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SOLVING POWER SERIES EQUATIONS, PART II : CHANGE OF GROUND FIELD

by Joseph BECKER (*)

In the study of algebraic geometry or local analytic geometry over a residue field k of characteristic zero, one usually ignores the residue field, especially if it is algebraically closed. One reduces all questions to purely algebraic statements in commutative algebra. If one wants to pass to an extension field F of k, we reduce the problem at hand to an appropriate exact sequence, tensor with F over k, and apply faithful flatness. In [7], I showed this technique is inadequate for analytic geometry, because analytic tensor product isn't flat. In particular there are examples of injective, local, algebra homomorphisms of convergent power series rings :

$$\varphi : \mathrm{A}\langle\langle x,y\rangle\rangle \to \mathrm{A}\langle\langle t_1,t_2,t_3,t_4,t_5\rangle\rangle,$$

where A is the field of algebraic numbers, such that the natural extension of

$$\varphi : \mathbf{C}\langle\langle x,y\rangle\rangle \to \mathbf{C}\langle\langle t_1,t_2,t_3,t_4,t_5\rangle\rangle,$$

where $\mathbf{C}$ is the field of complex numbers, is not injective.

In [9] we continued the investigation of the properties of local algebra homomorphisms begun in [2, 12, 14, 18], and showed that if $\varphi : \mathrm{R} \to \mathrm{S}$ is an algebra homomorphism of reduced analytic rings over $\mathbf{C}$ which is closed in the Krull topology, then φ is open in the Krull topology. This can be restated as follows : If φ is injective and $\varphi(\hat{\mathrm{R}}) \cap \mathrm{S} = \varphi(\mathrm{R})$, then $\hat{\varphi} : \hat{\mathrm{R}} \to \hat{\mathrm{S}}$ is injective, where hat denotes completion. This is a purely algebraic statement about formal and convergent power series, e.g.

$\{[\exists \text{ conv. } f \neq 0 \text{ with } f(\varphi)=0] \text{ and } [\forall \text{ formal } g \text{ with } g(\varphi) \text{ conv.}, \exists \text{ conv. } f \text{ with } f(\varphi)=g(\varphi)]\} \Rightarrow \{\exists \text{ formal } f \neq 0 \text{ with } f(\varphi)=0\}$,

the proof given in [9] uses topological techniques, and relies heavily on the completeness of the complex numbers. Since the above statement is purely

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algebraic, it is natural to ask whether it holds for series over a valued field k of characteristic zero. As we have mentioned, we can't answer this question by just reducing everything to a set of equations. Also, we can't just duplicate the proof, due to our essential use of functional analysis. Instead we will show that all the relevant topological properties of local algebra homomorphisms are invariant under change of the residue field and hence for subfields k of $\mathbf{C}$, closed maps are open. A long chase through the proofs of sections 1, 2, and 3 of this paper yields a purely algebraic, but far more technical, proof of the fact that closed maps are open.

1. Introduction.

We recall first the main result of [9]. Let $\varphi : R \to S$ be local $\mathbf{C}$ algebra homomorphism of reduced analytic rings. By the Krull topology, we mean the metric topology given by the powers of the maximal ideal; by the simple topology we mean the metric topology induced by the coefficients of the power series, i.e., the topology of convergence of coefficients. By the inductive topology, we mean the topology induced from the direct limit of the Banach algebras $B_r = \{f = \Sigma a_\alpha z^\alpha | \Sigma |a_\alpha| r^\alpha < \infty\}$. To say that R is closed in S in a given topology just means that R is a closed subset of S. To say that φ is open means that the map $R \to \varphi(R)$ is open where $\varphi(R)$ has the relative topology induced from S. If R injects into S, we say that R is a subspace of S if the topology on R induced from S is the same as the natural topology on R. We say that φ is strongly injective if the map of abelian groups $\hat{R}/R \to \hat{S}/S$ is injective. Consider the following conditions :

a) R is closed in S in the Krull topology.

b) R is closed in S in the simple topology.

c) R is closed in S in the inductive topology.

d) φ is open in the Krull topology.

e) φ is open in the simple topology.

f) φ is open in the inductive topology.

THEOREM [9, 2.8]. – *a, b, c, and f are equivalent; d and e are equivalent; a implies d; d does not imply f.*

We will generalize this result to analytic rings with more general residue field. We will employ the following technique : Start with the original equations over k, pass from k to $\mathbf{C}$, apply our results there, and deduce the desired result for k. More precisely if

$$R = k\langle\langle y_1,\ldots,y_m\rangle\rangle/I, \qquad S = k\langle\langle x_1,\ldots,x_n\rangle\rangle/J,$$

I generated by $f_1,\ldots,f_r$, J generated by $g_1,\ldots,g_s$, let

$$\bar{R} = \mathbf{C}\langle\langle (y_1,\ldots,y_m\rangle\rangle/\bar{I}, \qquad \bar{S} = \mathbf{C}\langle\langle x_1,\ldots,x_n\rangle\rangle/\bar{J},$$

where $\bar{I}$ and $\bar{J}$ are the ideals generated by the f_i, g_i respectively and $\bar{\varphi} : \bar{R} \to \bar{S}$ be the extension of $\varphi : R \to S$. (We may think of $\bar{R}$ as the analytic tensor product of R and $\mathbf{C}$ over k.) We must determine the relation-ship of φ being open, injective, closed, strongly injective, respectively.

To begin, note that if h is a formal power series over k, then h is a convergent power series over k if and only if h is a convergent power series when considered as a formal power series over $\mathbf{C}$. Let $i : k \to \mathbf{C}$ be the natural inclusion of fields. There is an additive k linear projection $\pi : \mathbf{C} \to k$ such that $\pi \cdot i =$ identity on k, and π extends to a projection $\mathbf{C}[[x]] \to k[[x]]$, where $x = (x_1,\ldots,x_n)$. Since π is constructed via a transcendence basis of $\mathbf{C}$ over k, it is highly discontinuous and does not carry $\mathbf{C}\langle\langle x\rangle\rangle$ to $k\langle\langle x\rangle\rangle$. It is elementary to check that if $f \in \mathbf{C}[[x]]$, $g \in k[[x]]$, then $\pi(f\cdot g) = \pi(f)\cdot g$. Also if

$$\varphi_i \in k[[x]], \qquad \varphi = (\varphi_1,\ldots,\varphi_n), \qquad \text{and} \qquad \psi \in \mathbf{C}[[y]],$$

then $\pi(\psi(\varphi)) = \pi(\psi)(\varphi)$. We check that R injects to $\bar{R}$: Suppose

$$\psi \in k\langle\langle y\rangle\rangle, \qquad f_i \in k\langle\langle y\rangle\rangle, \qquad h_i \in \mathbf{C}\langle\langle y\rangle\rangle, \qquad \text{and} \qquad \psi = \sum_{i=1}^{r} f_i h_i$$

is zero in $\bar{R}$. Then applying π, we have $\psi = \sum_{i=1}^{r} \pi(h_i) f_i$, $\pi(h_i) \in k[[y]]$. But then

$$\psi \in k\langle\langle y\rangle\rangle \cap (f_i)k\langle\langle y\rangle\rangle\hat{} = (f_i)k\langle\langle y\rangle\rangle$$

so $\exists \tilde{h}_i \in k\langle\langle y\rangle\rangle$ so $\psi = \sum_{i=1}^{r} \tilde{h}_i f_i$ is zero in R. Similarly it is trivial to show S injects in $\bar{S}$ and we get a commutative diagram :

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \big\downarrow i & & \big\downarrow i \\ \bar{R} & \xrightarrow{\varphi} & \bar{S} \end{array}$$

Furthermore if $\varphi : R_1 \to R_2$, and $\psi : R_2 \to R_3$ are local maps of analytic rings then $\overline{\psi\varphi} = \bar{\psi}\,\bar{\varphi}$.

We also employ a similar construction for complete rings. If

$$T = k[[x_1,\ldots,x_n]]/q$$

is a complete ring and F an extension field of k, we let $\bar{T} = T \hat{\otimes}_k F$ be the completed tensor product of T and F over k. This is constructed as follows : Let $F = \varinjlim F_\alpha$ be the direct limit of finitely generated field extensions F_α of k. Then $T \hat{\otimes}_k F = (\varinjlim T \otimes_k F_\alpha)^{\wedge}$. This extends in the obvious manner to finite modules over complete rings. We check that the functor $T \to T \hat{\otimes}_k F$ is flat : i.e. let p be an ideal of T, and show $p \hat{\otimes}_k F$ is an ideal of $T \hat{\otimes}_k F$. Since usual tensor product is exact, the exact sequence $0 \to p \to T$ induces an exact sequence $0 \to p \hat{\otimes}_k F_\alpha \to T \hat{\otimes}_k F_\alpha$. Since direct limit preserves exact sequences, this yields an exact sequence

$$0 \to p \hat{\otimes}_k F \to T \hat{\otimes}_k F.$$

It is trivial to check that if $\varphi : T_1 \to T_2, \psi : T_2 \to T_3$ are local maps of complete rings over k, then φ and ψ induce maps $\bar{\varphi} : \bar{T}_1 \to \bar{T}_2$, $\bar{\psi} : T_2 \to \bar{T}_3$, and we get a commutative diagram :

$$\begin{tikzcd}
T_1 \arrow[r, "\varphi"] \arrow[d] & T_2 \arrow[r, "\psi"] \arrow[d] & T_3 \arrow[d] \\
\bar{T}_1 \arrow[r, "\bar{\varphi}"] \arrow[rr, bend right, "\overline{\varphi\psi}"'] & \bar{T}_2 \arrow[r, "\bar{\psi}"] & \bar{T}_3
\end{tikzcd}$$

Next we note that if $T = k[[x_1,\ldots,x_n]]/q$, $q = (h_1,\ldots,h_t)$ is a complete ring and $\bar{T} = T \hat{\otimes}_k F$, that T injects into $\bar{T}$, π induces a map $\bar{T} \to T$, and $\pi(i) =$ identity on T. Only the second assertion requires a proof. Suppose ψ_1, $\psi_2 \in F[[x]]$ are two representatives of the same element of $\bar{T}$, then $\psi_1(x) - \psi_2(x) = \sum_{i=1}^{t} \sigma_i(x) h_i(x)$, $\sigma_i \in F[[x]]$, $h_i \in k[[x]]$. Applying π we have $\pi\psi_1 - \pi\psi_2 = \sum_{i=1}^{t} \pi(\sigma_i) h_i$ so $\pi\psi_1$ and $\pi\psi_2$ are the same in T.

