2.20) and (2.3) imply

$$|(\eta\cdot\nabla_\xi)^j a(x,\xi)| \leqslant C_j\,\Phi_w(\eta)^{j/2} . \tag{2.23}$$

Therefore

$$|k(x,y)| \leqslant C_M \int (1 + \varphi_{x,\xi}(x-y))^{-M} d\xi . \tag{2.24}$$

This is true for any $M \in \mathbf{N}$. Therefore Proposition 2.4 c) implies

$$|k(x,y)| \leqslant C'_M \int (1 + d(x,y)\,\delta(x,\xi)^{-1})^{-M} d\xi . \tag{2.25}$$

We use a partition of unity over sets as described in (2.20) to obtain (2.25) without the assumption (2.20). By Corollary 2.2, the integral on the right in (2.25) is equivalent to

$$\int_0^\infty (1 + t\,d(x,y))^{-M} dt = c_M\, d(x,y)^{-1} .$$

If $k = k_2$ instead, we argue as above with $\frac{1}{2}(x+y)$ in place of x, and note that taking $\lambda = \frac{1}{2}$ and $z = y - x$ in Proposition 2.4 d) gives

$$d\left(\frac{1}{2}(x+y), y\right) \approx d(x, y).$$

PROPOSITION 2.6. –

a) *If* $d(y, y+z) \leqslant d(x, y)$, *then*

$$|(z \cdot \nabla_y) k(x, y)| \leqslant C d(y, y+z)^\sigma d(x, y)^{-1-\sigma}.$$

b) *If* $d(x, x+z) \leqslant d(x, y)$, *then*

$$|(z \cdot \nabla_x) k(x, y)| \leqslant C d(x, x+z)^\sigma d(x, y)^{-1-\sigma}.$$

Here σ *is a positive constant depending only on* g.

Proof. – Suppose $k = k_1$. We proceed exactly as in the preceding proof, noting that differentiation $z \cdot \nabla_y$ introduces a factor $-i\langle \xi, z\rangle$. Now by definition of φ, assumption (2.10) and Proposition 2.4 c),

$$\begin{aligned} \langle \xi, z\rangle^2 \leqslant \varphi_{x,\xi}(z)\, \Phi_{x,\xi}(\xi) &\leqslant C\varphi_{x,\xi}(z) \\ &\leqslant C'\, \delta(x, \xi)^{-2\gamma} d(x, x+z)^{2\gamma}, \qquad (2.26) \end{aligned}$$

where we may need different positive exponents γ depending on whether $\delta(x, \xi)^{-1} d(x, x+z)$ is less than or greater than 1. Assume $d(y, y+z) \approx d(x, y)$; by Proposition 2.4 e) this implies $d(x, x+z) \approx d(x, y)$. In view of this and (2.26), the argument of the preceding proof gives

$$\begin{aligned} |(z \cdot \nabla_y) k(x, y)| &\leqslant C\left\{\int_0^d (td)^\alpha\, dt + \int_d^\infty (1+td)^{-M} (td)^\beta\, dt\right\} \\ &\leqslant C' d^{-1}, \quad d = d(x, y). \qquad (2.27) \end{aligned}$$

Here we have used the obvious generalization of Corollary 2.2 to integration over an interval. When $d(y, y+z) \leqslant d(x, y)$, choose $\lambda > 0$ so that $d(y, y+\lambda^{-1}z) = d(x, y)$. Then (2.27) with z replaced by $\lambda^{-1}z$, together with Proposition 2.4 d) with z replaced by $\lambda^{-1}z$, gives the desired inequality.

Next, consider $(z \cdot \nabla_x) k_1$. This differentiation leads to two terms, one with a factor $i\langle \xi, z\rangle$ and one with an x-differentiation of the symbol a. The latter introduces a factor $\leqslant \varphi_{x,\xi}(z)^{1/2}$ into the estimates, by (2.3) and (2.6). Therefore the argument proceeds as above. The proof for $k = k_2$ is essentially the same.

3. L^p estimates.

In this section we assume that g satisfies the conditions (2.6)-(2.12) of Section 2. We assume also that $a \in S(g, 1)$ implies that $a(x, D)^*$ is also an operator with symbol in $S(g, 1)$. This is automatic if we take for $a(x, D)$ the Weyl prescription (2.2).

THEOREM 3.1. – *Each operator with symbol in* $S(g, 1)$ *is bounded in* L^p, $1 < p < \infty$.

As in Section 2, let $\mathcal{K}$ be a bounded set in $S(g)$ and suppose $a \in \mathcal{K} \cap \mathcal{D}(W)$. Let $k = k_1$ or k_2, given by (2.16) and (2.17). In the following estimate, the constants depend only on $\mathcal{K}$, not on a.

