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HOLOMORPHIC FUNCTIONS ON LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES : I. LOCALLY CONVEX TOPOLOGIES ON $\mathcal{H}$ (U) (*)

by Seán DINEEN

In this article we study topological vector space structures on $\mathcal{H}(U)$, the set of all holomorphic functions on U where U is an open subset of the locally convex topological vector space E. We apply our results, in a further paper, to characterise certain pseudo domains.

These structures are of interest both intrinsically and because of the role they play in the theory of convolution equations and in the study of analytic continuation in infinite dimensions.

We discuss the following topologies on $\mathcal{H}(U)$:

1) The ported topology of Nachbin, $\mathcal{T}_\omega$. ([3], [28]).

2) The bornological topology associated with $\mathcal{T}_\omega$, $\mathcal{T}_{\omega,b}$. ([9], [11]).

3) The bornological topology associated with the compact open topology, $\mathcal{T}_{o,b}$. ([7], [9], [11], [17]).

4) The $\mathcal{T}_\delta$ topology introduced in [29] (a semi-norm p on $\mathcal{H}(U)$ is said to be $\mathcal{T}_\delta$ continuous if for each countable open cover of U, $(U_n)_n^\infty$, there exists $C > 0$ and n_1 a positive integer such that

$$p(f) \leqslant C \sup_{x \in \bigcup_{i=1}^{n_1} U_i} |f(x)|$$

for all $f \in \mathcal{H}(U)$).

For U balanced we apply the theory of generalised bases ([22], [23]) and deduce many of the properties of $(\mathcal{H}(U), \mathcal{T}_{\omega,b})$ and $(\mathcal{H}(U), \mathcal{T}_{\delta,b})$

(*) A number of the results contained in this paper were announced in C.R. Acad. S.C., Paris, t. 274, 544-547, 1972.

etc. from those of $(\mathcal{P}(^nE), \mathcal{T}_{\omega,b})$ and $(\mathcal{P}(^nE), \mathcal{T}_{o,b})$ ($\mathcal{P}(^nE)$ is the set of all continuous n-homogeneous polynomials on E). The following are some results obtained in such a fashion,

a) On $\mathcal{H}(U)$ the following are equivalent,

a.1) $(\mathcal{H}(U), \mathcal{T}_{\omega,b}) = (\mathcal{H}(U), \mathcal{T}_\delta)$

a.2) $(\mathcal{H}(U), \mathcal{T}_{\omega,b})$ is barrelled

a.3) $\mathcal{T}_{\omega,b}$ is the finest locally convex topology on $\mathcal{H}(U)$ for which the Taylor series converges absolutely and which induces on $\mathcal{P}(^nE)$ the $\mathcal{T}_\omega$ topology for each n.

b) $(\mathcal{H}(U), \mathcal{T}_{o,b})$ is barrelled if and only if $(\mathcal{P}(^nE), \mathcal{T}_{o,b})$ is barrelled for each n and $\mathcal{T}_{o,b}$ is the finest l.c. topology on $\mathcal{H}(U)$ for which the Taylor series converges absolutely and which induces on $\mathcal{P}(^nE)$ the $\mathcal{T}_{o,b}$ topology for each n.

c) $(\mathcal{H}(U), \mathcal{T}_\omega)$ is complete if and only if $(\mathcal{P}(^nE), \mathcal{T}_\omega)$ is complete for each n and if $P_n \in \mathcal{P}(^nE)$ for each n and $\sum_{n=0}^{\infty} p(P_n) < \infty$ for each $\mathcal{T}_\omega$- continuous semi-norm p on $\mathcal{H}(U)$, then $\sum_{n=0}^{\infty} P_n \in \mathcal{H}(U)$.

We also found this method of investigation useful in the construction of examples and counterexamples.

Unfortunately, it happens that $\mathcal{T}_\omega \neq \mathcal{T}_{\omega,b}$, for a large class of locally convex spaces E. In particular we find that if E is a Frechet space on which there exists no continuous norm then $\mathcal{T}_\omega \neq \mathcal{T}_{\omega,b}$ on $\mathcal{H}(U)$ (see also [15], [29]). This is unfortunate because if it were otherwise we would immediately obtain the equivalence of holomorphically convex domains and domains of holomorphy ([14], [15]).

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1. Definitions and Fundamental Properties of Locally Convex Topologies on $\mathcal{H}(U)$.

Our notation (unless otherwise stated) will be the same as [3],

[11] and [28]. However for convenience we state some of the more frequent and important definitions and theorems.

N, R and C will denote respectively the set of non-negative integers, real numbers and complex numbers. Let f be a complex valued function defined on a set A, then

$$\|f\|_A = \sup_{x \in A} |f(x)|$$

E will denote a locally convex Hausdorff(1) topological vector space (briefly LCS) and U will denote an open subset of E.

Let A be a vector space of continuous functions on the topological space X. $\mathcal{T}_0(A)$ *will denote the compact open topology on* A. *A semi-norm p on* A *is said to be ported* ([28], [29]) *by the compact subset* K *of* X *if for each open subset* V *of* X *containing* K *there exists* $C(V) > 0$ *such that* $p(f) \leqslant C(V) \|f\|_V$ *for all* f *in* A.

$\mathcal{T}_\omega(A)$ *denotes the locally convex topology on* A *generated by the semi-norms ported by the compact subsets of* X.

A semi-norm p on A *is said to be* $\mathcal{T}_\delta$*-continuous if for each increasing sequence of open subsets of* X, $(U_n)_{n=1}^\infty$, *such that* $\bigcup_{n=1}^\infty U_n = X$ *there exists* n_1 *a positive integer and* $C > 0$ *such that*

$$p(f) \leqslant C \|f\|_{U_{n_1}} \quad \textit{for all } f \textit{ in } A.$$

$\mathcal{T}_\delta(A)$ (*see* [29]) *denotes the locally convex topology on* A *generated by all the* $\mathcal{T}_\delta$*-continuous semi-norms.*

$\mathcal{T}_{o,b}(A)$ (*resp.* $\mathcal{T}_{\omega,b}(A)$) *will denote the bornological topology on* A *associated with* $\mathcal{T}_0(A)$ (*resp.* $\mathcal{T}_\omega(A)$(2), (3). *If* B *is a subspace of* A *and* $\mathcal{T}(A)$ *is any topology on* A *then* $T(A)|_B$ *will denote the topology on* B *induced by* $T(A)$. A *preliminary investigation of the various topologies yields.*

(1) This condition on E is not necessary for the development of the theory but its inclusion helps us to avoid many rather trivial situations.

(2) i.e. $\mathcal{T}_{o,b}(A)$ (resp. $\mathcal{T}_{\omega,b}(A)$) is the finest locally convex topology on A which has the same bounded sets as $\mathcal{T}_0(A)$ (resp. $\mathcal{T}_\omega(A)$) ([21]).

(3) We write $\mathcal{T}_0$ etc. in place of $\mathcal{T}_0(A)$ when there is no possible confusion as to which A is involved.

i) $\mathcal{T}_0 \leqslant \mathcal{T}_\omega \leqslant \mathcal{T}_{\omega,b} \leqslant \mathcal{T}_\delta$.

ii) $\mathcal{T}_0 \leqslant \mathcal{T}_{o,b} \leqslant \mathcal{T}_{\omega,b}$.

iii) $(\mathrm{A}, \mathcal{T}_\delta(\mathrm{A}))$ is a bornological locally convex space.

A is said to be *locally closed* if the following condition holds ; *if f is a function on X and for each $y \in \mathrm{X}$ there exists V_y open in X containing y and $f_{n,y} \in \mathrm{A}$ such that* $\|f_{n,y} - f\|_{\mathrm{V}_y} \to 0$ as $n \to \infty$ *then $f \in \mathrm{A}$.*

PROPOSITION 1.1. ([29]). – *If A is locally closed then $(\mathrm{A}, \mathcal{T}_\delta(\mathrm{A}))$ is barrelled.*

Proof. – Let $\mathfrak{A} = (\mathrm{V}_n)_{n=1}^\infty$ be an increasing sequence of open subsets of X such that $\bigcup_{n=1}^\infty \mathrm{V}_n = \mathrm{X}$.

Let $\mathrm{A}_\mathfrak{a} = \{ f \in \mathrm{A}, \|f\|_{\mathrm{V}_n} < \infty$ for each $n\}$.

$\mathrm{A}_\mathfrak{a}$ is a metrizable locally convex space when endowed with the topology generated by the semi-norms

$$\mathrm{P}_n(f) = \|f\|_{\mathrm{V}_n}$$

Since A is locally closed $\mathrm{A}_\mathfrak{a}$ is a Frechet space for each such $\mathfrak{A}$. Since A is a space of continuous functions on X we have $\mathrm{A} = \bigcup_\mathfrak{a} \mathrm{A}_\mathfrak{a}$. It is also immediate that the inductive limit topology, $\mathcal{T}_i$, is finer than the $\mathcal{T}_\delta(\mathrm{A})$ topology.

Now let p be a $\mathcal{T}_i$ continuous semi-norm on A. Let $\mathfrak{A}_1 = (\mathrm{V}_n)_{n=1}^\infty$ be an increasing open cover of X and suppose for each n there exists $f_n \in \mathfrak{A}_1$ such that $p(f_n) \geqslant n$ and

$$\|f_n\|_{\mathrm{V}_n} \leqslant 1$$

Let $\mathrm{W}_m = \{x \in \mathrm{X} \ ; \ |f_n(x)| < m$ for each $n\}$

Let W_n^0 be the interior of W_n. By construction $\mathfrak{A}_2 = (\mathrm{W}_n^0)_{n=1}^\infty$ is an increasing open cover of X. Since p is $\mathcal{T}_i$ continuous on A there exists n_1 a positive integer and $\mathrm{C} > 0$ such that

$$p(f) \leqslant \mathrm{C}\,\|f\|_{\mathrm{W}_{n_1}^0} \quad \text{for all} \quad f \in \mathrm{A}_{\mathfrak{a}_2}$$

Since $f_n \in A_{\alpha_2}$ for each n this contradicts the fact that

$$\sup_n p(f_n) = \infty$$

$\mathcal{P}(^n\mathrm{E})$(1) *(resp.* $\mathcal{L}(^n\mathrm{E})$) *will denote the set of all complex valued continuous n-homogeneous polynomials (resp. continuous symmetric n-linear forms on* E).

