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ON DEFINITIONS OF SUPERHARMONIC FUNCTIONS

by Seizô ITÔ

Dédié à Monsieur M. Brelot à l'occasion de son 70ᵉ anniversaire.

1. Introduction.

The classical definition of superharmonic functions by F. Riesz [3] (see also M. Brelot [1]) can be generalized in natural way to the case of the elliptic differential operator A of second order with variable coefficients (§ 2 of the present paper). On the other hand, L. Schwartz [4] has defined the superharmonicity with respect to the general elliptic differential operator in view-point of the theory of distribution and given an elegant proof to Riesz decomposition theorem. One may easily prove that the superharmonicity with respect to A (abreviated to A-*superharmonicity*) of the Riesz-Brelot sense implies that of Schwartz sense in case A is the ordinary Laplacian.

However, in the case of the elliptic differential operator A with variable coefficients, it seems not to be evident that the theory of distribution is applicable to A-superharmonic functions in the classical sense; in fact, even the local summability of an A-superharmonic function in the classical sense seems not to be trivial.

The purpose of the present paper is to prove that any A-superharmonic function in the Riesz-Brelot sense is locally summable and satisfies the A-superharmonicity in the sense of Schwartz distribution. The A-superharmonicity in Schwartz sense implies the Riesz decomposition formula as shown in [4], while one may easily see that any function represented by the Riesz decomposition formula is A-superharmonic in the Riesz-Brelot sense. Thus we may conclude the equivalence of the A-superharmonicity in the Riesz-Brelot sense, that of Schwartz sense and the Riesz decomposition formula for arbitrary elliptic differential operator A of second order with variable coefficients.

2. Main results.

Let Ω be a subdomain of an orientable m-dimensional C^∞-manifold ($m \geqslant 2$), and A be an elliptic differential operator of the form:

$$\begin{aligned} \mathrm{A}u(x) &= \operatorname{div}\,[\nabla u(x)] + (\boldsymbol{b}(x) . \nabla u(x)) + c(x)u(x) \\ &\equiv \sum_{i,j} \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i}\left[\sqrt{a(x)}\,a^{ij}(x)\,\frac{\partial u(x)}{\partial x^j}\right] + \sum_i b^i(x)\,\frac{\partial u(x)}{\partial x^i} + c(x)u(x), \end{aligned}$$

where $\|a^{ij}(x)\|$ is a contravariant tensor of class C^2 in Ω and is symmetric and strictly positive-definite for any $x \in \Omega$, $a(x) = \det \|a_{ij}(x)\| = \det \|a^{ij}(x)\|^{-1}$, $\boldsymbol{b}(x) \equiv \|b^i(x)\|$ is a contravariant vector of class C^2 in Ω, and $c(x)$ is a Hölder-continuous function satisfying $c(x) \leqslant 0$ in Ω. We shall denote by dx and $dS(x)$ respectively the volume element and the $m-1$ dimensional hypersurface element with respect to the Riemannian metric defined by the tensor $\|a_{ij}(x)\|$. The formally adjoint operator A^* of A is defined by

$$\mathrm{A}^*u(x) = \operatorname{div}\,[\nabla u(x) - \boldsymbol{b}(x)u(x)] + c(x)u(x).$$

By definition, a function $u(x)$ is said to be A-*harmonic* in Ω if it satisfies $\mathrm{A}u = 0$ in Ω, and is said to be A-*superharmonic* in Ω if it satisfies the following three conditions:

i) $-\infty < u(x) \leqslant \infty$ and $u(x) \not\equiv \infty$ in Ω,

ii) $u(x)$ is lower semi-continuous in Ω,

iii) if D is a domain with its compact closure $\overline{D} \subset \Omega$, and if $w(x)$ is continuous on $\overline{D}$, A-harmonic in D and satisfies $w(x) \leqslant u(x)$ on ∂D, then $w(x) \leqslant u(x)$ holds in D.

We shall prove the following two theorems in § 4.

THEOREM 1. — *Any A-superharmonic function in Ω is locally summable in Ω.*

THEOREM 2. — *Any A-superharmonic function $u(x)$ in Ω satisfies $Au \leqslant 0$ in Ω in the sense of distribution.*

3. Preliminary lemmas.

We shall use some properties of fundamental solutions of parabolic equations. The following facts are implied by the results of one of the author's previous papers [3].

