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EXISTENCE THEOREM FOR n-CAPACITIES [1]

by M. BRELOT.

1. For the study of inequalities in his theory of capacities [1], Choquet needed the existence of n disjoint compact sets of capacity 1, whose union is of capacity arbitrarily close to n; the capacities are in this case defined with respect to a fixed domain and its Green's function. While the existence proof is easy in the case of the whole R^τ space ($\tau \geqslant 3$), or in the case of a Euclidean domain whose boundary is « sufficiently regular » (in a sense to be defined), this is not so in the general case, which is treated here on Choquet's request. We will generalize somewhat on the conjectured theorem and deal with an arbitrary Green space $\mathcal{E}_0$, (that is, a space of type [2] $\mathcal{E}$ which has a Green's function, including in particular, the hyperbolic Riemann surfaces). Reference is made mainly to the paper [2] dealing with these spaces.

2. Consider a set E; let us denote its boundary by $\overset{*}{E}$, and its closure by $\overline{E}$. Also, we introduce an Alexandroff point $\mathcal{A}$ of $\mathcal{E}_0$ by means of which a compact space $\mathcal{E}'_0$ is obtained. We shall use normaly the topology of $\mathcal{E}_0$ and occasionally some notions in $\mathcal{E}'_0$ such as the boundary $\overset{*}{E}'$, and the closure $\overline{E}'$ of a set E. Certain obvious properties of the Dirichlet problem in open subsets Ω of $\mathcal{E}_0$ with the topology of $\mathcal{E}'_0$ will be utilized.

(1) This research was supported by the United States Air Force through the office of Scientific Research of the Air Research and Development Command.

(2) We recall [2] that to every point P of the Hausdorff space $\mathcal{E}$ there correspond an open neighborhood $\mathcal{V}_P$ and a homeomorphism from $\mathcal{V}_P$ onto an open set of R^τ (which is the Euclidean space compactified by a point at infinity) with a conformal ($\tau = 2$) or isometric ($\tau \geqslant 2$) structure.

We recall that if H_f^Ω is the solution corresponding to a function f given on the boundary $\overset{*}{\Omega}{}'$, then the restriction of H_f^Ω to an open subset ω is equal to the necessarily existing solution H_f^ω with respect to the function φ defined as follows:

$$\varphi = \begin{cases} f & \text{on} \quad \overset{*}{\omega}{}' \cap \overset{*}{\Omega}{}' \\ H_f^\Omega & \text{on} \quad \overset{*}{\omega} \cap \Omega \end{cases}$$

We derive from criterion B, No. 15, of [2] the

Lemma 1. — Suppose that K is a compact subset of $\mathcal{E}_0$. Then the inward flux into any regular[3] open set containing K is non-negative for every solution $u = H_f^{\mathcal{E}_0 - K}$, where f vanishes on $\mathcal{A}$ and is non-negative on $\overset{*}{K}$.

We recall that the equilibrium potential of K in $\mathcal{E}_0$ is equal, outside of K, to the solution $U = H_f^{\mathcal{E}_0 - K}$, where f vanishes at $\mathcal{A}$ and is equal to 1 on $\overset{*}{K}$; the extension of this function U, defined by assigning to it the value 1 on K, is a quasi-superharmonic function, equal, except at points at which K is thin, to a G-potential (Green-potential) of masses $\mu \geqslant 0$ situated on $\overset{*}{K}$, and whose total is the capacity of K. Except for a numerical factor, this capacity is therefore also equal to the flux of U passing into any regular open set containing K.

Let us now recall Lemma 9 of No. 22, article [2].

Lemma 2. — Suppose that K_n is a compact subset, varying in a fixed domain D_P^λ of $\mathcal{E}_0$ (where D_P^λ is defined by $G_P > \lambda$). If the equilibrium potential of K_n tends to zero at P, then the capacity tends to zero.

