ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 26, nº 4 (1976), p. 99-114

<http://www.numdam.org/item?id=AIF_1976__26_4_99_0>

Ann. Inst. Fourier, Grenoble
26, 4 (1976), 99-114.

ON VITALI-HAHN-SAKS-NIKODYM TYPE THEOREMS

by **Barbara T. FAIRES**

Vitali-Hahn-Saks-Nikodym type theorems have long been of interest to measure theorists. Starting with G. Vitali's now classical research [23] relating integral convergence and equi-absolutely integrable sequences and continuing through the work of H. Hahn [14], O. Nikodym [15], [16] and S. Saks [20], the following well-known results emerged.

NIKODYM'S CONVERGENCE THEOREM. — *If* (μ_n) *is a sequence of realvalued countably additive measures defined on a sigma algebra* Σ *and for each* $E \in \Sigma$, $\lim_n \mu_n(E)$ *exists, then* $\mu_0(E) = \lim_n \mu_n(E)$ *defines a countably additive measure on* Σ.

VITALI-HAHN-SAKS THEOREM. — *If* (μ_n), μ *are real-valued countably additive measures defined on a sigma algebra* Σ, *such that each* μ_n *is* μ*-continuous and* $\lim_n \mu_n(E)$ *exists for each* $E \in \Sigma$, *then* $\lim_{\mu(E)\to 0} \mu_n(E) = 0$ *uniformly in* n.

The first extension (related to the results of the present paper) of the Vitali-Hahn-Saks and Nikodym convergence theorems were to Banach space-valued measures. The measures μ_n were still required to be countably additive. A discussion of such extensions may be found, for example, in Dunford and Schwartz [12]. In the past few years, due in large part to the renewed interest in Banach space theory and the role played by vector measures in that theory, new attention has been focused upon extending the Vitali-Hahn-

Saks and Nikodym theorems to still more general cases. This recent effort has been marked with some real success.

The proper setting for theorems of the Vitali-Hahn-Saks and Nikodym type seems to be firmly established: these results are best formulated in terms of the strongly additive measures of C. E. Rickart [18]. If $\mathcal{A}$ is a Boolean algebra and X is a Banach space, then an additive map $\mu : \mathcal{A} \to X$ is said to be strongly additive if $\sum_n \mu(a_n)$ converges (unconditionally) for any sequence (a_n) of pairwise disjoint members of $\mathcal{A}$.

Both the Vitali-Hahn-Saks and Nikodym convergence theorems have been extended to the class of strongly additive set functions (with σ-complete domains) by J. K. Brooks and R. S. Jewett [3] and R. B. Darst ([4], [5]). It should be remarked that earlier T. Ando [1] had already proved a very general Vitali-Hahn-Saks theorem for scalar-valued bounded finitely additive measures on σ-complete Boolean algebras (both Brooks-Jewett and Darst work in σ-fields of sets); also, G. Seever [21] gave an extension of Ando's result to certain non-σ-complete algebras, again for scalar-valued measures. In the Brooks-Jewett-Darst extensions elegant " sliding hump " arguments are used instead of the category arguments involving the Frechet-Nikodym topologies. In this paper, " sliding hump " arguments will also be employed. There is an inherent advantage to be found in the " sliding hump " arguments: under the assumption that certain uniform conditions do not exist, " humps " behaving similarly to characteristic functions of disjoint sets (considered in l_∞) appear. Such considerations lead J. Diestel ([6], [7]) and J. Diestel and the author [8] to consider the relationship of the Vitali-Hahn-Saks and Nikodym convergence theorems to the Banach space results of A. Grothendieck [13], C. Bessaga and A. Pelczynski [2], A. Pelczynski [17], and H. P. Rosenthal [19]. In turn this motivated the problem dealt with in this paper: for which non-sigma complete Boolean algebras $\mathcal{A}$ does the Vitali-Hahn-Saks theorem hold? As one might expect the first response is: not all. An example of an algebra where the Vitali-Hahn-Saks theorem fails is given in section 2.

This paper proves that if $\mathcal{A}$ is a Boolean algebra possessing the interpolation property (property (I)) and (μ_n) is a sequence of strongly additive X-valued measures defined on $\mathcal{A}$ such that $\lim_n \mu_n(a)$ exists for each $a \in \mathcal{A}$, then $\mu_0(a) = \lim_n \mu_n(a)$ defines a strongly additive map from $\mathcal{A}$ to X *and* the additivity of the μ_n's is uniform. This result is closely related to the Nikodym convergence theorem and one can derive the Vitali-Hahn-Saks theorem from it. The theorem constitutes a natural extension of Seever's theorem; our proof is similar to Seever's in that we derive the result from an extension of another classical piece of measure theory: Nikodym's Boundedness Theorem. On the other hand, our proof of Nikodym's Boundedness Theorem differs greatly from Seever's on at least two counts. First, a Rosenthal-type lemma is proved (see [19], [22] Lemma 1); secondly, our proof shows clearly the role that property (I) plays in the proof. Seever's proof relied upon the fact that if $\mathcal{A}$ is a Boolean algebra with the property (I) and N is the ideal of null sets of a given measure, then $\mathcal{A}/N$ is a complete Boolean algebra and, therefore, the arguments could be made to depend upon formerly known results.