LEMMA 1.1. — *Let $\varphi : T_1 \to T_2$ be a local map of complete rings, then φ is injective if and only if $\bar{\varphi}$ is injective.*

Proof. — An elementary diagram chase.

$$
\begin{tikzcd}
\mathrm{T}_1 \arrow[r, "\varphi"] \arrow[d, shift right, "i_1"'] & \mathrm{T}_2 \arrow[d, shift left, "i_2"] \\
\bar{\mathrm{T}}_1 \arrow[r, "\bar{\varphi}"] \arrow[u, shift right, "\pi_1"'] & \bar{\mathrm{T}}_2 \arrow[u, shift left, "\pi_2"]
\end{tikzcd}
\qquad
\begin{aligned}
i_2\varphi &= \bar{\varphi} i_1 \\
\varphi\pi_1 &= \pi_2\bar{\varphi}
\end{aligned}
$$

If $\varphi : k[y] \to k[x]$ or $k[[y]] \to k[[x]]$ or more generally any local k algebra homomorphism of affine or complete rings, φ is injective if and only if $\bar{\varphi}$: $\mathbf{C}[y] \to \mathbf{C}[x]$, $\mathbf{C}[[y]] \to \mathbf{C}[[x]]$ is injective. Also it is easy to check that if $\varphi : \mathrm{R} \to \mathrm{S}$ is a map of k analytic rings and $\bar{\varphi} : \bar{\mathrm{R}} \to \bar{\mathrm{S}}$ is injective, then φ is injective. However the converse is false. This is despite the fact ([10, Satz 1.4.4, page 38] with the help of [11, 3.5.2, Theorem 1, page 227]) that $\bar{\mathrm{R}}$ is a flat R module, and hence an exact sequence $0 \to \mathrm{R} \to \mathrm{S}$ induces an exact sequence $0 \to \bar{\mathrm{R}} \to \mathrm{S} \,\bar{\otimes}_{\mathrm{R}}\, (\mathrm{R} \otimes_k \mathbf{C})$. This last module is not the same as $\mathrm{S} \,\bar{\otimes}_k\, \mathbf{C}$-analytic tensor product is not associative. If S is a finite R module, then $\mathrm{S} \otimes_k (\mathrm{R} \,\bar{\otimes}_k \mathbf{C}) = \mathrm{S} \,\bar{\otimes}_k\, \mathbf{C}$ and we get an exact sequence $0 \to \bar{\mathrm{R}} \to \bar{\mathrm{S}} \to 0$. However letting $\mathrm{R} = \mathrm{S} = \mathbf{Q}\langle\langle x, y\rangle\rangle$ and $\varphi(x) = x$, $\varphi(y) = xy$, it is easily checked that

$$
y \otimes \left(\sum_{r=1}^{\infty} x^r \otimes r^{1/r} \right) - \sum_{r=1}^{\infty} 1 \otimes yx^r \otimes r^{1/r}
$$

is a nonzero element of the kernel of the natural map $\mathrm{S} \otimes_{\mathrm{R}} (\mathrm{R} \,\bar{\otimes}_{\mathrm{Q}}\, \mathbf{C}) \to (\mathrm{S} \otimes_{\mathrm{R}} \mathrm{R}) \,\bar{\otimes}_{\mathrm{Q}} \mathbf{C}$.

On the other hand it is obvious that if $\varphi : \mathrm{R} \to \mathrm{S}$ is onto, then the induced map $\bar{\varphi} : \bar{\mathrm{R}} = \mathrm{R} \,\bar{\otimes}_k \mathbf{C} \to \mathrm{S} \,\bar{\otimes}_k \mathbf{C} = \bar{\mathrm{S}}$ is onto.

LEMMA 1.2. — *If* R *is an analytic ring, then* $\hat{\bar{\mathrm{R}}}$ *is isomorphic to* $\bar{\hat{\mathrm{R}}}$. *If* $\varphi : \mathrm{R} \to \mathrm{S}$, $\psi : \mathrm{S} \to \mathrm{T}$ *are local maps of analytic rings then there is a commutative diagram :*

$$
\begin{tikzcd}[column sep=small]
 & \hat{\mathrm{R}} \arrow[rrr] \arrow[ddr] & & & \hat{\mathrm{S}} \arrow[rrr] \arrow[ddr] & & & \hat{\mathrm{T}} \arrow[ddr] & \\
\mathrm{R} \arrow[ur] \arrow[rrr, dash] \arrow[ddr] & & & \mathrm{S} \arrow[ur] \arrow[rrr, dash] \arrow[ddr] & & & \mathrm{T} \arrow[ur] \arrow[ddr] & & \\
 & & \bar{\hat{\mathrm{R}}} \arrow[rrr] \arrow[dd] & & & \bar{\hat{\mathrm{S}}} \arrow[rrr] \arrow[dd] & & & \bar{\hat{\mathrm{T}}} \arrow[dd] \\
 & \bar{\mathrm{R}} \arrow[rrr] \arrow[dr] & & & \bar{\mathrm{S}} \arrow[rrr] \arrow[dr] & & & \bar{\mathrm{T}} \arrow[dr] & \\
 & & \hat{\bar{\mathrm{R}}} \arrow[rrr] & & & \hat{\bar{\mathrm{S}}} \arrow[rrr] & & & \hat{\bar{\mathrm{T}}}
\end{tikzcd}
$$

Proof. – Follows immediately from all the preceding properties. Let $\mathrm{R} = k\langle\langle x\rangle\rangle/(f_i)$; then

$$\bar{\mathrm{R}} = \mathbf{C}\langle\langle x\rangle\rangle/(f_i), \qquad \hat{\bar{\mathrm{R}}} = \mathbf{C}[[x]]/(f_i),$$
$$\hat{\mathrm{R}} = k[[x]]/(f_i), \qquad \text{and} \qquad \bar{\hat{\mathrm{R}}} = \mathbf{C}[[x]]/(f_i).$$

2. Descent from complete fields.

We now investigate what happens to the topological properties of φ when changing the residue field. We find that these properties are invariant under change of the residue field.

THEOREM 2.1. – *With the same notation as section* 1 *and all topologies being the Krull topology :*

a) $\bar{\varphi}$ *injective* $\Rightarrow$ φ *injective,*

b) R *is a subspace of* S $\Rightarrow$ $\bar{\mathrm{R}}$ *is a subspace of* $\bar{\mathrm{S}}$,

c) φ *open* $\Rightarrow$ $\bar{\varphi}$ *open,*

d) $\bar{\mathrm{R}}$ *is a subspace of* $\bar{\mathrm{S}}$ $\Rightarrow$ R *is a subspace of* S,

e) $\bar{\varphi}$ *open* $\Rightarrow$ φ *open,*

f) φ *strongly injective* $\Rightarrow$ $\bar{\varphi}$ *strongly injective,*

g) $\bar{\varphi}$ *strongly injective* $\Rightarrow$ φ *strongly injective,*

h) φ *closed* $\Rightarrow$ $\bar{\varphi}$ *closed.*

If S *is reduced, then we also have,*

i) $\bar{\varphi}$ *closed* $\Rightarrow$ φ *closed,*

j) φ *closed* $\Rightarrow$ φ *open,*

k) φ *injective and closed* $\Rightarrow$ φ *strongly injective.*

Proof. – We begin by reducing to the case where k is dense in $\mathbf{C}$. There are two possibilities-either $k \subset \mathbf{R}$ or k is dense in $\mathbf{C}$. If $k \subset \mathbf{R}$, let $i = \sqrt{-1}$ and $\mathrm{F} = k[i]$. Then F is dense in $\mathbf{C}$ and the projection $\mathrm{F} \to k$ is continuous in the Euclidean topology on $\mathbf{C}$ and hence the induced projection $\mathrm{F}[[x]] \to \mathrm{F}[[x]]$, carries $\mathrm{F}\langle\langle x\rangle\rangle$ to $k\langle\langle x\rangle\rangle$. Let $\varphi : \mathrm{R} \to \mathrm{S}$, and $\psi = \varphi \otimes 1 : \mathrm{R} \otimes_k \mathrm{F} \to \mathrm{S} \otimes_k \mathrm{F}$. It is now trivial to verify that ψ and φ have the same relevant properties.

