Annales de l'institut Fourier

Nguyen Phuong Các

The Dirichlet problem for a singular elliptic equation

Annales de l'institut Fourier, tome 26, nº 1 (1976), p. 205-224
<http://www.numdam.org/item?id=AIF_1976__26_1_205_0>

Ann. Inst. Fourier, Grenoble
26, 1 (1976), 205-224

THE DIRICHLET PROBLEM FOR A SINGULAR ELLIPTIC EQUATION

by Nguyen Phuong CÁC

1. Introduction.

Let G be a bounded domain with smooth boundary ∂G. Suppose that ∂G consists of two connected parts S_1 and S_2, $\partial G = S_1 \cup S_2$, of which S_2 may be empty. In this paper we propose to study the solvability of the Dirichlet problem for the elliptic operator $\mathcal{L}$:

$$\mathcal{L}[u] = a_{ij}(x) \frac{\partial^2 u}{\partial x_1 \partial x_j} + \frac{b_i(x)}{\theta(x)} \frac{\partial u}{\partial x_i} - c(x) u = f(x) \text{ in } G \tag{1}$$

$$u|_{\partial G} = \varphi|_{\partial G} \tag{2}$$

with $\theta(x) = 0$ *when* $x \in S_1$. In (1) we have used the customary summation convention : if an index is repeated then summation over that index from 1 to n is to be understood, unless other limits of summation are expressly indicated. Singular elliptic operators have been studied extensively. Many authors, among them Morel [9], Baouendi [1], Kohn-Nirenberg [5], Murthy-Stampacchia [11], used Sobolev's spaces, mostly weighted ones. This method has the elegance of the Hilbert space approach although the mechanism for determining the necessary estimates and the existence of the trace of a function in a weighted Sobolev's space may be quite complicated to develop. On the other hand, Schechter [12], using the Schauder estimates and the maximum principle, proves a very interesting result on the solvability of the Dirichlet problem for an equation similar to (1), when the portion S_1 of the boundary ∂G is contained in the hyperplane $x_n = 0$. However, it seems to us that the method and result of

Schechter cannot be applied immediately to the problem (1), (2) for an arbitrary domain G ; *moreover we feel that in general it is quite difficult to ascertain whether a given singular elliptic operator actually satisfies the condition of Schechter for the solvability of the Dirichlet problem and this is the main reason for our carrying out the investigation in this paper.* Recently, Lo [7], also using Schauder's estimates, the maximum principle together with suitably constructed barriers, again obtains conditions for the solvability of the Dirichlet problem for a particular case of equation (1) : the portion S_1 of ∂G is contained in $x_n = 0$, $\theta(x) = x_n$ and the coefficients $b_i(x)$, $1 < i < n-1$, are of the form $x_n b_i^*(x)$ with smooth b_i^*. Although Lo's result is apparently weaker than that of Schechter as we shall see in Proposition 6 later on, the barrier method is more readily adaptable to the general equation (1) and general domain G. We would also like to point out that Jamet and Parter [4] have also used barriers to study singular equations similar to (1) with S_1 contained in $x_n = 0$, by the method of finite difference of numerical analysis. Finally we mention that the prototypes of equation (1) are studied by Brousse and Poncin [2] and Huber [3].

In Section 3, our Propositions 1 and 2 give conditions for the solvability of the Dirichlet problem (1), (2) whereas Proposition 3 gives conditions under which the Dirichlet problem is not solvable in general. This proposition when applied to equations studied in [7] seems to give a stronger result than that obtained there. Our conditions for solvability and non-solvability do not exactly fit together, this seems to point toward the shortcomings of the barrier method that we use (or rather the special barriers that we construct). We also give examples of equations to which our results can be applied.

Section 4 discusses the relationship between our results and those of Lo [7] and Schechter [12].

2. Notations and basic assumptions.

For an integer $m \geqslant 0$ we denote by $C_m(G)$ $[C_m(\bar{G})]$ the set of all real functions with derivatives up to and including order m continuous in $G[\bar{G}]$. For a non-negative integer m and $0 < \alpha < 1$, let $C_{m+\alpha}(\bar{G})$ be the set of all functions in $C_m(\bar{G})$ whose derivatives of order m satisfy a Hölder condition with α in $\bar{G}$. We put

$$\|u\|^{G}_{m+\alpha} = \max_{\overline{G}} \sum_{k=0}^{m} |D^k u| + \sum_{k=m} \max_{P,Q\in\overline{G}} \frac{|D^k u(P) - D^k u(Q)|}{|P-Q|^{\alpha}}$$

for functions $u \in C_{m+\alpha}(\overline{G})$. $C_{m+\alpha}(G)$ denotes the subspace of $C_m(\overline{G})$ consisting of all functions belonging to $C_{m+\alpha}(\overline{\omega})$ for each domain ω with $\overline{\omega} \in G$.

In the sequel c_1, c_2, etc... denote various constants. For the equation

$$\mathcal{L}[u] = a_{ij}(x)\frac{\partial^2 u}{\partial x_i\, \partial x_j} + \frac{b_i(x)}{\theta(x)}\frac{\partial u}{\partial x_i} - c(x)u = f(x) \text{ in } G \tag{1}$$

$$u|_{\partial G} = \varphi|_{\partial G} \tag{2}$$

we put

$$\mathfrak{M}[u] = a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{b_i(x)}{\theta(x)}\frac{\partial u}{\partial x_i}$$

and make the following assumptions concerning its coefficients and nonhomogeneous term throughout the paper.

i) $\theta(x) \in C_2(\overline{G})$; $\theta(x)|_{S_1} = 0$; $\theta(x) > 0$ in $\overline{G} - S_1$; $|\mathrm{grad}\ \theta| = |\Delta\theta| > 0$ in $\overline{G} - S_1$.

ii) $a_{ij}(x)$, $b_i(x)$, $c(x)$, $f(x)$ belong to $C_\alpha(\overline{G})$ for a certain $0 < \alpha < 1$ and $c(x) \geqslant 0$ in $\overline{G}$, $(1 \leqslant i, j \leqslant n)$.

iii) $a_{ij}(x)\,\zeta_i\,\zeta_j > \nu\,|\zeta|^2$ in G for a constant $\nu > 0$ and for any vector $\zeta = (\zeta_1, \ldots, \zeta_n) \in R^n$.

iv) $\varphi \in C_0(\overline{G})$.