The derivation of the Vitali-Hahn-Saks theorem from the Nikodym Boundedness Theorem is of some interest in itself. In [9], J. Diestel, R. E. Huff and the author have studied the general problem of algebras with the Vitali-Hahn-Saks property and the Nikodym Boundedness property. In particular, it is shown there that whenever the Vitali-Hahn-Saks theorem holds, the Nikodym Boundedness theorem follows. The converse remains open; it is hoped that the derivation of the former from the latter, given here, will shed some light upon this problem.

The last section of this paper gives a proof of the Vitali-Hahn-Saks theorem for measures defined on an algebra with the property I and taking their values in a Hausdorff, topological commutative group.

Throughout the paper $\mathcal{A}$ will denote a Boolean algebra (with unit 1) with the *property* (I). $\mathcal{A}$ having the property (I) means that for any sequences (a_n) and (b_n) in $\mathcal{A}$ satisfying $a_n \leqslant b_m$ for all n, m there exists $b \in \mathcal{A}$ such that $a_n \leqslant b \leqslant b_n$ for all n. This condition is equivalent to the condition: given any sequences (a_n) and (b_n) in $\mathcal{A}$ with $a_n \wedge a_m = 0$, $b_n \wedge b_m = 0$ for $n \neq m$ and $a_n \wedge b_m = 0$ for all n, m, there exists an element a in $\mathcal{A}$ such that $a \geqslant a_n$ and $a \wedge b_n = 0$ for all n.

The symbol X denotes a Banach space and X^* its Banach space dual. A finitely additive $\mu : \mathcal{A} \to X$ is *bounded* whenever there exists $M > 0$ such that $\|\mu(b)\| \leqslant M$ for all $b \in \mathcal{A}$. A map $\mu : \mathcal{A} \to X$ is said to be *strongly bounded* if

$$\|\mu(e_n)\| \to 0$$

as $n \to \infty$ for each sequence (e_n) of pairwise disjoint elements in $\mathcal{A}$. A *strongly additive* $\mu : \mathcal{A} \to X$ is one which is finitely additive and strongly bounded. Rickart [18] showed that a bounded, finitely additive scalar valued measure is always strongly additive. A set H of strongly additive measures $\mu : \mathcal{A} \to X$ is *uniformly strongly additive* if for each sequence (b_n) of pairwise disjoint elements in $\mathcal{A}$,

$$\sup \{\|\mu(b_n)\| : \mu \in H\} \to 0 \quad \text{as} \quad n \to \infty.$$

If $\mu : \mathcal{A} \to X$, then for each $b \in \mathcal{A}$, $|\mu|(b)$ denotes the total variation of μ on b ([12], p. 97) and $\|\mu\|(b)$ denotes the semi-variation of μ on b ([12], p. 320). It is easily shown that $\mu : \mathcal{A} \to X$ is strongly additive if and only if $|\mu| : \mathcal{A} \to [0, \infty)$ (or $\|\mu\| : \mathcal{A} \to [0, \infty)$) is strongly bounded.

I wish to thank my advisor, Professor J. Diestel, for his suggestions in the preparation of this portion of my dissertation written at Kent State University. Also I thank Professor J. J. Uhl for the helpful discussions on this subject and for access to the preprint [22].

Some of the results in this paper are announced in [24].

Section 1.

LEMMA 1. 1. — *Let* K *be a set of bounded additive, scalar valued functions defined on* $\mathcal{A}$ *such that* $\sup_{\lambda \in K} |\lambda|(1) < +\infty$. *If* K *is not uniformly strongly additive, there is an* $\varepsilon > 0$, *a sequence* (μ_n) *in* K, *and an element* c *in* $\mathcal{A}$ *such that* $|\mu_n(c)| > \frac{\varepsilon}{2}$ *for all* $n \in \mathbf{N}$.

Proof. — If K is not uniformly strongly additive, then there is an $\varepsilon > 0$, a sequence (λ_n) in K, and a sequence (e_n) of pairwise disjoint elements in $\mathcal{A}$ such that $|\lambda_n(e_n)| > \varepsilon$ for all $n \in \mathrm{N}$. Let $i_1 = 1$. Partition $\mathrm{N}\setminus\{1\}$ into an infinite sequence of infinite disjoint sets (Π_n^1). Since $\mathcal{A}$ has the property (I), there is a sequence (b_n^1) of pairwise disjoint elements in $\mathcal{A}$ such that:

$$(a_1) \qquad b_n^1 \wedge e_1 = 0, \qquad n = 1, 2, \dots;$$

$$(b_1) \qquad b_n^1 \geqslant e_i \quad \text{for all} \quad i \in \Pi_n^1, \ n = 1, 2, \dots;$$

$$(c_1) \quad b_n^1 \wedge e_j = 0 \quad \text{for all} \quad j \in (\mathrm{N}\setminus\{1\})\setminus\left(\bigcup_{k=1}^{n-1} \Pi_k^1\right).$$

