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HOMOGENEOUS ALGEBRAS ON THE CIRCLE : II. — MULTIPLIERS, DITKIN CONDITIONS

by Colin BENNETT and John E. GILBERT

1. Introduction.

For a homogeneous Banach algebra $\mathfrak{A}$ on the circle group the translation operators $\{T(t) : 0 < t < \infty\}$ by definition give a strongly continuous representation of $(0, \infty)$ in $\mathfrak{A}$, i.e. $\{T(t) : 0 < t < \infty\}$ is a semi-group of contraction operators of class $(\mathcal{C}_0)$ (cf. §1 part 1). The infinitesimal generator of this semi-group is the differential operator $\frac{d}{d\xi}$; the domain of definition of powers $\left(\frac{d}{d\xi}\right)^k$, $k = 1, 2, \ldots,$ will be denoted by $\mathfrak{A}^{(k)}$. These spaces $\mathfrak{A}$, $\mathfrak{A}^{(k)}$ provide a natural setting for applications of methods from interpolation space theory, either real methods (cf. [5, 13]) or complex methods (cf. [6]) although only the K-method of Peetre will be used here.

If $\Lambda(\alpha, p ; \mathfrak{A})$, $\alpha > 0$, is the interpolation space

$$\Lambda(\alpha, p ; \mathfrak{A}) = (\mathfrak{A}, \mathfrak{A}^{(k)})_{\theta, p ; K} \quad , \quad \theta = \alpha/k \ , \ 1 \leq p \leq \infty,$$

then $\Lambda(\alpha, p ; \mathfrak{A})$ can be identified with the functions in $\mathfrak{A}$ satisfying

$$\left(\int_0^\infty (t^{-\alpha} \, \|(T(t) - I)^k f\|_{\mathfrak{A}} \,)^p \, \frac{dt}{t}\right)^{1/p} < \infty. \tag{1}$$

Thus $\Lambda(\alpha, p ; \mathfrak{A})$ is a Lipschitz subspace of $\mathfrak{A}$. By considering $\mathfrak{A}$, $\mathfrak{A}^{(k)}$ and $\Lambda(\alpha, p ; \mathfrak{A})$ within this framework of interpolation space theory we can exploit for arbitrary homogeneous algebras $\mathfrak{A}$ the fundamental theorems of stability (or re-iteration) and interpolation as well as the characterization (1). For instance, by (1) and the stability theorem, each space $\Lambda(\alpha, p ; \mathfrak{A})$ is a Banach algebra and, whenever $\mathfrak{A}^{(k)}$ is dense in $\Lambda(\alpha, p; \mathfrak{A})$ (in particular when $p < \infty$), $\Lambda(\alpha, p; \mathfrak{A})$ is a homogeneous Banach algebra on the circle group. More important for our purposes is the interpolation property enjoyed by all interpolation spaces. Using this theorem together with the stability theorem we derive estimates for the multiplier norm on closed primary ideals in $\Lambda(\alpha, p; \mathfrak{A})$ and $\mathfrak{A}^{(k)}$. With these estimates the various Ditkin conditions defined in part I of this series can be readily established. The difficulties involved here are very subtle because, as we show, the multiplier norm on a closed primary ideal is *not* equivalent to the multiplier norm on $\Lambda(\alpha, p; \mathfrak{A})$, $\alpha > 0$.

More generally, the interpolation space $\Lambda(\alpha, p; \mathfrak{X})$ can be constructed when $\{T(t) : 0 < t < \infty\}$ is any semi-group of contraction operators of class $(\mathcal{C}_0)$ on a Banach space $\mathfrak{X}$. Taking say $\mathfrak{X}$ as

$$L^p(T), \quad 1 \leq p < \infty \quad, \quad \mathcal{C}(T) \quad, \quad T \text{ circle group,}$$

and $\{T(t) : 0 < t < \infty\}$ the usual translation operators, or

$$\ell^p(Z) \quad, \quad 1 \leq p < \infty \quad, \quad c_0(Z),$$

and $\{\chi(t) : 0 < t < \infty\}$ multiplication by characters,

$$\chi(t) : \{a_n\} \to \{e^{int} a_n\}$$

we construct within this unifying framework of interpolation space theory two large classes of examples illustrating the general theory developed. Many algebras of interest previously studied in isolation are produced.

A complete discussion of many applications of interpolation space theory to Banach algebras more general than the theory given here appears in [7].

2. Applications of interpolation space theory.

The interpolation theory used in this paper will be that arising from the K-method of Peetre (i.e. a "real method"). For an excellent discussion of this method as well as of other material used here see [5] (esp. sections 1.1, 3.3, 3.4).

Let $(\mathfrak{X}, \|(.)\|\mathfrak{X})$ be a Banach space on which acts a semi-group $\{T(t) : 0 < t < \infty\}$ of contraction operators of class $(\mathcal{C}_0)$ and infinitesimal generator A. The domain of definition of powers A^k of A is

$$\mathfrak{X}^{(k)} = \{f : f, Af, \ldots, A^k f \in \mathfrak{X}\} \quad , \quad k \geq 1.$$

Under the "graph norm"

$$\|f\|_{(k)} = \|f\|_{\mathfrak{X}} + \|A^k f\|_{\mathfrak{X}} \quad , \quad f \in \mathfrak{X}^{(k)}, \tag{2}$$

$\mathfrak{X}^{(k)}$ becomes a Banach space on which

$$f \to \sum_{\ell=0}^{k} \|A^\ell f\|_{\mathfrak{X}} \quad , \quad f \in \mathfrak{X}^{(k)} \tag{3}$$

defines an equivalent norm ([4, p. 12]). For each t, $0 < t < \infty$, set

$$K(t, f) = K(t, f ; \mathfrak{X}, \mathfrak{X}^{(k)}) = \inf (\|f_0\|_{\mathfrak{X}} + t\|f_1\|_{(k)}) \quad , \quad f \in \mathfrak{X} ,$$

the infimum being taken over all representations $f = f_0 + f_1$ with $f_0 \in \mathfrak{X}$ and $f_1 \in \mathfrak{X}^{(k)}$.

(2.1) DEFINITION. — *Given any $\alpha > 0$ and integer $k > \alpha$ the interpolation space $\Lambda(\alpha, p ; \mathfrak{X})$, $1 \leqslant p \leqslant \infty$, is the subspace of $\mathfrak{X}$ of all f for which*

$$\|f\|_{\alpha, p ; \mathfrak{X}} = \left(\int_0^\infty (t^{-\theta} K(t, f ; \mathfrak{X}, \mathfrak{X}^{(k)})^p \frac{dt}{t}\right)^{1/p} \quad , \quad \theta = \alpha/k, \tag{4}$$

is finite (obvious modifications when $p = \infty$) (1).