LEMMA 3.2. – *There are positive constants* K *and* C *such that for any* $y \in V$ *and for any* $\delta > 0$, *if* $d(y, y_1) < \delta$ *then*

$$\int_{d(x,y)>K\delta} |k(x, y_1) - k(x, y)| \, dx \leqslant C\delta^{-1}. \tag{3.1}$$

Proof. – Let $y_s = y + s(y_1 - y)$, $0 \leqslant s \leqslant 1$. Then $d(y, y_s) < \delta$. By part b) of Proposition 2.4, if K is large enough then

$$d(x, y') \approx d(x, y) \quad \text{if} \quad d(x, y) \geqslant K d(y, y'). \tag{3.2}$$

Now

$$k(x, y_1) - k(x, y) = \int_0^1 ((y_1 - y) \cdot \nabla_y) \, k(x, y_s) \, ds.$$

Then (2.25) and (3.2) imply

$$|k(x, y_1) - k(x, y)| \leqslant C\delta^\sigma d(x, y)^{-1-\sigma}. \tag{3.3}$$

Now the set $\{x : d(x, y) \leqslant \epsilon\}$ has measure $O(\epsilon)$. Therefore the integral in (3.1) is dominated by

$$\int_{d(x,y)>K\delta} \delta^\sigma d(x, y)^{-1-\sigma} dx \approx \int_{K\delta}^\infty \delta^\sigma t^{-1-\sigma} dt = C(K, \sigma).$$

The proof of Theorem 3.1 now follows from trivial modifications in the version by Coifman and Weiss [4] of the Calderon-Zygmund theory. We sketch the argument here for completeness.

Let the ellipsoids $B_\delta(x)$ be as in section 3, with $K > 0$ chosen so that (3.1) is true and also so that

$$B_\delta(x) \cap B_\delta(y) \neq \phi \quad \text{implies} \quad B_\delta(y) \subset B_{K\delta}(x) .$$

If $j \in Z$, let

$$\delta_j = K^{-j}, \quad \mathcal{B}_j = \{B_\delta(x) : x \in V,\ \delta = \delta_j\},$$

and set $\mathcal{B} = \cup \mathcal{B}_j$.

If $f \in L^1(V)$, let f^* be the maximal function corresponding to $\mathcal{B}$:

$$f^*(x) = \sup \{m(|f|, B) : x \in B,\ B \in \mathcal{B}\},$$

where

$$m(g, B) = |B|^{-1} \int_B g .$$

LEMMA 3.3. – *If* $f \in L^1(V)$ *and* $\alpha > 0$, *then* $\{x : f^*(x) > \alpha\}$ *has measure* $\leqslant K\alpha^{-1} \int |f|$.

Proof. – Let j be the smallest integer such that $\alpha\delta_j > \int |f|$. Let C_j be a maximal collection of disjoint sets $B \in \mathcal{B}_j$ with the property

$$m(|f|, B) > \alpha . \tag{3.4}$$

After choosing $\mathcal{C}_j, \mathcal{C}_{j+1}, \ldots, \mathcal{C}_{j+m-1}$, let $\mathcal{C}_{j+m}$ be a maximal collection of disjoint sets $B \in \mathcal{B}_{j+m}$ which are disjoint from the previously chosen sets and satisfy (3.4). If $f^*(x) > \alpha$, then there is a set $B \in \mathcal{B}_i$ such that $x \in B$ and (3.4) is true. Necessarily $i \geqslant j$. Then B intersects a set $B_1 \in \mathcal{B}_\ell$, $i \geqslant \ell$. Therefore $B \subset B_1^*$, where we set

$$B_\delta(y)^* = B_{K\delta}(y) .$$

Thus

$$|\{x : f^*(x) > \alpha\}| \leqslant \sum_{\mathcal{C}} |B^*| = K \sum |B|$$

$$\leqslant K\alpha^{-1} \sum \int_B |f| \leqslant K\alpha^{-1} \int |f| .$$

COROLLARY 3.4. – *Suppose* $f \in L^1(V)$. *For almost every* $x \in V$,

$$f(x) = \lim_{|B| \to 0} m(f, B),$$

where the limit is taken over $B \in \mathcal{B}$ *such that* $x \in B$.

LEMMA 3.5. – *Suppose* $\Omega \subset V$ *is closed, and its complement* Ω^c *has finite measure. There is a family* $\mathcal{C} \subset \mathcal{B}$ *such that*

$$\bigcup_{\mathcal{C}} B = \Omega^c$$

and such that no point of V *is in more than* K^4 *distinct sets* $B \in \mathcal{C}$.

