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RADON-NIKODYM PROPERTY
FOR VECTOR-VALUED INTEGRABLE FUNCTIONS

by Surjit Singh KHURANA

It is proved in ([6], Theorem 1) that if a Banach space E possesses Radon-Nikodym (R − N) property, then the Banach space $L_p(E, \lambda)$, $1 < p < \infty$, of Bochner p-integrable functions also possesses this property. In this paper we give a new proof of the corresponding result when E is a Frechet space (i.e. a complete metrizable locally convex space [5]).

Let $(Y, \mathcal{B}, \nu)$ be a positive measure which is non-trivial (i.e., there exists a $B \in \mathcal{B}$ such that $0 < \nu(B) < \infty$) with $\mathcal{R} = \{A \in \mathcal{B} : (2(A)) < \infty\}$, E a Frechet space with $\{P_i\}$ an increasing sequence of semi-norms on E generating the topology of E, and $L_p(E, \lambda)$ the equivalence classes of strongly p-power integrable functions $Y \longrightarrow E, 1 \leqslant p < \infty$. (A function $f: X \longrightarrow E$ is called strongly p-power integrable if there exists a sequence $\{f_n\}$ of $\mathcal{R}$-simple E-valued functions of Y such that (i) $f_n \longrightarrow f$ a.e. $[\nu]$, and (ii) $\int [P_i(f_n - f)]_i^p \, d\nu \longrightarrow 0$, $\forall i$. The increasing sequence of semi-norms

$$N_{i,p}, \, N_{i,p}(f) = \left[\int [P_i(f)]^p \, d\nu \right]^{1/p}$$

makes $L_p(E, \lambda)$ a Frechet space. We use the definition of [4] for a Frechet space to have R − N property.

E = K, we denote $L_p(E, \nu)$ by $L_p(\nu)$ and the corresponding norm by $\| \cdot \|_p$.

THEOREM. − *Suppose* E *is a Frechet space with* R − N *property and* $(Y, \mathcal{B}, \nu)$ *a non-trivial positive measure space. Then* $L_p(E, \nu)$ *has* R − N *property for* $1 < p < \infty$.

Proof. — Using ([3], Theorem 5 (iv)) it is sufficient to prove the R − N property for every separable closed subspace; this means we can assume that E is separable ([3], Theorem 5). Let $(X, \mathcal{U}, \lambda)$ be a finite measure space, $\mu: \mathcal{U} \longrightarrow L_p(E, \nu)$ a measure of finite variation (i.e., $\forall i$, the variation of μ relative to $N_{p,i}$ is finite, [4]), absolutely continuous with respect to λ. Assume first that $\nu(Y) < \infty$ and let $\lambda \times \nu$ be the product of λ and ν on the σ-algebra $\mathcal{U} \times \mathcal{B}$.

For an $A \in \mathcal{U}, B \in \mathcal{B}$, define $\omega(A \times B) = \int_B \mu(A) \, d\nu \in E$ (since $\nu(Y) < \infty$, $P_i(\mu(A)) \in L_p(\nu)$, $\forall i$, implies $P_i(\mu(A)) \in L_1(\nu)$). Take $\{A_i \times B_i\}$ a disjoint sequence in $X \times Y$ $(A_i \in \mathcal{U}, B_i \in \mathcal{B})$ and let $\cup A_i \times B_i = A \times B$ $(A \in \mathcal{U}, B \in \mathcal{B})$. Fix an $f \in E'$. $f \circ \mu: \mathcal{U} \longrightarrow L_p(\nu)$ is of bounded variation and absolutely continuous relative to λ. Since $L_p(\nu)$ has R − N property, there exists a function $\phi: X \times Y \longrightarrow K$ such that

$$f \circ \mu(A) = \int_A \phi(x, y) \, d\lambda \, (x), \forall A \in \mathcal{U};$$

it is routine verification that $\phi(x, y) \in L_1(\lambda \times \nu)$. Thus

$$\int_{B_i} f \circ \mu(A_i) \, d\nu = \int_{A_i \times B_i} \phi(x, y) \, d(\lambda \times \nu)$$

(Fubini's theorem) and so $\sum \int_{B_i} f \circ \mu(A_i) \, d\nu = \int_B f \circ \mu(A) \, d\nu$ (unconditional convergence). Since $\langle f, \int_{B_i} \mu(A_i) \, d\nu \rangle = \int_{B_i} f \circ \mu(A_i) \, d\nu$, (simple verification), $\forall f \in E'$, by Pettis-Orlicz theorem,

$$\sum \int_{B_i} \mu(A_i) \, d\nu = \int_B \mu(A) \, d\nu.$$

Also for a finite disjoint collection $\{C_i \times D_i\}$ in $X \times Y$ $(C_i \in \mathcal{U}, D_i \in \mathcal{B})$,

$$\sum \int_{D_i} f \circ \mu(C_i) \, d\nu = \int_{\cup C_i \times D_i} \phi(x, y) \, d(\lambda \times \nu) \quad \text{(previous notation) and}$$