Indeed, if we let $a_i = e_i$ for all $i \in \Pi_1^1$ and $b_j = e_j$ for $j = i_1$, or $j \in (\mathrm{N}\setminus\{1\})\setminus\Pi_1^1$, then the sequences (a_i), (b_j) satisfy the conditions given in the definition of the property (I), (i.e. $a_i \wedge a_j = 0$, $b_i \wedge b_j = 0$ for $i \neq j$ and $a_i \wedge b_j = 0$ for all i, j). Thus, there is an element b_1^1 in $\mathcal{A}$ such that b_1^1 satisfies (a_1), (b_1), (c_1) for $n = 1$. Next, let $a_i = e_i$ for all $i \in \Pi_2^1$ and let (b_j) be the sequence in $\mathcal{A}$ with elements b_1^1, e_{i_1}, and e_j, $j \in (\mathrm{N}\setminus\{1\})\setminus(\Pi_1^1 \cup \Pi_2^1)$. The property (I) yields the existence of an element b_2^1 in $\mathcal{A}$ such that $b_2^1 \wedge b_1^1 = 0$ and b_2^1 satisfies (a_1), (b_1), (c_1) for $n = 2$. By continuing this process, we obtain a sequence (b_n^1) of pairwise disjoint elements in $\mathcal{A}$ as claimed.

The function λ_{i_1} is strongly additive so there is an $n_1 \in \mathrm{N}$ such that $|\lambda_{i_1}|(b_n^1) < \frac{\varepsilon}{2^3}$ for all $n \geqslant n_1$. Since

$$\sup_{\lambda \in \mathrm{K}} |\lambda| \, (1) < +\infty,$$

there is an $i_2 \in \Pi_{n_1}^1$, $i_2 > i_1$, such that $|\lambda_j(e_{i_2})| < \frac{\varepsilon}{2^3}$ for an infinite number of j in $\Pi_{n_1}^1$. If this were not the case (i.e. for every k in $\Pi_{n_1}^1$, $|\lambda_j(e_k)| < \frac{\varepsilon}{2^3}$ for only a finite number of j in $\Pi_{n_1}^1$), then for each $n \in \mathrm{N}$, there is a $j_n \in \Pi_{n_1}^1$ such that

$$|\lambda_{j_n}(e_{k_i})| > \frac{\varepsilon}{2^3} \quad \text{for} \quad i = 1, 2, \dots, n.$$

Thus $|\lambda_{j_n}|\,(1) > n\cdot\dfrac{\varepsilon}{2^3}$ for each $n \in \mathbf{N}$ (where $1 \in \mathcal{A}$ such that $1 \wedge a = a$ for all $a \in \mathcal{A}$). This is a contradiction. Let

$$\mathrm{N}_1 = \left\{ j \in \Pi^1_{n_1} : |\lambda_j(e_{i_2})| < \frac{\varepsilon}{2^3} \right\}.$$

Partition $\mathrm{N}_1 \setminus \{i_2\}$ into an infinite sequence of infinite disjoint sets (Π^2_n). Again, utilizing the property (I), we obtain a sequence (b^2_n) of pairwise disjoint elements in $\mathcal{A}$ satisfying the following:

(a_2) $\quad b^2_n \wedge (e_{i_1} \vee e_{i_2}) = 0, \quad n = 1, 2, \ldots;$

(b_2) $\quad b^2_n \geqslant e_i \quad$ for all $\quad i \in \Pi^2_n, \quad n = 1, 2, \ldots;$

(c_2) $\quad b^2_n \wedge e_j = 0 \quad$ for $\quad j \in (\mathrm{N}_1 \setminus \{i_2\}) \setminus \left(\bigcup_{k=1}^{n-1} \Pi^2_k \right).$

Choose $n_2 \in \mathbf{N}$, $n_2 > n_1$, such that $|\lambda_{i_2}|\,(b^2_n) < \dfrac{\varepsilon}{2^4}$ for all $n \geqslant n_2$. By the same reasoning as before, there is an

$$i_3 \in \Pi^2_{n_2},\ i_3 > i_2,$$

such that $|\lambda_j(e_{i_3})| < \dfrac{\varepsilon}{2^4}$ for an infinite number of $j \in \Pi^2_{n_2}$. Let $\mathrm{N}_2 = \left\{ j \in \Pi^2_{n_2} : |\lambda_j(e_{i_3})| < \dfrac{\varepsilon}{2^4} \right\}$. Notice that

$$|\lambda_{i_3}(e_{i_2})| < \frac{\varepsilon}{2^3}$$

since $i_3 \in \mathrm{N}_1$.