For any subdomain D of Ω with compact closure $\overline{D} \subset \Omega$ and with boundary ∂D of class C^3, there exists one and only one fundamental solution $U_D(t, x, y)$ of the initial-boundary value problem:

$$(3.1) \quad \frac{\partial u}{\partial t} = Au \text{ in } (0, \infty) \times D, \ u|_{t=0} = u_0, \ u|_{x\in\partial D} = \varphi.$$

The function $U_D(t, x, y)$ satisfies that

$$(3.2) \quad \begin{cases} U_D(t, x, y) \geqslant 0 \text{ for any} \\ \qquad \langle t, x, y\rangle \in (0, \infty) \times \overline{D} \times \overline{D}; \\ \text{the equality holds if and only if at least one} \\ \text{of } x \text{ and } y \text{ belongs to } \partial D \end{cases}$$

and that

$$(3.3) \qquad \frac{\partial U_D(t, x, y)}{\partial \boldsymbol{n}(y)} \leqslant 0 \text{ for any } t > 0, \ y \in \partial D$$

and $x \in D - \{y\}$ where $\dfrac{\partial}{\partial \boldsymbol{n}(y)}$ denotes the exterior normal derivative at $y \in \partial D$. For any continuous functions $u_0(x)$ on $\overline{D}$ and $\varphi(t, x)$ on $[0, \infty) \times \partial D$, there exists one and only one solution $u(t, x)$ of the initial boundary value problem (3.1) and it is given by

$$(3.4) \quad u(t, x) = \int_D U_D(t, x, y) u_0(y)\, dy - \int_0^t d\tau \int_{\partial D} \frac{\partial U_D(t-\tau, x, y)}{\partial \boldsymbol{n}(y)} \varphi(\tau, y)\, dS(y).$$

In particular, if $w(x)$ satisfies $Aw = 0$ in D and $w|_{\partial D} = \psi$ where ψ is continuous on ∂D, then

$$(3.5) \quad w(x) = \int_D U_D(t, x, y) w(y)\, dy - \int_0^t d\tau \int_{\partial D} \frac{\partial U_D(\tau, x, y)}{\partial \boldsymbol{n}_D(y)} \psi(y)\, dS(y).$$

LEMMA 1. — *Let Ω_0 be a subdomain of Ω with its compact closure $\overline{\Omega}_0 \subset \Omega$ and with boundary $\partial\Omega_0$ of class C^3, $u(x)$ be an A-superharmonic function on Ω such that $u(x) > 0$ on $\overline{\Omega}_0$ and $v(x)$ be a continuous function on $\overline{\Omega}_0$ such that $0 \leqslant v(x) < u(x)$ on $\overline{\Omega}_0$. Then $\int_{\Omega_0} U_{\Omega_0}(t, x, y) v(y)\, dy < u(x)$ on $(0, \infty) \times \overline{\Omega}_0$.*

Proof. — The function $v(t, x) = \int_{\Omega_0} U_{\Omega_0}(t, x, y) v(y)\, dy$ is the solution of the initial-boundary value problem (3.1) with $D = \Omega_0$, $u_0 = v$ and $\varphi = 0$. Suppose that

$$v(t, x) \geqslant u(x)$$

at some point $\langle t, x\rangle \in (0, \infty) \times \overline{\Omega}_0$, and put

$$t_1 = \inf\{t;\ v(t, x) \geqslant u(x) \text{ for some } x \in \overline{\Omega}_0\}.$$

Then

$$(3.6) \quad 0 \leqslant v(\tau, x) < u(x) \quad \text{whenever} \quad 0 < \tau < t_1$$

and $x \in \overline{\Omega}_0$. By means of the continuity of $v(t, x)$, lower semi-continuity of $u(x)$ and by the fact: $v(t, x) = 0$ for any $x \in \partial\Omega_0$, we may find a point $x_1 \in \Omega_0$ such that

$$(3.7) \quad v(t_1, x_1) = u(x_1) < \infty.$$

Since $u(x) - v(x)$ is positive and lower semi-continuous on

$\overline{\Omega}_0$, there exists a positive number δ such that

$$(3.8) \qquad 0 < v(x) + 3\delta < u(x) \quad \text{on} \quad \overline{\Omega}_0.$$

Further we may find a domain D with boundary ∂D of class C^3 such that $x_1 \in D \subset \Omega_0$ and that

$$v(x) < v(x_1) + \delta \quad \text{and} \quad u(x) > u(x_1) - \delta \quad \text{on} \quad \overline{D}.$$

Combining these inequalities with (3.8), we get

$$(3.9) \qquad v(x) + \delta < \inf_{x \in \overline{D}} u(x) \quad \text{on} \quad \overline{D}.$$

Let $\{u_n\}$ be a monotone increasing sequence of continuous functions on ∂D such that $\lim_{n\to\infty} u_n(y) = u(y)$ on ∂D. Then we may easily show that

$$(3.10) \qquad \lim_{n\to\infty} \Big[\inf_{y\in\partial D} u_n(y)\Big] = \inf_{y\in\partial D} u(y).$$

