ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 25, nº 2 (1975), p. 81-126

<http://www.numdam.org/item?id=AIF_1975__25_2_81_0>

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Ann. Inst. Fourier, Grenoble
25, 2 (1975), 81-126

THE DUAL OF WEAK L^p

by Michael CWIKEL (*)

1. Introduction.

For any given p in $(0, \infty)$, the space Weak L^p on a given measure space (X, Σ, μ) consists of those (equivalence classes of) real or complex valued measurable functions $f(x)$, whose distribution functions $f_*(\alpha)$,

$$f_*(\alpha) = \mu(\{x \mid |f(x)| > \alpha\})$$

satisfy
$$\sup_{\alpha > 0} \alpha^p f_*(\alpha) < \infty.$$

The quasi norm $|f|_p = \left(\sup_{\alpha>0} \alpha^p f_*(\alpha)\right)^{1/p}$ defines a topology on Weak L^p. L^p is continuously embedded in Weak L^p. Weak L^p claims our attention largely because of the Marcinkiewicz interpolation theorem, and the fact that several important operators such as the Hilbert transform and the Hardy-Littlewood maximal function which map L^p into L^p for $p > 1$, map L^1 into Weak L^1.

It is convenient for some purposes to consider L^p and Weak L^p within the more general framework of the Lorentz spaces $L(p, q)$. Let us recall the definition of these spaces.

For each measurable function f on X with distribution function $f_*(\alpha)$, we define the non-increasing rearrangement $f^*(t)$ of f by :

$$f^*(t) = \inf\{\alpha > 0 \mid f_*(\alpha) \leqslant t\} \quad \text{for all} \quad t > 0$$

(with the usual convention that the infimum of the empty set is $+\infty$).

(*) Supported in part by the C.N.R.S.

For $0 < p < \infty$, $0 < q < \infty$, define

$$\|f\|^*_{p,q} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \, dt/t\right)^{1/q}$$

and for $0 < p \leqslant \infty$, $q = \infty$,

$$\|f\|^*_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t).$$

Then for the above values of p and q the Lorentz spaces are defined by:

$$\mathrm{L}(p\,,q) = \{f \mid \|f\|^*_{p,q} < \infty\}.$$

It is easy to see that $\mathrm{L}(p\,,p) = \mathrm{L}^p$ and that $\sup_{\alpha>0} \alpha^p f_*(\alpha) = (\|f\|^*_{p,\infty})^p$, so that $\mathrm{L}(p\,,\infty) = \text{Weak } \mathrm{L}^p$.

$\|f\|^*_{p,q}$ is not a norm since the triangle inequality may fail. However for $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, the $\mathrm{L}(p\,,q)$ topology may be defined by the norm :

$$\|f\|_{\mathrm{L}(p,q)} = \left(\int_0^\infty (t^{1/p} f^{**}(t))^q \, dt/t\right)^{1/q} \quad \text{for} \quad q < \infty$$

$$= \sup_{t>0} t^{1/p} f^{**}(t) \quad \text{for} \quad q = \infty,$$

where $f^{**}(t)$ is defined by :

$$f^{**}(t) = \sup\left\{(\mu(\mathrm{E}))^{-1} \int_{\mathrm{E}} |f| \, d\mu \mid \mathrm{E} \in \Sigma,\ \mu(\mathrm{E}) \geqslant t\right\} \text{for } 0 < t < \mu(\mathrm{X})$$

$$= t^{-1} \int_{\mathrm{X}} |f| \, d\mu \qquad \text{for } t \geqslant \mu(\mathrm{X}).$$

If the measure space is non-atomic,

$$\sup\left\{(\mu(\mathrm{E}))^{-1} \int_{\mathrm{E}} |f| \, d\mu \mid \mathrm{E} \in \Sigma\ ,\ \mu(\mathrm{E}) \geqslant t\right\}$$

$$= \sup\left\{t^{-1} \int_{\mathrm{E}} |f| \, d\mu \mid \mathrm{E} \in \Sigma\ ,\ \mu(\mathrm{E}) = t\right\},$$

and consequently

$$\|f\|_{\mathrm{L}(p,\infty)} = \sup\left\{\mu(\mathrm{E})^{1/p-1} \int_{\mathrm{E}} |f| \, d\mu \mid \mathrm{E} \in \Sigma\right\}.$$

In the non-atomic case it also follows that :

$$f^{**}(t) = t^{-1} \int_0^t f^*(s) \, ds.$$

A detailed study of the Lorentz spaces has been made by Hunt [6]. It includes characterisations of their dual spaces for all values of p and q with q finite. $L(p,\infty)^*$ for $0 < p \leqslant 1$ has been studied in [2] and [4]. We summarise the results in Table 1.

Table 1

The dual space of $L(p,q)$ for various values of p and q.
$(1/p + 1/p' = 1 = 1/q + 1/q')$

<table>
<tr><th></th><th>$0 < p < 1$</th><th>$p = 1$</th><th>$1 < p < \infty$</th><th>$p = \infty$</th></tr>
<tr><td rowspan="2">$0 < q < 1$</td><td rowspan="4">$= \{0\}$ if the measure space is non atomic. [6]</td><td>$= L^{1*}$</td><td>$= L(p',\infty)$</td><td rowspan="4">$L(\infty, q)$ is undefined for all $q < \infty$.</td></tr>
<tr><td colspan="2">For $0 < q < 1$, $L(p,q)^* = L(p,1)^*$ This follows readily from theorem 2.5 of [6]</td></tr>
<tr><td>$q = 1$</td><td>$= L^{1*}$
$= L^\infty$ for a σ-finite measure space [5].</td><td>$= L(p',\infty)$ [6]</td></tr>
<tr><td>$1 < q < \infty$</td><td></td><td>$= L(p',q')$ [6]</td></tr>
<tr><td>$q = \infty$</td><td>$= \{0\}$ if measure space is non atomic. [2]</td><td>is non trivial, though functionals vanish on all simple functions [4]</td><td>$= L(p',1) \oplus S_0 \oplus S_\infty$ shown in this paper</td><td>$= L^{\infty *}$, the space of bounded finitely additive set functions. [5]</td></tr>
</table>

In this paper we consider $L(p,\infty)^*$ for $1 < p < \infty$. For this range of values of p, $L(p,\infty)$ is a Banach space with norm as defined above. The underlying measure space (X, Σ, μ) will be taken to be non atomic, except where explicitly stated otherwise. It will be seen that $L(p,\infty)^*$ can be expressed as the direct sum of three spaces, $L(p',1) \oplus S_0 \oplus S_\infty$, where $1/p + 1/p' = 1$. The spaces S_0 and S_∞ consist of functionals which exhibit a singular behavior similar to that seen in all elements of $L(1,\infty)^*$; they annihilate all functions which are bounded and supported on any set of finite measure. (There are also some essential differences between the elements of S_0 and S_∞ and those of $L(1,\infty)^*$, as shown in [3]).

Characterisations of S_0 and S_∞ are obtained, and both spaces are seen to be reflexive. Thus we readily deduce the form of the n^{th} dual of $L(p, \infty)$.

We note that, as indicated in Table 1, $L(p', 1)^* = L(p, \infty)$. It will be clear that the embedding of $L(p', 1)$ into $L(p', 1) \oplus S_0 \oplus S_\infty$ is the canonical embedding of $L(p', 1)$ into its double dual.

We shall use the notation $\overline{z}$ for the complex conjugate of any complex number (or complex valued function) z. The function *sgn* will be defined by

$$\begin{aligned} sgn(z) &= z/|z| \qquad \text{for all} \qquad z \neq 0 \\ &= 0 \qquad\quad \text{for} \qquad\quad z = 0. \end{aligned}$$

The open subinterval $(0, \mu(X))$ of the real line will be denoted by I.

2. Some technical preliminaries – continuously monotone families.

The definitions and consequent observations to be made in this section constitute a more careful look at the procedure which yields the non-increasing rearrangement function $f^*(t)$ from a measurable function f on X. Our aim is to have an apparatus with whose help we may work with f almost as easily as if it were itself a non-increasing function on the positive real axis. Roughly speaking we shall find a "direction" in X in which f is decreasing and looks rather like $f^*(t)$. The simple notion of a continuously monotone family of sets is central here, and it will also be important later, in describing the continuous linear functionals on $L(p, \infty)$.

LEMMA 1. – *Let g be a positive measurable function on* X, *such that $g_*(t)$ and $g^*(t)$ are both finite for all $t > 0$. Then*

$$t - \mu(\{x \mid g(x) = g^*(t)\}) \leqslant g_*(g^*(t)) \leqslant t$$

for all t, $0 < t < \mu(X)$.

Proof. – By the expanding sequence theorem,

$$\lim_{n\to\infty} g_*\left(\alpha + \frac{1}{n}\right) = g_*(\alpha)$$

for all $\alpha \in [0, \infty)$, and by the contracting sequence theorem,

$$\lim_{n\to\infty} g_*\left(\alpha - \frac{1}{n}\right) = \mu(\{x \mid g(x) \geqslant \alpha\}) = g_*(\alpha) + \mu(\{x \mid g(x) = \alpha\})$$

for all $\alpha \in (0, \infty)$. By definition of $g^*(t)$,

$$g_*\left(g^*(t) + \frac{1}{n}\right) \leqslant t \leqslant g_*\left(g^*(t) - \frac{1}{n}\right)$$

for each n. It remains only to pass to the limit as n tends to infinity. But to deal with those t for which $g^*(t) = 0$ (and consequently $g_*(0) \leqslant t$) we need the additional remark that

$$\mu(X) = \mu(\{x \mid g(x) = 0\}) + g_*(0)\,,$$

and so $\quad t - \mu(\{x \mid g(x) = 0\}) = t - \mu(X) + g_*(0) \leqslant g_*(0).$

DEFINITION. – A *continuously monotone family* (C.M.F.) *on* X *is a family* $(A_t)_{0 \leqslant t \leqslant \mu(X)}$ *of measurable subsets* A_t *of* X *such that* :

i) $A_0 = \emptyset$, $A_{\mu(X)} = X$.

ii) $t < s$ *implies* $A_t \subset A_s$.

iii) $\mu(A_t) = t$ *for each* t, $0 \leqslant t \leqslant \mu(X)$.

DEFINITION. – *Let* g *be a positive measurable function on* X. *Let* $(A_t)_{0 \leqslant t \leqslant \mu(X)}$ *be a C.M.F. on* X. *We shall say* (A_t) *carries* g *if, for all* t, $0 < t < \mu(X)$

$$\text{ess sup}\{g(x) \mid x \in X \setminus A_t\} \leqslant \text{ess inf}\{g(x) \mid x \in A_t\}.$$

(For example if $X = (0, \infty)$ and $A_t = (0, t)$ then (A_t) carries all non-increasing functions).

With the help of Lemma 1 we can verify the not very surprising fact that each function $g \geqslant 0$ with $g_*(t)$ and $g^*(t)$ both finite for all $t > 0$, is carried by a C.M.F. Given g, let $E_\lambda = \{x \mid g(x) = \lambda\}$

for each $\lambda \geqslant 0$. Since X is non-atomic, there exists a C.M.F. on E_λ, $(B_\theta^\lambda)_{0 \leqslant \theta \leqslant \mu(E_\lambda)}$. (Of course E_λ will have zero measure anyway for "most" values of λ, and $\mu(E_0)$ could be infinite). For each $t \in (0, \mu(X))$, let

$$A_t = \{x \mid g(x) > g^*(t)\} \cup B^{g^*(t)}_{t-g_*(g^*(t))}.$$

This is well defined since, by Lemma 1, $0 \leqslant t - g_*(g^*(t)) \leqslant \mu(E_{g^*(t)})$. To verify that (A_t) is a C.M.F. first note that

$$\mu(A_t) = g_*(g^*(t)) + (t - g_*(g^*(t))) = t.$$

For the inclusion property, take $t < s$. Either $g^*(t) = g^*(s)$ or $g^*(t) > g^*(s)$. In the former case $A_t \subset A_s$, since $B^\lambda_{t-g_*(\lambda)} \subset B^\lambda_{s-g_*(\lambda)}$ (where $\lambda = g^*(t) = g^*(s)$), and in the latter case again $A_t \subset A_s$, since

$$A_t \subset \{x \mid g(x) \geqslant g^*(t)\} \subset \{x \mid g(x) > g^*(s)\} \subset A_s.$$

It is obvious from the definition of (A_t) that it carries g.

We next consider an alternative definition of the non-increasing rearrangement (cf. [8]). Let $g(x)$ be any measurable function on an arbitrary measure space (X, Σ, μ). Define two non-increasing functions $g^R(t)$ and $g^L(t)$ on $(0, \infty)$.

$$g^R(t) = \inf_{E \in \Sigma, \mu(E) \leqslant t} \left(\operatorname{ess\,sup}_{x \in X \setminus E} |g(x)|\right) \quad \text{for} \quad t < \mu(X)$$

$$= 0 \quad \text{for} \quad t \geqslant \mu(X)$$

$$g^L(t) = \sup_{E \in \Sigma, \mu(E) \geqslant t} \left(\operatorname{ess\,inf}_{x \in E} |g(x)|\right) \quad \text{for} \quad t \leqslant \mu(X)$$

$$= 0 \quad \text{for} \quad t > \mu(X).$$

LEMMA 2. – *Let* $(A_t)_{0 \leqslant t \leqslant \mu(X)}$ *carry the positive measurable function g for which* $g_*(t) < \infty$, $g^*(t) < \infty$ *for all* $t > 0$. *Then*

$$g^R(t) = \operatorname{ess\,sup}\{g(x) \mid x \in X \setminus A_t\}$$

and

$$g^L(t) = \operatorname{ess\,inf}\{g(x) \mid x \in A_t\}.$$

Proof. – Let E be any measurable set with $\mu(E) \leqslant t$.

Then $$\operatorname*{ess\,sup}_{X\setminus E} g(x) = \max\,[\operatorname*{ess\,sup}_{X\setminus E\setminus A_t} g(x)\,,\ \operatorname*{ess\,sup}_{A_t\setminus E} g(x)]$$

and $$\operatorname*{ess\,sup}_{X\setminus A_t} g(x) = \max\,[\operatorname*{ess\,sup}_{X\setminus E\setminus A_t} g(x)\,,\ \operatorname*{ess\,sup}_{E\setminus A_t} g(x)].$$

Since (A_t) carries g, if $\mu(A_t\setminus E)$ and $\mu(E_t\setminus A)$ are positive,

$$\operatorname*{ess\,sup}_{E\setminus A_t} g(x) \leqslant \operatorname*{ess\,sup}_{X\setminus A_t} g(x) \leqslant \operatorname*{ess\,inf}_{A_t} g(x) \leqslant \operatorname*{ess\,sup}_{A_t\setminus E} g(x).$$

If $\mu(A_t\setminus E) = 0$, $A_t = E$ a.e. since $\mu(E) \leqslant t = \mu(A_t)$. If $\mu(E\setminus A_t) = 0$, then $A_t \supset E$ a.e. In all three cases we can deduce that

$$\operatorname*{ess\,sup}_{X\setminus A_t} g(x) \leqslant \operatorname*{ess\,sup}_{X\setminus E} g(x)$$

and therefore $g^R(t) = \operatorname*{ess\,sup}_{X\setminus A_t} g(x)$. The proof that $g^L(t) = \operatorname*{ess\,inf}_{A_t} g(x)$ is the exact dual of the above.

