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FRACTIONAL CARTESIAN PRODUCTS OF SETS

by Ron C. BLEI (*)

N -fold sums of « independent » sets serve in harmonic analysis as prototypical examples of $2N/(N + 1)$ -Sidon sets, and $\Lambda(q)$ sets whose $\Lambda(q)$ constants' growth is $\mathcal{O}(q^{N/2})$. Moreover, these features are exact : N -fold sums of independent sets are not $(2N/(N + 1) - \varepsilon)$ -Sidon and do not have $\Lambda(q)$ constants' growth asymptotic to $q^{(N/2-\varepsilon)}$, for any $\varepsilon > 0$ (see [4], [6] and [2]). In this paper, given any number $p \in (1, 2)$, we display a set that is p -Sidon but not $(p - \varepsilon)$ -Sidon for any $\varepsilon > 0$. The same pool of examples contains, for any number $a \in [1/2, \infty)$, a set whose $\Lambda(q)$ constants' growth is $\mathcal{O}(q^a)$ but not $\mathcal{O}(q^{a-\varepsilon})$ for any $\varepsilon > 0$. This answers questions raised in [4] and [6], and a question that is implicit in [2]. The type of sets displayed here exhibits « combinatorial » and « analytic » properties that one would expect « fractional » cartesian products (sums) of sets to possess, and hence the title of the paper. This class of sets naturally arises in the study of multidimensional extensions of Grothendieck's inequality ([1]); it is that study that led to the present work.

1. Definitions and main results.

We employ basic notation and facts of commutative harmonic analysis as presented and followed in [10]. Γ will be a countable discrete abelian group and $G = \Gamma$ will denote its compact dual group. Throughout, group operations in Γ and G will be designated by multiplicative notation.

We now define the type of sets that is the object of the present study. Let $J \geq K > 0$ be arbitrary integers, and

$$\mathcal{F} = \{1, \dots, J\}.$$

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For the sake of typographical convenience here and throughout the paper we

let $N = \binom{J}{K}$. Let

$$\{S_1, \dots, S_N\}$$

be the collection of all K -subsets of \mathcal{F} (sets containing K elements of \mathcal{F}), where each $S_\alpha \subset \mathcal{F}$ is enumerated as

$$S_\alpha = (\alpha_1, \dots, \alpha_K).$$

Let $\{P_1, \dots, P_N\}$ be the collection of projections from $(\mathbf{Z}^+)^J$ into $(\mathbf{Z}^+)^K$ defined as follows : For $1 \leq \alpha \leq N$, and $j = (j_1, \dots, j_j) \in (\mathbf{Z}^+)^J$

$$P_\alpha(j) = (j_{\alpha_1}, \dots, j_{\alpha_K}).$$

Next, let $F \subset \Gamma$ and

$$F = \{\gamma_i\}_{i \in (\mathbf{Z}^+)^K}$$

be a K -fold enumeration of F . Finally, define

$$(1.1) \quad F_{J,K} = \{(\gamma_{P_1(j)}, \dots, \gamma_{P_N(j)})\}_{j \in (\mathbf{Z}^+)^J} \subset F^N \subset \Gamma^N.$$

Throughout this paper, a set that is subscripted by J, K will denote the set defined by (1.1), for some fixed K -fold enumeration of F .

DEFINITION 1.1. — Let $M > 0$ be a fixed integer. $F = \{\gamma_j\}_{j=1}^\infty \subset \Gamma$ is M -independent if for any $L, L' > 0$ the relation

$$\prod_{j=1}^L \gamma_j^{\lambda_j} = \prod_{j=1}^{L'} \gamma_j^{\nu_j},$$

where the λ_j 's and ν_j 's are integers in $[-M, M]$, implies that $L = L'$ and $\lambda_j = \nu_j$ for $j = 1, \dots, L$. If F is M -independent for every M , then F is said to be independent. 1-independent sets are referred to as dissociate sets.

DEFINITION 1.2 (1.6.2 and 1.6.3 in [9]). — Let $F = \{\gamma_j\}_{j=1}^\infty \subset \Gamma, s \in \mathbf{Z}^+$ and $\gamma \in \Gamma$. Writing (formally) the Fourier series $h \sim \left(\sum_{j=1}^\infty \gamma_j\right)^s$, we define

$$r_s(F, \gamma) = \hat{h}(\gamma).$$

Equivalently, $r_s(F, \gamma)$ is the number of ways to write γ in the form of

$$(1.2) \quad \gamma = \gamma_{i_1} \dots \gamma_{i_s},$$

where $\gamma_{i_1}, \dots, \gamma_{i_s}$ are (not necessarily distinct) s elements in F , and where different permutations on the right hand side of (1.2) are counted as different representations.

For example, it is easy to see that if $F \subset \Gamma$ is independent then, for all $s > 0$, the J -fold cartesian product of F satisfies

$$(1.3) \quad \sup_{\gamma \in \Gamma^J} r_s(F_{J,1}, \gamma) = (s!)^J, \quad \text{for all } s \in \mathbf{Z}^+.$$

The following is an extension to (1.3) and is evidence that $F_{J,K}$ could be viewed as a J/K -fold cartesian product of F .

THEOREM 1.3. — *Let $F \subset \Gamma$ be an independent set. For all $J \geq K > 0$ and $s > 0$,*

$$(1.4) \quad 16^{-\binom{J}{K}s} s^{(J/K)s} \leq \sup_{\gamma \in \Gamma^N} r_s(F_{J,K}, \gamma) \leq s^{(J/K)s}.$$

We now list the « analytic » results that are based on the above « fractional cartesian products ». For $F \subset \Gamma$, $C_F(G)$ and $L_F^p(G)$, $1 \leq p \leq \infty$, will be the spaces of functions in $C(G)$ and $L^p(G)$, respectively, whose spectra lie in F . Recall that for $2 < q < \infty$ F is a $\Lambda(q)$ set if there is $A > 0$ so that for all $f \in L_F^2(G)$

$$(1.5) \quad A \|f\|_2 \geq \|f\|_q.$$

The « smallest » A for which (1.5) holds is the $\Lambda(q)$ constant of F and is denoted by $A(q,F)$. Of particular interest are sets $F \subset \Gamma$ for which $A(q,F)$ is $\mathcal{O}(q^a)$ for some $a \geq 1/2$. In fact, this growth condition can be neatly understood as follows: $A(q,F)$ is $\mathcal{O}(q^a)$ if and only if every $f \in L_F^2(G)$ also satisfies

$$\int_G \exp(\lambda |f|^{1/a}) < \infty, \quad \text{for all } \lambda > 0$$

(see Remarque on p. 350 of [2]).

DEFINITION 1.4. — *Let $\beta \in [1, \infty)$. $F \subset \Gamma$ is a Λ^β set if $A(q,F)$ is $\mathcal{O}(q^{\beta/2})$. F is said to be exactly Λ^β when F is Λ^a if and only if $a \in [\beta, \infty)$. F is said to be exactly non- Λ^β when F is Λ^a if and only if $a \in (\beta, \infty)$.*

DEFINITION 1.5. — *Let $p \in [1, 2)$. $F \subset \Gamma$ is a p -Sidon set if there is $D > 0$ so that for all $f \in C_F(G)$*

$$(1.8) \quad D \|f\|_\infty \geq \|\hat{f}\|_p.$$

The « smallest » D for which (1.8) holds is the p -Sidon constant of F and is denoted by $D(p,F)$. F is exactly p -Sidon when F is r -Sidon if and only if $r \in [p, 2)$. F is exactly non- p -Sidon when F is r -Sidon if and only if $r \in (p, 2)$.

J-fold cartesian products of dissociate sets are the classical examples of sets that are exactly Λ^J . These and other similar constructions of sets which are exactly Λ^J are studied extensively in [2]. The same J-fold products are also the simplest examples of sets which are exactly $2J/(J+1)$ -Sidon ([7], [4] and [6]). The gaps left open between the J and (J+1)-fold products of dissociate sets are filled by the « fractional cartesian products » that were defined at the outset.

THEOREM 1.6. — *Let $F \subset \Gamma$ be an independent set.*

a) $F_{J,K} \subset \Gamma^N$ is exactly $\Lambda^{J/K}$. Moreover, there is $\eta_{J,K} > 0$ so that for all $q > 2$

$$(1.9) \quad \eta_{J,K} q^{J/2K} \leq A(q, F_{J,K}) \leq q^{J/2K}.$$

b) $F_{J,K}$ is exactly $2J/(K+J)$ -Sidon. Moreover,

$$(1.10) \quad D\left(\frac{2J}{(K+J)}, F_{J,K}\right) \leq 2^{J/K}.$$

(Recall that $N = \binom{J}{K}$.)

