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KAI LAI CHUNG

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PROBABILISTIC APPROACH IN POTENTIAL THEORY TO THE EQUILIBRIUM PROBLEM

by Kai Lai CHUNG (1)

The problem of equilibrium is the first problem for the ancient period of potential theory recounted by Brelot in his recent historical article [1]. The existence of an equilibrium measure for the Newtonian potential was affirmed by Gauss as the charge distribution on the surface of a conductor which minimizes the electrostatic energy. But it was Frostman who rigorously established the existence in his noted thesis (1935), and extended it to the case of M. Riesz potentials. Somewhat earlier, F. Riesz had given his well-known decomposition for a subharmonic function which is a closely related result. For further history and references to the literature see Brelot's article. From the viewpoint of probability theory, the equilibrium problem in the Newtonian case takes the following from :

$$\underline{\underline{P}}^x\{T_B < \infty\} = \int_{\partial B} u(x, y)\, \mu_B(dy). \tag{1}$$

Here the underlying process is a Brownian motion $\{X_t,\ t \geqslant 0\}$ in R^3 ; $\underline{\underline{P}}^x\{\cdots\}$ denotes the probability (Wiener measure) when all paths issue from the point x ; B is a compact set (the conductor body) ; $T_B = T_B(\omega)$ is the hitting time of B by the path ω :

$$T_B(\omega) = \inf\{t > 0 \mid X_t(\omega) \in B\} ;$$

∂B is the boundary of B ; $u(x, y)$ is the associated potential density

$$u(x, y) = \frac{1}{2\pi |x - y|} ;$$

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and finally μ_B is the said equilibrium measure. Standard terminology and notation from the modern theory of Markov processes are used above. In the general setting to be considered here, the underlying process is a temporally homogeneous Markov process taking values in a topological space E which is locally compact and has a countable base with its Borel field $\mathcal{E}$. The transition semigroup will be assumed to be Borelian. However, we need not suppose the process to be a Hunt process ; in fact, strong Markov property will be used only peripherally toward the end, and quasi left continuity not at all. It is sufficient to assume that all paths are right continuous and have left limits in the time interval $[0, \infty)$. No overt duality assumptions are made in establishing the general formula (17) below.

A probabilistic proof of (1) is given in Ito-McKean [2, pp. 248ff.] which leans heavily on special analytic properties of the Brownian motion semigroup. In another paper, McKean [3] discussed a probabilistic interpretation of the result in a more general case, bringing in a number of things (capacity, Ueno's result, Weyl's lemma, etc.) which seem to obscure the real issue and leave the upshot unclear. For some reason the notion of a last exit time, which is manifestly involved in the arguments, would not be dealt with openly and directly. This may be partially due to the fact that such a time is not an "optional" (or "stopping") time, does not belong to the standard equipment, and so must be evaded at all costs. Actually the notion has been introduced to great advantage in Markov chains and the associated boundary theory, although it was only during the last few years that it became formalized (with considerable loss of intuition) under the name "co-optional". In the present approach it turns out to be a tame thing and leads very quickly to the classical results of Gauss-M. Riesz-Frostman, without any unnecessary complications. Moreover, pursuance of this simple idea yields a more complete solution of the equilibrium problem for a broad class of Markov processes. A historical note may be added here : a probabilistic solution to Dirichlet's problem was obtained by Doob (1954) by considering a first exit time ; here a similar solution to the so-called Robin's problem will be obtained by considering a last exit time.

Consider the general setting described in § 1. Call a Borel subset B of E "transient" iff for almost every path ω, there is a finite time $t^*(\omega)$ such that $X_t(\omega) \notin B$ for $t > t^*(\omega)$. Define

$$\Delta_B = \{\omega \in \Omega \mid \exists t > 0 : X_t(\omega) \in B\} ;$$

$$\gamma_B(\omega) = \begin{cases} \sup \{t > 0 \mid X_t(\omega) \in B\} & \text{if} \quad \omega \in \Delta_B ; \\ 0 , \text{ if} \quad \omega \in \Omega - \Delta_B . \end{cases}$$

Then B is transient if and only if $\gamma_B < \infty$ a.s. (almost surely). It is easy to see that γ_B is a random variable, called the *last exit time* from B.