A function $f : \mathrm{U} \to \mathbf{C}$ *is said to be holomorphic on* U *if for each* $\xi \in \mathrm{U}$ *there exists* $\mathrm{P}_{n,\xi} \in \mathcal{P}(^n\mathrm{E})$, $n = 0, 1 \ldots$ *such that*

$$f(x) = \sum_{n=0}^{\infty} \mathrm{P}_{n,\xi}(x - \xi) \tag{1.1}$$

for all x *uniformly in some neighbourhood of* ξ. We call (1.1) the Taylor series expansion of f at ξ and since this expansion is unique we denote $\mathrm{P}_{n,\xi}$ by $\dfrac{\hat{d}^n f(\xi)}{n!}$.

We shall frequently use the following result which is often taken as the definition of a holomorphic function ; $f : \mathrm{U} \to \mathbf{C}$ is holomorphic if and only if it is G-holomorphic (i.e. the restriction of f to each finite dimensional subspace of U is holomorphic as a function of several complex variables) and locally bounded (i.e. for each $x \in \mathrm{U}$ there exists a neighbourhood of x, $\mathrm{V}(x)$, contained in U such that the restriction of f to $\mathrm{V}(x)$ is bounded). $\mathcal{H}(\mathrm{U})$ *will denote the set of all complex valued holomorphic functions on* U. For the remainder of this paper U will denote (unless otherwise stated) a *balanced* open subset of E. In this case the Taylor series at 0 of $f \in \mathcal{H}(\mathrm{U})$ converges pointwise at all points of U to the function f.

The remainder of section 1 will be devoted to convergence of the Taylor series in various topologies on $\mathcal{H}(\mathrm{U})$ and to characterising a generating family of semi-norms for the different topologies on $\mathcal{H}(\mathrm{U})$.

Let $\mathcal{S}$ denote the set of all complex valued sequences, $(\alpha_n)_{n=0}^{\infty}$, such that

$$\lim_{n \to \infty} \sup |\alpha_n|^{1/n} \leqslant 1$$

(1) For $n = 0$, $\mathcal{P}(^0\mathrm{E})$ will be the set of constant functions from E to C.

LEMMA 1.1. – *Let* $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathcal{H}(U)$ *and* $\alpha = (\alpha_n)_{n=0} \in \mathcal{S}$ *then*

$$g = \sum_{n=0}^{\infty} \alpha_n \frac{\hat{d}^n f(0)}{n!} \in \mathcal{H}(U) .$$

Proof. – Let K be a compact balanced subset of U. Choose $\lambda > 1$ and V a convex balanced neighbourhood of 0 such that

$$\lambda K + \lambda V \subset U \quad \text{and} \quad \|f\|_{\lambda K + \lambda V} = M < \infty .$$

By Cauchy's inequalities ([3], [28])

$$\left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K+V} \leqslant \frac{1}{\lambda^n} M \quad \text{for each } n.$$

Hence if $|\alpha_n| < C_1 \left(\frac{1+\lambda}{2}\right)^n$ for each n where C_1 is some positive real number then

$$\sum_{n=0}^{\infty} |\alpha_n| \cdot \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K+V} \leqslant C_1 \cdot \sum_{n=0}^{\infty} M \left(\frac{1+\lambda}{2\lambda}\right)^n < \infty$$

Hence $$\|g\|_{K+V} < \infty \Rightarrow g \in \mathcal{H}(U) .$$

PROPOSITION 1.2. – *The Taylor series of* $f \in \mathcal{H}(U)$ *at* 0 *converges in* $(\mathcal{H}(U), \mathcal{T}_\delta)$ *(and hence in* $(\mathcal{H}(U), \mathcal{T}_{\omega,b})$ $(\mathcal{H}(U), \mathcal{T}_{o,b})$ *and* $(\mathcal{H}(U), \mathcal{T}_\omega)$*).*

Proof. – Let $g_n = n^2 \sum_{m=n}^{\infty} \frac{\hat{d}^m f(0)}{m!}$ for each n

Let

$$W_M = \{ x , |g_n(x)| \leqslant 1 \quad \text{for all} \quad n \geqslant M\}$$

and let V_m be the interior of W_m. By lemma 1.1. $\bigcup_{M=1}^{\infty} V_M = U$ and $V_M \subset V_{M+1}$ for each M. Let p be a $\mathcal{T}_\delta$-continuous semi-norm on $\mathcal{H}(U)$. Hence there exists n_1 a positive integer and $C > 0$ such that

$$p(f) \leqslant C \|f\|_{V_{n_1}} \quad \text{for all} \quad f \in \mathcal{H}(U) .$$

Hence $p(g_n) \leqslant C$ for all $n \geqslant n_1$, and thus $p\left(\sum_{m=n}^{\infty} \frac{\hat{d}^m f(0)}{m\,!}\right) \to 0$ as $n \to \infty$. Hence the Taylor series of f at 0 converges in $(\mathcal{H}(U), \mathcal{T}_\delta)$.

LEMMA 1.2. – *Let* B *be a bounded subset of* $(\mathcal{H}(U), \mathcal{T}_0)$ *(resp.* $(\mathcal{H}(U), \mathcal{T}_\omega)$*) and* $\alpha = (\alpha_n)_{n=0}^\infty \in \mathcal{S}$ *then the set*

$$\left(\alpha_n \frac{\hat{d}^n f(0)}{n\,!}\right)_{n=0,\, f\in B}^{\infty} \qquad \textit{is bounded in}$$

$(\mathcal{H}(U), \mathcal{T}_0)$ *(resp.* $(\mathcal{H}(U), \mathcal{T}_\omega)$*).*

Proof. – a) Without loss of generality we can suppose $B = (f_n)_{n=1}^\infty$. Let K be a compact balanced subset of U. There exists $\lambda > 1$ such that $\lambda K \subset U$. Hence there exists $M > 0$ such that

$$\sup_n .\ \| f_n \|_{\lambda K} = M < \infty .$$

By the Cauchy integral formula for each m and n we have

$$\left\| \frac{\hat{d}^n f_m(0)}{n\,!} \right\|_K \leqslant \frac{M}{\lambda^n}$$

Hence
$$\sup_n .\ \left\| \alpha_n \frac{\hat{d}^n f(0)}{n\,!} \right\|_K$$
$$\leqslant M \cdot \sup_n .\ \left(\frac{|\alpha_n|}{\lambda^n}\right) < \infty .$$

This completes the proof when B is a bounded subset of $(\mathcal{H}(U), \mathcal{T}_0)$.

b) Let p be a $\mathcal{T}_\omega$-continuous semi-norm on $\mathcal{H}(U)$. By the method used in lemma 1.1. one easily shows that the semi-norm p_1 defined by

$$p_1(f) = \sum_{n=0}^{\infty} |\alpha_n|\, p\left(\frac{\hat{d}^n f(0)}{n\,!}\right)$$

is always finite and $\mathcal{T}_\omega$-continuous on $\mathcal{H}(U)$. An application of this fact immediately gives the required result when B is a bounded subset of $(\mathcal{H}(U), \mathcal{T}_\omega)$.

PROPOSITION 1.3. – *Let p be a $\mathcal{T}_{o,b}$ (resp. $\mathcal{T}_{\omega}$, $\mathcal{T}_{\omega,b}$) continuous semi-norm on $\mathcal{H}(U)$, then there exists a $\mathcal{T}_{o,b}$ (resp. $\mathcal{T}_{\omega}$, $\mathcal{T}_{\omega,b}$) continuous semi-norm on $\mathcal{H}(U)$ such that*

i) $p_1 \geqslant p$

ii) $p_1(f) = \sum_{n=0}^{\infty} p_1\left(\frac{\hat{d}^n f(0)}{n!}\right)$

Proof. – *Let* $p_1(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right)$

then $p_1(f)$ is finite for all $f \in \mathcal{H}(U)$ and condition (ii) is satisfied.

If p is $\mathcal{T}_{\omega}$-continuous we have already seen that p_1 is $\mathcal{T}_{\omega}$-continuous so there remains only to consider the bornological topologies $\mathcal{T}_{o,b}$ and $\mathcal{T}_{\omega,b}$. Let B be a $\mathcal{T}_{o,b}$ (resp. $\mathcal{T}_{\omega,b}$) bounded subset of $\mathcal{H}(U)$. By lemma 1.2

$$\sup_{\substack{f \in B \\ n=1,2,\ldots}} p\left(n^2 \frac{\hat{d}^n f(0)}{n!}\right) = M < \infty .$$

Thus we have $\sup_{f \in B} \sum_{n=1}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqslant \sum_{n=1}^{\infty} M/n^2 < \infty .$

Hence p_1 is $\mathcal{T}_{o,b}$ (resp. $\mathcal{T}_{\omega,b}$) continuous. Since the Taylor series converges we also immediately get $p_1 \geqslant p$. Before discussing the same problem for the $\mathcal{T}_{\delta}$-topology on $\mathcal{H}(U)$ we discuss various topologies on $\mathcal{P}(^n E)$.

Let p be a $\mathcal{T}_{\delta}(\mathcal{P}(^n E))$ continuous semi-norm. For each $f \in \mathcal{H}(U)$ let

$$\widetilde{p}(f) = p\left(\frac{\hat{d}^n f(0)}{n!}\right)$$

Now if V is a convex balanced neighbourhood of 0 in U then $(nV)_{n=1}^{\infty}$ is an increasing sequence of open subsets of E which covers E. Hence there exists n_1 a positive integer and $C > 0$ such that

$$\widetilde{p}\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqslant C \cdot \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{n_1 V}$$

By the homogeneity of the elements of $\mathcal{P}(^n E)$ we get

$$\widetilde{p}(f) \leqslant C \cdot (n_1)^n \, \left\| \frac{\hat{d}^n f(0)}{n\,!} \right\|_V$$

$$\text{for all} \quad f \in \mathcal{H}(U)$$

Hence p is $\mathcal{T}_\omega(\mathcal{H}(U))|_{\mathcal{P}(^nE)}$ continuous on $\mathcal{P}(^nE)$ since $\widetilde{p}\,|_{\mathcal{P}(^nE)} = p$. We thus have the following relationship between various topologies on $\mathcal{P}(^nE)$.

