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ON A GENERALIZATION OF DE-RHAM LEMMA

by Kyoji SAITO

In this short note, we give a proof of a theorem (cf. § 1) which is a generalization of a lemma due to de-Rham [1] and which was announced and used in [2].

As no proof of this theorem was available in the literature, Lê Dũng Tráng pushed me to publish it : I am grateful to him.

1. Notations and formulations of the theorem.

Let R be a noetherian commutative ring with unit. The profondeur of an ideal $\mathfrak{A}$ of R is the maximal length q of sequences $a_1, \ldots, a_q \in \mathfrak{A}$ with :

i) a_1 is a non-zero-divisor of R.

ii) a_i is a non-zero-divisor of $R/a_1R + \cdots + a_{i-1}R$, $i=2, \ldots, q$.

Let M be a free R-module of finite rank n. We denote by $\overset{p}{\wedge} M$ the p-th exterior product of M (with $\overset{0}{\wedge} M = R$ and $\overset{-1}{\wedge} M = 0$).

Let $\omega_1, \ldots, \omega_k$ be given elements of M, and $(e_1, \ldots, e_n)$ be a free basis of M,

$$\omega_1 \wedge \cdots \wedge \omega_k = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} a_{i_1 \ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

We call $\mathfrak{A}$: the ideal of R generated by the coefficients $a_{i_1 \ldots i_k}$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$. (We put $\mathfrak{A} = R$, when $k = 0$.)

Then we define:

$$Z^p := \{\omega \in \bigwedge^{p} M : \omega \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0\} \qquad p = 0, 1, 2, \ldots$$

$$H^p := Z^p \Big/ \sum_{i=1}^{k} \omega_i \wedge \bigwedge^{p-1} M \qquad p = 0, 1, 2, \ldots$$

In the case when $k = 0$, we understand $Z^p = 0$, $H^p = 0$ for $p = 0, 1, 2, \ldots$.

THEOREM. — i) *There exists an integer* $m \geqslant 0$ *such that*:

$$\mathfrak{A}^m H^p = 0 \quad \textit{for} \quad p = 0, 1, 2, \ldots, n.$$

ii) $H^p = 0$ *for* $0 \leqslant p < \operatorname{prof}(\mathfrak{A})$.

2. Proof of the theorem.

Proof of i). — Since R is noetherian, we have only to show for any $\omega \in Z^p$ and any coefficients $a_{i_1 \ldots i_k}$,

$$1 \leqslant i_1 < \cdots < i_k \leqslant n,$$

there exists an integer $m \geqslant 0$ such that

$$(a_{i_1 \ldots i_k})^m \omega \in \sum_{i=1}^{k} \omega_i \wedge \bigwedge^{p-1} M.$$

If $a_{i_1 \ldots i_k}$ is nilpotent, then nothing is to show. Suppose $a_{i_1 \ldots i_k} = a$ is not nilpotent and let $R_{(a)}$ be the localization of R by the powers of $a = a_{i_1 \ldots i_k}$. There is a canonical morphism $R \to R_{(a)}$ and we denote by $[\omega]$ the image of $\omega \in \bigwedge^{p} M$ in $\left(\bigwedge^{p} M\right) \otimes_{R} R_{(a)} \left(= \bigwedge^{p} \left(M \otimes_{R} R_{(a)}\right)\right.$ because M is free over R).

Since the ideal in $R_{(a)}$ generated by the coefficients of $[\omega_1] \wedge \cdots \wedge [\omega_k]$ contains the image of $a = a_{i_1 \ldots i_k}$ in $R_{(a)}$, it coincides with $R_{(a)}$ and we may consider

$$[\omega_1], \ldots, [\omega_k]$$

as a part of free basis of $M \otimes_{R} R_{(a)}$. We add some other elements $[e_1], \ldots, [e_{n-k}]$ such that

$$[\omega_1], \ldots, [\omega_k], [e_1], \ldots, [e_{n-k}]$$

give a basis of $M \underset{R}{\otimes} R_{(a)}$. Then any element

$$[\omega] \in \overset{p}{\bigwedge} \left(M \underset{R}{\otimes} R_{(a)}\right)$$

can be developed in the form:

$$[\omega] = \sum_{l+m=p} \sum_{\substack{1 \leqslant i_1 < \cdots < i_l \leqslant k \\ 1 \leqslant j_1 < \cdots < j_m \leqslant n-k}} a_{i_1 \ldots i_k, j_1 \ldots j_{n-k}} [\omega_{i_1}] \wedge \cdots \wedge [\omega_{i_l}] \wedge [e_{j_1}] \wedge \cdots \wedge [e_{j_m}].$$