so $\left| f \circ \sum \int_{D_i} \mu(C_i) \, d\nu \right| \leqslant \int |\phi(x, y)| \, d(\lambda \times \nu)$. Combining these results we see that ω can be uniquely extended to a finitely additive set function $\omega: \theta \longrightarrow E$, θ being the algebra generated by $\{A \times B: A \in \mathcal{U}, B \in \mathcal{B}\}$. ω is countably additive, and $\omega(\theta)$ is bounded in E. Since E has R − N property it cannot contain a subspace isomorphic to c_0 ([1]; [3], Theorem 5). From this it easily follows that ω is exhaustive ([2], II; [7], Theorem 4). Thus ω can be uniquely extended to a countably additive measure on the σ-algebra $\mathcal{U} \times \mathcal{B}$ ([2], III). We claim that $\omega \ll \lambda \times \nu$. For an $f \in E'$, $A \in \mathcal{U}, B \in \mathcal{B}$, $f \circ \omega(A \times B) = \int_{A \times B} \phi(x, y) \, d(\lambda \times \nu)$ (pre-

vious notations) and so $f \circ \omega(H) = \int_H \phi \, d(\lambda \times \nu)$, $\forall H \in \mathcal{U} \times \mathcal{B}$.
If $(\lambda \times \nu)(H) = 0$ we get $f \circ \omega(H) = 0$ and so $\omega(H) = 0$.

We now prove that ω is of finite variation. Fix $i \in N$ and let $\lambda_0 = $ the finite variation of μ relative to the semi-norm $N_{i,p}$.
$$H = \{f \in E', |f(x)| \leqslant P_i(x), \forall x \in E\}$$
is a metrizable compact subset of $(E', \sigma(E', E))$ and so have a countable dense subset $\{f_j\}$. Let $\phi_j \in L_1(\lambda_0 \times \nu)$ such that
$$f_j \circ \mu(A) = \int_A \phi_j(x, y) \, d\lambda_0(x)$$
(same reasoning as before).

Let $\varphi_0 = \sup(|\varphi_1|, |\varphi_2|)$ and fix $x \in X$. Take
$$B_1 = \{y \in Y : |\varphi_1(x, y)| = \varphi_0(x, y)\} \text{ and } B_2 = Y \backslash B_1.$$
We claim the variation of $\xi = \chi_{B_1} f_1 \circ \mu + \chi_{B_2} f_2 \circ \mu$, in $L_p(\nu)$, does not exceed λ_0. From
$$|\xi(A)|^p = |(\chi_{B_1} f_1 \circ \mu + \chi_{B_2} f_2 \circ \mu)(A)|^p$$
$$\leqslant \chi_{B_1}(P_i(\mu(1)))^p + \chi_{B_2}(P_i(\mu(A)))^p = (P_i(\mu(A)))^p,$$
we get $\|\xi(A)\|_p \leqslant N_{i,p}(\mu(A)) \leqslant \lambda_0(A)$ and so the claim is established.
Now $\xi(A) = \int_A (\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2) \, d\lambda_0$. If $|\xi|$ is the variation of ξ relative to $L_p(\nu)$, then $|\xi|(A) = \int_A \|\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2\|_p \, d\lambda_0$.
If $\|\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2\|_p \geqslant 1 + \eta$ for some $\eta > 0$ on $A \in \mathcal{U}$, then $\lambda_0(A) \geqslant |\xi|(A) \geqslant (1 + \eta)\lambda_0(A)$ which means $\lambda_0(A) = 0$ and so $\|\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2\|_p \leqslant 1$ a.e. $[\lambda_0]$. Now
$$\|\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2\|_1 \leqslant \|\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2\|_p (\lambda_0(X))^{1/q} \leqslant (\lambda_0(X))^{1/q}$$

$\left(\text{Holder's inequality with } \dfrac{1}{p} + \dfrac{1}{q} = 1\right)$ means
$$\int \varphi_0(x, y) \, d\nu(y) \leqslant (\lambda_0(X))^{1/q}, \quad \text{a.e. } [\lambda_0].$$
Since $x \in X$ was arbitrary we see $\varphi_0(x, y) \in L_1(X \times Y, \lambda_0 \times \nu)$.
If $\varphi = \sup(|\varphi_1|, |\varphi_2|, \ldots)$, then proceeding in the same way we prove that $\int \varphi(x, y) \, d\nu(y) \leqslant (\lambda_0(X))^{1/q}$, a.e. $[\lambda_0]$ and so $\varphi \in L_1(X \times Y, \lambda_0 \times \nu)$ (the set where φ takes values $+\infty$ has measure zero; we put $\varphi \equiv 0$ on that set). Fix $\epsilon > 0$ and let $\{H_j\}$ be a finite disjoint collection in $\mathcal{U} \times \mathcal{B}$.
$$\sum_j P_i(\omega(H_j)) - \epsilon \leqslant \sum_j |f_{k(j)} \circ \omega(H_j)| \leqslant \sum \int_{H_j} |\varphi_{k(j)}| \, d(\lambda_0 \times \nu)$$
$$\leqslant \sum \int_{H_j} \varphi \, d(\lambda_0 \times \nu) \leqslant \int \varphi \, d(\lambda_0 \times \nu)$$

for some finite sequence $\{k(j)\} \subset N$. This proves ω is of finite variation. Since E has $R - N$ property we get a $g \in L_1(E, \lambda \times \nu)$ such that $\int_B \mu(A) d\nu = \int_B \int_A g(x, y) d\lambda(x) d\nu(y)$.