Proof. – Choose the smallest integer j such that there is a set $B \in \mathcal{B}$ with

$$B^{**} \subset \Omega^c, \quad B^{***} \cap \Omega \neq \phi . \tag{3.5}$$

Here B^* is defined as above, $B^{**} = (B^*)^*$, etc. Let $\mathcal{C}_j$ be a maximal collection of disjoint $B \in \mathcal{B}_j$ satisfying (3.5). Having chosen $\mathcal{C}_j, \mathcal{C}_{j+1}, \ldots, \mathcal{C}_{j+m-1}$, let $\mathcal{C}_{j+m}$ be a maximal collection of disjoint $B \in \mathcal{B}_{j+m}$, disjoint from the previously chosen sets and satisfying (3.5). Let $\mathcal{C} = \{B^* : B \in \mathcal{C}_i, \text{ some } i\}$. If $x \in \Omega^c$, then there is a $\delta = \delta_i$ such that $B = B_\delta(x)$ satisfies (3.5). Then B intersects some $B_1 \in \mathcal{C}_\ell$, $\ell \leqslant i$, so $x \in B_1^* \in \mathcal{C}$.

Suppose now that $x \in B_i^* \cap B_k^*$, where $B_i \in \mathcal{C}_i$ and $B_k \in \mathcal{C}_k$. We claim that $|i - k| \leqslant 2$. Suppose $i \leqslant k$, and let $\delta = \delta_{i-1}$. Then $B_\delta(x) \subset B_i^{**}$, so $B_\delta(x) \subset \Omega^c$. With $\epsilon = \delta_{k-4}$, we have $B_\epsilon(x) \supset B_k^{***}$, so $B_\epsilon(x) \cap \Omega = \phi$. Therefore $\epsilon > \delta$, which implies $k - 4 < i - 1$, or $k - i \leqslant 2$.

Finally, suppose $x \in \cap B_k^*$, where $B_k \in \mathcal{C}_{i(k)}$ and the B_k are distinct. Let $i = \inf \{i(k)\}$ and let $\delta = \delta_i - 2$. Then each $B_i \subset B_\delta(x)$. The B_i are disjoint with volumes at least $\delta_{i+2} = K^4 \delta$, since $i(k) \leqslant i + 2$ all k. On the other hand, $|B_\delta(x)| = \delta$, so the number of B_k's is at most K^4.

LEMMA 3.6. – *Suppose* $f \in L^1(V)$. *For each* $\alpha > 0$, *there is a family* $\mathcal{C} \subset \mathcal{B}$ *such that*

$$|f| \leqslant \alpha \quad \text{a.e. } on \;\; \Omega = \Big(\bigcup_{\mathcal{C}} B\Big)^c ; \tag{3.6}$$

$$m(|f|, B) \leqslant K^2 \alpha, \;\; all \;\; B \in \mathcal{C} ; \tag{3.7}$$

$$\sum_{\mathcal{C}} |B| \leqslant \alpha^{-1} K^5 \int |f| . \tag{3.8}$$

Proof. – Let $\Omega = \{x : f^*(x) \leqslant \alpha\}$ and let $\mathcal{C}$ be the family given by Lemma 3.5. Corollary (3.4) implies (3.6). If $B \in \mathcal{C}$, choose $x \in B^{**} \cap \Omega$. Then

$$\alpha \geqslant m(|f|, B^{**}) \geqslant |B|\, |B^{**}|^{-1}\, m(|f|, B) = K^{-2}\, m(|f|, B).$$

Finally, since there are at most K^4 overlaps among the sets $B \in \mathcal{C}$, (3.8) follows from (3.7) and Lemma 3.3.

Proof of Theorem 3.1. – By duality, it is enough to prove the result when $1 < p \leqslant 2$. Suppose $\mathcal{K}$ is a bounded set in $S(g)$ and suppose $a \in \mathcal{K} \cap \mathcal{D}(W)$. Let $A = A_1$ or A_2 be the corresponding operator. It is enough to show

$$\| Au \|_p \leqslant C_p \| u \|_p, \quad u \in \mathcal{D}(V), \quad 1 < p \leqslant 2, \tag{3.9}$$

where C_p depends only on $\mathcal{K}$. By assumption (2.12) and the closed graph theorem, (3.9) is true for $p = 2$. By the Marcinkiewicz interpolation theorem it is enough to show that for any $\alpha > 0$ and any $f \in L^1(V)$.

$$|\{x : |Tf(x)| > 2\alpha\}| \leqslant C_1\, \alpha^{-1} \int |f|. \tag{3.10}$$

Let Ω and $\mathcal{C}$ be as in Lemma 3.6 and its proof. Choose measurable functions f_B, $B \in \mathcal{C}$, such that

$$|f_B| \leqslant |f|, \quad f_B(x) = 0 \text{ if } x \notin B, \quad \sum_{\mathcal{C}} f_B = f \text{ on } \Omega^c. \tag{3.11}$$