Constructions analogous to (1.1) can be carried out within Γ by replacing the cartesian product operation with the group operation in Γ . Given (an infinite set) $F \subset \Gamma$, let \mathcal{P} be a N -partition of F :

$$\mathcal{P} = \{F_1, \dots, F_N\}$$

(that is, $F_1, \dots, F_N \subset \Gamma$ are mutually disjoint sets whose union is F), where each F_α is infinite. For each $1 \leq \alpha \leq N$, endow F_α with a K -fold enumeration

$$F_\alpha = \{\gamma_i^{(\alpha)}\}_{i \in (\mathbb{Z}^+)^K}.$$

Define

$$(1.11) \quad F_{\mathcal{P}}^{J,K} = \{\gamma_{P_1(i)}^{(1)} \dots \gamma_{P_N(i)}^{(N)}\}_{i \in (\mathbb{Z}^+)^J} \subseteq \Gamma.$$

In this work, a set that is superscripted by J , K and subscripted by \mathcal{P} will be the set defined in (1.11), where each member of \mathcal{P} is understood to have a K -fold enumeration. When \mathcal{P} is fixed and understood from the context, we write $F^{J,K}$ for $F_{\mathcal{P}}^{J,K}$. Observe that, for $F \subset \Gamma$, letting

$$F_\alpha = (1, \dots, 1, F, 1, \dots) \in \Gamma^N$$

↑
α-th coordinate

and

$$\bar{F} = \bigcup_{\alpha=1}^N F_{\alpha},$$

we have

$$F_{J,K} = \bar{F}_{\mathcal{P}}^{J,K} \subset \Gamma^N$$

where

$$\mathcal{P} = \{F_1, \dots, F_N\}.$$

COROLLARY 1.7. – *Let $F \subset \Gamma$ be a dissociate set and \mathcal{P} be any N -partition of F .*

a) $F_{\mathcal{P}}^{J,K} \subset \Gamma$ is exactly $\Lambda^{J/K}$. Moreover, there are constants $\eta_{J,K}, \zeta_{J,K} > 0$ so that for all $q > 2$

$$(1.12) \quad \eta_{J,K} q^{J/2K} \leq A(q, F_{\mathcal{P}}^{J,K}) \leq \zeta_{J,K} q^{J/2K}.$$

b) $F^{J,K}$ is exactly $2J/(K+J)$ -Sidon.

c) Suppose that Γ contains element with arbitrarily large order. Then, for every $\beta \in [1, \infty)$ there are $F_1, F_2 \subset \Gamma$ so that F_1 is exactly Λ^{β} and F_2 is exactly non- Λ^{β} .

d) Let Γ be any discrete abelian group. Then, for every $p \in [1, 2)$ there are $F_1, F_2 \subset \Gamma$ so that F_1 is exactly p -Sidon and F_2 is exactly non- p -Sidon.

The organization of the paper is as follows. In section 2, we prove the right hand inequality of (1.4) in Theorem 1.3. In section 3, fitted for $F^{J,K}$, appropriate Riesz products are developed for use in later sections. The Λ^{β} property is treated in section 4 where Theorem 1.6 (a) and Corollary 1.7 (a), (c) are proved. The left hand inequality of (1.9) in Theorem 1.6 is then used to establish the left hand inequality of (1.4) in Theorem 1.3. p -Sidonicity is treated in section 5, where the remaining parts of Theorem 1.6 and Corollary 1.7 are proved. We conclude in section 6 with some problems.

2. A combinatorial property of $F_{J,K}$.

Let $F \subset \Gamma$ be an independent set. We prove here the right hand inequality of (1.4) : For all $s \in \mathbf{Z}^+$ and $\gamma \in \Gamma^N$

$$(2.1) \quad r_s(F_{J,K}, \gamma) \leq s^{(J/K)s}.$$

We shall use an extension of Hölder's inequality which, to facilitate referencing here and in section 5, we state below.

2.1. *M-Hölder's inequality.* — Let X be a measure space, $M > 1$ be an arbitrary integer and $1 < p_1 < \dots < p_M < \infty$ be so that

$$\sum_{i=1}^M \frac{1}{p_i} = 1.$$

Then, for any f_1, \dots, f_M , measurable functions on X ,

$$\left| \int_X f_1, \dots, f_M \right| \leq \|f_1\|_{p_1}, \dots, \|f_M\|_{p_M}.$$

For typographical convenience, let

$$N = \binom{J}{K} \quad \text{and} \quad N_1 = \binom{J-1}{K-1}.$$

As usual, $l^{N_1}((\mathbf{Z}^+)^K)$ denotes all functions x on $(\mathbf{Z}^+)^K$ so that

$$\|x\|_{N_1} = \left(\sum_{i_1, \dots, i_K} |x(i_1, \dots, i_K)|^{N_1} \right)^{1/N_1} < \infty.$$

LEMMA 2.2. — Let $x_1, \dots, x_N \in l^{N_1}((\mathbf{Z}^+)^K)$. Then,

$$\left| \sum_{j \in (\mathbf{Z}^+)^J} x_1(P_1(j)) \dots x_N(P_N(j)) \right| \leq \|x_1\|_{N_1}, \dots, \|x_N\|_{N_1}.$$

Sketch of proof. — The key observation is that each $k \in \{1, \dots, J\}$ appears in precisely $N_1 = \binom{J-1}{K-1}$ distinct K -subsets of $\{1, \dots, J\}$. J successive applications of the N -Hölder inequality with $p_1 = \dots = p_N = 1/N_1$ yield the desired inequality. \square

Remark. — Another form of Lemma 2.2 was used in [1], where it served as a starting point for the study of the so-called « projectively bounded » multilinear forms on a Hilbert space (Lemma 1.2 of [1]).

Proof of (2.1). — First, by virtue of the canonical correspondence between $(\mathbf{Z}^+)^J$ and $F_{J,K}$, for notational simplicity we designate elements of $F_{J,K}$ as follows :

$$(\mathbf{Z}^+)^J \ni j \leftrightarrow \gamma(j) \in F_{J,K},$$

where

$$\gamma(j) = (\gamma_{P_1(j)}, \dots, \gamma_{P_N(j)}).$$

Let $\gamma(j_1), \dots, \gamma(j_s)$ be arbitrary elements of $F_{J,K}$ and write

$$(2.2) \quad \gamma = \gamma(j_1) \dots \gamma(j_s),$$

that is,

$$(2.2') \quad \gamma = \left(\prod_{k=1}^s \gamma_{P_1(j_k)}, \dots, \prod_{k=1}^s \gamma_{P_N(j_k)} \right)$$

(Recall that j_1, \dots, j_s need not be distinct.)

Let

$$\begin{aligned} L_1 &= \{P_1(j_1), \dots, P_1(j_s)\} \subset (\mathbf{Z}^+)^K \\ &\vdots \\ L_N &= \{P_N(j_1), \dots, P_N(j_s)\} \subset (\mathbf{Z}^+)^K. \end{aligned}$$

Next, define

$$V = \{j \in (\mathbf{Z}^+)^J : P_\alpha(j) \in L_\alpha \text{ for all } 1 \leq \alpha \leq N\}.$$

By the independence of F , the only way that γ can be obtained as a product of s elements from $F_{J,K}$ is for these elements to have in their 1st, ..., N^{th} coordinates the members of F that appear in the 1st, ..., N^{th} coordinates of (2.2'), respectively. That is, if $j'_1, \dots, j'_s \in (\mathbf{Z}^+)^J$ are so that

$$\gamma = \gamma(j'_1), \dots, \gamma(j'_s),$$

then $j'_1, \dots, j'_s \in V$. Therefore, the game plan is to estimate $|V|$, the « volume » of V , and exploit the fact that

$$(2.3) \quad r_s(F_{J,K}, \gamma) \leq |V|^s.$$

Let χ_1, \dots, χ_N be the characteristic functions of L_1, \dots, L_N in $(\mathbf{Z}^+)^K$. Clearly,

$$|V| = \sum_{j \in (\mathbf{Z}^+)^J} \chi_1(P_1(j)) \dots \chi_N(P_N(j)).$$

By Lemma 2.2,

$$(2.4) \quad \begin{aligned} |V| &\leq \|\chi_1\|_{N_1} \dots \|\chi_N\|_{N_1} \\ &\leq s^{N/N_1} = s^{J/K}. \end{aligned}$$