Let w_n be the solution of elliptic boundary value problem: $Aw_n = 0$ in D, $w_n|_{\partial D} = u_n$. Then $w_n(x) \leqslant u(x)$ in D by means of the A-superharmonicity of u, and the sequence $\{w_n\}$ is monotone increasing. Hence

$$w(x) = \lim_{n\to\infty} w_n(x) (\leqslant \infty)$$

exists and $w(x) \leqslant u(x)$ in D. Since $w_n(x) \geqslant \inf_{y\in\partial D} u_n(y)$ in D, we obtain from (3.10) and (3.9) that

$$(3.11) \qquad w(x) \geqslant \inf_{y\in\partial D} u(y) \geqslant v(x) + \delta \ \text{in D}.$$

On the other hand (cf. (3.5))

$$w_n(x) = \int_D U_D(t, x, y) w_n(y)\, dy + \int_0^t d\tau \int_{\partial D} \left\{ - \frac{\partial U_D(\tau, x, y)}{\partial \boldsymbol{n}(y)} \right\} u_n(y)\, dS(y).$$

Let $n \to \infty$, and we obtain

$$(3.12) \quad u(x) \geqslant w(x) = \int_D U_D(t, x, y) w(y)\, dy + \int_0^t d\tau \int_{\partial D} \left\{ - \frac{\partial U_D(\tau, x, y)}{\partial \boldsymbol{n}(y)} \right\} u(y)\, dS(y).$$

Applying (3.4) to $\varphi(t, x)$ restricted on $(0, \infty) \times \overline{\mathrm{D}}$, we get

$$\begin{aligned}\varphi(t, x) &= \int_{\mathrm{D}} \mathrm{U}_{\mathrm{D}}(t, x, y)\varphi(y)\, dy \\ &\quad + \int_0^t d\tau \int_{\partial \mathrm{D}} \left\{ - \frac{\partial \mathrm{U}_{\mathrm{D}}(t-\tau, x, y)}{\partial \boldsymbol{n}(y)} \right\} \varphi(\tau, y)\, d\mathrm{S}(y) \\ &\leqslant \int_{\mathrm{D}} \mathrm{U}_{\mathrm{D}}(t, x, y)[w(y) - \delta]\, dy \\ &\quad + \int_0^t d\tau \int_{\partial \mathrm{D}} \left\{ - \frac{\partial \mathrm{U}_{\mathrm{D}}(\tau, x, y)}{\partial \boldsymbol{n}(y)} \right\} u(y)\, d\mathrm{S}(y) \\ &\qquad\qquad \text{(from (3.11) and (3.6))} \\ &\leqslant u(x) - \delta \int_{\mathrm{D}} \mathrm{U}_{\mathrm{D}}(t, x, y)\, dy \\ &\qquad\qquad \text{(from (3.12))}.\end{aligned}$$

In particular $\varphi(t_1, x_1) \leqslant u(x_1) - \delta \int_{\mathrm{D}} \mathrm{U}_{\mathrm{D}}(t_1, x_1, y)\, dy$; this contradicts (3.7) since $\int_{\mathrm{D}} \mathrm{U}_{\mathrm{D}}(t_1, x_1, y)\, dy > 0$ by (3.2).

Remark. — Even the fact that $\{x | u(x) = \infty\}$ has no interior point is not guaranteed before Theorem 1 is proved. So, for instance, each term in (3.12) might be ∞ (where we use the usual convention rule: $\infty \geqslant \infty$, $\infty >$ any real number). However we do not have to care for such situations in the above proof since $\mathrm{U}_{\mathrm{D}}(t, x, y) \geqslant 0$ and $- \dfrac{\partial \mathrm{U}_{\mathrm{D}}(\tau, x, y)}{\partial \boldsymbol{n}(y)} \geqslant 0$.

LEMMA 2. — *Let* Ω_0 *and* $u(x)$ *be as in Lemma* 1. *Then*

$$\int_{\Omega_0} \mathrm{U}_{\Omega_0}(t, x, y) u(y)\, dy \leqslant u(x) \text{ on } (0, \infty) \times \overline{\Omega}_0.$$

Proof. — Let γ be a positive number less than $\min_{x \in \overline{\Omega}_0} u(x)$, and $\{\varphi_n\}$ be a monotone increasing sequence of continuous functions on $\overline{\Omega}_0$ such that $\varphi_n(x) \geqslant \gamma$ $(n = 1, 2, \ldots)$ and $\lim_{n \to \infty} \varphi_n(x) = u(x)$ on $\overline{\Omega}_0$. For every $n > \gamma^{-1}$, we apply Lemma 1 to the function $\varphi(x) = \varphi_n(x) - n^{-1}$ and obtain $\int_{\Omega_0} \mathrm{U}_{\Omega_0}(t, x, y)[\varphi_n(x) - n^{-1}]\, dy \leqslant u(x)$. Let $n \to \infty$, and we get the conclusion of Lemma 2.