Proof of 2.1*a*. – Trivial diagram chase.

Proof of 2.1*b*. — $\bar{R}$ is a subspace of $\bar{S} \Rightarrow \hat{R}$ injects into $\hat{S}$, by [6.1.12] $\Rightarrow \hat{\bar{R}}$ injects into $\hat{\bar{S}}$, by lemma 1.1 $\Rightarrow \hat{\bar{R}}$ injects into $\hat{\bar{S}}$, by lemma 1.2 $\Rightarrow \bar{R}$ is a subspace of $\hat{\bar{S}}$, since $\bar{R}$ is a subspace of its completion and the composition of subspaces is a subspace $\Rightarrow \bar{R}$ is a subspace of $\bar{S}$, by [6,1.8].

Proof of 2.1*c*. — $\varphi : R \to S$ open implies $R \to R/\ker\varphi$ is onto and $R/\ker\varphi$ is a subspace of S by [6,1.3]. Hence $\bar{R} \to (R/\ker\varphi)^-$ is onto and $(R/\ker\varphi)^-$ is a subspace of $\bar{S}$ by 2.1*b*. By [6, 1.1 and 1.6], φ is open.

LEMMA 2.2. — R *is a subspace of* $\bar{R}$.

Proof. — We recall the definition of subspace [6] :

$$\forall j \in N, \qquad \exists k \in N \qquad \text{so} \qquad m(R)^j \supset i^{-1}(m(\bar{R})^k).$$

By abuse of notation we will let $\sigma = y^k\eta$ denote the statement that $\sigma \in m(R)^k$. $\left(\text{We really should write } \exists n_\alpha \in R \text{ so } \sigma = \sum_{|\alpha| = k} y^\alpha\eta_\alpha.\right)$ Now the statement that R is a subspace of $\bar{R}$ becomes :

$$\forall j, \qquad \exists k, \qquad \forall\eta \in \bar{R}, \qquad \forall\psi \in R\ [\psi = y^k\eta \Rightarrow \exists\beta \in R \text{ so } \psi = y^j\beta].$$

In fact we can just let $k = j$. Then

$$\psi = y^k\eta \ \Rightarrow\ \psi = \pi(\psi) = y^k\pi(\beta) \ \Rightarrow\ \psi \in R \cap m(R)^k\hat{R} = m(R)^k.$$

Alternate proof of 2.2. — R is a subspace of $\hat{R}$ and $\hat{R}$ is a subspace of $\hat{\bar{R}} = \hat{R}$ by the Chevalley subspace theorem so [6, 1.7] R is a subspace of $\hat{\bar{R}}$. But $R \subset \bar{R} \subset \hat{\bar{R}}$ so [6, 1.8] R is a subspace of $\bar{R}$.

Proof of 2.1*d*. — $\bar{R}$ is a subspace of $\bar{S}$ so $\hat{\bar{R}}$ injects into $\hat{\bar{S}}$. Also $\hat{R}$ injects into $\hat{\bar{R}}$. An elementary diagram chase shows $\hat{R}$ injects into $\hat{S}$.

Proof of 2.1*e*. — We first show that it suffices to consider the case where R injects into S. Let $\varphi : R \to S$ and $R_0 = R/\ker\varphi$, then $R \to R_0 \to 0$ and $0 \to R_0 \to S$ are exact. It follows that $\bar{R} \to \bar{R}_0 \to 0$ is exact, and we have maps $\bar{R} \to \bar{R}_0 \to \bar{S}$. Clearly $\bar{\varphi}$ open implies the map $\bar{R}_0 \to \bar{S}$ is open. The special case would imply that the map $R_0 \to S$ is open. By [6, 1.6] applied to $R \to R_0 \to S$, φ is open.

We now assume φ is injective and $\bar{\varphi}$ is open. If $\bar{\varphi}$ were injective, then by [6, 1.2 and 1.1] and 6.1*d* we would have that $\bar{R}$ is a subspace of $\bar{S}$ so R is a subspace of S so φ is open. Let $0 \to K_1 \to \bar{R} \to \bar{S}$ be exact, with $K_1 \neq 0$. Applying π and lemmas 1.1 and 1.2 yields an exact sequence $0 \to K_2 \to \hat{R} \to \hat{S}$, with $K_2 \neq 0$. Let $R_1 = \bar{R}/K_1$ and $R_2 = \hat{R}/K_2$.

From the faithful flatness of completion applied to the sequence of finite $\bar{R}$ modules : $0 \to K_1 \to \bar{R} \to R_1 \to 0$, and from the faithful flatness of completed tensor product applied to the sequence $0 \to K_2 \to R \to R_2 \to 0$, we have that $\hat{R}_1 = \hat{\bar{R}}/\hat{K}_1$ and $\bar{R}_2 = \bar{\hat{R}}/\bar{K}_2$. From the flatness of completed tensor product over a field applied to the sequence $0 \to K_2 \to \hat{R} \to \hat{S}$, we have an exact sequence $0 \to \bar{K}_2 \to \bar{\hat{R}} \to \bar{\hat{S}}$. Since $\bar{\varphi}$ is open, [6, 1.11] the exact sequence $0 \to K_1 \to \bar{R} \to \bar{S}$ completes to an exact sequence $0 \to \hat{K}_1 \to \hat{\bar{R}} \to \hat{\bar{S}}$. Chasing canonical homomorphisms as in lemma 1.2, it follows that $K_1 = \bar{K}_2$. Let R_3 denote

$$\hat{R}_1 = \bar{R}_2 = C[[y_1,\ldots,y_n]]/(f_1,\ldots,f_q),$$

where $f_i \in k\langle\langle y\rangle\rangle$. Let $r = \dim R_3$, $r < n$; then $r = \dim R_1 = \dim R_2$ and for a generic linear combination $Z_1,\ldots,Z_r$ of $y_1,\ldots,y_n$ over k, $A = k\langle\langle Z_1,\ldots,Z_r\rangle\rangle$ injects into R, $\bar{A} \subset R_2$, $\hat{A} \subset R_2$, $\hat{\bar{A}} \subset R_3$, R_1 is finite over $\bar{A} = A_1$, R_2 is finite over $\hat{A} = A_2$, and R_3 is finite over $\hat{\bar{A}} = A_3$. Now z_{r+1} satisfies a nonzero polynomial P_1, P_2, P_3 in R_1, R_2, R_3 of minimal degree d_1, d_2, d_3 in z_{r+1} over A_1, A_2, A_3 respectively. (If the R_i are domains, this polynomial is monic because $\bar{A}$, $\hat{A}$, $\hat{\bar{A}}$ are each normal.) Considering the finite maps $A_i \to A_i[z_{r+1}]$, and applying the faithful flatness of completion and completed tensor product over a field, we find that $d_1 = d_3$, $d_2 = d_3$, (to be henceforth denoted by just d) and that $P_1 = P_3$, $P_2 = P_3$. Let

$$P_1 = \sum_{i=0}^{d} \bar{a}_i z_{r+1}^i, \qquad P_2 = \sum_{i=0}^{d} \hat{a}_i z_{r+1}^i, \qquad \bar{a}_i \in \bar{A},\ \hat{a}_i \in \hat{A}.$$

Then $P_1 - P_2$ is zero in R_3 and the coefficients $\bar{a}_i - \hat{a}_i$ are zero in $\hat{A}$. Hence $\bar{a}_i = \hat{a}_i \in \hat{A} \cap \bar{A} = A$. Hence $P_1 \in A[z_{r+1}]$. From the commutative diagram :

$$\begin{array}{ccc} R & \longrightarrow & \bar{R}/\bar{K}_1 \\ \downarrow & & \downarrow \\ S & \longrightarrow & \bar{S} \end{array}$$

we see that R injects into R_1. But P_1 is zero in R_1 so P_1 is zero in R. On the other hand, $A[z_{r+1}]$ injects into R so we have each $a_i = 0$, and we have a contradiction with the fact that P_1 is a nonzero polynomial.

LEMMA 2.3. — $R = \hat{R} \cap \bar{R}$, *where* $\hat{R}$ *and* $\bar{R}$ *are both considered as subsets of* $\hat{\bar{R}}$.

Proof. — If R is regular, the result is clear. Otherwise, let A be a regular analytic ring over k, and R a finite extension of A. Then $\bar{R}$ is a finite extension of $\bar{A}$, $\hat{R}$ is a finite extension of $\hat{A}$, and $\hat{\bar{R}}$ is a finite extension of $\hat{\bar{A}}$.

Let $f \in \hat{\mathrm{R}} \cap \bar{\mathrm{R}}$, then f satisfies a polynomial over $\hat{\mathrm{A}}, \bar{\mathrm{A}}, \bar{\hat{\mathrm{A}}}$, and the minimal degree polynomial $\hat{\mathrm{P}}, \bar{\mathrm{P}}, \bar{\hat{\mathrm{P}}}$ for f over $\hat{\mathrm{A}}, \bar{\mathrm{A}}, \bar{\hat{\mathrm{A}}}$ in $\hat{\mathrm{R}}, \bar{\mathrm{R}}, \bar{\hat{\mathrm{R}}}$ respectively, are all monic. Since maps of analytic and complete rings both satisfy the property that they are finite if and only if they are quasi finite, it follows easily that the degrees of $\hat{\mathrm{P}}, \bar{\mathrm{P}}, \bar{\hat{\mathrm{P}}}$ are the same, say $= r$. Now

$$\psi = \hat{\mathrm{P}} - \bar{\mathrm{P}} = \sum_{i=0}^{r-1} (\hat{a}_i - \bar{a}_i) f^i$$

is zero in $\hat{\mathrm{R}}$ and of degree less than the minimal polynomial $\bar{\hat{\mathrm{P}}}$. Hence each $\hat{a}_i - a_i = 0$. Since $\hat{\mathrm{A}} \cap \bar{\mathrm{A}} = \mathrm{A}, \bar{a}_i, \hat{a}_i \in \mathrm{A}$. Hence $f \in \hat{\mathrm{R}}$ and f is integral over A. Since the integral closure N of R is a finite module over R, R is closed in N; that is $\hat{\mathrm{R}} \cap \mathrm{N} = \mathrm{R}$, so $f \in \mathrm{R}$.