LEMMA 1.3. – *On $\mathcal{P}(^nE)$ the following topologies coïncide $\mathcal{T}_\omega(\mathcal{H}(U))|_{\mathcal{P}(^nE)}$, $\mathcal{T}_{\omega,b}(\mathcal{H}(U))|_{\mathcal{P}(^nE)}$, $\mathcal{T}_\delta(\mathcal{H}(U))|_{\mathcal{P}(^nE)}$, $\mathcal{T}_\omega(\mathcal{P}(^nE))$, $\mathcal{T}_{\omega,b}(\mathcal{P}(^nE))$, $\mathcal{T}_\delta(\mathcal{P}(^nE))$.*

We note also that $\mathcal{P}(^nE)$ and $\mathcal{H}(U)$ are both locally closed vector spaces and hence $(\mathcal{H}(U), \mathcal{T}_\delta)$ and $(\mathcal{P}(^nE), \mathcal{T}_\omega)$ are both barrelled and bornological. Similarly it is possible to show that on $\mathcal{P}(^nE)$ we have

$$\mathcal{T}_{o,b}(\mathcal{H}(U))\,|_{\mathcal{P}(^nE)} = \mathcal{T}_{o,b}(\mathcal{P}(^nE))$$

PROPOSITION 1.4. – *The $\mathcal{T}_\delta$ topology on $\mathcal{H}(U)$ is generated by all semi-norms p on $\mathcal{H}(U)$ which satisfy the following conditions*

i) $$p(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n\,!}\right)$$

ii) $p\,|_{\mathcal{P}(^nE)}$ *is $\mathcal{T}_\omega$-continuous for each n.*

Proof. – Let p be a semi-norm on $\mathcal{H}(U)$ which satisfies conditions i) and ii). Let $V = \{f, p(f) \leqslant 1\}$. V is convex balanced and absorbing. Now let $f_\alpha \in V$, $f_\alpha \to f$ in $(\mathcal{H}(U), \mathcal{T}_\delta)$. Since $p\,|_{\mathcal{P}(^nE)}$ is $\mathcal{T}_\omega$-continuous and hence $\mathcal{T}_\delta$ continuous we have $\sum_{n=0}^{k} p\left(\frac{\hat{d}^n f(0)}{n\,!}\right) \leqslant 1$ for all positive integers k, hence $p(f) \leqslant 1$. Thus V is a closed subset of $(\mathcal{H}(U), \mathcal{T}_\delta)$ and since this space is barrelled it is a neighbourhood of 0. Hence p is $\mathcal{T}_\delta$-continuous on $\mathcal{H}(U)$. Conversely if p is a $\mathcal{T}_\delta$-continuous semi-norm on $\mathcal{H}(U)$ then

$$p_1(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n\,!}\right)$$

satisfies conditions i) and ii). Hence p_1 is $\mathcal{T}_\delta$-continuous on $\mathcal{H}(U)$. Since the Taylor series converges in $(\mathcal{H}(U), \mathcal{T}_\delta)$ we have $p_1 \geqslant p$. This completes the proof.

COROLLARY 1.1. — *$\mathcal{T}_\delta$ is the finest locally convex topology on $\mathcal{H}(U)$ for which the Taylor series converges absolutely and which induces on $\mathcal{P}(^nE)$ the $\mathcal{T}_\omega$ topology for each n.*

COROLLARY 1.2. — *Let $(f_\alpha)_{\alpha \in A}$ be a bounded net in $(\mathcal{H}(U), \mathcal{T})$, $(\mathcal{T} = \mathcal{T}_{o,b}, \mathcal{T}_\omega, \mathcal{T}_{\omega,b}, \mathcal{T}_\delta)$, then $f_\alpha \to 0$ as $\alpha \to \infty$ if and only if $\dfrac{\hat{d}^n f_\alpha(0)}{n!} \to 0$ as $\alpha \to \infty$ in $(\mathcal{P}(^nE), \mathcal{T})$*

$$(\mathcal{T} = \mathcal{T}_{o,b}, \mathcal{T}_\omega \text{ for each } n).$$

Proof. — For $\mathcal{T} = \mathcal{T}_{o,b}$, $\mathcal{T}_\omega$ or $\mathcal{T}_{\omega,b}$ we use lemma 1.2 and proposition 1.3. For $\mathcal{T} = \mathcal{T}_\delta$ apply lemma 1.1 and proposition 1.4.

Remark. — We note that for the different topologies the continuous semi-norms of the form $p(f) = \sum_{n=0}^{\infty} p\left(\dfrac{\hat{d}^n f(0)}{n!}\right)$ generate the topology and also that such semi-norms are directed i.e., if p_1 and p_2 are two such semi-norms then there exists another such semi-norm p such that

$$p_i \leqslant p \quad \text{for} \quad i = 1,2$$

2. Generalised bases in $\mathcal{H}(U)$.

We now combine the results of the last section with results from the theory of generalised bases to find further relationships between the various topologies on $\mathcal{H}(U)$.

DEFINITION. — ([22], [23]) a) *A decomposition of a topological vector space E is a sequence of non-trivial subspaces of E, $(E_n)_{n=1}^{\infty}$, such that each x in E can be expressed uniquely in the form $x = \sum_{i=1}^{\infty} y_i$ where $y_i \in E_i$ for each i.*

b) *The decomposition is said to be Schauder if there exists a sequence of continuous orthogonal projections,* $(Q_n)_{n=1}^{\infty}$, *such that* $Q_n(E) = E_n$ *and* $x = \sum_{n=1}^{\infty} Q_n(x)$.

c) *If the projections* $R_n = \sum_{i=1}^{n} Q_i$ *are equicontinuous then the decomposition is said to be equiSchauder.*

d) *A Schauder decomposition,* $(E_n, Q_n)_{n=1}^{\infty}$, *is said to be shrinking if* $(E_n', Q_n')_{n=1}^{\infty}$ *is a Schauder decomposition for* $(E', \beta(E', E))$ *where* E_n' *is the topological dual of* E_n, Q_n' *is the adjoint of* Q_n *and* $\beta(E', E)$ *is the strong dual topology on* E'.

PROPOSITION 2.1. – $(\mathscr{P}(^nE), \mathscr{T})_{n=0}^{\infty}$ *is an equiSchauder shrinking decomposition for* $(\mathscr{H}(U), \mathscr{T})$ *where* $\mathscr{T} = \mathscr{T}_{o,b}, \mathscr{T}_{\omega}, \mathscr{T}_{\omega,b}$ or $\mathscr{T}_{\delta}$.

Proof. – Since the set of all $\mathscr{T}$-continuous semi-norms on $\mathscr{H}(U)$ which satisfy the condition $p(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right)$ generate the $\mathscr{T}$ topology on $\mathscr{H}(U)$ it is immediate that $(\mathscr{P}(^nE), \mathscr{T})_{n=0}^{\infty}$ is an équi-Schauder decomposition for $\mathscr{H}(U)$.

Now let $T \in (\mathscr{H}(U), \mathscr{T})'$ then $T = \sum_{n=0}^{\infty} T_n$ where $T_n \in (\mathscr{P}(^nE), \mathscr{T})'$ and $T(f) = \sum_{n=0}^{\infty} T_n\left(\frac{\hat{d}^n f(0)}{n!}\right)$. Hence $\widetilde{T}(f) = \sum_{n=0}^{\infty} \left|T_n\left(\frac{\hat{d}^n f(0)}{n!}\right)\right|$ is a $\mathscr{T}$-continuous semi-norm on $\mathscr{H}(U)$. An application of lemma 1.2 completes the proof for $\mathscr{T} = \mathscr{T}_{o,b}, \mathscr{T}_{\omega}, \mathscr{T}_{\omega,b}$.

There remains the case $\mathscr{T} = \mathscr{T}_{\delta}$. By lemma 1.1. and proposition 1.4. we get that

$$\widetilde{T}_1(f) = \sum_{n=0}^{\infty} n^2 \left|T_n\left(\frac{\hat{d}^n f(0)}{n!}\right)\right|$$

is a $\mathscr{T}_{\delta}$-continuous semi-norm on $\mathscr{H}(U)$. Hence if B is a $\mathscr{T}_{\delta}$ bounded subset of H(U) we have

$$\sup_{f \in B} \left|T(f) - \sum_{j=0}^{n} T_j\left(\frac{\hat{d}^j f(0)}{n!}\right)\right| \to 0 \quad \text{as } n \to \infty.$$

This completes the proof for $\mathscr{T} = \mathscr{T}_{\delta}$.

PROPOSITION 2.2. – *$(\mathcal{H}(U), \mathcal{T})$ $(\mathcal{T} = \mathcal{T}_{o,b}, \mathcal{T}_{\omega,b})$ is barrelled if and only if the following conditions hold.*

1) $(\mathcal{P}(^nE), \mathcal{T})$ is barrelled for each n.

2) If $T_n \in (\mathcal{P}(^nE), \mathcal{T})'$ for each n and $\sum_{n=0}^{\infty} T_n\left(\frac{\hat{d}^n f(0)}{n!}\right)$ converges for each $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathcal{H}(U)$

then
$$\sum_{n=0}^{\infty} T_n \in (\mathcal{H}(U), \mathcal{T})'.$$

Proof. – We apply corollary p. 381 [23]. Since $(\mathcal{H}(U), \mathcal{T})$ is bornological it has the Mackey topology. It remains to show our decomposition of $\mathcal{H}(U)$ is Schauder with respect to $\beta((\mathcal{H}(U), \mathcal{T})\,;\,(\mathcal{H}(U), \mathcal{T})')$. Let B be a $\sigma((\mathcal{H}(U), \mathcal{T})', \mathcal{H}(U))$ bounded subset of $(\mathcal{H}(U), \mathcal{T})'$. We must show

$$\sup_{T\in B} \left| \sum_{n=m}^{\infty} T\left(\frac{\hat{d}^n f(0)}{n!}\right)\right| \to 0 \quad \text{as } m \to \infty$$

for a given fixed $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathcal{H}(U)$. Suppose this were not true then there exists $\varepsilon > 0$ and $(n_j)_{j=1}^{\infty}$ an increasing sequence of positive integers and $T_{n_j} \in B$ such that,

$$\left| \sum_{n=n_j+1}^{n_{j+1}} T_{n_j} \left(\frac{\hat{d}^n f(0)}{n!}\right)\right| \geqslant \varepsilon \tag{*}$$

By using (*) and the fact that $\sum_{n=0}^{\infty} \left|T\left(\frac{\hat{d}^n f(0)}{n!}\right)\right| < \infty$ for each $T \in B$ we can construct a sequence $\alpha = (\alpha_n)_{n=0}^{\infty} \in \mathcal{S}$ such that

$$\sup_{T\in B} \left| \sum_{n=0}^{\infty} \alpha_n T\left(\frac{\hat{d}^n f(0)}{n!}\right)\right| = \infty$$

But this contradicts the fact that $\sum_{n=0}^{\infty} \alpha_n \frac{\hat{d}^n f(0)}{n!} \in \mathcal{H}(U)$ and hence completes the proof.