LEMMA 3. – a) *$g^R(t)$ is right continuous and $g^L(t)$ is left continuous.*

b) *Letting m denote Lebesgue measure on* $(0\,,\infty)$,

$$m(\{t \mid g^R(t) > \alpha\}) = g_*(\alpha) = m(\{t \mid g^L(t) > \alpha\})$$

for each $\alpha > 0$.

c) $$g^R(t) = g^*(t) \qquad \textit{for all } t,$$

and $$g^L(t) = g^*(t) \qquad \textit{for almost all } t.$$

Proof. – The definitions of $g^R(t)$ and $g^L(t)$ and all the statements of Lemma 3 are valid for an arbitrary measurable function g, on an arbitrary measure space (see [3]), but for conciseness we shall give the proof only for the case where the function g is carried by a C.M.F. $(A_t)_{0\leqslant t\leqslant \mu(X)}$. This case is sufficient for all our subsequent applications.

a) Since $g^R(t)$ is non increasing,

$$\begin{aligned}\lim_{n\to\infty} g^R(t + 1/n) &= \sup_{n\geqslant 1} g^R(t + 1/n)\\ &= \sup_{n\geqslant 1}\,(\operatorname{ess\,sup}\{g(x) \mid x \in X\setminus A_{t+1/n}\})\end{aligned}$$

$$\lim_{n\to\infty} g^{\mathrm{R}}(t+1/n) = \operatorname{ess\,sup}\left\{ g(x) \mid x \in \bigcup_{n=1}^{\infty} (\mathrm{X}\setminus \mathrm{A}_{t+1/n})\right\}$$

$$= \operatorname{ess\,sup}\{ g(x) \mid x \in \mathrm{X}\setminus \mathrm{A}_t\}$$

$$= g^{\mathrm{R}}(t)$$

and $g^{\mathrm{R}}(t)$ is consequently right continuous. The left continuity of $g^{\mathrm{L}}(t)$ is proved in a similar manner :

$$\inf_{n\geqslant 1} g^{\mathrm{L}}(t-1/n) = \operatorname{ess\,inf}\left\{ g(x) \mid x \in \bigcup_{n=1}^{\infty} \mathrm{A}_{t-1/n}\right\} = g^{\mathrm{L}}(t).$$

b) For each $t < \mu(\mathrm{X})$

$$g^{\mathrm{R}}(t) = \inf_{\mathrm{E}\in\Sigma,\,\mu(\mathrm{E})\leqslant t} \left(\operatorname{ess\,sup}_{x\in\mathrm{X}\setminus\mathrm{E}} |g(x)|\right).$$

But this infimum is actually attained (by the set $\mathrm{E} = \mathrm{A}_t$). Consequently

$$m(\{t \mid g^{\mathrm{R}}(t) > \alpha\}) = \inf\{t \mid g^{\mathrm{R}}(t) \leqslant \alpha\}$$

$$= \inf\left\{ \mu(\mathrm{E}) \mid \mathrm{E}\in\Sigma\,,\ \operatorname{ess\,sup}_{x\in\mathrm{X}\setminus\mathrm{E}} |g(x)| \leqslant \alpha\right\}.$$

Every measurable set E satisfying $\operatorname{ess\,sup}_{x\in\mathrm{X}\setminus\mathrm{E}} |g(x)| \leqslant \alpha$ must contain almost all of the set $\mathrm{E}_\alpha = \{x \mid |g(x)| > \alpha\}$, and E_α itself satisfies this same inequality. Therefore

$$m(\{t \mid g^{\mathrm{R}}(t) > \alpha\}) = \mu(\mathrm{E}_\alpha) = g_*(\alpha).$$

The argument for g^{L} is slightly different :

$$m(\{t \mid g^{\mathrm{L}}(t) > \alpha\}) = \sup\{t \mid g^{\mathrm{L}}(t) > \alpha\}$$

$$= \sup\left\{ \mu(\mathrm{E}) \mid \operatorname{ess\,inf}_{x\in\mathrm{E}} |g(x)| > \alpha\right\}.$$

Almost all of any set E satisfying $\operatorname{ess\,inf}_{x\in\mathrm{E}} |g(x)| > \alpha$ must be contained in E_α, and $\operatorname{ess\,inf}_{x\in\mathrm{E}_{\alpha+1/n}} |g(x)| > \alpha$ for each n. Consequently

$$m(\{t \mid g^{\mathrm{L}}(t) > \alpha\}) = g_*(\alpha)$$

for each $\alpha > 0$.

c) It is well known, and not hard to verify, t .at $g^*(t)$ is "unique" in the sense that any function $h(t)$ which is decreasing, right continuous and satisfies $m(\{t \mid h(t) > \alpha\}) = g_*(\alpha)$ for all $\alpha > 0$ must equal $g^*(t)$. Thus we have shown that $g^R(t) = g^*(t)$, and furthermore $g^L(t) = g^*(t)$ except on the necessarily countable (or even empty) set of points $(t_n)_{n=1}^\infty$ where $g^L(t)$ has jump discontinuities, because the function

$$\begin{aligned} f(t) &= g^L(t)\,, \quad \text{for} \quad t \notin (t_n)_{n=1}^\infty \\ &= \lim_{s \downarrow t_n} g^L(s)\,, \quad \text{for} \quad t = t_n \end{aligned}$$

is also right continuous with distribution function equal to g_*.

Remark 1. – If $(A_t)_{0 \leq t \leq \mu(X)}$ is a C.M.F. carrying the function $|f|$ where $f \in L(p, \infty)$, then $f^{**}(t) = t^{-1} \int_{A_t} |f|\, d\mu$, for if E is any set of measure t,

$$\int_{A_t} |f|\, d\mu = \int_{E \cap A_t} |f|\, d\mu + \int_{A_t \setminus E} |f|\, d\mu,$$

and
$$\int_E |f|\, d\mu = \int_{E \cap A_t} |f|\, d\mu + \int_{E \setminus A_t} |f|\, d\mu.$$

The sets $A_t \setminus E$ and $E \setminus A_t$ both have measure $t - \mu(A_t \cap E)$. On the former, $|f(x)| \geq f^*(t)$ a.e. whereas on the latter $|f(x)| \leq f^*(t)$ a.e. Thus $\int_{A_t} |f|\, d\mu \geq \sup_{\mu(E)=t} \int_E |f|\, d\mu$.

Remark 2. – From Remark 1, and the readily verified formula, $f^{**}(t) = t^{-1} \int_0^t f^*(s)\, ds$, it follows that $\int_{A_t} |f|\, d\mu = \int_0^t f^*(s)\, ds$. In fact, by similar arguments, $\int_{A_t} |f|^r\, d\mu = \int_0^t (f^*(s))^r\, ds$ for any exponent $r > 0$.

3. The decomposition $L(p, \infty)^* = L(p', 1) \oplus S_0 \oplus S_\infty$.

LEMMA 1. – *Given any $l \in L(p, \infty)^*$, there exists a set* E *of σ-finite measure, supporting l, that is, $l(\chi_E f) = l(f)$ for all $f \in L(p, \infty)$.*

Proof. – We consider real vector spaces, the extension to the complex case being obvious. Since $f(x) \leqslant g(x)$ a.e. implies

$$\|f\|_{L(p,\infty)} \leqslant \|g\|_{L(p,\infty)}$$

we can show in the usual way that $l = l^+ - l^-$, where l^+ and l^- are positive continuous functionals (see e.g. [2]). Thus it suffices to prove the lemma for l positive.

Given $\epsilon > 0$ there exists a positive function f_1 in $L(p, \infty)$ with $t^{1/p} f_1^{**}(t) \leqslant 1$, such that $l(f_1) \geqslant \|l\|(1-\epsilon)$. The set

$$E_1 = \{x \mid f_1(x) \neq 0\}$$

is σ-finite. Let g be any function which vanishes on E_1 with

$$\|g\|_{L(p,\infty)} = 1.$$

Then for each $t > 0$:

$$\begin{aligned}
(f_1 + g)^{**}(t) &= \sup_{A\in\Sigma,\mu(A)=t} t^{-1} \int_A |f_1 + g| \, d\mu \\
&= \sup_{A\in\Sigma,\mu(A)=t} t^{-1} \left[\int_{A\cap E_1} f_1 \, d\mu + \int_{A\setminus E_1} |g| \, d\mu\right] \\
&\leqslant \sup_{0\leqslant\alpha\leqslant t} t^{-1} [\alpha f_1^{**}(\alpha) + (t-\alpha)\, g^{**}(t-\alpha)] \\
&\leqslant \sup_{0\leqslant\alpha\leqslant t} t^{-1} [\alpha^{1-1/p} + (t-\alpha)^{1-1/p}] = 2^{1/p} t^{-1/p}.
\end{aligned}$$

So $\|f_1 + g\|_{L(p,\infty)} \leqslant 2^{1/p}$.

If g is positive, $|l(f_1 + g)| = l(f_1) + l(g) \leqslant 2^{1/p} \|l\|$.

So $\quad |l(g)| \leqslant 2^{1/p} \|l\| - l(f_1) \leqslant \|l\| (2^{1/p} - 1 + \epsilon)$,

and $\quad \|l(\chi_{X\setminus E_1} \cdot)\| = \sup_{0\neq f\in L(p,\infty)} |l(\chi_{X\setminus E_1} f)| / \|f\|_{L(p,\infty)}$

$$\leqslant \|l\| (2^{1/p} - 1 + \epsilon).$$

We repeat the same argument, choosing a function f_2 supported on $X\setminus E_1$ with $\|f_2\|_{L(p,\infty)} \leqslant 1$ such that $|l(f_2)| > \|l(\chi_{X\setminus E_1} \cdot)\|(1-\epsilon)$. Let E_2 be the support of f_2. It follows that

$$\|l(\chi_{X\setminus E_1\setminus E_2} \cdot)\| \leqslant (2^{1/p} - 1 + \epsilon)^2 \, \|l\|.$$

By further repetitions of the argument we can construct a sequence of σ-finite sets $(E_n)_{n=1}^{\infty}$ such that

$$\| l(\chi_{X \setminus E_1 \setminus E_2 \cdots \setminus E_{n-1} \setminus E_n} \cdot) \| \leqslant (2^{1/p} - 1 + \epsilon)^n \; \| l \|.$$

If the initial choice of ϵ was such that $\epsilon < 2 - 2^{1/p}$ then $E = \bigcup_{n=1}^{\infty} E_n$ is a σ-finite set supporting l.

LEMMA 2. – *If g is a measurable function, such that for every simple function f,*

$$\left| \int_X f g \, d\mu \right| \leqslant K \; \|f\|_{L(p,\infty)} \qquad (*)$$

where K is a constant independent of f, then $g \in L(p', 1)$ where $1/p + 1/p' = 1$.

Proof. – It is evident that $| \int f(\mathrm{Re}(g))\, d\mu | \leqslant K \; \|f\|_{L(p,\infty)}$ for all real valued simple functions f, and a similar inequality holds with $\mathrm{Im}(g)$. Therefore it suffices to consider real valued functions g and f.

In the case $X = (0, \infty)$ with Lebesgue measure, if g is a positive decreasing function we have only to choose a sequence (f_n) of simple functions such that $f_n(t) \uparrow t^{-1/p} = t^{1/p'-1}$ a.e. and we have immediately that $\int_0^\infty t^{-1/p} \, g^*(t) \, dt \leqslant K$, as required. This argument adapts to give a proof in general, with the technical help of a C.M.F. carrying $|g(x)|$. But before the existence of such a C.M.F. can be assumed, we must verify that $g^*(t) < \infty$ and $g_*(t) < \infty$, for all $t > 0$.

For each set E with $\mu(E) = t$,

$$\int_E |g| \, d\mu = \int sgn(g) \, \chi_E \, g \, d\mu \leqslant K \; \|sgn(g) \, \chi_E \|_{L(p,\infty)} \text{ by } (*)$$
$$\leqslant K \, t^{1/p}.$$

From this it follows that $t^{1/p'} g^{**}(t) \leqslant K$ for all t, and so $g \in L(p', \infty)$ and, in particular, $g^*(t)$ and $g_*(t)$ are finite for all positive t.

Now let $(A_t)_{0 \leqslant t \leqslant \mu(X)}$ be a C.M.F. carrying $|g|$, and let

$$s(x) = \sum_{k=-\infty}^{\infty} 2^{-k/p} \, \chi_{A_{2^{k+1}} \setminus A_{2^k}}(x).$$

Then
$$s^*(t) = \sum_{k=-\infty}^{\infty} 2^{-k/p} \chi_{[2^k, 2^{k+1})}(t)$$

and satisfies the inequality $t^{-1/p} \leqslant s^*(t) \leqslant 2^{1/p}\, t^{-1/p}$ for all positive t. One can quickly see that (A_t) carries both $s(x)$ and $s(x)\,|g(x)|$.

$$s^L(t)\, g^L(t) = \operatorname*{ess\,inf}_{x \in A_t} s(x) \cdot \operatorname*{ess\,inf}_{x \in A_t} |g(x)| \leqslant \operatorname*{ess\,inf}_{x \in A_t} s(x)\,|g(x)| = (sg)^L(t).$$

For each positive integer N, let
$$s_N(x) = \sum_{k=-N}^{N} 2^{-k/p} \chi_{A_{2^{k+1}} \setminus A_{2^k}}(x).$$

$$\begin{aligned}\|g\|_{L(p',1)} &\leqslant C \int_0^\infty t^{1/p'} g^*(t)\, dt/t = C \int_0^\infty t^{-1/p} g^*(t)\, dt \\ &\leqslant C \int_0^\infty s^*(t)\, g^*(t)\, dt \leqslant C \int_0^\infty (sg)^L(t)\, dt \\ &= C \int_X s(x)\,|g(x)|\, d\mu(x) = \lim_{N\to\infty} C \int_X s_N(x)\,|g(x)|\, d\mu(x) \\ &\leqslant CK \limsup_{N\to\infty} \|sgn(g) \,.\, s_N\|_{L(p,\infty)} \text{ by } (*) \\ &< \infty.\end{aligned}$$

Thus $g \in L(p', 1)$ and the lemma is proved.