Combining (2.4) and (2.3), we obtain (2.1). □

3. Riesz products for $F_{J,K}$.

Let $F \subset \Gamma$ be a dissociate set. Let $F_{\mathcal{P}}^{J,K} = F^{J,K}$ be defined by (1.11) where \mathcal{P} is an arbitrary N -partition of F (as usual, our convention is that $N = \binom{J}{K}$). For any $\gamma \in \Gamma$ and $\theta \in \mathbf{R}$, define

$$\cos(\gamma + \theta) = (e^{i\theta}\gamma + e^{-i\theta}\bar{\gamma})/2.$$

Next, let $\varphi_1, \dots, \varphi_N \in l^\infty((\mathbf{Z}^+)^K)$ be so that

$$\|\varphi_1\|_\infty, \dots, \|\varphi_N\|_\infty \leq 1,$$

and write, for $1 \leq \alpha \leq N$,

$$\varphi_\alpha(k) = |\varphi_\alpha(k)|e^{i\theta_\alpha(k)}, \quad k \in (\mathbf{Z}^+)^K.$$

We now consider the following Riesz product :

$$(3.1) \quad \mu \sim \left[\prod_{i \in (\mathbf{Z}^+)^K} (1 + |\varphi_1(i)|\cos(\gamma_i^{(1)} + \theta_1(i))) \right] \dots \left[\prod_{i \in (\mathbf{Z}^+)^K} (1 + |\varphi_N(i)|\cos(\gamma_i^{(N)} + \theta_N(i))) \right].$$

As usual, $\|\mu\| = 1$ and the spectral analysis of μ yields the following.

LEMMA 3.1. — *Let $\varphi_1, \dots, \varphi_N \in l^\infty((\mathbf{Z}^+)^K)$, $\|\varphi_1\|_\infty, \dots, \|\varphi_N\|_\infty \leq 1$. Then, there is $\mu \in M(G)$ so that*

$$(3.2) \quad \|\mu\| \leq 2^N,$$

and

$$(3.3) \quad \hat{\mu}(\gamma_{P_1(j)}^{(1)} \dots \gamma_{P_N(j)}^{(N)}) = \varphi_1(P_1(j)) \dots \varphi_N(P_N(j)),$$

for all $j \in (\mathbf{Z}^+)^J$.

If a higher degree of independence is assumed for $F \subset \Gamma$, then the norm estimate in (3.2) can be correspondingly improved. We illustrate this in

$\Gamma = \mathbf{Z}$. Let $M > 1$ and $F = \bigcup_{\alpha=1}^N F_\alpha \subset \mathbf{Z}$ be an M -independent set where

$$F_\alpha = \{\lambda_k^{(\alpha)}\}_{k \in (\mathbf{Z}^+)^K}$$

(as usual, the F_α 's are infinite and mutually disjoint). Let $K_M \in L^1(T)$ be the M^{th} Fejer kernel :

$$K_M(t) = \sum_{l=-M}^M \left(1 - \frac{|l|}{M+1}\right) e^{ilt}.$$

Let $\varphi_1, \dots, \varphi_N$ be unimodular functions in $l^\infty((\mathbb{Z}^+)^K)$ given by

$$(3.4) \quad \varphi_\alpha(k) = e^{i\theta_\alpha(k)}$$

for $k \in (\mathbb{Z}^+)^K$. Replacing (3.1) by

$$\mu \sim \left[\prod_{i \in (\mathbb{Z}^+)^K} K_M(\lambda_i^{(1)}t + \theta_1(i)) \right] \dots \left[\prod_{i \in (\mathbb{Z}^+)^K} K_M(\lambda_i^{(N)}t + \theta_N(i)) \right],$$

we obtain

LEMMA 3.2. — Let $\varphi_1, \dots, \varphi_N \in l^\infty((\mathbb{Z}^+)^K)$ be given by (3.4). Then, there is $\mu \in M(T)$ so that

$$(3.5) \quad \|\mu\| \leq (1 + 1/M)^N$$

and

$$(3.6) \quad \hat{\mu}(\lambda_{P_1(j)}^{(1)} + \dots + \lambda_{P_N(j)}^{(N)}) = \varphi_1(P_1(j)) \dots \varphi_N(P_N(j)),$$

for all $j \in (\mathbb{Z}^+)^J$.

DEFINITION 3.3. — $F_1, F_2 \subset \Gamma$ are said to be harmonically separated if there is $\mu \in M(G)$ so that

$$\hat{\mu} = \begin{cases} 1 & \text{on } F_1 \\ 0 & \text{on } F_2 \setminus F_1. \end{cases}$$

LEMMA 3.4. — Let $J > K > 0, J' > K' > 0,$ and $N = \binom{J}{K}, N' = \binom{J'}{K'}$. Let $F \subset \Gamma$ be dissociate, and $\mathcal{P}, \mathcal{P}'$ be N, N' -partitions of F , respectively. Then $F_{\mathcal{P}}^{J,K}$ and $F_{\mathcal{P}'}^{J',K'}$ are harmonically separated.

Proof. — If $N \neq N'$, then an application (whose details are left to the reader) of a Riesz product such as the one given by (3.1) yields the desired conclusion. Assume that $N = N'$, and let

$$\mathcal{P} = \{F_1, \dots, F_N\}$$

be an N -partition for F , where, as usual, F_α is enumerated

$$F_\alpha = \{\gamma_i^{(\alpha)}\}_{i \in (\mathbf{Z}^+)^K}.$$

Let $k = \binom{J-1}{K-1}$. $\otimes \mathbf{Z}_k = \Omega_{(k)}$ will denote the compact direct product of \mathbf{Z}_k , and $\oplus \mathbf{Z}_k = \hat{\Omega}_{(k)}$ will be its (discrete) dual group, the direct sum of \mathbf{Z}_k . $E^{(k)} = \{r_n^{(k)}\}_{n=1}^\infty$ will be the system of k -Rademacher functions realized as characters in $\hat{\Omega}_{(k)}$; The n^{th} k -Rademacher function $r_n^{(k)}$ is defined by

$$r_n^{(k)}(\omega) = \exp(2\pi i \omega(n)/k),$$

for all

$$\omega \in \Omega_{(k)} = \{(\omega(j))_{j=1}^\infty : \omega(j) \in \{0, 1, \dots, k-1\}\}.$$

Observe that $E^{(k)}$ is $(k-1)$ -independent and that for all $r \in E^{(k)}$, $r^m = 1$ iff $m \equiv 0 \pmod{k}$. In the sequel below, k is fixed and, for the sake of simplicity, will be omitted from all superscripts and subscripts. As usual, for $\gamma \in \Gamma$ and $\theta \in \mathbf{R}$, write

$$\cos(\gamma + \theta) = (e^{i\theta}\gamma + e^{-i\theta}\bar{\gamma})/2.$$

For each $\omega = (\omega_1, \dots, \omega_J) \in \Omega^J$ define the Riesz product

$$\begin{aligned} \mu_\omega \sim & \left\{ \prod_{i \in (\mathbf{Z}^+)^K} \left[1 + \cos \left(\gamma_i^{(1)} + 2\pi \sum_{m=0}^K \omega_{1_m}(i(m)) \right) \right] \right\} \dots \\ & \dots \left\{ \prod_{i \in (\mathbf{Z}^+)^K} \left[1 + \cos \left(\gamma_i^{(N)} + 2\pi \sum_{m=1}^K \omega_{N_m}(i(m)) \right) \right] \right\}, \end{aligned}$$

where $i = (i(1), \dots, i(K)) \in (\mathbf{Z}^+)^K$, and (see Section 1)

$$S_\alpha = (\alpha_1, \dots, \alpha_K) \subset \{1, \dots, J\},$$

for $1 \leq \alpha \leq N$. Next, we integrate over Ω^J the $M(G)$ -valued function whose value at $\omega \in \Omega^J$ is μ_ω :

$$(3.4.1) \quad \mu = \int_{\Omega^J} \mu_\omega d\omega \in M(G).$$

The spectral analysis of μ yields the following:

$$\begin{aligned} (3.4.2) \quad \hat{\mu}(\gamma_{i_1}^{(1)} \dots \gamma_{i_N}^{(N)}) = & 2^{-N} \int_{\Omega^J} \left[\prod_{m=1}^K r_{i_1(m)}(\omega_{1_m}) \right] \dots \\ & \dots \left[\prod_{m=1}^K r_{i_N(m)}(\omega_{N_m}) \right] d\omega_1 \dots d\omega_J, \end{aligned}$$

for all $i_1, \dots, i_N \in (\mathbf{Z}^+)^K$.