Proof of 2.1*g*. — By lemma 2.3, there is an injection of abelian groups $\hat{\mathrm{R}}/\mathrm{R} \to \hat{\hat{\mathrm{R}}}/\bar{\mathrm{R}}$. By hypothesis the natural map $\hat{\hat{\mathrm{R}}}/\bar{\mathrm{R}} \to \hat{\hat{\mathrm{S}}}/\bar{\mathrm{S}}$ is injective. The result follows from the commutative diagram :

$$\begin{tikzcd}
\hat{\mathrm{R}}/\mathrm{R} \arrow[r] \arrow[d] & \hat{\mathrm{S}}/\mathrm{S} \arrow[d] \\
\hat{\hat{\mathrm{R}}}/\bar{\mathrm{R}} & \hat{\hat{\mathrm{S}}}/\bar{\mathrm{S}}
\end{tikzcd}$$

LEMMA 2.4. — $\hat{\mathrm{R}} + \bar{\mathrm{R}} = \hat{\hat{\mathrm{R}}}$, *that is every formal power series over* C *can be written as the sum of a formal power series over* k *and a convergent power series overs* C.

Proof. — k is dense in C.

Proof of 2.1*f*. — We know $\hat{\mathrm{R}}/\mathrm{R}$ injects into $\hat{\mathrm{S}}/\mathrm{S}$. Suppose $\psi \in \hat{\hat{\mathrm{R}}}$ and $\psi(\varphi) \in \bar{\mathrm{S}}$; want to show $\psi \in \bar{\mathrm{R}}$. Write $\psi = \psi_1 + \psi_2$, $\psi_1 \in \hat{\mathrm{R}}$, $\psi_2 \in \bar{\mathrm{R}}$. Clearly $\psi_1(\varphi) \in \hat{\mathrm{S}}$, $\psi_2(\varphi) \in \bar{\mathrm{S}}$. Since $\psi_1(\varphi) = \psi(\varphi) - \psi_2(\varphi) \in \bar{\mathrm{S}}$, we have $\psi_1(\varphi) \in \hat{\mathrm{S}} \cap \bar{\mathrm{S}} = \mathrm{S}$. Now φ is strongly injective, $\psi_1(\varphi) \in \mathrm{S}$, and $\psi_1 \in \hat{\mathrm{R}}$ implies $\psi_1 \in \mathrm{R}$. Finally $\psi_1, \psi_2 \in \bar{\mathrm{R}}$ so $\psi \in \bar{\mathrm{R}}$.

Proof of 2.1*h*. — Assume φ is closed, $\psi \in \hat{\hat{\mathrm{R}}}$ and $\psi(\varphi) \in \bar{\mathrm{S}}$. Write $\psi = \psi_1 + \psi_2$, $\psi_1 \in \hat{\mathrm{R}}$, $\psi_2 \in \bar{\mathrm{R}}$. Clearly $\psi_1(\varphi) \in \hat{\mathrm{S}}$, $\psi_2(\varphi) \in \bar{\mathrm{S}}$. Since $\psi_1(\varphi) = \psi(\varphi) - \psi_2(\varphi) \in \bar{\mathrm{S}}$, we have $\psi_1(\varphi) \in \hat{\mathrm{S}} \cap \bar{\mathrm{S}} = \mathrm{S}$. Now φ closed, $\psi_1 \in \hat{\mathrm{R}}$ and $\psi_1(\varphi) \in \mathrm{S}$ implies there exists $\eta \in \mathrm{R}$ so $\eta(\varphi) = \psi_1(\varphi)$. Then $\eta + \psi_2 \in \bar{\mathrm{R}}$ and $(\eta + \psi_2)(\varphi) = \psi(\varphi)$.

Proof of 2.1*i*. — As in the proof of 2.1*e*, it is easy to check that it suffices to consider the case where φ is injective. (We omit the details.) Also from standard considerations, we know that S reduced implies $\bar{\mathrm{S}}, \hat{\mathrm{S}}$, and $\hat{\hat{\mathrm{S}}}$ are all reduced.

Now $\bar{\varphi}$ closed $\to$ $\bar{\varphi}$ open by [9, 2.13] $\to$ φ open, by 2.1*e*. Next φ open and injective $\to$ R is a subspace of S $\to$ $\bar{\mathrm{R}}$ is a subspace of $\bar{\mathrm{S}}$ by 6.1*b* $\to$ $\bar{\varphi}$ is injective. Finally $\bar{\varphi}$ injective and closed $\Rightarrow$ $\bar{\varphi}$ strongly injective by [9, 2.12] $\Rightarrow$ φ strongly injective by 2.1*g* $\Rightarrow$ φ closed.

Proof of 2.1*j*. — φ closed $\Rightarrow$ $\bar{\varphi}$ closed by 2.1*h* $\Rightarrow$ $\bar{\varphi}$ open by [9, 2.12] $\Rightarrow$ φ open by 2.1*e*.

Proof of 2.1*k*. — φ closed and injective $\Rightarrow$ φ closed, open, and injective by 2.1*j* $\to$ φ strongly injective by [6, 1.2, 1.12, and 2.2].

THEOREM 2.5. — *The following are equivalent :*

(i) φ *is open in the Krull topology,*

(ii) φ *is open in the simple topology,*

(iii) $\bar{\varphi}$ *is open in the Krull topology,*

(iv) $\bar{\varphi}$ *is open in the simple topology.*

Proof. — (iii) $\Leftrightarrow$ (iv) by [9, 2.8] ; (i) $\Leftrightarrow$ (iii) by theorem 2.1; (i) $\Rightarrow$ (ii) by the proof of [9, 1.5]; (ii) $\Rightarrow$ (i) because φ open in the simple topology implies the completion of φ in the simple topology is injective so by the proof of [9, 2.7]. $\hat{\bar{\mathrm{R}}} \to \hat{\bar{\mathrm{S}}}$ is injective. Hence by lemma 1.1, $\hat{\mathrm{R}} \to \hat{\mathrm{S}}$ is injective. Hence φ is open in the Krull topology.

THEOREM 2.6. — *Assume* S *is reduced. Then we have the following set of implications. This shows all the properties listed are equivalent.*

$$\begin{tikzcd}
\varphi \textit{ closed Krull} \arrow[r, leftrightarrow] \arrow[d, shift left, "②"] & \bar{\varphi} \textit{ closed Krull} \arrow[d, leftrightarrow] \\
\varphi \textit{ closed simple} \arrow[u, shift left, "①"] \arrow[d, "③"] & \bar{\varphi} \textit{ closed simple} \arrow[d] \\
\varphi \textit{ closed inductive} \arrow[d, "⑧"] & \varphi \textit{ closed inductive} \arrow[l, "⑥"'] \\
\varphi \textit{ open inductive} \arrow[d, "④"] & \bar{\varphi} \textit{ open inductive} \arrow[l, "⑦"'] \arrow[d] \\
rk\ \varphi = \dim \mathrm{R}/\ker \varphi \arrow[r, leftrightarrow, "⑤"] & rk\ \bar{\varphi} = \dim \bar{\mathrm{R}}/\ker \bar{\varphi}. \arrow[uuuu, bend right=60]
\end{tikzcd}$$

Here for the rank of φ, we can't use the usual geometric definition which works when k is algebraically closed, but must resort to a purely algebraic definition. Let $D(S/R)$ be the module of differentials of S over R and

$$\Omega(S/R) = D(S/R)/\bigcap_{i=1}^{\infty} m^i D(S/R),$$

where m is the maximal ideal of S. By the rank of φ we mean the rank of the S module $\Omega(S/R)$. It is elementary to check the following : If R and S are analytic rings and $i : R \to S$ the natural inclusion, then $rk\, i \leqslant \dim R$. If $\psi : R \to S$, $\eta : S \to T$ are local maps of analytic rings and $rk\, \eta = \dim S = \dim T$, then $rk\, \psi = \dim R$ if and only if $rk\, \eta\psi = \dim R$.

The proof of all the unnumbered implications has already been given in [9] or theorem 2.1.

Proof of (1) *and* (3). — Obvious since the Krull topology is weaker than the simple topology and the simple topology is stronger than the inductive topology.