Some of the above hypotheses are unnecessarily restrictive. For example, in Proposition 1 actually we only need to require that a_{ij}, b_i, c, f, belong to $C_\alpha(\overline{\omega})$ for every domain ω with $\overline{\omega} \subset \overline{G}$ and bounded away from S_1 and that $a_{ij}(x)\,\zeta_i\,\zeta_j > 0$ for $x \in \overline{G} - S_1$. But in order not to complicate the statement of hypotheses we prefer not to point out these variations which are not difficult to detect. The statements of some results obtained in the next section might have been simpler if we assume $|\Delta\theta| > 0$ in $\overline{G}$ instead of $|\Delta\theta| > 0$ in $\overline{G} - S_1$. However this assumption will exclude some interesting equations from our study.

3. Main results.

In contrast to the particular case where $\theta(x) = x_n$ and the portion S_1 of ∂G is contained in $x_n = 0$ studied by Lo [7], for equation (1) in general the condition for the solvability of the Dirichlet problem (1), (2) is much simpler if $\varphi(x)$ is constant on S_1. This case also has interesting application. Accordingly, we prove

PROPOSITION 1. – *Suppose that* $\varphi|_{S_1} = \varphi_1 =$ *constant and suppose also that there is a neighbourhood* U *of* S_1 *such that in* $G \cap U$ *the condition*

$$\text{(I)} \qquad (1-\beta)\, a_{ij} \frac{\partial\theta}{\partial x_i} \frac{\partial\theta}{\partial x_j} - \theta\,\mathfrak{M}[\theta] \geqslant c_1 \theta\,,$$

where β *and* c_1 *are positive constants with* $\beta < 1$, *is satisfied. Then the Dirichlet problem* (1), (2) *has a unique solution* $u \in C_{2+\alpha}(G) \cap C_0(\overline{G})$.

Proof. – The uniqueness of the solution is an immediate consequence of the maximum principle (cf., for example, [10]).

We first consider the case when $\varphi \in C_{2+\alpha}(\overline{G})$.

For each $n = 1, 2, \ldots$, let G_n be

$$G_n = \{x \mid x \in G\,, d(x\,, S_1) > n^{-1}\}$$

Let $u_n(x)$ be the unique solution (cf. [6,8]) in $C_{2+\alpha}(\overline{G}_n)$ of the Dirichlet problem

$$\begin{cases} \mathfrak{L}[u] = f \text{ . in } G_n \\ u|_{\partial G_n} = \varphi|_{\partial G_n} \end{cases}$$

(It may be necessary to smooth off small portions of the part of the boundary of G_n contained in G to insure existence of u_n).

We first show that the sequence $\{u_n(x)\}_{n=1}^{\infty}$ *is uniformly bounded.* (In Lo [7], proof of Theorem 2, it is just stated without justification that the sequence $u_n(x)$ is uniformly bounded. However it seems to us that this uniform boundedness is an immediate consequence of the maximum principle only if $f \equiv 0$ in G ; otherwise it is not obvious because $\theta^{-1}(x)$ is not bounded in G). If we can construct a function $w(x) \in C_0(\bar{G})$ such that $\mathcal{L}[w] \leqslant -1$ in G then the boundedness follows. In fact, choose $k > 0$ sufficiently large such that

$$k\,\mathcal{L}[w] \leqslant -|f(x)|$$

in G and let

$$\mu = \max_{\bar{G}} |\varphi(x)| + k \max_{\bar{G}} |w(x)|$$

Then

$$\mathcal{L}[u_n + kw + \mu] \leqslant 0 \quad \text{in} \quad G_n$$

$$u_n + kw + \mu|_{\partial G_n} \geqslant 0$$

since $\mu \geqslant 0$ and $c(x) \geqslant 0$. By the maximum principle

$$u_n + kw + \mu \geqslant 0 \quad \text{in} \quad G_n$$

i.e.,

$$u_n \geqslant -kw - \mu \qquad \text{in} \quad G_n$$

Similarly,

$$\mathcal{L}[u_n - kw - \mu] \geqslant 0 \quad \text{in} \quad G_n$$

$$u_n - kw - \mu|_{\partial G_n} \leqslant 0$$

and hence

$$u_n \leqslant kw + \mu \quad \text{in} \quad G_n$$

Construction of $w(x) \in C_0(\bar{G})$ *such that* $\mathcal{L}[w] \leqslant -1$ *in* G.

Consider the function $v(x) = \exp\{-\lambda\theta^\beta(x)\}$ where β is the constant in condition (I) of the hypotheses and λ is a positive constant to be determined later. By elementary calculation we obtain

$$\mathcal{L}[v] = \lambda\beta\theta^{\beta-2}\left\{[\lambda\beta\theta^\beta - (\beta-1)]\,a_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j} - \theta\mathfrak{M}[\theta]\right\}$$

$$\exp\{-\lambda\theta^\beta\} - c\exp\{-\lambda\theta^\beta\}$$

By condition (I) of the hypotheses, since $\lambda\beta\theta^\beta(x) \geqslant 0$,

$$\mathcal{L}[v] \geqslant [\lambda\beta c_1 \theta^{\beta-1}(x) - c(x)] \exp\{-\lambda\theta^\beta(x)\} \text{ in } G \cap U.$$