Under the norm $\|f\|_{\alpha, p ; \mathfrak{X}}$, $\Lambda(\alpha, p ; \mathfrak{X})$ becomes a Banach space. As is usual in the subject we adopt the convention that, for Banach spaces, $\mathfrak{N}_1 \subset \mathfrak{N}_0$ means the identity embedding $\mathfrak{N}_1 \to \mathfrak{N}_0$ is continuous. Then

$$\mathfrak{X}^{(k)} \subset \Lambda(\alpha, p ; \mathfrak{X}) \subset \mathfrak{X} \quad , \quad k > \alpha,$$

(1) With the notation of [5 p. 167], $\Lambda(\alpha, p ; \mathfrak{X}) = \mathfrak{X}, \mathfrak{X}^{(k)})_{\theta, p ; k}$

and $\mathfrak{X}^{(k)}$ is dense in $\Lambda(\alpha, p ; \mathfrak{X})$ at least when $p \neq \infty$. We denote by $\lambda(\alpha, \infty ; \mathfrak{X})$ the closure of $\mathfrak{X}^{(k)}$ in $\Lambda(\alpha, \infty ; \mathfrak{X})$. The important *stability (*or *re-iteration)* theorem for interpolation spaces shows that $\Lambda(\alpha, p ; \mathfrak{X})$ is independent (up to norm equivalence) of the integer k in (4) (cf. [5, pp. 198-202]). Important also is the *interpolation* theorem : suppose T is a bounded linear operator

$$T : \mathfrak{X} \to \mathfrak{X} \quad , \quad T : \mathfrak{X}^{(k)} \to \mathfrak{X}^{(k)}$$

with respective norms M_0 , M_1. Then T is a bounded linear operator

$$T : \Lambda(\alpha, p ; \mathfrak{X}) \to \Lambda(\alpha, p ; \mathfrak{X})$$

whose norm satisfies the convexity inequality

$$\|T\| \leqslant M_0^{1-\theta} \ M_1^{\theta} \quad , \quad \theta = \alpha/k.$$

The Lipschitz character of $\Lambda(\alpha, p ; \mathfrak{X})$ is well-known : $\Lambda(\alpha, p ; \mathfrak{X})$ consists of all f in $\mathfrak{X}$ for which

$$\left(\int_0^\infty (t^{-\alpha} \ \|(T(t) - I)^k f\|_{\mathfrak{X}})^p \ \frac{dt}{t}\right)^{1/p} \quad , \quad k > \alpha, \tag{5}$$

is finite ([5, theorem 3.4.2]). A very useful estimate is

$$\|f\|_{\alpha,p ; \mathfrak{X}} \leqslant \text{const.} \ (\|f\|_{\mathfrak{X}})^{1-\alpha/k} \ (\|f\|_{(k)})^{\alpha/k} \quad , \quad f \in \mathfrak{X}^{(k)} \tag{6}$$

(essentially this is [5, theorems 3.2.12, 3.2.36], see also [13, pp. 12-13] for a proof for an equivalent real interpolation method).

Now suppose $\mathfrak{A}$ is a Banach algebra such that

(a) $\mathfrak{X}$ *is a right Banach module over* $\mathfrak{a}$ *under an operation* $\circ$,

(b) $\{T(t) : 0 < t < \infty\}$ *acts on* $\mathfrak{a}$ *and*

$$T(t)\,(f \circ g) = (T(t) f) \circ (T(t) g) \quad , \quad f \in \mathfrak{X} \quad , \quad g \in \mathfrak{a}$$

(2.2) THEOREM. – *When* $\mathfrak{X}$ *and* $\mathfrak{A}$ *satisfy conditions (a), (b)*

$$\|f \circ g\|_{\alpha,p ; \mathfrak{X}} \leq \text{const.}\{\|f\|_{\mathfrak{A}} \|g\|_{\alpha,p ; \mathfrak{X}} + \|f\|_{\alpha,p ; \mathfrak{X}} \|g\|_{\mathfrak{A}}\} \tag{7}$$

for all f, g *in* $\mathfrak{A} \cap \Lambda(\alpha, p ; \mathfrak{X})$.

Proof. – (cf. [10, proof lemma (1.5)]). Use the stability theorem for $\Lambda(\alpha, p; \mathfrak{X})$ and the Leibnitz formula

$$(\mathrm{T}(t) - 1)^k (f \circ g) = \sum_{j=0}^{k} \binom{k}{j} [(\mathrm{T}(t) - 1)^j f] \circ [(\mathrm{T}(t) - 1)^{k-j} \mathrm{T}(t)^j g]$$

in the characterization (5).

It will be convenient to speak of a Banach space which is an algebra with jointly continuous multipliciation as a *Banach algebra* without renorming the space so that the norm necessarily is submultiplicative.

(2.3) COROLLARY. – *Suppose* $\Lambda(\alpha_0, 1; \mathfrak{X}) \subset \mathfrak{A}$ *for some* α_0. *Then* $\Lambda(\alpha, p; \mathfrak{X})$ *is a Banach subalgebra of* $\mathfrak{A}$ *whenever*

$$\alpha \geq \alpha_0 \quad , \quad p = 1 \quad , \quad \alpha > \alpha_0 \quad , \quad 1 \leq p \leq \infty. \tag{8}$$

When (8) holds

$$\|f \circ g\|_{\alpha, p; \mathfrak{X}} \leqslant \text{const. } \|f\|_{\alpha, p; \mathfrak{X}} \|g\|_{\alpha, p; \mathfrak{X}} \tag{9}$$

for all $f, g \in \Lambda(\alpha, p; \mathfrak{X})$.

Proof. – Inequality (9) follows form (7) provided $\Lambda(\alpha, p; \mathfrak{X}) \subset \mathfrak{A}$, in particular when $\alpha = \alpha_0$, $p = 1$. But $\Lambda(\alpha, p; \mathfrak{X}) \subset \Lambda(\alpha_0, 1; \mathfrak{X})$ if $\alpha > \alpha_0$, $1 \leq p \leq \infty$, and so $\Lambda(\alpha, p; \mathfrak{X})$ is a Banach subalgebra of $\mathfrak{a}$ whose norm satisfies (9) when (8) holds.

(2.4) COROLLARY. – *If* $\mathfrak{X}$ *is a Banach algebra for which*

$$\mathrm{T}(t)(f \circ g) = \mathrm{T}(t) f \circ \mathrm{T}(t) g$$

then $\Lambda(\alpha, p; \mathfrak{X})$ *is a Banach algebra for all* $\alpha > 0$, $1 \leq p \leq \infty$, *whose norm satisfies* (9).