COROLLARY 2.1. – *$(\mathcal{H}(U), \mathcal{T}_{o,b})$ is barrelled if and only if the following conditions hold ;*

1) $(\mathcal{P}(^nE), \mathcal{T}_{o,b})$ is barrelled for each n,

2) $\mathcal{T}_{o,b}$ is the finest locally convex topology on $\mathcal{H}(U)$ for which the Taylor series converges absolutely and which, induces on $\mathcal{P}(^nE)$ the $\mathcal{T}_{o,b}$ topology for each n.

Proof. – By the proceeding proposition if $(\mathcal{H}(U), \mathcal{T}_{o,b})$ is barrelled then 1) is satisfied.

Let p be a semi-norm on $\mathcal{H}(U)$ for which the Taylor series converges absolutely and which induces on $\mathcal{P}(^nE)$ the $\mathcal{T}_{o,b}$ topology for each n. The set

$$V = \left\{ f \in \mathcal{H}(U) \mid \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leq 1 \right\}$$

is a closed convex balanced and absorbing subset of $(\mathcal{H}(U), \mathcal{T}_{o,b})$ and hence a neighbourhood of zero. Hence 2) is also satisfied.

Now suppose 1) and 2) are satisfied and let $T_n \in (\mathcal{P}(^nE), \mathcal{T}_{o,b})'$ for each n be such that

$$\sum_{n=0}^{\infty} T_n\left(\frac{\hat{d}^n f(0)}{n!}\right)$$

is convergent for each $f \in \mathcal{H}(U)$. By lemma 1.1 $\sum_{n=0}^{\infty} \left| T_n\left(\frac{\hat{d}^n f(0)}{n!}\right) \right| < \infty$ for all $f \in \mathcal{H}(U)$. Hence $\sum_{n=0}^{\infty} T_n \in (\mathcal{H}(U), \mathcal{T}_{o,b})'$ and an application of the proceeding proposition completes the proof.

PROPOSITION 2.3. – *The following are equivalent ;*

1) $(\mathcal{H}(U), \mathcal{T}_\delta) = (\mathcal{H}(U), \mathcal{T}_{\omega,b})$.

2) $(\mathcal{H}(U), \mathcal{T}_{\omega,b})$ is barrelled.

3) $\mathcal{T}_{\omega,b}$ is the finest locally convex topology on $\mathcal{H}(U)$ for which the Taylor series converges absolutely and which induces on $\mathcal{P}(^nE)$ the $\mathcal{T}_\omega$ topology for each n.

Proof. – We have already seen the 3) $\Rightarrow$ 1) $\Rightarrow$ 2) and the same method as in the previous corollary can be used to show 2) $\Rightarrow$ 3).

We now turn to be problem of completeness for $(\mathcal{H}(U), \mathcal{T})$.

DEFINITION. – *Let $\mathcal{T}$ be a locally convex topology on $\mathcal{H}(U)$ then U is T.S.$\mathcal{T}$ (T.S. = Taylor series) complete if whenever $(P_n)_{n=0}^{\infty}$ is a sequence of continuous n-homogeneous polynomials on E and $\sum_{n=0}^{\infty} p(P_n) < \infty$ for any $\mathcal{T}$-continuous semi-norm p on $\mathcal{H}(U)$ then*

$$\sum_{n=0}^{\infty} P_n \in \mathcal{H}(U).$$

PROPOSITION 2.4. – *$(\mathcal{H}(U), \mathcal{T})$ $(\mathcal{T} = \mathcal{T}_\omega, \mathcal{T}_{\omega,b}, \mathcal{T}_{o,b}, \mathcal{T}_\delta)$ is complete (resp. quasi complete, sequentially complete) if and only if the following conditions hold*

1) *$(\mathcal{P}(^nE), \mathcal{T})$ is complete (resp. quasi-complete, sequentially complete) for each n.*

2) *U is T.S.$\mathcal{T}$ complete.*

Proof. – Apply the theorem of [22] and the fact that our decomposition is equiSchauder.

Remark. – 1) U is T.S.$\mathcal{T}_{o,b}$ (resp. T.S.$\mathcal{T}_{\omega,b}$) complete if and only if it is T.S.$\mathcal{T}_0$ (resp. T.S.$\mathcal{T}_\omega$) complete.

2) If $\mathcal{T}_\alpha \geqslant \mathcal{T}_\beta$ on $\mathcal{H}(U)$ then if U is T.S.$\mathcal{T}_\beta$ complete it is also T.S.$\mathcal{T}_\alpha$ complete.

COROLLARY 2.2. – *$(\mathcal{H}(U), \mathcal{T}_{\omega,b})$ is complete if and only if $(\mathcal{H}(U), \mathcal{T}_\omega)$ is complete.*

PROPOSITION 2.5. – *$(\mathcal{H}(U), \mathcal{T})$ $(\mathcal{T} = \mathcal{T}_\omega, \mathcal{T}_{\omega,b}, \mathcal{T}_{o,b}, \mathcal{T}_\delta)$ is semi-reflexive if and only if the following conditions hold.*

1) *$(\mathcal{P}(^nE), \mathcal{T})$ is semi-reflexive for each n.*

2) *U is T.S.$\mathcal{T}$ complete.*

Proof. – Apply theorem 3.2 of [23].

Condition 2) of the above proposition is not surprising since $(\mathcal{H}(U), \mathcal{T})$ $(\mathcal{T} = \mathcal{T}_{o,b}, \mathcal{T}_{\omega,b}, \mathcal{T}_\delta)$ is reflexive if and only if it is semi-reflexive and a reflexive space is always quasi-complete.

It is also possible to use this method to give necessary and sufficient conditions for $(\mathcal{H}(U), \mathcal{T})$ to be Montel, Schwartz, etc. We restrict ourselves to stating some results of this kind.

PROPOSITION 2.6. – a) $(\mathcal{H}(U), \mathcal{T})$ $(\mathcal{T} = \mathcal{T}_{o,b}, \mathcal{T}_{\omega,b}, \mathcal{T}_\omega, \mathcal{T}_\delta)$ *is a Schwartz space if and only if* $(\mathcal{P}(^nE), \mathcal{T})$ *is a Schwartz space for each n.*

b) $(\mathcal{H}(U), \mathcal{T})$ $(\mathcal{T} = \mathcal{T}_{o,b}, \mathcal{T}_{\omega,b}, \mathcal{T}_\delta)$ *is a Montel space if and only if* $(\mathcal{P}(^nE), \mathcal{T})$ *is Montel for each n and* U *is* T.S.$\mathcal{T}_\omega$*-complete.*

3. Examples and Counterexamples on $\mathcal{H}(U)$.

In the last section we found different necessary and sufficient conditions on $\mathcal{H}(U)$ for the various topologies to coincide. In this section we see how difficult it is to find necessary and sufficient conditions on E for the topologies to coincide. We provide at the same time a number of interesting examples of the different topologies which seem to indicate that if such conditions exist they will not be of the usual functional analytic kind (e.g. barrelled, bornological or complete) but may possibly be of the form, there exists a continuous norm on E or the compact subsets of E are all finite dimensional. We hope to return to this classification problem at a future date. For the moment we concentrate on special examples related to the following questions.

1) When is $\mathcal{T}_{o,b} = \mathcal{T}_{\omega,b}$?

2) When is $\mathcal{T}_{\omega,b} = \mathcal{T}_\delta$?

3) When is $\mathcal{T}_\omega = \mathcal{T}_{\omega,b}$?

The following is a useful starting point and we shall also need this result in Paper II.

LEMMA 3.1. – *If* U *is an arbitrary open subset of a metrizable space then*

$$(\mathcal{H}(U), \mathcal{T}_{o,b}) = (\mathcal{H}(U), \mathcal{T}_{\omega,b}) = (\mathcal{H}(U), \mathcal{T}_\delta)$$

Proof. – Let B be a bounded subset of $(\mathcal{H}(U), T_0)$. For each n let

$$V_n = \{x \in U, \ |f(x)| < n \quad \text{for all} \quad f \in B\}$$

Let W_n be the interior of V_n. Since E is metrizable $\bigcup_{n=1}^{\infty} W_n = U$. Hence B is a bounded subset of $(\mathcal{H}(U), \mathcal{T}_\delta)$. Since $(\mathcal{H}(U), \mathcal{T}_{o,b})$ is bornological and $\mathcal{T}_\delta \geqslant \mathcal{T}_{o,b}$ this implies $(\mathcal{H}(U), \mathcal{T}_{o,b}) = (\mathcal{H}(U), \mathcal{T}_\delta)$.

An examination of the proof suggests we proceed in the following direction.

DEFINITION. – *A subset* B *of* H(U) *is said to be equibounded if it is uniformly bounded in a neighbourhood of each point.*

By the method of the last lemma we see immediately that if every $\mathcal{T}_0$-bounded subset of $\mathcal{H}(U)$ is equibounded then

$$(\mathcal{H}(U), \mathcal{T}_{o,b}) = (\mathcal{H}(U), \mathcal{T}_{\omega,b}) = (\mathcal{H}(U), \mathcal{T}_\delta).$$

LEMMA 3.2. – *If* E *is a Baire space then every $\mathcal{T}_0$-bounded subset of $\mathcal{H}(E)$ is equibounded.*

Proof. – Let B be a bounded subset of $(\mathcal{H}(E), \mathcal{T}_0)$. By lemma 1.2. $\left(\dfrac{\hat{d}^n f(0)}{n!}\right)_{n=0, f \in B}^{\infty}$ is also a bounded subset of $(\mathcal{H}(E), \mathcal{T}_0)$. For each m let

$$V_n = \left\{x, \ \left|\frac{\hat{d}^n f(0)}{n!}(x)\right| \leqslant m \ \text{for all } n \text{ and } f \in B\right\}$$

Then $\bigcup_{m=1}^{\infty} V_m = E$ and since E is Baire there exists n_1 a positive integer, $x \in E$ and V a convex neighbourhood of 0 in E such that

$$x + V \subset V_{n_1}.$$

By the method used in lemma 1 [13] we immediately get that $V \subset V_{n_1}$. Hence

$$\sup_{f \in B} \|f\|_{\frac{1}{2}V} < \infty .$$

This means B is bounded in a neighbourhood of 0. Since the point 0 was arbitrarily chosen we get that B is equibounded.

LEMMA 3.3. – *If* E *is complete and every* $\mathcal{T}_0$ *bounded subset of* $\mathcal{H}(U)$ *is equi-bounded then* E *is barrelled.*

Proof. – Let V be a closed convex balanced absorbing subset of E.