Since every $l \in \{1, \dots, J\}$ appears in precisely $k = \binom{J-1}{K-1} S_\alpha$'s, and

$$\int_{\Omega} r^m = \begin{cases} 1 & \text{if } m \equiv 0 \pmod{k} \\ 0 & \text{otherwise} \end{cases}, \quad r \in E^{(k)}$$

we obtain from (3.4.2) that

$$\hat{\mu}(\gamma_{i_1}^{(1)} \dots \gamma_{i_N}^{(N)}) = \begin{cases} 2^{-N} & \text{if there is } j \in (\mathbf{Z}^+)^J \text{ so that} \\ & P_\alpha(j) = i_\alpha \text{ for all } 1 \leq \alpha \leq N; \\ 0 & \text{otherwise.} \end{cases}$$

□

4. The Λ^β property.

Proof of Theorem 1.6, part (a). — The right hand inequality in (1.9) follows immediately from (2.1) via the following.

LEMMA 4.1. (Théorème 3 in [2]). — *Let s be a positive integer. Then, for $F \subset \Gamma$,*

$$A(2s, F) \leq [\sup_{\gamma \in F} r_s(F, \gamma)]^{1/2s}.$$

We now prove that there is $\eta_{J,K} > 0$ so that

$$\eta_{J,K} q^{J/2K} \leq A(q, F_{J,K}),$$

where $F \subset \Gamma$ is independent (the idea for the argument that follows originates — as far as we can determine — in [5]). Let $n > 0$ be arbitrary, and denote

$$V_n = \{j = (j_1, \dots, j_J) \in (\mathbf{Z}^+)^J : 1 \leq j_1, \dots, j_J \leq n\}.$$

Let g be the trigonometric polynomial defined by

$$g = \sum_{j \in V_n} (\gamma_{P_1(j)} \dots \gamma_{P_N(j)})$$

(as always, $N = \binom{J}{K}$).

Clearly,

$$(4.1) \quad \|g\|_2 = n^{J/2}.$$

Next, let h be the Riesz product defined by

$$h(t_1, \dots, t_N) = \left[\prod_{j \in U_n} (1 + \cos \gamma_j(t_1)) \right] \dots \left[\prod_{j \in U_n} (1 + \cos \gamma_j(t_N)) \right],$$

where $U_n = \{i = (i_1, \dots, i_K) \in (\mathbf{Z}^+)^K : 1 \leq i_1, \dots, i_K \leq n\}$,

$\cos \gamma = (\gamma + \bar{\gamma})/2$, and $t_1, \dots, t_N \in G$. We observe that

$$(4.2) \quad \|h\|_1 = 1,$$

and

$$(4.3) \quad \|h\|_2 \leq \|h\|_\infty \leq 2^{Nn^K}.$$

Combining (4.2) and (4.3), we obtain for any $1 < R < 2$

$$(4.4) \quad \|h\|_R \leq 2^{Nn^K/R'}$$

($1/R + 1/R' = 1$).

Also, note that

$$(4.5) \quad \hat{h} = 1/2^N \quad \text{on} \quad \{(\gamma_{P_1(i)}) \dots; \gamma_{P_N(i)}\}_{i \in V_n}.$$

Letting $R' = n^K$ and applying (4.4) and (4.5), we deduce

$$\begin{aligned} 2^{-Nn^j} = g * h(0) &\leq \|g\|_{n^K} \|h\|_{(n^K)}, \\ &\leq \|g\|_{n^K} \cdot 2^N. \end{aligned}$$

Therefore (from (4.1)),

$$(4.6) \quad 2^{-2N} \|g\|_2 n^{j/2} \leq \|g\|_{n^K}.$$

n was arbitrary, and the left hand inequality of (1.9) follows. □

Completion of the proof of Theorem 1.3. – The left hand inequality in (1.4) follows from Lemma 4.1 and (4.6). □

Exploiting (1.9) of Theorem 1.6 via an « averaging » procedure, we now prove part (a) of Corollary 1.7 :

Let $F \subset \Gamma$ be a dissociate set, and $F_{\mathcal{F}}^{j,K} = F^{j,K}$ be defined by (1.11). Let $f \in L_{F^{j,K}}^2(G)$ be given by

$$f \sim \sum_{j \in (\mathbf{Z}^+)^j} a_j \gamma_{P_1(i)}^{(1)} \dots \gamma_{P_N(i)}^{(N)}.$$

We select $E = \{e_i\}_{i \in (\mathbf{Z}^+)^K}$, an infinite independent set in some $\Gamma_0 (= \hat{G}_0)$,

and « randomize » the Fourier coefficients of f as follows. Let $t = (t_1, \dots, t_N) \in G_0^N$, and

$$f_t \sim \sum_{j \in \mathbf{Z}^{+J}} a_j e_{P_1(j)}(t_1) \dots e_{P_N(j)}(t_N) \gamma_{P_1(j)}^{(1)} \dots \gamma_{P_N(j)}^{(N)}.$$

For each $t = (t_1, \dots, t_N) \in G_0^N$, let $\mu_t \in M(G)$ be so that

$$\hat{\mu}(\gamma_{P_1(j)}^{(1)} \dots \gamma_{P_N(j)}^{(N)}) = \overline{e_{P_1(j)}(t_1) \dots e_{P_N(j)}(t_N)},$$

and

$$(4.7) \quad \|\mu_t\| \leq 2^N$$

(Lemma 3.1).

Observe that

$$f = f_t * \mu_t.$$

Therefore, for all $t \in G_0^N$ and $q > 2$,

$$(4.8) \quad \|f\|_q^q \leq \|f_t\|_q^q \|\mu_t\|^q \leq 2^{Nq} \|f_t\|_q^q.$$

It follows from (4.8) that

$$\|f\|_q^q \leq 2^{Nq} \int_{G_0^N} \left(\int_G |f_t|^q \right) dt.$$

Interchanging the order of integration, and applying the right hand inequality of (1.9), we deduce that

$$(4.9) \quad \begin{aligned} \|f\|_q &\leq 2^N A(q, E_{J,K}) \|f\|_2 \\ &\leq 2^N q^{J/2K} \|f\|_2. \end{aligned}$$

(4.9) proves the right hand inequality in (1.12). To show the left hand inequality, we follow a computation identical to the one used in showing the left hand inequality of (1.9). The proof of 1.7 (a) is complete. \square

We assume now that Γ is such that for every $M \geq 1$, Γ contains an infinite M -independent set. In fact, for concreteness' sake, assume $\Gamma = \mathbf{Z}$.

LEMMA 4.3. — *Let $J > K > 0$ be arbitrary. There exists $F \subset \mathbf{Z}$ so that for any \mathcal{P} , an N -partition of F , $\eta_{J,K} q^{J/2K} \leq A(q, F_{\mathcal{P}}^{J,K}) \leq 4q^{J/2K}$, for some $\eta_{J,K} > 0$ and all $q > 2$.*

Proof. — Let $M > 0$ be so that

$$[(M+1)/M]^N \leq 4$$

$\left(N = \binom{J}{K} \right)$, and let $F \subset Z$ be an M -independent set. Run through the argument leading to (4.9), where an application of Lemma 3.2 replaces that of 3.1. The result is an improved estimate (over (4.9)) :

$$A(q, F_{\mathcal{F}}^{J,K}) \leq 4q^{J/2K}. \quad \square$$

Proof of Corollary 1.7, part (c). — Again, without loss of generality, we shall work in Z . Let $\beta \in [1, \infty)$ be arbitrary, and $J_n > K_n > 0$, $n = 1, \dots$, be so that $(J_n/K_n)_{n=1}^\infty$ is a monotonically increasing sequence converging to β . Predictably, the strategy is to let $E_n \subset Z$ be the « fractional » sum corresponding to $J_n > K_n > 0$ given by Lemma 4.3, and then select finite sets $F_n \subset E_n$ with the following properties :

$$(i) \quad A\left(q, \bigcup_{k=1}^\infty l_k F_k\right) \leq 8q^{\beta/2},$$

where $l_{k+1} \gg l_k > 0$ are chosen appropriately

$$(l_n F_n = \{l_n \lambda_j^{(n)} : \lambda_j^{(n)} \in F_n\});$$

(ii) the F_n 's are sufficiently « thick » so that $\bigcup_{k=1}^\infty l_k F_k$ is not $\Lambda^{\beta-\varepsilon}$ for any $\varepsilon > 0$.