Set

$$g_0 = f \cdot 1_\Omega, \quad g_B = m(f_B, B)\, 1_B, \quad g = g_0 + \sum g_B$$

where 1_S is the characteristic function of the set S. Then

$$|g_0| \leqslant \alpha \quad \text{a.e.}; \quad |g_B| \leqslant K^2 \alpha. \tag{3.12}$$

Because of the bound on the number of overlaps, we get

$$|g| \leqslant K^6 \alpha. \tag{3.13}$$

Thus

$$|\{x : |Tg(x)| > \alpha\}| \leqslant \alpha^{-2} \int |g|^2 \leqslant K^6 \alpha^{-1} \int |g|$$
$$\leqslant K^{10} \alpha^{-1} \int |f|. \tag{3.14}$$

Now let $h_B = f_B - g_B$, $h = f - g = \sum h_B$. It remains to be proved that

$$|\{x : |Th(x)| > \alpha\}| \leqslant C\alpha^{-1} \int |f|. \tag{3.15}$$

Let $\Omega_1 = \left(\bigcup_{\mathcal{C}} B^*\right)^c$. Then $|\Omega_1^c|$ satisfies an inequality of the form

(3.15), so it is enough to consider the intersection of the set in (3.15) with Ω_1. It suffices to show

$$\int_{\Omega_1} |Th| \leqslant C \int |f| , \tag{3.16}$$

and to prove this it is enough to show

$$\int_{(B^*)^c} |Th_B| \leqslant C \int_B |h_B| , \quad B \in \mathcal{C} . \tag{3.17}$$

Now $\int h_B = 0$, so letting y_B be the center of B we have

$$Th_B(x) = \int (k(x,y) - k(x,y_B))\, h_B(y)\, dy .$$

Thus the integral on the left in (3.17) is dominated by

$$\int_{y \in B} |h_B(y)| \int_{x \notin B^*} |k(x,y) - k(x,y_B)|\, dx\, dy .$$

Thus (3.17) follows from Lemma 3.2.

4. Estimates in weighted Hölder spaces.

Let g, Φ, and φ be as in Section 2. We assume here (2.6)-(2.11). If $a \in S(g)$, we consider $A = a(x, D)$ to be defined by (2.1).

It is useful for some applications to allow modifications of the distance function $d(x,y)$ of Section 2. Let

$$d_1(x,y) = f(x, d(x,y)) . \tag{4.1}$$

We assume that f is a positive continuous function on $V \times \mathbf{R}_+$, with a continuous partial derivative in the scalar variable, such that

$$cf(x,\lambda) \leqslant f(x',\lambda) \leqslant Cf(x,\lambda) \quad \text{if} \quad d(x',x) \leqslant \lambda ; \tag{4.2}$$

$$cf(x,\lambda) \leqslant \lambda \frac{\partial f}{\partial \lambda}(x,\lambda) \leqslant Cf(x,\lambda) , \tag{4.3}$$

where again c and C denote positive constants.

If s is positive and u is a continuous function on V, set

$$|u|_s = \sup |u(x') - u(x)|\, d_1(x',x)^{-s} . \tag{4.4}$$

The quotient of $\{u = |u|_s < \infty\}$ by the constant functions is a Banach space which we denote by $\Lambda_s(d_1)$.

THEOREM 4.1. – *There is a positive index* σ *such that each operator* A *with symbol* $a \in S(1, g)$ *is bounded in* $\Lambda_s(d_1)$ *if* $0 < s < \sigma$.

The principal tools in the proof of Theorem 4.1 are a splitting of the symbol a and a corresponding refinement of the inequalities in Proposition 2.5 and 2.6 b). To split a smoothly, we first smooth $\delta(x, \xi)$. Let

$$g_{1,w}(y, \eta) = \varphi_w(y) + \Phi_w(\eta).$$

The inequalities (2.23) and (2.26) did not use (2.3) in full force, but only (2.3) with g_1 in place of g. Because of (2.7) and (2.8), we may (as in [5], via a partition of unity) replace $\delta(x, \xi)$ by a function which is equivalent in the sense of Section 2 and which satisfies

$$|\delta_w^{(k)}(w_1, w_2, \dots, w_k)| \leqslant C_k\, \delta(w) \Pi\, g_{1,w}(w_j)^{1/2}. \tag{4.5}$$

Choose $\psi \in C^\infty(R_+)$ such that $\psi(\lambda) = 0$, $\lambda \leqslant 1$, $\psi(\lambda) = 1$, $\lambda \geqslant 2$. Given $a \in S(g)$ and $\lambda > 0$, set

$$a_{1,\lambda}(w) = a(w)\, \psi(\lambda^{-1}\delta(w)), \quad a_{2,\lambda} = a - a_{1,\lambda}. \tag{4.6}$$

Thus $a_{1,\lambda}$ vanishes for $\delta(w) \leqslant \lambda$ and $a_{2,\lambda}$ vanishes for $\delta(w) \geqslant 2\lambda$. Moreover, if $\mathcal{K}$ is a bounded set in $S(g)$, then $\{a_{1,\lambda} : a \in \mathcal{K}, \lambda > 0\}$ and $\{a_{2,\lambda} : a \in \mathcal{K}, \lambda > 0\}$ are bounded sets in $S(g_1)$. In what follows, we shall assume $a \in \mathcal{K} \cap \mathcal{D}(W)$ and derive estimates depending only on $\mathcal{K}$.

