ANDRÉ UNTERBERGER
JULIANNE UNTERBERGER

Hölder estimates and hypoellipticity

Annales de l'institut Fourier, tome 26, nº 2 (1976), p. 35-54
<http://www.numdam.org/item?id=AIF_1976__26_2_35_0>

Ann. Inst. Fourier, Grenoble
26, 2 (1976), 35-54.

HOLDER ESTIMATES AND HYPOELLIPTICITY

by A. and J. UNTERBERGER

When one has proved an estimate of the kind

$$\|u\|_{s_1} \leqslant C(\|u\|_{s_2} + \|Pu\|_{s_3}),$$

in order to show that this implies the sought after regularity theorem, there still remains to be carried a shift in the s_j's, possibly a localization, and a regularization: a secondary objective of this paper is to help in this humble task.

Our chief concern will be with « Hölder estimates », a kind which was introduced by F. John [7] and used also by L. Hörmander [4, 5]. As a comparison between theorem 5.1 in [4] and theorem 3.1 in [11] may suggest, there is an obvious link between Hölder estimates and Carleman estimates of a rather loose type.

The main tool in this paper is a generalization of theorem 2.4.1 of L. Hörmander [3], another generalization of which was systematically used by A. Unterberger [10, 11].

To illustrate on a well-known example the flexibility as a tool of Hölder estimates, especially when induction is needed, we give a new proof of L. Hörmander's theorem on hypoelliptic second-order operators [6], somewhat inspired by that of J. Kohn [8], but slightly shorter, and perhaps easier to generalize, as an example will show.

1. Mollifiers, regularization and Sobolev-norms.

We use mollifiers associated with multiple symbols $\varphi(x, \eta, y)$ by the formula

$$Op(\varphi)u(x) = \int \varphi(x, \eta, y) e^{-2i\pi\langle y-x, \eta\rangle} u(y)\, dy\, d\eta.$$

The authors take this opportunity to thank Aarhus University for the wonderful hospitality they enjoyed during the academic year 1974-75.

We always assume

(S) φ is a C^∞ function on $\mathbf{R}^{3n}$, and for all multi-indices $\alpha_1, \beta, \alpha_2$ and every $M > 0$, there exists $C > 0$ such that

$$|D_x^{\alpha_1} D_\eta^{\beta} D_y^{\alpha_2} \varphi(x, \eta, y)| \leqslant C(1 + |\eta|)^{-M}$$

for all $(x, \eta, y) \in \mathbf{R}^{3n}$.

On the subject of multiple symbols, the reader may consult K. O. Friedrichs [2], or H. Kumano-go [9], or K. Watanabe [12], or forthcoming lecture notes by A. Unterberger at Aarhus University.

As is well-known, one may also write

$$Op(\varphi)u(x) = \int \tilde{\varphi}(x, \xi)\hat{u}(\xi)e^{2i\pi\langle x, \xi\rangle}\, d\xi,$$

where the symbol

$$\tilde{\varphi}(x, \xi) = \int dz \int \varphi(x, \xi + \zeta, x + z)e^{-2i\pi\langle z, \zeta\rangle}\, d\zeta$$

belongs to $S^{-\infty}$, the usual notation for a class of symbols of order $-\infty$; it is then clear, using an integration by parts, that $Op(\varphi)$ operates continuously from the space $\mathscr{E}'(\mathbf{R}^n)$ to the space $\mathscr{S}(\mathbf{R}^n)$.

We also consider, for $t > 0$, the one-parameter family of mollifiers $Op(\varphi_t)$, where $\varphi_t(x, \eta, y) = \varphi(x, t\eta, y)$.

The main reason for the introduction of these mollifiers is the following commutation theorem:

THEOREM 1.1. — *Let φ satisfy* (S); *let X be a first-order differential operator whose coefficients are defined and C^∞ in an open subset Ω of $\mathbf{R}^n$; let Ω' be a relatively compact open subset of Ω, and assume that there exists a compact subset L of Ω with the following property:*

for every $y \in \Omega'$, the set of $x \in \mathbf{R}^n$ such that $\varphi(x, \eta, y) \neq 0$ for some $\eta \in \mathbf{R}^n$ is contained in L.

Then there exists ψ satisfying (S) *such that, for every $u \in \mathscr{E}'(\Omega')$, and every $t > 0$, one has*

$$[X, Op(\varphi_t)]u = Op(\psi_t)u.$$

Proof. — First note that for every $u \in \mathcal{E}'(\Omega')$, the support of $\mathrm{O}p(\varphi_t)u$ is included in L, so that $[X, \mathrm{O}p(\varphi_t)]u$ is well defined and has its support contained in L; changing X outside L, it is no loss of generality to assume that the coefficients of X extend as functions in $\mathcal{S}(\mathbf{R}^n)$; also, one may assume that $u \in \mathcal{D}(\Omega')$.

Let $X = \sum_k a_k \frac{\partial}{\partial x_k} + b$. One has

$$\mathrm{O}p(\varphi_t)Xu(x) = \sum_k \int \varphi(x, t\eta, y) e^{-2i\pi\langle y-x, \eta\rangle} a_k(y) u'_k(y)\, dy\, d\eta + \int \varphi(x, t\eta, y) e^{-2i\pi\langle y-x, \eta\rangle} b(y) u(y)\, dy\, d\eta$$

and, after an integration by parts, the first term may be written as

$$\sum_k \int e^{-2i\pi\langle y-x, \eta\rangle} u(y)\, dy\, d\eta \left[-\frac{\partial \varphi}{\partial y_k}(x, t\eta, y) a_k(y) + 2i\pi \eta_k \varphi(x, t\eta, y) a_k(y) - \varphi(x, t\eta, y) \frac{\partial a_k}{\partial y_k}(y) \right].$$

With the straightforward expression for $X\, \mathrm{O}p(\varphi_t)u(x)$, one gets

$$[X, \mathrm{O}p(\varphi_t)]u(x) = \int e^{-2i\pi\langle y-x, \eta\rangle} u(y)\, dy\, d\eta \left[\sum_k a_k(x) \frac{\partial \varphi}{\partial x_k}(x, t\eta, y) + a_k(y) \frac{\partial \varphi}{\partial y_k}(x, t\eta, y) + 2i\pi \sum_k (a_k(x) - a_k(y)) \eta_k \varphi(x, t\eta, y) + \sum_k \varphi(x, t\eta, y) \frac{\partial a_k}{\partial y_k}(y) + (b(x) - b(y)) \varphi(x, t\eta, y) \right].$$

With smooth functions c_{jk} chosen such that

$$a_k(x) - a_k(y) = \sum_j c_{jk}(x, y)(x_j - y_j)$$

hewnever $y \in \Omega'$ and $x \in \mathrm{L}$, one may, after an integration by parts, rewrite the bothering term in the middle as

$$-\sum_{jk} \int e^{-2i\pi\langle y-x, \eta\rangle} c_{jk}(x, y) \frac{\partial}{\partial \eta_j} (\eta_k \varphi(x, t\eta, y)) u(y)\, d\eta\, dy,$$

which proves Theorem 1.1 if we choose

$$\begin{aligned}\psi(x, \eta, y) = &\sum_k a_k(x) \frac{\partial\varphi}{\partial x_k}(x, \eta, y) + a_k(y) \frac{\partial\varphi}{\partial y_k}(x, \eta, y)\\ &- \sum_k c_{kk}(x, y)\varphi(x, \eta, y)\\ &- \sum_{jk} c_{jk}(x, y)\eta_k \frac{\partial\varphi}{\partial \eta_j}(x, \eta, y)\\ &+ \sum_k \varphi(x, \eta, y) \frac{\partial a_k}{\partial y_k}(y) + (b(x) - b(y))\varphi(x, \eta, y).\end{aligned}$$