THEOREM. — *Let* N_0 *and* N_∞ *be seminorms on* $L(p, \infty)$ *defined by* :
$$N_0(f) = \limsup_{t\to 0} t^{1/p} f^{**}(t)$$
$$N_\infty(f) = \limsup_{t\to\infty} t^{1/p} f^{**}(t)\,.$$

Let S_0 *be the subspace of* $L(p, \infty)^*$ *of functionals* l *satisfying* $|l(f)| \leqslant C\; N_0(f)$ *for some constant* C, *and analogously let* S_∞ *be the subspace of functionals "subordinate" to* $N_\infty(\cdot)$. *Then*
$$L(p, \infty)^* = L(p', 1) \oplus S_0 \oplus S_\infty\,.$$

Proof. — Given $l \in L(p, \infty)^*$, let E be a σ-finite set supporting l. Define the set function $\nu(F)$ for all sets $F \in \Sigma$, by
$$\nu(F) = l(\chi_F) = l(\chi_{E\cap F}).$$

One can readily verify that ν is σ-additive and absolutely continuous with respect to the σ-finite measure μ_E.

$$(\mu_E(F) = \mu(E \cap F)).$$

Thus ν has a locally integrable Radon-Nikodym derivative g, which vanishes off E, and for all simple functions f, $l(f) = \int_X f\, g\, d\mu$.

By Lemma 2 $g \in L(p', 1)$. Furthermore, in view of the generalized Hölder inequality $|\int fg\, d\mu| \leqslant \|f\|^*_{p,\infty} \cdot \|g\|^*_{p',1}$ of which a proof is given in [6], the functional l_g, $l_g(f) = \int_X fg\, d\mu$, belongs to $L(p, \infty)^*$. In fact each $l \in L(p, \infty)^*$ uniquely defines such an l_g. Let l_s be the "singular" part of l, $l_s = l - l_g$. We have shown that

$$L(p, \infty)^* = L(p', 1) \oplus S,$$

where S is the space of "singular" functionals on $L(p, \infty)$, which vanish on the closure in $L(p, \infty)$ of the simple functions. In particular, singular functionals vanish on all functions which are bounded and supported on sets of finite measure. To show that $S = S_0 \oplus S_\infty$, write any $l_s \in S$ in the form $l_s = l_0 + l_\infty$, where $l_0(f) = l_s(f\chi_{\{x \mid |f(x)| > \alpha\}})$, and $l_\infty(f) = l_s\ (f\chi_{\{x \mid |f(x)| \leqslant \alpha\}})$ for some number $\alpha > 0$. To show that the definitions of l_0 and l_∞ are independent of the choice of α, and that l_0 and l_∞ are linear, one has only to observe that the following four functions are bounded and supported on sets of finite measure, and thus annihilated by l_s :

$$f\chi_{\{x \mid |f(x)| > \alpha\}} - f\chi_{\{x \mid |f(x)| > \beta\}},$$

$$f\chi_{\{x \mid |f(x)| \leqslant \alpha\}} - f\chi_{\{x \mid |f(x)| \leqslant \beta\}},$$

$$(f+g)\chi_{\{x \mid |f(x)+g(x)| > \alpha\}} - f\chi_{\{x \mid |f(x)| > \alpha\}} - g\chi_{\{x \mid |g(x)| > \alpha\}},$$

$$(f+g)\chi_{\{x \mid |f(x)+g(x)| \leqslant \alpha\}} - f\chi_{\{x \mid |f(x)| \leqslant \alpha\}} - g\chi_{\{x \mid |g(x)| \leqslant \alpha\}},$$

for all f, $g \in L(p, \infty)$ and all positive numbers α, β. From this it also follows that :

$$l_0(f) = \lim_{\alpha \to \infty} l_s(f\chi_{\{x \mid |f(x)| > \alpha\}})$$

and

$$l_\infty(f) = \lim_{\alpha \to 0} l_s(f\chi_{\{x \mid |f(x)| \leqslant \alpha\}}).$$

Consequently $l_0 \in S_0$ since $l_0(f) \leqslant \|l\| \sup_{0<t<\epsilon} t^{1/p} f^{**}(t)$ for every $\epsilon > 0$. To see that $l_\infty \in S_\infty$, note that $l_\infty(f) = l_\infty(f_\alpha)$ for all $\alpha > 0$

where
$$f_\alpha(x) = f(x) \quad \text{if} \quad |f(x)| \leqslant \alpha$$
$$= \alpha \quad \text{if} \quad |f(x)| > \alpha$$

and
$$\sup_{t>0} \ t^{1/p} f_\alpha^{**}(t) \leqslant \sup_{t \geqslant f_*(\alpha)} t^{1/p} f^{**}(t).$$

Either $f_*(\alpha) \to \infty$ as $\alpha \to 0$, or $f_*(0) < \infty$ which means that $l_s(f_\alpha) = 0$ for all $\alpha > 0$. This completes the proof of the theorem.

Remark 1. – From the above it is clear that $l \in S_0$ annihilates all bounded functions and that each $l \in S_\infty$ annihilates all functions which vanish on the complement of a set of finite measure. Of course if $\mu(X) < \infty$, $S_\infty = \{0\}$.

Remark 2. – The results of this section remain true if X has atoms.

4. Representation of the singular functionals.

Let the measurable subsets of X be partitioned into equivalence classes by the relation $A \sim B$ iff $\mu(A \Delta B) = 0$. Henceforth we will use latin' capitals, A, B, C... to denote such equivalence classes as well as sets, and $\mu(A)$ to denote the μ measure of any and every set in the class A. We shall not always be very pedantic in distinguishing between a set and its equivalence class, but nor is there need to be for our purposes.

DEFINITIONS. –

i) *Let* Σ_F *be the metric space of "sets", (that is, equivalence classes of sets)* A, *for which* $0 < \mu(A) < \infty$, *equipped with the metric* $d(A, B) = \mu(A \Delta B)$.

ii) *Let* Φ *be the unit ball of* $L^\infty(X, \Sigma, \mu)$, *and let* Φ_1 *be the subset of functions* $\phi \in \Phi$, *for which* $|\phi(x)| = 1$ *μ-almost everywhere.*

iii) *Let* $\Omega = \Sigma_F \times \Phi$. Ω *is a metric space with the metric,* $d((A, \phi), (B, \psi)) = \mu(A \Delta B) + \|\phi - \psi\|_{L^\infty}$.

iv) *Let* $C(\Omega)$ *be the Banach space of all bounded continuous complex valued (or real valued) functions on* Ω *equipped with the supremum norm.*

Each $f \in L(p, \infty)$ defines a function in $C(\Omega)$, which we may also write unambigously as f. For each $(A, \phi) \in \Omega$,

$$f(A, \phi) = (\mu(A))^{1/p-1} \int_A f\phi \, d\mu.$$

The linear map of $L(p, \infty)$ into $C(\Omega)$ given by $f(x) \to f(A, \phi)$, is readily seen to be an isometry onto a closed subspace of $C(\Omega)$:

$$\sup_{\Omega} |f(A, \phi)| = \sup_{A \in \Sigma_F} (\mu(A))^{1/p-1} \int_A |f| \, d\mu = \sup_{t>0} t^{1/p} f^{**}(t).$$

Since a metric space is completely regular, the Stone-Čech compactification $\beta\Omega$ of Ω exists, and $C(\Omega)$ is isometric to $C(\beta\Omega)$. From here on the notation (A, ϕ) will be used in an extended sense to denote points of $\beta\Omega$. We can now remark that each continuous linear functional l on $L(p, \infty)$ may be represented, via a (non-unique) Hahn Banach extension, as a bounded regular Borel measure λ on $\beta\Omega$.

$$l(f) = \int_{\beta\Omega} f(A, \phi) \, d\lambda(A, \phi).$$

This representation as it stands is very unwieldy as compared to the straightforward and canonical representation obtained for at least the $L(p', 1)$ part of $L(p, \infty)^*$. Nevertheless it is the starting point for a characterisation of S_0 and S_∞. Given any l in S_0 or S_∞, we shall see that the corresponding measure λ on $\beta\Omega$ can be taken to be concentrated on a very much simpler subset of $\beta\Omega$, specifically the closure, with respect to a suitable topology, of the "curve" $\{(A_t, \phi) \mid 0 < t < \mu(X)\}$, where ϕ is a fixed element of Φ, and $(A_t)_{0 \leq t \leq \mu(X)}$ is a C.M.F. Each element l in S_0 or S_∞ can thus be represented in the form :

$$l(f) = \int_I f(A_t, \phi) \, d\nu(t) \tag{1}$$

where ν is a finitely additive bounded set function on the Borel subsets of the open interval $I = (0, \mu(X))$, such that for all $g \in L^\infty(I)$

$$\int_I g(t) \, d\nu(t) \leq \text{const.} \lim_{n\to\infty} \operatorname*{ess\,sup}_{t \in (0, 1/n)} |g(t)| \tag{2}$$

if $l \in S_0$, and alternatively

$$\int_I g(t)\, d\nu(t) \leqslant \text{const.} \lim_{n\to\infty} \operatorname*{ess\,sup}_{t\in(n,\infty)} |g(t)| \tag{3}$$

if $l \in S_\infty$.

We shall say "ν concentrates at zero" to indicate that a finitely additive set function ν satisfies (2), and similarly (3) will be indicated by saying "ν concentrates at infinity". Such "concentrating" set functions must exist of course, as elements of $L^\infty(I)^*$ dominated by the semi-norms on the right hand sides of (2) and (3).

It is obvious that an l of form (1) is in S_0 or S_∞. A much longer argument will be needed to show that each element of S_0 and of S_∞ can be thus represented. We shall show first (Theorem 1) that the representation is valid for the functionals in a norm dense subset of S_0 and of S_∞. Thus every singular functional l is the limit of a sequence $(l_n)_{n=1}^\infty$

$$l_n(f) = \int_I f(A_{n,t}, \phi_n)\, d\nu_n(t).$$

We then show that the three sequences (ν_n), $(A_{n,t})$ and (ϕ_n) have subsequences which converge to limits ν, (A_t) and φ in a sufficiently strong manner to imply that

$$l(f) = \int_I f(A_t, \phi)\, d\nu.$$

This is done in three stages ; Theorem 2 (convergence of (ν_n)), Theorem 3 (convergence of $(A_{n,t})$) and Theorem 4 (convergence of (ϕ_n)). It will be seen that ν can always be taken to be a positive set function with $\nu(I) = \|l\|$.

THEOREM 1. – *Let* $l \in S_\alpha$, $\alpha = 0$ or ∞, *such that for some* $f \in L(p, \infty)$, $l(f) = \|l\| \; \|f\|_{L(p,\infty)}$.

Then there exists a positive finitely additive set function $\nu \in L^\infty(I)^*$ *which concentrates at* α, *with* $\|\nu\| = \nu(I) = \|l\|$, *such that*

$$l(g) = \int_I g(F_t, \phi)\, d\nu(t) \qquad \text{for all} \qquad g \in L(p, \infty),$$

where $(F_t)_{t\in I}$ *is a C.M.F. carrying* $|f|$ *and* $\phi(x) = sgn(\overline{f(x)}) \in \Phi$.

Proof. – Let us first deal with the case $\alpha = 0$. Let l be a functional in S_0 such that for some f in $L(p, \infty)$, $l(f) = \|l\| \, \|f\|_{L(p,\infty)}$. It is convenient to suppose that $\|l\| = \|f\|_{L(p,\infty)} = 1$. As shown above, we may write

$$l(g) = \int_{\beta\Omega} g(A, \phi) \, d\lambda(A, \phi) \qquad \text{for all} \qquad g \in L(p, \infty)$$

where λ is a regular Borel measure on $\beta\Omega$ with total variation 1. Since $l(f) = 1$, it follows that any Borel subset of $\beta\Omega$ disjoint from $\{(A, \phi) \mid |f(A, \phi)| = 1\}$ must have zero λ-measure.

Let $(F_t)_{t \in I}$ be a C.M.F. carrying the positive function $|f(x)|$. Then the function $f_n = f \cdot \chi_{F_{1/n}}$ differs from f by a bounded function, and so $l(f_n) = 1$ for all n. Thus any Borel subset of $\beta\Omega$ disjoint from $\Lambda = \bigcap_{n=1}^{\infty} \{(A, \phi) \mid |f_n(A, \phi)| = 1\}$ has zero λ-measure.

Since Ω is dense in $\beta\Omega$, each "point" (B, ψ) in Λ is the limit in $\beta\Omega$ of a net $(B_\gamma, \psi_\gamma)_{\gamma \in \Gamma}$ of elements (B_γ, ψ_γ) in Ω. Clearly $\lim_{\gamma \in \Gamma} |f_n(B_\gamma, \psi_\gamma)| = 1$ for each integer n. We now show furthermore that $\lim_{\gamma \in \Gamma} \mu(B_\gamma) = 0$.

$$|f_n(B_\gamma, \psi_\gamma)| \leqslant (\mu(B_\gamma))^{1/p-1} \int_{B_\gamma} |f_n| \, d\mu \leqslant (\mu(B_\gamma))^{1/p-1} \int_{F_{1/n}} |f| \, d\mu$$

$$\leqslant (\mu(B_\gamma)/\mu(F_{1/n}))^{1/p-1} \, \|f\|_{L(p,\infty)} = (n\mu(B_\gamma))^{1/p-1} \, \|f\|_{L(p,\infty)}.$$

So $\quad 1 = \liminf_{\gamma \in \Gamma} |f_n(B_\gamma, \psi_\gamma)| \leqslant \|f\|_{L(p,\infty)} / (n \limsup_{\gamma \in \Gamma} \mu(B_\gamma))^{1-1/p}$

for all n. This is a contradiction if $\limsup_{\gamma \in \Gamma} \mu(B_\gamma) > 0$. Henceforth we shall find it convenient to use the notation $|\gamma| = \mu(B_\gamma)$ for each $\gamma \in \Gamma$.

By definition of the net $(B_\gamma, \psi_\gamma)_{\gamma \in \Gamma}$,

$$g(B, \psi) = \lim_{\gamma \in \Gamma} g(B_\gamma, \psi_\gamma) \qquad \text{for all} \qquad g \in L(p, \infty).$$

We now wish to deduce that :

$$g(B, \psi) = \lim_{\gamma \in \Gamma} g(F_\gamma, \phi) \qquad \text{for all} \qquad g \in L(p, \infty), \qquad (4)$$

where each F_γ is a member of the C.M.F. $(F_t)_{t\in I}$ with

$$0 < \mu(F_\gamma) \leqslant \mu(B_\gamma),$$

and $\phi \in \Phi$ is the function $\phi(x) = f(B, \psi)\ sgn\ (\overline{f(x)})$. But before verifying (4), let us show how it enables the proof of the theorem to be completed for the case $\alpha = 0$.

For all $g \in L(p, \infty)$,

$$|l(g)| \leqslant (\text{var}\ \lambda)\ \sup\{|g(B, \psi)| \mid (B, \psi) \in \Lambda\}$$

$$\leqslant \limsup_{t\to 0} |g(F_t, sgn(\overline{f(x)}))|$$

by (4) and the fact that $\lim_{\gamma\in\Gamma} \mu(B_\gamma) = 0$ for each net $(B_\gamma, \psi_\gamma)_{\gamma\in\Gamma}$ in Ω tending to a point (B, ψ) in Λ. $l(g)$ is thus completely determined by the function $\tilde{g}$ on $(0, \mu(X))$, $\tilde{g}(t) = g(F_t, sgn\ (\overline{f(x)}))$, and, via a Hahn Banach extension, we have l in the form (1), where ν is a bounded finitely additive measure with total variation $v(\nu, I) = 1$. Clearly ν concentrates at zero.