Proof of (2). — Suppose φ is injective and φ and $\bar{\varphi}$ are closed in the Krull topology, i.e. $\bar{\varphi}(\hat{\bar{R}}) \cap \bar{S} = \bar{\varphi}(\bar{R})$ and $\varphi(\hat{R}) \cap S = \varphi(R)$. We then show φ is closed in the simple topology, i.e. $\varphi(\hat{\bar{R}}) \cap S = \varphi(R)$. Clearly $\varphi(\hat{\bar{R}}) \cap S \subset \bar{\varphi}(\hat{\bar{R}}) \cap \bar{S} \subset \bar{\varphi}(\bar{R}) \cap S$. Let $\eta \in S$, $\psi \in \bar{R}$, and $\psi(\varphi) = \eta$. Then $\eta = \pi(\eta) = (\pi\psi)\varphi \in \varphi(\hat{R})$. So

$$\eta \in \varphi(\hat{R}) \cap S \subset \varphi(R).$$

Proof of (4). — This goes exactly as in [9, theorem 2.8$f \to d$] *since the completeness of the field was not usued there.*

Note that in each of (5), (6), and (7), we have from [9, 2.8] that $\bar{\varphi}$ is open in the Krull topology. By 2.1e, φ is open in the Krull topology. By the habitual argument, it suffices to consider the case where φ is injective-then R is a subspace of S and by a now standard argument, $\bar{\varphi}$ is also injective.

Proof of (5). — Since $rk\, \varphi = rk\, \bar{\varphi}$, this is obvious.

Proof of (6). — Let η be in the closure of R in S in the inductive topology, then η is in the closure of $\bar{R}$ in $\bar{S}$ in the inductive topology. Hence $\eta \in \bar{S}$. Now $\eta \in \bar{R} \cap S$ so applying the projection π, we have $\eta \in \hat{R} \cap S$. Since φ is closed in the Krull topology, $\eta \in R$.

LEMMA 2.7. — R *is a subspace of* $\bar{R}$ *in the inductive topology.*

Sketch of proof. – If R is regular, this is obvious since the Banach norms $\|-\|_n$ on R are just the restrictions of the corresponding Banach norms on $\bar{R}$. Although the inductive topology on R is not metric, we may still consider the completion of R with respect to Cauchy nets, denoted IC(R). The desired result if just the statement that $IC(R) = \bar{R}$, which is of course true for regular rings. Clearly $\bar{R}$ is complete in this topology. If $R = k\langle\langle y \rangle\rangle / I$, then

$$IC(\bar{R}) = IC(\mathbf{C}\langle\langle y \rangle\rangle)/IC(\bar{I}). \quad \text{and} \quad IC(R) = IC[k\langle\langle y \rangle\rangle)/IC(I).$$

The last equality is not as clear. One needs to check that the inductive topology on I induced from being a subset of $k\langle\langle y \rangle\rangle$ is the same as the inductive topology on I considered as a finite module over $k\langle\langle y \rangle\rangle$. (Over the real or complex field this follows trivially from the open mapping theorem. For more general fields this technique is not available, but this result is still true [13, Kapitel II, section 2.7, Satz 9, page 97].)

We have an exact sequence $0 \to I \to k\langle\langle y \rangle\rangle \to R \to 0$ and an induced commutative diagram :

$$\begin{array}{ccccccc} 0 \to & IC(I) & \to & IC(k\langle\langle y \rangle\rangle) & \to & IC(R) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & I & \to & \mathbf{C}\langle\langle y \rangle\rangle & \to & R & \to 0 \end{array}$$

By induction on the homological dimension of I, we may assume $IC(I) = \bar{I}$ (since it's true for principal ideals) and hence the first two columns of the diagram are isomorphisms. By a elementary diagram chase, the last column is also an isomorphism.

The technique employed here is similar to that used in [9, theorem 2.7]. A more thorough explanation can be found there.

Proof of (7). – We have a commutative diagram with $\varphi, \bar{\varphi}$ injective.

$$\begin{array}{ccc} \bar{R} & \xrightarrow{\bar{\varphi}} & \bar{S} \\ \uparrow & & \uparrow \\ R & \xrightarrow{\varphi} & S \end{array}$$

Now $\bar{\varphi}$ is open [9, 3.*i*] $\Rightarrow \bar{R}$ is a subspace of $\bar{S} \Rightarrow R$ is subspace of $\bar{S}$, by 2.7 and [9, 3.v] $\Rightarrow$ R is a subspace of S by [9, 3.vi].

Part 8. – This proof is long and tedious and is deferred to section 3.

3. The Final Step.

This section is devoted to proving part 8 of theorem 2.6. Although the result holds even when S has nilpotents, for the sake of simplicity we will consider only the case where S is a domain.

In this section we will consider analytic rings R and S whose residue fields k_R, k_S respectively are not necessarily equal. By the statement $\varphi : R \to S$ is a local algebra homomorphism, we also mean that $\varphi(M_R) \subset M_S$, $\varphi(k_R) \subset k_S$ and $[k_S : k_R] < \infty$. The general theory of such mappings is carried out in [2, section 1] and to the best of my knowledge, everything comes out exactly as in the usual case. This generalization is not considered for its own sake, but because of the change of residue field which occurs in lemma 3.2.

Since we already have that φ open in inductive topology is equivalent to $rk\ \varphi = \dim R/\ker \varphi$, it is sufficient to show that φ closed in inductive topology implies that φ is open in inductive topology or $rk\ \varphi = \dim R/\ker \varphi$. The proof will be divided into 3 sections :

(i) Reduction to the case where φ is injective, S is regular, and $\dim S = rk\ \varphi$.

(ii) Proof that if f_i is a sequence in R and $\varphi(f_i)$ converges in the inductive topology on S, then f_i converges in the simple topology on R.

iii) Assuming $\dim R > \dim S$, the construction of a sequence f_i in R which violates the above conditions.

In part (i) we use the rank condition as our conclusion.

Observation 3.1. – If $\psi : R \to T$ and $\eta : T \to S$ are local maps of analytic rings (which do not necessarily all have the same residue field), ψ and η are closed in the inductive topology, and η is injective and open in the inductive topology, then $\eta(\psi)$ is closed in the inductive topology. Note that this does not seem at all obvious if we drop the hypothesis that η is open, and we do not yet have that closed implied open.

LEMMA 3.2. – *Reduction of part* 8 *to the special case where* φ *is injective and* S *is regular.*

Proof. – Clearly $\varphi : R \to S$ is closed iff the induced map $R/\ker \varphi \to S$ is closed. Also $rk\ \varphi$ equals the rank of the induced map. So we may assume φ is injective.

Now consider the scheme B over S formed by blowing up the maximal ideal of R and N be the scheme formed from B by taking the normalization (integral closure in full ring of quotients) of each affine piece of B. Clearly in any affine piece B is a finite algebra over S. That N is a finite B module follows from the following facts : [16, section 36]. A ring A is pseudogeometric if A is Noetherian and if for every prime ideal p of A and ring E, $A/p \subset E$, with the quotient field of E a finite extension of the quotient field of A/p, and E integral over A/p, we have that E is a finite A/p module. In characteristic zero, analytic rings are pseudogeometric. If A is pseudogeometric, then every localization of a finite algebra over A is pseudogeometric.

Let

$$M(S)=(x_1,\ldots,x_n)S,\ B=S[x_2/x_1,\ldots,x_n/x_1],\ I=(x_1,\ldots,x_n)B=x_1B,$$

and IN be the extension of I to N. Now I is principal, and a proper ideal so $ht\ I = 1$. Similarly the height of IN in N is one. Let J be the jacobian ideal of N. By the Serre criteria for normality, $ht\ J \geqslant 2$. Since $ht\ IN = 1$, $J \not\subseteq IN$, and $J' = J/IN$ is not zero in $N' = N/IN$. Clearly $B' = B/I$ is an affine ring. Now N is a finite B module so N' is a finite B' module. A finite extension of an affine ring is affine so N' is is an affine ring. Hence there exists a maximal ideal M' of N' with $J' \not\subset M'$. (This is easily seen by tensoring N' with the algebraic closure of $\bar{k}$ over the residue field F of T, finding a maximal ideal M_0 of $N' \otimes_F k$ with $J' \otimes_F \bar{k} \not\subset M_0$ via the Hilbert Nullstellensatz and letting $M' = M_0 \cap N'$). Let M be the contraction of M' to N. By the jacobian criterian $T = N_M$ is regular. Since N_M is a spot over S, there is [2, 1.11] a smallest analytic ring T^* containing T with $\hat{T} = \hat{T}^*$. This ring T^* is just the analytic ring gotten by blowing up and normalizing in the category of analytic rings. Since T is a finite algebra over S, the residue field of T is finite over the residue field of S. Since T and T^* has the same completion, they have the same residue field. It can be shown from [6, section 3], theorem 2.1 and the previous parts of theorem 2.6, that the quadratic transform $i : S \to T^*$ is injective and open and closed in the inductive topology. By 3.1 the composition $\sigma : R \to S \to T^*$ is closed in the inductive topology. By the special case, $rk\ \sigma = \dim R$. Since $rk\ i = \dim S = \dim T$, it follows that $rk\ \varphi = \dim R$.

Remark. — Obviously one would also like to reduce to the case where R is regular via a finite map $A \to R$ where A is regular. However we don't know the composition $A \to R \to S$ is closed in the inductive topology as 3.1

is not applicable. We will get around this difficulty in another way in proposition 3.7.

