ANNALES DE L'INSTITUT FOURIER

ERIC T. SAWYER

Good-irreducible inner functions on a polydisc

Annales de l'institut Fourier, tome 29, nº 2 (1979), p. 185-210

<http://www.numdam.org/item?id=AIF_1979__29_2_185_0>

Ann. Inst. Fourier, Grenoble
29, 2 (1979), 185-210.

GOOD/IRREDUCIBLE INNER FUNCTIONS ON A POLYDISC ([1])

by Eric SAWYER

In [8] Rudin and Ahern have studied to what extent the classical factorization of a bounded holomorphic function on the unit disc U carries over to the case of the unit polydisc U^n $(n>1)$. They show that every bounded holomorphic function on U^n factors as a product of a zero-free function in $H^\infty(U^n)$ (notation is as in [7] and [8]) and at most countably many irreducible functions in the unit ball of $H^\infty(U^n)$. In contrast with the case of the unit disc, they show that this factorization need not be unique. However, the question of the uniqueness of this factorization remains open in the case of a good inner function. Theorem 2.3 below bears on this question. It is shown that if $\psi(w,z)$ is an inner function on $U \times U^n$ $(n \geqslant 1)$ and if « enough » of the boundary sections $\lim_{r\to 1} \psi(rw,.)$ are rational functions, then the Rudin-Ahern factorization of ψ is unique.

In connection with the factorization problem, it seems desirable to develop methods of constructing good and/or irreductible inner functions on a polydisc. Theorem 3.5 below shows that if f, g, and R are non-constant inner functions on U^m, U^n, and U^2 respectively and if R is a product of rational inner functions and is not divisible by a finite Blaschke product on U^2, then the function $\varphi_{f,g,R}$ defined by

$$U^m \times U^n \ni (w,z) \to R(f(w),g(z))$$

is a good inner function on $U^m \times U^n$. The case $m = 1, f(w) = w$, and

$$R(x,y) = \frac{xy - \alpha}{1 - \bar{\alpha}xy} \quad (\alpha \in U)$$

occurs in Theorem 3.6 of [8].

([1]) Most of the material in this article is taken from the author's thesis ([11]).

Theorem 4.5 gives necessary and sufficient conditions for the irreducibility of $\varphi_{f,g,R}$ in the case $m = 1$, f is a Blaschke product of degree two, and $R(x,y) = \dfrac{xy - \alpha}{1 - \bar{\alpha}xy}$. (The case where f is a Blaschke product of degree one is settled in Theorem 3.6 of [8]). The theory of the backward shift operator on $H^2(U)$ (see [3]) plays a decisive role here.

Two other results of independent interest (used in proving the above-mentioned theorems) will be noted here. One is a generalization of a theorem of Frostman on good inner functions ([4]). Theorem 3.2 below shows that if $h(\alpha,z)$ is bounded and holomorphic on $U \times U^n$ $(n \geqslant 1)$ and if the closure of $\{h(.,z);\, z \in U^n\}$ in the topology of uniform convergence on compact subsets does not contain the identically zero function, then $h(\alpha,.)$ is a good function for α in U, except possibly on a set of capacity zero. Rudin has given a generalization of Frostman's theorem in a somewhat different direction (see [7]; Theorem 3.6.2).

The other result is a representation theorem for certain Nevanlinna class functions on a polydisc. Theorem 1.4 shows that if $h(w,z)$ is in $N(U^m \times U^n)$ $(m,n \geqslant 1)$ and if « enough » of its boundary sections $\lim_{r\to 1} h(rw,.)$ $(w \in T^m)$ are rational functions, then h can be expressed as a rational function of z with coefficients in $N(U^m)$. Additional information is supplied in the case that h is an inner function.

We use the notation of [7] and [8]. Denote by U the open unit disc in C (the complex numbers) and by T the boundary of U. Let $U^n(T^n)$ be the Cartesian product of n copies of $U(T)$. The letter m will be used to denote $2n$-dimensional Lebesgue measure on C^n and normalized Haar measure on T^n for $n = 1,2,\ldots$. Denote by $H^p(U^n)$ and $N(U^n)$ the usual Hardy and Nevanlinna spaces of holomorphic functions on U^n. Denote by $N_*(U^n)$ the class of functions f in $N(U^n)$ for which the functions $\log^+|f_r|$, $(f_r(z) = f(rz))$, form a uniformly integrable family $(0<r<1)$. If f is in $N(U^n)$, then $f^*(z) = \lim_{r\to 1} f(rz)$ exists for almost every z in T^n. If f is in $H^\infty(U^n)$ and $|f^*(z)| = 1$ for almost every z in T^n, then f is said to be an inner function.

If f is in $N_*(U^n)$, then

$$\lim_{r\to 1} \int_{T^n} \log|f(rz)|\, dm(z) \leqslant \int_{T^n} \log|f^*(z)|\, dm(z)$$

([7]; Theorems 3.2.4 and 3.3.5). If equality holds, we say that f is a good function. Note that if f is an inner function on U^n, then this definition of a good inner function coincides with that given in [7].

Suppose f is holomorphic on $U^m \times U^n$. If w is in U^m, then the function $U^n \ni z \longmapsto f(w,z)$ is referred to as a section of f and is denoted by $f(w,.)$. If w is in T^m and $\lim_{r \to 1} f(rw,.)$ exists uniformly on compact subsets of U^n, then this holomorphic limit function is referred to as a boundary section of f and is denoted also by $f(w,.)$.

Finally, for $n \geqslant 1$, we define $P^\infty(U^n)$ to be the collection of f in $H^\infty(U^n)$ such that there exists φ inner on U^n and g in $H^\infty(U^n)$ satisfying $\varphi^* \bar{f}^* = g^*$ almost everywhere on T^n. See [3] for a detailed discussion of this class of functions in the case $n = 1$.

1. A Representation theorem.

1.1. LEMMA. – *Suppose m and n are positive integers and $0 < p \leqslant \infty$. If f is in $N(U^m \times U^n)$ (respectively $H^p(U^m \times U^n)$), then $f(w,.)$ is in $N(U^n)$ (respectively $H^p(U^n)$) for every w in U^m.*

Proof. – The assertion concerning H^∞ is trivial, so assume $0 < p < \infty$. Denote by φ_0 and φ_p the functions $\log^+|f(.)|$ and $|f(.)|^p$ on $U^m \times U^n$ for $0 < p < \infty$. For each z in U^n, $\varphi_p(.,z)$ is either multiply subharmonic or identically $-\infty$ on U^m. Thus, if ξ is in $(rU)^m$ for some $0 < r < 1$ and if z is in U^n and $0 \leqslant p < \infty$, then (see [7]; paragraph 3.2.1)

$$\varphi_p(\xi,z) \leqslant \int_{T^m} \varphi_p(rw,z)\, P_\xi^r(w)\, dm(w) \tag{1}$$

where $P_\xi^r(w)$ is the usual Poisson kernel for the polydisc $(rU)^m$. Thus

$$\begin{aligned} \int_{T^n} \varphi_p(\xi,rz)\, dm(z) &\leqslant \int_{T^n} \left(\int_{T^m} \varphi_p(rw,rz) P_\xi^r(w)\, dm(w) \right) dm(z) \\ &\leqslant \sup\{P_\xi^r(w);\ w \in T^m\} \int_{T^{m+n}} \varphi_p(r\eta)\, dm(\eta). \end{aligned} \tag{2}$$

Now fix ξ in U^m and let $r \to 1$. Inequality (2) yields the remaining assertions of the lemma.

Suppose α and β are multi-indices of degree t, i.e. $\alpha, \beta \in (Z_+)^t$. If $z \in U^t$, define $z^\alpha = z_1^{\alpha_1}, \ldots, z_t^{\alpha_t}$. Define $\alpha \leqslant \beta$ if and only if $\alpha_i \leqslant \beta_i$ for

$1 \leqslant i \leqslant t$. Put $|\alpha| = \sum_{i=1}^{t} \alpha_i$. Part (A) of the following lemma is elementary and part (B) follows from chapter 5 of [7].

1.2. Lemma. — *Let* $R\ (\not\equiv 0)$ *be a rational function on* C^t *with no pole in* U^t. *Then*

A) R can be expressed uniquely in the form

$$R(z) = \frac{\sum_{0 \leqslant |\alpha| \leqslant m} a_\alpha z^\alpha}{1 + \sum_{1 \leqslant |\beta| \leqslant n} b_\beta z^\beta} \qquad z \in U^t$$

where (i) m *and* n *are non-negative integers;* $a_\alpha, b_\beta \in C$ *for* $0 \leqslant |\alpha| \leqslant m$, $1 \leqslant |\beta| \leqslant n$; *and there are multi-indices* α *and* β *with* $|\alpha| = m$ *and* $|\beta| = n$ *such that* $a_\alpha \neq 0$, $b_\beta \neq 0$, *and* (ii) *the polynomials*

$$P(z) = \sum_{0 \leqslant |\alpha| \leqslant m} a_\alpha z^\alpha \qquad \textit{and} \qquad Q(z) = 1 + \sum_{1 \leqslant |\beta| \leqslant n} b_\beta z^\beta$$

are relatively prime in $C[z_1, \ldots, z_t]$.

B) *Set* $a_\alpha = b_\beta = 0$ *if* $|\alpha| > m$, $|\beta| > n$. *If in addition* R is an inner function on U^t, *then* $m \geqslant n$, *there exists a multi-index* τ *with* $|\tau| = m$ *such that* $a_\alpha = b_\alpha = 0$ *if* α *does not satisfy* $\alpha \leqslant \tau$, *and* $|a_\tau| = 1$ *and* $a_\tau \overline{b_\beta} = a_{\tau-\beta}$ *for* $0 \leqslant \beta \leqslant \tau$.