Let us begin by supposing all paths continuous. It then follows that

$$X(\gamma_B) \in \partial B \quad \text{a.s.} \tag{2}$$

From now on we fix B and write γ for γ_B.

It is well-known that a compact set is transient for Brownian motion in R^3 ; thus the setting above includes the Newton-Gauss case described above. It also includes, e.g. the case of a Brownian motion in R^2, terminated after the path leaves an open ball. In this case the state space E should be the open ball with a one-point compactification ∂, and we must assume that $\partial \notin \bar{B}$, the closure of B.

To study the distribution of the last exit position $X(\gamma)$, we put

$$L(x , A) = \underline{\underline{P}}^x\{\gamma > 0 ; X(\gamma) \in A\} , \quad x \in E , \quad A \in \mathcal{B}. \tag{3}$$

We are going to determine this by calculating

$$\int_{\partial B} L(x , dy) f(y) = \underline{\underline{E}}^x\{\gamma > 0 ; f(X(\gamma))\} \tag{4}$$

for every x and every $f \in C_b$ (the class of bounded continuous functions on E), where $\underline{\underline{E}}^x$ denotes the mathematical expectation corresponding to $\underline{\underline{P}}^x$. This is done by a little device as follows. Take any $\varepsilon > 0$ and consider the "mixed approximation" :

$$\frac{1}{\varepsilon}\int_0^\infty \underline{\underline{E}}^x\{f(X_t) \quad ; \quad t < \gamma \leqslant t+\varepsilon\}\,dt$$

$$= \frac{1}{\varepsilon}\int_0^\infty \underline{\underline{E}}^x\{f(X_t)\,\underline{\underline{P}}^{X_t}[0 < \gamma \leqslant \varepsilon]\}\,dt. \quad (5)$$

Setting

$$\psi_\epsilon(x) = \frac{1}{\varepsilon}\,\underline{\underline{P}}^x\,[0 < \gamma \leqslant \varepsilon]$$

we may rewrite (5) as

$$\underline{\underline{E}}^x\left\{\int_0^\infty (f\psi_\epsilon)\,(X_t)\,dt\right\} = \int_0^\infty \frac{1}{\varepsilon}\int_{\gamma\in(t,t+\epsilon)} f(X_t)\,d\underline{\underline{P}}^x\,dt$$

$$= \int_{[\gamma>\epsilon]} \frac{1}{\varepsilon}\int_{\gamma-\epsilon}^{\gamma} f(X_t)\,dt\,d\underline{\underline{P}}^x + \int_{[0<\gamma\leqslant\epsilon]} \frac{1}{\varepsilon}\int_0^\gamma f(X_t)\,dt\,d\underline{\underline{P}}^x\ .$$

The last-written integral is bounded by

$$\int_{[0<\gamma\leqslant\epsilon]} \frac{\gamma}{\varepsilon}\,\|f\|\,d\underline{\underline{P}}^x \leqslant \|f\|\,\underline{\underline{P}}^x\{0 < \gamma \leqslant \varepsilon\},$$

which converges to 0 as $\varepsilon \downarrow 0$. On the other hand,

$$\lim_{\epsilon\downarrow 0}\int_{\gamma-\epsilon}^{\gamma} f(X_t)\,dt = f(X_\gamma) \quad (7)$$

boundedly, by the continuity of $t \to f(X_t)$. Hence as $\varepsilon \downarrow 0$ the first integral in the last member of (6) converges to $\int_{[\gamma>0]} f(X_\gamma)d\underline{\underline{P}}^x$, which is just the number in (4).