The kernel k of the operator $a(x, D)$ is given by

$$k(x, y) = \int e^{i\langle \xi, x-y\rangle} a(x, \xi)\, d\xi. \tag{4.7}$$

Therefore

$$\int k(x, y)\, dy = a(x, 0) = 0. \tag{4.8}$$

Similarly, derivatives of k have mean value zero with respect to y, and the same is true for the kernels $k_{j,\lambda}$ corresponding to the symbols $a_{j,\lambda}$, $j = 1,2$.

LEMMA 4.2. – $f(x, \mu)\, f(x, \lambda)^{-1} \sim \mu\lambda^{-1}$.

Proof. – By (4.3),

$$\log(f(x, \mu)\, f(x, \lambda)^{-1}) = \int_\lambda^\mu \frac{\partial f}{\partial \lambda}(x, s)\, f(x, s)^{-1}\, ds \approx \log(\mu\lambda^{-1}).$$

LEMMA 4.3. – *If* $\lambda = d(x, x')$, *then for any* $M > 0$,

$$|k_{2,\lambda}(x, y)| \leqslant C_M d(x, y)^{-1} (1 + \lambda_1^{-1} d_1(x, y))^{-M}, \qquad (4.9)$$

where $\quad \lambda_1 = d_1(x, x') = f(x, \lambda)$.

Proof. – By the argument given in the proof of Proposition 2.5, the left side of (4.9) is dominated by

$$\int_{\delta(x,\xi)\leqslant 2\lambda} (1 + d(x, y)\,\delta(x, \xi)^{-1})^{-N-1}\, d\xi$$

$$\approx \int_{\lambda^{-1}}^{\infty} (1 + t\,d(x, y))^{-N-1}\, dt \approx d(x, y)^{-1} (1 + \lambda^{-1} d(x, y))^{-N} .$$

Here we again have used the generalization of Corollary 2.2 to integration over a subinterval. By Lemma 4.2, we may choose N so that the last expression is dominated by the right side of (4.9).

LEMMA 4.4. – *If* $d(x, x + z) \approx \lambda$, *there are positive* C *and* σ *such that*

$$|(z \cdot \nabla_x)\, k_{1,\lambda}(x, z)| \leqslant C d(x, y)^{-1} (1 + d_1(x, y)\,\lambda_1^{-1})^{-\sigma}, (4.10)$$

where $\quad \lambda_1 = f(x, \lambda)$.

Proof. – By the argument given in the proof of Proposition 2.6, the left side of (4.10) is dominated by

$$\int_{\delta(x,\xi)>\lambda} (1 + d(x, y)\,\delta(x, \xi)^{-1})^{-N}\, \lambda^{\alpha}\, \delta(x, \xi)^{-\alpha}\, d\xi$$

$$\approx \int_0^{\lambda^{-1}} (1 + t d(x, y))^{-N}\, \lambda^{\alpha}\, t^{\alpha}\, dt .$$

When $d(x, y) \leqslant \lambda$, this last expression is $\approx \lambda^{-1} \leqslant d(x, y)^{-1}$. When $d(x, y) > \lambda$, it is equivalent to

$$d(x, y)^{-1}\, \lambda^{\sigma}\, d(x, y)^{-\alpha} \leqslant d(x, y)^{-1}\, \lambda_1^{\sigma}\, d_1(x, y)^{-\sigma},$$

by Lemma 4.2.

LEMMA 4.5. – *If* h *is a function of slow variation on* $\mathbf{R}_+$, *then*

$$\int h(d_1(x, y))\, d(x, y)^{-1}\, dy \approx \int_0^{\infty} h(t)\, t^{-1}\, dt . \qquad (4.11)$$

Proof. – The measure of the set $\{y : d(x, y) \in (2^j, 2^{j+1})\}$ is $\approx 2^j$. Therefore the left side of (4.11) is equivalent to

$$\int_0^\infty h(f(x,\lambda))\,\lambda^{-1}\,d\lambda$$
$$\approx \int_0^\infty h(f(x,\lambda))\,f(x,\lambda)^{-1}\,\frac{\partial f}{\partial \lambda}(x,\lambda)\,d\lambda\,. \qquad (4.12)$$

Here we have used (4.3). The integrals on the right in (4.11) and (4.12) are the same.