Definition. — *Suppose that* R *is as in Lemma* 1.2. *The degree of* R, *denoted* deg R, *is defined to be the ordered pair of integers* (m,n) *determined uniquely by* R *as in Lemma* 1.2. *When* $R \equiv 0$, *define* $\deg R = (-1,0)$.

1.3. Lemma. — *Suppose that* R_j *is holomorphic on* U^t *and rational of degree* $\leqslant (m,n)$ *for* $1 \leqslant j < \infty$. ($(i,j) \leqslant (m,n) \Leftrightarrow i \leqslant m$ *and* $j \leqslant n$). *If* R_j *tends uniformly on compact subsets of* U^t *to* h *in* $H(U^t)$ *as* $j \to \infty$, *then* h *is a rational function of degree* $\leqslant (m,n)$.

Proof. — Suppose $R_j = P_j/Q_j$ where P_j and Q_j are polynomials of degrees equal to or less than m and n respectively and $Q_j(0) = 1$ for all j. Lemma 1 of [1] shows that $\sup \{|Q_j(z)| ; z \in K, 1 \leqslant j < \infty\} < \infty$ for any compact subset K of U^t. Since $R_j \to h$ uniformly on compact subsets, we also have $\sup \{|P_j(z)| ; z \in K, 1 \leqslant j < \infty\} < \infty$ for any compact subset K of

U^t. Thus there are polynomials P and Q of degrees equal to or less than m and n respectively such that $P_j \to P$ and $Q_j \to Q$ as $j \to \infty$. Hence $hQ = P$ where $Q(0) = 1$ and this completes the proof of the lemma.

The next theorem shows that if $h(w,z)$ is in $N(U^q \times U^t)$ and has « enough » rational boundary sections in z, then $h(w,.)$ is rational of fixed degree for almost every w in T^q and h can be expressed as a « rational » function of z with coefficients in $N(U^q)$.

1.4. THEOREM. — *Suppose that* $h(w,z)$ $(\not\equiv 0)$ *is in* $N(U^q \times U^t)$ $(q \geqslant 1, t \geqslant 1)$ *and that there exists a set* E *contained in* T^q *with positive measure such that* $h(w,.)$ *is a rational function for* w *in* E.

Then there exist unique non-negative integers m *and* n *and functions* h_α, k_β $(0 \leqslant |\alpha| \leqslant m, 1 \leqslant |\beta| \leqslant n)$ *in* $N(U^q)$ *such that*

a) $\deg h(w,.) = (m,n)$ *for almost every* w *in* T^q.

b) $h(w,z)\,(1 + \sum_{1 \leqslant |\beta| \leqslant n} k_\beta(w) z^\beta) = \sum_{0 \leqslant |\alpha| \leqslant m} h_\alpha(w) z^\alpha \quad (w \in U^q, z \in U^t)$.

Furthermore the following conditions hold :

(i) $1 + \sum_{1 \leqslant |\beta| \leqslant n} k_\beta(w) z^\beta \neq 0$ *for* (w,z) *in* $U^q \times U^t$ *and the functions* k_β *are in* $H^\infty(U^q)$.

(ii) *The functions* h_α, k_β *are rational combinations (over* C*) of finitely many of the functions* $\{h(.,z) ; z \in U^t\}$.

(iii) *If* h *is in* $H^p(U^q \times U^t)$, *then* h_α *is in* $H^p(U^q)$ *for* $0 \leqslant |\alpha| \leqslant m$ *and* $1 \leqslant p \leqslant \infty$.

If in addition h *is an inner function on* $U^q \times U^t$, *then*

(iv) $m \geqslant n$ *and there exists a multi-index* τ *with* $|\tau| = m$ *such that* $h_\alpha \equiv k_\alpha \equiv 0$ *if* α *does not satisfy* $\alpha \leqslant \tau$ *(set* $h_\alpha \equiv k_\beta \equiv 0$ *if* $|\alpha| > m$, $|\beta| > n$*). Moreover the functions* h_α, k_β *are in* $P^\infty(U^q)$, h_τ *is inner on* U^q, *and* $\overline{h_\tau^* k_\beta^*} = h_{\tau-\beta}^*$ *almost everywhere on* T^q *for* $0 \leqslant \beta \leqslant \tau$.

Proof. — Let

$$\delta(g) = \begin{cases} \deg g & \text{if } g \text{ is a rational function on } U^t \\ (\infty,\infty) & \text{otherwise.} \end{cases}$$

There exists a Borel set $\mathscr{E}$ contained in T^q with $m(\mathscr{E}) = 1$ such that the map

$$(3) \qquad \mathscr{E} \ni w \to h(w,.) \in H(U)$$

is a Borel function ([13]; chapter 17). Lemma 1.3 now shows that the function $\mathscr{E} \ni x \to \delta(h(w,.))$ is Borel. Thus we can find a pair of integers m and n and a Borel set E_1 contained in T^q having positive measure such that $\deg h(w,.) = (m,n)$ for each w in E_1 and such that the index (m,n) satisfies the following minimal property;

(4) If $\deg h(w,.) = (i, j)$ for w in some Borel subset E_2 of T^q and if $(i,j) \leqslant (m, n)$ but $(i,j) \neq (m,n)$ then $m(E_2) = 0$.

Note that since $h \not\equiv 0$, we must have $m \geqslant 0$. By Lemma 1.2(A) we have the following unique representation of $h(w,.)$ for w in E_1;

$$(5) \qquad h(w,z) = \frac{\sum_{0 \leqslant |\alpha| \leqslant m} a_\alpha(w) z^\alpha}{1 + \sum_{1 \leqslant |\beta| \leqslant n} b_\beta(w) z^\beta} \qquad (w \in E_1, z \in U^t).$$

Our aim now is to show that the functions $a_\alpha(.)$, $b_\beta(.)$ defined on E_1 by (5) can be extended to quotients of Nevanlinna class functions on U^q in the sense that $a_\alpha(.)$, $b_\beta(.)$ give the radial limit values on E_1 of these quotients. One would hope to accomplish this by substituting a finite number of values for z in (5) and solving the resulting system of equations by Cramer's rule. One of the problems arising with this approach is that in order to apply Cramer's rule, it is necessary that the determinant of the system of equations be non-zero. In order to establish that this determinant is non-zero for an appropriate choice of z's, it seems necessary to introduce the « nested » sequence of matrices $\{M_\lambda\}$ defined below. Actually we shall not follow exactly the argument outlined above, but rather a slight variant of it.

We totally order the multi-indices of degree t as follows; $u \prec v \Leftrightarrow$ *either* $|u| < |v|$ *or* $|u| = |v|$ and there exists s $(1 \leqslant s \leqslant t)$ such that $u_s < v_s$ and $u_r = v_r$ for $1 \leqslant r < s$. Let M (respectively N) denote the cardinality of $\{v \in (Z_+)^t ; |v| \leqslant m \text{ (respectively } n)\}$ and define $v : \{1,2,\ldots,M+N\} \to (Z_+)^t$ to be such that $v(1), v(2), \ldots, v(M)$ are the first M multi-indices arranged according to $\prec$ and such that $v(M+1), v(M+2), \ldots, v(M+N)$ are the first N multi-indices arranged according to $\prec$.

For λ (an integer) such that

$$1 \leqslant \lambda \leqslant M + N, \qquad \text{and} \qquad x = (x_1,\ldots,x_\lambda)$$

in $U^t \times \ldots U^t = U^{\lambda t}$, and w in $U^q \cup \mathscr{E}$ (where $\mathscr{E}$ is as in (3)) define the matrix $M_\lambda = M_\lambda(w,x)$ to be the $\lambda \times \lambda$ matrix whose ij^{th} entry is

$$(6) \qquad \begin{cases} x_i^{v(j)} & \text{if} \quad 1 \leqslant j \leqslant M \\ h(w,x_i)\, x_i^{v(j)} & \text{if} \quad M+1 \leqslant j \leqslant \lambda . \end{cases}$$

Note that when $\lambda > 1$, the matrix $M_{\lambda-1}$ is obtained from the matrix M_λ by deleting its last column and its bottom row.

For fixed x in $U^{\lambda t}$, Lemma 1.1 shows that the entries in the matrix in (6), when considered as functions of w in U^q, are Nevanlinna class functions. Since $N(U^q)$ is a ring it follows that the function

$$(7) \qquad U^q \ni w \rightarrow \det M_\lambda(w,x)$$

is in $N(U^q)$ for each x in $U^{\lambda t}$. (« det » stands for determinant). The definition of $\mathscr{E}$ together with (6) shows that the radial limits of the function defined in (7) exist at w in $\mathscr{E}$ and are given by $\det M_\lambda(w,x)$ (recall that M_λ is defined for w in $\mathscr{E}$).

From equation (5) we obtain that $\det M_{M+N}(w,x) = 0$ for w in E_1 and x in $U^{(M+N)t}$. Thus for each x in $U^{(M+N)t}$, the radial boundary values of $\det M_{M+N}(.,x)$ vanish on E_1. Now E_1 is a set of positive measure in T^q and $\det M_{M+N}(.,x)$ is in $N(U^q)$. Thus $\det M_{M+N}(.,x) \equiv 0$ for each x in $U^{(M+N)t}$, i.e. $\det M_{M+N} \equiv 0$.