Remark 3.3. — The maximal ideal M above is not necessarily a k rational point. More precisely let

$$\mathrm{R} = k\langle\langle \mathrm{X}_1,\ldots,\mathrm{X}_n\rangle\rangle/\mathrm{I} = k\langle\langle x_1,\ldots,x_n\rangle\rangle,$$
$$\mathrm{B} = k\langle\langle x_1,\ldots,x_n\rangle\rangle[z_2,\ldots,z_n],$$

where $z_i = x_i/x_1$ for $i \geqslant 2$, and $\mathrm{B}' = k[z_2,\ldots,z_n]$. Then the maximal ideal $\mathrm{M} \cap \mathrm{B}$ is generated by x_1 and $\mathrm{M}' \cap \mathrm{B}'$, but $\mathrm{M}' \cap \mathrm{B}'$ is not necessarily of the form $(z_2-a_2,\ldots,z_n-a_n)$ for some $a_i \in k$. B′ is just the initial ideal of I dehomogenized with respect to x_1 so is likely to be an example of the type below. If the maximal ideal $\mathrm{M} \cap \mathrm{B}$ is not a k rational point the residue field of $\mathrm{B}_{\mathrm{M} \cap \mathrm{B}}$ will be bigger than k.

We have used implicitly the following Hilfsatz in the above discussion. Let $\mathrm{R}_1 \subset \mathrm{R}_2$ be rings with R_2 integral over R_1 and M be a maximal ideal in R_2. Then $\mathrm{R}_1 \cap \mathrm{M}$ is a maximal ideal in R_1.

Proof. — $\mathrm{D} = \mathrm{R}_1/\mathrm{R}_1 \cap \mathrm{M} \subset \mathrm{R}_2/\mathrm{M} = \mathrm{F}$ so D is domain and F is a field and F is integral over D. If $x \in \mathrm{D}$ and

$$\left(\frac{1}{x}\right)^n + a_{n-1}\left(\frac{1}{x}\right)^{n-1} + \cdots + a_1\left(\frac{1}{x}\right) + a_0 = 0, \quad a_i \in \mathrm{D},$$

then $-1 = x(a_{n-1}+a_{n-2}+\ldots+a_0x^{n-1})$ so $\dfrac{1}{x}$ is in D. Hence D is a field.

Remark. — By restricting to an irreductible component of $\hat{\mathrm{B}}_{\mathrm{M} \cap \mathrm{B}}$ and the corresponding component of N, and making a judicious choice of the maximal ideal M, we may assume that the residue fields of N_M and $\mathrm{B}_{\mathrm{M} \cap \mathrm{B}}$ are equal. This is easily seen from the lemma below (since the residue fields of $\mathrm{B}_{\mathrm{M} \cap \mathrm{B}}$ and $\mathrm{B}'_{\mathrm{M}' \cap \mathrm{B}'}$ are clearly equal) which says that outside a set of codim one, the map $\operatorname{Spec} \mathrm{N}' \to \operatorname{Spec} \mathrm{B}'$ is unramified.

LEMMA 3.4. — *Let* A *and* C *be reduced affine rings over a field* k *and* $\mathrm{A} \subset \mathrm{C}$ *be a finite integral extension. Then there is an ideal* $\mathrm{I} \subset \mathrm{A}$ *with* $ht\ \mathrm{I} \geqslant 1$ *such that for every maximal ideal* M *of* A *with* $\mathrm{I} \not\subset \mathrm{M}$ *and maximal ideal* $\bar{\mathrm{M}}$ *of* C *with* $\bar{\mathrm{M}} \cap \mathrm{A} = \mathrm{M}$, *we have* $\mathrm{MC}_{\bar{\mathrm{M}}} = \bar{\mathrm{M}}\mathrm{C}_{\bar{\mathrm{M}}}$ *(i.e. the extension* $\mathrm{A}_\mathrm{M} \subset \mathrm{C}_{\bar{\mathrm{M}}}$ *is unramified).*

Proof. – Recall that if N is a finite R module that Supp N is the set of prime ideals p in R such that $N_p \neq 0$, or equivalently, the set of primes in R which contain the annihilator of N, where $\mathrm{Ann}\, N = \{r \in R : rN = 0\}$.

Let $\Omega(C/A)$ be the module of differentials of C over A. Then $\Omega(C/A)$ is a finite C module and $\mathrm{Ann}\,\Omega(C/A)$ is an ideal in C. Let I be its contraction to A. Let M be any maximal ideal in A with $I \not\subset M$ and $\bar{M}$ any maximal ideal of C with $\bar{M} \cap A = M$. Clearly $MC \subset \bar{M}$ and $\mathrm{Ann}\,\Omega(C/A) \not\subset \bar{M}$, so $\Omega(C/A)_{\bar{M}} = 0$. We have an exact sequence :

$$0 \to C \otimes_A \Omega(A/k) \to \Omega(C/k) \to \Omega(C/A) \to 0.$$

Localizing at $\bar{M}$ and utilizing the functorial properties of the module of differentials yields :

$$0 \to C_{\bar{M}} \otimes_A \Omega(A_M/k) \to \Omega(C_{\bar{M}}/k) \to 0.$$

The residue field F of $C_{\bar{M}}$ is a finite (possibility trivial) extension of k so $\Omega(F/k) = 0$. It follows that $\Omega(C_{\bar{M}}/F) = \Omega(C_{\bar{M}}/k) = 0$. Now if $x_1, \ldots, x_j$ are in the maximal ideal of the localization of an affine ring, then $x_1, \ldots, x_j$ generate that maximal ideal if and only if $dx_1, \ldots, dx_j$ generate the module of differentials of that local ring over its residue field. Hence the last exact sequence implies that MA_M generates $\bar{M}C_{\bar{M}}$.

It remains to show $ht\, I \geqslant 1$. If k is algebraically closed then the inclusion $A \subset C$, induces a finite map $\mathrm{Spec}\, C \to \mathrm{Spec}\, A$, and any regular point of Spec C where the map has maximal jacobian rank will be a point not in I so $ht\, I \geqslant 1$. We now reduce to the above case. Let $\bar{k}$ be an algebraic closure of k. Then

$$\Omega(C \otimes_k \bar{k}/A \otimes_k \bar{k}) = \Omega(C/A) \otimes_k \bar{k},\quad \mathrm{Ann}\,(\Omega(C/A)) \otimes_k \bar{k} = \mathrm{Ann}\,(\Omega(C/A) \otimes_k \bar{k}),$$

$A \otimes_k \bar{k} \subset C \otimes_k \bar{k}$ is a finite extension, tensoring with $\bar{k}$ over k preserves height, $A \otimes_k \bar{k}$ and $B \otimes_k \bar{k}$ are reduced. Hence the general case follows.

Remark 3.5. – Assume $[F : k] = d < \infty$ and $\eta \in F$ is a primitive element for the field extension. Any element of F can be written uniquely as a sum $\sum_{i=0}^{d-1} a_i \eta^i$, where $a_i \in k$. Applying this process to the coefficients of a power series yields a k linear additive map $\sigma : F[[z_1, \ldots, z_n]] \to k[[t_1, \ldots, t_{nd}]]$, where the t_l are the $\eta^i z_j$ for $0 \leqslant i \leqslant d-1$, $1 \leqslant j \leqslant n$. Now σ does not in general, carry $F\langle\langle z\rangle\rangle$ to $k\langle\langle t\rangle\rangle$.

In lemma 3.2, we constructed a map $\psi : S \to T^* = F\langle\langle z\rangle\rangle$. For technical reasons arising in proposition 3.7, we desire this map to go into $k\langle\langle t\rangle\rangle$ instead. We can't just apply σ; however we will show that in fact because of the special construction of the map $S \to T^*$, we can assume $\psi(S) \subset k\langle\langle t\rangle\rangle$. By 3.4, there is no change of residue field in passing from B to N, so it suffices to show the image of S in $B_{M \cap B}$ lies in a k analytic ring. The maximal ideal $\tilde{M} = M \cap B$ can be constructed as follows: $B' = B/I = k[z_2,\ldots,z_n]$, $z_i = x_i/x_1$, is an affine ring over k, choose a maximal ideal $\tilde{M}_0$ of $B' \otimes_k \bar{k} = \bar{B}'$, $\tilde{M}_0 = (z_i - a_i)\bar{B}$, $a_i \in \bar{k}$, and let $\tilde{M}' = \tilde{M}_0 \cap B'$ and $\tilde{M}$ be the contraction of $\tilde{M}'$ to B. Then $F = k(a_i)$ is a finite extension of k, F is the residue field of $B_{\tilde{M}}$, and z_1, $z_2 - a_2, \ldots, z_n - a_n$ generate the maximal ideal of $B_{\tilde{M}}$. To show $\psi(S) \subset k\langle\langle t\rangle\rangle$ it is sufficient to show each $\psi(x_i) \in k\langle\langle t\rangle\rangle$. But $x_i = z_1 z_i = z_1(z_i - a_i) + a_i z_1$; since this is a polynomial over F it obviously gives rise to a convergent power series over k.

Remark 3.6. – It remains to check that the topological conditions of part 8 of theorem 2.6 are preserved when passing from $F\langle\langle z\rangle\rangle$ to $k\langle\langle t\rangle\rangle$ in remark 3.5. Since a norm of $\sigma(g)$ in $k\langle\langle t\rangle\rangle$ is not less than the corresponding norm of g in $F\langle\langle z\rangle\rangle$, it is trivial to check that S closed in the inductive topology on $F\langle\langle z\rangle\rangle$ implies that S closed in the inductive topology on $k\langle\langle t\rangle\rangle$. Also the ranks of both maps $S \to F\langle\langle z\rangle\rangle$ and $S \to k\langle\langle t\rangle\rangle$ are the same and equal to dim S so the rank of the map $R \to S \to F\langle\langle z\rangle\rangle$ is unchanged by this change of residue field.