We next show that ν is positive. For every Borel subset E of I, denote by $v(\nu, E)$, the total variation of ν on E (as in [5] Chapter III).

Let $\tilde{f}(t) = f(F_t, sgn\,(\overline{f(x)}) = t^{1/p-1} \int_{F_t} |f|\, d\mu$, for all $t \in I$.

Then $\quad 1 = l(f) = \int_I \tilde{f}(t)\, d\nu(t)$

$$= \text{Re}\left[\int_E \tilde{f}(t)\, d\nu(t)\right] + \text{Re}\left[\int_{I\setminus E} \tilde{f}(t)\, d\nu(t)\right]$$

$$\leqslant |\nu(E)| \cos\theta + (1 - v(\nu, E))$$

where $\quad \nu(E) = |\nu(E)|\, e^{i\theta}$

$$\leqslant 1 + |\nu(E)|\ (\cos\theta - 1)$$

so θ must be zero, and $\nu(E) \geqslant 0$ for each E.

It remains to prove (4). This will be done in four steps.

Step 1 : $\lim_{\gamma\in\Gamma} (\mu(F_{|\gamma|} \,\Delta\, B_\gamma)/|\gamma|) = 0.$

Proof. – Since $\mu(F_{|\gamma|}) = |\gamma| = \mu(B_\gamma)$, we have that

$$\mu(B_\gamma \cap F_{|\gamma|}) = |\gamma| - \mu(B_\gamma \backslash F_{|\gamma|}) = |\gamma| - \mu(F_{|\gamma|} \backslash B_\gamma)$$

and so, $$\mu(B_\gamma \backslash F_{|\gamma|}) = \mu(F_{|\gamma|} \backslash B_\gamma) = \frac{1}{2}\, \mu(F_{|\gamma|}\ \Delta\ B_\gamma).$$

Let us denote $\epsilon(\gamma) = \mu(B_\gamma \backslash F_{|\gamma|})$. Recalling Remark 1 of Section 2, and the fact that (F_t) carries $|f|$, we note that $\int_{B_\gamma} |f|\, d\mu \leqslant \int_{F_{|\gamma|}} |f|\, d\mu$. Therefore

$$\int_{B_\gamma} |f|\, d\mu \leqslant \frac{1}{2} \left(\int_{B_\gamma} |f|\, d\mu + \int_{F_{|\gamma|}} |f|\, d\mu \right)$$

$$= \frac{1}{2} \left(\int_{B_\gamma \cap F_{|\gamma|}} |f|\, d\mu + \int_{B_\gamma \cup F_{|\gamma|}} |f|\, d\mu \right)$$

$$\leqslant \frac{1}{2}\, \|f\|_{L(p,\infty)}\, [(\mu(B_\gamma \cap F_{|\gamma|}))^{1-1/p} + (\mu(B_\gamma \cup F_{|\gamma|}))^{1-1/p}]$$

$$= \frac{1}{2}\, [(|\gamma| - \epsilon(\gamma))^{1-1/p} + (|\gamma| + \epsilon(\gamma))^{1-1/p}].$$

Consequently,

$$|\gamma|^{1/p-1} \int_{B_\gamma} |f|\, d\mu \leqslant \frac{1}{2}\, [(1 - \epsilon(\gamma)/|\gamma|)^{1-1/p} + (1 + \epsilon(\gamma)/|\gamma|)^{1-1/p}].$$

For $s \in [0\,,\,1]$, let $w(s)$ be the function,

$$w(s) = \frac{1}{2}\, [(1-s)^{1-1/p} + (1+s)^{1-1/p}].$$

Then $$\liminf_{\gamma\in\Gamma} w(\epsilon(\gamma)/|\gamma|) \geqslant \liminf_{\gamma\in\Gamma} |\gamma|^{1/p-1} \int_{B_\gamma} |f|\, d\mu$$

$$\geqslant \lim_{\gamma\in\Gamma} |\gamma|^{1/p-1} \left| \int_{B_\gamma} f\, \psi_\gamma\, d\mu \right| = |f(B\,,\,\psi)| = 1.$$

But the function $w(s)$ is strictly decreasing for $s \in (0\,,\,1)$ and $w(0) = 1$. Therefore $\lim_{\gamma\in\Gamma} \epsilon(\gamma)/|\gamma| = 0$ and step 1 is proved. The above chain of inequalities also implies that :

$$\lim_{\gamma\in\Gamma} |\gamma|^{1/p-1} \int_{F_{|\gamma|}} |f|\, d\mu = 1. \tag{5}$$

Step 2. – $\lim_{\gamma\in\Gamma} g(B_\gamma, \psi_\gamma) = \lim_{\gamma\in\Gamma} g(F_{|\gamma|}, \psi_\gamma)$ for all $g \in L(p,\infty)$.

Proof. – $|g(B_\gamma, \psi_\gamma) - g(F_{|\gamma|}, \psi_\gamma)|$

$$\leqslant |\gamma|^{1/p-1} \int_{B_\gamma \Delta F_{|\gamma|}} |g\,\psi_\gamma|\, d\mu \leqslant [\mu(B_\gamma \,\Delta\, F_{|\gamma|})/|\gamma|]^{1-1/p}\, \|g\|_{L(p,\infty)}.$$

Therefore, using step 1, $\lim_{\gamma\in\Gamma} |g(B_\gamma, \psi_\gamma) - g(F_{|\gamma|}, \psi_\gamma)| = 0$.

Step 3. – For each $\gamma \in \Gamma$ there exists a positive number $\eta(\gamma) \leqslant |\gamma|$, such that

i) $$\lim_{\gamma\in\Gamma} g(F_{|\gamma|}, \psi_\gamma) = \lim_{\gamma\in\Gamma} g(F_{\eta(\gamma)}, \psi_\gamma)$$

ii) $$\limsup_{\gamma\in\Gamma} (\eta(\gamma))^{1/p}\, f^L(\eta(\gamma)) \geqslant (1 - 1/p)/2.$$

Proof. – In view of (5) there exists $\gamma_0 \in \Gamma$, such that $\gamma \geqslant \gamma_0$, implies that $|\gamma|^{1/p-1} \int_{F_{|\gamma|}} |f|\, d\mu > \frac{1}{2}$.

Let $\eta(\gamma) = \sup\{s \,|\, 0 < s \leqslant |\gamma|,\ s^{1/p} f^L(s) \geqslant (1 - 1/p)/2\}$ for each $\gamma \geqslant \gamma_0$. The set over which the supremum is taken is non-empty and the supremum is positive, for if not, $s^{1/p} f^L(s) < (1 - 1/p)/2$ for $s \in (0, |\gamma|]$ and by Remark 2.2,

$$|\gamma|^{1/p-1} \int_{F_{|\gamma|}} |f|\, d\mu = |\gamma|^{1/p-1} \int_0^{|\gamma|} f^L(s)\, ds \leqslant \frac{1}{2},$$

which is a contradiction. By the left continuity of $f^L(s)$,

$$(\eta(\gamma))^{1/p} f^L(\eta(\gamma)) \geqslant (1 - 1/p)/2 \qquad \text{for all} \qquad \gamma \geqslant \gamma_0,$$

and part ii) of step 3 is proved.

For each $\gamma \geqslant \gamma_0$,

$$\begin{aligned}
|\gamma|^{1/p-1} \int_{F_{|\gamma|}} |f|\, d\mu &= |\gamma|^{1/p-1} \left(\int_0^{\eta(\gamma)} f^L(s)\, ds + \int_{\eta(\gamma)}^{|\gamma|} f^L(s)\, ds \right) \\
&\leqslant |\gamma|^{1/p-1} \left(\int_{F_{\eta(\gamma)}} |f|\, d\mu + (1 - 1/p)/2 \int_{\eta(\gamma)}^{|\gamma|} s^{-1/p}\, ds \right) \\
&\leqslant |\gamma|^{1/p-1} \left[(\eta(\gamma))^{1-1/p} + (|\gamma|^{1-1/p} - (\eta(\gamma))^{1-1/p})/2 \right] \\
&= [1 + (\eta(\gamma)/|\gamma|)^{1-1/p}]/2 \leqslant 1.
\end{aligned}$$

In view of (5), it follows that

$$\lim_{\gamma\in\Gamma} (\eta(\gamma)/|\gamma|) = 1. \tag{6}$$

Now to show part i), let $g \in L(p, \infty)$.

$$|g(F_{|\gamma|}, \psi_\gamma) - g(F_{\eta(\gamma)}, \psi_\gamma)|$$
$$= \left| |\gamma|^{1/p-1} \int_{F_{|\gamma|}} g\psi_\gamma d\mu - (\eta(\gamma))^{1/p-1} \int_{F_{\eta(\gamma)}} g\psi_\gamma d\mu \right|$$
$$\leqslant \left| |\gamma|^{1/p-1} - (\eta(\gamma))^{1/p-1} \right| \int_{F_{|\gamma|}} |g|\, d\mu +$$
$$+ (\eta(\gamma))^{1/p-1} \int_{F_{|\gamma|}\setminus F_{\eta(\gamma)}} |g|\, d\mu$$
$$\leqslant ((\eta(\gamma)/|\gamma|)^{1/p-1} - 1) \|g\|_{L(p,\infty)} +$$
$$+ (\eta(\gamma))^{1/p-1}(|\gamma| - \eta(\gamma))^{1-1/p} \|g\|_{L(p,\infty)}$$
$$= \|g\|_{L(p,\infty)} ((\eta(\gamma)/|\gamma|)^{1/p-1} - 1 + (|\gamma|/\eta(\gamma) - 1)^{1-1/p}).$$

By (6) this has generalized limit zero with respect to the directed set Γ, and so part i) is proved.

Step 4. – $\lim_{\gamma\in\Gamma} g(F_{\eta(\gamma)}, \psi_\gamma) = \lim_{\gamma\in\Gamma} g(F_{\eta(\gamma)}, \phi)$ for all $g \in L(p, \infty)$ where $\phi(x) = f(B, \psi)\, sgn(\overline{f(x)})$.

Proof. – On the set $F_{\eta(\gamma)}$, for each $\gamma \geqslant \gamma_0$, f is non-zero and therefore $|\phi| = 1$. For any number $r > 1$ let

$$H(\gamma) = (\eta(\gamma))^{1/p-1} f^L(\eta(\gamma)) \int_{F_{\eta(\gamma)}} |\phi - \psi_\gamma|^r\, d\mu$$

$$H(\gamma) \leqslant 2^{r-1}(\eta(\gamma))^{1/p-1} \int_{F_{\eta(\gamma)}} |f|\, |\phi - \psi_\gamma|\, d\mu, \text{ since } \|\phi - \psi_\gamma\|_{L^\infty} \leqslant 2,$$
$$\leqslant 2^{r-1}(\eta(\gamma))^{1/p-1} \int_{F_{\eta(\gamma)}} |f|\, |1 - \psi_\gamma\overline{\phi}|\, d\mu$$
$$\leqslant 2^{r-1}(\eta(\gamma))^{1/p-1} \int_{F_{\eta(\gamma)}} |f|\, |1 - \mathrm{Re}(\psi_\gamma\overline{\phi})|\, d\mu$$
$$+ 2^{r-1}(\eta(\gamma))^{1/p-1} \int_{F_{\eta(\gamma)}} |f|\, |\mathrm{Im}(\psi_\gamma\overline{\phi})|\, d\mu$$
$$= h_1(\gamma) + h_2(\gamma).$$

We shall show that both these terms have generalized limit zero with respect to Γ.

$$|1 - \mathrm{Re}(\psi_\gamma \overline{\phi})| = 1 - \mathrm{Re}(\psi_\gamma \overline{\phi})$$

and $$|f| \, \mathrm{Re}(\psi_\gamma \overline{\phi}) = \mathrm{Re}(\overline{f(\mathrm{B}, \psi)} \, f \psi_\gamma) \quad \text{on} \quad \mathrm{F}_{\eta(\gamma)} .$$

In view of steps 1, 2, 3,

$$\lim_{\gamma \in \Gamma} (\eta(\gamma))^{1/p-1} \int_{\mathrm{F}_{\eta(\gamma)}} f \psi_\gamma d\mu = f(\mathrm{B}, \psi).$$

So $$\lim_{\gamma \in \Gamma} (\eta(\gamma))^{1/p-1} \int_{\mathrm{F}_{\eta(\gamma)}} \mathrm{Re}(\overline{f(\mathrm{B}, \psi)} \, f \psi_\gamma) \, d\mu = 1.$$

Also, by much the same argument as in the proof of equation (5),

$$\lim_{\gamma \in \Gamma} (\eta(\gamma))^{1/p-1} \int_{\mathrm{F}_{\eta(\gamma)}} |f| \, d\mu = 1,$$

and consequently $\lim_{\gamma \in \Gamma} h_1(\gamma) = 0$. But further, since

$$\mathrm{Re}(\overline{f(\mathrm{B}, \psi)} \, f \psi_\gamma) \leqslant |\mathrm{Re}(\overline{f(\mathrm{B}, \psi)} \, f \, \psi_\gamma)| \leqslant |f| ,$$

$$\lim_{\gamma \in \Gamma} (\eta(\gamma))^{1/p-1} \int_{\mathrm{F}_{\eta(\gamma)}} |\mathrm{Re}(\overline{f(\mathrm{B}, \psi)} \, f \psi_\gamma)| \, d\mu = 1 .$$

For the second term,

$$|\mathrm{Im}(\psi_\gamma \overline{\phi})|^2 = |\psi_\gamma \overline{\phi}|^2 - |\mathrm{Re}(\psi_\gamma \overline{\phi})|^2 \leqslant 2(1 - |\mathrm{Re}(\psi_\gamma \overline{\phi})|) .$$

$$(\eta(\gamma))^{1/p-1} \int_{\mathrm{F}_{\eta(\gamma)}} |f| \, |\mathrm{Im}(\psi_\gamma \overline{\phi})| \, d\mu$$

$$\leqslant (\eta(\gamma))^{1/p-1} \left(\int_{\mathrm{F}_{\eta(\gamma)}} |f| \, d\mu \right)^{1/2} \left(\int_{\mathrm{F}_{\eta(\gamma)}} |f| \, |\mathrm{Im}(\psi_\gamma \overline{\phi})|^2 d\mu \right)^{1/2}$$

$$\leqslant \left[2(\eta(\gamma))^{1/p-1} \int_{\mathrm{F}_{\eta(\gamma)}} |f| \, (1 - |\mathrm{Re}(\psi_\gamma \overline{\phi})|) \, d\mu \right]^{1/2}$$

which has generalized limit zero. Therefore $\lim_{\gamma \in \Gamma} \mathrm{H}(\gamma) = 0$. For each $\gamma \geqslant \gamma_0$, and each $g \in \mathrm{L}(p, \infty)$

$$|g(F_{\eta(\gamma)}, \psi_\gamma) - g(F_{\eta(\gamma)}, \phi)|$$

$$\leqslant [\eta(\gamma)]^{1/p-1} \int_{F_{\eta(\gamma)}} |g|\, |\phi - \psi_\gamma|\, d\mu$$

$$\leqslant [\eta(\gamma)]^{1/p-1} \left[\int_{F_{\eta(\gamma)}} |g|^{r'}\, d\mu\right]^{1/r'} \left[\int_{F_{\eta(\gamma)}} |\phi - \psi_\gamma|^{r}\, d\mu\right]^{1/r}$$

$$\leqslant [\eta(\gamma)]^{1/p-1} \left[\int_0^{\eta(\gamma)} (g^L(s))^{r'}\, ds\right]^{1/r'} [H(\gamma)\,\eta(\gamma)/(\eta(\gamma))^{1/p}\, f^L(\eta(\gamma))]^{1/r}$$

$$\leqslant [\eta(\gamma)]^{1/p-1/r'} \left[\int_0^{\eta(\gamma)} (\text{const.}) s^{-r'/p}\, ds\right]^{1/r'} [2\,H(\gamma)/(1 - 1/p)]^{1/r}$$

using ii) of step 3. If we choose r so that $r' < p$, then the above expression has generalized limit zero, and we have proved step 4, demonstrated the validity of equation (4), and thus completed the proof of the theorem for $\alpha = 0$.