Since $\theta(x)$ is continuous in $\overline{G}$, $\theta|_{S_1} = 0$, $0 < \beta < 1$ and $c(x)$ is bounded in G, there is a neighbourhood U_1 of S_1 and a constant $c_2 > 0$ such that $\mathcal{L}[v] \geqslant c_2$ in $G \cap U_1$. Now let $V = \overline{G} - U_1$ and

$$M_1 = \max_{x\in V} |(1-\beta)a_{ij}(x)\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j} - \theta\mathfrak{M}[\theta]| < \infty$$

$$M_2 = \max_{x\in V} \theta(x) < \infty$$

$$m = \min_{x\in V} \theta(x) > 0$$

Then

$$\mathcal{L}[v] \geqslant \{\lambda\beta M_2^{\beta-1}[\lambda\beta m^\beta \nu |\Delta\theta|^2 - M_1] - \max_{\overline{G}} c(x)\} \exp(-\lambda M_2^\beta)$$

Since by hypothesis $\min_V |\nabla\theta(x)| > 0$, by choosing λ sufficiently large we obtain

$$\mathcal{L}[v] \geqslant c_3 > 0 \text{ in } V.$$

Thus

$$\mathcal{L}[v] \geqslant \min(c_2, c_3) > 0 \text{ in } G$$

It then suffices to take $w(x) = c_4 \exp\{-\lambda\theta^\beta(x)\}$ with $c_4 < 0$ sufficiently small to obtain $\mathcal{L}[w] \leqslant -1$ in G.

Proof that there is a solution $u \in C_{2+\alpha}(G \cup S_2) \cap C_0(\overline{G})$ of the Dirichlet problem (1), (2).

By the well known Schauder estimates up to the boundary [6,8]

$$\|u_n\|_{2+\alpha}^{G_n} \leqslant K(n)[\|f\|_\alpha + \|\varphi\|_{2+\alpha}]$$

where $K(n)$ is a constant depending on the ellipticity constant ν of $\mathcal{L}$ as well as the $\|.\|_\alpha$-norms of its coefficients in G_n. From these estimates and the diagonal process we can extract a subsequence of $\{u_n\}$ which for convenience we still denote by $\{u_n\}$ which converges in $C_2(\overline{G}_n)$ for each n to a function $u(x)$. This function $u(x)$ belongs to each $C_{2+\alpha}(\overline{G}_n)$ and $u(x)|_{S_2} = \varphi(x)|_{S_2}$. It remains to show that if φ_1 is the constant value of $\varphi(x)$ on S_1 then for every point $x^0 \in S_1$, $u(x) \to \varphi_1$ as $x \to x^0$. It suffices to show that there is a neighbourhood of S_1 and a function $\omega(x)$ continuous in $\overline{G}$, belonging to $C_2(G)$ such that $\omega(x)|_{S_1} = 0$ and $\omega(x) > 0$, $\mathcal{L}[\omega] < -1$ in the intersection of this neighbourhood and G. In fact, suppose that such a function ω exists. For every $\epsilon > 0$, there is a neighbourhood of S_1 in which

$$\varphi_1 - \epsilon \leqslant \varphi(x) \leqslant \varphi_1 + \epsilon$$

Thus there would exist a neighbourhood $U_2(\epsilon)$ of S_1 such that $\omega(x)|_{S_1} = 0$ and $\omega(x) > 0$, $\mathcal{L}[\omega] < -1$ in $(\overline{U}_2 \cap \overline{G}) - S_1$ and

$$\varphi_1 - \epsilon \leqslant \varphi(x) \leqslant \varphi_1 + \epsilon \quad \text{in} \quad \overline{U}_2 \cup \overline{G}$$

Let $S_{1,n} = \partial G_n - S_2$. If n is sufficiently large then $S_{1,n}$ will be contained in this neighbourhood U_2. Choose k_1 positive, independent of n such that

$$\mathcal{L}[k_1 \omega(x) + \varphi_1 + \epsilon - u_n(x)] \leqslant 0 \quad \text{in} \quad G_n \cap U_2$$

$$k_1 \omega(x) + \varphi_1 + \epsilon - u_n(x)|_{\partial(G_n \cap U_2)} \geqslant 0$$

It is possible to find such a k_1 because $f(x)$ is bounded in $\overline{G}$ and on the portion of ∂U_2 contained in G, min $\omega(x) > 0$, and the $|u_n(x)|$ are uniformly bounded in G ; and on the remaining portion of $\partial(G_n \cap U_2)$, $u_n(x) = \varphi(x) \leqslant \varphi_1 + \epsilon$ by construction of the u_n's. Thus by the maximum principle

$$k_1 \omega(x) + \epsilon - u_n(x) \geqslant 0$$

i.e., $$u_n(x) \leqslant k_1 \omega(x) + \varphi_1 + \epsilon \quad \text{in} \quad G_n \cap U_2$$

Similarly, by considering $-k_2 \omega(x) + \varphi_1 - \epsilon - u_n(x)$ for a suitable $k_2 > 0$ we deduce

$$-k_2 \omega(x) + \varphi_1 - \epsilon \leqslant u_n(x) \quad \text{in} \quad G_n \cap U_2$$

Thus for all n sufficiently large so that $S_{1,n}$ is contained in $U_2(\epsilon)$ we have

$$-k_2 \omega(x) + \varphi_1 - \epsilon \leqslant u_n(x) \leqslant k_1 \omega(x) + \varphi_1 + \epsilon \quad \text{in} \quad G_n \cap U_2$$

Let $n \to \infty$ and then $x \to x_0 \in S_1$ we obtain

$$\varphi_1 - \epsilon \leqslant u(x_0) \leqslant \varphi_1 + \epsilon$$

Since $\epsilon > 0$ is arbitrary, we deduce $u(x_0) = \varphi_1$ for all $x_0 \in S_1$.

Construction of the function $\omega(x)$.