Remark. — If X is reduced to its zero-order term b, and $b(x) - b(y) = \Sigma d_j(x, y)(x_j - y_j)$, one may also write $[\mathrm{X}, \mathrm{O}p(\varphi_t)] = t\mathrm{O}p(\chi_t)$, with

$$\chi(x, \eta, y) = \frac{1}{2i\pi} \sum_j d_j(x, y) \frac{\partial\varphi}{\partial\eta_j}(x, \eta, y).$$

Theorem 1.2. — *Let* φ *satisfy* (S), *and* $s \in \mathbf{R}$. *Then*

(i) *for* $0 < t \leqslant 1$, $\mathrm{O}p(\varphi_t)$ *remains in a bounded subset of the space of continuous linear endomorphisms of* $\mathrm{H}^s(\mathbf{R}^n)$ *with the operator-norm topology.*

(ii) *for every* $u \in \mathrm{H}^s(\mathbf{R}^n)$, $\mathrm{O}p(\varphi_t)u$ *converges in the space* $\mathrm{H}^s(\mathbf{R}^n)$, *as* $t \to 0$, *to the product of* u *by the function* $\varphi(x, 0, x)$.

Proof. — As, for $0 < t \leqslant 1$,

$$|\mathrm{D}_x^{\alpha_1}\mathrm{D}_\eta^{\beta}\mathrm{D}_y^{\alpha_2}(\varphi(x, t\eta, y))| \leqslant \mathrm{C}t^{|\beta|}(1 + t|\eta|)^{-|\beta|} \leqslant \mathrm{C}(1 + |\eta|)^{-|\beta|},$$

with a constant C independent of t, the simple symbol associated with φ_t remains in a bounded subset of the space of symbols S^0, which proves (i).

When $t \to 0$, $\varphi(x, t\eta, y)$ converges in the standard (local-type) topology of $\mathrm{C}^\infty(\mathbf{R}^{3n})$ to $\varphi(x, 0, y)$, which is a multiple symbol of the operator of multiplication by $\varphi(x, 0, x)$; together with the already remarked boundedness of $\{\varphi_t\}$ in the space of multiple symbols of order 0, this suffices to imply (ii): this easy, but useful, argument is implicit in all treatments of pseudo-differential operators, when a reduction to compactly supported (possibly multiple) symbols is needed; it is stated explicitly, for instance, in R. Beals and C. Fefferman ([1], corollary, p. 4).

The next theorem generalizes theorem 2.4.1. of L. Hörmander [3].

THEOREM 1.3. — *Let* φ *satisfy* (S), *and* $s \in \mathbf{R}$, $m \in \mathbf{R}$. *Assume that for some* $\sigma > s$, *and all multi-indices* α_1, β, α_2, *the estimate* $|D_x^{\alpha_1} D_\eta^\beta D_y^{\alpha_2} \varphi(x, \eta, y)| \leqslant C|\eta|^{\sigma-|\beta|}$ *is valid for some* $C > 0$, *and all* (x, η, y) *with* $|\eta| \leqslant 1$. *Then there exists* $C_1 > 0$ *such that for every* $u \in \mathcal{S}(\mathbf{R}^n)$, *one has*

$$\int_0^1 t^{-2s} \| Op(\varphi_t) u \|_m^2 \frac{dt}{t} \leqslant C_1 \| u \|_{s+m}^2.$$

Assume moreover that for a certain open subset Ω *of* $\mathbf{R}^n$, *there is no point* (x, η) *with* $x \in \Omega$ *and* $\eta \in \mathbf{R}^n$, $\eta \neq 0$, *such that* $\varphi(x, \lambda\eta, x) = 0$ *for every* $\lambda > 0$.

Then, for every compact subset K *of* Ω, *there exist two constants* $C_2 > 0$ *and* $C_3 > 0$ *such that, for every* $u \in \mathcal{D}_K(\Omega)$ *one has*

$$\| u \|_{s+m}^2 \leqslant C_2 \int_0^1 t^{-2s} \| Op(\varphi_t) u \|_m^2 \frac{dt}{t} + C_3 \| u \|_{s+m-\frac{1}{2}}^2.$$

Proof. — It is no loss of generality to assume

$$s < \sigma \leqslant s + \frac{1}{2}.$$

One has $\| Op(\varphi_t) u \|_m = \| \Lambda^m Op(\varphi_t) u \|$, with

$$\Lambda^m = \left(1 - \frac{\Delta}{4\pi^2}\right)^{m/2}.$$

As

$$|D_x^{\alpha_1} D_\eta^\beta D_y^{\alpha_2}(\varphi(x, t\eta, y))| \leqslant Ct^\sigma |\eta|^{\sigma-|\beta|},$$

$t^{-\sigma}\varphi_t$ remains in a bounded subset of the space of (classical) multiple symbols of order σ, so that $t^{-\sigma}[\Lambda^m, Op(\varphi_t)]$ remains, for $t > 0$, in a bounded subset of the space of operators of order $m + \sigma - 1$; with some constant $C > 0$, one may then write, for every $u \in \mathcal{S}(\mathbf{R}^n)$:

$$\int_0^1 t^{-2s} \| [\Lambda^m, Op(\varphi_t)] u \|^2 \frac{dt}{t} \leqslant C \int_0^1 t^{-2s+2\sigma} \| u \|_{\sigma+m-1}^2 \frac{dt}{t} \leqslant C_1 \| u \|_{s+m-\frac{1}{2}}^2.$$

In this way, the proof of Theorem 1.3 is reduced to the case when $m = 0$. One has

$$\mathrm{Op}(\varphi_t)^* u(x) = \int \bar{\varphi}(y, t\eta, x) e^{-2i\pi\langle y-x, \eta\rangle} u(y)\, dy\, d\eta,$$

and $\|\mathrm{Op}(\varphi_t)u\|^2 = (\mathrm{Op}(\varphi_t)^*\mathrm{Op}(\varphi_t)u, u)$, with

$$\mathrm{Op}(\varphi_t)^*\mathrm{Op}(\varphi_t)u(x) = \int \bar{\varphi}(y, t\eta, x)\varphi(y, t\xi, z) e^{-2i\pi[\langle y-x, \eta\rangle + \langle z-y, \xi\rangle]} u(z)\, dy\, d\eta\, dz\, d\xi,$$

so that $\mathrm{Op}(\varphi_t)^*\mathrm{Op}(\varphi_t)$ is an operator with multiple symbol $b(x, \eta, y, \xi, z)$ of a type considered by Watanabe [12] and possibly, previously, by Kumano-go [9], a paper which was unfortunately unavailable to us.