Proof of Theorem 4.1. – Let a and k be as above, and suppose $|u|_s < \infty$. Given $\lambda > 0$, let $A_{1,\lambda}$ and $A_{2,\lambda}$ be the operators corresponding to the symbols $a_{1,\lambda}$ and $a_{2,\lambda}$. Let

$$v_\lambda = A_{1,\lambda}\,u\,, \qquad w_\lambda = A_{2,\lambda}\,u\,.$$

Given x, $x' \in V$, let $\lambda = d(x,x')$. It suffices to show

$$|w_\lambda(x)| + |w_\lambda(x')| \leqslant C\,|u|_s\,d_1(x,x')^s \qquad (4.13)$$

$$|v_\lambda(x') - v_\lambda(x)| \leqslant C\,|u|_s\,d_1(x,x')^s\,, \qquad (4.14)$$

where as usual C depends only on the bounded set $\mathcal{K}$. Note that by (4.2), $d_1(x,x') \approx d_1(x',x)$. Therefore it is enough to deal with either term on the left in (4.13). Now since $k_{2,\lambda}$ satisfies (4.8),

$$w_\lambda(x) = \int k_{2,\lambda}(x,y)\,u(y)\,dy = \int k_{2,\lambda}(x,y)\,(u(y) - u(x))\,dy\,.$$

By 4.9,

$$|w_\lambda(x)| \leqslant \left(C_M \int (1 + d_1(x,x')^{-1}\,d_1(x,y))^{-M}\,d_1(x,y)^s\,dy\right)|u|_s\,. \qquad (4.15)$$

By Lemma 4.5, the integral in (4.15) is equivalent to

$$\int (1 + d_1(x,x')^{-1}\,t)^{-M}\,t^{s-1}\,dt = c_s\,d_1(x,x')^s\,,$$

if $M > s$. This proves (4.13).

To prove (4.14), we write

$$v(x') - v(x) = \int_0^1 v(x,x',r)\,dr\,,$$

where

$$v(x,x',r) = \int ((x'-x)\cdot\nabla_x)\,k(x_r,y)\,u(y)\,dy\,,$$
$$x_r = x + r(x'-x)\,. \qquad (4.16)$$

Now the derivative of k occurring here also has mean value zero with respect to y. Therefore we may replace $u(y)$ by $u(y) - u(x_r)$ in (4.16). Note that for $0 \leqslant r \leqslant 1/2$, $d(x_r, x') \approx d(x, x')$, so $d(x_r, x_r + (x' - x)) \approx d(x, x')$; when $1/2 \leqslant r \leqslant 1$, $d(x_r, x) \approx d(x', x)$, so $d(x_r, x_r - (x' - x)) \approx d(x', x)$; here we are using Proposition 2.5 d) and e). It follows from these remarks and (4.10) that

$$|v(x, x', r)|$$

$$\leqslant C \int d(x_r, y)^{-1} (1 + d_1(x_r, y)\lambda_1^{-1})^{-\sigma} d_1(x_r, y)^s \, dy \times |u|_s, \quad (4.17)$$

where $\lambda_1 = d_1(x, x')$. By Lemma 4.5, the integral in (4.17) is equivalent to

$$\int_0^\infty (1 + \lambda_1^{-1} t)^{-\sigma} t^{s-1} \, dt = c_{s,\sigma} \lambda_1^s$$

provided $0 < s < \sigma$. This proves (4.14), and completes the proof of Theorem 4.1.

A result analogous to Theorem 4.1 is true for the space of functions with norm

$$|u|_s^* = \sup |u(x)| + |u|_s$$

if we require of the symbol $a \in S(g)$ that it vanish for $\delta(x, \xi) \geqslant \delta_0 > 0$ (which is to say, for suitably small ξ). In fact the argument used to prove (4.13) shows that with this restriction,

$$|Au(x)| \leqslant C_s \delta_0^s |u|_s .$$

5. Examples.

Here we consider two examples of metrics associated to certain hypoelliptic operators, and indicate the corresponding L^p and Schauder estimates.