Let $g(x) = \theta^{\beta}(x)$ where $0 < \beta < 1$ is the constant in condition (I) of the hypotheses. Elementary calculation gives

$$\mathcal{L}[g] = \beta\theta^{\beta-2} \left\{ (\beta - 1)\, a_{ij}(x) \frac{\partial\theta}{\partial x_i} \frac{\partial\theta}{\partial x_j} - \theta\mathfrak{M}[\theta] \right\} = c(x)\,\theta^{\beta}$$

$$\leqslant - c_1 \beta\theta^{\beta-1}(x).$$

Since $\theta(x)$ is continuous in $\overline{G}$, $\theta(x)|_{S_1} = 0$, $\theta(x) > 0$ in $\overline{G} - S_1$ and $0 < \beta < 1$ we see that $\mathcal{L}[g] \leqslant - c_5 < 0$ in a small neighbourhood of S_1. It then suffices to take $\omega(x) = k_3 g(x)$ where k_3 is a suitably large positive constant.

Thus the Dirichlet problem (1), (2) has been solved in the case $\varphi \in C_{2+\alpha}(\overline{G})$. *We now consider the case* $\varphi \in C_0(\overline{G})$.

We can find a sequence $\{\varphi_l\}_{l=1}^{\infty}$ of functions of $C_{2+\alpha}(\overline{G})$ converging in $C_0(\overline{G})$ to φ. Let $u_l \in C_{2+\alpha}(G \cup S_2) \cap C(\overline{G})$ be the solution of the Dirichlet problem

$$\mathcal{L}[u_l] = f \text{ in } G$$

$$u_l|_{\partial G} = \varphi_l|_{\partial G}$$

By the maximum principle,

$$\max_{\overline{G}} |u_l(x) - u_k(x)| \leqslant \max_{\partial G} |\varphi_l - \varphi_k| \to 0$$

Thus $\{u_l(x)\}_{l=1}^{\infty}$ converges uniformly in $C_0(\overline{G})$ to a function $u \in C_0(\overline{G})$ and

$$u(x)|_{\partial G} = \varphi(x)|_{\partial G}$$

Moreover, in each domain $\mathcal{D}$ with $\overline{\mathcal{D}} \subset G$ we have the Schauder interior estimates (cf., e.g., [6]).

$$\|u_l\|_{2+\alpha}^{\mathcal{D}} \leqslant K(\mathcal{D})\, [\|f\|_{\alpha} + \|u_l\|_0]$$

where $K(\mathcal{D})$ is a constant depending on the ellipticity ν of $\mathcal{L}$, the $\|\cdot\|_{\alpha}$-norms of its coefficients in $\mathcal{D}$ and the distance from $\mathcal{D}$ to ∂G

only. Since $\max_{\overline{G}} |u_l(x) - u_k(x)| \to 0$ we deduce $||u_l - u_k||^{\varpi}_{2+\alpha} \to 0$. Therefore $u(x) \in C_{2+\alpha}(\overline{\varpi})$, i.e., $u(x) \in C_{2+\alpha}(G)$ and it is obvious that $\mathcal{L}[u] = f$ in G. Thus we have shown that the Dirichlet problem (1), (2) has a solution $u \in C_{2+\alpha}(G) \cap C_0(\overline{G})$. Q.E.D.

Comments on the proof of theorem 1. – Besides our proof of the uniform boundedness of the sequence $\{u_n(x)\}_{n=1}^{\infty}$ that we have mentioned earlier, our "passing to the limit" for the sequence $\{u_n\}$ is different from that of [7], Theorem 2. It follows Schechter [12] instead. In [7] it is done immediately for $\varphi \in C_0(\overline{G})$. But it seems to us that the estimate (inequality (9), page 340 in [7])

$$||u||_{2+\alpha}^{G_\eta} < \widetilde{K}\,(||f||_{\alpha}^{G_\eta} + ||u||_{0}^{G_\eta}$$

is not known for a domain G_η in such generality, and only known if $||u||_{2+\alpha}^{G_\eta}$ is replaced by $||u||^{\varpi}_{2+\alpha}$ where ϖ is an interior subdomain of G. If this is the case then it is not immediately clear that the limit function $u(x)$ of the sequence $\{u_\eta\}$ in [7] belong to $C_0(G \cup S_2)$.

Example. – Proposition 1 seems to be particularly suitable for the following class of Dirichlet problem. Suppose without loss of generality that the boundary ∂G contains the origin 0. Consider, where $\rho = OM$, the operator

$$\mathcal{L}[u] = a_{ij}(x)\,\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \frac{b_i(x)}{\rho^2}\,\frac{\partial u}{\partial x_i} - c(x)u \tag{3}$$

Then $S_1 = \{0\}$ and $\varphi(x)$ is certainly constant on S_1. In particular, the following operator has been considered by Morel [9] :

$$\mathcal{L}[u] = \Delta u + \frac{\delta}{\rho}\frac{\partial u}{\partial \rho} = \Delta u + \frac{\delta}{\rho^2}\,x_i\,\frac{\partial u}{\partial x_i} \tag{4}$$

where δ is a constant. For it, condition (I) in Proposition 1 becomes

$$2\rho^2\,[2(1-\beta) - n - \delta] \geqslant c_1\rho^2$$

$$2(1-\beta) - n - \delta \geqslant c_1 > 0$$

for some $0 < \beta < 1$ and $c_1 > 0$. This is satisfied if

$$\delta < 2 - n \tag{5}$$

We thus reobtain a condition of Morel ([8], page 395).

If the boundary function $\varphi(x)$ in the Dirichlet problem (1), (2) is not constant on S_1 then we need an additional condition to those of Proposition 1 to guarantee the solvability of the Dirichlet problem.

DEFINITION – *Let* $P \in S_1$. *A function* $w(M)$ *is called a local barrier for the operator* $\mathfrak{L}$ *at the point* P *if there exists a small neighbourhood* N *of* P *such that* $w(M)$ *is twice continuously differentiable in* $G \cap N$, *continuous in* $\overline{G} \cap N$ *and*

a) $w(M) > 0$ *in* $(\overline{G} \cap N) - \{P\}$,

b) $w(P) = 0$,

c) $\mathfrak{L}[w] \leqslant -1$ *in* $G \cap N$.