Let $\Lambda = \min\{\lambda; 1 \leqslant \lambda \leqslant M+N \text{ and } \det M_\lambda \equiv 0\}$. One can easily show that $\det M_\lambda \not\equiv 0$ for $1 \leqslant \lambda \leqslant M$ (if $1 \leqslant \lambda \leqslant M$ and $\det M_\lambda \equiv 0$ expand $\det M_\lambda$ according to the bottom row to obtain a polynomial in x_λ. The coefficient of $x_\lambda^{v(\lambda)}$, which must be identically zero, is precisely $M_{\lambda-1}$. Continuing in this fashion, we eventually obtain $M_1 \equiv 0$, a contradiction). Thus $\Lambda \geqslant M+1$. Let $x = (x_1, \ldots, x_{\Lambda-1})$ and $z = x_\Lambda$. Using (6) expand $\det M_\Lambda$ according to the bottom row to obtain

$$(9) \qquad 0 = \det M_\Lambda(w,(x,z))$$

$$= h(w,z)\left(\sum_{\lambda=M+1}^{\Lambda} H_\lambda(w,x) z^{v(\lambda)}\right) + \sum_{\lambda=1}^{M} H_\lambda(w,x) z^{v(\lambda)}$$

for x in $U^{(\Lambda-1)t}$, z in U^t, and w in U^q and where (for any matrix M, denote by $[M]_{i,j}$ the matrix obtained from M by deleting the i^{th} row and j^{th} column)

$$(10) \qquad H_\lambda(w,x) = (-1)^{\Lambda+\lambda} \det [M_\Lambda(w,(x,z))]_{\Lambda,\lambda} \qquad (1 \leqslant \lambda \leqslant \Lambda).$$

Clearly the functions H_λ are holomorphic on $U^q \times U^{(\Lambda-1)t}$.

Now $H_\Lambda = \det M_{\Lambda-1}$ (see the comment following (6)) and by the definition of Λ, is not identically zero. Thus there exists x_0 in $U^{(\Lambda-1)t}$ such that $H_\Lambda(.,x_0) \not\equiv 0$. The existence of such an x_0 is the crux of the proof. Now substitute x_0 for x in (9). Changing notation we obtain

$$(11)\quad h(w,z)\left(\sum_{0\leqslant|v|\leqslant n} B_v(w)z^v\right) = \sum_{0\leqslant|u|\leqslant m} A_u(w)z^u \qquad (w\in U^q, z\in U^t)$$

for $A_u(w) = - H_\lambda(w,x_0)$ $(u=v(\lambda),\lambda\leqslant M)$ and $B_v(w) = H_\lambda(w,x_0)$ $(v=v(\lambda),\lambda>M)$. Equation (10) shows that the functions A_u and B_v are in $N(U^q)$ by the same argument used to show that (7) is in $N(U^q)$. Since $B_{v(\Lambda)} = H_\Lambda(.,x_0) \not\equiv 0$, equation (11) shows that $\deg h(w,.)$ is at most (m,n) for almost every w in T^q. The minimal property of (m,n) given in (4) then shows that $\deg h(w,.) = (m,n)$ for almost every w in T^q.

Suppose, in order to derive a contradiction, that $B_{(0,\ldots,0)} \equiv 0$. Then $A_{(0,\ldots,0)} \equiv 0$ and for each w in $\mathscr{E}$, equation (11) and Theorem 1.3.2 of [7] show that the polynomials $\sum_{0\leqslant|v|\leqslant n} B_v(w)z^v$ and $\sum_{0\leqslant|u|\leqslant m} A_u(w)z^u$ have a common factor in $C[z_1,\ldots,z_t]$. Together with (11), this shows that $\deg h(w,.) \leqslant (m-1,n-1)$ for almost every w in T^q, contradicting (4). Thus $B_{(0,\ldots,0)} \not\equiv 0$. Henceforth we shall write B_0 for $B_{(0,\ldots,0)}$.

At this point we appeal to a theorem of Ahern ([1]; Theorem 2) to conclude that the meromorphic functions A_u/B_0 are actually holomorphic on U^q, that the meromorphic functions B_v/B_0 are actually bounded and holomorphic on U^q, and that the function

$$U^q \times U^t \ni (w,z) \to 1 + \sum_{1\leqslant|v|\leqslant n} (B_v(w)/B_0(w))z^v$$

is zero-free. Thus equation (11) can be written

$$(12)\qquad h(w,z)\left(1 + \sum_{1\leqslant|v|\leqslant n} k_v(w)z^v\right) = \sum_{0\leqslant|u|\leqslant m} h_u(w)z^u$$

where the functions h_u, k_v are holomorphic, the functions k_v are bounded, and

$$1 + \sum_{1\leqslant|v|\leqslant n} k_v(w)z^v \neq 0$$

for (w,z) in $U^q \times U^t$.

We now show that h_u is in $N(U^q)$. Indeed

$$\log^+|h_u| \leqslant \log^+|A_u| + \log^+|B_0| - \log|B_0|.$$

Now $B_0 \not\equiv 0$ and hence $\lim_{r\to 1} \int_{T^q} \log|B_0(rw)|\, dm(w) > -\infty$. Since both A_u and B_0 are in $N(U^q)$, it now follows that h_u is in $N(U^q)$.

Thus far we have established (modulo the uniqueness) assertions *a*) and *b*) of Theorem 1.4. Clearly the integers m and n in *a*) are uniquely determined by h. We now show that the Nevanlinna class functions h_u, k_v in (12) are uniquely determined by h. Suppose that h'_u, k'_v are in $N(U^q)$ and that

$$(12)' \qquad h(w,z)\left(1 + \sum_{1\leqslant|v|\leqslant n} k'_v(w)z^v\right) = \sum_{0\leqslant|u|\leqslant m} h'_u(w)z^u.$$

Now suppose that w in T^q is such that $\deg h(w,.) = (m,n)$ and that each of the functions h_u, k_v, h'_u, k'_v has a radial limit at w. The uniqueness assertion of Lemma 1.2(A) together with (12) and (12)′ shows that $h_u(w) = h'_u(w)$ and $k_v(w) = k'_v(w)$ for $0 \leqslant |u| \leqslant m, 1 \leqslant |v| \leqslant n$. Since the set of such w in T^q has measure one, we must have $h_u = h'_u$ and $k_v = k'_v$. This completes the proof of assertions *a*) and *b*).

Assertion (i) has already been established and assertion (ii) is an immediate consequence of (6), (10), and the definition of the functions h_u, k_v.

We now prove (iii). Let $P(w,z) = \sum_{0\leqslant|u|\leqslant m} h_u(w)z^u = \sum_{k=0}^{m} g_k(w,z_2,\ldots,z_t)z_1^k$ and $Q(w,z) = 1 + \sum_{1\leqslant|v|\leqslant n} k_v(w)z^v$ where $g_k(w,z_2,\ldots,z_t)$ is in $N(U^q)[z_2,\ldots,z_t]$ $(0\leqslant k\leqslant m)$. Since Q is bounded, it follows that P is in $H^p(U^q\times U^t)$ whenever h is $(1\leqslant p\leqslant\infty)$. Lemma 1.1. shows that $g_0(.) = P(.,(0,\cdot,\ldots\cdot))$ is in $H^p(U^q\times U^{t-1})$. Thus the function

$$U^q \times U^t \ni (w,z) \to P(w,z) - g_0(w,z_2,\ldots,z_t)$$

is in $H^p(U^q\times U^t)$. Let $P_1(w,z) = \sum_{k=1}^{m} g_k(w,z_2,\ldots,z_t)z_1^{k-1}$. Since

$$z_1P_1(w,z) = P(w,z) - g_0(w,z_2,\ldots,z_t),$$

it follows easily that P_1 is in $H^p(U^q\times U^t)$. Another appeal to Lemma 1.1 shows that $g_1(.) = P_1(\cdot,(0,\cdot,\ldots,\cdot))$ is in $H^p(U^q\times U^{t-1})$. Continuing in this manner we obtain that g_k is in $H^p(U^q\times U^{t-1})$ for $0 \leqslant k \leqslant m$. Now write each g_k as a polynomial in z_2 with coefficients in $N(U^q)[z_3,\ldots,z_t]$. As in the above we can show that these coefficients actually lie in $H^p(U^q \times U^{t-2})$. Continuing with this argument t times we finally obtain that h_u is in $H^p(U^q)$ for $0 \leqslant |u| \leqslant m$.

It remains to prove (iv). If h is inner, then by (i) and (iii) the h_α, k_β are bounded. Furthermore, for almost every w in T^q, $h(w,\cdot)$ is inner on U^t and rational of degree (m,n). Lemma 1.2(B) now establishes assertion (iv) and this completes the proof of Theorem 1.4.

The following corollary is an easy consequence of parts b) and (ii) of Theorem 1.4. We omit the proof. This corollary generalizes a result due to Ahern ([1]; Theorem 5). Compare also with Theorem 2.1 of [9] where a similar result is proved with boundary sections replaced by slices.