For $\alpha = \infty$ the proof is almost the same. Steps 1 to 4 are valid irrespective of whether $\lim_{\gamma\in\Gamma} |\gamma| = 0$ or not, so all that is needed is to show that in this case $\lim_{\gamma\in\Gamma} \eta(\gamma) = \infty$. Since $l \in S_\infty$, $l(f_n) = 1$ for each n, where $f_n = f \cdot \chi_{X\setminus F_n}$.

$$|f_n(F_{|\gamma|}, \psi_\gamma)| \leqslant |\gamma|^{1/p-1} \int_{F_{|\gamma|}\setminus F_n} |f|\, d\mu$$

$$\leqslant (\max[|\gamma| - n, 0]/|\gamma|)^{1-1/p}.$$

By step 2, $$\liminf_{\gamma\in\Gamma} |f_n(F_{|\gamma|}, \psi_\gamma)| = 1,$$

and so $$\lim_{\gamma\in\Gamma} |\gamma| = \infty \cdot \text{ Furthermore } \lim_{\gamma\in\Gamma} \eta(\gamma) = \infty$$

also, since $$\lim_{\gamma\in\Gamma} (\eta(\gamma)/|\gamma|) = 1.$$

Remark 1. – In proving the theorems which follow, it will be convenient to assume that the function ϕ appearing in the conclusion of Theorem 1 is in Φ_1, that is, $|\phi(x)| = 1$ for almost all $x \in X$. To justify such an assumption we have only to redefine ϕ on the set E where $f(x) = 0$, and to check that the functional $l(g) = \int_I g(F_t, \phi)\, d\nu$

is unchanged by this redefinition. Certainly l is unchanged by any alterations to ϕ on the set $X \setminus \bigcup_{t \in I} F_t$. If $\mu(E \cap F_t) > 0$ for any t then f vanishes almost everywhere on $X \setminus F_t$. This can only happen if $l \in S_0$, and in this case we can change f on the set

$$\{x \mid |f(x)| \leqslant 1\} \cap \left[\bigcup_{t \in I} F_t \right]$$

without changing the hypotheses or conclusions of Theorem 1. Thus we can arrange that $f(x) \neq 0$ for all $x \in \bigcup_{t \in I} F_t$.

THEOREM 2. – *Let $l \in S_\alpha$, $\alpha = 0$ or ∞. Then l is the strong limit of a sequence of functionals $l_n \in S_\alpha$,*

$$l_n(f) = \int_I f(A_{n,t}, \phi_n)\, d\nu$$

where ν is a positive finitely additive set function on the Borel subsets of I such that $\nu(I) = \|l\|$ and ν concentrates at α, and where for each n, $(A_{n,t})_{t \in I}$ is a C.M.F. and $\phi_n \in \Phi_1$.

Proof. – Bishop and Phelps [1] proved that the continuous linear functionals which attain their norms on the unit ball of a Banach space, are a norm dense subset of the dual space. Consequently, from Theorem 1 and Remark 1, $l = \text{strong} \lim_{n \to \infty} \widetilde{l}_n$, where

$$\widetilde{l}_n(f) = \int_I f(A_{n,t}, \phi_n)\, d\nu_n$$

where each ν_n is a positive finitely additive set function which concentrates at α, with $\nu_n(I) = \|\widetilde{l}_n\|$, and where $\phi_n \in \Phi_1$.

For each n there exists a function f_n in the unit ball of $L(p, \infty)$ for which $\widetilde{l}_n(f_n) = \|\widetilde{l}_n\|$. We may of course assume that $\|\widetilde{l}_n\| = \|l\|$ for all n, and let us again take $\|l\| = 1$ for convenience.

In order to prove the theorem, it will be shown that the sequence (ν_n) tends to a limit ν in a sufficiently strong topology.

For any $g \in L(p, \infty)$ and any C.M.F. $(A_t)_{t \in I}$,

$$g(A_t) = g(A_t, 1) = t^{1/p-1} \int_{A_t} g(x)\, d\mu(x)$$

is a bounded continuous function on I.

Let $C(p, \infty)$ denote the space of all such bounded continuous functions on I obtained as g ranges over all of $L(p, \infty)$, and $(A_t)_{t \in I}$ ranges over all C.M.F.s. $C(p, \infty)$ will be normed by :

$$\|f\|_{C(p,\infty)} = \inf\{\|g\|_{L(p,\infty)} \mid g(A_t) = f(t) \text{ for all } t \in I \text{ and some C.M.F. } (A_t)\}.$$

LEMMA 2A. – *Let* $f(t) \in C(p, \infty)$, *and let* $(A_t)_{t \in I}$ *be an arbitrary C.M.F. Then for any positive* ϵ, *there exists a function* $g \in L(p, \infty)$ *such that* $\sup_{t \in I} |g(A_t) - f(t)| < \epsilon$ *and* $\|g\|_{L(p,\infty)} \leqslant \|f\|_{C(p,\infty)}$.

Proof. – Take any function $h \in L(p, \infty)$ and C.M.F. $(B_t)_{t \in I}$ such that

$$f(t) = t^{1/p-1} \int_{B_t} h \, d\mu.$$

Let $(t(n))_{n=-\infty}^{\infty}$ be an increasing sequence of points in I and let

$$g(x) = \sum_{n=-\infty}^{\infty} \left[(t(n+1) - t(n))^{-1} \int_{B_{t(n+1)} \setminus B_{t(n)}} h \, d\mu \right] \chi_{A_{t(n+1)} \setminus A_{t(n)}}(x).$$

For any measurable set E,

$$\int_E |g| \, d\mu \leqslant \sum_{-\infty}^{\infty} \mu(E \cap A_{t(n+1)} \setminus A_{t(n)}) (t(n+1) - t(n))^{-1} \int_{B_{t(n+1)} \setminus B_{t(n)}} |h| \, d\mu$$

$$\leqslant \sum_{-\infty}^{\infty} \mu(E \cap A_{t(n+1)} \setminus A_{t(n)}) \, \mu(C_n)^{-1} \int_{C_n} |h| \, d\mu$$

where $C_n \subset B_{t(n+1)} \setminus B_{t(n)}$ is the set of measure $\mu(E \cap A_{t(n+1)} \setminus A_{t(n)})$ in a C.M.F. carrying $|h| \, \chi_{B_{t(n+1)} \setminus B_{t(n)}}$. Then $C = \bigcup_{n=1}^{\infty} C_n$ has the same measure as E, and $\int_E |g| \, d\mu \leqslant \int_C |h| \, d\mu$. Consequently

$$\|g\|_{L(p,\infty)} \leqslant \|h\|_{L(p,\infty)}.$$

Also by construction, $g(A_{t(n)}) = f(t(n))$ for all n. $f(t)$ is uniformly continuous on all closed subintervals of I and $t^{1-1/p}g(A_t)$ is the linear interpolation of its values at the points $t(n)$. Thus it is clear that by choosing a sufficiently "dense" sequence $(t(n))$, one can satisfy $\sup_{t\in I} |g(A_t) - f(t)| < \epsilon$, and the lemma is proved.

(We remark that $C(p, \infty)$ is closed under pointwise multiplication of functions, and that the $C(p, \infty)$ norm topology is strictly finer than the supremum norm topology).

Let $M(p, \infty)$ be the dual space of $C(p, \infty)$. Each element ρ of $L^\infty(I)^*$, when restricted to $C(p, \infty)$, defines an element of $M(p, \infty)$, with $\|\rho\|_{M(p,\infty)} \leqslant v(\rho, I)$, since $\|f\|_{L^\infty(I)} \leqslant \|f\|_{C(p,\infty)}$ for all $f \in C(p, \infty)$. In particular each ν_n is in $M(p, \infty)$.

We recall that there exists a function f_n in the unit ball of $L(p, \infty)$ with $\widetilde{l}_n(f_n) = 1$. In other words $g_n(t) = f_n(A_{n,t}, \phi_n)$ is in the unit ball of $C(p, \infty)$, and $\int_I g_n d\nu_n = 1$. It follows that $\|\nu_n\|_{M(p,\infty)} = \nu_n(I) = 1$.

It will be shown that (ν_n) is a Cauchy sequence in the $M(p, \infty)$ norm and consequently it has a strong limit in $M(p, \infty)$. This limit can also be thought of as a finitely additive set function, since the sequence (ν_n) must have a weak star convergent subsequence in $L^\infty(I)^*$, with limit ν and the restriction of ν to $C(p, \infty)$ must be the limit of the sequence (ν_n) in $M(p, \infty)$. It is clear that ν must be a positive set function with $\nu(I) = \|\nu\|_{M(p,\infty)} = 1$. ν concentrates at α since each ν_n concentrates at α.

Before passing to the proof that (ν_n) is Cauchy in $M(p, \infty)$ norm, we remark that if $l_n(f) = \int_I f(A_{n,t}, \phi_n) d\nu$ for each f in $L(p, \infty)$, where $\nu = \lim_{n\to\infty} \nu_n$, then

$$\begin{aligned} |l_n(f) - \widetilde{l}_n(f)| &\leqslant \|f(A_{n,t}, \phi_n)\|_{C(p,\infty)} \, \|\nu - \nu_n\|_{M(p,\infty)} \\ &\leqslant \|f\|_{L(p,\infty)} \, \|\nu - \nu_n\|_{M(p,\infty)}. \end{aligned}$$

Thus (l_n) and $(\widetilde{l}_n)$ must both converge strongly to the same limit l and Theorem 2 follows.

The following lemma immediately implies that (ν_n) is a Cauchy sequence.

LEMMA 2B. – *Let l_1 and l_2 be two singular functionals of form* (1)

$$l_1(f) = \int_I f(A_t, \phi)\, d\nu$$

$$l_2(f) = \int_I f(B_t, \psi)\, d\rho$$

where ν and ρ are positive finitely additive set functions with $\nu(I) = \rho(I) = 1 = \|l_1\| = \|l_2\|$, and ϕ and ψ are functions in Φ_1. Let $\epsilon = \|l_1 - l_2\|$, then $\|\nu - \rho\|_{M(p,\infty)} = 0(\epsilon)$, where $0(\epsilon)$ denotes any function depending only on ϵ, which tends to zero as ϵ tends to zero.

Proof. – For each integer m, let

$$g_m(x) = \sum_{n=-\infty}^{\infty} \lambda_n \chi_{A_{t(n+1)} \setminus A_{t(n)}}(x)$$

where $(\lambda_n)_{-\infty}^{\infty}$ is a decreasing sequence, and $(t(n))_{-\infty}^{\infty}$ an increasing sequence such that the step function s_m, on I,

$$s_m(t) = \sum_{-\infty}^{\infty} \lambda_n \chi_{[t(n), t(n+1))}(t)$$

satisfies $\sup_{t \in I} |s_m(t) - (1 - 1/p)\, t^{-1/p}| \leqslant 1/m$. Then the sequence $(g_m(x))$ converges uniformly μ almost everywhere to a limit $g(x)$ which is carried by (A_t) with $g^*(t) = (1 - 1/p)\, t^{-1/p}$ and $g(A_t) = 1$ for all t. So $\|g\|_{L(p,\infty)} = 1$, and

$$\int_I g(A_t)\, d\nu = \int_I g(A_t)\, d\rho = 1.$$

$$\epsilon \geqslant |l_1(g\bar{\phi}) - l_2(g\bar{\phi})| \geqslant 1 - \left| \int_I g(B_t, \psi\bar{\phi})\, d\rho \right| \geqslant 1 - \int_I g(B_t)\, d\rho.$$

So $1 - \epsilon \leqslant \int_I g(B_t)\, d\rho$.

Exactly as in step 1 of the proof of Theorem 1,

$$g(B_t) \leqslant w(\mu(B_t \setminus A_t)/t)$$

where $w(s) = \frac{1}{2}\left[(1 - s)^{1-1/p} + (1 + s)^{1-1/p}\right]$ for $s \in [0, 1]$.

Let $$P_s = \{t \in I \mid \mu(B_t \backslash A_t)/t > s\}$$

$$Q_s = I \backslash P_s = \{t \in I \mid \mu(B_t \backslash A_t)/t \leqslant s\}$$

$$1 - \epsilon \leqslant \int_{P_s} g(B_t)\, d\rho + \int_{Q_s} g(B_t)\, d\rho \leqslant \rho(P_s)\, w(s) + (1 - \rho(P_s))$$

since $w(s)$ is a decreasing function.