Define the potential U by

$$U\varphi(x) = \underline{\underline{E}}^x\left\{\int_0^\infty \varphi(X_t)\,dt\right\}$$

where φ is any positive measurable function. The result of calculations in (6) is then as follows :

$$\lim_{\epsilon\downarrow 0} U(f\psi_\epsilon)\,(x) = \int_{\partial B} L(x\ ,dy)\,f(y). \quad (8)$$

We now make the following assumptions on the potential kernel U. There exists a σ-finite measure ξ such that

i) $U(x, A) = \int_A u(x, y)\, \xi(dy)$ where $u(x, y) \geqslant 0$, for $x \in E$, $A \in \mathcal{E}$;

ii) $y \to u(x, y)^{-1}$ is finite continuous, for $x \in E$;

iii) $u(x, y) = +\infty$ if and only if $x = y$.

The key formula (8) will now be recorded as follows :

$$\forall f \in C_b : \lim_{\epsilon \downarrow 0} \int_E u(x, y)\, \psi_\epsilon(y)\, f(y)\, \xi(dy) = \int_{\partial B} L(x, dy)\, f(y) \quad (9)$$

Set

$$M_\epsilon(A) = \int_A \psi_\epsilon(y)\, \xi(dy) , \qquad A \in \mathcal{E}. \quad (10)$$

For any $x \in E$ and $\varphi \in C_K$ (class of continuous functions on E with compact supports), the function

$$y \to \varphi(y)\, u(x, y)^{-1}$$

belongs to C_K by assumption ii) above. Substituting this for f in (9), we have

$$\lim_{\epsilon \downarrow 0} \int_E \varphi(y)\, M_\epsilon(dy) = \int_{\partial B} L(x, dy)\, u(x, y)^{-1} \varphi(y).$$

This being true for every $\varphi \in C_K$, we conclude first that the measures M_ϵ converge vaguely to a measure $\mu (= \mu_B)$ as $\varepsilon \downarrow 0$; and secondly that this vague limit is identified as follows :

$$\forall x \in E : \mu(dy) = L(x, dy)\, u(x, y)^{-1}. \quad (11)$$

It follows from assumption ii) that μ is a σ-finite measure in $\mathcal{E}$. Since $L(x, \cdot)$ is concentrated on ∂B, so is μ. Since $u(x, y) < \infty$ for $x \neq y$ by assumption iii), we have if $x \notin \partial B$:

$$L(x, dy) = u(x, y)\, \mu(dy). \quad (12)$$

This means if $x \notin \partial B$ and $A \subset \partial B$, $A \in \mathcal{E}$, we have

$$L(x, A) = \int_A u(x, y)\, \mu(dy). \quad (13)$$

Now it is clear that for a transient set B we have

$$\{0 \leqslant T_B < \infty\} = \{0 < \gamma_B < \infty\}.$$

Thus if $x \notin \partial B$, the Gauss-Riesz-Frostman formula (1) is just the particular case of (13) for A = E. The measure μ_B is called the *equilibrium measure* for B, and its total mas $\mu_B(\partial B)$ the *capacity* of B, up to a multiplicative constant.

We shall establish (13), and consequently (1), for all $x \in E$. Taking $x = y$ in (1) and using assumption iii), we see that μ is atomless, namely for every $y \in E$:

$$\mu(\{y\}) = 0.$$

Next we have again by iii) and (11), if $x \neq y$:

$$L(x, \{y\}) = u(x, y)\, \mu(\{y\}) = 0. \tag{14}$$

Finally,

$$L(x, A\backslash\{x\}) = \int_{A\backslash(x)} u(x, y)\, \mu(dy) = \int_A u(x, y)\, \mu(dy),$$

by the usual convention $\infty \cdot 0 = 0$ when the integrand takes the value $+\infty$ at a point while the corresponding mass at the point is 0. Thus (13) will hold true if and only if

$$\forall x \in E : L(x, \{x\}) = 0. \tag{15}$$

A point x in E is called a *holding point* iff almost every path starting at x must remain at x for some strictly positive time. We show that if x is nota holding point, namely if for each $\delta > 0$ we have