1.5. COROLLARY. – *Suppose that* $h\ (\not\equiv 0)$ *is in* $N(U^n)\ (n \geqslant 2)$ *and that for each* k *between* 1 *and* n *inclusive there exists a subset* E_k *of* T^{n-1} *having positive measure such that the boundary section*

$$U \ni z_k \to h(w_1,\ldots,w_{k-1},z_k,w_{k+1},\ldots,w_n)$$

is rational for every $(w_1,\ldots,w_{k-1},w_{k+1},\ldots,w_n)$ *in* E_k. *Then* h *is rational.*

2. The Rudin-Ahern factorization.

2.1. DEFINITION. – Suppose Q is in $P^\infty(U)[z_1,\ldots,z_t]$, i.e. $Q(w,z) = \sum_{0 \leqslant |\beta| \leqslant n} k_\beta(w) z^\beta \quad (k_\beta \in P^\infty(U))$, that $Q(w,0) \equiv 1$, and that $Q(w,z) \neq 0$ for (w,z) in $U \times U^t$. Let τ be the least multi-index such that $\tau - \beta \geqslant 0$ for all β such that $k_\beta \not\equiv 0$ and let h_τ be the inner function such that

$$h_\tau H^2(U) = \{h \in H^2(U)\,;\ h^*\overline{k_\beta^*} \in H^2(T) \quad \text{for} \quad 1 \leqslant |\beta| \leqslant n\}.$$

Such an h_τ exists and is unique up to multiplication by a constant of modulus one by Beurling's theorem ([2]). Define

$$Q^\#(w,z) = h_\tau(w) z^\tau + \sum_{1 \leqslant |\beta| \leqslant n} h_{\tau-\beta}(w) z^{\tau-\beta}$$

where $h_{\tau-\beta}$ is bounded and holomorphic in U and satisfies $h^*_{\tau-\beta} = h^*_\tau \overline{k^*_\beta}$. $Q^\#$ is uniquely determined up to multiplication by a constant of modulus one.

The quotient $Q^{\#}/Q$ is an inner function on $U \times U^t$ ([1]; the discussion following Theorem 3).

The following remark will be useful in § 2.2 and § 2.3.

Remark. – Suppose that P and Q are polynomials in $P^{\infty}(U)[z_1,\ldots,z_t]$ of degrees m and n respectively (and where Q is as in the above definition). Suppose also that $P(w,z)/Q(w,z)$ defines an inner function on $U \times U^t$ such that $\deg(P(w,\cdot)/Q(w,\cdot)) = (m,n)$ for almost every w in T. Then there exists an inner function φ on U and a multi-index γ such that $P(w,z) = \varphi(w)z^{\gamma}Q^{\#}(w,z)$. Furthermore P is a constant multiple of $Q^{\#}$ if and only if $m = n$ and there is no non-constant inner function on U which simultaneously divides each coefficient of P. Note that ([3]; Theorem 2.2.8) if f, g are in $P^{\infty}(U)$, then g divides f in $P^{\infty}(U)$ if and only if g divides f in $H^{\infty}(U)$.

The proof is an easy application of Theorem 1.4, the definition of $Q^{\#}$, and Beurling's theorem. We now present some algebraic preliminaries that will be needed in § 2.2 and § 2.3.

Let R be a ring. For a and b in R, we shall say that b divides a (in R), written $b|a$ (in R), if there exists c in R such that $a = bc$. A subset S of R is said to be a *prime set for* R if whenever $a \in S$, b and c are in R, and $a|bc$, then there exists $a_0 \in S$ such that $a_0|a$ and a_0 divides at least one of b and c. For example, the collection of non-constant inner functions on U is a prime set for both $H^{\infty}(U)$ and $P^{\infty}(U)$.

The following lemma may be viewed as an analogue of Gauss' lemma. The elementary proof is omitted.

LEMMA A. – *Let* R *be a ring and suppose that* $S(\subset R)$ *is a prime set for* R. *Then* S *is a prime set for* $R[z_1,\ldots,z_t]$, $t = 1,2,\ldots$.

LEMMA B. – *Let* R *denote either* $H^{\infty}(U)$ *or* $P^{\infty}(U)$ *and denote by* F *the quotient field of* R. *Let* Q_0 *be in* $R[z_1,\ldots,z_t]$ *and let* Q_1 *and* Q_2 *be in* $F[z_1,\ldots,z_t]$. *Suppose that* $Q_0 = Q_1Q_2$ *and* $Q_1(0) = Q_2(0) = 1$. *Then* Q_1 *and* Q_2 *are in* $R[z_1,\ldots,z_t]$.

Proof. – Consider the case $R = H^{\infty}(U)$ first. Let S be the collection of all non-invertible elements (non-units) of $H^{\infty}(U)$. It is easy to see, using the classical factorization of bounded holomorphic functions on U, that S is a prime set for $H^{\infty}(U)$. Moreover, the classical factorization shows that there exist g_1 and g_2 in $H^\infty(U)$ such that both of the following two conditions hold (the assumption $Q_1(0) = Q_2(0) = 1$ is used here).

(i) $g_k Q_k \in H^\infty(U)[z_1,\ldots,z_t]$ for $k = 1,2$,

(ii) $g_k Q_k$ is not divisible (in $H^\infty(U)[z_1,\ldots,z_t]$) by a member of S, $k = 1,2$.

If $g_1 g_2 \in S$, then applying Lemma A to $g_1 g_2 Q_0 = (g_1 Q_1)(g_2 Q_2)$, we obtain that some element of S divides either $g_1 Q_1$ or $g_2 Q_2$, contradicting condition (ii). Thus $g_1 g_2 \notin S$, i.e. g_1 and g_2 are invertible. It follows that Q_1 and Q_2 are in $H^\infty(U)[z_1,\ldots,z_t]$.

The case $R = P^\infty(U)$ follows easily from the above case with the aid of Theorem 2.2.8 of [3] which shows that if f and g are in $P^\infty(U)$ and f/g is in $H^\infty(U)$, then f/g is actually in $P^\infty(U)$. This completes the proof of the lemma.

The next lemma follows easily from Lemma B.

LEMMA C. — *Let* R *denote either* $H^\infty(U)$ *or* $P^\infty(U)$. *Let* Q *be in* $R[z_1,\ldots,z_t]$ *and suppose that* $Q(0) = 1$. *Then* Q *factors uniquely* (*up to reordering of the factors*) *as* $Q = \prod_{j=1}^{k} Q_j$ *where* Q_j *is an irreducible element in*

$$R[z_1,\ldots,z_t] \quad \textit{and} \quad Q_j(0) = 1 \quad (1 \leqslant j \leqslant k).$$

2.2. LEMMA. — *Suppose* $\psi(w,z)$ *is an inner function on* $U \times U^t$ *of the form given in Theorem* 1.4 *with* $q = 1$ *and* $n \geqslant 1$. *That is*

$$(1) \qquad \psi(w,z) = \frac{h_\tau(w)z^\tau + \sum_{1 \leqslant |\beta| \leqslant n} h_{\tau-\beta}(w) z^{\tau-\beta}}{1 + \sum_{1 \leqslant |\beta| \leqslant n} k_\beta(w) z^\beta} \qquad (w \in U, z \in U^t).$$

Then with the notation of Theorem 1.4 *and letting* $P(w,z)$ *and* $Q(w,z)$ *denote the numerator and denominator respectively on the right side of* (1),

A) ψ *is irreducible as an inner function on* $U \times U^t$ *if and only if both of the following two conditions hold :*

a) τ *is the least multi-index such that* $\tau - \beta \geqslant 0$ *for all* β *such that* $k_\beta \not\equiv 0$ *and* h_τ *is the « smallest » inner function on* U *such that*

$h_\tau^* \overline{k_\beta^*} \in H^\infty(T)$ *for all* β, i.e. $h_\tau H^2(U) = \{f \in H^2(U); f^* \overline{k_\beta^*} \in H^2(T)$ *for all* $\beta\}$.

b) Q *is irreducible as an element of the polynomial ring* $P^\infty(U)[z_1,\ldots,z_t]$.

B) *If* $Q = \prod_{j=1}^{k} Q_j$ *where each* Q_j *is an irreducible element of* $P^\infty(U)[z_1,\ldots,z_t]$, *then there exist elements* P_j *in* $P^\infty(U)[z_1,\ldots,z_t]$ *and an inner function* φ *on* U *and a multi-index* γ *such that* $\psi_j = P_j/Q_j$ *is an irreducible inner function on* $U \times U^t$ *for* $1 \leqslant j \leqslant k$ *and*

$$\psi(w,z) = \varphi(w) z^\gamma \prod_{j=1}^{k} \psi_j(w,z) \qquad (w \in U, z \in U^t).$$

Proof. – The proof of (A) is split into three parts.

(I) ψ *irreducible* $\Rightarrow$ *(a)* : Let σ be the least multi-index such that $\sigma - \beta \geqslant 0$ for all β such that $k_\beta \not\equiv 0$. If $\tau \neq \sigma$, then $\tau > \sigma$ and the non-constant inner function $z^{\tau-\sigma}$ divides $\psi(w,z)$, i.e. ψ is reducible. Let Y be the closed invariant (for the forward shift operator) subspace $\{f \in H^2(T); f^* \overline{k_\beta^*} \in H^2(T)$ for all $\beta\}$ of $H^2(U)$. Let h be an inner function that generates Y (Beurling's theorem), i.e. $Y = hH^2(U)$. If $Y \neq h_\tau H^2(U)$, then since $h_\tau \in Y$, there exists a non-constant inner function φ on U such that $h_\tau = \varphi h$. Thus

$$\psi(w,z) = \varphi(w) z^\gamma Q^{\#}(w,z)/Q(w,z)$$

for some γ in $(Z_+)^t$ where $Q^{\#}$ is as in § 2.1 and $Q^{\#}/Q$ is a non-constant inner function $(n \geqslant 1)$ on $U \times U^t$. Thus ψ is reducible.