Put $\psi = \mu(A) - \int_A g(x, y) d\lambda(x)$. We get $\int_B \psi d\nu = 0$, $\forall B \in \mathcal{B}$. Fix $i \in N$ and let $\{f_j\}$ be a countable dense set in the compact metric space

$$H = \{f \in E': |f(x)| \leqslant P_i(x), \ \forall x \in E\} \subset (E', \sigma(E', E)).$$

We get $\int_B f_j \circ \psi = 0$, $\forall B \in \mathcal{B}$ and so $P_i(\psi) = 0$ a.e. $[\nu]$. Thus $\psi = 0$ a.e. $[\nu]$. Thus $\mu(A) = \int_A g(x, y) d\lambda(x)$. It is easy to verify that $g(\cdot, x) \in L_1(L_p(E, \nu), \lambda)$.

Now we consider the case when $\nu(Y) = + \infty$. By ([3], Theorem 5) it is enough to prove the result for every closed separable subspace of $L_p(E, \nu)$. Let F be a closed separable subspace of $L_p(E, \nu)$. It is a simple verification that there exists a $B \in \mathcal{B}$ with σ-finite ν-measure such that $f = 0$ a.e. $[\nu]$ outside B, $\forall f \in F$. Thus ([3], Theorem 5) we can assume that ν is σ-finite. Let $\{K_n\}$ be a $\mathcal{B}$-measurable partition of Y, such that $0 < \nu(K_n) < \infty$, $\forall n$. Define $\nu_n = \chi_{K_n} \nu$, $\nu_n: \mathcal{B}_n \longrightarrow [0, \infty)$, $\mathcal{B}_n = \mathcal{B} \cap K_n$. Given μ as before, we get $\mu_n: \mathcal{U} \longrightarrow L_p(E, \nu_n)$, $\mu_n(A) = \chi_{K_n} \mu(A) \in L_p(E, \nu_n)$. It is easy to verify that μ_n is of finite variation relative to $L_p(E, \nu_n)$ and absolutely continuous relative to λ. Proceeding as before we get $g_n: X \times K_n \longrightarrow E$ such that

$$\mu_n(A) = \int_A g_n(x, y) d\lambda(x) = \int_A \chi_{K_n} g_n(x, y) d\lambda(x).$$

Define $g(x, y) = g_n(x, y)$, $y \in K_n$, we claim

$$\mu(A) = \int_A g(x, y) d\lambda(x).$$

If $\mu(A) = f \in L_p(E, \nu)$, then

$$(P_i(\mu(A) - \sum_{j=1}^k \mu_j(A)))^p \leqslant (P_i(f))^p$$

and so by dominated convergence theorem $\sum_{j=1}^k \mu_i(A)$ converges to $\mu(A)$ in $L_p(E, \nu)$. Let $|\mu|$ and $\left|\sum_{j=1}^k \mu_j\right|$ be the variations of μ and $\sum_{j=1}^k \mu_j$ relative to $N_{i,p}$. Then

$$|\mu|(A) \geqslant \left|\sum_{j=1}^{\infty}\mu_j\right|(A) = \int_A N_{i,p}\left(\sum \chi_{K_j}g_j\right)d\lambda.$$

By monotone convergence theorem $N_{i,p}(g) < \infty$, a.e. $[\lambda]$ and $\int N_{i,p}(g)d\lambda < \infty$. On the set where $N_{i,p}(g) = +\infty$ we change its value and the value of each of g_n to 0 and so $g(\cdot, x) \in L_p(E, \nu)$, $\forall x \in X$. Now it is easy to verify that $h_n = \sum_{j=1}^{n}\chi_{K_j}g_j$ converges to g in $L_p(E, \nu)$, a.e. $[\lambda]$ and $N_{i,p}(g - h_n)$, as a function of x, is decreasing as n increases. By monotone convergence theorem, $\int N_{i,p}(g - h_n)d\lambda \longrightarrow 0$. Thus

$$N_{i,p}\left(\int_A gd\lambda - \int_A h_n d\lambda\right) \leqslant \int N_{i,p}(g - h_n)d\lambda \longrightarrow 0$$

and so $\int_A h_n d\lambda \longrightarrow \int_A gd\lambda$ in $L_p(E, \lambda)$. But

$$\int_A h_n d\lambda = \sum_{j=1}^{n}\mu_j(A) \longrightarrow \mu(A)$$

and so $\mu(A) = \int_A gd\lambda$. The result now follows easily.

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