The first example is connected with an operator A with symbol $a \in C^\infty(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$,

$$a(x, \xi) = \xi_1^2 + x_1^{2k} p_2(x, \xi),$$

where p_2 is a second order partial differential operator which is non-negative and is elliptic with respect to the variables $x' = (x_2, x_3, \ldots, x_n)$. There is an a priori estimate valid for A,

$$\| D_1^2 u \| + \| x_1^{2k} \Delta u \| + \| u \|_{2\epsilon} \leqslant C_K (\| Au \| + \| u \| , \tag{5.1}$$

$u \in \mathcal{D}_K$, where the norms are L^2 and $H^{2\epsilon}$-norms, Δ denotes the laplacian, and $\epsilon = (k+1)^{-1}$. Corresponding to this estimate is the function

$$p(x, \xi) = |\xi_1| + |x_1|^k |\xi| + |\xi|^\epsilon . \tag{5.2}$$

The function p satisfies (locally in x) the estimates

$$|\nabla_x p| \leqslant Cp \cdot (|\xi|^{1/k} p^{-1/k}) , \tag{5.3}$$

$$|\nabla_\xi p| \leqslant C = Cp \cdot p^{-1} \tag{5.4}$$

which suggests the metric

$$g^0_{x,\xi}(y, \eta) = |\xi|^{2/k} p^{-2/k} |y|^2 + p^{-2} |\eta_1|^2 + |\xi|^{-2} |\eta'|^2 . \tag{5.5}$$

This metric satisfies the conditions of [5] ; therefore the operators with symbols in $S(g^0)$ are bounded in L^2. In the region where $p \approx x_1^k |\xi| + |\xi|^\epsilon$, the partial gradient with respect to ξ' satisfies the more precise estimate

$$|\nabla_{\xi'} p| \leqslant Cp \cdot \{ |x_1|^k p^{-1} + p^{-k-1} \} ,$$

which suggest the larger metric

$$g_{x,\xi}(y, \eta) = |\xi|^{2/k} p^{-2/k} |y|^2 + p^{-2} \eta_1^2 p^{-2} (|x_1|^k + p^{-k})^2 |\eta'|^2 . \tag{5.6}$$

This metric is not "slowly warying", so the results of [5] do not apply directly. However $S(g) \subset S(g^0)$, and the asymptotic expansion for composition [8] implies that the corresponding operators are a subalgebra of the algebra of operators with symbols in $S(g^0)$. The metric g given by (5.6) satisfies (locally in x) the conditions of section 2. In fact, $p^{2k+1} \geqslant |\xi|$, so

$$\varphi_{x,\xi}(y) \geqslant p^2 |y|^2 \geqslant |\xi|^{2/k} p^{-2/k} |y|^2 = g_{x,\xi}(y, 0) ,$$

proving (2.6). Condition (2.7) is a consequence of (5.4), while (2.9) and (2.10) are easily checked, and the local version of (2.12) follows from the inclusion $S(g) \subset S(g_0)$. To check (2.8), suppose that for some $z \in V \backslash (0)$, $\varphi_{y,\xi}(z) = \lambda^2 \varphi_{x,\xi}(z)$, where $\lambda \geqslant 2$. This implies that either

$$p(y, \xi) \geqslant \lambda p(x, \xi) \quad \text{or} \tag{5.7}$$

$$|x_1|^k p(x, \xi)^{-1} + p(x, \xi)^{-k-1} \geqslant \lambda (|y_1|^k p(y, \xi)^{-1} + p(y, \xi)^{-k-1}) . \tag{5.8}$$

Taking the k -th root in (5.7) leads to

$$|\xi|^{1/k}\,|y_1 - x_1| \geqslant c\lambda^{1/k}\, p(x\,,\xi)^{1/k}\,, \tag{5.9}$$

or

$$\lambda \leqslant g_{x,\xi}(y - x\,, 0)^k \leqslant \varphi_{x,\xi}(y - x)^k\,. \tag{5.10}$$

Suppose, on the other hand, that (5.8) is true. Then either $p(y\,,\xi) \geqslant \lambda^\epsilon p(x\,,\xi)$, which leads to an inequality like (5.10), or else

$$\begin{aligned} |x_1|^k\, p(x\,,\xi)^{-1} &\geqslant c\lambda(|y_1|^k\, p(y\,,\xi)^{-1} + p(y\,,\xi)^{-k-1}) \\ &\geqslant |y_1|^k\, p(x\,,\xi)^{-1} + c\lambda p(x\,,\xi)^{-k-1}\,, \end{aligned}$$

if λ is large enough. Then

$$\lambda^{1/k} \leqslant c_1\, p(x\,,\xi)^{-1/k}\,|x_1 - y_1| \leqslant c_1\,\varphi_{x,\xi}(y - x)\,.$$

Thus (2.8) is satisfied. Finally, condition (2.11) is clear, since $\Phi_{x,\xi}$ depends on ξ only through the function $p(x\,,\xi)$, so $\delta(x\,,\xi) = \delta(x\,,\eta)$ is equivalent to $p(x\,,\xi) = p(x\,,\eta)$.