Let $$B = \{\varphi \in E' , \|\varphi\|_V \leqslant 1\}$$

Since V is a barrel in E, V absorbs ([21] p. 208) every closed balanced convex complete subset of E. Since E is complete V absorbs all compact subsets of E. Hence B is a $\mathcal{T}_0$-bounded subset of $\mathcal{H}(U)$. By hypothesis there exists W a neighbourhood of 0 in E such that

$$\sup_{\varphi \in B} \|\varphi\|_W \leqslant 1$$

The Hahn-Banach theorem implies that $W \subset V$. Hence V is a neighbourhood of 0 and E is barrelled.

LEMMA 3.1. implies that the completeness condition is necessary in lemma 3.3.

We now ask ourselves a series of questions all arising from the above and all of which we answer.

Q.1 Is every $\mathcal{T}_\delta$-bounded subset of $\mathcal{H}(E)$ equibounded ?

Q.2 If E is complete and barrelled is $\mathcal{T}_\delta = \mathcal{T}_{\omega,b} = \mathcal{T}_{o,b}$ on $\mathcal{H}(E)$?

Q.3 If E is complete but not a Baire space can we have

$$\mathcal{T}_\delta = \mathcal{T}_{\omega,b} = \mathcal{T}_{o,b} \text{ on } \mathcal{H}(E) \text{ ?}$$

Q.4 If $(\mathcal{H}(E_i), \mathcal{T}_\delta) = (\mathcal{H}(E_i), \mathcal{T}_{\omega,b}) = (\mathcal{H}(E_i), \mathcal{T}_{o,b})$ for $i = 1, 2$, is $\mathcal{T}_\delta = \mathcal{T}_{o,b} = \mathcal{T}_{\omega,b}$ on $\mathcal{H}(E_1 \times E_2)$?

These questions will be answered in the examples which we give in the remainder of this section.

Let $E = \sum_{i=1}^{\infty} E_i$ where each E_i is a Banach space and let E be endowed with the direct sum topology.

We let $F_n = \sum_{i=n+1}^{\infty} E_i$ and $F^n = \sum_{i=1}^{n} E_i$.

LEMMA 3.4. – *For* $E = \sum_{i=1}^{\infty} E_i$ *and* $P \in \mathcal{P}(^m E)$, *if we let* P_n *be defined by*

$$P_n(x + y) = P_n(x) \text{ for all } x \in F_n \text{ and } y \in F^n ,$$

then $P_n \to P$ *uniformly in a neighbourhood of each point of* E.

Proof. – It is immediate that we only need consider a neighbourhood of 0. Choose V_1 open convex balanced and bounded in E_i such that $\|P\|_{V_1} \leqslant 1$.

For $x_i \in E_i$, $i = 1, 2$.

$$P_2(x_1 + x_2) = P(x_1 + x_2) = \sum_{k=0}^{m} \binom{m}{k} A(x_1)^k (x_2)^{n-k}$$

where A is the symmetric n-linear form associated with P ([28]). Hence there exists V_2 open balanced convex and bounded in E_2 such that

$$\sup_{\substack{x_i \in V_i \\ i=1,2}} \sum_{0 < k < m} \binom{m}{k} |A(x_1)^k (x_2)^{n-k}| \leqslant \frac{1}{2}$$

$$\text{i.e.} \quad |P_2(x_1 + x_2) - P_1(x_1 + x_2)| \leqslant \frac{1}{2}$$

By induction choose V_n open convex balanced and bounded in E_n such that

$$|P_n\Big(\sum_{i=1}^{n} x_i\Big) - P_{n-1}\Big(\sum_{i=1}^{n-1} x_i\Big)| \leqslant \frac{1}{2^n}$$

$$\text{for} \quad \sum_{i=1}^{n} x_i \in \sum_{i=1}^{n} V_i$$

$V = \sum_{i=1}^{\infty} V_i$ is a neighbourhood of 0 in E.

Now for $\sum_{i=1}^{k} x_i \in V$,

$$|P(x) - P_n(x)| = \left|P\left(\sum_{i=1}^{k} x_i\right) - P\left(\sum_{i=1}^{n} x_i\right)\right| \leqslant$$

$$\leqslant \sum_{j=n+1}^{k} \left|P\left(\sum_{i=1}^{j} x_i\right) - P\left(\sum_{i=1}^{j-1} x_i\right)\right|$$

$$\leqslant \sum_{j=n+1}^{\infty} \frac{1}{2^j} \leqslant \frac{1}{2^n}$$

Hence $\| P - P_n \|_V \to 0$ as $n \to \infty$.

Proposition 3.1. – *Let* $E = \sum_{i=1}^{\infty} E_i$ *where each* E_i *is a Banach space then* $\mathcal{T}_{o,b} = \mathcal{T}_{\omega,b} = \mathcal{T}_\delta$ *on* $\mathcal{H}\left(\sum_{i=1}^{\infty} E_i\right)$.

Proof. – Let p be a $\mathcal{T}_\delta$-continuous semi-norm on $\mathcal{H}\left(\sum_{i=1}^{\infty} E_i\right)$. We can suppose (by proposition 1.1.) that

$$p(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n\,!}\right) \quad \text{for all} \quad f \in \mathcal{H}(E)$$

We claim there exists m a positive integer such that if $f \in \mathcal{H}(E)$ and $f|_{F^m} = 0$ then $p(f) = 0$. If this were not true then by using the proceeding lemma and the fact that polynomials are dense in $(\mathcal{H}(E), \mathcal{T}_\delta)$ we can construct $(P_n)_{n=1}$ a sequence of homogeneous polynomials on E such that for some increasing sequence of integers $(k_n)_{n=1}^{\infty}$ we have the following

1) $P_n|_{F^{k_n}} = 0$

2) $P_n(x + y) = P_n(x)$

$$\text{for all} \quad x \in F^{k_{n+1}}, \; y \in F_{k_{n+1}}$$

3) $p(P_n) \neq 0$ for any n.

We claim $(\lambda_n P_n)_{n=1}^{\infty}$ is a T_δ-bounded sequence in $\mathcal{H}(E)$ for any sequence of scalars $(\lambda_n)_{n=1}^{\infty}$. We prove this by showing the sequence is equibounded. Let $x \in E$ be arbitrary then $x \in \sum_{i=1}^{l} E_i$ for some positive integer l. Without loss of generality we can suppose $l \leqslant n_1$.

Choose V_{n_1} a convex balanced bounded neighbourhood of 0 in F^{n_1} which contains x.

Let
$$\alpha = 2\,|\lambda_1|\,\|P_1\|_{V_{n_1}}$$

Now for $x_1 \in F^{n_1}$, $x_2 \in \sum_{i=n_1+1}^{n_2} E_i$ and $x_3 \in F_{n_2}$ we have

$$P_2(x_1 + x_2 + x_3) = P_2(x_1 + x_2) =$$

$$= \sum_{0 \leqslant k < \gamma_2} \binom{\gamma_2}{k} A_2(x_1)^k (x_2)^{j_2 - k} \quad (*)$$

where γ_2 is the order of P_2 and A_2 is the symmetric γ_2-linear mapping associated with P_2.

Since (*) does not contain the summation $k = \gamma_2$ we can choose a convex balanced bounded neighbourhood of 0 in $\sum_{i=n_1+1}^{n_2} E_i$, V_{n_2}, such that

$$|\lambda_2| \cdot \|P_2\|_{V_{n_1}+V_{n_2}} < \alpha$$

By using expansions similar to (*) and an inductive argument we find V a neighbourhood of x in E such that

$$|\lambda_n| \cdot \|P_n\|_V < \alpha \quad \text{for each } n.$$

Hence $(\lambda_n P_n)_{n=0}^{\infty}$ is a T_δ-bounded sequence.

If we now take $\lambda_n = \dfrac{n}{p(P_n)}$ we get a contradiction to the fact

that $\qquad (\lambda_n P_n)_{n=1}^{\infty}$ is $\mathcal{T}_\delta$-bounded.

Hence there exists m a positive integer such that if

$$f \in \mathcal{H}(\mathrm{E})\,,\ f\,|_{\mathrm{F}^m} = 0$$

then

$$p(f) = 0\,.$$

Let B be a $\mathcal{C}_0$-bounded subset of $\mathcal{H}(\mathrm{E})$. For each $f \in \mathrm{B}$ we let $\widetilde{f} \in \mathcal{H}(\mathrm{E})$ denote the element of $\mathcal{H}(\mathrm{E})$ defined by

$$\widetilde{f}(x+y) = f(x) \quad \text{for } x \in \mathrm{F}^m\,,\ y \in \mathrm{F}_m$$

Hence $(\widetilde{f})_{f \in \mathrm{B}}$ is an equibounded subset of $\mathcal{H}(\mathrm{E})$. This implies that $\sup\limits_{f \in \mathrm{B}}.\ p(\widetilde{f}) < \infty$. For each $f \in \mathrm{B}$ we have $f - \widetilde{f}\,|_{\mathrm{F}^m} = 0$, hence

$$p(f - \widetilde{f}) = 0 \text{ i.e. } \quad p(f) = p(\widetilde{f})\,.$$

Hence $\sup\limits_{f \in \mathrm{B}}.\ p(f) < \infty$. This implies that p is $\mathcal{C}_{o,b}$ continuous on $\mathcal{H}(\mathrm{E})$ and on $\mathcal{H}\left(\sum\limits_{i=1}^{\infty} \mathrm{E}_i\right)$

$$\mathcal{C}_\delta = \mathcal{C}_{\omega,b} = \mathcal{C}_{o,b}\,.$$

Remarks. –

1) This answers Q.3 since $\sum\limits_{i=1}^{\infty} \mathrm{E}_i$ is not Baire if $\mathrm{E}_i \neq 0$ for an infinite number of i's.

2) By using the fact that $\mathcal{C}_{o,b} = \mathcal{C}_\delta$ on $\mathcal{H}\left(\sum\limits_{i=1}^{\infty} \mathrm{E}_i\right)$ one easily shows that

$$\mathcal{C}_0 = \mathcal{C}_{o,b} = \mathcal{C}_\omega = \mathcal{C}_{\omega,b} = \mathcal{C}_\delta \quad \text{on} \quad \mathcal{H}\left(\sum_{i=1}^{\infty} \mathrm{E}_i\right)$$

if E_i is a finite dimensional space for each i.