PROPOSITION 2. – *The Dirichlet problem* (1), (2) *has a unique solution* $u \in C_{2+\alpha}(G) \cap C_0(\overline{G})$ *if*

(I) *There is a neighbourhood* U *of* S_1 *such that in* $G \cap U$ *the condition*

$$(1-\beta)\, a_{ij}(x) \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} - \theta\, \mathfrak{M}[\theta] \geqslant c_1 \theta,$$

where β *and* c_1 *are positive constants with* $\beta < 1$, *is satisfied.*

(II) *For every point* $P \in S_1$ *there exists a neighbourhood* N(P) *and a unit vector* $\mu(P)$ *originating from* P *such that if* M *is any point in* $N \cap G$ *and* Q *is the orthogonal projection of* M *on* $\mu(P)$ *then* $\mathbf{HM} \cdot \mathbf{b}(M) \leqslant \widetilde{k}\theta(M)$ *where* $\widetilde{k}$ *is a constant independent of* M *and* $\mathbf{b}(M)$ *is the vector* $(b_1(M), \ldots, b_n(M))$. *It is also required that the support line of* $\mu(P)$ *intersects* ∂G *at a finite number of points in any neighbourhood of* P.

Example. – Although condition (II) above is cumbersome, it is simple to check that it is verified in interesting cases. For example, let G be the unit sphere in R^n. Consider the operator

$$\mathfrak{L}[u] = a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{x_i}{\theta(x)} \frac{\partial u}{\partial x_i} - c(x)u$$

where $\theta(x)$ is any function satisfying i) in Section II. It satisfies condition (II) of Proposition 2 : it suffices to take as $\mu(P)$ the unit vector of the direction PO.

Proof of proposition 2. – We use the same notation as in the proof of Proposition 1 : For each $n = 1,2, \dots$ let u_n be the solution of the Dirichlet problem

$$\mathcal{L}[u] = f(x) \text{ in } G_n$$

$$u|_{\partial G_n} = \varphi|_{\partial G_n}$$

The passing to the limit of the sequence $\{u_n\}$ is first carried out for $\varphi \in C_{2+\alpha}(\bar{G})$ to obtain $u \in C_{2+\alpha}(G \cup S_2)$. Again, condition (I) of the hypothesis guarantees that $\{u_n(x)\}$ is uniformly bounded in G. This, together with the existence of a local barrier at every point $P \in S_1$ are used to show that $u|_{S_1} = \varphi|_{S_1}$. Here we only show that conditions (I) and (II) of the hypothesis enable one to construct a local barrier at every point $P \in S_1$. For other details we refer the reader to Lo [7].

Let $\mu(P) = (\mu_1, \mu_2, \dots, \mu_n)$ and let

$$e(x) = \theta^\beta(x) + \sum_{i=1}^n x_i^2 - \Big(\sum_{i=1}^n \mu_i x_i\Big)^2 \tag{6}$$

By elementary calculation

$$\begin{aligned}\mathcal{L}[e(x)] = {} & 2\left[\sum_{i=1}^n a_{ii}(x) - \sum_{i,j=1}^n a_{ij}\mu_i\mu_j\right] \\ & + \beta\theta^{\beta-2}\left[(\beta-1)\sum_{i,j=1}^n a_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j} + \theta\,\mathfrak{M}[\theta]\right] \\ & + \frac{2}{\theta(x)}\sum_{i=1}^n b_i\left[x_i - \mu_i\Big(\sum_{j=1}^n \mu_j x_j\Big)\right] - c(x)\,e(x)\end{aligned}$$

Thus there is a neighbourhood N of P such that in $N \cap G$,

$$\mathcal{L}[e(x)] \leqslant 2\left[\sum_i a_{ii}(x) - \sum_{i,j} a_{ij}\mu_i\mu_j\right] - c_1\beta\theta^{\beta-1} + 2\tilde{k},$$

by conditions (I) and (II) and the fact that $c(x) \geqslant 0$, $e(x) > 0$ in G. Since the $a_{ij}(x)$, $1 \leqslant i,j \leqslant n$ are bounded in G, $|\mu| = 1$, $c_1 > 0$, $0 < \beta < 1$ and $\theta(P) = 0$ so there exists a small neighbourhood of P and a constant c_6 such that in that neighbourhood

$$\mathcal{L}[e(x)] \leqslant c_6 < 0$$

It then suffices to take as a local barrier at $P \in S_1$, $w(x) = c_7 e(x)$ where $c_7 > 0$ is a sufficiently large constant.

Q.E.D.

Proposition 3 below when applied to the special case of equation (1) considered in [7] gives what seems to us a stronger result than that obtained in [7].

For Proposition 3, instead of assuming that $a_{ij}, b_i, c, f \in C_\alpha(\overline{G})$ we assume that $a_{ij}(x), b_i(x)$ and $c(x)$ are continuous and $c(x) \geqslant 0$ in $\overline{G}$. Otherwise the general assumptions made in Section II remain unchanged.

PROPOSITION 3. – *There is at most one solution of the equation*

$$\mathcal{L}[u] = f$$

belonging to $C_2(G) \cap C(\overline{G})$ *and assuming given values on* S_2 *if there exists a neighbourhood of* S_1 *in which the condition*

$$\text{(III)} \qquad (1 - c_7 \theta^\gamma)\, a_{ij}(x) \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} \leqslant \theta \mathfrak{M}[\theta]$$

is satisfied for positive constants c_7 *and* γ.

Before proving this proposition, perhaps it is worthwhile to point out that condition (III) above and condition (I) in Propositions 1 and 2 cannot be satisfied simultaneously as can be seen without difficultly. Thus, as it should be, Proposition 3 on one hand and Propositions 1 and 2 on the other are mutually exclusive.