We start with

$$E_n = \{\lambda_{P_1(j)}^{(L_n)} + \dots + \lambda_{P_{N_n}(j)}^{(N_n)}\}_{j \in (Z^+)^{J_n}},$$

a « fractional » sum corresponding to $J_n > K_n > 0$ as in Lemma 4.3 $\left(N_n = \binom{J_n}{K_n} \right)$. Next, fix a positive sequence $(\varepsilon_n)_{n=1}^\infty$ that is monotonically converging to 0. For each $n > 0$ select $L_n \in Z^+$ so that

$$(4.10) \quad L_n^{\varepsilon_n K_n} \geq n 16^{N_n}.$$

Let

$$F_n = \{\lambda_{P_1(j)}^{(L_n)} + \dots + \lambda_{P_{N_n}(j)}^{(N_n)}\}_{j \in V_{L_n}} \subset E_n,$$

where

$$V_{L_n} = \{j = (j_1, \dots, j_{J_n}) \in (Z^+)^{J_n} : 1 \leq j_\alpha \leq L_n\}.$$

Observe that by running through an argument identical to the one that led to (4.6), we obtain

$$g_n \in L_{F_n}^2(T)$$

so that

$$(4.11) \quad \|g_n\|_{(L_n)^{K_n}} \geq 4^{-N_n} (L_n)^{J_n/2} \|g_n\|_2.$$

We now determine by induction that « dilation » factors of the F_n 's (l_n in (i) above) : Let $l_1 = 1$, and suppose that l_1, \dots, l_k were determined so that

$$(4.12) \quad A\left(q, \bigcup_{n=1}^k l_n F_n\right) \leq 8q^{J_k/2K_k},$$

for all $2 < q < \infty$. Observe that the cardinality of $\bigcup_{n=1}^k l_n F_n \cup F_{k+1}$ is

$\sum_{n=1}^{k+1} (L_n)^{J_n}$, and, therefore, however we choose l_{k+1} , we will have

$$(4.13) \quad A\left(q, \bigcup_{n=1}^{k+1} l_n F_n\right) \leq \left[\sum_{n=1}^{k+1} (L_n)^{J_n}\right]^{1/2},$$

for all $2 < q < \infty$. Guided by (4.13), we choose M_k so that

$$(4.14) \quad \left[\sum_{n=1}^{k+1} (L_n)^{J_n}\right]^{1/2} \leq 8M_k^{J_{k+1}/2K_{k+1}}.$$

Finally, select l_{k+1} so that

$$(4.15) \quad (\lambda_1 \pm \dots \pm \lambda_r) + (v_1 \pm \dots \pm v_{r'}) \neq 0$$

for all $\{\lambda_i\}_{i=1}^r \subset \bigcup_{n=1}^k l_n F_n$ and $\{v_j\}_{j=1}^{r'} \subset l_{k+1} F_{k+1}$, where $r, r' \leq 2M_k$ (we allow repetitions in $\{\lambda_i\}_{i=1}^r$ and $\{v_j\}_{j=1}^{r'}$).

Claim. $A\left(q, \bigcup_{n=1}^{k+1} l_n F_n\right) \leq 8q^{J_{k+1}/2K_{k+1}}$, for all $2 < q < \infty$.

Let f be a trigonometric polynomial with spectrum in $\bigcup_{n=1}^{k+1} l_n F_n$ and write

$f = f_1 + f_2$ where

$$\text{spectrum}(f_1) \subseteq \bigcup_{n=1}^k l_n F_n,$$

$$\text{spectrum}(f_2) \subseteq l_{k+1} F_{k+1}.$$

It follows from (4.15) that

$$(4.16) \quad \|f_1 + f_2\|_{2m}^{2m} = \| |f_1|^2 + |f_2|^2 \|_m^m,$$

for all $m \leq M_k/2$. Therefore, it follows from (4.16) that

$$\|f_1 + f_2\|_{2m}^2 \leq \|f_1\|_{2m}^2 + \|f_2\|_{2m}^2.$$

Applying the induction hypothesis, and the monotonicity of $(J_n/K_n)_{n=1}^\infty$, we obtain

$$(4.17) \quad \|f_1 + f_2\|_{2m} \leq 8(2m^{J_{k+1}/2K_{k+1}})\|f_1 + f_2\|_2.$$

Combining (4.14) and (4.17), we obtain the claim. Combining the « claim »,

(4.11) and (4.10), we obtain that $\bigcup_{n=1}^\infty I_n F_n$ is exactly Λ^β .

By choosing $J_n > K_n > 0$, where $(J_n/K_n)_{n=1}^\infty$ is a monotonically decreasing sequence converging to β , and following a procedure similar to the above, we obtain a set in \mathbf{Z} which is exactly non- Λ^β . The details are left to the reader. □

5. The p -Sidonicity property.

Proof of Theorem 1.6, part (b). — First, for the sake of economy in notation, we adopt the following conventions : Let

$$U = \{i_1, \dots, i_M\} \subset \{1, \dots, J\} = F, \quad \text{and} \quad P_U : (\mathbf{Z}^+)^J \rightarrow (\mathbf{Z}^+)^M$$

be the projection defined by

$$P_U((j_1, \dots, j_J)) = (j_{i_1}, \dots, j_{i_M}).$$

\sum_U will denote summation over $P_U((\mathbf{Z}^+)^J)$. For example,

$$\sum_U a_{j_1, \dots, j_J} = \sum_{j_{i_1}, \dots, j_{i_M}} a_{j_1, \dots, j_J}.$$

Also, in what follows γ_{S_x} will mean $\gamma_{P_x(i)}$ whenever these occur in summands. For example,

$$\sum_{j=(j_1, \dots, j_J) \in (\mathbf{Z}^+)^J} \gamma_{P_1(i)} \cdots \gamma_{P_N(i)} \equiv \sum_{j_1, \dots, j_J} \gamma_{S_1} \cdots \gamma_{S_N}.$$

Let $E \subset \Gamma$ be an independent set, and $E_{J,K} \subset \Gamma^N$ be as in (1.1) (as usual, $N = \binom{J}{K}$).

LEMMA 5.1. — Let f be a trigonometric polynomial in $C_{E_{J,K}}(G^N)$ given by

$$f = \sum_{j_1, \dots, j_J} a_{j_1, \dots, j_J}(\gamma_{S_1}, \dots, \gamma_{S_N}).$$

Then,

$$2^{J/K} \|f\|_\infty \geq \sum_{S_\alpha} \left(\sum_{\sim S_\alpha} |a_{j_1, \dots, j_J}|^2 \right)^{1/2}$$

for all $\alpha = 1, \dots, N$.

Proof. — Let $1 \leq \alpha \leq N$ be arbitrary. Since E is a 1-Sidon set with Sidon constant = 1, it is easy to see that by a proper choice of $t_\alpha \in G$ we obtain

$$(5.1.1) \quad \|f\|_\infty \geq \sup_{t_1, \dots, t_{\alpha-1}, t_{\alpha+1}, \dots, t_N} \sum_{S_\alpha} \left| \sum_{\sim S_\alpha} a_{j_1, \dots, j_J} \gamma_{S_1}(t_1) \dots \gamma_{S_N}(t_N) \right|.$$

Since the sup-norm dominates the L^1 -norm, it follows from (5.1.1) that

$$\|f\|_\infty \geq \sum_{S_\alpha} \int_{G^{N-1}} \left| \sum_{\sim S_\alpha} a_{j_1, \dots, j_J}(\gamma_{S_1}, \dots, \gamma_{S_{\alpha-1}}, \gamma_{S_{\alpha+1}}, \dots, \gamma_{S_N}) \right|$$

(we make the obvious modification for $\alpha = 1$ and $\alpha = N$). Since $E_{J,K}$ is a $\Lambda(2)$ set whose $\Lambda(2)$ constant is bounded by $2^{J/K}$ (this follows from part (a) of Theorem 1.6), we obtain

$$2^{J/K} \|f\|_\infty \geq \sum_{S_\alpha} \left(\sum_{\sim S_\alpha} |a_{j_1, \dots, j_J}|^2 \right)^{1/2}. \quad \square$$

In a previous version of this manuscript, by following a multidimensional version of Littlewood’s rearrangement argument (see p. 168 of [7] and Lemma 3 of [6]) we proved that

$$(5.1) \quad \sum_{\alpha=1}^N \sum_{S_\alpha} \left(\sum_{\sim S_\alpha} |a_{j_1, \dots, j_J}|^2 \right)^{1/2} \geq \frac{1}{K!} \left(\sum_{j_1, \dots, j_J} |a_{j_1, \dots, j_J}|^{\frac{2J}{K+J}} \right)^{\frac{K+J}{2J}}.$$

Then, combining Lemma 5.1 and (5.1), we deduce that $E_{J,K}$ is a $2J/(K+J)$ -Sidon and that

$$(5.2) \quad D\left(\frac{2J}{K+J}, E_{J,K}\right) \leq 2^{J/K} \binom{J}{K} K!.$$

The estimate in (5.2), however, is not as tight as we would like it to be. In order to obtain the existence of sets that are exactly p -Sidon, for any $p \in (1,2)$, we

require the sharper Lemma 5.3 below. We are grateful to S. Kaijser at Uppsala University for pointing out to us that Littlewood's classical inequality (the case $J = 2$ and $K = 1$ in (5.1)) can be proved without Littlewood's rearrangement argument. Indeed, this is a key observation in the demonstration below. In the course of the proof of 5.3 we greatly benefited also from stimulating conversations with M. Benedicks at the Institut Mittag-Leffler.