3) We see later that this example also provides a solution to Q.1 but we first proceed to construct an example which provides a solution to questions 2 and 4. Let $\mathrm{E}_1 = \sum\limits_{i=1}^{\infty} \mathrm{C}$ and $\mathrm{E}_2 = \prod\limits_{i=1}^{\infty} \mathrm{C}$. Since E_2 is Frechet $\mathcal{C}_{o,b} = \mathcal{C}_{\omega,b} = \mathcal{C}_\delta$ on $\mathcal{H}(\mathrm{E}_2)$ and proposition 3.1 shows that the same is true of $\mathcal{H}(\mathrm{E}_1)$. We let $((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty})$ denote the generic point of $\mathrm{E}_1 \times \mathrm{E}_2$ (note this implies $x_n = 0$ for all n sufficiently

large. U_n (resp. V_n) will denote the element of E_1 (resp. E_2) for which $x_m = 0$ for $n \neq m$ (resp. $y_m = 0$ for $n \neq m$) and $x_n = 1$ (resp. $y_n = 1$).

PROPOSITION 3.2. – $(\mathcal{H}(E_1 \times E_2), \mathcal{T}_{o,b}) \neq (\mathcal{H}(E_1 \times E_2), \mathcal{T}_{\omega,b})$.

Proof. – Let Φ_n be the 2-homogeneous polynomials on $E_1 \times E_2$ defined as follows :

$$\Phi_n (\{x_m\}_{m=1} ; \{y_m\}_{m=1}) = x_n y_n$$

We claim $(\Phi_n)_{n=1}^\infty$ is a $\mathcal{T}_0$-bounded subset of $\mathcal{P}(^2E_1 \times E_2)$. Let K be compact in $E_1 \times E_2$. Then $K \subset K_n \times \prod_{i=1}^\infty L_i$ where K_n is a compact subset of $\sum_{i=1}^\infty C$ and hence is contained in some finite dimensional subspace of $\sum_{i=1}^\infty C$ and L_i is compact in C for each i.

Hence $\|\Phi_n\|_K = 0$ for all n sufficiently large and thus we have

$$\sup_n \|\Phi_n\|_K < \infty .$$

This implies $(\Phi_n)_{n=1}^\infty$ is a $\mathcal{T}_0$-bounded subset of $\mathcal{P}(^2E_1 \times E_2)$. We now show $(\Phi_n)_{n=1}^\infty$ is not a $\mathcal{T}_\omega$-bounded subset of $\mathcal{P}(^2E_1 \times E_2)$. For each $P \in \mathcal{P}(^2E_1 \times E_2)$ let

$$T(P) = \sum_{n=1}^\infty |P(U_n, nv_n) - P(U_n, 0)|$$

By the same method as used in [26] for $\prod_{i=1}^\infty C$ one shows that T(P) is finite for each $P \in \mathcal{P}(^2E_1 \times E_2)$. Since $(\mathcal{P}(^2E_1 \times E_2), \mathcal{T}_\omega)$ is barrelled this implies that T is a $\mathcal{T}_\omega$-continuous semi-norm on $\mathcal{P}(^2E_1 \times E_2)$.

We have however

$$T(\Phi_n) = n$$

and hence $\sup_n T(\Phi_n) = \infty$. Hence $(\Phi_n)_{n=1}^\infty$ is not $\mathcal{T}_\omega$-bounded on $\mathcal{P}(^2E_1 \times E_2)$ and thus $\mathcal{T}_{\omega,b} \neq \mathcal{T}_{o,b}$ on $\mathcal{P}(^2E_1 \times E_2)$ and on $\mathcal{H}(E_1 \times E_2)$.

Remarks. –

1) $E_1 \times E_2$ is a complete barrelled space but $\mathcal{T}_{\omega,b} \neq \mathcal{T}_{o,b}$ on $\mathcal{H}(E_1 \times E_2)$ and this provides an answer to Q.2.

2) Q.4 is also answered in the negative by this proposition.

3) $E_1 \times E_2$ is an example of a barrelled space which is not C-barrelled in the sense of Lelong ([24], [25]).

4) In the above examples we either have $\mathcal{T}_{o,b} \neq T_{\omega,b}$ or $\mathcal{T}_{o,b} = \mathcal{T}_{\omega,b} = \mathcal{T}_\delta$. We do not yet have an example in which $\mathcal{T}_{\omega,b} \neq \mathcal{T}_\delta$ for H(U) (see also [29]).

5) An example is given in [29] in which $\mathcal{T}_{\omega,b} \neq \mathcal{T}_{o,b}$.

We now discuss the question of when $\mathcal{T}_\omega = \mathcal{T}_{\omega,b}$. We commence by discussing bounding sets since if there exists a non-compact bounding set then $\mathcal{T}_\omega \neq \mathcal{T}_\delta$ ([12], [13], [14]).

DEFINITION. – *A closed subset* C *of* E *is said to be bounding if*

$$\|f\|_C < \infty$$

for each $f \in \mathcal{H}(E)$.

PROPOSITION 3.3. – *Let* E *be a locally convex space such that* (E , *p*) *is separable or reflexive for each continuous semi-norm* *p* *on* E *then the bounding subsets of* E *are precompact.*

Proof. – If $C \subset E$ is not precompact then there exists p a continuous semi-norm on E such that C is not precompact in (E , *p*). An application of the result from Banach spaces ([12], [17]) now completes the proof.

COROLLARY 3.1. – *If* E *is complete and separable then the bounding subsets of* E *are compact.*

One of the interesting questions concerning bounding sets is to characterise the E for which they are relatively compact. This problem is intimately related to the following problem, if E is a locally convex topological vector space with completion $\hat{E}$ and F is the largest subspace of $\hat{E}$ such that all $f \in \mathcal{H}(E)$ can be extended (uniquely) to elements of $\mathcal{H}(F)$ is $\hat{E} = F$? This in turn involves one in characterising domains of holomorphy. We refer to [17], [34] for further details.

PROPOSITION 3.4. — *Let* E_i *be a Banach space with an unconditional basis for each* $i = 1, 2, \ldots$ *then on* $\mathcal{H}\left(\sum_{i=1}^{\infty} E_i\right)$.

$$\mathcal{T}_\omega = \mathcal{T}_{\omega,b} = \mathcal{T}_{o,b} = \mathcal{T}_\delta$$

Proof. — We have already seen that

$$\mathcal{T}_{\omega,b} = \mathcal{T}_{o,b} = \mathcal{T}_\delta \quad \text{on} \quad \mathcal{H}\left(\sum_{i=1}^{\infty} E_i\right)$$

Let p be a $\mathcal{T}_{o,b}$ continuous semi-norm on $\mathcal{H}(E)$ $\left(E = \sum_{i=1}^{\infty} E_i\right)$. For each n let $V_n = \left\{x \in E, x = \sum_{i=1}^{n} x_i, \sum_{i=1}^{n} \|x_i\|_i \leq n\right\}$ where $\| \ \|_i$ is a fixed norm on E_i for each i. Since $\bigcup_{n=1}^{\infty} V_n = E$ we can find(¹) $C > 0$ and n_1 a positive integer such that

$$p(f) \leq C \|f\|_{V_{n_1}} \quad (*)$$
$$\text{for all} \quad f \in \mathcal{H}(E)$$

Since $\sum_{i=1}^{n} E_i$ has an unconditional basis for each positive integer n we can apply the method of [13] to (*) to complete the proof.

We now find however that for a large class of locally convex spaces E

$$\mathcal{T}_\omega \neq \mathcal{T}_{\omega,b} \quad \text{on} \quad \mathcal{H}(E).$$

DEFINITION. — *A sequence* $(x_n)_{n=1}^{\infty}$ *of non zero elements of* E *(a locally convex space) is very strongly convergent to* 0 *if* $\lambda_n x_n \to 0$ *as* $n \to \infty$ *for any sequence of scalars* $(\lambda_n)_{n=1}^{\infty}$(²).

(¹) It is true in general that if $(V_n)_{n=1}^{\infty}$ is an increasing sequence of subsets of U such that each compact set is contained in some V_n and p is a $\mathcal{T}_{o,b}$ continuous semi-norm on $\mathcal{H}(U)$ then we can find $C > 0$ and n_1 a positive integer such that $p(f) \leq C \|f\|_{V_{n_1}}$ for all $f \in \mathcal{H}(U)$.

(²) It is easy to prove that $(x_n)_{n=1}^{\infty}$ is very strongly convergent to zero if and only if for each continuous semi-norm p on E we have $p(x_n) = 0$ for all sufficiently large n.

PROPOSITION 3.5. – *If* E *contains a very strongly convergent sequence then*

$$(\mathcal{H}(E), \mathcal{T}_\omega) \neq (\mathcal{H}(E), \mathcal{T}_{\omega,b}).$$

Proof. – For each $f \in \mathcal{H}(E)$.

Let
$$p(f) = \sum_{n=0}^{\infty} |f(ny + x_n) - f(ny)|$$

where y is a fixed non-zero element of E. Since each element of $\mathcal{H}(E)$ is continuous and each bounded holomorphic function is constant this immediately implies that $p(f)$ is finite for each $f \in \mathcal{H}(E)$.

Let B be a $\mathcal{T}_0$-bounded subset of $\mathcal{H}(E)$. Hence the set

$$\left(\frac{\hat{d}^n f(0)}{n!}\right)_{n=0, f\in B}^{\infty}$$

is a $\mathcal{T}_0$-bounded subset of $\mathcal{H}(E)$. Since the point 0 is not special to the definition of $\mathcal{T}_0$ we have also that

$$\left(\frac{\hat{d}^m}{m!}\left(\frac{\hat{d}^n f(0)}{n!}\right)(y)\right)_{m,n=0, f\in B}^{\infty}$$

is a bounded subset of $(\mathcal{H}(E), \mathcal{T}_0)$. The sequence $(\lambda_n x_n)_{n=1}^{\infty}$ is compact in E for any sequence of scalars $(\lambda_n)_{n=1}$ and we thus have

$$\sup_{\substack{f\in B \\ n,m=0,1,\ldots \\ k=1,2,}} \left|\left(\frac{\hat{d}^m}{m!}\left(\frac{\hat{d}^n f(0)}{n!}\right)(y)\right)(\lambda_k x_k)\right| < \infty$$

i.e.
$$\sup_{\substack{f\in B \\ n_1 m=0,1, \\ m\leqslant n \\ k=1,2,\ldots}} \left||\lambda_k|^{n-m}\left(\frac{\hat{d}^m}{m!}\left(\frac{\hat{d}^n f(0)}{n!}\right)(y)\right)(x_k)\right| < \infty$$

This means there exists k_1 a positive integer such that

$$\left(\frac{\hat{d}^m}{m!}\left(\frac{\hat{d}^n f(0)}{n!}\right)(y)\right)(x_k) = 0$$

for $\quad k \geqslant k_1 \quad$ and $\quad n > m$.