(II) ψ *irreducible* $\Rightarrow$ *(b)* : Assume that $Q = Q_1 Q_2$ where Q_j is a polynomial in $P^\infty(U)[z_1,\ldots,z_t]$ of positive degree $(j = 1,2)$. Clearly we may assume that $Q_1(0) = Q_2(0) = 1$. Let S be the collection of all non-constant inner functions on U. The Remark in § 2.1 shows that $Q_j^{\#}$ is not divisible by an element of S, hence by Lemma A, neither is $Q_1^{\#} Q_2^{\#}$. The Remark in § 2.1 thus shows that $Q^{\#} = Q_1^{\#} Q_2^{\#}$ and that there exists an inner function φ on U and a multi-index γ such that

$$P(w,z) = \varphi(w) z^\gamma Q_1^{\#}(w,z) Q_2^{\#}(w,z).$$

Thus

$$\psi(w,z) = \varphi(w) z^\gamma (Q_1^{\#}(w,z)/Q_1(w,z))(Q_2^{\#}(w,z)/Q_2(w,z)).$$

Since Q_1 and Q_2 have positive degree, the inner functions $Q_1^{\#}/Q_1$ and $Q_2^{\#}/Q_2$ are non-constant and ψ is reducible.

(III) *(a) and (b)* $\Rightarrow$ ψ *irreducible :* Assume that (a) and (b) hold. Suppose that $\psi = \psi_1\psi_2$ where each ψ_j is an inner function on $U \times U^t$. We must show that either ψ_1 or ψ_2 is constant.

Now for almost every w in T, $\psi(w,\cdot) = \psi_1(w,\cdot)\psi_2(w,\cdot)$ is a rational inner function on U^t. It follows that for such w, both $\psi_1(w,\cdot)$ and $\psi_2(w,\cdot)$ are rational inner functions (if f and g are inner on U^t and fg is rational, then for a.e. $z \in T^t, f_z g_z$ is a rational inner function on U. Theorem VIII.32 of [12] shows that both of the slices f_z and g_z are rational for such z. Theorem 5.2.4 of [7] then shows that f and g are rational inner functions). Thus Theorem 1.4 applies. Let $\psi_j(w,z) = P_j(w,z)/Q_j(w,z)$ where P_j, Q_j are in $P^\infty(U)[z_1,\ldots,z_t]$ be the representation of ψ_j as given in Theorem 1.4 $(j=1,2)$. Now $\psi = \psi_1\psi_2$ and

$$\deg \psi_1(w,\cdot) + \deg \psi_2(w,\cdot) = \deg(\psi_1(w,\cdot)\psi_2(w,\cdot))$$

whenever both $\psi_1(w,\cdot)$ and $\psi_2(w,\cdot)$ are inner functions (this is certainly true for rational inner functions on the disc as their zeroes and poles lie in disjoint sets, namely U and $C^\infty\backslash\bar{U}$ respectively. For the general case, apply this result to slices and use Theorem 5.2.2 of [7]). Thus the uniqueness assertion of Theorem 1.4 implies that $Q = Q_1Q_2$ and $P = P_1P_2$.

Condition (b) (Q is irreducible) shows that one of Q_1 and Q_2 has degree zero as a polynomial in $P^\infty(U)[z_1,\ldots,z_t]$, say Q_1. Since $Q_1(\cdot,0) \equiv 1$, we have $Q_1(\cdot,\cdot) \equiv 1$. Thus there exists an inner function φ on U and a multi-index γ such that $P_1(w,z) = \psi_1(w,z) = \varphi(w)z^\gamma$ for (w,z) in $U \times U^t$. Thus $P(w,z) = \varphi(w)z^\gamma P_2(w,z)$. Condition (a) and the Remark in § 2.1 show that $\gamma = 0$ and that φ is constant. Thus ψ_1 is constant and the proof of part (A) of the lemma is complete.

B) Let $\psi_j(w,z) = Q_j^{\#}(w,z)/Q_j(w,z)$ where $Q_j^{\#}$ is as in § 2.1 $(1 \leq j \leq k)$. By part (A) of this lemma, ψ_j is an irreducible inner function on $U \times U^t$. Arguing as in (II) above, we can show that there exists an inner function φ on U and a multi-index γ such that

$$\psi(w,z) = \varphi(w)z^\gamma \prod_{j=1}^{k} \psi_j(x,z)$$

and this completes the proof of the lemma.

2.3. THEOREM. — *Suppose that* $\psi(w,z)$ *is an inner function on* $U \times U^t$ *and that* $\psi(w,\cdot)$ *is rational for* w *in some subset of positive measure in* T. *Then the*

Rudin-Ahern factorization of ψ is unique, i.e. ψ factors uniquely as a product of a zero-free inner function and at most countably many irreducible inner functions.

Proof. – Theorem 1.4 and Lemma 2.2(B) show that ψ has the factorization

$$\psi(w,z) = \varphi(w)z^{\gamma} \prod_{j=1}^{k} \psi_j(w,z)$$

where ψ_j is an irreducible inner function on $U \times U'$ for $1 \leqslant j \leqslant k$ and $z^{\gamma} = z_1^{\gamma_1}, \ldots, z_t^{\gamma_t}$ where z_τ is an irreducible inner function on U' for $1 \leqslant s \leqslant t$. It is wellknown that the Rudin-Ahern factorization for φ is unique. Thus it suffices to show that if X_1 is any irreducible inner function on $U \times U'$ that divides ψ, then either X_1 is independent of z and divides φ or X_1 is a constant multiple of one of the functions $\psi_1, \ldots, \psi_k, z_1, \ldots, z_{\prime}$. So suppose that $\psi = X_1X_2$ where X_1 and X_2 are inner functions on $U \times U'$ and X_1 is irreducible. Let $\psi = P/Q(\psi_j = P_j/Q_j)$ be the representation of $\psi(\psi_j)$ as given in Theorem 1.4. Arguing as in the second paragraph in (III) above (see the third part of the proof of Lemma 2.2(A)) we obtain that X_j has the representation $X_j = A_j/B_j(A_j,B_j \in P^{\infty}(U)[z_1,\ldots,z_t])$ as given in Theorem 1.4 and that $P = A_1A_2$ and $Q = B_1B_2$. Now if B_1 has degree zero as a polynomial in $P^{\infty}(U)[z_1,\ldots,z_{\prime}]$, then

$$B_1 \equiv 1 \qquad \text{and} \qquad X_1(w,z) = A_1(w,z) = \lambda(w)Z^{\delta}$$

for some inner function λ and multi-index δ. Since X_1 is an irreducible inner function, it follows easily that either X_1 is a constant multiple of one of $z_1, \ldots, z_t$ or X_1 divides φ. If B_1 has positive degree, then since B_1 is irreducible (Lemma 2.2(A)), Lemma C of § 2.1 now shows that B_1 is one of the Q_j's, say Q_1. Lemma 2.2(A) now shows that X_1 is a multiple of ψ_1 and this completes the proof of the theorem.

3. Good Inner Functions.

3.1. LEMMA. – *Let μ be a positive finite Borel measure on C with compact support contained in the unit disc U. Suppose that the function defined by*

$$C \ni w \to \int \log|z-w| \, d\mu(z)$$

is real-valued and continuous. Then, if $\{f_n\}_{n=0}^{\infty} \subset H(U)$ and $f_n \not\equiv 0 \quad \forall n \geqslant 0$, we have

(i) $\displaystyle\int \log|f_n(z)| \, d\mu(z) > -\infty, \qquad \forall n \geqslant 0.$

(ii) *If* $f_n \to f_0$ *uniformly on compact subsets of* U *as* $n \to \infty$ *then*

$$\lim_{n\to\infty} \int \log|f_n(z)|\, d\mu(z)$$

exists and equals $\int \log|f_0(z)|\, d\mu(z)$.

Proof. – Choose $0 < r < 1$ such that $\operatorname{supp}\mu$ is contained in

$$B(0,r) = \{z \in C\,;\, |z| < r\}$$

and such that f_0 has no zeros on $\partial B(0,r)$. Suppose f_0 has k zeros (counting multiplicities) in $B(0,r)$. Then there exists an integer N such that f_n has no zeros on $\partial B(0,r)$ and has k zeros (counting multiplicities) in $B(0,r)$ for all $n \geqslant N$. Without loss of generality we may assume that $N = 1$. Clearly, for each $n \geqslant 0$, we may enumerate the k zeros, $\{a_n^j\}_{1\leqslant j\leqslant k}$, of f_n in $B(0,r)$ in such a way that $\lim_{n\to\infty} a_n^j = a_0^j$ for $1 \leqslant j \leqslant k$. Now define

$$(1) \qquad h_n(z) = f_n(z) \prod_{j=1}^{k} \left(\frac{r^2 - \overline{a_n^j} z}{r(z - a_n^j)}\right) \qquad z \in U,\ n \geqslant 0.$$

Clearly h_n is holomorphic and bounded away from zero on $B(0,r)$ for each $n \geqslant 0$. Also $|h_n(z)| = |f_n(z)|$ for $|z| = r$, $n \geqslant 0$. Since

$$\log|h_n(\cdot)| = \log|f_n(\cdot)| \to \log|f_0(\cdot)| = \log|h_0(\cdot)|$$

uniformly on $\partial B(0,r)$ as $n \to \infty$ and $\log|h_n(\cdot)|$ is bounded and harmonic in $B(0,r)$ $\forall n \geqslant 0$, we must have that $\log|h_n(\cdot)| \to \log|h_0(\cdot)|$ uniformly on $B(0,r)$. Hence

$$(2) \qquad \lim_{n\to\infty} \int \log|h_n(z)|\, d\mu(z) = \int \log|h_0(z)|\, d\mu(z).$$

From equation (1) we obtain for all $n \geqslant 0$

$$(3) \quad \int \log|f_n(z)|\, d\mu(z) = \int \log|h_n(z)|\, d\mu(z) + k\|\mu\| \log r + \sum_{j=1}^{k} \left(\int \log|z - a_n^j|\, d\mu(z) - \int \log|r^2 - \overline{a_n^j} z|\, d\mu(z) \right).$$

Conclusions (i) and (ii) now follow easily from (2), (3), and the hypothesis on the measure μ. This completes the proof of the lemma.