We require two basic inequalities : Minkowski's inequality which, to facilitate referencing, we state below, and the M-Hölder inequality that is given in section 2 ((2.1)).

(5.2) *Minkowski's inequality.* — Let X and Y be measure spaces and g be a measurable function on $X \times Y$. For any $1 < r < \infty$,

$$(5.3) \quad \left(\int_X \left(\int_Y |g(x,y)| \right)^r \right)^{1/r} \leq \int_Y \left(\int_X |g(x,y)|^r \right)^{1/r}.$$

LEMMA 5.3. — Let $J > K > 0$.

$$(5.4) \quad \left(\sum_{j_1, \dots, j_J} |a_{j_1, \dots, j_J}|^{\frac{2J}{K+J}} \right)^{\frac{K+J}{2J}} \leq \prod_{\alpha=1}^N \left[\sum_{S_\alpha} \left(\sum_{\sim S_\alpha} |a_{j_1, \dots, j_J}|^2 \right)^{1/2} \right]^{1/N} \\ \left(N \equiv \binom{J}{K} \right).$$

Proof. — We prove (5.4) by induction on J . We start with the case $J = 2$ and $K = 1$. Write

$$(5.3.1) \quad \sum_{i,j} |a_{ij}|^{4/3} = \sum_{i,j} |a_{ij}|^{2/3} |a_{ij}|^{2/3}.$$

By applying 2-Hölder's inequality to the sum over j with $p_1 = 3/2$ and $p_2 = 3$ for the first and second factors, respectively, in the summand of the right side of (5.3.1), we obtain

$$(5.3.2) \quad \sum_{i,j} |a_{ij}|^{4/3} \leq \sum_i \left(\sum_j |a_{ij}| \right)^{2/3} \left(\sum_j |a_{ij}|^2 \right)^{1/3}.$$

Next, applying 2-Hölder's inequality to the sum over i with p_1 and p_2 for the second and first factors, respectively, in the summand of the right side of

(5.3.2.), we obtain

$$(5.3.3) \quad \sum_{i,j} |a_{ij}|^{4/3} \leq \left[\sum_i \left(\sum_j |a_{ij}| \right)^2 \right]^{1/3} \left[\sum_i \left(\sum_j |a_{ij}|^2 \right) \right]^{2/3}.$$

From Minkowski's inequality, we have

$$\left(\sum_i \left(\sum_j |a_{ij}| \right)^2 \right)^{1/2} \leq \sum_j \left(\sum_i |a_{ij}|^2 \right)^{1/2},$$

and, therefore (from (5.3.3)), we deduce

$$(5.3.4) \quad \left(\sum_{i,j} |a_{ij}|^{4/3} \right)^{3/4} \leq \left[\sum_i \left(\sum_i |a_{ij}|^2 \right)^{1/2} \right]^{1/2} \left[\sum_i \left(\sum_j |a_{ij}|^2 \right)^{1/2} \right]^{1/2}.$$

(5.3.4) starts the inductive proof of (5.4), and we now assume that $J > 2$ and that (1.4) holds for all $J_1 < J$. We write

$$(5.3.5) \quad \sum_{j_1, \dots, j_J} |a_{j_1, \dots, j_J}|^{2J} = \sum_{j_1, \dots, j_J} |a_{j_1, \dots, j_J}|^{2} \dots |a_{j_1, \dots, j_J}|^{2}.$$

For notational reasons, we shall write

$$f_n(j_1, \dots, j_J) = a_{j_1, \dots, j_J}$$

for the n -th factor in the summand of the right hand side of (5.3.5) (even though $f_1 \equiv \dots \equiv f_J$), and whenever there is no confusion we shall write merely f_n :

$$(5.3.5') \quad \sum_{j_1, \dots, j_J} |a_{j_1, \dots, j_J}|^{2J} = \sum_{j_1, \dots, j_J} |f_1|^{2} \dots |f_J|^{2}.$$

Now, apply J-Hölder's inequality to the sum over j_j with

$$p_1 \equiv p = (K + J)/2$$

for f_1 and $p_2 = \dots = p_J \equiv q = \frac{(J-1)(K+J)}{(K+J-2)}$ for f_2, \dots, f_J and obtain that the right side of (5.3.5') is majorized by

$$(5.3.6) \quad \sum_{j_1, \dots, j_{J-1}} \left(\sum_{j_J} |f_1| \right)^{\frac{2}{(K+J)}} \prod_{n=2}^J \left(\sum_{j_J} |f_n| \right)^{\frac{2(J-1)}{(K+J-2)}}^{\frac{K+J-2}{(J-1)(K+J)}}.$$

Next, apply J-Hölder's inequality to the sum over j_{j-1} with p for the factor containing f_2 and q for the remaining factors in the summand of (5.3.6), and obtain that (5.3.6) is majorized by

$$(5.3.7) \quad \sum_{j_1, \dots, j_{J-2}} \left\{ \left[\sum_{j_{j-1}} \left(\sum_{j_j} |f_{j_1}| \right)^{\frac{2(J-1)}{(K+J-2)}} \right]^{\frac{(K+J-2)}{(J-1)(K+J)}} \cdot \left[\sum_{j_{j-1}} \left(\sum_{j_j} |f_{j_2}| \right)^{\frac{2(J-1)}{(K+J-2)}} \right]^{\frac{(K+J-2)}{2(J-1)}} \right]^{\frac{2}{(K+J)}} \cdot \prod_{n=3}^J \left[\sum_{j_{j-1}, j_j} |f_{j_n}|^{\frac{2(J-1)}{(K+J-2)}} \right]^{\frac{(K+J-2)}{(J-1)(K+J)}} \right\}.$$

We continue in this fashion : At then n -th step, $n > 2$, we apply the J-Hölder inequality to the sum over j_{j-n+1} with p for the factor that contains f_n and with q for the remaining factors. After J such operations we obtain that (5.3.5') is majorized by

$$(5.3.8) \quad \left[\sum_{j_1, \dots, j_{j-1}} \left(\sum_{j_j} |f_{j_1}| \right)^{\frac{2(J-1)}{(K+J-2)}} \right]^{\frac{(K+J-2)}{(J-1)(K+1)}} \dots \left[\sum_{j_1, \dots, j_{j-n}} \left(\sum_{j_{j-n+1}} \left(\sum_{j_{j-n+2}, \dots, j_j} |f_{j_n}|^{\frac{2(J-1)}{(K+J-2)}} \right)^{\frac{(K+J-2)}{2(J-1)}} \right)^{\frac{2(J-1)}{(K+J-2)}} \right]^{\frac{(K+J-2)}{(J-1)(K+1)}} \dots \left[\sum_{j_1} \left(\sum_{j_2, \dots, j_j} |f_{j_1}|^{\frac{2(J-1)}{(K+J-2)}} \right)^{\frac{(K+J-2)}{2(J-1)}} \right]^{\frac{2}{(K+J)}}.$$

Now, apply Minkowski's inequality to each of the first $(J - 1)$ factors in (5.3.8) as follows : To the first factor, apply Minkowski's inequality (as stated above) with $|g| = |f_1|$,

$$\int_X = \sum_{j_1, \dots, j_{j-1}}, \quad \int_Y = \sum_{j_j}, \quad \text{and} \quad r = \frac{2(J-1)}{(K+J-2)}.$$

To the n -th factor, $1 < n < J$, apply Minkowski's inequality with

$$|g| = \sum_{j_{j-n+2}, \dots, j_j} |f_{j_n}|^{\frac{2(J-1)}{(K+J-2)}},$$

$$\int_X = \sum_{j_1, \dots, j_{j-n}}, \quad \int_Y = \sum_{j_{j-n+1}},$$

and again with $r = \frac{2(J-1)}{(K+J-2)}$.