Then

$$\int_0^1 t^{-2s}\|\mathrm{Op}(\varphi_t)u\|^2 \frac{dt}{t} = (\mathrm{R}u, u),$$

where R is the operator with multiple symbol

$$r(x, \eta, y, \xi, z) = \int_0^1 t^{-2s}\bar{\varphi}(y, t\eta, x)\varphi(y, t\xi, z) \frac{dt}{t}.$$

On $\mathbf{R}^{2n}$, let $1 = \beta(\eta, \xi) + \alpha_1(\eta, \xi) + \alpha_2(\eta, \xi)$ where $\beta \in \mathscr{D}(\mathbf{R}^{2n})$, α_1 and α_2 are two C^∞ functions whose supports do not contain the origin and which are moreover homogeneous of degree 0 for $|\xi|^2 + |\eta|^2 \geqslant 1$, and satisfy the following two conditions: $|\eta| > \frac{|\xi|}{2}$ on the support of α_1, and $|\xi| > |\eta|$ on the support of α_2.

Then, with obvious changes of variables, one may write

$$\begin{aligned} r(x, \eta, y, \xi, z) &= \beta \int_0^\infty \cdots - (1-\beta)\int_1^\infty \cdots + \alpha_1 \int_0^\infty \cdots + \alpha_2 \int_0^\infty \cdots \\ &= \beta(\eta, \xi) \int_0^1 t^{-2s}\bar{\varphi}(y, t\eta, x)\varphi(y, t\xi, z)\frac{dt}{t} \\ &\quad - (1-\beta(\eta, \xi)) \int_1^\infty t^{-2s}\bar{\varphi}(y, t\eta, x)\varphi(y, t\xi, z)\frac{dt}{t} \\ &\quad + \alpha_1(\eta, \xi)|\eta|^{2s} \int_0^\infty t^{-2s}\bar{\varphi}\left(y, t\frac{\eta}{|\eta|}, x\right)\bar{\varphi}\left(y, t\frac{\xi}{|\eta|}, z\right)\frac{dt}{t} \\ &\quad + \alpha_2(\eta, \xi)|\xi|^{2s} \int_0^\infty t^{-2s}\bar{\varphi}\left(y, t\frac{\eta}{|\xi|}, x\right)\varphi\left(y, t\frac{\xi}{|\xi|}, z\right)\frac{dt}{t}. \end{aligned}$$

With the notations of Watanabe, it is clear that the first two terms are multiple symbols in the class $S_{1,0}^{-\infty,-\infty}$, and that the last two terms belong respectively to $S_{1,0}^{2s,0}$ and $S_{1,0}^{0,2s}$, so that R is a pseudo-differential operator of order $2s$: the first part of Theorem 1.3. follows.

Also, the simple symbol defining the same operator as the myltiple symbol $r(x, \eta, y, \xi, z)$ differs by an error term in S^{2s-1} from $r(x, \xi, x, \xi, x) = \int_0^1 t^{-2s}|\varphi(x, t\xi, x)|^2 \frac{dt}{t}$, a function which, up to an error term in $S^{-\infty}$, may be written for large $|\xi|$ as $\int_0^\infty t^{-2s}|\varphi(x, t\xi, x)|^2 \frac{dt}{t} = |\xi|^{2s}\int_0^\infty t^{-2s}\left|\varphi\left(x, t\frac{\xi}{|\xi|}, x\right)\right|^2 \frac{dt}{t}$.

The second part of Theorem 1.3 is then a consequence of the (non-sharp) Garding inequality.

2. How to derive classical estimates from Hölder estimates.

Hölder estimates are, generally speaking, estimates of the kind $p(u) \leqslant C(q(u))^\delta(r(u))^{1-\delta}$, where p, q, r are semi-norms on a vector space, and $0 \leqslant \delta \leqslant 1$; more factors may be allowed.

On this subject, the reader may consult the papers of F. John and L. Hörmander mentioned in the introduction.

As, unless $\delta = 0$ or 1, the right-hand side of a Hölder estimate is generally not a sublinear function of u, such an estimate may carry a lot more information than it seems. As a first example, let us show that an estimate

$$p(u) \leqslant C\|u\|_{s_1}^{\delta}\|u\|_{s_2}^{1-\delta},$$

assumed to be valid for every $u \in \mathscr{D}(\mathbf{R}^n)$, is almost as good as an estimate $p(u) \leqslant C\|u\|_{\delta s_1+(1-\delta)s_2}$.

It is clearly weaker, due to the logarithmic convexity of the function $s \longmapsto \|u\|_s$, and as a matter of fact, strictly weaker in general, as the elementary estimate

$$|u(0)|^2 \leqslant \|u\|\;\|u\|_1,$$

valid for $u \in \mathscr{D}(\mathbf{R})$, together with the fact that the Dirac measure on $\mathbf{R}$ does not belong to $H^{-\frac{1}{2}}(\mathbf{R})$, shows.

However, let us introduce, for $s \in \mathbf{R}$ and $k \in \mathbf{R}$, the semi-norm $\| \quad \|_{s,k}$ defined, for $u \in \mathcal{D}(\mathbf{R}^n)$, by

$$\|u\|_{s,k}^2 = \int (1 + |\xi|^2)^s (1 + \mathrm{Log}\,(1 + |\xi|))^{2k} |\hat{u}(\xi)|^2 \, d\xi.$$

Then one has the following result:

Proposition 2.1. — *Let* $\delta \in]0, 1[$ *and* $s_1 < s_2$; *let*

$$s = \delta s_1 + (1 - \delta) s_2;$$

let $k > \frac{1}{2}$. *There exists a constant* $C > 0$ *depending only on* n, δ, s_1, s_2 *and* k *such that, for every semi-norm* p *on* $\mathcal{D}(\mathbf{R}^n)$ *satisfying* $p(u) \leqslant \|u\|_{s_1}^{\delta} \|u\|_{s_2}^{1-\delta}$ *for every* $u \in \mathcal{D}(\mathbf{R}^n)$, *one has, for every* $u \in \mathcal{D}(\mathbf{R}^n)$:

$$p(u) \leqslant C \|u\|_{s,k}.$$

Proof. — By Hahn-Banach's theorem one may assume that $p(u) = |\langle f, u \rangle|$ for a certain $f \in H^{-s_2}(\mathbf{R}^n)$; one may in fact even assume that $f \in \mathcal{D}(\mathbf{R}^n)$: for let $\psi \in \mathcal{D}(\mathbf{R}^{2n})$ satisfy $\psi(0) = 1$, and let, for $0 < \varepsilon \leqslant 1$, R_ε be the pseudo-differential operator with symbol $\psi(\varepsilon x, \varepsilon \xi)$. Then, as $\varepsilon \to 0$, $R_\varepsilon f$ converges weakly to f, and $\langle R_\varepsilon f, u \rangle = \langle f, {}^t R_\varepsilon u \rangle$, where the operators ${}^t R_\varepsilon$ are uniformly bounded either as endomorphisms of $H^{s_1}(\mathbf{R}^n)$ or as endomorphisms of $H^{s_2}(\mathbf{R}^n)$.

Thus assume that for some $f \in \mathcal{D}(\mathbf{R}^n)$ and all $u \in \mathcal{D}(\mathbf{R}^n)$ one has $|\langle f, u \rangle| \leqslant \|u\|_{s_1}^{\delta} \|u\|_{s_2}^{1-\delta}$.