Then the fact $[\omega] \wedge [\omega_1] \wedge \cdots \wedge [\omega_k] = 0$ is equivalent to the existence of some $\eta_i' \in \overset{p-1}{\bigwedge} \left(M \underset{R}{\otimes} R_{(a)}\right)$ $i = 1, \ldots, k$ with $[\omega] = \sum_{i=1}^{k} \eta_i' \wedge [\omega_i]$. Let us take $\eta_i \in \overset{p-1}{\bigwedge} M$ and $m_1 \geqslant 0$ with $\eta_i' = a^{-m_1}[\eta_i]$ $i = 1, \ldots, k$.

Then we have:

$$\left[a^{m_1}\omega - \sum_{i=1}^{k} \eta_i \wedge \omega_i\right] = a^{m_1}[\omega] - \sum_{i=1}^{k} [\eta_i] \wedge [\omega_i] = 0.$$

By the definition of $R_{(a)}$, there exists some $m_2 \geqslant 0$ such that

$$a^{m_2} \left\{a^{m_1}\omega - \sum_{i=1}^{k} \eta_i \wedge \omega_i\right\} = 0 \quad \text{in} \quad \overset{p}{\bigwedge} M.$$

This completes the proof of i).

Proof of ii). We prove it by double induction on (p, k) for $p, k \geqslant 0$.

a) In the case $k = 0$, the assertion is trivially true by the definition of H^p.

b) *Case* $p = 0$.

Let $\omega \in \overset{0}{\bigwedge} M = R$ with $\omega \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0$. The fact $p = 0 < \text{prof}\ (\mathfrak{A})$ implies the existence of $a \in \mathfrak{A}$, which is non-zero-divisor of R. Since $a\omega = 0$, we get $\omega = 0$.

c) *Case* $0 < p < \textit{prof}\,(\mathfrak{A})$ *and* $0 < k$.

The induction hypothesis is then, that for $(p - 1, k)$ and $(p, k - 1)$ the assertion ii) of the theorem is true.

Let $a \in \mathfrak{A}$ be a non-zero-divisor of R. According to i), there exists an integer $m > 0$ with $a^m H^p = 0$. Since $a^m \in \mathfrak{A}$ is again a non-zero-divisor of R, we may assume that $m=1$.

We denote by $\bar{\omega}$ the image of $\omega \in \bigwedge^{p} \mathrm{M}$ in

$$\Big(\bigwedge^{p} \mathrm{M}\Big) \otimes_{\mathrm{R}} \mathrm{R}/a\mathrm{R} \simeq \bigwedge^{p} \Big(\mathrm{M} \otimes_{\mathrm{R}} \mathrm{R}/a\mathrm{R}\Big).$$

For $\omega \in Z^p$, we have a presentation:

$$(*) \quad a\omega = \sum_{i=1}^{k} \eta_i \wedge \omega_i, \quad \text{with} \quad \eta_i \in \bigwedge^{p-1} \mathrm{M}.$$

We have then: $0 = \sum_{i=1}^{k} \bar{\eta}_i \wedge \bar{\omega}_i$.

For any $1 \leqslant j \leqslant k$, we get:

$$\bar{\eta}_j \wedge \bar{\omega}_1 \wedge \cdots \wedge \omega_k = \Big(\sum_{i=1}^{k} \bar{\eta}_i \wedge \bar{\omega}_i\Big) \wedge ((-1)^{j-1}\bar{\omega}_1 \wedge \cdots \wedge \hat{\bar{\omega}}_j \wedge \cdots \wedge \bar{\omega}_k) = 0.$$