We therefore obtain

$$(5.3.9) \quad \left(\sum_{j_1, \dots, j_J} |a_{j_1, \dots, j_J}|^{\frac{2J}{K+J}} \right)^{\frac{(K+J)}{2J}} \leq \prod_{n=1}^J \left(\sum_{j_n} \left(\sum_{\sim j_n} |a_{j_1, \dots, j_J}|^{\frac{2(J-1)}{K+J-2}} \right)^{\frac{(K+J-2)}{2(J-1)}} \right)^{1/J}$$

$\left(\sum_{\sim j_n} \right)$ denotes summation with respect to indices in $\{j_1, \dots, j_J\} \setminus \{j_n\}$. If $K = 1$, (5.3.9) reduces to (5.4) and the lemma is proved. We now assume $K > 1$ and apply the induction hypothesis that (5.4) holds for $J - 1$ and $K - 1$ for each of the J factors on the right hand side of (5.3.9). Namely, for each $1 \leq n \leq J$ we have

$$(5.3.10) \quad \sum_{j_n} \left(\sum_{\sim j_n} |a_{j_1, \dots, j_J}|^{\frac{2(J-1)}{K+J-2}} \right)^{\frac{(K+J-2)}{2(J-1)}} \leq \sum_{j_n} \prod_{\alpha=1}^{N_1} \left(\sum_{S_\alpha^n} \left(\sum_{\sim S_\alpha^n} |a_{j_1, \dots, j_J}|^2 \right)^{1/2} \right)^{1/N_1}$$

[Recall that $N_1 \equiv \binom{J-1}{K-1} \cdot \{S_1^n, \dots, S_{N_1}^n\}$ denotes the collection of all $(K-1)$ -subsets of $\{1, \dots, J\} \setminus \{n\}$; $\sum_{S_\alpha^n}$ denotes summation with respect to the indices $j_{\alpha_1}, \dots, j_{\alpha_{K-1}}$, where $S_\alpha^n = (\alpha_1, \dots, \alpha_{K-1})$; $\sum_{\sim S_\alpha^n}$ denotes summation with respect to the remaining indices (except j_n).] We now apply N_1 -Hölder inequality, on the right hand side of (5.3.10), to the sum over j_n with $p_1 = \dots = p_{N_1} = 1/N_1$ for each of the $N_1 \equiv \binom{J-1}{K-1}$ factors in the summand. Combining this application with (5.3.9), we obtain that the left hand side of (5.3.9) is majorized by

$$(5.3.11) \quad \prod_{n=1}^J \prod_{\alpha=1}^{N_1} \left[\sum_{j_n} \sum_{S_\alpha^n} \left(\sum_{\sim S_\alpha^n} |a_{j_1, \dots, j_J}|^2 \right)^{1/2} \right]^{1/N_1}.$$

Finally, observe that

$$(5.3.12)$$

$$\{\{1\} \cup S_1^1, \{1\} \cup S_2^1, \dots, \{1\} \cup S_{N_1}^1, \dots, \{n\} \cup S_1^n, \dots, \{n\} \cup S_{N_1}^n, \dots, \{J\} \cup S_1^J, \dots, \{J\} \cup S_{N_1}^J\} = \{S_1, \dots, S_N\}.$$

Also, notice that each set in the enumeration of the collection on the left hand side of (5.3.12) occurs precisely K times. Therefore, the expression in (5.3.11) equals

$$\prod_{\alpha=1}^{N_1 \cdot J/K} \left(\sum_{S_\alpha} \left(\sum_{\sim S_\alpha} |a_{j_1, \dots, j_J}|^2 \right)^{1/2} \right)^{\frac{K}{N_1 \cdot J}} = \prod_{\alpha=1}^N \left(\sum_{S_\alpha} \left(\sum_{\sim S_\alpha} |a_{j_1, \dots, j_J}|^2 \right)^{1/2} \right)^{1/N},$$

and the proof of the Lemma is complete. □

Combining Lemmas 5.1 and 5.3, we deduce that

$$D(2J/(K+J), E_{J,K}) \leq 2^{J/K}.$$

We now proceed to show that

$$D([2J/(K+J)] - \varepsilon, E_{J,K}) = \infty$$

for all $\varepsilon > 0$. The argument that follows is similar to the one used in showing that $E_{J,K}$ is not $\Lambda^{J/K-\varepsilon}$ for any $\varepsilon > 0$ (and is an adaptation of a proof used in 2.7 of [4]). We use the fact that if $F \subset \Gamma$ is p -Sidon, then there is $B > 0$ so that for all $f \in L_F^1(G)$

$$(5.5) \quad \|f\|_b \leq B \sqrt{b} \|\hat{f}\|_a$$

for all $1 < b < \infty$, where

$$a = 2p/(3p - 2)$$

(see (9) in [3]). Let $n > 0$ be arbitrary, and let

$$V_n = \{j = (j_1, \dots, j_J) \in (\mathbb{Z}^+)^J : 1 \leq j_1, \dots, j_J \leq n\}.$$

Let

$$g = \sum_{j \in V_n} (\gamma_{P_1(j)} \dots \gamma_{P_N(j)}).$$

Clearly,

$$(5.6) \quad \|\hat{g}\|_a = n^{J/a}.$$

Next, let $U_n = \{i = (i_1, \dots, i_K) \in (\mathbb{Z}^+)^K : 1 \leq i_1, \dots, i_K \leq n\}$ and define

$$h(t_1, \dots, t_N) = \left[\prod_{i \in U_n} (1 + \cos \gamma_i(t_1)) \right] \dots \left[\prod_{i \in U_n} (1 + \cos \gamma_i(t_N)) \right].$$

As in section 4, we conclude (see (4.2)-(4.4)) that for any $1 < R < 2$

$$\|h\|_R \leq 2^{\frac{Nn^K}{R}},$$

and

$$\hat{h} = 1/2^N \quad \text{on} \quad \{(\gamma_{P_1(j)} \dots \gamma_{P_N(j)})\}_{j \in V_n}.$$

Therefore, as in section 4,

$$(5.7) \quad 2^{-Nn^J} = |g * h(0)| \leq \|g\|_{n^k} \|h\|_{(n^k)}, \\ \leq \|g\|_{n^k} \cdot 2^N.$$

If $E_{j,k}$ is p -Sidon, it follows from (5.5), (5.6) and (5.7) that

$$(5.8) \quad 2^{-2Nn^J} \leq Bn^{\binom{K}{2} + \binom{J}{a}}.$$

But, (5.8) holds for all $n > 0$ only if $p \geq \frac{2J}{(K+J)}$. The proof of part *b*) of Theorem 1.6 is complete. □

Proof of Corollary 1.7, part (b). — Let $F \subset \Gamma$ be dissociate and $F_{\mathcal{P}}^{j,k} = F^{j,k}$ be given by (1.11), for an arbitrary \mathcal{P} . Let f be a trigonometric polynomial in $C_{F,j,k}(G)$ given by

$$f = \sum_{j \in (\mathbb{Z}^+)^J} a_j \gamma_{P_1(j)}^{(1)} \cdots \gamma_{P_N(j)}^{(N)}.$$

Select $E = \{e_i\}_{i \in (\mathbb{Z}^+)^K}$, an infinite independent set in some $\Gamma_0 (= \hat{G}_0)$. Let $t = (t_1, \dots, t_N) \in G_0^N$ be arbitrary, and $\mu_t \in M(G)$ be so that

$$\hat{\mu}_t(\gamma_{P_1(j)}^{(1)} \cdots \gamma_{P_N(j)}^{(N)}) = e_{P_1(j)}(t_1) \cdots e_{P_N(j)}(t_N)$$

for all $j \in (\mathbb{Z}^+)^J$, and

$$\|\mu_t\| \leq 2^N$$

(Lemma 3.1).