$$\underline{\underline{P}}^x\{X(t) = x \quad \text{for} \quad t \in [0, \delta]\} = 0, \tag{16}$$

then (15) is true. The following proof requires only that the process be separable. To simplify notation we may then suppose that the dyadic numbers $\left\{ \frac{n}{2^m} ; m \geqslant 0, n \geqslant 0 \right\}$ are a separability set. For each n we define

$$S_n = \min \left\{ \frac{m}{2^n} \mid X\left(\frac{m}{2^n}\right) \neq x \right\}$$

with the convention that $S_n = +\infty$ when the set on the right side is empty. If x is not a holding point, then

$$\underline{\underline{P}}^x\{S_n \downarrow 0\} = 1$$

by separability. We have, therefore, using only ordinary Markov property because S_n is countably-valued :

$$\underline{\underline{P}}^x\{\gamma > S_n \ ; X(\gamma) = x\} \leqslant \underline{\underline{E}}^x\{P^{X(S_n)}[\gamma > 0 \ ; X(\gamma) = x]\} = 0$$

by (14), since $X(S_n) \neq x$. Letting $n \to \infty$ we obtain (15).

On the other hand, if x is a holding point, then

$$U(x, \{x\}) = u(x, x)\, \xi(\{x\}) > 0 \ ,$$

hence $\xi(\{x\}) > 0$ and so $U(x, \{x\}) = \infty$ by assumption iii) above. [I owe this observation to Hans Föllmer, which enabled me to deal with a holding point.] This implies that the singleton $\{x\}$ constitutes a "recurrent set" in the sense that starting from x almost every path will hit $\{x\}$ after an arbitrarily large t. A familiar argument then shows that almost no path can lead from x to the transient set B. Thus by definition

$$L(x, \{x\}) \leqslant P^x\{T_B < \infty\} = 0$$

and so (15) is also true for a holding point x.

We have therefore established the fundamental result (13) for every x and every A under the hypotheses specified.

Note that the existence of the measure μ_B for the representation given in (1) has been established for every transient set B, without any regularity condition whatever on ∂B. However, the hitting probability on the left side of (1) need not be equal to one for all $x \in B$, as required by the classical definition of equilibrium potential. Herein lies the necessity of a condition like Poincaré's to ensure that every x on ∂B is regular for B, or the exception of a set of points on ∂B which are irregular for B.

Finally, formula (11) gives an explicit solution to Robin's problem of determining the equilibrium measure. Indeed, by a suitable choice of the arbitrary point x there, the probabilistic expression may even yield a deterministic one. A trivial case in point is when B is a ball and x its center.

The method of proof will now be extended to the case where the paths are right continuous with left limits. Here are the necessary changes. Relation (2) is replaced by

$$X(\gamma_B -) \in \overline{B},$$

where $\overline{B}$ is the closure of B. The last exit distribution in (3) is redefined by using the left limit $X(\gamma -)$ instead of $X(\gamma)$. Both $L(x, \cdot)$ and μ are now concentrated on $\overline{B}$ instead of ∂B. If we replace $X(\gamma)$ by $X(\gamma -)$ and ∂B by $\overline{B}$ in the obvious places, all the steps go through as before. The final result is, for each Borel Set $A \subset \overline{B}$, and each $x \in E$:

$$\underline{\underline{P}}^x\{X(\gamma_B -) \in A\} = L(x, A) = \int_A u(x, y)\, \mu_B(dy) ; \qquad (17)$$

in particular

$$\underline{\underline{P}}^x\{T_B < \infty\} = L(x, \overline{B}) = \int_{\overline{B}} u(x, y)\, \mu_B(dy).$$

In this form the result covers the M. Riesz-Frostman potentials where $u(x, y) = c/|x - y|^\alpha$, α real > 0, c = constant. As is known, the corresponding Markov process is a stable process whose paths may be assumed to be right continuous and have left limits.