We want to show that for some constant C depending only on n, δ, s_1, s_2 and k, one has

$$\|f\|_{-s,-k} \leqslant C.$$

By Young's inequality, one has, for every $u \in \mathcal{D}(\mathbf{R}^n)$ and $t > 0$:

$$\begin{aligned} |\langle f, u \rangle|^2 \leqslant \|u\|_{s_1}^{2\delta} \|u\|_{s_2}^{2(1-\delta)} &= t^{-2\delta(1-\delta)(s_2-s_1)} \|u\|_{s_1}^{2\delta} t^{2\delta(1-\delta)(s_2-s_1)} \|u\|_{s_2}^{2(1-\delta)} \\ &\leqslant \delta t^{-2(1-\delta)(s_2-s_1)} \|u\|_{s_1}^2 + (1 - \delta) t^{2\delta(s_2-s_1)} \|u\|_{s_2}^2, \end{aligned}$$

hence

$$t^{4s} |\langle f, u \rangle|^2 \leqslant \delta t^{2(s+s_1)} \|u\|_{s_1}^2 + (1 - \delta) t^{2(s+s_2)} \|u\|_{s_2}^2.$$

Added in proof: Prop. 2.1. is a consequence of (IV.1.1) in Lions-Peetre: Sur une classe d'espaces d'interpolation, IHES n° 19, 1964.

Let $\varphi \in \mathscr{D}(\mathbf{R}^n)$ be real valued and satisfy $\hat{\varphi}(\xi) = 0(|\xi|^\sigma)$ as $|\xi| \to 0$ for some sufficiently large σ and, with

$$\varphi_t(x) = t^{-n}\varphi\left(\frac{x}{t}\right)$$

and $\check{\varphi}_t(x) = \varphi_t(-x)$, apply this inequality to

$$u = \check{\varphi}_t * \varphi_t * \overline{f}$$

and integrate from 0 to $\frac{1}{2}$ with respect to the measure $\left(\log \frac{1}{t}\right)^{-2k} \frac{dt}{t}$. One gets

$$\begin{aligned}\int_0^{\frac{1}{2}} \left(\log \frac{1}{t}\right)^{-2k} t^{4s}\|\varphi_t * f\|^4 \frac{dt}{t} &\leqslant \delta \int_0^{\frac{1}{2}} \left(\log \frac{1}{t}\right)^{-2k} t^{2(s+s_1)}\|\varphi_t * \overline{f}\|^2_{s_1} \frac{dt}{t} \\ &+ (1-\delta) \int_0^{\frac{1}{2}} \left(\log \frac{1}{t}\right)^{-2k} t^{2(s+s_2)}\|\varphi_t * \overline{f}\|^2_{s_2} \frac{dt}{t}.\end{aligned}$$

By theorem 1.1 of [11], the right-hand side is less than $c_1\|f\|^2_{-s,-k}$ for some constant c_1 depending only on n, φ, s_1, s_2, δ and k.

Using this theorem again, and with constants depending only on n, φ, s_1, s_1, δ and k, one has

$$\begin{aligned}\|f\|^4_{-s,-k} &\leqslant c_2\left[\int_0^{\frac{1}{2}} \left(\log \frac{1}{t}\right)^{-2k} t^{2s}\|\varphi_t * f\|^2 \frac{dt}{t}\right]^2 + c_3\|f\|^4_{-s_2} \\ &\leqslant c_2\left(\int_0^{\frac{1}{2}} \left(\log \frac{1}{t}\right)^{-2k} \frac{dt}{t}\right)\left(\int_0^{\frac{1}{2}} \left(\log \frac{1}{t}\right)^{-2k} t^{4s}\|\varphi_t * f\|^4 \frac{dt}{t}\right) \\ &+ c_3\|f\|^4_{-s_2} \\ &= c_4 \int_0^{\frac{1}{2}} \left(\log \frac{1}{t}\right)^{-2k} t^{4s}\|\varphi_t * f\|^4 \frac{dt}{t} + c_3\|f\|^4_{-s_2}.\end{aligned}$$

Now, obviously, $\|f\|_{-s_2} \leqslant 1$. Hence

$$\|f\|^4_{-s,-k} \leqslant c_4c_1\|f\|^2_{-s,-k} + c_3.$$

and $\|f\|^2_{-s,-k} \leqslant \frac{1}{2}\left[c_4c_1 + (c_4^2c_1^2 + 4c_3)^{\frac{1}{2}}\right]$, which concludes the proof of Proposition 2.1.

The next theorem shows how regularity theorems may be proved via Hölder estimates (which may in some instances be easier to prove than standard ones): its principal defect is that we have to assume that P dominates, in a certain sense, some operators Q_k which are not intrinsically attached to P, but depend on a certain representation of P.

Let P belong to $\mathcal{A}_m$, the free associative algebra over **C** on m generators $X_1, \ldots, X_m$. Adding a new generator Φ free from the X_j's, one has the identities

$$[X_{i_1} \ldots X_{i_k}, \Phi] = X_{i_1} \ldots X_{i_{k-1}}[X_{i_k}, \Phi] + X_{i_1} \ldots [X_{i_{k-1}}, \Phi]X_{i_k} + \cdots + [X_{i_1}, \Phi]X_{i_2} \ldots X_{i_k}.$$

Also

$$X_{i_1} \ldots X_{i_{r-1}}[X_{i_r}, \Phi]X_{i_{r+1}} \ldots X_{i_k} = X_{i_1} \ldots X_{i_{r-2}}[X_{i_r}, \Phi]X_{i_{r-1}}X_{i_{r+1}} \ldots X_{i_k} + X_{i_1} \ldots X_{i_{r-2}}[X_{i_{r-1}}, [X_{i_r}, \Phi]]X_{i_{r+1}} \ldots X_{i_k},$$

from which it is easily seen by induction that there exists a finite number of elements Q_k of $\mathcal{A}_m$ such that one has the identity

$$[P(X_1, \ldots, X_m), \Phi] = \sum_k [X_{i_1^k}[X_{i_2^k}, \ldots, [X_{i_{r_k}^k}, \Phi] \ldots]]Q_k(X_1, \ldots, X_m).$$

Note that the « degrees » of the Q_k's are strictly less than the degree of P, so that, enlarging the set $\{Q_k\}$, one may assume that a like identity holds with P replaced by any of the Q_k's.

Now one may, in these identities, substitute operators for the letters $X_1, \ldots, X_m, \Phi$, provided that all the words containing at most once the letter Φ be well-defined as operators.

THEOREM 2.2. — *Let the operator* P *be expressed as* $P(X_1, \ldots, X_m)$, *where the* X_j*'s are smooth first-order differential operators in an open subset* Ω *of* $\mathbf{R}^n$.

Let the differential operators Q_k *be expressed as* $Q_k(X_1, \ldots, X)_m$, *and assume that whenever* R *is either* P *or one of the* Q_k*'s, one has formally, as just explained, the following identity with complex coefficients depending on* R :

$$(1)\quad [R(X_1, \ldots, X_m), \Phi] = \Sigma a^k_{\{i\}}[X_{i_1}, [X_{i_2}, \ldots, [X_{i_r}, \Phi] \ldots]]\, Q_k(X_1, \ldots, X_m).$$

Assume that for some relatively compact subset Ω' *of* Ω, *there exist real numbers* s_1, s_2, r_1, r_2, s' *and* r', *a finite set* $\{\mu\}$, *numbers* $\beta_{1,\mu}$, $\beta_{2,\mu}$, $\gamma_{1,\mu}$, $\gamma_{2,\mu}$, *all* $\geqslant 0$ *and satisfying* $\beta_{1,\mu} + \beta_{2,\mu} + \gamma_{1,\mu} + \gamma_{2,\mu} = 1$, *and a number* $C > 0$ *such that, for every* $u \in \mathcal{D}(\Omega')$, *one has*

$$\|u\|_{s'} + \sum_k \|Q_k u\|_{r'} \leqslant C \sum_\mu \|u\|_{s_1}^{\beta_{1,\mu}} \|u\|_{s_2}^{\beta_{2,\mu}} \|Pu\|_{r_1}^{\gamma_{1,\mu}} \|Pu\|_{r_2}^{\gamma_{2,\mu}}.$$

Then, if for every μ *one has*

$$(\beta_{1,\mu} + \beta_{2,\mu})s' + (\gamma_{1,\mu} + \gamma_{2,\mu})r' > \beta_{1,\mu}s_1 + \beta_{2,\mu}s_2 + \gamma_{1,\mu}r_1 + \gamma_{2,\mu}r_2,$$

the operator P *is hypoelliptic in* Ω'.