Proof of Proposition 3. – By using the maximum principle on the homogeneous equation (for the detail we refer the reader to the proof of Theorem 1 in Lo [7]), it suffices to show that there is a barrier function $w(x)$ which is

a) Twice continuously differentiable and nonnegative in $\overline{G} - S_1$;

b) For every given number $A > 0$, there exists a neighbourhood N of S_1 such that $w(x) > A$ in $G \cap N$;

c) $\mathcal{L}[w(x)] \leqslant 0$ in G.

Our construction of this barrier function is significantly different from that of Lo in [7]. Let

$$E = 2 \max_{\bar{G}} \theta(x)$$

and consider the function

$$w(x) = \log \left(\log \frac{E}{\theta(x)}\right) - \exp\{\lambda\theta^2(x)\} + k \tag{7}$$

where λ, k are positive constants to be determined later. By elementary calculation,

$$\mathcal{L}[w] = \frac{\left(a_{ij} \dfrac{\partial\theta}{\partial x_i} \dfrac{\partial\theta}{\partial x_j} - \theta\mathfrak{M}[\theta]\right) \log (E/\theta) - a_{ij} \dfrac{\partial\theta}{\partial x_i} \dfrac{\partial\theta}{\partial x_j}}{\theta^2 \log^2 (E/\theta)}$$

$$- 2\lambda \left\{ (2\lambda\theta^2 + 1)\, a_{ij} \frac{\partial\theta}{\partial x_i} \frac{\partial\theta}{\partial x_j} + \theta\mathfrak{M}[\theta] \right\} \exp\{\lambda\theta^2\} - cw \tag{8}$$

By hypothesis (III) of the Proposition, in a neighbourhood of S_1

$$a_{ij} \frac{\partial\theta}{\partial x_i} \frac{\partial\theta}{\partial x_j} + \theta\mathfrak{M}[\theta] \geqslant (2 - c_7\theta^\gamma)\, a_{ij} \frac{\partial\theta}{\partial x_i} \frac{\partial\theta}{\partial x_j} \ ;$$

since $\theta|_{S_1} = 0$, there is a neighbourhood of S_1 in which $2 - c_7\theta^\gamma > 0$. Then there is a neighbourhood of S_1 in which

$$\mathcal{L}[w] = \frac{\{c_7\theta^\gamma \log (E/\theta) - 1\}\, a_{ij} \dfrac{\partial\theta}{\partial x_i} \dfrac{\partial\theta}{\partial x_j}}{\theta^2 \log^2 (E/\theta)} - c(x)\, w(x)$$

Since $\theta^\gamma \log(E/\theta) \to 0$ as $\theta \to 0$ because $\gamma > 0$, there is a neighbourhood N' of S_1 in which

$$\mathcal{L}[w] \leqslant - c(x)\, w(x)$$

Now let

$$M_3 = \max_{\bar{G}-N'} \left| \frac{\left\{ a_{ij} \dfrac{\partial\theta}{\partial x_i} \dfrac{\partial\theta}{\partial x_j} - \theta\mathfrak{M}[\theta] \right\} \log (E/\theta) - a_{ij} \dfrac{\partial\theta}{\partial x_i} \dfrac{\partial\theta}{\partial x_j}}{\theta^2 \log^2 (E/\theta)} \right| < \infty$$

$$m_1 = \min_{\bar{G}-N'} \theta^2\, a_{ij} \frac{\partial\theta}{\partial x_i} \frac{\partial\theta}{\partial x_j} > 0$$

(because of condition i), Section II, on $\theta(x)$),

$$M_4 = \max_{\bar{G}-N'} |\theta \mathfrak{M}[\theta]| < \infty$$

We first choose λ such that (cf. equation (8) above)

$$2\lambda m_1 - M_4 > 1, \qquad \text{i.e.,} \qquad \lambda > (M_4 + 1)/2m_1$$

Since $e^{\lambda\theta^2} \geqslant 1$ in $\bar{G}$, we have

$$\mathfrak{L}[w] \leqslant M_3\, 2\lambda - c(x)\, w(x) \quad \text{in } \bar{G} - N'$$

So if we choose

$$\lambda > \max\{(M_3/2), (M_4 + 1)/2m_1\}$$

then

$$\mathfrak{L}[w] \leqslant - c(x)\, w(x) \quad \text{in } \bar{G} - N'$$

Once λ has been chosen we choose k such that $w(x) > 0$ in G, this is certainly possible because (cf. expression (7) above)

$$\log (E/\theta) \geqslant \log 2$$

The function $w(x)$ with λ, k so chosen satisfies all the three requirements (a), (b) and (c) above.

Q.E.D.

4. A special case.

In this section we will show how our results can be applied to the equations studied in [7]. Proposition 6 deals with the relationship between the result of Schechter [12] and that of Lo [7].

Throughout this section, let G be a bounded domain with smooth boundary, G is contained in the strip $0 < x_n < t_0$ and a portion S_1 of the boundary ∂G is contained in the hyperplane $x_n = 0$. Lo [7] considers the following Dirichlet problem.

$$[u] = a_{ij}(x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n-1} b_i^*(x) \frac{\partial u}{\partial x_i} + \frac{h(x)}{x_n} \frac{\partial u}{\partial x_n} - c(x)\, u = f(x) \text{ in G} \quad (9)$$

$$u|_{\partial G} = \varphi|_{\partial G} \tag{10}$$

with the normalization $a_{nn}(x) \equiv 1$.