The continuation of this process yields an infinite sequence of infinite subsets of $\mathbf{N}$, $\mathrm{N}_1 \supset \mathrm{N}_2 \supset \ldots$, an increasing sequence $i_1 < i_2 < \cdots$ of positive integers such that if $k \geqslant 3$, then $i_k \in \mathrm{N}_{k-1}$, and a sequence $(b^k_{n_k}) = (b_k)$ of elements in $\mathcal{A}$ such that:

(1) $\quad b_k \geqslant e_{i_j} \quad$ for all $\quad j > k;$

(2) $\quad b_k \wedge e_{i_j} = 0 \quad$ for $\quad 1 \leqslant j \leqslant k;$

(3) $\quad |\lambda_{i_k}|\,(b_k) < \dfrac{\varepsilon}{2^{2+k}}, \quad k = 1, 2, \ldots;$

(4) $\quad |\lambda_j(e_{i_k})| < \dfrac{\varepsilon}{2^{1+k}} \quad$ for all $\quad j \in \mathrm{N}_{k-1};$

(5) $\quad |\lambda_{i_k}(e_{i_k})| > \varepsilon, \quad k = 1, 2, \ldots.$

Notice that the choice of the N_i's and (4) imply that

$$|\lambda_{i_k}(e_{i_j})| < \frac{\varepsilon}{2^{1+j}}$$

for $2 \leqslant j < k$. Let $h_k = b_k \vee \left(\bigvee_{j=2}^{k} e_{i_j}\right)$. Then $h_k \geqslant e_{i_j}$ for all $k, j \geqslant 2$. Applying the property (I), we can choose $c \in \mathcal{A}$ such that $h_k \geqslant c \geqslant e_{i_k}$ for all $k \geqslant 2$. For the remainder of the proof let $\lambda_{i_k} = \lambda_k$, $e_{i_k} = e_k$, and assume $k \geqslant 3$. Since $\lambda_k(c) = \lambda_k(h_k - e_k) - \lambda_k(h_k \setminus c) + \lambda_z(e_k)$, we have

$$\lambda_k(e_k) + \lambda_k\left[\left(b_k \vee \left(\bigvee_{j=2}^{k} e_j\right)\right) \wedge e'_k\right] - \lambda_k\left[b_k \vee \left(\bigvee_{j=2}^{k} e_j\right)\right) \wedge c'\right]$$

which by (2) and the disjointness of the e_k's is

$$= \lambda_k(e_k) + \lambda_k\left(b_k \vee \left(\bigvee_{j=2}^{k-1} e_j\right)\right) - \lambda_k(b_k \wedge c')$$
$$= \lambda_k(e_k) + \lambda_k(b_k) + \sum_{j=2}^{k-1} \lambda_k(e_j) - \lambda_k(b_k \wedge c').$$

Thus

$$|\lambda_k(c)| \geqslant |\lambda_k(e_k)| - |\lambda_k(b_k)| - \sum_{j=2}^{k-1} |\lambda_k(e_j)| - |\lambda_k(b_k \wedge c')|$$

which by (3), (4), and (5) is

$$\geqslant \varepsilon - \frac{\varepsilon}{2^{2+k}} - \sum_{j=2}^{k-1} \frac{\varepsilon}{2^{1+j}} - \frac{\varepsilon}{2^{2+k}} > \frac{\varepsilon}{2}.$$

Section 2.

THEOREM 2.1. — (Nikodym Boundedness) *Let* K *be a set of bounded, additive functions* $\lambda : \mathcal{A} \to X$, *and suppose* K *is elementwise bounded on* $\mathcal{A}$; *i.e. for every*

$$b \in \mathcal{A}, \quad \sup_{\lambda \in K} \|\lambda(b)\| < \infty.$$

Then K *is uniformly bounded on* $\mathcal{A}$; *i.e.*

$$\sup_{b \in \mathcal{A}} \sup_{\lambda \in K} \|\lambda(b)\| < \infty.$$

Proof. — If K is not uniformly bounded, then neither is the set $\{f\lambda : \lambda \in K, f \in X^*, \|f\| \leqslant 1\}$. Thus we may assume without loss of generality that the functions in K are scalar-valued. We shall use the terminology that an element e in $\mathfrak{A}$ is unfriendly [10] whenever $\sup_{\lambda \in K} |\lambda| (e) = \infty$. Suppose $\mathfrak{A}$ contains an unfriendly element b. Two cases arise:

(1) b is not the supremum of two disjoint unfriendly elements in $\mathfrak{A}$.

(2) Every unfriendly element in $\mathfrak{A}$ can be written as the supremum of two disjoint unfriendly elements.

We plan to show that both (1) and (2) are impossible. First, we show that if we assume (1), then we reach a contradiction.

Let $b \in \mathfrak{A}$ be an unfriendly element which is not the supremum of two disjoint unfriendly elements in $\mathfrak{A}$. Let $\lambda_1 \in K$ be such that $|\lambda_1| (b) > 1 + 8 \sup_{\lambda \in K} |\lambda(b)|$. Then we can choose $e \in \mathfrak{A}$, $e \leqslant b$, with the property that $|\lambda_1(e)| > \frac{|\lambda_1| (b)}{4}$. Then

$$|\lambda_1(b \wedge e')| = |\lambda_1(b) - \lambda_1(e)| \geqslant |\lambda_1(e)| - |\lambda_1(b)| \geqslant \frac{|\lambda_1|(b)}{4} - \frac{|\lambda_1|(b) - 1}{8} > \frac{|\lambda_1|(b)}{8}.$$

If e is not unfriendly, let $e_1 = e$. If e is unfriendly, let $e_1 = b \setminus e$. Then, in either case, $|\lambda_1(e_1)| \geqslant \frac{|\lambda_1|(b)}{8}$.