From this the following fundamental lemma is obtained:

Lemma 3. — In $\mathcal{E}_0$ consider an increasing sequence Ω_n of domains tending to $\mathcal{E}_0$ as limit, such that each Ω_n is relatively compact in $\mathcal{E}_0$ (that is, such that the closure $\overline{\Omega}_n$ of each Ω_n in $\mathcal{E}_0$ is compact). We note that the set $\overline{D}_P^\lambda$, where $G_P \geqslant \lambda$, is not necessarily compact.

It follows from the hypothesis that the capacity of the compact set $\alpha_n = \overline{D}_P^\lambda \cap \overset{*}{\Omega}_n$ tends to zero, and the capacity of the compact set $K_n = \overline{D}_P^\lambda \cap \overline{\Omega}_n$ tends to $1/\lambda$.

(3) Cf. [2] for precise definition.

We remark that in $(\mathcal{E}_0 - K_n)$ the function $1/\lambda.\ G_P^{(M)}$ is the solution of the Dirichlet problem for boundary values vanishing at $\mathcal{A}$, equal to 1 on $(\overset{*}{D}_P \cap \overline{\Omega_n})$ and less on α_n than a constant $h > 0$ independent of n (for n sufficiently large). Hence:

$$\text{eq. pot. of } K_n \leqslant 1/\lambda\, G_P \leqslant \text{eq. pot. of } K_n + h.(\text{eq. pot. of } \alpha_n)$$

Analogous inequalities hold, in the same order, for the respective quantities of flux entering an open set containing K_n, by virtue of lemma 1. Thus:

$$\text{cap. } K_n \leqslant 1/\lambda \leqslant \text{cap. } K_n + h.\text{cap. } \alpha_n$$

We shall show that the capacity of α_n tends to zero; in view of lemma 2, it will be sufficient to show that the equilibrium potential V_n of α_n tends to zero at each fixed point.

Now, set $\beta_n = D_P^\lambda \cap C\Omega_n$ (this set contains α_n) and denote by ω_n the connected component of $C\beta_n$ which contains P. V_n is dominated by the solution ϖ_n of the Dirichlet problem for ω_n respective to boundary values which vanish at $\mathcal{A}$, and are equal to 1 elsewhere. Now, we may prove easily, for instance, by application of theorem 6′-13 (No. 16 of [2]), that the quantity $(G_P^{\mathcal{E}_0} - G_P^{\omega_n})$ is equal to the solution of the Dirichlet problem in ω_n, respective to boundary values vanishing at $\mathcal{A}$ and equal to $G_P^{\mathcal{E}_0}$ elsewhere; thus, this quantity dominates $\lambda\varpi_n$. The proof is completed by observing that $G_P^{\omega_n}$ tends to $G_P^{\mathcal{E}_0}$.

3. The simplest method of establishing the proposed theorem seems to involve the following result obtained independently by N. Aronszajn and K. T. Smith.

Lemma 4. — If K_0 is any compact set (not containing any point at infinity of $\mathcal{E}_0$ when $\tau \geqslant 3$) (4), and if λ is a non-negative number not exceeding the capacity γ_0 of K_0, then there exists a compact subset of K_0 whose capacity equals λ (5).

By means of sufficiently small neighborhoods of each point

(4) A point at infinity of $\mathcal{E}_0$ ($\tau \geqslant 3$) forms a set of strictly positive capacity; this necessitates the elimination of such points.

(5) Choquet has pointed out that the following proof can be extended to any numerical function $f(K) \geqslant 0$ (of a compact set K) which is continuous on the right, subadditive and such that $f(K) = 0$ for any finite K.