Since

$$\frac{\hat{d}^n f(0)}{n!}(ny + x_k) = \sum_{m=0}^{n} \left(\frac{\hat{d}^m}{m!}\left(\frac{\hat{d}^n f(0)}{n!}\right)(ny)\right)(x_k) =$$

$$= \sum_{m=0}^{n} n^m \left(\frac{\hat{d}^m}{m!}\left(\frac{\hat{d}^n f(0)}{n!}(y)\right)(x_k)\right)$$

this implies

$$\frac{\hat{d}^n f(0)}{n!}(ny + x_k) = \frac{\hat{d}^n f(0)}{n!}(ny)$$

$$\text{for all} \quad k \geqslant k_1 \,,\; n = 0, 1, \ldots$$

$$\text{and} \quad f \in \mathrm{B}\,.$$

Hence

$$f(ny + x_k) = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}(ny + x_k) = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}(ny) = f(ny)$$

for all $k \geqslant k_1$ and $f \in \mathrm{B}$.

This means that

$$\sup_{f \in \mathrm{B}} p(f) = \sup_{f \in \mathrm{B}} \sum_{j=0}^{k} |f(jy + x_j) - f(jy)|$$

$$< \infty$$

Hence p is $\mathcal{T}_{o,b}$ (and hence $\mathcal{T}_{\omega,b}$) continuous on $\mathcal{H}(\mathrm{E})$.

Now suppose p was ported by the compact subset K of E. Choose $\Phi \in \mathrm{E}'$ such that $\Phi(y) \neq 0$ and $\Phi(x_n) = 0$ for all n (by using subsequences if necessary). Again by using subsequences if necessary we can choose a sequence of elements of E$'$ such that

1) $\Phi_n(x_m) \neq 0$ if and only if $n = m$.

2) $\Phi_n(y) = 0$ for all n.

Let $\varphi_{n,m} = \Phi^n \cdot \Phi_m$ for each pair of integers n and m. We now have

$$p(\varphi_{n,m}) = |m^n \Phi(y)^n \Phi_m(x_m)|$$

If V is a neighbourhood of K in E we have by assumption $\mathrm{C(V)} > 0$ such that

$$p(f) \leqslant \mathrm{C(V)} \, \|f\|_{\mathrm{V}} \quad \text{for all} \quad f \in \mathcal{H}(\mathrm{E})$$

Hence

$$m^n \,|\Phi(y)|^n \,|\Phi_m(x_m)| \leqslant C(V) \,\|\Phi\|_V^n \,\|\Phi_m\|_V$$

Taking n^{th} roots of both sides and letting $n \to \infty$ we get

$$m \cdot |\Phi(y)| \leqslant \|\Phi\|_V$$

Now letting $V \to K$ (in the obvious way) we get

$$m \cdot |\Phi(y)| \leqslant \|\Phi\|_K$$

This is impossible since m is arbitrary. Hence $(\mathcal{H}(E), \mathcal{T}_\omega)$ is not bornological.

COROLLARY 3.2. – *If* E *is a metrizable space on which there exists no continuous norm then* $(\mathcal{H}(E), \mathcal{T}_\omega)$ *is not bornological.*

Proof. – Let $(p_n)_{n=1}^\infty$ be an increasing sequence of semi-norms of E such that $p_n \leqslant p_{n+1}$ for each n and $(p_n)_{n=1}$ defines the topology on E. Choose $x_n \in E$, $x_n \neq 0$ such that $p_n(x_n) = 0$ for each n. Then it is easily seen that $(x_n)_{n=1}^\infty$ is a very strongly convergent sequence in E. Hence $(\mathcal{H}(E), \mathcal{T}_\omega)$ is not bornological by the proceeding proposition.

Examples :

1) $E = \prod_{i \in A} E_i$ where A is any infinite indexing set and E_i is a locally convex space for each $i \in A$(1).

2) $E = \mathcal{C}^p(\Omega)$ where Ω is any non empty open subset of $\mathbf{R}^n$ and $\mathcal{C}^p(\Omega)$ is the set of p-times continuously differentiable functions on Ω $(1 \leqslant p \leqslant \infty)$

3) $E = \mathcal{L}^p_{\mathrm{loc}}(\mathbf{R}^n)$.

We note that the existence of a continuous norm on E is not sufficient to ensue that $(\mathcal{H}(E), \mathcal{T}_\omega)$ is bornological (see [13] where it is shown that $(\mathcal{H}(l_\infty), \mathcal{T}_\omega)$ is not bornological).

(1) This generalises the following result announced in [29] ; $\left(\mathcal{H}\left(\prod_{i=1}^\infty C\right), \mathcal{T}_\omega\right)$ is not bornological.

We now discuss the question of completeness for $(\mathcal{H}(E), \mathcal{T})$. We have seen already that $(\mathcal{H}(E), \mathcal{T})$ $(\mathcal{T} = \mathcal{T}_\omega, \mathcal{T}_{\omega,b}, \mathcal{T}_{o,b}, \mathcal{T}_\delta)$ is complete if and only if $(\mathcal{P}(^nE), \mathcal{T})$ is complete for each n and if $P_n \in \mathcal{P}(^nE)$ for each n and $\sum_{n=0}^{\infty} p(\mathcal{P}_n) < \infty$ for all $\mathcal{T}$-continuous semi-norms p on $\mathcal{H}(E)$ implies $\sum_{n=0}^{\infty} P_n \in \mathcal{H}(E)$. In particular we immediately encounter the following problem, if $\sum_{n=0}^{\infty} \| P_n \|_K < \infty$ for all compact subsets K of E does $f = \sum_{n=0}^{\infty} P_n \in \mathcal{H}(E)$. A stronger problem has been considered in the literature where it has been solved for Banach spaces [36], Frechet spaces [33], Baire spaces [27] and for a variety of spaces in [19]. In particular we have

PROPOSITION 3.6. – *If* E *is a Frechet space then* $(\mathcal{H}(E), \mathcal{T}_{o,b})$ *is complete if and only if* $(\mathcal{P}(^nE), \mathcal{T}_\omega)$ *is complete for each n.*

For E Frechet $(\mathcal{P}(^nE), \mathcal{T}_\omega)$ is the inductive limit of a sequence of Banach space and one can apply the results of [21] to solve particular cases. By such a method we get ;

COROLLARY 3.3. – *If* $E = \prod_{i=1}^{\infty} E_i$ *where each* E_i *is a Banach space then*

$(\mathcal{H}(E), \mathcal{T}_{o,b})$ *and* $(\mathcal{H}(E), \mathcal{T}_\omega)$ *are both complete.*

We now discuss completeness of $(\mathcal{H}(E), \mathcal{T}_{o,b})$ where E is a countable direct sum of Banach spaces. We employ our earlier notation for direct sums. For E a Banach space $\mathcal{H}_b(E) = \{ f \in \mathcal{H}(E), \| f \|_B < \infty$ for each bounded subset B of E$\}$.

PROPOSITION 3.7. – $\left(\mathcal{H}\left(\sum_{i=1}^{\infty} E_i\right), \mathcal{T}\right)$ $(\mathcal{T} = \mathcal{T}_{o,b}, \mathcal{T}_{\omega,b}, \mathcal{T}_\omega, \mathcal{T}_\delta)$ *is complete if and only if* $\mathcal{H}_b(F^n) = \mathcal{H}(F^n)$ *for each n where* E_i *is a Banach space for each i and* $F^n = \sum_{i=1}^{n} E_i$ *for each n.*

COROLLARY 3.4. – *If* E_i *is a separable Banach space for each i then* $\left(\mathcal{H}\left(\sum_{i=1}^{\infty} E_i\right), \mathcal{T}\right)$ $(\mathcal{T} = \mathcal{T}_\omega, \mathcal{T}_{\omega,b}, \mathcal{T}_{o,b}, \mathcal{T}_\delta)$ *is complete if and only if* E_i *is a finite dimensional space for each i.*

Proof. – (of proposition). Let $E = \sum_{i=1}^{\infty} E_i$.

Since $\mathcal{T}_{o,b} = \mathcal{T}_{\omega,b} = \mathcal{T}_\delta$ on $\mathcal{H}(E)$ (by proposition 3.1) and since (corollary 2.2). $(\mathcal{H}(E), \mathcal{T}_{\omega,b})$ is complete if and only if $(\mathcal{H}(E), \mathcal{T}_\omega)$ is complete we need only consider the problem for $(\mathcal{H}(E), \mathcal{T}_\omega)$. Suppose without loss of generality that $\mathcal{H}_b(E_1) \neq \mathcal{H}(E_1)$.

Let $f_n \in \mathcal{H}(E_i)$ for each n have radius of boundedness $\frac{1}{n}$ ([28]) and suppose $f_n(0) = 0$ for all n. Let j_n denote the projection of E onto E_n. For each $n > 1$ let φ_n be a continuous linear form on E_n of norm 1. Now let $f(x) = \sum_{n=2}^{\infty} \varphi_n(j_n(x))\, f_n(j_1(x))$ for all $x \in E$. Since $j_n(x) = 0$ for all but a finite number of n this sum is always finite and hence also G-analytic. Let

$$V_m = \left\{ x \in E \mid \|j_n(x)\|_{E_1} \leqslant 1 \quad \text{and} \quad \|j_n(x)\|_{E_i} \leqslant \inf\left(1, \frac{1}{2^n \left\| \dfrac{\hat{d}^m f_n(0)}{m!} \right\|_{E_1}}\right) \right\}$$

($\| \ \|_{E_i}$ denotes a fixed norm on E_i for each i). Then V_m is a neighbourhood of 0 in E and if $x \in V_m$

$$\left| \frac{\hat{d}^{m+1} f(0)}{(m+1)!}(x) \right| = \left| \sum_{n=2}^{\infty} \varphi_n(j_n(x)) \cdot \frac{\hat{d}^m f_n(0)}{m!}(j_1(x)) \right|$$

$$\leqslant \sum_{n=2}^{\infty} \left\| \frac{\hat{d}^m f_n(0)}{m!} \right\|_{E_1} \cdot \frac{1}{2^n \left\| \dfrac{\hat{d}^m f_n(0)}{m!} \right\|_{E_1}}$$

$$\leqslant 1$$

Hence $\dfrac{\hat{d}^m f(0)}{m\,!}$ is continuous for each m. Now suppose there exists a convex balanced neighbourhood of 0 in such that

$$\sup_{x \in V} |f(x)| \leqslant 1 .$$

Hence there exists $\delta_n > 0$ such that

$$V \supset \{x \in E_n \ , \ \|x\|_{E_n} \leqslant \delta_n\}$$

for each integer n.