For the remainder of this section and all of §3 we shall take for $\mathfrak{X}$ a Banach algebra $\mathcal{A}$ homogeneous on T in the sense of Silov, i.e. satisfying (H.0), (H.1), (H.2), (H.3) in §1 of part 1([2]). As in part I we assume the translation operators $\{T(t) : 0 < t < \infty\}$ are contractions and that

$$\mathcal{C}^{\infty}(T) \subset \mathfrak{A} \subset \mathcal{C}(T) : \tag{10}$$

the infinitesimal generator of the semi-group $\{T(t) : 0 < t < \infty\}$ is $D = \dfrac{d}{d\xi}$. From (10) it follows that

$$\mathcal{C}^{\infty}(T) \subset \mathfrak{A}^{(k)} \subset \Lambda(\alpha, p; \mathfrak{A}) \subset \mathcal{C}(T) \quad , \quad k > \alpha > 0. \tag{11}$$

(2.5) THEOREM. – *Let $\mathfrak{A}$ be a Banach algebra homogeneous on the circle group T. Then for $\alpha > 0$, $1 \leqslant p \leqslant \infty$ the spaces $\Lambda(\alpha, p; \mathfrak{A})$, $\lambda(\alpha, \infty; \mathfrak{A})$ and $\mathfrak{A}^{(k)}$ are all Banach algebras. Furthermore, $\Lambda(\alpha, p; \mathfrak{A})$, $\lambda(\alpha, \infty; \mathfrak{A})$ and $\mathfrak{A}^{(k)}$ are homogeneous Banach algebras on T provided $\alpha > 0$, $1 \leqslant p < \infty$.*

Proof. – The first assertion is clear (cf. corollary (2.4) and (9)). That T is the maximal ideal space of $\mathfrak{A}^{(k)}$ has been proved by Loy ([14, p. 312]). But. in view of the inner inclusion relation in (11), the spectral radius norm on $\Lambda(\alpha, p; \mathfrak{A})$ and $\lambda(\alpha, p; \mathfrak{A})$ is dominated by that on $\mathfrak{A}^{(k)}$. Hence. since $\mathfrak{A}^{(k)}$ is dense in $\Lambda(\alpha, p; \mathfrak{A})$, $p \neq \infty$, and in $\lambda(\alpha, \infty; \mathfrak{A})$, it follows that T is the maximal ideal space of $\Lambda(\alpha, p; \mathfrak{A})$ and $\lambda(\alpha, p; \mathfrak{A})$ also when $p \neq \infty$ ([14, p. 312]). On the other hand, the operators $\{T(t) : 0 < t < \infty\}$ are strongly continuous on $\mathfrak{A}^{(k)}$ and hence, by (6), on the dense subspace $\mathfrak{A}^{(k)}$ of $\Lambda(\alpha, p; \mathfrak{A})$, $\lambda(\alpha, \infty, \mathfrak{A})$, $p \neq \infty$. Thus

$$\Lambda(\alpha, p; \infty) \ , \ \lambda(\alpha, p; \mathfrak{A})$$

and $\mathfrak{A}^{(k)}$ are homogeneous on T.

For well-known technical reasons of no interest here, the spaces $\Lambda(\alpha, p; \mathfrak{A})$, $\lambda(\alpha, \infty; \mathfrak{A})$ with α an integer, i.e. $\alpha = [\alpha]$, will not be considered ; instead, $\mathfrak{A}^{(k)}$, $k = 1, 2, \ldots$, will be discussed, albeit briefly.

By a multiplier on a Banach Function algebra $\mathfrak{A}$ we shall always mean a complex-valued function σ such that the pointwise product

([2]) the notation changes from Part I : $T(t)$ always denotes an operator from some semi-group $\{T(t) : 0 < t < \infty\}$ sometimes specified sometimes arbitrary, but T (sic) denotes the circle group originally denoted by ∂D. D will here be reserved for the differential operator $\dfrac{d}{d\xi}$.

σf belongs to $\mathfrak{A}$ for all $f \in \mathfrak{A}$. The operator $f \to \sigma f$ necessarily is bounded (closed-graph theorem). On $\mathfrak{A}$ and $\Lambda(\alpha, p; \mathfrak{A})$, of course, the multipliers are just $\mathfrak{A}$, $\Lambda(\alpha, p; \mathfrak{A})$ respectively and the multiplier norm is equivalent to $\|(.)\|_{\mathfrak{A}}$ $\|(.)\|_{\alpha, p; \mathfrak{A}}$. Also, any $\sigma \in \Lambda(\alpha, p; \mathfrak{A})$ is a multiplier on any closed ideal I in $\Lambda(\alpha, p; \mathfrak{A})$.

For $\alpha > 0$ and $k \geq 0$ set

$$I_{\alpha, p}(\xi_0) = \{f \in \Lambda(\alpha, p; \mathfrak{A}) : f(\xi_0) = \cdots = D^{[\alpha]} f(\xi_0) = 0\},$$

$$I_{\mathfrak{A}}^{(k)}(\xi_0) = \{f \in \mathfrak{A}^{(k)} : f(\xi_0) = \cdots = D^k f(\xi_0) = 0\},$$

where $e^{i\xi_0} \in T$. Thus $I_{\alpha,p}(\xi_0)$ and $I_{\mathfrak{A}}^{(k)}(\xi_0)$ are closed primary ideals in $\Lambda(\alpha, p; \mathfrak{A})$, $\mathfrak{A}^{(k)}$ respectively (though not necessarily the smallest closed primary ideal in the appropriate maximal ideal if functions in $\mathfrak{A}$ are all sufficiently smooth). For two quite general classes of algebras we shall derive precise estimates for the multiplier norm on $I_{\alpha,p}(\xi_0)$, $I_{\mathfrak{A}}^{(k)}(\xi_0)$ (cf. theorems (3.2), (4.1)). By homogeneity it is enough to consider $\xi_0 = 0$ only. The Strong Ditkin and Strong Analytic Ditkin conditions for $\Lambda(\alpha, p; \mathfrak{A})$, $\lambda(\alpha, \infty; \mathfrak{A})$ and $\mathfrak{A}^{(k)}$, $p \neq \infty$, follow provided the smoothness of functions in $\mathfrak{A}$ is suitably restricted. Interest in these results arises on the one hand from the importance of the two Ditkin conditions in the harmonic analysis of $\Lambda(\alpha, p; \mathfrak{A})$, $\mathfrak{A}^{(k)}$ (cf. part I and [12, §39] [20, chaps. 1, 2]) and on the other from the fact that the multiplier norm on $I_{\alpha,p}(\xi_0)$ must be markedly different from the multiplier norm on $\Lambda(\alpha, p; \mathfrak{A})$, $\alpha > 0$. In contrast, for $\mathfrak{A}$ itself, the multipliers on $I_{\mathfrak{A}}(\xi_0)$ $(= I_{\mathfrak{A}}^{(0)}(\xi_0))$ may well coincide with $\mathfrak{A}$ as Meyer has shown for the Wiener algebra $\mathfrak{F}(\ell^1(Z))$ ([15]). The proofs given are also of technical interest ; for we use a group of automorphisms on T (introduced by one of us in [1]) with play exactly the same rôle for the circle group as the dilation group does for the real line. Amusingly enough, although these operators do not give automorphisms of, say $\mathfrak{F}(\ell^1(Z))$, classical results such as Bernstein's theorem enable us to derive all the results we need (cf. (4.2), (4.3), (4.4)).