Choquet has proved also that if K is any compact set of greenian capacity $\lambda > 0$, for every μ such that $0 < \mu < \lambda$, there exists a family with the power of continuum, of compact subsets whose capacity is equal to μ.

of K_0, this set may be covered with a finite number of compact sets α_i, each of capacity less than ε, where ε is an arbitrarily preassigned positive number. We may then form a finite sequence of some intersections $(K_0 \cap \alpha_i)$, taken in arbitrary order, whose union possesses a capacity just exceeding γ. This union is a compact set whose capacity lies between λ and $(\lambda + \varepsilon)$. From this compact set we can then obtain in the same manner another one contained in it, whose capacity lies between λ and $(\lambda + \varepsilon/2)$, and so on. We thus form a decreasing sequence of compact sets K_n whose intersection is indeed a set of capacity γ. This follows from the « continuity on the right » of the capacity, which may be proved, for instance, by using the well-known « continuity from inside » of the solution of the Dirichlet problem.

Instead of Lemma 4, we may also use the following lemma which permits the choice of a simple form for the compact sets of the proposed theorem.

Lemma 5. — [6] Every open set ω_0 of capacity $\gamma_0 > 0$ (least upper bound of the capacities of all compact subsets of ω_0) eontains a compact set of pre-assigned capacity λ $(0 < \lambda < \gamma_0)$ such that this compact set is « *simple* », i. e., is the union of finitely many disjoint compact sets which may be assumed « *elementary* » in the following sense : each of them is contained in a neighborhood $\mathcal{V}_P$ and its image in $\mathcal{V}'_P$ is a closed sphere or disc in the case of isometric structure; in the case of conformal structure, it is a domain whose boundary is a Jordan analytic curve.

Let us consider first for the case of a 2-dimensional space $(\tau = 2)$ a very regular [7] open set ω_1 and an open set ω_2, relatively compact [8] in $\mathcal{E}_0$, such that $\bar{\omega}_1 \subset \omega_0$, $\bar{\omega}_1 \subset \omega_2$ and that the capacity γ_1 of $\bar{\omega}_1$ exceeds λ. By covering the boundary $\overset{*}{\omega}_1$ with a finite number of neighborhoods $\mathcal{V}_P$ which are dealt with in succession, we form a simple set E which does not meet $\overset{*}{\omega}_2$, and which cover $\overset{*}{\omega}_1$ except for a set e of length less than a prescribed number l [9].

[6] In this lemma the points at infinity cause no difficulty. By eliminating them, the capacity of the open set under consideration is not changed.

[7] For exact definition see [2] No. 6.

[8] i.e., whose closure in $\mathcal{E}_0$ is compact.

[9] In the case of conformal structure, this length is defined in a pre-assigned metric, the definition of which is given in [2] No. 5.

Now the equilibrium potential of E is a majorant for the solution of the Dirichlet problem in $\mathcal{E} - \bar{\omega}_1$ for values vanishing on $\mathcal{A}$ and e, and equal to 1 elsewhere. On ω_2^*, this solution is, for l sufficiently small, arbitrarily close to the equilibrium potential V of $\bar{\omega}_1$, by virtue of the nature of the harmonic measure in $\mathcal{E} - \bar{\omega}_1$. Thus, the elementary compact sets of E can be chosen in such a way that on $\overset{*}{\omega}_2$ the equilibrium potential of their union exceeds θV, where $\theta < 1$ is arbitrarily close to 1. Hence by virtue of Lemma 1, E may be chosen in such a way, that its capacity is greater than $\gamma_1 - \varepsilon$, with $\varepsilon > 0$ arbitrarily prescribed.

Thus, a simple set E whose capacity exceeds λ can be obtained.

Let us consider the elementary compact sets of such an E in any fixed order; we terminate the sequence as soon as the total capacity obtained is not less than λ.