It follows from this that an operator with symbol in $S(g)$ maps L^p_c to L^p_{loc} if $1 < p < \infty$. Using this, it is possible to construct a scale of weighted L^p spaces associated to the operator A and derive corresponding estimates. We cite here only the L^p version of (5.1) :

$$\|D_1^2\, u\|_p + \|x_1^{2k}\,\Delta' u\|_p + \|u\|_{p,2\epsilon} \leqslant C_K(\|Au\|_p + \|u\|_p)\,, \tag{5.11}$$

$u \in \mathcal{D}_K$, where $\epsilon = (k+1)^{-1}$ again, and $\|\ \ \|_{p,2\epsilon}$ denotes the norm in the Bessel potential space $L^p_{2\epsilon}$ of Calderon [2].

The function $\delta(x\,,\xi)$ associated with the metric g is given (up to a multiplicative constant) by

$$p^{-n-1}(|x_1|^k + p^{-k})^n\,, \qquad p = p(x\,,\xi).$$

Thus $p(x\,,\xi)^{-1} = f(x\,,\delta(x\,,\xi))$, for suitable f. It is convenient to use this function to define a distance function d_1 and the associated spaces of Hölder type. Thus $d_1(x\,,y) \approx \eta$ if

$$g_{x,\xi}(y - x) = 1\,, \qquad p(x\,,\xi)^{-1} = \eta\,. \tag{5.12}$$

By considering separately the cases $|x_1| > \eta$ and $|x_1| \leqslant \eta$, one finds

$$\begin{aligned} d_1(x\,,y) \approx\ & |y_1 - x_1| \\ & + \min\{|y' - x'|^{\frac{1}{k+1}}\,,\ |x_1|^{-k}\,|y' - x'|\}. \end{aligned} \tag{5.13}$$

Thus for $x_1 \neq 0$ and y close to x, the metric is euclidean with distorting factor $|x_1|^{-k}$, while for $x_1 = 0$ the metric is parabolic. The corresponding Hölder spaces are (locally) preserved by operators with symbols in $S(g)$. The functions in such a space $\Lambda_s(d_1)$ are locally in the ordinary Hölder space C^s away from the hyperplane $x_1 = 0$ $(0 < s < 1)$, but are smoother in the normal than in the tangential variable on this hyperplane.

The second example is associated to the operator B with symbol $b \in C^\infty(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$,

$$b(x, \xi) = x_1^k p_2(x, \xi') + p_1(x, \xi') + i\xi_1 + p_0(x, \xi)$$

where the p_j are classical symbols of order j and where $p_2(x, \xi') \geqslant c|\xi'|$ for $|\xi'|$ large, with $c > 0$. When $k = 1$, this is essentially Kannai's example of a hypoelliptic operator without hypoelliptic adjoint [6] ; see also [7], [8]. It is not difficult to obtain the L^2 estimate

$$\|D_1 u\| + \|x_1^k \Delta' u\| + \|u\|_\epsilon \leqslant C_K(\|Bu\| + \|u\|), \qquad (5.14)$$

$u \in \mathcal{D}_K$, where $\epsilon = (k+1)^{-1}$. We set

$$q(x, \xi)^2 = \xi_1^2 + (|\xi'|^4 + \xi_1^2) x_1^{2k} + |\xi|^{2\epsilon}, \qquad (5.15)$$

so

$$q \approx |\xi_1| + |x_1|^{k/2} |\xi'| + |\xi'|^\epsilon .$$

Examination of derivatives of q leads as above to metrics

$$g^0_{x,\xi}(y, \eta) = |\xi|^{4/k} q^{-4/k} y_1^2 + |y'|^2 + q^{-4}\eta_1^2 + |\xi|^{-2} |\eta'|^2, \qquad (5.16)$$

$$g_{x,\xi}(y, \eta) = |\xi|^{4/k} q^{-4/k} y_1^2 + |y'|^2 + q^{-4}\eta_1^2 + q^{-2}(|y_1|^{k/2} + q^{-k})^2 |\eta'|^2 . \qquad (5.17)$$

Again, g^0 satisfies the conditions of [5], locally, and operators with symbols in $S(g^0)$ map L^2_c to L^2_{loc}. As above, g satisfies conditions (2.6)-(2.12), and the results of the present paper apply. The L^p analogue of the estimate (5.14) is

$$\|D_1 u\|_p + \|x_1^k \Delta' u\|_p + \|u\|_{p,\epsilon} \leqslant C_K(\|Bu\| + \|u\|), \qquad (5.18)$$

$u \in \mathcal{D}_K$, $\epsilon = (k+1)^{-1}$. The natural distance function in this case is

$$d_1(x, y) \approx |y_1 - x_1|^{1/2} + \min\{|y' - x'|^{1/k+1}, |x_j|^{-k/2} |y' - x'|\}. \qquad (5.19)$$

Thus functions in the corresponding Hölder spaces are required to be smoother in the x'-directions than in the x_1-direction away from the hyperplane $x_1 = 0$, but smoother (equally smooth, if $k = 1$) in the x_1-direction on the critical hyperplane.

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