As the proofs for the two classes differ only in detail some preliminary explanation will be instructive. Define $\nu(\xi)$ on T by $\nu(\xi) = e^{i\xi} - 1$ and set

$$\Theta = \frac{d}{dv} = -ie^{-i\xi}\,D\,;$$

thus

$$D^k = \sum_{n=1}^{k} \beta_n(\xi)\,\Theta^n \tag{12}$$

with $\beta_n \in \mathcal{C}^\infty(T)$. When $\Phi : f \to \Phi f$ is defined by

$$\Phi f(\xi) = \frac{1}{v}\,(f(\xi) - f(0)) \quad , \quad f \in \mathcal{C}(T),$$

it is easy to check that

(a) *for each f in $\mathcal{A}^{(m)}$*

$$f(\xi) - \sum_{k=0}^{m-1} \frac{1}{k\,!}\, v^k(\xi)\,\Theta^k f(0) = v^m\,\Theta^m(f) \in I_{\mathcal{A}}^{(m-1)}(0)\ ,\ m \geq 1, \tag{13}$$

(b) *for all f in $\mathcal{A}^{(m)}$ and σ in $\mathcal{C}^{(n)}(T)$, $n \leq m$,*

$$\Theta^n\,[v^m\,\Phi^m\,(f)\,\sigma] = v^{m-n}\Big\{\sum_{\ell=0}^{n} \binom{n}{\ell}\,\Phi^{m-\ell}\,(\Theta^\ell f)\,(v^{n-\ell}\,\Theta^\ell\sigma)\Big\}. \tag{14}$$

In particular,

$$f \in I_{\mathcal{A}}^{(m-1)}(0) \Rightarrow f = v^m\,\Phi^m\,(f)\,. \tag{15}$$

Now, by the stability theorem for $\Lambda(\alpha, p\,; \mathcal{A})$,

$$\Lambda(\alpha, p\,; \mathcal{A}) = (\mathcal{A}^{([\alpha])}, \mathcal{A}^{([\alpha]+1)})_{\theta, p; K} \quad , \quad \theta = \alpha - [\alpha], \tag{16}$$

provided $\alpha \neq [\alpha]$. Suppose for the moment we have established

(A) *for all $f \in \mathcal{A}^{([\alpha])}$ and $\sigma \in \mathcal{C}^\infty(T)$*

$$\Big\|\Big\{f - \sum_{k=0}^{[\alpha]} \frac{1}{k\,!}\, v^k\,\Theta^k f(0)\Big\}\sigma\Big\|_{([\alpha])} \leq \text{const.}\ \|f\|_{([\alpha])}\,M_0(\sigma),$$

(B) *for all* $f \in \mathfrak{A}^{([\alpha]+1)}$ *and* $\sigma \in \mathcal{C}^\infty(\mathrm{T})$

$$\left\| \left\{ f - \sum_{k=0}^{[\alpha]} \frac{1}{k!} \nu^k \Theta^k f(0) \right\} \sigma \right\|_{([\alpha]+1)} \leq \text{const. } \|f\|_{([\alpha]+1)} \mathrm{M}_1(\sigma).$$

Then, by (15) and the interpolation theorem applied to (16),

$$\|f\sigma\|_{\alpha,p\,;\,\mathfrak{A}} \leqslant \text{const. } \|f\|_{\alpha,p\,;\,\mathfrak{A}} \; \mathrm{M}_0(\sigma)^{[\alpha]+1-\alpha} \mathrm{M}_1(\sigma)^{\alpha-[\alpha]} \tag{17}$$

for $f \in \mathrm{I}_{\alpha,p}(0)$, $\alpha \neq [\alpha]$. This estimates the multiplier norm on $\mathrm{I}_{\alpha,p}(0)$, hence on $\mathrm{I}_{\alpha,p}(\xi_0)$.

We deal first with estimate (B). The left hand side is

$$\|\nu^b \Phi^b(f)\sigma\|_{\mathfrak{A}} + \|\mathrm{D}^b[\nu^b \Phi^b(f)\sigma]\|_{\mathfrak{A}} \quad , \quad f \in \mathfrak{A}^{(b)},$$

where $b = [\alpha] + 1$. Using (12) and (14) we can thus reduce the proof of (B) to the establishing of

$$\sum_{n=0}^{b} \sum_{\ell=0}^{n} \|\nu^{b-n}\Phi^{b-\ell}(\Theta^\ell f)\,\nu^{n-\ell}\,\Theta^{n-\ell}\sigma\|_{\mathfrak{A}} \leq \text{const. } \|f\|_{(b)} \mathrm{M}_1(\sigma). \tag{18}$$

The proof for (A) is identical except for one additional (but trivial) technical complication : with $a = [\alpha]$ write

$$f - \sum_{k=0}^{a} \frac{1}{k!} \nu^k \Theta^k f(0) = \nu^a \Phi^a(f) - \frac{1}{a!} \nu^a \Theta^a f(0).$$

Hence, just as for (B), (A) reduces to the establishing of

$$\sum_{n=0}^{a} \sum_{\ell=0}^{n} \|\nu^{a-n}\Phi^{a-\ell}(\Theta^\ell f)\;\nu^{n-\ell}\Theta^{n-\ell}\sigma\|_{\mathfrak{A}}$$

$$+ \|f\|_{(a)} \left\{ \sum_{k=0}^{a} \|\nu^k \mathrm{D}^k \sigma\|_{\mathfrak{A}} \right\} \leqslant \text{const. } \|f\|_{(a)} \mathrm{M}_0(\sigma). \tag{19}$$

Since $\mathfrak{A}$ is a Banach algebra we can see that what is now wanted is an estimate for $\|\nu^m \Phi^n(\Theta^\ell f)\|_{\mathfrak{A}}$ or for some similar expression. This is the heart of the proof.

The multiplier norm on the primary ideal $I^{(k)}_{\mathfrak{A}}(0)$ in $\mathfrak{A}^{(k)}$ is estimated by (A) alone with $k = [\alpha]$.

3. Lipschitz algebras on T.

Let ϕ_r, $-1 < r < 1$, be the linear fractional transformation

$$\phi_r(\xi) = \frac{e^{i\xi} - r}{1 - re^{i\xi}} \quad , \qquad e^{i\xi} \in \mathrm{T}.$$

Define $\Phi_r : \mathcal{C}(\mathrm{T}) \to \mathcal{C}(\mathrm{T})$ by $\Phi_r(f) = f \circ \phi_r$. Throughout this section we shall impose on $\mathfrak{A}$ the following condition.

CONDITION (R). – *For r, $-1 < r < 1$, Φ_r is a linear operator on $\mathfrak{A}$ with*

$$\|\Phi_r(1 + \cos \xi) f)\|_{\mathfrak{A}} \leqslant \rho_{\mathfrak{A}}(r)\, \|f\|_{\mathfrak{A}} \quad , \quad f \in \mathfrak{A}, \tag{20}$$

for some function $\rho_{\mathfrak{A}}$.