Let us examine the non-trivial case when the total capacity of our partial sequence Σ actually exceeds it. Denoting the last set of Σ by S_0, we consider S_0 on its local image. If the image is not a circle, then we consider a variable compact set S contained in S_0, such that, under the conformal mapping of S_0 on an circle, S corresponds to a circle of radius ρ, concentric with the one on which S_0 is mapped. It is easy to see that the capacity of the union of $\Sigma - S_0$ and the variable compact set is a continuous function of ρ. The continuity on the right (S considered decreasing) is obvious (in view of the continuity on the right of the capacity, or of the continuity from inside of the solution of the Dirichlet problem). As regards continuity on the left, (S considered increasing to S_0, $\bar{S} \subset \dot{S}_0$) we observe that the global equilibrium potential, which is majorated by the equilibrium potential of Σ, is an increasing superharmonic function whose limit is 1 in the interior of S_0, and hence on its boundary (as well as on the remaining elementary compact sets). It follows that this limit is a majorant for the equilibrium potential of Σ, and hence is equal to it. It is obvious that the respective capacities converge correspondingly. The continuous dependence on ρ makes it possible to choose the variable compact set in such a way that the global capacity is exactly λ.

In the case of a τ-dimensional space ($\tau \geqslant 3$), the choice of the elementary compact sets which almost cover $\overset{*}{\omega}_1$ is more difficult. However, we can take for ω_1 a union of spherical domains, eliminate a sufficiently small neighborhood of the intersection of their surfaces, and cover almost all the remainder of $\mathring{\omega}_1$ by means of small spheres orthogonal to the preceding ones. The rest of the argument is as before.

EXISTENCE THEOREM. — *Given a positive integer* N, *and* $\varepsilon > 0$, *then there exist in* $\mathcal{E}_0$ N *mutually disjoint compact sets, each of capacity* 1, *such that the capacity of their union differs from* N *by less than* ε, *and these compact sets may even be chosen to be simple.*

Let us consider an increasing sequence of domains Ω_n tending to $\mathcal{E}_0$, and assumed relatively compact in $\mathcal{E}_0$.

Recall the notations : D_P^λ denotes the domain in which $G_P > \lambda$; Σ_P^λ denotes its boundary, on which $G_P = \lambda$; $\overline{D}_P^\lambda$ denotes the union of these two sets, that is, the closure of D_P^λ. We choose a first value of λ, say, λ_1. We note first that the capacity of $\overline{D}_P^{\lambda_1} \cap \overline{\Omega}_n$ or of its boundary $(\Sigma_P^{\lambda_1} \cap \Omega_n) \cup (\Omega_n^* \cap \overline{D}_P^{\lambda_1})$ tends to $1/\lambda_1$ as that of $(\Omega_n^* \cap D_P^{\lambda_1})$ tends to zero. Hence the capacity of $\Sigma_P^\lambda \cap \overline{\Omega}_n$ exceeds 1 for n sufficiently large.

In $(\Sigma_P^\lambda \cap \overline{\Omega}_n)$ we may choose in accordance with Lemma 4 a compact set α_1 of capacity 1, provided that λ_1 has been chosen so that $\Sigma_P^{\lambda_1}$ does not contain a point at infinity.

We choose an Ω_{n_1} containing α_1; the ratio of the values of a variable positive harmonic function in Ω_{n_1} taken at P and at an arbitrary point of α_1 is always between two positive numbers r_1 and $1/r_1 (r_1 < 1)$.

Observe now that $(1/\lambda_1 . G_P)$ is a majorant for the equilibrium potential of α_1.

We then choose λ_2 small enough to satisfy :

a) $\lambda_2 < \varepsilon_0 \lambda_1$, $\lambda_2 < \varepsilon_0 r_1$ (ε_0 arbitrarily chosen > 0)

b) $D_P^{\lambda_2}$ contains Ω_{n_1}

c) $\Sigma_P^{\lambda_2}$ does not contain a point at infinity.

Consider n sufficiently large so that $(\Sigma_P^{\lambda_2} \cap \overline{\Omega}_n)$ is of capacity > 1; in this set we fix a compact set α_2 of capacity 1. On this set, the equilibrium potential of α_1 will be less than ε_0; and its own equilibrium potential will be equal to λ_2 at P, hence will be majorated by $\dfrac{\lambda_2}{r_1} < \varepsilon_0$ on α_1.