Therefore,

$$\sup_{t \in G_0^N} \left| \sum_{j \in (\mathbb{Z}^+)^J} a_j e_{P_1(j)}(t_1) \cdots e_{P_N(j)}(t_N) \right| = \sup_{t \in G_0^N} |\langle f, \mu_t \rangle| \leq 2^N \|f\|_{\infty}.$$

By Theorem 1.6, part *(b)*, we have

$$(5.9) \quad \left(\sum_{j \in (\mathbb{Z}^+)^J} |a_j|^{\frac{2J}{(K+J)}} \right)^{\frac{(K+J)}{2J}} \leq 2^{J/K} \cdot 2^N \|f\|_{\infty}.$$

This proves that $F^{j,k}$ is $\frac{2J}{(K+J)}$ – Sidon. To show that $F^{j,k}$ is exactly

$\frac{2J}{(K+J)}$ – Sidon we follow the same route that was used to prove the corresponding fact for $E_{j,k}$. □

COROLLARY 5.4. — Let $J \geq K > 0$ and $J' \geq K' > 0$ be arbitrary. Let $F \subset \Gamma$ be dissociate and $\mathcal{P}, \mathcal{P}'$ be $\begin{pmatrix} J \\ K \end{pmatrix}, \begin{pmatrix} J' \\ K' \end{pmatrix}$ — partitions of F respectively. Then $F_{\mathcal{P}}^{J,K} \cup F_{\mathcal{P}'}^{J',K'}$ is exactly a p -Sidon set, where

$$p = \max \left\{ \frac{2J}{(K+J)}, \frac{2J'}{(K'+J')} \right\}.$$

Proof. — Apply Lemma 3.4. □

Proof of Corollary 1.7, part (d). — We consider two cases.

I. Γ contains elements with arbitrarily large order. As usual, we shall assume that $\Gamma = \mathbf{Z}$. First, observe that for any $J > K > 0$ the (M-independent) set $F \subset \mathbf{Z}$ given in Lemma 4.3 has the property

$$(5.10) \quad D(2J/(K+J), F^{J,K}) \leq 16 \cdot 2^{J/K}.$$

(5.10) is achieved by applying, en route to (5.9), Lemmas 3.2 and 4.3 in place of Lemma 3.1 and Theorem 1.6 (b), respectively, whence 2^N is replaced by 4 and $2^{J/K}$ by $4 \cdot 2^{J/K}$ in (5.9). Let $p \in [1, \infty)$, and $J_n > K_n > 0$ be so that $(J_n/K_n)_{n=1}^\infty$ is a monotonically increasing sequence converging to $p/(2-p)$. For each $n > 0$, let $F_n \subset \mathbf{Z}$ be the set given in Lemma 4.3 corresponding to $J_n > K_n > 0$. Select finite sets $E_n \subset F_n$ so that

$$(5.11) \quad D([2J_n/(J_n + K_n)] - 1/n, E_n) > n.$$

Next, determine $d_n \in \mathbf{Z}^+$ with the following property : Whenever f is a trigonometric polynomial with spectrum in $\bigcup_{n=1}^\infty d_n E_n$, then

$$(5.12) \quad 6 \|f\|_\infty \geq \sum_{n=1}^\infty \|f_n\|_\infty,$$

where $f = \sum_{n=1}^\infty f_n$ and $\text{spectrum}(f_n) \subset d_n E_n$. Combining (5.10), (5.11), and

(5.12), we conclude that $\bigcup_{n=1}^\infty d_n E_n$ is exactly p -Sidon.

To obtain a set in \mathbf{Z} which is exactly non p -Sidon, we choose $J_n > K_n > 0$ so that $(J_n/K_n)_{n=1}^\infty$ is a monotonically decreasing sequence to $p/(2-p)$, and carry out a construction similar to the one above. Details are left to the reader.

II. Γ contains $\oplus \mathbf{Z}_a$ for some $a \in \mathbf{Z}^+$. Clearly, we may assume that for any given $J > K > 0$, there is a dissociate set

$$E = \{\gamma_{ij}\}_{i \in (\mathbf{Z}^+)^K} \subset \oplus \mathbf{Z}_a$$

so that

$$(5.13) \quad E_{J,K} = \{(\gamma_{P_1(j)} \cdots \gamma_{P_N(j)})\}_{j \in (\mathbf{Z}^+)^J} \subset \oplus \mathbf{Z}_a = (\oplus \mathbf{Z}_a)^N.$$

SUBLEMMA. — Let $E_{J,K} \subset \oplus \mathbf{Z}_a$ be given by (5.13). Then,

$$A(q, E_{J,K}) \leq (\mathbf{B}q)^{[J/K]^2},$$

where $[J/K]$ = least integer equal to or greater than J/K , and \mathbf{B} is the 1-Sidon constant of E .

Proof. — Let $g \in L^2_{E_{J,K}}(\otimes \mathbf{Z}_a)$, $\|g\|_2 = 1$, be given by

$$g \sim \sum_{j \in (\mathbf{Z}^+)^J} a_j (\gamma_{P_1(j)} \cdots \gamma_{P_N(j)}).$$

Observe that there are $[J/K]$ subsets of $\{1, \dots, J\}$, $S_1, \dots, S_{[J/K]}$, so that $\bigcup_{n=1}^{[J/K]} S_n = \{1, \dots, J\}$. Therefore, we may estimate the q^{th} -norm of g as follows :

$$\begin{aligned} \|g\|_q^q &= \int_{(\omega_1, \dots, \omega_N) \in (\otimes \mathbf{Z}_a)^N} \left| \sum_{j \in (\mathbf{Z}^+)^J} a_j \gamma_{P_1(j)}(\omega_1) \cdots \gamma_{P_N(j)}(\omega_N) \right|^q d\omega_1 \cdots d\omega_N \\ &= \int_{(\otimes \mathbf{Z}_a)^{N-[J/K]}} \int_{(\otimes \mathbf{Z}_a)^{[J/K]}} \left| \sum_j a_j \gamma_{P_1(j)} \cdots \gamma_{P_N(j)} \right|^q \leq \int_{(\otimes \mathbf{Z}_a)^{N-[J/K]}} (\mathbf{B}q)^{[J/K]^2} \\ &\leq (\mathbf{B}q)^{[J/K]^2} \end{aligned}$$

(the appearance of \mathbf{B} above is explained by (2) of 5.7.7 in [10]). □

To prove Corollary 1.7 (d), in the present context, we follow a route identical to the one followed in Case I where the use of (5.10) is replaced by a use of the Sublemma. □

6. Problems.

The « J/K -fractional » product of a dissociate set $E \subset \Gamma$ was defined in this work as a subset of the $\binom{J}{K}$ -fold product of E . Subsequently, through

the use of Riesz products, $\binom{J}{K}$ appeared in some of the estimates. The first four problems focus on replacing $\binom{J}{K}$ by J/K in the various computations.

Problem 6.1. – Improve the lower bound on $\sup_{\gamma \in \Gamma^N} r_s(F_{J,K}, \gamma)$ in (1.4) of Theorem 1.3. In particular, is there $C > 0$ so that for all $s \in \mathbf{Z}^+$

$$C^{-J/K} s^{(J/K)s} \leq \sup_{\gamma \in \Gamma^N} r_s(F_{J,K}, \gamma),$$

where $F \subset \Gamma$ is independent ?

Problem 6.2. – Improve the norm estimate in (3.2) of Lemma 3.1. In particular, is there $\mu \in M(G)$ fulfilling (3.3) and so that

$$\|\mu\| \leq C^{J/K},$$

for some (universal) $C > 0$?

Problem 6.3. – Given any rational $\beta \in [1, \infty]$, we constructed in this paper $E \subset \Gamma$ for which there are $\eta_\beta, \zeta_\beta > 0$ so that for all $q > 2$

$$(6.4.1) \quad \eta_\beta q^{\beta/2} \leq A(q, E) \leq \zeta_\beta q^{\beta/2}.$$

While (6.4.1) is a stronger statement than « E is exactly Λ^β », in Corollary 1.7 (c) we deduced for all $\beta \in [1, \infty)$ no more than the existence of sets that are exactly Λ^β . Given any $\beta \in [1, \infty)$, can we find sets (say in \mathbf{Z}) for which (6.4.1) holds ?

Problem 6.4. – Prove Corollary 1.7 (c) for $\Gamma = \bigoplus \mathbf{Z}_a, a \in \mathbf{Z}^+$.

Problem 6.5. – A classical theorem due to Rudin ([10]) states that every 1-Sidon set is Λ^1 . The converse was recently established by Pisier ([9]). Could it be that $E \subset \Gamma$ is a Λ^β set if and only if E is $2\beta/(1+\beta)$ -Sidon ?

Problem 6.6. – In Section 5, an essentially probabilistic method was employed to show that $E_{J,K}$ is not $(2J/(K+J)-\varepsilon)$ -Sidon for any $\varepsilon > 0$. In the case $J = 2$ and $K = 1$, an explicit construction of trigonometric polynomials (bounded bilinear forms) displaying this fact is given on p. 172 of [8]. Replace the probabilistic procedure in Section 5 by a constructive one.

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