REMARK. — Due to the logarithmic convexity of the function $s \longmapsto \|u\|_s$, it would not be a greater generality to allow s_1, s_2, r_1 and r_2 to depend on μ.

Proof of Theorem 2.2. — Let $\varepsilon > 0$ be the minimum for all μ of

$$(\beta_{1,\mu} + \beta_{2,\mu})s' + (\gamma_{1,\mu} + \gamma_{2,\mu})r' - \beta_{1,\mu}s_1 - \beta_{2,\mu}s_2 - \gamma_{1,\mu}r_1 - \gamma_{2,\mu}r_2.$$

We are going to show first :

for every $\tau \in \mathbf{R}$ and every compact subset K of Ω' there exists $C > 0$ such that, for every $u \in \mathcal{D}_K(\Omega')$, one has

$$(2)\quad \|u\|_{s'+\tau} + \sum_k \|Q_k u\|_{r'+\tau} \leqslant C[\|u\|_{s'+\tau-\varepsilon} + \|Pu\|_{r'+\tau-\varepsilon}].$$

There exists a finite decreasing sequence $((\mathcal{Q}_q)_{1 \leqslant q \leqslant p}$ of subsets of $\{Q_k\}$ with the following two properties :

(i) $\mathcal{Q}_1 = \{Q_k\}$, and $\mathcal{Q}_p$ contains only constants.

(ii) for every $q \leqslant p - 1$, and every $R \in \mathcal{Q}_p$, an identity such as (1) holds, where in the right-hand side occur only elements of $\mathcal{Q}_{q+1}$.

For convenience, we set $\mathcal{Q}_{p+1} = \{0\}$.

For the typographer's benefit, we delete everywhere the sign $\sum\limits_{\mu}$ as well as the subscript μ in the following proof of (2). Also, we discard temporarily the terms for which either $\beta_1 + \beta_2 = 0$ or $\gamma_1 + \gamma_2 = 0$.

Let

$$\beta = (s_2 - s_1)\frac{\beta_1\beta_2}{\beta_1 + \beta_2}, \qquad \gamma = (r_2 - r_1)\frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2},$$

$$s = \frac{\beta_1 s_1 + \beta_2 s_2}{\beta_1 + \beta_2}, \qquad r = \frac{\gamma_1 r_1 + \gamma_2 r_2}{\gamma_1 + \gamma_2},$$

and

$$\delta = (\beta_1 + \beta_2)(\gamma_1 + \gamma_2)(r - r' - s + s').$$

By the hypothesis of Theorem 2.2, for every $u \in \mathcal{D}(\Omega')$ and all values of the parameters $\lambda > 1$ and $t > 0$, one has, with a constant $C > 0$ independent of u, λ and t:

$$\begin{aligned}\|u\|_{s'}^2 + \sum_{1\leqslant q\leqslant p} \lambda^{2q} \sum_{Q\in\mathcal{Q}_q} \|Qu\|_{r'}^2 \\ \leqslant C\left(\lambda^{\frac{2p\beta_1}{\beta_1+\beta_2}} t^{-2\beta-\frac{2\beta_1\delta}{\beta_1+\beta_2}} \|u\|_{s_1}^{2\beta_1}\right)\left(\lambda^{\frac{2p\beta_2}{\beta_1+\beta_2}} t^{2\beta-\frac{2\beta_2\delta}{\beta_1+\beta_2}} \|u\|_{s_2}^{2\beta_2}\right) \\ \left(t^{-2\gamma+\frac{2\gamma_1\delta}{\gamma_1+\gamma_2}} \|Pu\|_{r_1}^{2\gamma_1}\right)\left(t^{2\gamma+\frac{2\gamma_2\delta}{\gamma_1+\gamma_2}} \|Pu\|_{r_2}^{2\gamma_2}\right).\end{aligned}$$

Using the concavity of the logarithm (i.e. Young's inequality with exponents $\frac{1}{\beta_1}, \frac{1}{\beta_2}, \frac{1}{\gamma_1}$ and $\left.\frac{1}{\gamma_2}\right)$, one gets

$$\begin{aligned}(3) \quad \|u\|_{s'}^2 + \sum_{1\leqslant q\leqslant p} \lambda^{2q} \sum_{Q\in\mathcal{Q}_q} \|Qu\|_{r'}^2 \\ \leqslant C\left\{\lambda^{\frac{2p}{\beta_1+\beta_2}}\left[t^{-\frac{2\beta}{\beta_1}-\frac{2\delta}{\beta_1+\beta_2}} \|u\|_{s_1}^2 + t^{\frac{2\beta}{\beta_2}-\frac{2\delta}{\beta_1+\beta_2}} \|u\|_{s_2}^2\right]\right. \\ \left. + t^{-\frac{2\gamma}{\gamma_1}+\frac{2\delta}{\gamma_1+\gamma_2}} \|Pu\|_{r_1}^2 + t^{\frac{2\gamma}{\gamma_2}+\frac{2\delta}{\gamma_1+\gamma_2}} \|Pu\|_{r_2}^2\right\}.\end{aligned}$$

Observe that

$$\begin{aligned}s_1 + \frac{\beta}{\beta_1} + \frac{\beta}{\beta_1 + \beta_2} &= s_2 - \frac{\beta}{\beta_2} + \frac{\delta}{\beta_1 + \beta_2} \\ &= s' + (\beta_1 + \beta_2)(s - s') + (\gamma_1 + \gamma_2)(r - r') \leqslant s' - \varepsilon\end{aligned}$$

and

$$r_1 + \frac{\gamma}{\gamma_1} - \frac{\delta}{\gamma_1 + \gamma_2} = r_2 - \frac{\gamma}{\gamma_2} - \frac{\delta}{\gamma_1 + \gamma_2}$$
$$= r' + (\beta_1 + \beta_2)(s - s') + (\gamma_1 + \gamma_2)(r - r') \leqslant r' - \varepsilon.$$

Now choose $\varphi(x, \eta, y) = \alpha(x, y)|\eta|^{2\sigma}e^{-\pi|\eta|^2}$, where σ is a large integer and the smooth function α is such that $\alpha(x, y)$ is zero for all y when x is outside some compact subset of Ω', and $\alpha(x, x) = 1$ for all x in some neighbourhood of K.

Applying Theorem 1.1, whenever R is P or one of the Q_k's, one has, for some nice mollifiers ψ_k^R and every $t > 0$

$$(4) \qquad [R, Op(\varphi_t)] = \sum_k Op((\psi_k^R)_t)Q_k$$

where, moreover, if $R \in \mathcal{Q}_q$ $(1 \leqslant q \leqslant p)$, only Q_k's belonging to $\mathcal{Q}_{q+1}$ occur in the right-hand side.