Suppose $e_1, e_2, \ldots, e_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ have been chosen such that $e_i \in \mathfrak{A}$, $\lambda_i \in K$, $\bigvee_{j=1}^{n} e_j \leqslant b$, and $b \setminus \left(\bigvee_{j=1}^{n} e_j \right)$ is unfriendly. By the assumption of case 1, $\bigvee_{j=1}^{n} e_j$ is not unfriendly. Therefore, $\sup_{\lambda \in K} |\lambda| \left(\bigvee_{j=1}^{n} e_j \right) = \alpha < +\infty$. By the hypothesis $\sup_{\lambda \in K} \left| \lambda \left(b \setminus \left(\bigvee_{j=1}^{n} e_j \right) \right) \right| = \beta < +\infty$. Choose λ_{n+1} in K such that $|\lambda_{n+1}|(b) > (n+1) + 10(\alpha + \beta)$. Then

$$|\lambda_{n+1}| \left(b \setminus \left(\bigvee_{j=1}^{n} e_j \right) \right) \geqslant |\lambda_{n+1}|(b) - |\lambda_{n+1}| \left(\bigvee_{j=1}^{n} e_j \right) \geqslant |\lambda_{n+1}|(b) - \alpha$$
$$\geqslant |\lambda_{n+1}|(b) - \frac{|\lambda_{n+1}|(b)}{10} + \frac{n+1}{10} + \beta \geqslant \frac{9}{10} |\lambda_{n+1}|(b).$$

Now choose $e \leqslant b \setminus \left(\bigvee_{j=1}^{n} e_j \right)$ such that

$$|\lambda_{n+1}(e)| \geqslant |\lambda_{n+1}|(b)/5.$$

Then

$$\left|\lambda_{n+1}\left(b \setminus \left(\bigvee_{j=1}^{n} e_j \vee e\right)\right)\right| \geqslant |\lambda_{n+1}(e)| - \left|\lambda_{n+1}\left(b \setminus \left(\bigvee_{j=1}^{n} e_j\right)\right)\right|$$
$$\geqslant \frac{|\lambda_{n+1}|(b)}{5} - \beta \geqslant \frac{|\lambda_{n+1}|(b)}{5} - \frac{|\lambda_{n+1}|(b)}{10} + \frac{(n+1)}{10} + \alpha$$
$$\geqslant |\lambda_{n+1}|(b)/10.$$

If e is not unfriendly, let $e_{n+1} = e$. If e is an unfriendly set, let $e_{n+1} = b \setminus \left(\bigvee_{j=1}^{n} e_j \vee e \right)$. In either case,

$$|\lambda_{n+1}(e_{n+1})| \geqslant |\lambda_{n+1}|(b)/10.$$

We have now constructed a sequence (e_n) of pairwise disjoint elements in $\mathcal{A}$ and a sequence (λ_n) of members of K such that for each $n \in \mathrm{N}$, $|\lambda_n(e_n)| \geqslant \frac{|\lambda_n|(b)}{10} \geqslant \frac{n}{10}$. Since

$$\sup_n \frac{|\lambda_n|(b)}{|\lambda_n(e_n)|} \leqslant 10$$

and each $e_n \leqslant b$, we can apply Lemma 1.1 to the set

$$\left\{ \frac{\lambda_n}{|\lambda_n(e_n)|} : n \in \mathrm{N} \right\}.$$

This yields a number $\varepsilon > 0$, an element c in $\mathcal{A}$ and a subsequence of $\left(\frac{\lambda_n}{|\lambda_n(\mathrm{E}_n)|} \right)$ $\left(\text{which we still denote } \left(\frac{\lambda_n}{|\lambda_n(e_n)|} \right) \right)$ such that for $n \geqslant 3$, $\frac{|\lambda_n(c)|}{|\lambda_n(e_n)|} > \frac{\varepsilon}{2}$. Thus

$$|\lambda_n(c)| \geqslant \frac{\varepsilon}{2} |\lambda_n(e_n)| \geqslant \frac{n\varepsilon}{20}$$

and $\sup_n |\lambda_n(c)| = \infty$. This eliminates case (1).

Now, assume case (2); i.e. every unfriendly element can be written as the supremum of two disjoint unfriendly elements.