(3.1) *Remarks.* – Clearly each Φ_r is an isometric automorphism of $\mathcal{C}(\mathrm{T})$ hence bounded on $\mathfrak{A}$ if Φ_r maps $\mathfrak{A}$ into $\mathfrak{A}$. However, (20) is a more convenient (and natural) formulation of this boundedness requirement. As later examples show, $\rho_{\mathfrak{A}}(r)$ is closely related to the index function associated with L^p-spaces and general rearrangement-invariant Banach Function spaces on T where the index function is important in interpolation theory (cf. [4] ; in [1] there is a systematic study of properties stemming from condition (R)). Here we show that $\rho_{\mathfrak{A}}(r)$ is crucial in describing the properties of $\mathfrak{A}$ and its Banach subalgebras $\Lambda(\alpha, p ; \mathfrak{A})$, $\lambda(\alpha, \infty ; \mathfrak{A})$. Throughout we shall assume

$$\rho_{\mathfrak{A}}(r) \geq 1 \qquad , \qquad -1 < r \leq 0 ; \tag{21}$$

in practice (21) always is satisfied.

The next theorem and theorem (3.6) are the main results of this section.

(3.2) THEOREM. – *Let* $\mathfrak{A}$ *be a homogeneous Banach algebra on* T *satisfying condition* (R). *Then, provided* $\alpha \neq [\alpha]$,

$$\|f\sigma . (e^{i\xi} + 1)^s\|_{\alpha,p\,;\,\mathfrak{A}} < \text{const.} \left(\int_0^1 \rho_{\mathfrak{A}}(-r)\,dr\right)^{\alpha} \|f\|_{\alpha,p\,;\,\mathfrak{A}} \, \mathrm{N}(\sigma,\alpha)$$

for all f *in the primary ideal* $\mathrm{I}_{\alpha,p}(0)$ *in* $\Lambda(\alpha, p\,;\mathfrak{A})$, $1 \leq p \leq \infty$, *and* σ *in* $\mathcal{C}^{\infty}(\mathrm{T})$, *where*

(i) $\mathrm{N}(\sigma,\alpha) = \sum_{k=0}^{[\alpha]+1} \|\sin^k \xi \, \mathrm{D}^k \sigma\|$,

(ii) $s \geq ([\alpha] + 1)([\alpha] + 2)$, *s integer.*

(3.3) *Remarks.* – (i) The method of proof of (3.2) can be used to show : when $s \geq (m + 1)(m + 2)$

$$\|f\sigma . (e^{i\xi} + 1)^s\|_{(m)} \leq \text{const.}\left(\int_0^1 \rho_{\mathfrak{A}}(-r)\,dr\right)^s \|f\|_{(m)} \, \mathrm{N}(\sigma,r)$$

for all f in $\mathrm{I}_{\mathfrak{A}}^{(m)}(0)$ in $\mathfrak{A}^{(m)}$ and σ in $\mathcal{C}^{\infty}(\mathrm{T})$ where

$$\mathrm{N}(\sigma,r) = \sum_{k=0}^{m+1} \|\sin^k \xi \, \mathrm{D}^k \sigma\|_{\mathfrak{A}} .$$

(ii) Under the usual mapping $z \to i\left(\frac{z-1}{z+1}\right)$ of the unit disk onto the upper half-plane, $z = e^{-i\pi} = -1$ maps onto ∞ and ϕ_r to the dilation operator $x \to \left(\frac{1+r}{1-r}\right)x$. For any Banach algebra $\mathcal{B}$ with $(-\infty,\infty)$ as maximal ideal space, automatically functions in $\mathcal{B}$ vanish at $\pm\infty$; in contrast, for Banach algebras $\mathfrak{A}$ on T, functions in $\mathrm{I}_{\alpha,p}(0)$ do not all vanish at $z = -1$. The factor $(e^{i\xi} + 1)^s$ builds in the "zero at -1" property seemingly necessary for all applications of the operator Φ_r.

(iii) On $\mathcal{C}(\mathrm{T})$ $\{\Phi_r : -1 < r < 1\}$ is a group of contraction operators of class $(\mathcal{C}_0)$. In view of the expressions for $\mathrm{N}(\sigma, \alpha)$ and $\mathrm{N}(\sigma, r)$ it is interesting to note that the infinitesimal generator of this group is $(\sin \xi)\dfrac{d}{d\xi}$.

(iv) For the practical significance of estimate (3.2) (i) see (29) and (31) where $\sigma_r = \Phi_r(\sigma)$, $\sigma \in \mathcal{C}^\infty(\mathrm{T})$.

Because of the discussion in §2 we have only to show :

(A)′ for $f \in \mathfrak{A}^{(a)}$

$$\sum_{n=0}^{a} \sum_{\ell=0}^{n} \|\nu^{a-n}\, \Phi^{a-\ell}(\Theta^{\ell} f)\, \nu^{n-\ell}\, \Theta^{n-\ell}((e^{i\xi}+1)^s \sigma)\|_{\mathfrak{a}}$$

$$+ \|f\|_{(a)} \left\{ \sum_{k=0}^{a} \|\nu^k \mathrm{D}^k ((e^{i\xi}+1)^s \sigma)\|_{\mathfrak{a}} \right\} \leqslant \text{const.} \|f\|_{(a)}\, \mathrm{M}_0(\sigma);$$

(B)′ for $f \in \mathfrak{A}^{(b)}$

$$\sum_{n=0}^{b} \sum_{\ell=0}^{n} \|\nu^{b-n}\, \Phi^{b-\ell}(\Theta^{\ell} f)\, \nu^{n-\ell}\, \Theta^{n-\ell}((e^{i\xi}+1)^s \sigma)\|_{\mathfrak{a}} \leqslant \text{const.} \|f\|_{(b)}\, \mathrm{M}_1(\sigma)$$

with

$$\mathrm{M}_0(\sigma) = \left(\int_0^1 \rho_{\mathfrak{a}}(-r)\, dr\right)^a \sum_{k=0}^{a} \|\sin^k \xi \mathrm{D}^k \sigma\|_{\mathfrak{a}}$$

$$\mathrm{M}_1(\sigma) = \left(\int_0^1 \rho_{\mathfrak{a}}(-r)\, dr\right)^b \sum_{k=0}^{b} \|\sin^k \xi \mathrm{D}^k \sigma\|_{\mathfrak{a}}$$

(recall $a = [\alpha]$, $b = [\alpha] + 1$). Now by (12)

$$\nu^{n-\ell}\, \Theta^{n-\ell}((e^{i\xi}+1)^s \sigma) = (e^{i\xi}+1)^{s+\ell-n} \left\{ \sum_{k=0}^{n-\ell} \eta_k(\xi) \sin^k \xi \mathrm{D}^k \sigma \right\}$$

with $\eta_k \in \mathcal{C}^\infty(\mathrm{T})$, η_k independent of σ. Consequently, both (A)′ and (B)′ follow easily from the second of the following two lemmas.