3.2. THEOREM. – *Suppose that* h *is in* $H^\infty(U^n \times U)$ (*where* $n \geqslant 1$). *Denote by* K *the closure in* H(U) (*with respect to the topology of uniform convergence on compact subsets of* U) *of the set of functions* $\{h(w,\cdot)\,;\, w \in U^n\}$. *If* 0, *i.e. the identically zero function, is not in* K, *then* $h(\cdot,z)$ *is a good function for quasi-every* z *in* U, *i.e. except on a set of logarithmic capacity zero.*

Proof. – Without loss of generality we may assume that $|h| \leqslant 1$ on $U^n \times U$. Let $B = \{z \in U\,;\, h(\cdot,z) \text{ is not good}\}$. Now B is Borel and hence capacitable. Assume, in order to derive a contradiction, that the logarithmic capacity of B is positive. Then there exists a positive finite non-trivial Borel measure μ supported in B such that the function defined by $C \ni \xi \to \int \log|z-\xi|\, d\mu(z)$ is real-valued and continuous ([5] ; 7.33 and 7.37). Clearly K is metrizable and since it is contained in a bounded (with respect to the supremum norm) subset of H(U), it follows that K is also compact. Now define $T : K \to R$ by $T(f) = \int \log|f(z)|\, d\mu(z)$ for f in K. The measure μ satisfies the hypotheses of Lemma 3.1 and K does not contain the identically zero function by hypothesis. Thus Lemma 3.1 implies that T is real-valued and, since K is metrizable, that T is continuous. In particular we have

$$\lim_{r\to 1} \int_U \log|h(rw,z)|\, d\mu(z) = \int_U \log|h(w,z)|\, d\mu(z) \tag{4}$$

for w in $\mathscr{E} = \{w \in T^n\,;\, h(w,\cdot) = \lim_{r\to 1} h(rw,\cdot)$ exists uniformly on compact subsets of $U\}$, thus for almost every w in T^n. Since K is compact, T is bounded and hence

$$\sup_{0<r<1}\ \sup_{w\in T^n} \left| \int_U \log|h(rw,z)|\, d\mu(z) \right| < \infty . \tag{5}$$

Thus by Fatou's lemma and Fubini's theorem

$$\int_U \left(\lim_{r\to 1} \int_{T^n} \log|h(rw,z)|\, dm(w) \right) d\mu(z)$$

$$\geqslant \overline{\lim_{r\to 1}} \int_{T^n} \left(\int_U \log|h(rw,z)|\, d\mu(z) \right) dm(w)$$

and by (4), (5), the bounded convergence theorem, and Fubini's theorem again

$$\geqslant \int_{U}\left(\int_{T^n} \log|h(w,z)|\, dm(w)\right) d\mu(z).$$

Thus we have

$$(6) \quad \int_{U}\left(\int_{T^n} \log|h(w,z)|\, dm(w) - \lim_{r\to 1}\int_{T^n} \log|h(rw,z)|\, dm(w)\right) d\mu(z) \leqslant 0.$$

Fatou's lemma shows that the quantity in parentheses in (6) is non-negative for each z in U and hence it follows from (6) that

$$\int_{T^n} \log|h(w,z)|\, dm(w) = \lim_{r\to 1}\int_{T^n} \log|h(rw,z)|\, dm(w)$$

for μ-almost every z in U. This says that $h(\cdot,z)$ is good for μ-almost every z in U and contradicts the fact that μ is a non-trivial measure with support in $B = \{z \in U; h(\cdot,z) \text{ is not good}\}$. This completes the proof of Theorem 3.2.

The next result is a simple generalization of a result due to Nordgren and Ryff in the disc ([6], [10]). Indeed the proof given by Nordgren in [6] applies almost verbatim to the more general situation considered below. The details can be found in [11].

3.3. LEMMA. — *Let* $\varphi_1, \ldots, \varphi_m$ *be non-constant inner functions on* $U^{k_1}, \ldots, U^{k_m}$ *respectively and define*

$$\varphi : U^n \to U^m \quad (n = k_1 + \cdots + k_m) \qquad by \qquad \varphi(z) = (\varphi_1(z_1), \ldots, \varphi_m(z_m))$$

for $z = (z_1, \ldots, z_m)$ *in* $U^{k_1} \times \cdots \times U^{k_m}$. *Then*

a) $H^2(U^m) \ni f \to f \circ \varphi$ *defines a bounded linear map from* $H^2(U^m)$ *into* $H^2(U^n)$.

b) $(f \circ \varphi)^* = f^* \circ \varphi^*$ *almost everywhere on* T^n *for* f *in* $H^2(U^m)$.

DEFINITION. — *A finite Blaschke product on* U^n *is a function of the form*

$$U^n \ni (z_1, \ldots, z_n) \to c \prod_{k=1}^{N_1}\left(\frac{z_1 - a_k^1}{1 - \overline{a_k^1} z_1}\right) \times \cdots \times \prod_{k=1}^{N_n}\left(\frac{z_n - a_k^n}{1 - \overline{a_k^n} z_n}\right)$$

where $|c| = 1$ *and the* a_k^j *are in* U.

3.4. THEOREM. — *Let* R *be a non-constant irreducible rational inner function on* U^2 *that is not a finite Blaschke product on* U^2 *(see above definition). If* f *and* g *are non-constant inner functions on* U^m *and* U^n *respectively, then the function* $\varphi_{f,g,R}$ *defined by* $U^m \times U^n \ni (z,w) \to R(f(z),g(w))$ *is a good inner function on* $U^m \times U^n$.

Proof. — Lemma 3.3 shows that $\varphi_{f,g,R}$ is an inner function on $U^m \times U^n$. Chapter 5 of [7] shows that there exists a polynomial Q in $C[x,y]$ with no zeros in U^2 and a monomial M in $C[x,y]$ with coefficient of modulus one such that

$$P(x,y) = M(x,y)\tilde{Q}\left(\frac{1}{x},\frac{1}{y}\right)$$

($\tilde{Q}$ denotes the polynomial obtained from Q by replacing the coefficients of Q by their complex conjugates) is a polynomial in x and y that is relatively prime to Q and such that

$$(7) \qquad R(x,y) = \frac{P(x,y)}{Q(x,y)} = \frac{M(x,y)\tilde{Q}\left(\frac{1}{x},\frac{1}{y}\right)}{Q(x,y)} \qquad (x,y) \text{ in } U^2$$

Lemma 2.2(A) shows that Q is irreducible as a polynomial on C^2. We claim also that P is irreducible. Indeed, suppose (in order to derive a contradiction) that $P = P_1P_2$ where each P_j is a non-constant polynomial. Now

$$Q(x,y) = \tilde{P}\left(\frac{1}{x},\frac{1}{y}\right)\bigg/\tilde{M}\left(\frac{1}{x},\frac{1}{y}\right) = M(x,y)\tilde{P}\left(\frac{1}{x},\frac{1}{y}\right)$$

and clearly we can factor M as $M = M_1M_2$ where M_j is a monomial such that

$$Q_j(x,y) = M_j(x,y)\tilde{P}_j\left(\frac{1}{x},\frac{1}{y}\right)$$

is a polynomial for $j = 1,2$. Since P is not divisible by a non-constant monomial (Lemma 2.2(A)), it follows that Q_j is a non-constant polynomial for $j = 1,2$. Since $Q = Q_1Q_2$, this contradicts the irreducibility of Q. Thus P is an irreducible polynomial.

Now if the quotient of two good functions is in N_*, then the quotient is again good and hence the proof of the theorem will be complete once we show that the function $\varphi_{f,g,P}(z,w) = P(f(z),g(w))$ is a good function whenever P is an irreducible polynomial in two complex variables.

It will now be shown that the function defined by

$U \times U^n \ni (\alpha,w) \to P(2\alpha,g(w))$ satisfies the hypothesis of Theorem 3.2 (the « 2 » here could be replaced by any constant greater than one). Suppose (in order to derive a contradiction) that 0 is in the closure of the set of functions $\{P(2\cdot,g(w)); w \in U^n\} \subset H(U)$. Then there exists a sequence $\{w_k\}_{k=1}^{\infty}$ contained in U^n such that $P(2\cdot,g(w_k)) \to 0$ uniformly on compact subsets of U as $k \to \infty$. If y_0 is a cluster point of $\{g(w_k)\}_{k=1}^{\infty}$ in $\bar{U}$, then the continuity of P forces $P(2\cdot,y_0)$ to be identically zero on U and hence $P(\cdot,y_0) \equiv 0$ on C. Since $P(x,y)$ is not independent of x (by (7) and the hypothesis on R), it is easily seen that P is a reducible polynomial and this yields the desired contradiction. We now conclude from Theorem 3.2 that $P(2\alpha,g(\cdot))$ is a good function for quasi-every $\alpha \in U$. Since sets of logarithmic capacity zero are invariant under dilations, we have in particular that $P(\alpha,g(\cdot))$ is good for q.e. $\alpha \in \bar{U}$.