Thus $$\rho(P_s) \leqslant \epsilon(1 - w(s))^{-1}.$$

Since $w'(0) = 0$, and $w''(s)$ is negative and bounded away from zero on $(0, 1)$, it follows that $1 - w(s) \geqslant Cs^2$ on $[0, 1]$ where C is a positive constant depending only on p. Putting $s = \epsilon^{1/4}$ we obtain

$$\rho(P_{\epsilon^{1/4}}) \leqslant C^{-1} \sqrt{\epsilon}. \tag{7}$$

We shall need a second estimate,

$$\left| \int_I f(B_t, \phi) - f(B_t, \psi)\, d\rho \right| \leqslant 0(\epsilon)\, \|f\|_{L(p,\infty)} \tag{8}$$

for all $f \in L(p, \infty)$. This is shown by methods similar to steps 3 and 4 of Theorem 1. For some $r > p'$, let

$$H(t) = (1 - 1/p)\, t^{-1} \int_{A_t \cap B_t} |\phi - \psi|^r\, d\mu$$

$$= t^{1/p-1}\, g^L(t) \int_{A_t \cap B_t} |\phi - \psi|^r\, d\mu$$

$$\leqslant t^{1/p-1} \int_{A_t \cap B_t} g|\phi - \psi|^r\, d\mu\,, \quad \text{since } (A_t) \text{ carries } g,$$

$$\leqslant 2^{r-1}\, t^{1/p-1} \int_{B_t} g\,|\phi - \psi|\, d\mu$$

$$\leqslant 2^{r-1}\, t^{1/p-1} \int_{B_t} g\,|\mathrm{Re}(1 - \psi\bar{\phi})|\, d\mu$$

$$+ 2^{r-1}\, t^{1/p-1} \int_{B_t} g\,|\mathrm{Im}(\psi\bar{\phi})|\, d\mu$$

$$\leqslant 2^{r-1}\, [g(B_t) - g(B_t, \mathrm{Re}(\psi\bar{\phi}))]$$

$$+ 2^{r-1} \left(t^{1/p-1} \int_{B_t} g\, d\mu\right)^{1/2} \left(t^{1/p-1} \int_{B_t} g[\mathrm{Im}(\psi\bar{\phi})]^2\, d\mu\right)^{1/2}$$

$$H(t) \leqslant 2^{r-1} \operatorname{Re}(g(B_t) - g(B_t, \psi\bar{\phi}))$$

$$+ 2^{r-1}\left(t^{1/p-1} \int_{B_t} 2g[1 - \operatorname{Re}(\psi\phi)]\, d\mu\right)^{1/2}$$ (as in step 4 of Theorem 1).

$$\int_I H(t)\, d\rho \leqslant 2^{r-1} \operatorname{Re}\left(\int_I g(B_t) - g(B_t, \psi\bar{\phi})\, d\rho\right)$$

$$+ 2^r \left(\int_I \operatorname{Re}(g(B_t) - g(B_t, \psi\bar{\phi}))\, d\rho\right)^{1/2}$$

by Schwarz' inequality.

$$\left|\int_I g(B_t) - g(B_t, \psi\bar{\phi})\, d\rho\right|$$

$$\leqslant \left|\int_I g(B_t)\, d\rho - 1\right| + \left|1 - \int_I g(B_t, \psi\bar{\phi})\, d\rho\right|$$

$$= \left|\int_I (g(B_t) - g(A_t))\, d\rho\right| + \left|\int_I g(A_t)\, d\nu - \int_I g(B_t, \psi\bar{\phi}) d\rho\right|$$

$$\leqslant 2\rho(P_{\epsilon^{1/4}}) + \int_{Q_{\epsilon^{1/4}}} |g(A_t) - g(B_t)|\, d\rho + |l_1(g\bar{\phi}) - l_2(g\bar{\phi})|$$

$$\leqslant 2C^{-1}\sqrt{\epsilon} + (2\epsilon^{1/4})^{1-1/p} + \epsilon = 0(\epsilon).$$

We have shown that

$$\int_I H(t)\, d\rho = 0(\epsilon).$$

Then for any $f \in L(p, \infty)$,

$$\left|\int_I f(B_t, \phi) - f(B_t, \psi)\, d\rho\right|$$

$$\leqslant \int_{Q_{\epsilon^{1/4}}} \left(t^{1/p-1} \int_{B_t} |f|\, |\phi - \psi|\, d\mu\right) d\rho + 0(\epsilon)\, \|f\|_{L(p,\infty)}$$

$$\leqslant \int_{Q_{\epsilon^{1/4}}} t^{1/p-1} \left(\int_{B_t} |f|^{r'}\, d\mu\right)^{1/r'} \left(\int_{B_t} |\phi - \psi|^r\, d\mu\right)^{1/r} d\rho$$

$$+ 0(\epsilon)\|f\|_{L(p,\infty)}$$

$$\leqslant \int_{Q_{\epsilon^{1/4}}} t^{1/p-1} \left(\int_{F_t} |f|^{r'}\, d\mu\right)^{1/r'} \left(\frac{t\, H(t)}{1 - 1/p} + 2^r t \epsilon^{1/4}\right)^{1/r} d\rho$$

$$+ 0(\epsilon)\, \|f\|_{L(p,\infty)}$$

where (F_t) carries $|f|$,

$$= \int_{Q_{\epsilon^{1/4}}} t^{1/r+1/p-1} \left[\int_0^t (f^L(s))^{r'} \, ds \right]^{1/r'} \left(\frac{H(t)}{1-1/p} + 2^r \epsilon^{1/4} \right)^{1/r} d\rho$$

$$+ \, 0(\epsilon) \, \|f\|_{L(p,\infty)}$$

$$\leqslant \|f\|^*_{p,\infty} \left[\int_I \left(t^{(1/r+1/p-1)r'} \int_0^t s^{-r'/p} ds \right) d\rho \right]^{1/r'}$$

$$\left[\int_I \frac{H(t)}{1-1/p} + 2^r \epsilon^{1/4} d\rho \right]^{1/r}$$

$$+ \, 0(\epsilon) \, \|f\|_{L(p,\infty)}$$

$= 0(\epsilon) \, \|f\|_{L(p,\infty)}$, since $r' < p$ and $\|\cdot\|_{L(p,\infty)}$ and $\|\cdot\|^*_{p,\infty}$ are equivalent. Thus (8) is proved.

$$\|\rho - \nu\|_{M(p,\infty)} = \sup \left\{ \left| \int_I h(t) \, d\nu - \int_I h(t) \, d\rho \right| \; | \; h \in C(p, \infty), \; \|h\|_{C(p,\infty)} \leqslant 1 \right\}$$

$$\leqslant \sup \left\{ \left| \int_I f(A_t) \, d\nu - \int_I f(A_t) \, d\rho \right| \; | \; f \in L(p, \infty), \; \|f\|_{L(p,\infty)} \leqslant 1 \right\},$$

using Lemma 2 A. Now for each $f \in L(p, \infty)$,

$$\left| \int_I f(A_t) \, d\nu - \int_I f(A_t) \, d\rho \right|$$

$$\leqslant |l_1(f\bar{\phi}) - l_2(f\bar{\phi})| + \left| \int_I f(B_t, \psi\bar{\phi}) \, d\rho - \int_I f(B_t) \, d\rho \right|$$

$$+ \left| \int_I f(B_t) - f(A_t) \, d\rho \right|.$$

The first term is dominated by $\epsilon \|f\|_{L(p,\infty)}$. The second equals :

$$\left| \int_I (f\bar{\phi}) \; (B_t, \psi) \, d\rho - \int_I (f\bar{\phi}) \; (B_t, \phi) \, d\rho \right| \leqslant 0(\epsilon) \, \|f\|_{L(p,\infty)},$$

from (8).

For the third,

$$\left|\int_I f(B_t) - f(A_t)\,d\rho\right| \leqslant \int_{Q_{\epsilon^{1/4}}} |f(B_t) - f(A_t)|\,d\rho + \int_{P_{\epsilon^{1/4}}} |f(B_t) - f(A_t)|\,d\rho$$

$$\leqslant (2\epsilon^{1/4})^{1-1/p}\|f\|_{L(p,\infty)} + 2C^{-1}\sqrt{\epsilon}\|f\|_{L(p,\infty)}.$$

Therefore $\|\rho - \nu\|_{M(p,\infty)} = 0(\epsilon)$ and the proof of Lemma 2B and thus of Theorem 2, is complete.

THEOREM 3. – *Every $l \in S_\alpha$, $\alpha = 0$ or ∞, is the strong limit of a sequence of functionals (l_n) in S_α*

$$l_n(f) = \int_I f(A_t, \phi_n)\,d\nu \quad \textit{for all} \quad f \in L(p,\infty),$$

where $(A_t)_{t\in I}$ is a C.M.F., ν a positive finitely additive set function with $\nu(I) = \|\nu\|_{M(p,\infty)} = \|l\|$ which concentrates at α, and (ϕ_n) is a sequence in Φ_1.

Proof. – As before we normalise to have $\|l\| = 1$. By the preceding theorem, l is the strong limit of a sequence $(\tilde{l}_n)$ where

$$\tilde{l}_n(f) = \int_I f(A_{n,t}, \phi_n)\,d\nu.$$

We shall construct a C.M.F. $(A_t)_{t\in I}$, which is the "limit" of the sequence $\{(A_{n,t})_{t\in I}\}_{n=1}^{\infty}$ in the sense that

$$\lim_{n\to\infty} \int_I \mu(A_{n,t}\setminus A_t)/t\,d\nu = 0. \qquad (9)$$

From this the theorem will follow readily, since

$$|l_n(f) - \tilde{l}_n(f)| \leqslant \|f\|_{L(p,\infty)} \int_I [\mu(A_{n,t}\,\Delta\,A_t)/t]^{1-1/p}\,d\nu,$$

as in step 2 of Theorem 1,

$$\leqslant \|f\|_{L(p,\infty)} \left(\int_I 2\mu(A_{n,t}\setminus A_t)/t\,d\nu\right)^{1/p'}$$

and so $\lim_{n\to\infty} \|l_n - \tilde{l}_n\| = 0.$

LEMMA 3A. – *Let* $\widetilde{l}_n$, ν *and* $(A_{n,t})$ *be as above, then*

$$\int_I \mu(A_{n,t}\backslash A_{m,t})/t\, d\nu = O(\|\widetilde{l}_n - \widetilde{l}_m\|)$$

for each pair of integers n and m.

Proof. – Fix an n and let g be the positive function with $\|g\|_{L(p,\infty)} = 1$ carried by $(A_{n,t})_{t\in I}$ such that $g(A_{n,t}) = 1$ for all t. (as in the proof of Lemma 2B).

By the same reasoning as in the proof of Lemma 2B and in step 1 of Theorem 1,

$$g(A_{m,t}) \leqslant w(\mu(A_{n,t}\backslash A_{m,t})/t)$$

and $g(A_{n,t}) - g(A_{m,t}) \geqslant 1 - w(\mu(A_{n,t}\backslash A_{m,t})/t) \geqslant C(\mu(A_{n,t}\backslash A_{m,t})/t)^2$.

$$\text{So } \left[\int_I \mu(A_{n,t}\backslash A_{m,t})/t\, d\nu\right]^2 \leqslant C^{-1}\int_I g(A_{n,t}) - g(A_{m,t})\, d\nu$$

$$\leqslant C^{-1}\left|\int_I g(A_{n,t}) - g(A_{m,t}\,, \phi_m\overline{\phi}_n)\, d\nu\right|$$

$$\leqslant C^{-1}\, \|\widetilde{l}_n - \widetilde{l}_m\|\,.$$

This proves the lemma.

In all that follows we will suppose that

$$\epsilon(n) = \int_I \mu(A_{n,t}\backslash A_{n+1,t})/t\, d\nu$$

satisfies $\sum_{n=1}^{\infty} \epsilon(n) < \infty$. It is clear from Lemma 3A, that this can always be guaranteed, by passing if necessary to a subsequence of $(\widetilde{l}_n)$.

Let us first consider the case $l \in S_0$.

Roughly speaking, the idea for the construction of the "limit" C.M.F. $(A_t)_{t\in I}$ is to take a sequence of positive numbers $(a(n))_{n=1}^{\infty}$ which tends monotonically to zero, and thus partitions the part of I near zero into disjoint intervals $(a(n+1)\,,\,a(n)]$ $n = 1, 2\ldots$ Then we could define $A_t = A_{n,t}$ for $t \in (a(n+1)\,, a(n)]$ for each n. In fact our definition will be almost this, but adjustments are necessary to ensure that $A_t \subset A_s$ whenever $t \leqslant s$. A certain condition will have to be imposed on the sequence $(a(n))_{n=1}^{\infty}$ to permit these adjustments. Subsequently we shall show that if the convergence of $a(n)$ to zero is sufficiently rapid, then the corresponding C.M.F. $(A_t)_{t\in I}$ satisfies (9).

Since ν concentrates at 0, we can find, for each fixed n, a positive sequence $(a(k))_{k=1}^{\infty}$ tending to zero such that

$$\mu(A_{n,a(k)} \setminus A_{n+1,a(k)})/a(k) < 2\,\epsilon(n)$$

for all k. Consequently, given any positive sequence $(b(k))_{k=1}^{\infty}$ tending to zero we can find a strictly decreasing sequence $(a(k))_{k=1}^{\infty}$ with $0 < a(k) < b(k)$ for all k, such that :

$$\mu(A_{k,a(k+1)} \setminus A_{k+1,a(k+1)})/a(k+1) < 2\,\epsilon(k) \quad \text{for all} \quad k. \tag{10}$$

All positive sequences which decrease strictly with limit zero, and satisfy (10) will be termed *admissible.*

Let $(a(k))_{k=1}^{\infty}$ (we shall also use the notation $a(\cdot)$) be an admissible sequence. Fix k, and for each $t \in [a(k+1)\,, a(k)]$ define the set B_t :

$$B_t = [A_{k,t} \setminus (A_{k,a(k+1)} \setminus A_{k+1,a(k+1)} \setminus C_{k,t})] \cup \Big[\bigcup_{m \geq k} (A_{m+1,a(m+1)} \setminus A_{m,a(m+1)})\Big]$$

where $(C_{k,t})_{a(k+1) \leq t \leq a(k)}$ is a "rescaled C.M.F." on the set

$$A_{k,a(k+1)} \setminus A_{k+1,a(k+1)},$$

that is :

i) $C_{k,a(k+1)} = \emptyset\,,\ C_{k,a(k)} = A_{k,a(k+1)} \setminus A_{k+1,a(k+1)}$

ii) $s \leq t$ implies $C_{k,s} \subset C_{k,t}$

iii) $\mu(C_{k,t}) = \left(\dfrac{t - a(k+1)}{a(k) - a(k+1)}\right) \mu(A_{k,a(k+1)} \setminus A_{k+1,a(k+1)})$

for all $t \in [a(k+1), a(k)]$.

Clearly for $s\,, t$ in $[a(k+1), a(k)]$, $s \leq t$ implies $B_s \subset B_t$.

$$B_{a(k)} = A_{k,a(k)} \cup \Big[\bigcup_{m \geq k} (A_{m+1,a(m+1)} \setminus A_{m,a(m+1)})\Big]$$

and

$$B_{a(k+1)} = [A_{k,a(k+1)} \backslash (A_{k,a(k+1)} \backslash A_{k+1,a(k+1)})] \cup \Big[\bigcup_{m \geq k} (A_{m+1,a(m+1)} \backslash A_{m,a(m+1)}) \Big]$$

$$= (A_{k,a(k+1)} \cap A_{k+1,a(k+1)}) \cup (A_{k+1,a(k+1)} \backslash A_{k,a(k+1)}) \cup \Big[\bigcup_{m \geq k} (A_{m+1,a(m+1)} \backslash A_{m,a(m+1)}) \Big]$$

$$= A_{k+1,a(k+1)} \cup \Big[\bigcup_{m \geq k+1} (A_{m+1,a(m+1)} \backslash A_{m,a(m+1)}) \Big] .$$

We can now extend the definition of B_t to all t, $0 \leq t \leq a(1)$. On each interval $[a(k+1), a(k)]$ we use the same formula as above, and k ranges over all the positive integers. When $t = a(m)$ is the common end point of two adjacent intervals, $[a(m+1), a(m)]$ and

$$[a(m), a(m-1)],$$

there are two "rival" definitions for B_t, but these definitions coincide, in view of the consistent expressions obtained above for $B_{a(k)}$ and $B_{a(k+1)}$.