When these have been proved (the first lemma is the most important step in the proof of the second) theorem (3.2) will finally have been established.

(3.4) LEMMA. – *Suppose* Φ *is defined by* $\Phi f = \frac{1}{\nu}(f(\xi) - f(0))$.

Then, for each integer $m \geq 0$.

$$\|(e^{i\xi} + 1)^{2m+1}\Phi(f)\|_{(m)} \leq \text{const.} \left(\int_0^1 \rho_{\mathfrak{a}}(-r)\,dr\right) \|f\|_{(m+1)}$$

for all f *in* $\mathfrak{A}^{(m+1)}$.

(3.5) LEMMA. – *For each integer* $m \geq 0$

$$\|(e^{i\xi} + 1)^n \Phi^m(f)\|_{\mathfrak{a}} \leqslant \text{const.} \left(\int_0^1 \rho_{\mathfrak{a}}(-r)\,dr\right)^m \|f\|_{(m)} \tag{22}$$

for all $f \in \mathfrak{A}^{(m)}$ *whenever* $n \geqslant m^2$.

Proof of (3.4). – We have to estimate

$$\|(e^{i\xi} + 1)^{2m+1}\Phi(f)\|_{\mathfrak{a}} + \|D^m[(e^{i\xi}+1)^{2m+1}\Phi(f)]\|,\ f \in \mathfrak{A}^{(m+1)}. \tag{23}$$

Now

$$\Phi f(\xi) = \frac{1}{\nu}\int_0^\xi (Df)(t)\,dt = i\int_0^1 \Phi_{-r}(Df)\frac{e^{i\xi}+1}{(e^{i\xi}+r)(1+re^{i\xi})}\,dr$$

changing variables so that $e^{i\xi} = \phi_r(e^{it})$. Thus

$$(e^{i\xi} + 1)\,\Phi f = 2i\int_0^1 \Phi_{-r}((1 + \cos\xi)Df)\frac{dr}{(1+r)^2}\,. \tag{24}$$

When $m = 0$ condition (R) gives

$$\|(e^{i\xi} + 1)\,\Phi f\|_{\mathfrak{a}} \leq 2\int_0^1 \|\Phi_{-r}((1 + \cos\xi)\,Df)\|_{\mathfrak{a}}\,dr$$

$$\leq 2\left(\int_0^1 \rho_{\mathfrak{a}}(-r)\,dr\right)\|Df\|_{\mathfrak{a}}\ ,$$

and so

$$\|(e^{i\xi} + 1)\,\Phi f\|_{\mathfrak{a}} \leqslant 2\left(\int_0^1 \rho_{\mathfrak{a}}\,(-r)dr\right)\ \|f\|_{(1)}\,. \tag{25}$$

For the case $m > 0$ only the second of the terms in (23) requires further proof since, by (25),

$$\|(e^{i\xi} + 1)^{2m+1}\ \Phi f\,\|_{\mathfrak{a}} \leqslant \text{const.}\ \left(\int_0^1 \rho_{\mathfrak{a}}\,(-r)\,dr\right)\ \|f\|_{(m+1)} \tag{26}$$

whenever $f \in \mathfrak{A}^{(m+1)}$. But

$$\mathrm{D}\,[\Phi_{-r} f] = ie^{i\xi}\,\frac{1 - r^2}{(1 + re^{i\xi})^2}\ \Phi_{-r}\,(\mathrm{D}f).$$

Thus

$$[(e^{i\xi} + 1)^2\ \Theta]\ \Phi_{-r}(f) = \left(\frac{1 - r}{1 + r}\right)\ \Phi_{-r}((e^{i\xi} + 1)^2\,\mathrm{D}f).$$

This last result coupled with a routine induction argument shows that for each integer k and $g \in \mathcal{C}^{\infty}(\mathrm{T})$ there exist $a_0, \ldots, a_k \in \mathrm{L}^{\infty}(0,1)$ and $g_0, \ldots, g_k, h_0, \ldots, h_k \in \mathcal{C}^{\infty}(\mathrm{T})$ such that

$$[(e^{i\xi} + 1)^2\,\Theta]^k\ \Phi_{-r}(gf) = \sum_{\ell=0}^{k} a_\ell(r)\,h_\ell\,(\xi)\ \Phi_{-r}((1 + \cos\xi)g_\ell\,\mathrm{D}^\ell f).$$

But then, by (12) and (24) there exist functions $\gamma_0, \ldots, \gamma_m$ in $\mathcal{C}^{\infty}(\mathrm{T})$ so that

$$\|\mathrm{D}^m\ [(e^{i\xi} + 1)^{2m+1}\ \Phi f]\|_{\mathfrak{a}} \leqslant \text{const.}\ \sum_{\ell=0}^{m} \int_0^1\ \|\Phi_{-r}((1 + \cos\xi)g_\ell\ \mathrm{D}^\ell f)\|\,dr.$$

This combined with (26) gives

$$\|(e^{i\xi} + 1)^{2m+1}\ \ \Phi f\|_{(m)} \leqslant \text{const.}\left(\int_0^1\ \rho_{\mathfrak{a}}(-r)dr\right)\|f\|_{(m+1)}$$

because of condition (R). The proof is now complete

Proof of (3.5). – We have

$$\Phi[(e^{i\xi}+1)^q\ \Phi^k(f)] = (e^{i\xi}+1)^q\ \Phi^{k+1}(f) + \gamma(\xi)\ \Theta^k f(0)$$

with γ a function in $\mathcal{C}^\infty(\mathrm{T})$. Thus, if $g(\xi) = (e^{i\xi}+1)$,

$$g^s\ \Phi^m(f) = g\,\Phi\,[(e^{i\xi}+1)^{s-1}\ \Phi^{m-1}(f)] + \delta(\xi)\ \Theta^{m-1}f(0)$$

($\delta \in \mathcal{C}^\infty(\mathrm{T})$) and so

$$\|g^s\Phi^m(f)\|_{\mathcal{A}} \leq \text{const.}\left(\int_0^1 \rho_{\mathcal{A}}(-r)\,dr\right)\{\|g^{s-1}\ \Phi^{m-1}(f)\|_{(1)} + \|f\|_{(m-1)}\}$$

using (21) and lemma (3.4). Continuing this proof we see by induction that

$$\|g^s\ \Phi^m(f)\|_{\mathcal{A}} \leqslant \text{const.}\left(\int_0^1 \rho_{\mathcal{A}}(-r)\,dr\right)^p$$

$$\times\{\|g^{s-s_p}\ \Phi^{m-p}(f)\|_{(p)} + \|f\|_{(m-1)}\}$$

for $f \in \mathcal{A}^{(m)}$ and integer p, $1 \leqslant p \leqslant m$, where

$$s_p = 1 + (2+1) + (4+1) + \ldots (2(p-1)+1) = p^2.$$

Inequality (22) follows easily.