It will now be shown that each of the functions

$$P(f^*(z),g(\cdot)) \qquad \text{and} \qquad P(f(rz),g(\cdot)) \quad (0<r<1)$$

is good for almost every z in T^m. To this end, let $B = \{\alpha \in \bar{U}; P(\alpha,g(\cdot))$ is not good$\}$. The set B has logarithmic capacity zero and hence there exists a positive superharmonic function v defined in a neighbourhood of $\bar{U}$ that takes the value $+\infty$ on B ([5]; Theorem 7.33). Since $v \circ f$ is a positive multiply superharmonic function on U^m we have for $0 < r < 1$

$$\int_{T^m} v(f(rz))\, dm(z) < \infty \qquad \text{and} \qquad \int_{T^m} \varliminf_{r\to 1} (v \circ f)(rz)\, dm(z) < \infty .$$

The first inequality shows that $\{z \in T^m; f(rz) \in B\}$ is contained in a set of measure zero for $0 < r < 1$. The lower semi-continuity of v shows that $\varliminf_{r\to 1} (v \circ f)(rz) = +\infty$ on $\{z \in T^m; f^*(z) \in B\}$ and hence the second inequality shows that $\{z \in T^m; f^*(z) \in B\}$ is contained in a set of measure zero. Thus we have shown that if $0 < r \leqslant 1$, then

$$(8) \quad \lim_{s\to 1} \int_{T^n} \log|P(f(rz),g(sw))|\, dm(w) = \int_{T^n} \log|P(f(rz),g^*(w))|\, dm(w)$$

for almost every z in T^m where when $r = 1$, $f(rz)$ is to be interpreted as $f^*(z)$.

Now fix $0 < r \leqslant 1$ momentarily and consider the following family of measurable functions on T^m indexed by s $(0<s<1)$

$$(9) \qquad T^m \ni z \to \int_{T^n} \log|P(f(rz),g(sw))|\, dm(w).$$

This family is uniformly bounded above and increases pointwise with s. Hence if we let h_s denote the function in (9) corresponding to s, we have that $|h_s(z)| \leqslant |h_{\frac{1}{2}}(z)| + C$ for $\frac{1}{2} < s < 1$ where C is a constant such that $h_s(z) \leqslant C$ for all $0 < s < 1$ and all z in T^m. Now using (8) we apply the dominated convergence theorem ($|h_{\frac{1}{2}}(\cdot)| + C$ is integrable with respect to Haar measure on T^m) to conclude that

$$(10) \quad \lim_{s\to 1} \int_{T^m} \left(\int_{T^n} \log |P(f(rz),g(sw))|\, dm(w) \right) dm(z) = \int_{T^m} \left(\int_{T^n} \log |P(f(rz),g^*(w))|\, dm(w) \right) dm(z)$$

for $0 < r \leqslant 1$ where again, when $r = 1$, $f(rz)$ is to be interpreted as $f^*(z)$. By symmetry we also have for $0 < s \leqslant 1$

$$(11) \quad \lim_{r\to 1} \int_{T^n} \left(\int_{T^m} \log |P(f(rz),g(sw))|\, dm(z) \right) dm(w) = \int_{T^n} \left(\log |P(f^*(z),g(sw))|\, dm(z) \right) dm(w).$$

Let $A = \int_{T^{m+n}} \log |\varphi^*_{f,g,P}|\, dm$ and suppose that $\varepsilon > 0$ is given. By (10) with $r = 1$ there exists $0 < s_0 < 1$ such that

$$\int_{T^{m+n}} \log |P(f^*(z),g(s_0 w))|\, dm(z,w) > A - \varepsilon$$

(the continuity of P has been used here) and now by (11) with $s = s_0$ there exists $0 < r_0 < 1$ such that

$$(12) \qquad \int_{T^{m+n}} \log |P(f(r_0 z),g(s_0 w))|\, dm(z,w) > A - \varepsilon.$$

Using the multiple subharmonicity of $\log |\varphi_{f,g,P}|$, we obtain from (12) that for all t satisfying $\max(r_0,s_0) \leqslant t < 1$

$$\int_{T^{m+n}} \log|\varphi_{f,g,P}(t\xi)|\, dm(\xi) > A - \varepsilon.$$

This shows that $\varphi_{f,g,P}$ is good and completes the proof of Theorem 3.4.

3.5. Theorem. — *Let* R *be a finite or infinite (converging uniformly on compact subsets) product of rational inner functions on* U^2. *Suppose that* R *is not divisible by a non-constant finite Blaschke product on* U^2. *If* f *and* g *are non-constant inner functions on* U^m *and* U^n *respectively, then* $\varphi_{f,g,R}$ *(see the statement of Theorem* 3.4) *is a good inner function on* U^{m+n}.

Proof. — Without loss of generality R is non-constant. Now every rational inner function on U^2 factors as a finite product of irreducible rational inner functions on U^2 (Lemma 2.2(B)). Hence there exist non-constant irreducible rational inner functions R_i on U^2 such that $R = \prod_{i \in I} R_i$ where I is an at most countable index set. By hypothesis, no R_i is a finite Blaschke product on U^2 and hence Theorem 3.4 implies that φ_{f,g,R_i} is a good inner function on U^{m+n} for every $i \in I$. Now

$$(\varphi_{f,g,R})_w = \prod_{i \in I} (\varphi_{f,g,R_i})_w$$

for all w in T^{m+n}. For almost every w in T^{m+n}, $(\varphi_{f,g,R_i})_w$ is a Blaschke product for every $i \in I$ (Theorem 5.3.2 of [7]) and thus $(\varphi_{f,g,R})_w$ is a Blaschke product for almost every w in T^{m+n}. Theorem 5.3.2 of [7] now implies that $\varphi_{f,g,R}$ is a good inner function on U^{m+n} and this completes the proof of Theorem 3.5.

4. Reducibility of certain inner functions.

The purpose of this section is to indicate a technique for investigating the reducibility of certain inner functions. Complete proofs will not always be given. However, detailed proofs of all the results stated here may be found in [11]. The following lemma follows easily from Theorems 2.2.1, 2.2.3, 2.2.4, and 2.4.4 of [3].

4.1. Lemma. — A) *Suppose* τ *is in* $H^\infty(U^n)$ $(n \geq 1)$. *If* σ *is an inner function on* U^n *and* $\sigma\tau$ *is in* $P^\infty(U^n)$, *then* τ *is in* $P^\infty(U^n)$.

B) *Suppose* f *is in* $H^\infty(U)$, f *is not rational, and* f^2 *is rational. Then* f *is not in* $P^\infty(U)$.

C) *If* φ *is a non-constant inner function on* U^n $(n \geq 1)$ *and if* f *is in* $H^\infty(U)$, *then* f *is in* $P^\infty(U)$ *if and only if* $f \circ \varphi$ *is in* $P^\infty(U^n)$.

4.2. LEMMA. — *Suppose that* F *is in* $H(U^n)$ $(n \geqslant 2)$ *and that* m *is an integer greater than one. Then* F *admits a holomorphic* m^{th} *root in* U^n *if and only if* $F(w_1, \ldots, w_{k-1}, \cdot, w_{k+1}, \ldots, w_n)$ *admits a holomorphic* m^{th} *root in* U *for every choice of* w_i *in* U *and* $1 \leqslant k \leqslant n$.

The lemma is proved easily by induction using the following claim.

Claim. — Suppose F is in $H(U^{q+1})(q \geqslant 1)$, m is an integer greater than one, and $F(w,\cdot)$ admits a holomorphic m^{th} root in U for each w in U^q and $F(\cdot,z)$ admits a holomorphic m^{th} root in U^q for each z in U. Then F admits a holomorphic m^{th} root in U^{q+1}.

Proof of Claim. — Without loss of generality $F \not\equiv 0$. Fix w_0 in U^q such that $F(w_0,\cdot) \not\equiv 0$ and let φ be a holomorphic m^{th} root of $F(w_0,\cdot)$ in U. Let $G = U \backslash Z(\varphi)$ (if f is a function, then $Z(f)$ denotes its zero-set). Clearly G is an open, connected, and dense subset of U. Now for each z in G, there exists a unique holomorphic m^{th} root g_z of $F(\cdot,z)$ in U^q satisfying $g_z(w_0) = \varphi(z)$. Using this together with the fact that the family of functions $\{g_z\}_{z \in K}$ is a normal family on U^q whenever K is a subset of G that is relatively compact in U, it is easy to see that the function $G \ni z \to g_z \in H(U^q)$ is continuous.

We now claim that the function $G \ni z \to g_z(w)$ is a holomorphic function on G for each fixed w in U^q, that it admits a (necessarily unique) holomorphic extension to U and, denoting this extension by h_w, that $h_w(\cdot)^m = F(w,\cdot)$. To see this, fix w in U^q momentarily. If $g_z(w) = 0$ for every z in G, let h_w be identically zero. If $g_{z_0}(w) \neq 0$ for some z_0 in G, then by hypothesis there exists a holomorphic m^{th} root f of $F(w,\cdot)$ in U such that $f(z_0) = g_{z_0}(w)$. Let $G' = G \backslash Z(f) = U \backslash Z(\varphi f)$. Clearly G' is an open, connected, and dense subset of G. Now if $\varepsilon = e^{2\pi i/m}$ and z is in G', then

$$0 = F(w,z) - F(w,z) = f(z)^m - g_z(w)^m = R(z)S(z)$$

where $R(z) = f(z) - g_z(w)$ and $S(z) = \prod_{k=1}^{m-1} (f(z) - \varepsilon^k g_z(w))$ for z in G'. Both R and S are continuous functions on G' and thus their zero-sets, $Z(R)$ and $Z(S)$, are closed relative to G'. Furthermore, $Z(R) \cup Z(S) = G'$ and $Z(R) \cap Z(S) = \varnothing$. Now $Z(R) \neq \varnothing$ since z_0 is in $Z(R)$ and thus the connectedness of G' forces $Z(R) = G'$. Hence $f(z) = g_z(w)$ for all z in G'. Since both sides of this equation are continuous for z in G and since G' is dense in G, it follows that $f(z) = g_z(w)$ for all z in G. Thus the claim made at the beginning of this paragraph holds upon letting $h_w = f$.