$(B_t)_{0 \leq t \leq a(1)}$ is almost a C.M.F. on $B_{a(1)}$. We certainly have $B_s \subset B_t$ whenever $s \leq t$, but $\mu(B_t)$ is not necessarily equal to t. Instead we have only approximate equality as t tends to zero. To show this, observe that because of the admissibility of $a(\cdot)$,

$$t - 2a(k+1)\,\epsilon(k) \leq \mu(B_t) \leq t + 2 \cdot \sum_{m \geq k} \epsilon(m)\, a(m+1)$$

for all $t \in [a(k+1), a(k)]$.

Thus, as t tends to zero (which corresponds to k tending to infinity), $\lim_{t \to 0} \mu(B_t)/t = 1$.

It is clear that $\mu(B_t)$ is a continuous non decreasing function of t on $[0, a(1)]$ (define $B_0 = \emptyset$) and so for each number $t \in [0, \mu(B_{a(1)})]$ there exists an $s(t) \in [0, a(1)]$ such that $\mu(B_{s(t)}) = t$. Let us then define the C.M.F. $(A_t)_{t \in I}$ by $A_t = B_{s(t)}$ for all $t \in [0, \mu(B_{a(1)})]$. For $t > \mu(B_{a(1)})$ we may define A_t quite arbitrarily since we are working with an $l \in S_0$, and a ν which concentrates at zero. Similarly we may consider (B_t) defined for all $t \in I$.

We shall henceforth refer to (B_t) as a *quasi-C.M.F. generated by* $a(\cdot)$, and to (A_t) as a *C.M.F. generated by* $a(\cdot)$. In fact for our purposes it is easier to work with the quasi C.M.F.. If we can show that there exists an admissible sequence $a(\cdot)$ which generates a quasi-C.M.F. (B_t) for which (9) holds :

$$\lim_{n\to\infty} \int_I \mu(A_{n,t}\backslash B_t)/t \, d\nu = 0$$

then we can readily deduce that the corresponding C.M.F. (A_t) generated by $a(\cdot)$ also satisfies (9) by noting that

$$\mu(A_{n,t}\backslash A_t) \leqslant \mu(A_{n,t}\backslash B_t) + \mu(B_t\backslash A_t) \leqslant \mu(A_{n,t}\backslash B_t) + |\mu(B_t) - t|$$

and that $\int_I |\mu(B_t) - t|/t \, d\nu = 0$ since the integrand tends to zero as t tends to zero, and ν concentrates at zero.

Let δ be a positive number less than $\mu(X)$, let $I_0 = (0\,,\delta]$, and let βI_0 denote the Stone-Čech compactification of I_0. Clearly there exists a Borel measure λ supported on $\beta I_0\backslash I_0$, with total variation $v(\lambda\,,\beta I_0) = v(\nu\,,I)$, such that

$$\int_{I_0} f \, d\nu = \int_{I_0} \beta[f] \, d\lambda$$

for every continuous bounded function f on I_0, where $\beta[f]$ is the continuous extension of f to βI_0.

For each pair of real numbers $a\,,b$, with $0 < a < b$ let

$$\begin{aligned} V_{a,b}(t) &= 0 \quad \text{for} \quad t \geqslant b \\ &= \frac{b-t}{b-a} \quad \text{for} \quad a \leqslant t \leqslant b \\ &= 1 \quad \text{for} \quad 0 < t \leqslant a. \end{aligned}$$

Let n be an arbitrary fixed integer, and let $a(\cdot)$ be an admissible sequence generating the quasi-C.M.F. (B_t).

$$\mu(A_{n,t}\backslash B_t)/t \leqslant 1 \quad \text{for all} \quad t,$$

and if $t \in [a(k+1)\,,a(k)]$ where $k \geqslant n$

$$\begin{aligned}\mu(A_{n,t}\setminus B_t)/t &\leqslant \mu(A_{n,t}\setminus A_{k,t})/t + \mu(A_{k,t}\setminus B_t)/t \\ &\leqslant \mu(A_{n,t}\setminus A_{k,t})/t + \mu(A_{k,k+1}\setminus A_{k+1,a(k+1)}\setminus C_{k,t})/t \\ &\leqslant \sum_{m=n}^{k-1} \mu(A_{m,t}\setminus A_{m+1,t})/t + 2\epsilon(k) \\ &= \sum_{m=n}^{k-1} V_{a(m+1),a(m)}(t)\, \mu(A_{m,t}\setminus A_{m+1,t})/t + 2\epsilon(k).\end{aligned}$$

Thus for all $t \in (0\,,\, a(n)]$

$$\mu(A_{n,t}\setminus B_t)/t \leqslant f_n(t\,;a(\cdot))$$

where

$$f_n(t\,;a(\cdot)) = \min\left[1,\ \sum_{m \geqslant n} V_{a(m+1),a(m)}\, \mu(A_{m,t}\setminus A_{m+1,t})/t + \theta(t)\right],$$

$\theta(t)$ being a continuous function on I_0 which tends to zero as t tends to zero, such that

$$\theta(t) \geqslant 2\epsilon(k) \quad \text{on} \quad [a(k+1),\, a(k)] \quad \text{for each} \quad k = 1, 2 \ldots$$

For each admissible sequence $a(\cdot)$ and each integer n, $f_n(t\,;a(\cdot))$ is continuous and bounded on I_0, and so $\beta[f_n(t\,;a(\cdot))]$ is defined as a continuous bounded function on βI_0.

We now regard the set D of admissible sequences $a(\cdot)$ as a directed set with the partial ordering $a(\cdot) \geqslant b(\cdot)$ iff $a(k) \leqslant b(k)$ for all k.

LEMMA 3B. – *Let n be an arbitrary fixed integer, then*

$$\lim_{a(\cdot)\in D} \beta\,[f_n(t\,;a(\cdot))] \leqslant \sum_{m \geqslant n} \beta\,[\mu(A_{m,t}\setminus A_{m+1,t})/t]$$

pointwise on $\beta I_0 \setminus I_0$.

Proof. – Let x be a point of $\beta I_0 \setminus I_0$. Then x is the limit of a net $(t_\gamma)_{\gamma\in\Gamma}$ of points in I_0 in the sense that $\beta[f](x) = \lim_{\gamma\in\Gamma} f(t_\gamma)$ for every bounded continuous function f on I_0.

First note that $\lim_{\gamma\in\Gamma} t_\gamma = 0$ in the topology of $[0\,,\,\delta]$ for if not a subnet of $(t_\gamma)_{\gamma\in\Gamma}$ would converge to a point in $(0\,,\,\delta]$ which cannot be x.

Choose an arbitrary $\epsilon > 0$. For each integer m, there exists a $\gamma'_m \in \Gamma$ such that for all $\gamma \geqslant \gamma'_m$

$$\mu(A_{m,t_\gamma} \backslash A_{m+1,t_\gamma})/t_\gamma \leqslant \beta[\mu(A_{m,t} \backslash A_{m+1,t})/t]\,(x) + \epsilon/2^m.$$

For all $m \geqslant n$ define $\gamma_m = \sup_{n \leqslant r \leqslant m} \gamma'_r$.

Define the subnet Γ_n of Γ to consist of the sequence $\gamma_n, \gamma_{n+1}, \gamma_{n+2}, \ldots$ together with all elements $\gamma \in \Gamma$ for which $\gamma \geqslant \gamma_m$ for all $m = n, n+1, \ldots$ Now construct an admissible sequence $b(\cdot)$ with $0 < b(m) \leqslant \min_{n \leqslant r \leqslant m-1} t_{\gamma_r}$ for each $m \geqslant n$. Then for each $\gamma \in \Gamma_n$,

$$f_n(t_\gamma \,;\, b(\cdot)) \leqslant \sum_{m \geqslant n} V_{b(m+1),b(m)}(t_\gamma)\, \mu(A_{m,t_\gamma} \backslash A_{m+1,t_\gamma})/t_\gamma + \theta(t_\gamma)$$

$$\leqslant \sum_{m \geqslant n} \beta[\mu(A_{m,t} \backslash A_{m+1,t})/t]\,(x) + \epsilon + \theta(t_\gamma).$$

Passing to the limit with respect to the net Γ_n,

$$\beta[f_n(t\,;b(\cdot))]\,(x) \leqslant \sum_{m \geqslant n} \beta[\mu(A_{m,t} \backslash A_{m+1,t})/t]\,(x) + \epsilon.$$

For all $c(\cdot)$ and $d(\cdot) \in D$ with $c(\cdot) \geqslant d(\cdot)$, we have

$$f_n(t\,;\,c(\cdot)) \leqslant f_n(t\,;d(\cdot)) \quad \text{for all} \quad t \in I_0.$$

Thus $$\lim_{a(\cdot) \in D} \beta[f_n(t\,;a(\cdot))]\,(x) = \inf_{a(\cdot) \in D} \beta[f_n(t\,;a(\cdot))]\,(x)$$

for each $x \in \beta I_0 \backslash I_0$, and satisfies the required inequality.

LEMMA 3C. – *Let n be an arbitrary fixed integer, and λ a Borel measure induced an $\beta I_0 \backslash I_0$ by ν in the manner described above. Then*

$$\lim_{a(\cdot) \in D} \int_{\beta I_0} \beta[f_n(t\,;a(\cdot))]\,d\lambda \leqslant \int_{\beta I_0} \sum_{m \geqslant n} \beta[\mu(A_{m,t} \backslash A_{m+1,t})/t]\,d\lambda.$$

Proof. – First let us note that λ must be a positive measure since $\int_{\beta I_0} f\,d\lambda = \int_{I_0} f\,d\nu \geqslant 0$ for every positive bounded continuous function f on βI_0.

Let

$$F(x\,;a(\cdot)) = \max\left\{0\,,\beta\,[f_n(t\,;a(\cdot))]\,(x) - \sum_{m\geqslant n}\beta\,[\mu(A_{m,t}\backslash A_{m+1,t})/t]\,(x)\right\}$$

$F(x\;;\,a(\cdot))$ is an upper semi continuous function on $\beta I_0\backslash I_0$ for each $a(\cdot)\in D$. $a(\cdot)\geqslant b(\cdot)$ implies $F(x\,;a(\cdot))\leqslant F(x\,;b(\cdot))$, and by the previous lemma, $\lim_{a(\cdot)\in D} F(x\,;a(\cdot)) = 0$ for all $x\in\beta I_0\backslash I_0$.

For each $\epsilon > 0$, $\bigcap_{a(\cdot)\in D}\{x \mid F(x\,;a(\cdot))\geqslant\epsilon\} = \emptyset$

and, since each of the sets $\{x \mid F(x\,;a(\cdot))\geqslant\epsilon\}$ is compact, there exists a finite collection of admissible sequences

$$a_1(\cdot),\ a_2(\cdot)\ldots a_k(\cdot),\quad\text{such that}$$

$$\bigcap_{j=1}^{k}\{x \mid F(x\;;\;a_j(\cdot))\geqslant\epsilon\} = \emptyset$$

Thus for all $a(\cdot)\geqslant\sup_{j=1}^{k} a_j(\cdot)$, $0\leqslant F(x\,;a(\cdot)) < \epsilon$ for all $x\in\beta I_0\backslash I_0$, and $F(x\,;a(\cdot))$ converges to zero uniformly. This argument is of course nothing other than Dini's theorem. The proof of the lemma now follows obviously.

We are now ready to construct an admissible sequence $b(\cdot)$ which generates a quasi C.M.F. (B_t) satisfying (9). For each n we can find, using Lemma 3C, an admissible sequence $a_n(\cdot)$ such that

$$\int_{\beta I_0}\beta[f_n(t\,;a_n(\cdot))]\,d\lambda\leqslant\int_{\beta I_0}\sum_{m\geqslant n}\beta\,[\mu(A_{m,t}\backslash A_{m+1,t})/t]\,d\lambda + 1/n.$$

Let $b(\cdot)$ be an admissible sequence such that

$$0 < b(m) < \min_{1\leqslant n\leqslant m} a_n(m)\quad\text{for each}\quad m.$$

Observe that $f_n(t\,;a(\cdot))$ depends only on the members of the sequence $a(m)$ with $m\geqslant n$, and that $b(m) < a_n(m)$ for all $m\geqslant n$. Thus $f_n(t\,;b(\cdot))\leqslant f_n(t\,;a_n(\cdot))$ for each n and all $t\in I_0$.

Let (B_t) be the quasi C.M.F. generated by $b(\cdot)$. Then, for each n, as shown earlier, $\mu(A_{n,t}\backslash B_t)/t\leqslant f_n(t\,;b(\cdot))$ for $t\in(0\,,b(n)]$. Consequently,

$$\begin{aligned}\int_I \mu(A_{n,t}\backslash B_t)/t\, d\nu &\leqslant \int_{I_0} f_n(t\,;a_n(\cdot))\, d\nu \\ &= \int_{\beta I_0} \beta[f_n(t\,;a_n(\cdot))]\, d\lambda \\ &\leqslant \sum_{m\geqslant n} \int_{\beta I_0} \beta[\mu(A_{m,t}\backslash A_{m+1,t})/t]\, d\lambda + 1/n \\ &= \sum_{m\geqslant n} \int_I \mu(A_{m,t}\backslash A_{m+1,t})/t\, d\nu + 1/n \\ &= \sum_{m\geqslant n} \epsilon(m) + 1/n,\end{aligned}$$

which tends to zero as n tends to infinity. Thus (B_t) satisfies (9) and the proof of Theorem 3 is complete for the case $l \in S_0$.

We now turn to the proof for $l \in S_\infty$.

A sequence $(a(k))_{k=1}^{\infty}$ of positive numbers will be called ∞-*admissible* if it is strictly increasing with limit $+\infty$ and

$$\lim_{k\to\infty} \frac{1}{a(k)} \sum_{m=1}^{k-1} a(m) = 0 \tag{11}$$

$$\mu(A_{k,a(k)}\backslash A_{k-1,a(k)})/a(k) < 2\,\epsilon(k-1) \quad \text{for each} \quad k. \tag{12}$$

For $t \in [a(k)\,,\, a(k+1)]$ define :

$$B_t = [A_{k,t}\backslash(A_{k,a(k)}\backslash A_{k-1,a(k)}\backslash C_{k,t})] \cup \Big[\bigcup_{m\leqslant k} (A_{m-1,a(m)}\backslash A_{m,a(m)})\Big]$$

where $(C_{k,t})_{a(k)\leqslant t\leqslant a(k+1)}$ is a rescaled C.M.F. on $A_{k,a(k)}\backslash A_{k-1,a(k)}$ which means that :

i) $C_{k,a(k)} = \emptyset$, $C_{k,a(k+1)} = A_{k,a(k)}\backslash A_{k-1,a(k)}$

ii) $s \leqslant t$ implies that $C_{k,s} \subset C_{k,t}$

iii) $\mu(C_{k,t}) = \left(\dfrac{t-a(k)}{a(k+1)-a(k)}\right) \mu(A_{k,a(k)}\backslash A_{k-1,a(k)})$.