Thus it is possible to manufacture a sequence (e_n) of pairwise disjoint elements in $\mathcal{A}$ with each e_n unfriendly. Choose $\lambda_1 \in K$ such that $|\lambda_1|(e_1) \geqslant 1$ and let $b_1 \leqslant e_1$ be an element in $\mathcal{A}$ for which $|\lambda_1(b_1)| \geqslant |\lambda_1|(e_1)/4$. Let $i_1 = 1$. Partition $N \setminus \{i_1\}$ into an infinite sequence (Π_n^1) of infinite disjoint sets. Apply the property (I) to obtain a sequence (a_n^1) of pairwise disjoint elements in $\mathcal{A}$ such that:

$$(a_1) \qquad a_n^1 \wedge e_{i_1} = 0, \qquad n = 1, 2, \ldots;$$
$$(b_1) \qquad a_n^1 \geqslant e_i \quad \text{for all} \quad i \in \Pi_n^1;$$
$$(c_1) \qquad a_n^1 \wedge e_j = 0 \quad \text{for all} \quad j \in (N \setminus \{i_1\}) \Big\backslash \left(\bigcup_{k=1}^{n-1} \Pi_k^1 \right).$$

Choose $n_1 \in N$ such that $|\lambda_1|(a_{n_1}^1) < 1$.

Let i_2 be the smallest element in $\Pi_{n_1}^1$. Choose $\lambda_2 \in K$ such that $|\lambda_2|(e_{i_2}) \geqslant 1 + 4 \sup_{\lambda \in K} |\lambda(b_1)|$. Let $b_2 \leqslant e_{i_2}$ be an element in $\mathcal{A}$ chosen so that $|\lambda_2(b_2)| \geqslant |\lambda_2|(e_{i_2})/4$. Partition $\Pi_{n_1}^1 \setminus \{i_2\}$ into an infinite sequence (Π_n^2) of infinite pairwise disjoint sets. As in the proof of Lemma 1.1, we can choose a sequence (a_n^2) of pairwise disjoint members of $\mathcal{A}$ such that:

$$(a_2) \qquad a_n^2 \wedge (e_{i_1} \vee e_{i_2}) = 0, \qquad n = 1. 2, \ldots;$$
$$(b_2) \qquad a_n^2 \geqslant e_i \quad \text{for all} \quad i \in \Pi_n^2;$$
$$(c_2) \qquad a_n^2 \wedge e_j = 0 \quad \text{for all} \quad j \in (\Pi_{n_1}^1 \setminus \{i_2\}) \Big\backslash \left(\bigcup_{k=1}^{n-1} \Pi_k^2 \right).$$

Choose $n_2 > n_1$ such that $|\lambda_2|(a_{n_2}^2) < 1$.

If we proceed in this manner, we obtain a sequence (λ_n) in K, a subsequence (e_{i_k}) of (e_n), a sequence (b_n) in $\mathcal{A}$, and a sequence $(a_{n_k}^k) = (a_k)$ in $\mathcal{A}$ satisfying:

$$(1) \qquad b_k \leqslant e_{i_k}, \qquad k = 1, 2, \ldots;$$
$$(2) \qquad |\lambda_k(b_k)| \geqslant \frac{|\lambda_k|(e_{i_k})}{4} \geqslant \frac{k}{4} + \sum_{j=1}^{k-1} \sup_{\lambda \in K} |\lambda(b_j)|;$$
$$(3) \qquad |\lambda_k|(a_k) < 1, \qquad k = 1, 2, \ldots;$$
$$(4) \qquad a_k \geqslant e_{i_j}, \qquad j > k;$$
$$(5) \qquad a_k \wedge e_{i_j} = 0, \qquad 1 \leqslant j \leqslant k.$$

Let $h_k = a_k \vee \left(\bigvee_{i=1}^{k} b_i \right)$. Then $h_k \geqslant b_j$ for all j, k. Choose $c \in \mathcal{A}$ such that $h_k \geqslant c \geqslant b_k$ for all k (an application of the property (I)). Then

$$
\begin{aligned}
|\lambda_k(c)| &\geq |\lambda_k(b_k)| - |\lambda_k(a_k)| - \sum_{j=1}^{k-1} |\lambda_k(b_j)| - |\lambda_k(a_k \wedge c)| \\
&\geq \frac{k}{4} + \sum_{j=1}^{k-1} |\lambda_k(b_k)| - 1 - \sum_{j=1}^{k-1} |\lambda_k(b_j)| - 1 \\
&\geq \frac{k}{4} - 2 \to \infty \quad \text{as} \quad k \to \infty.
\end{aligned}
$$

Thus case (2) is impossible also.

We have shown that $\mathcal{A}$ does not contain an element e such that $\sup_{\lambda \in K} |\lambda|(e) = \infty$. Therefore,

$$\sup_{\lambda \in K} \sup_{e \in \mathcal{A}} |\lambda(e)| = \sup_{\lambda \in K} |\lambda|(1) < + \infty.$$

Corollary 2.2. — *If* K *is a set of bounded, additive functions* $\lambda : \mathcal{A} \to X$ *satisfying* $\sup_{\lambda \in K} \|\lambda(b)\| < \infty$ *for each* $b \in \mathcal{A}$, *then* $\sup_{\lambda \in K} \|\lambda\|(b) < \infty$ *for each* $b \in \mathcal{A}$.