On the other hand, observe that in the last part of Theorem 1.3 one may obviously replace the error term $C_3\|u\|^2_{s+m-\frac{1}{2}}$ by $C_4\|u\|^2_{-N}$, however large N.

Applying the estimate (3) with u replaced by $Op(\varphi_t)u$, and integrating from 0 to 1 with respect to the measure $t^{-2\tau}\frac{dt}{t}$, one gets, applying Theorem 1.3 and choosing p' as the maximum of all the numbers $\frac{p}{\beta_1 + \beta_2}$:

there exist two constants $h > 0$ and $C > 0$ such that, for every $u \in \mathcal{D}_K(\Omega')$ and every $\lambda > 1$, one has

$$(5) \quad h\Big[\|u\|^2_{s'+\tau} + \sum_{1\leqslant q\leqslant p} \lambda^{2q} \sum_{Q\in\mathcal{Q}_q} \|Qu\|^2_{r'+\tau}\Big]$$
$$- C\sum_{1\leqslant q\leqslant p} \lambda^{2q} \sum_{Q\in\mathcal{Q}_{q+1}} \|Qu\|^2_{r'+\tau}$$
$$\leqslant C\Big\{\lambda^{2p'}\|u\|^2_{s'+\tau-\varepsilon} + \|Pu\|^2_{r'+\tau-\varepsilon} + \sum_{Q\in\mathcal{Q}_1} \|Qu\|^2_{r'+\tau-\varepsilon}\Big\}.$$

This yields (2) if λ is chosen large enough.

However, we did not take into account the terms for which $\gamma_2 + \gamma_2 = 0$ or $\beta_1 + \beta_2 = 0$, and one may verify that, though the preceding proof takes care also of the terms with $\gamma_1 + \gamma_2 = 0$, it is not adapted for terms with $\beta_1 + \beta_2 = 0$.

Noting that in this case $r_1 + \frac{\gamma}{\gamma_1} = r_2 - \frac{\gamma}{\gamma_2} = r \leqslant r' - \varepsilon$, we get on the right-hand side of (5) terms of the form

$$C\lambda^{2p}\Big[\|Pu\|^2_{r'+\tau-\varepsilon} + \sum_{Q\in\mathcal{Q}_1} \|Qu\|^2_{r'+\tau-\varepsilon}\Big].$$

As on the left-hand side of (5) we have a term $h\lambda^2 \sum_{Q\in\mathcal{Q}_1} \|Qu\|^2_{r'+\tau}$, it is obviously enough, in order that (5), hence (2), be valid, that one may write, for some p' :

$$\lambda^{2p} \sum_{Q\in\mathcal{Q}_1} \|Qu\|^2_{r'+\tau-\varepsilon} \leqslant C\Big[\lambda \sum_{Q\in\mathcal{Q}_1} \|Qu\|^2_{r'+\tau} + \lambda^{2p'}\|u\|^2_{s'+\tau-\varepsilon}\Big].$$

But now, d being the order of the operator P, this is a consequence of the estimate, valid for $\varphi \in \mathcal{D}(\mathbf{R}^n)$ and every $\lambda > 0$:

$$\|\varphi\|^2_{r'+\tau-\varepsilon} \leqslant C(\lambda^{1-2p}\|\varphi\|^2_{r'+\tau} + \lambda^{2(p'-p)}\|\varphi\|^2_{s'+\tau-\varepsilon-d+1}),$$

in its turn a consequence of the logarithmic convexity of the function $s \longmapsto \|\varphi\|_s$.

Thus (2) is proved in general.

Finally, assume that for some $u \in \mathcal{E}'(\Omega)$, some $\tau \in \mathbf{R}$ and some open subset Ω'' of Ω', u is $H^{s'+\tau-\varepsilon}_{\text{loc}}$ in Ω'', Pu is $H^{r'+\tau-\varepsilon}_{\text{loc}}$ and Qu is $H^{r'+\tau-\varepsilon}_{\text{loc}}$ in Ω'' for every $Q \in \mathcal{Q}_1$; then we shall show that u is $H^{s'+\tau}_{\text{loc}}$ in Ω'' and that Qu is $H^{r'+\tau}_{\text{loc}}$ in Ω'' for every $Q \in \mathcal{Q}_1$, which will prove Theorem 2.2 by induction.

For every compact subset L of Ω'', there exists a compact subset K of Ω'' and a smooth function $\alpha(x, y)$ with support in $K \times K$ such that $\alpha(x, x) = 1$ for all x in some neighbourhood of L. Choose this time $\varphi(x, \eta, y) = \alpha(x, y)e^{-\pi|\eta|^2}$, so that the distribution $\alpha(x, x)u(x)$ is the weak limit, as $t \to 0$, of $Op(\varphi_t)u$; also, for every $Q \in \mathcal{Q}_1$, $\alpha(x, x)Qu(x)$ is the weak limit, as $t \to 0$, of $Op(\varphi_t)Qu$. Thus it suffices to show that, as $t \to 0$, $Op(\varphi_t)u$ remains in a bounded subset of $H^{s'+\tau}(\mathbf{R}^n)$ and that, for every $Q \in \mathcal{Q}_1$, $Op(\varphi_t)u$ remains in a bounded subset of $H^{r'+\tau}(\mathbf{R}^n)$. One may apply the estimate (2) with u replaced by $Op(\varphi_t)u$, and use the identities (4) to express the commutators with $Op(\varphi_t)$: note that the formula given in the proof of Theorem 1.1 shows that the mollifiers ψ^R_k also have their (x, y)-supports contained in $K \times K$.

Using the first part of Theorem 1.2, one gets immediately that $Op(\varphi_t)u$ remains in a bounded subset of $H^{s'+\tau}(\mathbf{R}^n)$. If $R \in \mathcal{Q}_q$, we also get that $Op(\varphi_t)Ru$ remains in a bounded subset of $H^{r'+\tau}(\mathbf{R}^n)$, provided that we know already that all the Qu's are $H^{r'+\tau}_{loc}$ in Ω'' when $Q \in \mathcal{Q}_{q+1}$, so that this last part is proved by induction on q, starting from $q = p + 1$.

This concludes the proof of Theorem 2.2.

3. A short proof of L. Hörmander's theorem on hypoelliptic second-order operators.

Let $P = -\sum_{j\geq 1} X_j^2 + X_0 + f$, where the X_j's $(j \geq 0)$ are C^∞ real vector fields on an open subset Ω of $\mathbf{R}^n$, and f is a C^∞ complex-valued function on Ω.

Denote by $\mathcal{F}_p (p \geq 0)$ the set of all iterated brackets $[X_{i_1}[X_{i_2} \ldots [X_{i_q}, X_{i_{q+1}}] \ldots]]$ with $0 \leq q \leq p$.

In [6], L. Hörmander proved the following theorem : Assume that condition (H) holds :

(H) For every compact subset K of Ω, there exists $p \geq 0$ such that at every point of K the linear space of all vectors is generated by the set of values at this point of the fields belonging to $\mathcal{F}_p$.

Then P is hypoelliptic in Ω.

We shall prove that for every compact subset K of Ω, there exist $\delta \in]0, 1]$ and $C > 0$ such that the following two estimates hold for every $u \in \mathcal{D}_K(\Omega)$:

(1) $\|u\|_1 \leq C(\|u\|_1 + \|Pu\|_1)^{1-\delta}\|u\|^{\delta/2}(\|u\| + \|Pu\|)^{\delta/2}$.