Here the symbol ^ means, we omit the corresponding term. Since the ideal of $\mathrm{R}/a\mathrm{R}$ generated by the coefficients of $\bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k$ is equal to $\mathfrak{A}/a\mathrm{R}$ and

$$\operatorname{prof} \mathfrak{A}/a\mathrm{R} = \operatorname{prof} \mathfrak{A} - 1 \geqslant p - 1 \geqslant 0,$$

we can apply to $\bar{\eta}_j$ the induction hypothesis for $(p-1, k)$; there exist $\xi_{ji} \in \bigwedge^{p-2} \mathrm{M}$, $j, i = 1, \ldots, k$, such that

$$\bar{\eta}_j = \sum_{i=1}^{k} \bar{\xi}_{ji} \wedge \bar{\omega}_i, \qquad j = 1, \ldots, k.$$

Lifting back this relation to $\bigwedge^{p-1} \mathrm{M}$, we find some $\zeta_j \in \bigwedge^{p-1} \mathrm{M}$, $j = 1, \ldots, k$, such that

$$\eta_j = \sum_{i=1}^{k} \xi_{ji} \wedge \omega_i + a\zeta_j \qquad j = 1, \ldots, k.$$

Replacing η_j in the presentation (*) by this, we obtain:

$$a\Big(\omega - \sum_{j=1}^{k} \zeta_j \wedge \omega_j\Big) = \sum_{i,j=1}^{k} \xi_{ji} \wedge \omega_i \wedge \omega_j.$$

Multiplying by $\omega_2 \wedge \cdots \wedge \omega_k$, we have:

$$a\Big(\omega - \sum_{i=1}^{k} \zeta_i \wedge \omega_i\Big) \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0.$$

Since a is a non-zero-divisor of R, we have :

$$\left(\omega - \sum_{i=1}^{k} \zeta_i \wedge \omega_i\right) \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0.$$

Now since the ideal $\mathfrak{A}'$ generated by the coefficients of $\omega_2 \wedge \cdots \wedge \omega_k$ contains the ideal $\mathfrak{A}$, we have prof $\mathfrak{A}' \geqslant$ prof $\mathfrak{A} > p$. Again by the induction hypothesis for $(p, k-1)$, we find some $\theta_j \in \bigwedge^{p-1} M$, $j = 2, \ldots, k$ with

$$\omega - \sum_{i=1}^{k} \zeta_i \wedge \omega_i = \sum_{j=2}^{k} \theta_j \wedge \omega_i.$$

This ends the proof of ii).

3. Remark.

We can formulate the theorem in § 2, for a more general class of modules M than the one of free modules, as follows.

Let M be a R-finite module with homological dimension $hd_R(M) \leqslant 1$, and $\omega_1, \ldots, \omega_k$ be elements of M. Since $hd_R(M) \leqslant 1$, we have a free resolution :

$$0 \to L_1 \to L_2 \to M \to 0.$$

Let $\tilde{\omega}_1, \ldots, \tilde{\omega}_k$ be some liftings of $\omega_1, \ldots, \omega_k$ in L_2 and $\tilde{e}_1, \ldots, \tilde{e}_m$ be images in L_2 of a free basis $e_1, \ldots, e_m$ of L_1. Let $\mathfrak{A}$ be the ideal of R generated by coefficients of $\tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_k \wedge \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_m$.

Since $\mathfrak{A}$ can be considered as a Fitting ideal of the following resolution :

$$L_1 \oplus R^k \to L_2 \to M\Big/\sum_{i=1}^{k} R\omega_i \to 0.$$

we obtain the following lemma.

LEMMA. — $\mathfrak{A}$ does only depend on M and $\omega_1, \ldots, \omega_k$ and does depend neither on the choice of $\tilde{\omega}_1, \ldots, \tilde{\omega}_k$ and $e_1, \ldots, e_m$ nor on the resolution of M, we have used.

Let us define again :

$$H^p = \left\{\omega \in \bigwedge^{p} M : \omega \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0\right\}\Big/\sum_{i=1}^{k} \omega_i \wedge \bigwedge^{p-1} M.$$

Then we obtain again: i) $\mathfrak{A}^m H^p = 0$, $p = 0, 1, 2, \ldots$ for some $m > 0$ and ii) $H^p = 0$ for $0 \leqslant p < \operatorname{prof} \mathfrak{A}$.

For the proof we have only to apply the theorem to L_2 and $\tilde{\omega}_1, \ldots, \tilde{\omega}_k, \tilde{e}_1, \ldots, \tilde{e}_m$.

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