Corollary 2.3. — (Dieudonné-Grothendieck boundedness theorem) *Let* $\mu : \mathcal{A} \to X$ *be any function. Suppose* $H \subseteq X^*$ *is total and* $h\mu$ *is bounded and additive for each* $h \in H$. *Then* μ *is bounded and additive.*

Theorem 2.4. — *Let* $\mu_n : \mathcal{A} \to X$ *be strongly additive for each* $n \in N$. *If* $\mu_n(b) \to 0$ *for each* $b \in \mathcal{A}$, *then* $\{\mu_n : n \in N\}$ *is uniformly strongly additive.*

Proof. — By a result in ([9], Theorem 2.1) it suffices to give the proof for μ_n taking values in the scalar field. For each $b \in \mathcal{A}$, $\sup \{|\mu_n(b)| : n \in N\} < \infty$. Hence by Theorem 2.1, $\sup \{|\mu_n|(b) : n \in N\} < \infty$ for each $b \in \mathcal{A}$. If the set $\{\mu_n : n \in N\}$ is not uniformly strongly additive, then by Lemma 1.1, there is an $\varepsilon > 0$, a subsequence (μ_{n_k}) of (μ_n) and an element c in $\mathcal{A}$ such that $|\mu_{n_k}(c)| > \frac{\varepsilon}{2}$ for all $k \in N$. Thus $\mu_n(c) \to 0$ a contradiction.

The following corollary is immediate from Theorem 2.4. A proof can be seen in ([3], corollary 1.2).

COROLLARY 2.5. — *Let* $\mu_n : \mathcal{A} \to X$ *be strongly additive for* $n = 1, 2, \ldots$. *If* $\lim_n \mu_n(b) = \mu(b)$ *exists for each* $b \in \mathcal{A}$, *then* μ *is strongly additive and the* μ_n, $n \in N$, *are uniformly strongly additive.*

It is known that Theorem 2.4 holding for an algebra $\mathcal{A}$ is equivalent to the Vitali-Hahn-Saks Theorem holding for $\mathcal{A}$ (see [3], [9], [11]). Thus we have the next result.

THEOREM 2.5. — *Let* $\gamma : \mathcal{A} \to [0, \infty)$ *be a bounded, monotone function and for each* $n \in N$, $\mu_n : \mathcal{A} \to X$ *a strongly additive function with* $\mu_n \ll \gamma$. *If* $\lim_n \mu_n(b)$ *exists for each* $b \in \mathcal{A}$, *then* $\{\mu_n : n \in N\}$ *is uniformly absolutely continuous with respect to* γ.

As promised we now give an example of an algebra for which the Vitali-Hahn-Saks theorem does not hold.

Example 2.6. — Let A be the algebra of finite and cofinite subsets of the natural numbers N and for each $n \in N$, let μ_n denote the point mass at n. Then for every $E \in \mathcal{A}$

$$\lim_n \mu_n(E) = \mu(E) = \begin{cases} 0 & \text{if } E \text{ is finite} \\ 1 & \text{if } N \setminus E \text{ is finite.} \end{cases}$$

Since any infinite sequence of pairwise disjoint elements in $\mathcal{A}$ consists of finite subsets of N, each μ_n is strongly additive. However the μ_n's are not uniformly strongly additive since $\sup_n \|\mu_n\{i\}\| = 1$ for each element i in N.

Section 3.

In this section G denotes a Hausdorff topological commutative group with η a base for the neighborhoods of 0 in G consisting of symmetric elements. The meaning of a measure $\mu : \mathcal{A} \to G$ being strongly additive is clear. A notion for group-valued measures (similar to variation for vector-valued measures) is defined for each $b \in \mathcal{A}$ by

$$\mu((b)) = \{\mu(b \wedge e) : e \in \mathcal{A}\}.$$

It is a short exercise to show that $\mu : \mathcal{A} \to G$ is strongly additive if and only if given a sequence (b_n) of pairwise disjoint elements in $\mathcal{A}$ and a neighborhood V of 0 in G, there is an $n_0 \in N$ such that $\mu((b_n)) \subset V$ for all $n \geqslant n_0$.

THEOREM 3.1. — *For each* $n \in N$, *suppose* $\mu_n : \mathcal{A} \to G$ *is strongly additive. If* $\mu_n(b) \to 0$ *as* $n \to \infty$ *for each* $b \in \mathcal{A}$, *then* $\{\mu_n : n \in N\}$ *is uniformly strongly additive.*

Remark. — The proof presented here proceeds as that of Lemma 1.1. In fact, the observation that elements in η can be chosen to behave in a designated manner yields the proof. We give the details.

Proof. — Suppose the conclusion does not hold. Then there is a sequence (e_n) of pairwise disjoint elements in $\mathcal{A}$, a symmetric element V in η, and a sequence $m_1 < m_2 < \cdots$ of positive integers such that for each $n \in N$, $\mu_{m_n}(e_n) \notin V$. To simplify notation, let $\mu_{m_n} = \mu_n$.