(2) for every $j \geq 1$

$$\|X_j u\|_{1-\delta} \leq C(\|u\|_1 + \|Pu\|_1)^{1-\delta}\|u\|^{\delta/2}(\|u\| + \|Pu\|)^{\delta/2}.$$

This will imply the result by Theorem 2.2.

In the following estimates, claimed for $u \in \mathcal{D}_K(\Omega)$, the constant C may depend on K; as $X_j^* = -X_j + a_j$, a_j a C^∞ real-valued function, an obvious integration by parts yields

(3) $$\operatorname{Re}(Pu, u) = \sum_{j\geq 1} \|X_j u\|^2 + \operatorname{Re}(u, (g + f)u),$$

where g is a function depending only on the X_j's, so that, adding if necessary a constant to f, we may assume without loss of generality that

$$\text{(4)} \qquad Re(Pu, u) \geqslant \sum_{j\geqslant 1} \|X_j u\|^2 + \|u\|^2.$$

LEMMA 3.1. — *For any three real vector fields* A, B_1, B_2 :

$$|Re(Au, B_1B_2u)| \leqslant C\{\|u\|_1(\|B_1u\| + \|B_2u\|) + \|Au\| \, \|[B_1, B_2]u\|\}.$$

Proof. — Neglecting small terms, one has

$$Re(Au, B_1B_2u) = Re(B_1^*Au, B_2u) \sim - Re(B_1Au, B_2u) \sim - Re(AB_1u, B_2u),$$

and

$$Re(Au, B_1B_2u) \sim Re(Au, B_2B_1u) \sim - Re(AB_2u, B_1u) \sim Re(B_2u, AB_1u).$$

LEMMA 3.2. — *For any two real vector fields* Z_1, Z_2 :

$$|Re(PZ_1u, Z_2u)| \leqslant C\{(\|u\|_1 + \|Pu\|_1)((Re(Pu, u))^{\frac{1}{2}} + \|Z_2u\|) + \sum_{j\geqslant 1} \|[X_j, Z_1]u\| \, \|[X_j, Z_2]u\|\}.$$

Proof. — $Re(PZ_1u, Z_2u) = Re([P, Z_1]u, Z_2u) + Re(Z_1Pu, Z_2u)$. The second term is less in absolute value than $C\|Pu\|_1\|Z_2u\|$. Also

$$\begin{aligned} Re([P, Z_1]u, Z_2u) &= -2Re \sum_{j\geqslant 1} (X_j[X_j, Z_1]u, Z_2u) + Re(T_1u, Z_2u) \\ &= 2Re \sum_{j\geqslant 1} ([X_j, Z_1]u, X_jZ_2u) + Re(T_2u, Z_2u), \end{aligned}$$

where T_1 and T_2 are first order differential operators, and one uses Lemma 3.1 and (4).

LEMMA 3.3. — *For every real vector field* Y, *and* $j \geqslant 1$:

$$\|[X_j, Y]u\| \leqslant C(\|u\|_1 + \|Pu\|_1)^{\frac{1}{2}}((Re(Pu, u))^{\frac{1}{2}} + \|Yu\|)^{\frac{1}{2}}.$$

Proof. — We first remark that for every real vector field Z :

$$\text{(5)} \qquad \|X_jZu\| \leqslant C(\|u\|_1 + \|Pu\|_1)$$

and

$$(6) \qquad \|ZX_j u\| \leqslant C(\|u\|_1 + \|Pu\|_1);$$

for (6) is an obvious consequence of (5), and

$$\sum_{j\geqslant 1} \|X_j Zu\|^2 \leqslant Re(PZu, Zu) \leqslant C(\|u\|_1 + \|Pu\|_1)\|u\|_1$$

by Lemma 3.2. Now

$$\|[X_j, Y]u\|^2 = (X_j Yu, [X_j, Y]u) - (YX_j u, [X_j, Y]u),$$

and

$$\begin{aligned} Re(X_j Yu, [X_j, Y]u &\leqslant C\|Yu\|\,\|u\|_1 - Re(Yu, X_j[X_j, Y]u) \\ &\leqslant C\|Yu\|(\|u\|_1 + \|Pu\|_1) \end{aligned}$$

by (5). Also

$$\begin{aligned} - Re(YX_j u, [X_j, Y]u) & \\ &\leqslant C\|X_j u\|\,\|u\|_1 + Re(X_j u, [X_j, Y]Yu) \\ &\leqslant C\|X_j u\|\,\|u\|_1 - Re([X_j, Y]X_j u, Yu) \\ &\leqslant C\{(Re(Pu, u))^{\frac{1}{2}}\|u\|_1 + \|Yu\|(\|u\|_1 + \|Pu\|_1)\}, \end{aligned}$$

by (6).

LEMMA 3.4. — $\|X_0 u\| \leqslant C(\|u\|_1 + \|Pu\|_1)^{\frac{1}{2}}((Re(Pu, u))^{\frac{1}{2}})^{\frac{1}{2}}$.

Proof. — $$\begin{aligned} \|X_0 u\|^2 &= (X_0 u, Pu) + \sum_{j\geqslant 1} (X_0 u, X_j^2 u) - Re(X_0 u, fu) \\ &\leqslant C\|u\|(\|u\|_1 + \|Pu\|_1) + \sum_{j\geqslant 1} Re(X_0 u, X_j^2 u), \end{aligned}$$

and by Lemma 3.1:

$$Re(X_0 u, X_j^2 u) \leqslant C\|u\|_1\|X_j u\| \leqslant C\|u\|_1(Re(Pu, u))^{\frac{1}{2}}.$$

LEMMA 3.5. — *For every real vector field* Y:

$$\|[X_0, Y]u\| \leqslant C(\|u\|_1 + \|Pu\|_1)^{3/4}((Re(Pu, u))^{\frac{1}{2}} + \|Yu\|)^{1/4}.$$

Proof. — Let $Z = [X_0, Y]$.

$$\|Zu\|^2 = (X_0 Yu, Zu) - (YX_0 u, Zu).$$

We have

$$Re(X_0 Yu, Zu) \leqslant -Re(Yu, X_0 Zu) + C\|Yu\|\,\|u\|_1,$$

and

$$\begin{aligned} - Re(YX_0 u, Zu) &\leqslant C\|X_0 u\|\,\|u\|_1 + Re(X_0 u, ZYu) \\ &\leqslant C(\|X_0 u\| + \|Yu\|)\|u\|_1 - Re(X_0 Zu, Yu). \end{aligned}$$

Using Lemma 3.4, Lemma 3.5 is reduced to estimating $-\operatorname{Re}(X_0Zu, Yu)$ by the right-hand side of the claimed inequality. Now

$$-\operatorname{Re}(X_0Zu, Yu) = -\operatorname{Re}(PZu, Yu) - \sum_{j\geqslant 1} \operatorname{Re}(X_j^2Zu, Yu) + \operatorname{Re}(fZu, Yu),$$

and the third term is trivial; the first one is taken care of by Lemmas 3.2 and 3.3.