Now define $\psi : U^q \times U \to C$ by $\psi(w,z) = h_w(z)$ for (w,z) in $U^q \times U$. Clearly $\psi(w,\cdot)$ is holomorphic on U for every w in U^q. Using once more the fact that $\{g_z\}_{z\in K}$ is a normal family on U^q whenever $K \subset G$ and $\bar{K} \subset U$, it is easy to see that $\psi(\cdot,z)$ is holomorphic on U^q for every z in U. Hartogs' theorem now shows that ψ is holomorphic on $U^q \times U$ and ψ obviously satisfies $\psi^m = F$. This completes the proof of the claim.

4.3. Corollary. – *Let m be an integer greater than one. Suppose f and g are in $H(U^n)$ and $Z(f) \bigcap Z(g) = \varnothing$. If fg admits a holomorphic m^{th} root in U^n, then both f and g do also.*

Define $\varphi_\alpha(z) = \dfrac{z-\alpha}{1-\bar{\alpha}z}$ *for* $z \in \bar{U}, \alpha \in U$.

4.4. Lemma. – *If α and β are in U, then there exist γ and δ in U and θ in $[0,2\pi)$ such that*

$$\varphi_\alpha(z)\varphi_\beta(z) = e^{i\theta}\varphi_\delta(\varphi_\gamma(z)^2), \qquad z \in U. \tag{1}$$

Proof. – Without loss of generality, we may assume that $\alpha \neq \beta$. Clearly there exists γ in U such that $\varphi_\gamma(\alpha) = -\varphi_\gamma(\beta)$. Let $\delta = \varphi_\gamma(\alpha)^2 = \varphi_\gamma(\beta)^2$. The function $\varphi_\delta(\varphi_\gamma(\cdot)^2)$ is holomorphic in a neighbourhood of $\bar{U}$, inner on U, has zeros of multiplicity one at α and β, and has no other zeros in U. Thus, up to multiplication by a constant of modulus one, $\varphi_\delta(\varphi_\gamma(\cdot)^2)$ must be the function $\varphi_\alpha(\cdot)\varphi_\beta(\cdot)$ ([12]; Theorem VIII.32). This completes the proof of the lemma.

4.5. Theorem. – *If g is a non-constant inner function on U^n and $0 < |\mu| < 1$, then the inner function on U^{n+1} defined by*

$$U^n \times U \ni (w,z) \to \frac{\varphi_\alpha(z)\varphi_\beta(z)g(w) - \mu}{1 - \bar{\mu}\varphi_\alpha(z)\varphi_\beta(z)g(w)} \tag{2}$$

is reducible if and only if $\alpha = \beta$ and g admits a holomorphic square root in U^n.

Proof. – Denote the function defined in (2) by H.

(*if*) Assume $\alpha = \beta$ and $g = h^2$ where h is holomorphic on U^n. Clearly h must be an inner function. Let ν in U be such that $\nu^2 = \mu$. Then

$$H(w,z) = \left(\frac{\varphi_\alpha(z)h(w) - \nu}{1 - \bar{\nu}\varphi_\alpha(z)h(w)}\right)\left(\frac{\varphi_\alpha(z)h(w) + \nu}{1 + \bar{\nu}\varphi_\alpha(z)h(w)}\right).$$

The two functions on the right side are non-constant inner functions on U^{n+1} (they are also irreducible; see the proof of Theorem 3.6 in [8]) and this shows that H is a reducible inner function.

(*only if*) Define $\hat{H}(w,z) = H(w,\varphi_{-\gamma}(z))$ where γ is related to α and β as in Lemma 4.4. Using (2) and Lemma 4.4 we obtain after some elementary algebra that

$$\hat{H}(w,z) = \frac{P(w,z)}{Q(w,z)} = \frac{A(w)z^2 - B(w)}{1 - K(w)z^2} \tag{3}$$

where

$$A(w) = \varphi_{-\mu\delta}(g_1(w)), \qquad B(w) = \varphi_{-\mu}(\delta g_1(w)),$$
$$K(w) = \varphi_{-\delta}(\bar{\mu} g_1(w)), \qquad g_1 = e^{i\theta} g. \tag{4}$$

Note that A is an inner function and that B and K are in $P^\infty(U^n)$ (indeed $\overline{A^*B^*} = K^*$ almost everywhere on T^n). Now assume that H is reducible. Then so is $\hat{H}$, say $\hat{H} = H_1H_2$ where H_1 and H_2 are non-constant inner functions on U^{n+1}. Now neither H_1 nor H_2 can be independent of z. (Indeed, if H_1 is independent of z and if $H_1(w_0,z_0) = 0$, then $H_1(w_0,\cdot) \equiv 0$. Since $\hat{H} = H_1H_2$ it follows that $\hat{H}(w_0,\cdot) \equiv 0$. This is clearly impossible from an inspection of (2). Thus H_1 is zero-free. Now $\hat{H}$ is a good inner function by Theorem 3.4 and hence H_1 is also a good inner function. This forces H_1 to be constant, a contradiction, and we conclude that H_1 cannot be independent of z. This line of reasoning appears in the proof of Theorem 3.6 in [8]).

Arguing as in the second paragraph in part (III) of the proof of Lemma 2.2(A), we obtain that H_j has a representation $H_j = P_j/Q_j$ ($P_j,Q_j \in P^\infty(U^n)[z]$) as given by Theorem 1.4 and that $P = P_1P_2$ and $Q = Q_1Q_2$. Since H_j is not independent of z, it follows that the degrees of P_1 and P_2 (as polynomials in $P^\infty(U^n)[z]$) must each be one. An inspection of (2) shows that $\hat{H}$ is not divisible by z and thus we conclude that the degrees of Q_1 and Q_2 are also each one. Thus both P and Q, considered as quadratic polynomials in $P^\infty(U^n)[z]$, factor into linear factors. Hence the discriminants of P and Q, respectively 4AB and 4K (see (3)), each have a square root in $P^\infty(U^n)$. We claim that B also has a square root in $P^\infty(U^n)$. Indeed, the zero-sets of the functions A and B are disjoint (from an inspection of (4)) and so Corollary 4.3 implies that both A and B have square roots in $H^\infty(U^n)$, say a and b respectively. Now ab is in $P^\infty(U^n)$ and a is inner (since A is inner) and thus Lemma 4.1(A) implies that b is in $P^\infty(U^n)$.

Assume (in order to derive a contradiction) that $\delta \neq 0$. If $|\delta| \leqslant |\mu|$ then the function defined by $U \ni \xi \to \varphi_{-\mu}(\delta\xi)$ is bounded, rational, and zero-free on U. Let S be a holomorphic square root of this function in U. Now S^2 is rational and S is a bounded holomorphic function on U that is not rational. Thus Lemma 4.1(B) shows that S is not in $P^\infty(U)$ and Lemma 4.1(C) then shows that $S \circ g_1$ is not in $P^\infty(U^n)$, a contradiction since $S \circ g_1 = \pm b$. Similarly, by considering K in place of B, a contradiction is derived from the assumptions $\delta \neq 0$, $|\mu| \leqslant |\delta|$. Thus the assumption that $\delta \neq 0$ is false. So $\delta = 0$ and from (4), $K = \bar{\mu} g_1$.

Now $\delta = 0$ implies that $\alpha = \beta$ (Lemma 4.4) and the equation $K = \bar{\mu} g_1 = \bar{\mu} e^{i\theta} g$ implies that g admits a holomorphic square root in U^n. This completes the proof of Theorem 4.5.

BIBLIOGRAPHY

[1] P. R. AHERN, Singular sets of inner functions, *Indiana Univ. Math. J.*, 21 (1971), 147-155.

[2] A. BEURLING, On two problems concerning linear transformations in Hilbert space, *Acta. Math.*, 81 (1949), 239-255.

[3] R. G. DOUGLAS, H. S. SHAPIRO, and A. L. SHIELDS, Cyclic vectors and invariant subspaces for the backward shift operator, *Ann. Inst. Fourier*, Grenoble, 20, 1 (1970), 37-76.

[4] O. FROSTMAN, Potentiel d'équilibre et capacité des ensembles..., *Lunds Univ. Mat. Sem. d.*, (1935), 1-118.

[5] L. L. HELMS, Introduction to Potential Theory, Wiley-Interscience, 1969.

[6] E. A. NORDGREN, Composition operators, *Can. J. Math.*, 20 (1968), 442-449.

[7] W. RUDIN, Function Theory in Polydiscs, Benjamin, 1969.

[8] W. RUDIN and P. R. AHERN, Factorizations of bounded holomorphic functions, *Duke Math. J.*, 39 (1972), 767-777.

[9] W. RUDIN and E. L. STOUT, Boundary properties of functions of several complex variables, *J. Math. Mech.*, 14 (1965), 991-1006.

[10] J. V. RYFF, Subordinate H^p functions, *Duke Math. J.*, 33 (1966), 347-354.

[11] E. SAWYER, Inner functions on polydiscs, Ph. D. thesis, McGill University (1977).

[12] M. TSUJI, Potential Theory in Modern Function Theory, Chelsea Publ. Co.

[13] A. ZYGMUND, Trigonometric Series, 2nd ed., Cambridge Univ. Press, 1959.

Manuscrit reçu le 18 janvier 1978
Révisé le 24 juillet 1978.

Eric SAWYER,
University of Wisconsin
Madison, Wisc. (USA).