Important applications of theorem (3.2) follow from the next theorem. Frequently we shall require σ to be a function in $\mathcal{C}^\infty(\mathrm{T})$ satisfying

$$\left(\frac{e^{i\xi}-1}{e^{i\xi}+1}\right)^m \sigma \in \mathcal{C}^\infty(\mathrm{T}) \quad , \quad 0 \leqslant m \leqslant [\alpha]+3.$$

(3.6) THEOREM. – *For any* $\sigma \in \mathcal{C}^\infty(\mathrm{T})$ *satisfying* (25) set

$$\tau_r = (e^{i\xi}+1)^s\ \Phi_r(\sigma) \quad , \quad s = ([\alpha]+1)^2.$$

Suppose that the function $\rho_{\mathfrak{a}}(r)$ *satisfies*

$$\text{(a)} \quad \int_0^1 \rho_{\mathfrak{a}}(-r)\,dr < \infty \quad , \quad \text{(b)} \quad \rho_{\mathfrak{a}}(r) = O(1), r \to 1-.$$

Then

$$\lim_{r \to 1-} \|f\tau_r\|_{\alpha, p; \mathfrak{a}} = 0 \quad , \quad \alpha \neq [\alpha], \tag{28}$$

for all f *in the ideal* $\mathrm{I}_{\alpha,p}(0)$ *in* $\Lambda(\alpha, p; \mathfrak{A})$, $p \neq \infty$, *or in* $\lambda(\alpha, \infty; \mathfrak{A})$.

We postpone for the moment the proof of (3.6).

(3.7) Corollary. – *Under hypotheses (a) and (b) of theorem (3.6) for* $\alpha \neq [\alpha]$ *the algebras* $\Lambda(\alpha, p; \mathfrak{A})$, $p \neq \infty$, *and* $\lambda(\alpha, \infty; \mathfrak{A})$ *satisfy the Strong Ditkin condition. In fact there exists a sequence*

$$\{\tau_n\} \subset \mathcal{C}^\infty(\mathrm{T})$$

such that

(i) $\tau_n(\xi) = 1$ *in a neighborhood* of 0,

(ii) $\lim\limits_{n \to \infty} \|f\tau_n\|_{\alpha, p; \mathfrak{a}} = 0 \quad , \quad f \in \mathrm{I}_{\alpha,p}(0).$

Proof of (3.7). – Choose any $\sigma \in \mathcal{C}^\infty(\mathrm{T})$ such that $\sigma(\xi) = 1$ in a neighborhood of 0 while $\sigma(\xi) = 0$ in a neighborhood of π. Then for $0 < r < 1$,

(i)′ $\Phi_r(\sigma) = 1$ in a neighborhood of 0 (depending on r),

(ii)′ $\Phi_r(\sigma) = 0$ in a neighborhood of π (*independent* of r).

Because of (ii)′ there exists $\gamma \in \mathcal{C}^\infty(\mathrm{T})$, such that

$$\gamma\Phi_r(\sigma) = \Phi_r(\sigma) \quad , \quad 0 < r < 1 \quad , \quad (e^{i\xi} + 1)^{-k}\,\gamma \in \mathcal{C}^\infty(\mathrm{T}) \quad , \quad k \geqslant 0.$$

Now, with $r_n = 1 - 1/n$, set

$$\tau_n = \Phi_{r_n}(\sigma) = (e^{i\xi} + 1)^s\,((e^{i\xi} + 1)^{-s}\,\gamma)\,\Phi_{r_n}(\sigma).$$

Then $\{\tau_n\}$ has property (i) and, by (28), since (27) automatically is satisfied,

$$\lim_{n\to\infty} \|f\tau_n\|_{\alpha,p\,;\,\mathfrak{A}} = \lim_{r\to 1-} \|(f.(e^{i\xi}+1)^{-s}\gamma)(e^{i\xi}+1)^s\Phi_r(\sigma)\|_{\alpha,p\,;\,\mathfrak{A}} = 0$$

so that $\{\tau_n\}$ also has property (ii).

(3.8) COROLLARY. – *Under hypotheses (a) and (b) of theorem (3.6) for $\alpha \neq [\alpha]$ the algebras $\Lambda(\alpha, p\,; \mathfrak{A})$, $p \neq \infty$, and $\lambda(\alpha, \infty; \mathfrak{A})$ satisfy the Strong Analytic Ditkin condition. In fact there exists a sequence $\{\tau_n\} \subseteq \mathcal{C}^\infty(\mathrm{T})$ satisfying*

(i) τ_n *has an analytic extension into the open unit disk,*

(ii) $\tau_n(0) = 1$ *for all n,*

(iii) $\lim_{n\to\infty} \|f\tau_n\|_{\alpha,p\,;\,\mathfrak{A}} = 0 \quad , \quad f \in \mathrm{I}_{\alpha,p}(0).$

Proof. – Choose any $\sigma \in \mathcal{C}^\infty(\mathrm{T})$ satisfying (27) and such that $\sigma(0) = 1$. If σ also has an analytic extension into the open unit disk then, with

$$\tau_n = 2^{-s}\,(e^{i\xi}+1)^s\,\Phi_{r_n}(\sigma) \quad , \quad r_n = 1 - 1/n,$$

clearly $\{\tau_n\}$ has the required properties.

The practical importance of the estimates in theorem (3.2) lies in the next two theorems from which theorem (3.6) will follow easily.

(3.9) THEOREM. – *For any function $\sigma \in \mathcal{C}^\infty(\mathrm{T})$ the function σ_r,*

$$\sigma_r = \Phi_r(\sigma) \quad , \quad 0 < r < 1,$$

satisfies

$$\|\sin^k \xi\, \mathrm{D}^k \sigma_r\|_{\mathfrak{A}} \leqslant \text{const.}\ \rho_{\mathfrak{A}}(r) \left(\sum_{m=0}^{k+1} \left\| \left(\frac{e^{i\xi}-1}{e^{i\xi}+1}\right) \mathrm{D}^m \sigma \right\|_{\mathfrak{A}} \right) \quad (29)$$

uniformly in r, $0 < r < 1$.