Choose $x_n \in E_n$ such that $\varphi_n(x_n) \neq 0$ then for $x_1 \in E_1$ $\|x_1\| \leqslant \delta_1$, $\frac{1}{2} x_1 + \frac{1}{2} x_n \in V$.

Hence $1 \geqslant \|f\|_V \geqslant \sup\limits_{\|x_1\| < 1} |\varphi_n\left(\frac{1}{2} x_n\right) f_n\left(\frac{x_1}{2}\right)|$ which implies that

$$\sup_{\substack{x \in E_1 \\ \|x\|_{E_1} < \frac{1}{2}\delta_1}} |f_n(x)| < \infty$$

Since f_n has radius of boundedness $\frac{1}{n}$ this leads to a contradiction.

Hence $f \notin \mathcal{H}(E)$.

Now $f|_{F^n} \in \mathcal{H}(F^n)$ for any n and thus $\sum\limits_{m=0}^{\infty} \left\| \dfrac{\hat{d}^n f(0)}{n\,!} \right\|_K < \infty$ for any K compact in E. This means that $(\mathcal{H}(E), \mathcal{T}_{o,b})$ is not complete.

Now suppose $\mathcal{H}_b\left(\sum\limits_{i=1}^{n} E_i\right) = \mathcal{H}\left(\sum\limits_{i=1}^{n} E_i\right)$ for all i. Let $f = \sum\limits_{n=0}^{\infty} P_n$ where $P_n \in \mathcal{P}(^nE)$ for each n and $\sum\limits_{n=0}^{\infty} \|P_n\|_K < \infty$ for all K compact in E.

Choose $\delta_1 > 0$ such that

$$\|P_n\|_{\{x \in E_1 ,\, \|x_1\|_{E_1} < \delta_1\}} \leqslant \frac{1}{2} \quad \text{for all} \quad n \geqslant 1 .$$

Now suppose $\delta_1, \ldots, \delta_n > 0$ have been choosen such that

$$\|P_m\|_{\left\{x\in F^n,\, x=\sum_{i=1}^{n} x_i,\, \|x_i\|_{E_i}\right\}} \leqslant \sum_{i=1}^{n}\left(\frac{1}{2}\right)^i$$

for all $m \geqslant 1$.

Let $Q_m = P_m - \widetilde{P}_m$ where $\widetilde{P}_m$ is defined on E by

$$\widetilde{P}_m(x+y) = P(x) \quad \text{for} \quad x \in F^n\ ,\ y \in F_n = \sum_{i=n+1}^{\infty} E_i\ .$$

Then $$\widetilde{f} = \sum_{m=0}^{\infty} \widetilde{P}_m \in \mathcal{H}(E) \quad \text{and} \quad \widetilde{f}|_{F^n} = f|_{F^n}\ .$$

Hence $$P_m(x+y) = \widetilde{P}_m(x) + \sum_{0<R<m} \binom{m}{R} A_m(x)^R\,(y)^{m-R}$$

for $x \in F^n$, $y \in E_{n+1}$ and A_m is the m-linear form on E associated with P_m for each m.

Since $\mathcal{H}(F^n) = \mathcal{H}_b(F^n)$ we can choose $\delta_{n+1} > 0$ such that

$$\sum_{0<R<m} \binom{m}{R} \sup_{\substack{x=\sum_{i=1}^{n} x_i,\, y\in E_{n+1} \\ \|x_i\|_{E_i}<\delta_i,\, \|y\|<\delta_{n+1} \\ x_i\in E_i}} |A_m(x)^R\,(y)^{m-R}| \leqslant \left(\frac{1}{2}\right)^{n+1}$$

for all $m > 0$,

This means

$$\|P_m\|_{\left\{x=\sum_{i=1}^{n+1} x_i,\, x_i\in E_i,\, \|x_i\|_{E_i}<\delta_i\right\}} \leqslant \sum_{i=1}^{n+1} \frac{1}{2^i} \quad \text{for each } m.$$

Hence there exists a neighbourhood V of 0 such that $\|\mathcal{P}_m\|_V \leqslant 1$ for all m. Since it is possible to use the same method about any point of E we have provėd that $f \in \mathcal{H}(E)$. Hence $\left(\mathcal{H}\left(\sum_{i=1}^{\infty} E_i\right), \mathcal{T}_\delta\right)$ is complete.

(*Proof* of Corollary). – Apply the proposition and the result that $\mathcal{H}_b(E) = \mathcal{H}(E)$ for E a separable Banach space if and only if E is a finite dimensional space.

COROLLARY 3.5. – *If f is a G-holomorphic function on* $\sum_{i=1}^{\infty} C$ *then f is holomorphic.*

Remarks 1). – This proposition also provides an answer to Q.2 for if f is the function discussed in the first part of the proposition then $\left(\dfrac{\hat{d}^n f(0)}{n!}\right)_{n=0}^{\infty}$ is a $\mathcal{T}_\delta$-bounded sequence in $\mathcal{H}\left(\sum_{i=1}^{\infty} E_i\right)$ but it is not equibounded for then we would have $f \in \mathcal{H}\left(\sum_{i=1}^{\infty} E_i\right)$.

2) By means of the sequence $\left(\dfrac{\hat{d}^n f(0)}{n!}\right)_{n=0}^{\infty}$ discussed above we can also show that $\sum_{i=1}^{\infty} E_i$ is a barrelled space which is not C-barrelled in the sense of Lelong ([24], [25]).

3) An examination of the proof also yields the following facts :

i) $\mathcal{P}\left({}^n \sum_{i=1}^{\infty} E_i\right), \mathcal{T}_\omega\Big)$ is complete for each n.

ii) $P \in \mathcal{P}\left({}^m \sum_{i=1}^{\infty} E_i\right)$ if and only if $P|_{\sum_{i=1}^{n} E_i} \in \mathcal{P}\left({}^m \sum_{i=1}^{n} E_i\right)$ for each integer n.

4) If $(\mathcal{H}(E), \mathcal{T}_{o,b})$ is complete then it is barrelled. By proposition 1.3. we thus have $\left(\mathcal{H}\left(\sum_{n=0}^{\infty} C \times \prod_{n=0}^{\infty} C\right), \mathcal{T}_{o,b}\right)$ is not complete and we have already seen that $\left(\mathcal{H}\left(\sum_{n=0}^{\infty} C\right), \mathcal{T}_{o,b}\right)$ and $\mathcal{H}\left(\prod_{n=0}^{\infty} C\right)$, $\mathcal{T}_{o,b}\Big)$ are both complete.

5) The method used in proposition 3.7. can also be used to show

$$\left(\mathcal{H}\left(\sum_{i=0}^{\infty} C \times \prod_{i=0}^{\infty} C\right), \mathcal{T}_\delta\right) \quad \text{is complete.}$$

A related problem is also encountered in the literature ([19], [27], [33]) ; if $P_n \in \mathcal{P}(^nE)$ for each n and $\sum_{n=0}^{\infty} P_n(x)$ converges for all $x \in E$ does $\sum_{n=0}^{\infty} P_n \in H(E)$? In terms of the terminology we have developed we could state this as follows. Is E T.S.$\mathcal{T}_p$ complete where $\mathcal{T}_p$ denotes the topology of pointwise convergence on $\mathcal{H}(E)$? This problem arises in extending Hartog's theorem to infinite dimensional spaces. We have the following proposition whose proof is immediate.

PROPOSITION 3.8. – *If* E_i *for* $i = 1, 2$ *are* T.S.$\mathcal{T}_p$ *complete L.C.S. and each separately continuous polynomial on* $E_1 \times E_2$ *is continuous then every separatly holomorphic function on* $E_1 \times E_2$ *is holomorphic if and only if* $E_1 \times E_2$ *is* T.S.$\mathcal{T}_p$ *complete.*

If E_1 is a separable infinite dimensional Banach space and $E_2 = \sum_{i=1}^{\infty} C$ then E_i for $i = 1, 2$ is T.S. $\mathcal{T}_p$ complete and separately continuous polynomials on $E_1 \times E_2$ are continuous. However, we have seen that $(\mathcal{H}(E_1 \times E_2), \mathcal{T}_\delta)$ is not complete and hence $E_1 \times E_2$ is not T.S. $\mathcal{T}_p$ complete.

We finish this section by giving an example of when $(\mathcal{H}(E), \mathcal{T}_{\omega,b})$ is a Montel space.

PROPOSITION 3.9. – *Let* E *be a Frechet Nuclear space then* $(\mathcal{H}(E), \mathcal{T}_{\omega,b})$ *is a Montel space.*

Proof. – By [35] p. 525.

$$\underbrace{(E \hat{\otimes} E \hat{\otimes} \cdots \hat{\otimes} E)'}_{n \text{ times}} = L_n(E)$$

By [35] p. 520 $E \hat{\otimes} E \hat{\otimes} \cdots E$ is Montel and hence $(\mathcal{L}_n(E), \beta)$ is also Montel where β denotes the strong topology on $\mathcal{L}_n(E)$. Let

$$F = \{x \in E \hat{\otimes} E \cdots \hat{\otimes} E \mid f(x) = 0 \quad \text{for all} \quad f \in \mathcal{P}(^nE)\}.$$

F is a closed subspace of $E \hat{\otimes} \cdots \hat{\otimes} E$ and

$$\mathcal{P}(^nE) = \{f \in \mathcal{L}_n(E) \mid f(x) = 0 \text{ for } x \in F\}$$

By a slight modification of Corollary p. 285 [21] (extend it to Frechet Montel spaces) we find that the topology on $\mathcal{P}(^nE)$ induced from $(\mathcal{L}_n(E), \beta)$ and the strong dual topology of $(E \hat{\otimes} \cdots \hat{\otimes} E|_F)'$ coincide, Hence $(\mathcal{P}(^nE), \beta)$ is the strong dual of $E \hat{\otimes} \cdots \hat{\otimes} E|_F$. Thus $(\mathcal{P}(^nE), \beta)$ is Montel and hence barrelled. But the strong dual of a metrizable space is barrelled if and only if it is bornological ([35] p. 39).

Hence $(\mathcal{P}(^nE), \beta)$ is bornological. Since E is Frechet $\mathcal{T}_\omega = \mathcal{T}_{o,b}$ and $\mathcal{T}_{o,b} \geqslant \beta$ and this implies $\mathcal{T}_\omega = \beta$. An application of proposition 2.6 completes the proof.

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