Let $i_1 = 1$. Partition the set $N \setminus \{1\}$ into an infinite number of infinite disjoint sets $(\Pi_n^1)_{n=1}^{\infty}$. There exists a sequence (b_n^1) of pairwise disjoint elements in $\mathcal{A}$ such that:

(a_1) $\quad b_n^1 \geqslant e_i$ for all $i \in \Pi_n^1$, $n = 1, 2, \ldots$;

(b_1) $\quad b_n^1 \wedge e_{i_1} = 0$, $\quad n = 1, 2, \ldots$;

(c_1) $\quad b_n^1 \wedge e_j = 0$ for all $j \in (N \setminus \{1\}) \Big\backslash \left(\bigcup_{i=1}^{n} \Pi_i^1 \right)$.

That such a sequence (b_n^1) exists follows from the property (I).

By the continuity of addition, we can choose a symmetric element V_0 in η such that $V_0 + V_0 + V_0 + V_0 \subset V$ and for each $k \in N$, a $V_k \in \eta$ such that $V_{k+1} + V_{k+1} \subset V_k$. V_1 is chosen so that $V_1 + V_1 \subset V_0$. Then $\sum_{i=1}^{k} V_i \subset V_0$ for all k in N. Since μ_{i_1} is strongly additive, there is an $n_1 \in N$ such that $\mu_{i_1}((b_{n_1}^1)) \subset V_0$. Recall that

$$\mu_{i_1}((b_{n_1}^1)) = \{\mu_{i_1}(b_{n_1}^1 \wedge e) : e \in \mathcal{A}\}.$$

The $\lim\limits_{n} \mu_n(e_{i_1}) = 0$, so there exists an element i_2 in $\Pi_{n_1}^1$,

$i_2 > i_1$, such that $\mu_{i_2}(e_{i_1}) \in V_0$. Partition the set $\Pi^1_{n_1} \setminus \{i_2\}$ into an infinite number of infinite disjoint sets (Π^2_n). An application of the definition of property (I) yields a sequence (b^2_n) of pairwise disjoint elements in A such that:

$$(a_2) \quad b^2_n \geqslant e_i \quad \text{for all} \quad i \in \Pi^2_n, \quad n = 1, 2, \ldots;$$
$$(b_2) \quad b^2_n \wedge (e_{i_1} \vee e_{i_2}) = 0, \quad n = 1, 2, \ldots;$$
$$(c_2) \quad b^2_n \wedge e_j = 0 \quad \text{for} \quad j \in (\Pi^1_{n_1} \setminus \{i_2\}) \setminus \left(\bigcup_{i=1}^{n} \Pi^2_i\right).$$

Choose $n_2 \in N$, $n_2 > n_1$, such that $\mu_{i_2}((b^2_{n_2})) \subset V_0$ and choose $i_3 \in \Pi^2_{n_2}$, $i_3 > i_2$, such that $\mu_{i_3}(e_{i_1}) \in V_1$ and $\mu_{i_3}(e_{i_2}) \in V_1$.

Proceed in this construction to obtain a sequence

$$(b^k_{n_k}) = (b_k)$$

of elements in $\mathfrak{A}$ and a sequence $i_1 < i_2 < \ldots$ of positive integers such that:

$$(1) \quad b_n \geqslant e_{i_k}, \quad k > n;$$
$$(2) \quad b_n \wedge e_{i_k} = 0, \quad 1 \leqslant k \leqslant n;$$
$$(3) \quad \mu_{i_k}((b_k)) \in V_0, \quad k = 1, 2, \ldots;$$
$$(4) \quad \mu_{i_n}(e_{i_k}) \in V_{n-2}, \quad 1 \leqslant k < n;$$
$$(5) \quad \mu_{i_k}(e_{i_k}) \in V, \quad k = 1, 2, \ldots.$$

Let $h_n = b_n \vee \left(\bigvee_{k=1}^{n} e_{i_k}\right)$. Then $h_n \geqslant e_{i_k}$ for all n, k. Choose $c \in \mathfrak{A}$ such that $h_n \geqslant c \geqslant e_{i_n}$ for all n. As in the proof of Lemma 1.1,

$$\mu_{i_n}(c) = \mu_{i_n}(e_{i_n}) + \mu_{i_n}(b_n) + \sum_{k=1}^{n-1} \mu_{i_n}(e_{i_k}) - \mu_{i_n}(b_n \wedge c').$$

Since $\lim_n \mu_{i_n}(c) = 0$, there is an $n_0 \in N$ such that $\mu_{i_n}(c) \in V_0$ for all $n \geqslant n_0$. Thus for all $n \geqslant n_0$,

$$\mu_{i_n}(e_{i_n}) = \mu_{i_n}(c) - \mu_{i_n}(b_n) - \sum_{k=1}^{n-1} \mu_{i_n}(e_{i_k}) - \mu_{i_n}(b_n \wedge c')$$
$$\varepsilon V_0 + V_0 + V_0 + V_0 \subset V.$$

This contradicts (5).

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Manuscrit reçu le 10 juillet 1975
Proposé par G. Choquet.

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