Also, by (5)

$$|\operatorname{Re}(X_j^2Zu, Yu)| \leqslant |\operatorname{Re}(X_jZu, X_jYu)| + C(\|u\|_1 + \|Pu\|_1)\|Yu\|,$$

and

$$|\operatorname{Re}(X_jZu, X_jYu)| \leqslant \|X_jZu\|\,\|X_jYu\| \leqslant (\operatorname{Re}(PZu, Zu))^{\frac{1}{2}}(\operatorname{Re}(PYu, Yu))^{\frac{1}{2}}.$$

Finally, by Lemma 3.2:

$$|\operatorname{Re}(PZu, Zu)| \leqslant C(\|u\|_1 + \|Pu\|_1)^2,$$

and by Lemmas 3.2 and 3.3:

$$|\operatorname{Re}(PYu, Yu)| \leqslant C(\|u\|_1 + \|Pu\|_1)((\operatorname{Re}(Pu, u))^{\frac{1}{2}} + \|Yu\|).$$

Proof of (1) *and* (2). — As $\|X_ju\|_{1-\delta} \leqslant \|X_ju\|_1^{1-\delta}\|X_ju\|^{\delta}$, (2) is a consequence of (6) and (4).

If $F \in \mathscr{F}_p$ $(p > 0)$, (4) and Lemmas 3.4, 3.3 and 3.5 show by induction that

$$\|Fu\| \leqslant C(\|u\|_1 + \|Pu\|_1)^{1-2^{-2p-1}}((\operatorname{Re}(Pu, u))^{\frac{1}{2}})^{2^{-2p-1}}.$$

Together with the hypothesis (H), this obviously implies (1), which completes this proof of L Hörmander's theorem.

Remark 1. — Using, in Lemma 3.4,

$$|(X_0u, Pu)| \leqslant C\|u\|_{\frac{1}{2}}\|Pu\|_{\frac{1}{2}} \leqslant C\|u\|^{\frac{1}{2}}\|u\|_1^{\frac{1}{2}}\|Pu\|^{\frac{1}{2}}\|Pu\|_1^{\frac{1}{2}},$$

and trivial modifications elsewhere, one may easily improve estimates (1) and (2) to

$$\|u\|_1 + \sum_{j\geqslant 1}\|X_ju\|_{1-\delta} \leqslant C\|u\|_1^{\frac{1-\delta}{2}}\|u\|^{\frac{\delta}{2}}(\|u\|_1 + \|Pu\|_1)^{\frac{1-\delta}{2}}(\|u\| + \|Pu\|)^{\frac{\delta}{2}},$$

for which the « gain » ε occuring in the proof of Theorem 2.2 is $\frac{\delta}{2}$ rather than $\frac{\delta^2}{2}$.

Remark 2. — λ being an integer $\geqslant 1$, the preceding proof applies equally well to the (principal type) operator on $\mathbf{R}^2$:

$$Q = \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial x_1} + i x_1^{2\lambda}\frac{\partial}{\partial x_2}\right)$$
$$= \left(\frac{\partial}{\partial x_1}\right)^2 + \left(x_1^\lambda \frac{\partial}{\partial x_2}\right)^2 + i\frac{\partial}{\partial x_2}\left(x_1^{2\lambda}\frac{\partial}{\partial x_2}\right) - i\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}.$$

With $X_1 = \frac{\partial}{\partial x_1}$ and $X_2 = x_1^\lambda \frac{\partial}{\partial x_2}$, so that

$$(adX_1)^\lambda X_2 = \lambda!\,\frac{\partial}{\partial x_2},$$

the starting point is the identity

$$-\operatorname{Re}(Qu, u) = \|X_1 u\|^2 + \|X_2 u\|^2 - \operatorname{Re}(\lambda i X_2 u, x_1^{\lambda-1}u),$$

which yields

$$\|X_1 u\| + \|X_2 u\| \leqslant C\|u\|^{\frac{1}{2}}(\|u\| + \|Qu\|)^{\frac{1}{2}}.$$

With $Z_1 = \frac{\partial}{\partial x_j}$, $Z_2 = \frac{\partial}{\partial x_k}$ $(j, k = 1, 2)$, so that Z_1 commutes with the bothering therm $i\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}$ of Q, Lemma 3.2 is proved to be valid for Q in a straightforward way, and so is Lemma 3.3, when in both lemmas $(\operatorname{Re}(Pu, u))^{\frac{1}{2}}$ is replaced by $\|u\|^{\frac{1}{2}}(\|u\| + \|Qu\|)^{\frac{1}{2}}$.

With $\delta = 2^{-\lambda}$, one then gets, by induction:

$$\|u\|_1 + \|X_1 u\|_{1-\delta} + \|X_2 u\|_{1-\delta} \leqslant C(\|u\|_1 + \|Qu\|_1)^{1-\delta}\|u\|^{\frac{\delta}{2}}(\|u\| + \|Qu\|)^{\frac{\delta}{2}}.$$

Finally, despite the presence of the term $-i\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}$ in Q, the proof of Theorem 2.2 applies in this case too provided we take only mollifiers $\varphi(x, \eta, y)$ of the form $\alpha(x_2, y_2)\hat{\psi}(\eta)$

(with $\psi \in \mathscr{D}(\mathbf{R}^2)$), so that $Op(\varphi_t)$ will commute with $\frac{\partial}{\partial x_1}$, and $\frac{\partial}{\partial x_2}$ will not occur among the Q_k's.

In this way, we prove only global hypoellipticity in strips $a < x_2 < b$, but hypoellipticity follows since Q is elliptic for $x_1 \neq 0$.

BIBLIOGRAPHY

[1] R. Beals and C. Fefferman, Spatially inhomogeneous pseudo-differential operators I, *Comm. Pure Appl. Math.*, 27 (1974), 1-24.

[2] K. O. Friedrichs, with the assistance of R. Vaillancourt, Pseudo-differential operators, *Lecture Notes*, N. Y. Univ., 1968.

[3] L. Hörmander, Linear partial differential operators, Springer Verlag, 1963.

[4] L. Hörmander, On the singularities of solutions of partial differential equations with constant coefficients, *Symp. on linear partial differential operators*, Jerusalem, June 1972.

[5] L. Hörmander, On the existence and regularity of solutions of linear pseudo-differential equations, *L'Enseignement Mathématique*, 17 (2) (1971), 99-163.

[6] L. Hörmander, Hypoelliptic second-order differential equations, *Acta Math.*, 119 (1967), 147-171.

[7] F. John, Continuous dependance on data for solutions of partial differential equations with a prescribed bound, *Comm. Pure Appl. Math.*, 13 (1960), 551-585.

[8] J. Kohn, Pseudo-differential operators and hypoellipticity, *Proc. Symp. Pure Math.*, 23 (1973), 61-69.

[9] H. Kumano-go, Algebras of pseudo-differential operators, *J. Fac. Sci. Univ. Tokyo*, 17 (1970), 31-50.

[10] A. Unterberger, Résolution d'équations aux dérivées partielles dans des espaces de distributions d'ordre de régularité variable, *Ann. Inst. Fourier*, 21 (1971), 85-128.

[11] A. Unterberger, Ouverts stablement convexes par rapport à un opérateur différentiel, *Ann. Inst. Fourier*, 22 (1972), 269-290.

[12] K. Watanabe, On the boundedness of pseudo-differential operators of type ρ, δ with $0 \leqslant \rho = \delta < 1$, *Tôhoku Math. J.*, 25 (1973), 339-345.

Manuscrit reçu le 5 avril 1975
Proposé par B. Malgrange.

A. et J. Unterberger,
Département de Mathématiques
U.E.R. des Sciences
Moulin de la Housse
51062 Reims Cedex.