This equation is a special case of equation (1) with

$$b_i(x) = x_n b_i^*(x) \quad (1 \leqslant i \leqslant n-1) \ , \ b_n(x) = h(x) \ , \ \theta(x) = x_n .$$

Thus

$$a_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} = 1, \quad \theta \mathfrak{M}[\theta] = h(x)$$

and condition (I) in Proposition 2 becomes

(I′). For small $x_n > 0$,

$$(1 - \beta) - h(x) \geqslant c_1 x_n$$

where β and c_1 are positive constants with $\beta < 1$. If $h(x)$ is continuous, as assumed in [7], Theorem 2, this condition is satisfied if

$$1 - h(x_1, \ldots, x_{n-1}, 0) > 0 \quad \text{for} \quad (x_1, \ldots, x_{n-1}, 0) \in S_1$$

Condition (II) in Proposition 2 is automatically satisfied with, for any $P \in S_1$, $\mu(P)$ being the unit vector originating at P and parallel to the x_n-axis. Our barrier function (6) is then reduced to that constructed by Lo [7]. For reference we list the following result :

PROPOSITION 4 [Lo]. – *Suppose that* $a_{ij}(x)$, $b_i^*(x)$, $h(x)$, $c(x)$ *belong to* $C_\alpha(\overline{G})$ *and* $c(x) \geqslant 0$ *in* $\overline{G}$ *and* $\varphi \in C_0(\overline{G})$. *Then there exists a unique solution* $u \in C_{2+\alpha}(G) \cap C_0(\overline{G})$ *of the Dirichlet problem* (9), (10) *if*

$$h(x_1, \ldots, x_{n-1}, 0) < 1$$

for all points $(x_1, \ldots, x_{n-1}, 0) \in S_1$.

From Proposition 3 we deduce the following result for equation (9) which is stronger than a similar result in [7] (Theorem 1).

PROPOSITION 5. – *Suppose that* $a_{ij}(x)$, $b_i^*(x)$, $h(x)$ *and* $c(x)$ *are continuous in* $\overline{G}$ *and* $c(x) \geqslant 0$ *in* $\overline{G}$. *Suppose also that* $h(x)$ *is defined in* $B \times [0, t_0]$ *where* B *is the projection of* $\overline{G}$ *on the plane* $x_n = 0$ *and is lipschitzian with the same Lipschitz constant at each point of* B. *Then* *equation* (9) *has at most one solution* $u \in C_2(G) \cap C_0(\overline{G})$ *taking given values on* S_2 *if* $h(x_1, \ldots, x_{n-1}, 0) \geqslant 1$ *for all* $(x_1, \ldots, x_{n-1}, 0) \in B$.

Proof. – It remains to verify that condition (III) in Proposition 3 is satisfied, i.e., that for x_n small we have

$$1 - h(x) \leqslant c_7 x_n \quad (x \in \overline{G}) \tag{11}$$

for a constant c_7. But for $x \in \overline{G}$, $(x_1, \ldots, x_{n-1}, 0) \in B$ and

$$1 - h(x) = 1 - h(x_1, \ldots, x_{n-1}, 0) + h(x_1, \ldots, x_{n-1}, 0) - h(x)$$
$$\leqslant h(x_1, \ldots, x_{n-1}, 0) - h(x)$$

Because $h(x)$ is uniformly lipschitzian at points of B, there exists a constant c_7 such that for all $x \in G$

$$|h(x) - h(x_1, \ldots, x_{n-1}, 0| < c_7 x_n$$

and (11) follows.

Note. – For the validity of Proposition 5, Theorem 1 of [7] requires that $h(x)$ *be in* $C_2(\overline{G})$, *even in* x_n and $h(x_1, \ldots, x_{n-1}, 0) > 1$ which is more restrictive than our condition. We also wish to point out that a careful reading of the proof of Theorem 1 of [7] seems to indicate that there also, $h(x)$ needs to be defined not only in $\overline{G}$ but in $B \times [0, t_0]$ instead, because $x \in \overline{G}$ does not necessarily imply $(x_1, \ldots, x_{n-1}, 0) \in S_1$.

We now prove that Theorem 2 of [7] which is listed as Proposition 4 above can be deduced from a result of M. Schechter. In [12], Schechter proves that the boundary value problem (with the normalization $a_{nn}(x) \equiv 1$)

$$\mathcal{L}[u] = a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + a_i(x) \frac{\partial u}{\partial x_i} - c(x) u = f \text{ in } G \tag{12}$$

$$u|_{\partial G} = \varphi|_{\partial G} \tag{13}$$

has a unique solution $u \in C_{2+\alpha}(G) \cap C_0(\overline{G})$ if $\varphi \in C(\overline{G})$ and if in any subset $\overline{G}'$ of $\overline{G}$ bounded away from S_1 the operator $\mathcal{L}$ is uniformly elliptic and a_{ij}, a_i, c, f belong to $C_\alpha(\overline{G}')$, $c(x) \geqslant 0$ in $\overline{G}$ and if the following crucial condition is satisfied.

(S) *There are functions* $q(t) \geqslant 0$, $p(t)$ *continuous in* $0 < t \leqslant t_0$ *such that*

$$|a_{ij}|\ ,\ |a_i|\ ,\ |c|\ ,\ |f| \leqslant q(x_n) \tag{14}$$

$$a_n(x) \leqslant p(x_n) \tag{15}$$

and the integral

$$\mathrm{R}(p\ ,\ q) = \int_0^{t_0} \exp\{\mathrm{P}(t)\} \int_t^{t_0} \exp\{-\mathrm{P}(s)\}\, q(s)\, ds dt \tag{16}$$

exists where

$$\mathrm{P}(t) = \int_t^{t_0} p(s)\ ds$$

We prove

PROPOSITION 6. – *If the coefficients of the partial differential equation* (9) *satisfy all conditions of Proposition 4 then they satisfy conditions* (S) *of M. Schechter as applied to equation* (9).