DEFINITION. – *Let* F *be a valued field and* $Z = (Z_1,\ldots,Z_n)$ *be indeterminents over* F. *A local map* $H : F\langle\langle Z\rangle\rangle \to F\langle\langle Z\rangle\rangle$ *is called a monomial map if* H *is of the form* $(Z_1, M_2Z_2, \ldots, M_nZ_n)$ *where each* M_j *is a nonzero monomial in only the variables* $Z_1, \ldots, Z_{j-1}$. *A routine computation similar to* [6, *Remark* 4] *shows that* H *is injective, open in the Krull topology, and strongly injective. Also it is easy to check* H *is open and closed in the inductive topology. Also* $rk\, H = n$.

LEMMA 3.7. – *Reduction of part* 8 *to the special case where* $rk\ \varphi = \dim S$.

Proof. – Suppose $Y = (Y_1,\ldots,Y_r)$; $x = (x_1,\ldots,x_n)$, $\varphi : k\langle\langle Y\rangle\rangle \to k\langle\langle x\rangle\rangle$, and rank $\varphi < n$. Then Eakin and Harris [14, section 4] have proven there exists an isomorphism $H_1 : k\langle\langle Y\rangle\rangle \to k\langle\langle Y\rangle\rangle$, and a map $H_2 : k\langle\langle x\rangle\rangle \to k\langle\langle x\rangle\rangle$ which is the composition of monominal maps such that

$$\psi = H_2(\varphi)H_1 : k\langle\langle Y\rangle\rangle \to k\langle\langle x_1,\ldots,x_{n-1}\rangle\rangle \text{ i.e.}$$

you can transform off one of the variables. From 3.1, it follows that ψ is closed in the inductive topology on $k\langle\langle x_1,\ldots,x_n\rangle\rangle$ induces the inductive topology on $k\langle\langle x_1,\ldots,x_{n-1}\rangle\rangle$, ψ is also closed as a map into $k\langle\langle x_1,\ldots,x_{n-1}\rangle\rangle$. Also $rk\,\psi = rk\,\varphi$ since $rk\,\mathrm{H} = n$.

PROPOSITION 3.7. — *Assume* $\dim \mathrm{R} = r > n = \dim \mathrm{S} = rk\,\varphi$, S *is regular,* $\varphi : \mathrm{R} \to \mathrm{S}$ *is closed in the inductive topology. Then there exists a sequence* $h_j \in \mathrm{R}$ *such that* $\varphi(h_j) \to 0$ *in the inductive topology on* S, *but* $h_j \nrightarrow 0$ *in the simple topology on* R.

Proof. — Suppose $\varphi : \mathcal{O}_r \to \mathcal{O}_n$ is injective and $r > n$. By generalizing [14, lemma 4.2] the construction used in Gabrielov's example, we have that there exists a sequence of polynomials $f_\nu \in k[\mathrm{Y}_1,\ldots,\mathrm{Y}_r]$ of degree ν so that

(i) $\max\{|b| : b \text{ is a coefficient of } f_\nu\} = 1$

(ii) $\lim\limits_{\nu\to\infty} \|f_\nu(\varphi_1,\ldots,\varphi_r)\|_t^{1/\nu} = 0$ for some $t > 0$.

Let $\mathrm{P}_k = \sum\limits_{i=0}^{k} \dfrac{1}{\varepsilon_{i\,!}} f_{i\,!}(\mathrm{Y}_1,\ldots,\mathrm{Y}_r)\mathrm{Y}_1^{i\,!}$ and $\mathrm{P} = \lim\limits_{k\to\infty} \mathrm{P}_k$, where $\varepsilon_\nu = \|f_\nu(\varphi)\|_t$. For each i, $f_{i\,!}(\mathrm{Y}_1,\ldots,\mathrm{Y}_r)\mathrm{Y}_1^{i\,!}$ is of degree $2(i\,!)$ and order $\geqslant i\,!$; thus P is well defined. Moreover, P is divergent because for each i, $f_{i\,!}$ has a coefficient 1, which cannot be cancelled by a coefficient from any other position, and for any k with $i\,! \leqslant k \leqslant 2(i\,!)$, $(1/\varepsilon_{i\,!})^{1/k} \to \infty$ as $i \to \infty$. Hence the sequence P_k of polynomials converges in no Banach norm in $\mathcal{O}_r$. However,

$$\|\mathrm{P}(\varphi)\|_t \leqslant \sum_{i=0}^{\infty} \frac{1}{\varepsilon_{i\,!}} \|f_{i\,!}(\varphi)\|_t \|\varphi_1\|_t^{i\,!} \leqslant \sum_{i=0}^{\infty} \|\varphi_1\|_t^{i\,!} < \infty,$$

provided t is chosen so $\|\varphi_1\|_t < 1$. It follows that $\mathrm{P}_k(\varphi)$ converges in the inductive topology on $\mathcal{O}_n$.

Now R is a finite extension of a regular ring $\mathrm{A} = k\langle\langle \mathrm{Y}_1,\ldots\mathrm{Y}_r\rangle\rangle \subset \mathrm{R}$. Let φ also denote the restriction of φ to $\mathcal{O}_r$, and let P_j denote the image of the above polynomials in R. Since $\varphi(\mathrm{P}_j)$ converges to $\varphi(\mathrm{P})$ in the inductive topology and R is closed in S, $\varphi(\mathrm{P}) \in \varphi(\mathrm{R})$. That is there exists $g \in \mathrm{R}$, $g = \lim g_j$, g_j polynomial of degree j, with $\varphi(g) = \varphi(\mathrm{P})$. Since g is convergent, $g_j \to g$ in the inductive topology, and so $\varphi(g_j) \to \varphi(g)$ in the inductive topology. Letting $h = g - \mathrm{P}$, $h_j = g_j - \mathrm{P}_j$, we have $\varphi(h_j) \to 0$ in the inductive topology on S and $h_j \to h$ in the simple topology on R. But $h \neq 0$ since $g \in \mathrm{R}$, and $\mathrm{P} \notin \mathrm{R}$ (because $\mathrm{P} \in \hat{\mathrm{A}} - \mathrm{A}$ and $\hat{\mathrm{A}} \cap \mathrm{R} = \mathrm{A}$).

LEMMA 3.7. — *Let* R *be an analytic domain. Then for every* $l > 0$, *there is a countable set* $\{p_i\}_{i=1}^{\infty}$ *of prime ideals in* R, *with each* $\dim R/p_i = l$, *and a function* $n : \mathrm{N} \to \mathrm{N}$, *with* $\lim_{j\to\infty} n(j) = \infty$, *and so that* $q_{n(j)} \subset m^j$, *where* $q_j = \bigcap_{i=1}^{j} p_i$, *and* m *is the maximal ideal of* R.

Proof. — By a trivial backwards induction on l, it suffices to consider the case where $l = \dim R - 1$.

Now let R be finite extension of a regular ring $A = k\langle\langle Y_1,\ldots,Y_r\rangle\rangle$. Then $\hat{R}$ is a finite extension of $\hat{A}$. Take any countable distinct set of hyperplanes H_i in k^r and I_i be the ideal of H_i in A. Clearly $\bigcap_{i=1}^{\infty} I_i = (0)$ and $\bigcap_{i=1}^{\infty} \hat{I}_i = (0)$. Now I_iR is not usually prime, but there exist a finite number of primes p_i in R so that the contraction to A of each of them is just I_i. Let $\{p_i\}$ be the set of all such resulting primes. Since

$$\left(\bigcap_{i=1}^{\infty} p_i\right) \cap A = \left(\bigcap_{i=1}^{\infty} I_i\right) \cap A = (0)$$

and R is a domain, we have that $\bigcap_{i=1}^{\infty} p_i = (0)$. Since $\hat{p}_i$ is a height one prime, $\hat{p}_i$ is also a height one prime; hence $\hat{p}_i \cap \hat{A}$ is a height one prime. But $p_i = \hat{p}_i \cap R$, $I_i = p_i \cap A$, and $I_i = \hat{p}_i \cap A$, so $\hat{I}_i \subset \hat{p}_i \cap \hat{A}$. Since $\hat{I}_i$ and $\hat{p}_i \cap \hat{A}$ are both height one primes, they are equal. Since $\hat{R}$ is a domain, the previous line of reasoning show that $\bigcap_{i=1}^{\infty} \hat{p}_i = (0)$. Hence

$$\bigcap_{j=1}^{\infty} \hat{q}_j = \bigcap_{j=1}^{\infty} \bigcap_{i=1}^{j} \hat{p}_i = \bigcap_{i=1}^{\infty} \hat{p}_i = (0).$$

By the Chevalley subspace theorem $\forall j \exists n$ so $\hat{q}_n \subset \hat{m}^j$. Then

$$q_n = \hat{q}_n \cap R \subset \hat{m}^j \cap R = m^j.$$

LEMMA 3.8. — *Let* R *be an analytic domain and* f_h *be a sequence in* R. *Then* $f_h \to 0$ *in the simple topology on* R *if and only if the images of* f_h *in* R/p *converge to zero in the simple topology, for every prime* p *of* R *with* $\dim R/p = 1$.