4. Proof of Theorems.

Let $u(x)$ be an arbitrary A-superharmonic function on Ω. For any given subdomain Ω_0 of Ω with compact closure $\overline{\Omega}_0 \subset \Omega$ and with boundary $\partial\Omega_0$ of class C^3, we may assume in proofs of Theorems 1 and 2 that $u(x) > 0$ on $\overline{\Omega}_0$ because, if $\inf_{x \in \overline{\Omega}_0} u(x) = \alpha \leqslant 0$, we may replace $u(x)$ by

$$u(x) + (1 - \alpha)u_0(x)$$

where u_0 is the solution of the elliptic boundary value problem: $Au_0 = 0$ in Ω_0, $u_0|_{\partial\Omega_0} = 1$.

Proof of Theorem 1. — Let D be an arbitrary subdomain of Ω with compact closure $\overline{D}$. By the A-superharmonicity of u, we may find a point $x_0 \in \Omega$ where $u(x_0) < \infty$. Let Ω_0 be a subdomain of Ω such that $\overline{D} \cup \{x_0\} \subset \Omega_0$, $\overline{\Omega}_0$ is compact and $\partial\Omega_0$ is of class C^3. Then, as we have noticed above, we may assume that $u(x) > 0$ on $\overline{\Omega}_0$. Hence

$$(4.1) \quad u(x) \geqslant \int_{\Omega_0} U_{\Omega_0}(t, x, y)u(y)\, dy \text{ on } (0, \infty) \times \overline{\Omega}_0$$

by Lemma 2. We fix a positive number t_0. Then, since $U_{\Omega_0}(t_0, x, y) > 0$ on $\Omega_0 \times \Omega_0$ and $\beta \equiv \min_{y \in \overline{D}} U_{\Omega_0}(t_0, x_0, y) > 0$ by (3.2), it follows from (4.1) that $u(x_0) \geqslant \int_D \beta u(y)\, dy$, which implies $\int_D u(y)\, dy \leqslant u(x_0)/\beta < \infty$, q.e.d.

Proof of Theorem 2. — Let $\varphi(x)$ be an arbitrary non-negative valued function of class C^2 and with compact support in Ω, and Ω_0 be a subdomain of Ω containing the support of φ and such that $\overline{\Omega}_0$ is compact and $\partial\Omega_0$ is of class C^3. It suffices to prove that

$$\int_{\Omega_0} u(x) . A^*\varphi(x)\, dx \leqslant 0.$$

We may assume that $u(x) > 0$ on $\overline{\Omega}_0$ as we have noticed above. Hence we have by Lemma 2

$$(4.2) \quad \int_{\Omega_0} u(y) \left\{ \int_{\Omega_0} \varphi(x) U_{\Omega_0}(t, x, y)\, dx - \varphi(y) \right\} dy = \int_{\Omega_0} \varphi(x) \left\{ \int_{\Omega_0} U_{\Omega_0}(t, x, y) u(y) dy - u(x) \right\} dx \leqslant 0.$$

On the other hand, since

$$\frac{\partial}{\partial t}\int_{\Omega_0}\varphi(x)U_{\Omega_0}(t, x, y)\,dx = \int_{\Omega_0}\varphi(x)\,\frac{\partial U_{\Omega_0}(t, x, y)}{\partial t}\,dx$$
$$= \int_{\Omega_0}\varphi(x)\,.\,A_x U_{\Omega_0}(t, x, y)\,dx = \int_{\Omega_0} A^*\varphi(x)\,.\,U_{\Omega_0}(t, x, y)\,dx$$

(the subscript x to A indicates to operate A to $U_{\Omega_0}(t, x, y)$ as a function of x), we get

$$\lim_{t\downarrow 0}\frac{\partial}{\partial t}\int_{\Omega_0}\varphi(x)U_{\Omega_0}(t, x, y)\,dx = A^*\varphi(y)$$

boundedly in $y \in \Omega_0$; accordingly

$$\lim_{t\downarrow 0}\frac{1}{t}\left\{\int_{\Omega_0}\varphi(x)U_{\Omega_0}(t, x, y)\,dx - \varphi(y)\right\}$$
$$= \lim_{t\downarrow 0}\frac{1}{t}\int_0^t\left\{\frac{\partial}{\partial\tau}\int_{\Omega_0}\varphi(x)U_{\Omega_0}(\tau, x, y)\,dx\right\}d\tau = A^*\varphi(y)$$

boundedly in y. Combining this result with (4.2), we obtain

$$\int_{\Omega_0} A^*\varphi(y)\,.\,u(y)\,dy \leqslant 0,$$

q.e.d.

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