Proof. – For the function $q(t)$ in condition (S) of Schechter we take (cf. equation (9))

$$q(t) \equiv \max_{\overline{\mathrm{G}}}\{\,|a_{ij}(x)|\ ,\ (1 \leqslant i\ ,j \leqslant n)\ ;$$
$$|b_i^*(x)|\ ,\ (1 \leqslant i \leqslant n-1)\ ,\ |c(x)|\ ,\ |f(x)|\}$$

We construct a function $p(t)$ as follows : Since $h(x_1, \ldots, x_{n-1}, 0) < 1$ and $h(x)$ is continuous, there is $\eta > 0$ such that if we denote

$$\mathrm{T}_{2\eta} = \{x \,|\, x \in \overline{\mathrm{G}}\ ,\ 0 \leqslant x_n < 2\eta\}$$

then we have

$$\mathrm{M}_5 = \max_{x \in \overline{\mathrm{T}}_{2\eta}} h(x) < 1$$

Let d be the diameter of the set $\overline{\mathrm{G}}$ and $\tau = d/\eta$. Then for all

$$x \in \overline{\mathrm{G}}_{2\eta} = \overline{\mathrm{G}} - \mathrm{T}_{2\eta}$$

we have

$$h(x) \leqslant M_5 + \|h\|_\alpha^{\bar{G}_{2\eta}} \,.\, d^\alpha = M_5 + \|h\|_\alpha^{\bar{G}_{2\eta}} \,.\, \tau^\alpha \eta^\alpha$$
$$h(x) \leqslant M_5 + \|h\|_\alpha^{\bar{G}_{2\eta}} \,.\, \tau^\alpha (x_n - \eta)^\alpha$$

because if $x \in \bar{G}_{2\eta}$ then $x_n \geqslant 2\eta$.

We define a function $\pi(t)$ as follows

$$\pi(t) = M_5 \quad \text{for } 0 \leqslant t \leqslant \eta$$
$$\pi(t) = M_5 + \|h\|_\alpha^{\bar{G}_{2\eta}} \,.\, \tau^\alpha (t - \eta)^\alpha \quad \text{for } \eta \leqslant t \leqslant t_0$$

Then $\pi(t)$ is continuous in $[0\,,\, t_0]$ and $h(x) \leqslant \pi(x_n)$ for all $x \in \bar{G}$. It remains to show that condition (16) of Schechter is satisfied with $q(t) \equiv$ constant and $p(t) = \dfrac{\pi(t)}{t}$, i.e., to verify that

$$I = \int_0^{t_0} \exp\{P(t)\} \int_t^{t_0} \exp\{-P(s)\}\, ds dt < \infty$$

where

$$P(t) = \int_t^{t_0} P(s)\, ds = \int_t^{t_0} \frac{\pi(s)}{s}\, ds$$

But

$$I = \int_0^{\eta} \exp\{P(t)\} \int_t^{\eta} \exp\{-P(s)\}\, ds\, dt$$
$$+ \Big(\int_0^{\eta} \exp\{P(t)\}\, dt\Big)\Big(\int_\eta^{t_0} \exp\{-P(s)\}\, ds\Big)$$
$$+ \int_\eta^{t_0} \exp\{P(t)\} \int_t^{t_0} \exp\{-P(s)\}\, ds dt$$

Now it is obvious that

$$\int_\eta^{t_0} \exp\{P(t)\} \int_t^{t_0} \exp\{-P(s)\}\, ds dt\,,\ \int_\eta^{t_0} \exp\{-P(s)\}\, ds < \infty$$

Because of the fact that for $0 < t \leqslant \eta$, $p(t) = \dfrac{M_5}{t} < \dfrac{1}{t}$, it can be verified by elementary calculation that

$$\int_0^{\eta} \exp\{P(t)\} \int_t^{\eta} \exp\{-P(s)\}\, ds dt\,, \int_0^{\eta} \exp\{P(t)\} dt < \infty$$

Q.E.D.

BIBLIOGRAPHY

[1] M.S. BAOUENDI, Sur une classe d'opérateurs elliptiques dégénérés, *Bull. Soc. Math. France,* 95 (1967), 45-87.

[2] P. BROUSSE and H. PONCIN, Quelques résultats généraux concernant la détermination de solutions d'équations elliptiques par les conditions aux frontières, Jubilé Scientifique de M.P. Riabonchinsky, *Publ. Sci. et techn. du Ministère de l'Air,* Paris, 1954.

[3] A. HUBER, Some results on generalized axially symmetric potentials, *Proceedings of the Conference on Differential Equations,* University of Maryland, 1955.

[4] P. JAMET and S.V. PARTER, Numerical methods for elliptic differential equations whose coefficients are singular on a portion of the boundary, *Siam J. Numer. Anal.,* 4 (1967), 131-146.

[5] J.J. KOHN and L. NIRENBERG, Non coercive boundary value problems, *Comm. Pure Appl. Math.,* 18 (1965), 443-492.

[6] O.A. LADYZHENSKAYA and N.N. URALTSEVA, Linear and quasilinear and quasilinear elliptic equations, Translated from the Russian, Academic Press, New York 1968.

[7] C.Y. LO, Dirichlet problems for singular elliptic equations, *Proc. Amer. Math. Soc.,* 39 (1973), 337-342.

[8] C. MIRANDA, Partial Differential Equations of Elliptic Type, Springer-Verlag, New York 1970.

[9] H. MOREL, Introduction de poids dans l'étude des problèmes aux limites, *Ann. Inst. Fourier, Grenoble,* 12 (1962), 299-414.

[10] M.H. PROTTER and H.F. WEINBERGER, Maximum Principles in Differential Equations, Prentice Hall, Englewood Cliffs, New Jersey 1967.

[11] M.K.V. MURTHY and G. STAMPACCHIA, Boundary value problems for some degenerate elliptic operators, *Ann. Math. Pura Appl.* 80 (1968), 1-122.

[12] M. SCHECHTER, On the Dirichlet problem for second order equations with coefficients singular at the boundary, *Comm. Pure Appl. Math.*, 13 (1960), 321-328.

Manuscrit reçu le 16 janvier 1974 .

Nguyen Phuong CÁC,
Department of Mathematics
101 MacLean Hall
The University of Iowa
Iowa City, Iowa 52 242 (USA).