In like manner λ_3 may be found, such that on $\Sigma_P^{\lambda_3}$ a compact set α_3 of capacity 1 can be chosen, whose equilibrium potential is less than ε_0 on α_1 and α_2, while the equilibrium potentials of α_1 and α_2 are less than ε_0 on α_3.

Thus we successively form N compact sets of capacity 1, such that each of them has equilibrium potential less than ε_0 on each of the other sets. Therefore, the sum of the equilibrium potentials is on each of them quasi-everywhere between 1 and $(1 + (N - 1)\varepsilon_0)$. The value of this sum lies therefore between the equilibrium potential of the union and the quantity obtained by multiplying this potential with $(1 + (N - 1)\varepsilon_0)$. It follows from Lemma 1 that the capacity of the union differs from N by less than $N(N - 1)\varepsilon_0$.

To prove the existence of *simple* compacts the above reasoning must be slightly modified. Instead of choosing a compact set α_1 on $\Sigma_P^{\lambda_1}$ we shall consider the open set on which $\lambda_1' < G_P < \lambda_1''$, and its intersection with one of the Ω_n, choosing λ_1' and λ_1'' close to $\lambda_1 < 1$, and n sufficiently large; in the open set thus obtained we choose a compact set in accordance with Lemma 5, etc.

Extensions. — 1). The reasoning can easily be modified to extend the results somewhat: *It is possible to form a sequence of compacts* K_n (even *simple* compact sets) *each of capacity* 1, *with the following property*: *given* $\varepsilon > 0$, *there exists an index* n_0 *such that the union of any number* N *of compact sets of the sequence with indices* $> n_0$ *is a compact set of capacity lying between* N *and* $(N - N\varepsilon)$, *or even, according to an improvement of N. Aronszajn, between* N *and* $N - \varepsilon$.

We can choose the λ_i and compact sets K_i of capacity 1 on $\Sigma_P^{\lambda_i}$ or between two surfaces $\Sigma_P^{\lambda_i'}$ and $\Sigma_P^{\lambda_i''}$ (with λ_i' and λ_i'' close to λ_i), in such a way that, on each K_n, the equilibrium potentials of all the other K_i have a sum smaller than the term ε_n of an arbitrarily given sequence of positive numbers with finite sum.

In order to obtain the result of Aronszajn, we must use his more subtle argument, as follows: Let μ be the capacitary distributions of the systems S of N compact sets $K_{n_1} K_{n_2} \ldots K_{n_N}$, and μ_{n_i} the capacitary distributions of the compact K_{n_i}.

If U^ν denotes the Green's potential of ν, we may write

$$\int U^{\mu} \Sigma \, d\mu_{n_i} = \int U^{\Sigma \mu_{n_i}} d\mu = \int \Sigma U^{\mu_{n_i}} d\mu.$$

The first integral is obviously equal to N; the last one is not greater than $\int_S d\mu + \sum_{i=1}^{N} \int_{K_{n_i}} \varepsilon_{n_i} d\mu$. The first term is equal to N. Now, on K_{n_i} the total amount of μ is not greater than the capacity of K_{n_i}, i. e. 1. (Compare, for instance, the equilibrium potential of S and of K_{n_i} in the neighborhood of K_{n_i}; one may use Lemma 1 or a similar one.) $\sum_{i=1}^{N} \int_{K_{n_i}} \varepsilon_{n_i} d\mu$ is therefore not greater than $\sum_{i=1}^{N} \varepsilon_{n_i} \leqslant \sum_{p=n_i}^{\infty} \varepsilon_p$ which is arbitrarily small when n_i is large enough.

2) An obvious modification of the reasoning extends the results to compact sets K_i of capacity not equal to 1 but to an arbitrary number $\theta > 0$.

3) We may also consider capacities *which are not necessarily* equal, and we may form, for instance, N *disjoint simple compact* sets having given positive capacities, and such that their union is of capacity arbitrarily close to the sum of the capacities.

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