Proof. — Let p_i, q_j be as in previous lemma. Now $f_h \to 0$ in R iff $\forall k$, $f_h \to 0$ in R/m^k. Hence $[\forall i,\ f_h \to 0 \text{ in } R/p_i] \Rightarrow [\forall j, f_h \to 0 \text{ in } R/q_j]$ because

$$R/q_j \to \bigoplus_{i=1}^{j} R/p_i$$

is an injection of finite R modules $\Rightarrow [\forall l, f_h \to 0 \text{ in } R/m^l]$ because $q_j \subset m^l \Rightarrow f_h \to 0$ in R.

Part 8 of theorem 2.6 will follow immediately from propositions 3.4 and 3.9.

PROPOSITION 3.9. — *Assume* $\varphi : R \to S$ *is injective and closed in the inductive topology. If* S *is the inductive limit of the algebras* B_r, h_j *is a sequence in* R, *with* $\varphi(\{h_j\}_{j=1}^{\infty})$ *contained in a fixed* B_r *and* $\varphi(h_j) \to 0$ *in* B_r, *then* $h_j \to 0$ *in the simple topology on* R.

Remark. — We use the hypothesis that each $\varphi(h_j) \in B_r$, rather than just saying $\varphi(h_j) \to 0$ in the inductive topology on S because without the compactness of the inclusions $B_t \to B_s$, $s < t$, this condition does not seem to follow automatically from the fact that $\varphi(h_j)$ converges.

Proof of prop. — Let B denote one of the algebras B_r, with r small enough so that each $\varphi(h_j) \in B$ and each $\varphi(y_i) \in B$ where $y_1, \ldots, y_n$ generate the maximal ideal of R. Let $\psi : B \cap \varphi(R) \to R$ denote the inverse of $\varphi | R \cap \varphi^{-1}(B)$. We give $B \cap \varphi(R)$ the topology from the norm $\|-\|$ on B, and R the simple topology, and show ψ is continuous.

By lemma 3.8 it suffices to prove that for every prime p in R with $\dim R/p = 1$, the composition $B \cap \varphi(R) \to R \to R/p$ is continuous, where R/p has the simple topology. Let N be the integral closure of R/p in its field of quotients. Since $\dim R/p = 1$, N is normal and of dim one so regular. Hence there is a finite field extension F of k so that $N = F\langle\langle t\rangle\rangle$. Clearly it also suffices to show the composition, also denoted by ψ, $B \cap \varphi(R) \to R \to R/p \to F\langle\langle t\rangle\rangle$ is continuous.

Any $h \in F\langle\langle t\rangle\rangle$ has a representation $\sum_{n=0}^{\infty} h_n t^n$ with $h_n \in F$. For some y_i, the image $\bar{y}_i$ of y_i in $F\langle\langle t\rangle\rangle$ is a nonzero nonunit. Without loss of generality, we may assume $\|\bar{y}_i\| < 1$. Let f denote $\varphi(y_i)$ and $\sum_{n \geqslant p} C_n t^n$ be the expansion of $\psi(f) = \bar{y}_i$, where $p > 0$ is the order of $\psi(f)$. Let

$$\psi(f)^k = \sum_{n \geqslant kp} C_{kn} t^n,$$

where $C_{kn} \in F$ and $C_{k,kp} \neq 0$. Clearly f lies in the maximal ideal of S.

We will now prove the following statement by induction on l (the desired result is the totality of all these statements) :

For every integer l, there exists a real number M_l so that for every

$$(*) \qquad g \in R \cap \varphi(B), \qquad |\psi(g)_l| \leqslant M_l \|g\|.$$

Since φ does not change the constant term of a power series, and $\|g\| \geqslant$ the absolute value of its constant term, we may pick $M_0 = 1$, and the $l = 0$ stage of the induction is done. Assume the induction hypothesis for $j = 0, \ldots, l-1$, and its negation for $j = l$; we will reach a contradiction. We have : $\forall 0 \leqslant j \leqslant l-1$, $\exists M_j$ so

$$\forall g \in B \cap \varphi(R), \quad |\varphi(g)_j| \leqslant M_j \|g\| \quad \text{and} \quad \forall M_l, \ \exists g \in B \cap \varphi(R)$$

with $|\varphi(g)_l| > M_l \|g\|$.

Now pick inductively $g_i \in B \cap \varphi(R)$ so that $\|g\| < 1$ and $|\psi(g_i)_l|$ is arbitrarily large (exact formula to be given later). Then $\sigma = \sum_{i=1}^{\infty} g_i f^i$ converges in the Krull topology on S so σ is also a formal power series over k. Since $\|f\| < 1$, the sum also converges in the norm on B; hence $\sigma \in B$. Finally the partial sums all lie in $\varphi(R)$ and converge in B and hence in the inductive topology on S; since $\varphi(R)$ is closed in S, we have $\sigma \in \varphi(R)$. Similarly for any $n \geqslant 0$, $\sum_{i=1}^{n} g_{i+n} f^i$ converges to an element of $B \cap \varphi(R)$.

I now claim that $\psi\left(\sum_{k=1}^{\infty} g_k f^k\right) = \sum_{k=1}^{\infty} \psi(g_k)\psi(f)^k$ as formal power series in t. To see this, recall that two elements of $F[[t]]$ are equal if and only if their coefficients are all equal and note that

$$\psi\left(\sum_{k=1}^{\infty} g_k f^k\right) = \sum_{k=1}^{n} \psi(g_k)\psi(f)^k + \psi(f)^n \psi\left(\sum_{k=n+1}^{\infty} g_k f^{k-n}\right) = R_1 + R_2,$$

and

$$\sum_{k=1}^{\infty} \psi(g_k)\psi(f)^k = \sum_{k=1}^{n} \psi(g_k)\psi(f)^k + \psi(f)^n \sum_{k=n+1}^{\infty} \psi(g_k)\psi(f)^{k-n} = S_1 + S_2.$$

The first n coefficients of R_2 and of S_2 are all zero so the n th order expansion of $R_1 + R_2$ equals the n th order expansion of R_1, and the n th order expansion of $S_1 + S_2$ equals the n th order expansion of S_1. Also $R_1 = S_1$, so the n th order expansion of R_1 equals the n th order expansion of S_1. Hence the n th order expansion of $R_1 + R_2$ equals the

n th order expansion of $S_1 + S_2$, for all n. Hence

$$R_1 + R_2 = S_1 + S_2 \quad \text{and} \quad \sum_{k=1}^{\infty} \psi(g_k)\psi(f)^k$$

is a convergent power series in t.

Let $[r]$ denote the greatest integer less than or equal to r and let $h = [(m - l)/p]$. Let $\psi(g_k) = \sum_{i=0}^{\infty} A_{ki}t^i$, $\psi(g_k)_i = A_{ki}$, and compute

$$\begin{aligned}
\psi(\sigma) &= \sum_{k=1}^{\infty} \psi(g_k)\psi(f)^k = \sum_{k=1}^{\infty}\left(\sum_{i=0}^{\infty} A_{ki}t^i\right)\left(\sum_{n\geqslant kp} C_{kn}t^n\right) \\
&= \sum_{m=1}^{\infty} t^m\left(\sum_{k=1}^{m}\sum_{j=0}^{m-pk} A_{kj}C_{k,m-j}\right) \\
&= \sum_{m-1}^{\infty} t^m \left[\begin{array}{l} \displaystyle\sum_{k=1}^{h-1}\sum_{j=0}^{m-pk} A_{kj}C_{k,m-j} + \sum_{j=0}^{m-ph-1} A_{hj}C_{h,m-j} \\ \displaystyle + A_{kl}C_{h,m-l} + \sum_{j=0}^{l-1}\sum_{k=h+1}^{m} A_{kj}C_{k,m-j} \end{array}\right].
\end{aligned}$$

Recall that $|A_{kj}| = |\psi(g_k)_j| \leqslant M_j\|g_k\| \leqslant M_j$ for $0 \leqslant j \leqslant l - 1$ and $k \geqslant j$. Hence for $m \geqslant l(p + 1)$, we have

$$|\varphi(\sigma)_m - A_{hl}C_{h,m-l}| \leqslant \left|\sum_{k=1}^{h-1}\sum_{j=0}^{m-pk} A_{kj}C_{k,m-j}\right| + \sum_{j=0}^{l-1}\sum_{k=h}^{m} M_j|C_{k,m-j}|.$$

It is possible to inductively pick for $m = kp + l$, $h = k$, the $\varphi(g_h)_l$ so that

$$|\varphi(g_h)_l| \geqslant |C_{h,m-l}|^{-1}\left[\left|\sum_{k=1}^{h-1}\sum_{j=0}^{m-pk} A_{kj}C_{k,m-j}\right| + \sum_{j=0}^{l-1}\sum_{k=h}^{m} M_j|C_{k,m-j}| + m^m\right]$$

because $C_{k,kp} \neq 0$, the first sum runs from $k = 1, \ldots, h - 1$ so depends on previously picked A_{kj}'s and the second sum runs from $j = 0, \ldots, l - 1$, so also depends on previously picked data. Making these choices, we have $|\psi(\sigma)_m| \geqslant m^m$. The radius of convergence of the power series $\Sigma\, a_n z^n$ is $\liminf |a_n|^{-1/n}$, so we have a contradiction with the fact that $\psi(\sigma)$ is convergent. Line (*) is now proven.

Q.E.D.

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