The assumption ii) is expedient for our method and remains to be analysed. Next, we examine some possibilities of relaxing the assumption iii) to illustrate the relationship between the poles of u and the polar-like sets of the process. We shall shun the standard duality assumptions but lead up to them. Let us put

$$N_x = \{y \in E \mid u(x, y) = +\infty\},$$

$$\widetilde{N}_y = \{x \in E \mid u(x, y) = +\infty\}.$$

Thus our previous assumption iii) amounts to the simplest of its kind :

$$N_x = \widetilde{N}_x = \{x\}.$$

We will not assume this, but confine ourselves to a Hunt process below in order to avail ourselves of standard arguments and results of the theory.

0) If for every x, $N_x = \emptyset$, then (17) is true by (11).

I) Suppose that for each $y \in \overline{B}$, $\widetilde{N}_y \neq \emptyset$. Then in order that (17) be true as asserted, it is necessary that μ_B is atomless ; and it is necessary and sufficient that

$$\forall x \in \overline{B} : L(x , N_x \cap \overline{B}) = 0 . \tag{18}$$

Proof. – Take $y \in \overline{B}$, $x \in \widetilde{N}_y$, $A = \{y\}$ in (17) :

$$L(x , \{y\}) = u(x , y)\, \mu_B(\{y\}) ,$$

hence $\mu_B(\{y\}) = 0$. On the other hand, if (18) holds, then for every $A \subset \overline{B}$:

$$L(x , A) = L(x , A \backslash (N_x \cap \overline{B})) = \int_{A \backslash (N_x \cap \overline{B})} u(x , y)\, \mu(dy)$$

$$= \int_A u(x , y)\, \mu(dy) ,$$

where the second equation is true by (11), the third because $\mu(N_x) = 0$ from

$$1 \geqslant L(x , N_x) = \int_{N_x} (+\infty)\, \mu(dy) .$$

The necessity of (18) is shown in the same way.

II) Suppose for each $y \in \overline{B}$, $\widetilde{N}_y \neq \emptyset$ and $E - \widetilde{N}_y$ is finely dense (in particular if $\widetilde{N}_y$ is of null U-potential), then for each x, $L(x , \cdot)$ is atomless. If in addition, the set $N_x \cap \overline{B}$ is countable for each $x \in \overline{B}$, then (17) is true as asserted.

Proof. – It follows as before that if $x \notin \widetilde{N}_y$, then $L(x , \{y\}) = 0$. This is then true for every x by an argument similar to that given after (16), based on the fact that x is regular for $E - \widetilde{N}_y$. When $N_x \cap \overline{B}$ is countable, (18) follows.

III) Suppose that for each x , $y \to u(x , y)$ is an excessive function, and $E - N_x$ is finely dense. Then (17) is true as asserted.

Proof. – For each $y \notin N_x$, the process $\{u(x , X_t)$, $t \geqslant 0\}$ is a supermartingale under $\underline{\underline{P}}^y$. Hence by a well known result of Doob's :

$$\underline{\underline{P}}^y\{\exists t > 0 : X_t \in N_x\} = 0 ;$$

this is then also true if we replace X_t by X_{t-}, by a result of Hunt's. Furthermore, the result is true for every y, since $E - N_x$ is finely dense. It follows that $L(x, N_x) = \underline{\underline{P}}^x\{\gamma > 0 \ ; \ X(\gamma -) \in N_x\} = 0$ for every x.

Of course the assumption $y \to u(x, y)$ is "wrong", since generally it is $x \to u(x, y)$ that should be excessive, whereas $y \to u(x, y)$ should be co-excessive. However, if $u(x, y)$ is symmetric in (x, y) this makes sense. Let us also remark that if $y \to u(x, y)$ is locally integrable with respect to the *reference* measure ξ, then $\xi(N_x) = 0$. Consequently, N_x is of null potential and so $E - N_x$ is indeed finely dense as assumed.

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K.L. CHUNG
Department of Mathematics
Stanford University
Stanford, California 94305 (USA)