As before we have monotonicity of B_t on the interval $[a(k)\,,\, a(k+1)]$

$$B_{a(k+1)} = A_{k,a(k+1)} \cup \Big[\bigcup_{m\leqslant k} (A_{m-1,a(m)}\backslash A_{m,a(m)})\Big]$$

and

$$B_{a(k)} = [A_{k,a(k)} \cap (A_{k-1,a(k)} \cup (X \setminus A_{k,a(k)}))] \cup (A_{k-1,a(k)} \setminus A_{k,a(k)}) \cup \Big[\bigcup_{m \leqslant k-1} (A_{m-1,a(m)} \setminus A_{m,a(m)}) \Big]$$

$$= A_{k-1,a(k)} \cup \Big[\bigcup_{m \leqslant k-1} (A_{m-1,a(m)} \setminus A_{m,a(m)}) \Big].$$

The consistency of the expressions for $B_{a(k)}$ and $B_{a(k+1)}$ permits us to define B_t for all $t \in [a(1), \infty)$.

For $t \in [a(k), a(k+1)]$ we have, using the ∞-admissibility of $a(\cdot)$,

$$t - 2a(k)\,\epsilon(k-1) \leqslant \mu(B_t) \leqslant t + 2 \sum_{m \leqslant k} a(m)\,\epsilon(m-1).$$

In view of (11) it follows that $\lim_{t \to \infty} \mu(B_t)/t = 1$.

From here the proof is an obvious analogue of that for $l \in S_0$. We pass to a Borel measure λ on $\beta[1, \infty) \setminus [1, \infty)$. The role of $V_{a,b}(t)$ is played by $\Lambda_{a,b}(t) = 1 - V_{a,b}(t)$, and the analogue of the function $\theta(t)$ will be a continuous function $\theta_\infty(t)$ which tends to zero as t tends to infinity and for which

$$\inf_{a(k) \leqslant t \leqslant a(k+1)} \theta_\infty(t) \geqslant 2\,\epsilon(k-1)$$

for each integer k.

THEOREM 4. – *Each $l \in S_\alpha$, $\alpha = 0$ or ∞, has the form*

$$l(f) = \int_I f(A_t, \phi)\, d\nu$$

where $(A_t)_{t \in I}$ is a C.M.F. $\phi \in \Phi_1$, and ν is a positive finitely additive set function concentrating at α, with $\|\nu\|_{M(p,\infty)} = \nu(I) = \|l\|$.

Proof. – Much of the proof proceeds along similar lines to that of Theorem 3.

Our starting point is, naturally enough, the sequence $(l_n)_{n=1}^\infty$ of Theorem 3, which converges strongly to l,

$$l_n(f) = \int_I f(A_t, \phi_n)\, d\nu.$$

We shall construct a function $\phi \in \Phi$ which is a "limit" of the sequence $(\phi_n)_{n=1}^{\infty}$ in the sense that

$$\lim_{n\to\infty} \int_I \left[t^{-1} \int_{A_t} |\phi_n - \phi|^r \, d\mu \right]^{1/r} d\nu = 0 \tag{13}$$

where, as in the proofs of Theorems 1 and 2, r is a fixed number in $(1, \infty)$ such that $r' < p$.

Given such a ϕ, we have, for each $f \in L(p, \infty)$

$$\left| l_n(f) - \int_I f(A_t, \phi) \, d\nu \right| \leqslant \int_I \left[t^{1/p-1} \int_{A_t} |f| \, |\phi_n - \phi| \, d\mu \right] d\nu$$

$$\leqslant \int_I \left[t^{-1} \int_{A_t} |\phi_n - \phi|^r \, d\mu \right]^{1/r} \left[t^{r'/p-1} \int_{A_t} |f|^{r'} \, d\mu \right]^{1/r'} d\nu \, .$$

The second factor in the integrand is bounded uniformly for all $t \in I$ since $r' < p$ and

$$\int_{A_t} |f|^{r'} \, d\mu \leqslant \int_0^t [f^*(s)]^{r'} \, ds$$

$$\leqslant (\|f\|_{p,\infty}^*)^{r'} \, (1 - r'/p)^{-1} \, t^{1-r'/p} .$$

Thus from (13) it will follow that $l = \lim_{n\to\infty} l_n$ must have the desired form

$$l(f) = \int_I f(A_t, \phi) \, d\nu .$$

LEMMA 4A. – *Let the sequence of functionals* $(l_n)_{n=1}^{\infty}$ *and the number r be as defined above. Then for each pair of integers m and n,*

$$\int_I \left[t^{-1} \int_{A_t} |\phi_n - \phi_m|^r \, d\mu \right]^{1/r} d\nu = O(\|l_n - l_m\|) .$$

Proof. – Using a function g carried by $(A_t)_{t\in I}$ with $g(A_t) = 1$ for each t we proceed almost exactly as for the estimating of $H(t)$ and $\int_I H(t) \, d\rho$ (proof of (8) in Lemma 2B) and show that

$$\int_I \left[t^{-1} \int_{A_t} |\phi_n - \phi_m|^r \, d\mu \right] d\nu = O(\|l_n - l_m\|).$$

(The argument is in fact even simpler than that for (8) since we may take $\rho = \nu$ and $(A_t)_{t\in I} = (B_t)_{t\in I}$).

We have only to apply Hölder's inequality and the proof is complete.

Let $\epsilon(n) = \int_I \left[t^{-1} \int_{A_t} |\phi_{n+1} - \phi_n|^r \, d\mu \right]^{1/r} d\nu.$

Lemma 4A allows us to assume that $\sum_{n=1}^{\infty} \epsilon(n) < \infty$, by passing if necessary to a subsequence of $(l_n)_{n=1}^{\infty}$.

CASE 1 : $l \in S_0$.

For each strictly decreasing sequence $(a(k))_{k=1}^{\infty}$ we define ϕ to be *the function generated by* $a(\cdot)$ if

$$\phi = \sum_{k=1}^{\infty} \phi_k \, \chi_{A_{a(k)} \setminus A_{a(k+1)}}.$$

A sequence $a(\cdot)$ tending strictly monotonically to zero will be termed Φ-*admissible* if

i) $a(1) < 1$, ii) $a(k+1) < a(k)^2$ for each k.

Let $a(\cdot)$ be Φ-admissible, and suppose it generates ϕ. Then for $t \in [a(k+1), a(k)]$ with $k \geqslant n$,

$$\left[t^{-1} \int_{A_t} |\phi_n - \phi|^r \, d\mu \right]^{1/r}$$

$$\leqslant \left[t^{-1} \int_{A_t} |\phi_n - \phi_k|^r \, d\mu \right]^{1/r} + \left[t^{-1} \int_{A_t} |\phi_n - \phi_{k+1}|^r \, d\mu \right]^{1/r} + \left[t^{-1} \int_{A_{a(k+2)}} |\phi_n - \phi|^r \, d\mu \right]^{1/r}$$

$$\leqslant 2 \sum_{m=n}^{k} \left[t^{-1} \int_{A_t} |\phi_m - \phi_{m+1}|^r \, d\mu \right]^{1/r} + 2\,[a(k+2)/a(k+1)]^{1/r}$$

$$\leqslant 2 \sum_{m=n}^{k} V_{a(m),a(m-1)}(t) \left[t^{-1} \int_{A_t} |\phi_m - \phi_{m+1}|^r \, d\mu \right]^{1/r} + 2\,[a(k+1)]^{1/r}$$

where $V_{a,b}(t)$ is as defined for Theorem 3.

Thus for all $t \in (0, a(n)]$, $\left[t^{-1} \int_{A_t} |\phi_n - \phi|^r \, d\mu\right]^{1/r}$ is dominated by the continuous bounded function :

$$h_n(t\,; a(\cdot)) = \min\left\{ 2, 2 \sum_{m \geqslant n} V_{a(m),a(m-1)}(t) \left[t^{-1} \int_{A_t} |\phi_m - \phi_{m+1}|^r \, d\mu\right]^{1/r} + 2t^{1/r} \right\}.$$

As in theorem 3 we let λ be the Borel measure on $\beta I_0 \backslash I_0$ induced by ν and by analogous reasoning to Lemmas 3B, 3C, 3D etc. we construct a Φ-admissible sequence $b(\cdot)$ such that for each n

$$\int_{\beta I_0} \beta[h_n(t\,; b(\cdot))] \, d\lambda \leqslant 2 \sum_{m \geqslant n} \int_{\beta I_0} \beta\left[t^{-1} \int_{A_t} |\phi_m - \phi_{m+1}|^r \, d\mu\right]^{1/r} d\lambda + 1/n.$$

Thus, if ϕ is the function generated by $b(\cdot)$

$$\int_I \left[t^{-1} \int_{A_t} |\phi_n - \phi|^r \, d\mu\right]^{1/r} d\nu \leqslant 2 \sum_{m \geqslant n} \epsilon(m) + 1/n$$

and (13) follows.

CASE 2 : $l \in S_\infty$.

A strictly increasing sequence $(a(k))_{k=1}^{\infty}$ *generates* the function ϕ, where

$$\phi = \sum_{k=1}^{\infty} \phi_k \, \chi_{A_{a(k+1)} \backslash A_{a(k)}}.$$

Such a sequence $a(\cdot)$ is Φ-∞-*admissible* if

i) $a(1) > 1$, and ii) $a(k+1) > a(k)^2$ for each k.

Let ϕ be the function generated by the Φ-∞-admissible sequence $a(\cdot)$. Then for $t \in [a(k), a(k+1))$ with $k \geqslant n$,

$$\left[t^{-1}\int_{A_t}|\phi_n-\phi|^r\,d\mu\right]^{1/r}$$

$$\leqslant\left[t^{-1}\int_{A_t}|\phi_n-\phi_k|^r\,d\mu\right]^{1/r}+\left[t^{-1}\int_{A_t}|\phi_n-\phi_{k-1}|^r\,d\mu\right]^{1/r}$$

$$+\left[t^{-1}\int_{A_{a(k-1)}}|\phi_n-\phi|^r\,d\mu\right]^{1/r}$$

$$\leqslant 2\sum_{m=n}^{k-1}\left[t^{-1}\int_{A_t}|\phi_m-\phi_{m+1}|^r\,d\mu\right]^{1/r}+2\,[a(k-1)/a(k)]^{1/r}$$

$$\leqslant 2\sum_{m=n}^{k-1}\Lambda_{a(m),a(m+1)}(t)\left[t^{-1}\int_{A_t}|\phi_m-\phi_{m+1}|^r d\mu\right]^{1/r}+2[a(k)]^{-1/2r}$$

where $\Lambda_{a,b}(t)$ is defined as for Theorem 3.

Let $\theta_\infty(t)$ be a continuous bounded function on $(0\,,\infty)$ such that $\lim_{t\to\infty}\theta_\infty(t)=0$, and $\inf_{a(k)\leqslant t\leqslant a(k+1)}\theta_\infty(t)\geqslant 2\,[a(k)]^{-1/2r}$ for each integer k. Then for all $t\in[a(n)\,,\infty)$

$$\left[t^{-1}\int_{A_t}|\phi_n-\phi|^r\,d\mu\right]^{1/r}\leqslant\min\left\{2,2\sum_{m\geqslant n}\Lambda_{a(m),a(m+1)}(t)\right.$$

$$\left.\left[t^{-1}\int_{A_t}|\phi_m-\phi_{m+1}|^r\,d\mu\right]^{1/r}+\theta_\infty(t)\right\}.$$

From this (13) follows in much the same way as before. This completes the proof of Theorem 4.

Remark 2. – As it stands our characterisation of the elements of S_0 and S_∞ is not canonical, in the sense that one can have

$$\int_I f(A_t\,,\phi)\,d\nu=\int_I f(B_t\,,\psi)\,d\rho$$

for all $f\in L(p\,,\infty)$ where $\rho\neq\nu$, $\phi\neq\psi$ and $(A_t)_{t\in I}\neq(B_t)_{t\in I}$. However each functional l, $l(f)=\int_I f(A_t\,,\phi)\,d\nu$ can be seen to be in one to one correspondence with a triple $([\nu],\ [(A_t)_{t\in I}]\,,[\phi])$ where the square brackets denote equivalence classes in the sets of *positive* bounded finitely additive set functionals, of C.M.F. *s*, and of functions in Φ_1 respectively.

These three classes are defined by the equivalence relations :

i) $\rho \sim \nu$ iff $\|\rho - \nu\|_{M(p,\infty)} = 0$

ii) $(A_t)_{t\in I} \sim (B_t)_{t\in I}$ iff $\int_I \mu(A_t \ \Delta B_t)/t \ d\nu = 0$

for any $\nu \in [\nu]$, (and thus for every $\nu \in [\nu]$)

iii) $\phi \sim \psi$ iff $\int_I \left[t^{-1} \int_{A_t} |\phi - \psi|^r \, d\mu\right]^{1/r} d\nu = 0$

for any $\nu \in [\nu]$ and any $(A_t) \in [(A_t)]$ (and thus for every $\nu \in [\nu]$ and every $(A_t) \in [(A_t)]$.

An apparent shortcoming of our characterisation is that there seems to be no way in which it reflects the linear structure of S_α. Let $l_j(f) = \int_I f(A_{j,t}, \phi_j) \, d\nu_j$ $j = 1, 2, 3$ with $l_3 = l_2 + l_1$. There does not seem to be a "recipe" for defining ν_3, $(A_{3,t})$ and ϕ_3 (or their equivalence classes) directly in terms of ν_1, ν_2, $(A_{1,t})$,, $(A_{2,t})$, ϕ_1 and ϕ_2.

Remark 3. – From Theorem 4 one can see that every functional in S_α attains its norm on the unit ball of $L(p, \infty)$. Let F_α be the quotient space of $L(p, \infty)$ defined by the equivalence relation

$$f \sim g \quad \text{iff} \quad N_\alpha(f - g) = 0$$

where N_0 and N_∞ are the semi norms which define S_0 and S_∞ (section 3). Then S_α is the dual of F_α and each functional in S_α attains its norm on the unit ball of F_α. Applying a theorem of James [7], we see that F_α must be reflexive. Let $F_{\alpha,k}$ denote the direct sum of k copies of F_α, and $S_{\alpha,k}$ the direct sum of k copies of S_α, then for each n the $(2n - 1)$ th dual of $L(p, \infty)$ is

$$L(p', 1) \oplus S_{0,n} \oplus S_{\infty,n}$$

and the $2n$th dual is $L(p, \infty) \oplus F_{0,n} \oplus F_{\infty,n}$.

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Manuscrit reçu le 2 juillet 1974
accepté par J.P. Kahane.

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