Proof. – Now

$$(e^{2i\xi} - 1)^k \, D^k \, \sigma_r = \sum_{m=0}^{k} \gamma_m(\xi) \, [(e^{2i\xi} - 1) \, D]^m \, \sigma_r \tag{30}$$

where $\gamma_0, \ldots, \gamma_k$ are functions in $\mathcal{C}^\infty(T)$. But

$$D \, \sigma_r = ie^{i\xi} \frac{(1 - r^2)}{(1 - re^{i\xi})^2} \, \Phi_r(D\sigma)$$

so

$$(e^{2i\xi} - 1) D \, \sigma_r = ie^{i\xi} \, \Phi_r \, ((e^{2i\xi} - 1) D \, \sigma).$$

Using this last result and (30) we deduce that

$$\|\sin^k \xi \, D^k \, \sigma_r\|_{\mathfrak{a}} \leqslant \text{const.} \sum_{m=0}^{k} \|\Phi_r \, ([e^{2i\xi} - 1) \, D]^m \, \sigma)\|_{\mathfrak{a}}$$

$$< \text{const.} \, \rho_{\mathfrak{a}}(r) \left(\sum_{m=0}^{k+1} \left\| \left(\frac{e^{i\xi} - 1}{e^{i\xi} + 1}\right) D^m \, \sigma \right\|_{\mathfrak{a}} \right).$$

This establishes (29).

(3.10) THEOREM. – *For any* $\sigma \in \mathcal{C}^\infty(T)$

$$\|\sin^k \xi \, D^k \, \{(e^{i\xi} - 1) \, \sigma_r\}\|_{\mathfrak{a}}$$

$$\leqslant \text{const.} \left(\frac{1 - r}{1 + r}\right) \rho_{\mathfrak{a}}(r) \left(\sum_{m=0}^{k+1} \left\| \left(\frac{e^{i\xi} - 1}{e^{i\xi} + 1}\right)^2 D^m \, \sigma \right\|_{\mathfrak{a}} \right) \tag{31}$$

uniformly in r, $0 < r < 1$.

Proof. – Now

$$(e^{2i\xi} - 1)^k \, D^k \, \{(e^{i\xi} - 1) \, \sigma_r\} = \sum_{m=0}^{k} \delta_m(\xi) \, (e^{i\xi} - 1)[(e^{2i\xi} - 1) \, D]^m \, \sigma_r$$

with $\delta_0, \ldots, \delta_k$ in $\mathcal{C}^\infty(T)$. Since

$$(e^{i\xi} - 1) \, \Phi_r \, (.) = (e^{i\xi} + 1)\left(\frac{1 - r}{1 + r}\right) \Phi_r \left(\frac{e^{i\xi} - 1}{e^{i\xi} + 1} (.)\right)$$

we obtain (31) exactly as in (3.9).

In the next theorem $\mathfrak{X}^*$ denotes the dual space of $\mathfrak{X}$ and $D^k\delta_0$ the (distributional) derivative of the Dirac measure δ_0 at 0.

(3.11) THEOREM. – *The distributions* $\delta_0, D\delta_0, \ldots, D^{[\alpha]}\delta_0$ *belong to* $\Lambda(\alpha, p; \mathfrak{A})^*$ *for all* $\mathfrak{A}$ *but if* $\rho_{\mathfrak{A}}(r) = O(1)$ as $r \to 1-$ *then* $D^{[\alpha]+1}\delta_0$ *does not belong to* $\Lambda(\alpha, p; \mathfrak{A})^*$, $1 \leq p \leq \infty$.

Proof. – Since $\Lambda(\alpha, p; \mathfrak{A}) \subset \Lambda(\alpha, p; \mathcal{C}(\mathrm{T})) \subset \mathcal{C}^{[\alpha]}(\mathrm{T})$ the first assertion is clear. Now set

$$\sigma_r(\xi) = (e^{i\xi} + 1)^{2[\alpha]+2}\, \Phi_r((e^{i\xi} + 1)^2)$$

$$= (1 - r)^2\, (e^{i\xi} + 1)^{2[\alpha]+4}\, (1 - re^{i\xi})^{-2}.$$

Then

$$\|\sigma_r\|_{\mathfrak{A}} \leq \text{const.}\ \|\Phi_r((e^{i\xi} + 1)^2)\|_{\mathfrak{A}} \leq \text{const.}\ \rho_{\mathfrak{A}}(r).$$

But

$$(e^{i\xi} + 1)^2\, D\Phi_r(.) = ie^{i\xi}\left(\frac{1 + r}{1 - r}\right)\Phi_r\,([(e^{i\xi} + 1)^2\, D](.)).$$

Hence (much as in the proof of (3.9)) we obtain

$$\|D^k\sigma_r\|_{\mathfrak{A}} \leq \text{const.}\ \rho_{\mathfrak{A}}(r)\sum_{m=0}^{R}\left(\frac{1 + r}{1 - r}\right)^m < \frac{\text{const.}}{(1 - r)^k}\,\rho_{\mathfrak{A}}(r)\ ,\ k \geq 0.$$

Consequently (cf. (6))

$$\|\sigma_r\|_{\alpha, p; \mathfrak{A}} \leq \text{const.}\ (1 - r)^{-\alpha}\,\rho_{\mathfrak{A}}(r). \qquad (32)$$

Now define $\{\beta_r\}_{0<r<1}$ in $\Lambda(\alpha, p; \mathfrak{A})$ by $\beta_r = (1 - r)^{\alpha}\sigma_r$. By (32) this is a uniformly norm bounded family in $\Lambda(\alpha, p; \mathfrak{A})$ whenever $\rho_{\mathfrak{A}}(r) = O(1)$, $r \to 1-$. But

$$\limsup_{r\to 1-} < \beta_r, D^{[\alpha]+1}\delta_0 > = \limsup_{r\to 1-} D^{[\alpha]+1}\beta_r(0)$$

$$= \text{const.}\ \limsup_{r\to 1-}(1 - r)^{\alpha+2}\, D^{[\alpha]+1}\,[1/(1 - \mathrm{r}e^{i\xi})^2]\,(0) = \infty.$$

Hence $D^{[\alpha]+1}\delta_0 \in \Lambda(\alpha, p; \mathfrak{A})^*$.

(3.12) COROLLARY. — *When* $\rho_{\mathfrak{a}}(r) = O(1)$, $r \to 1-$, *the family*

$$\{(e^{i\xi} - 1)^{|\alpha|+1}\phi : \phi \in \mathcal{C}^\infty(T)\}$$

is dense in $I_{\alpha,p}(0)$, $p \neq \infty$, *and in* $I_{\alpha,\infty}(0) \cap \lambda(\alpha, \infty; \mathfrak{A})$.

We can finally prove theorem (3.6).

Proof of (3.6). Choose any σ in $\mathcal{C}^\infty(T)$ satisfying (27). Then for each f in $I_{\alpha,p}(0)$

$$\|f\tau_r\|_{\alpha,p;\mathfrak{a}} = \|f.(e^{i\xi}+1)^s\sigma_r\|_{\alpha,p;\mathfrak{a}} \leqslant \text{const.}\,\rho_{\mathfrak{a}}(r)\,\|f\|_{\alpha,p;\mathfrak{a}} \qquad (33)$$

uniformly in r, $0 < r < 1$, substituting (29) in theorem (3.2). Given $\varepsilon > 0$ choose $\phi \in \mathcal{C}^\infty(T)$ with

$$\|f - (e^{i\xi} - 1)^{|\alpha|+2}\phi\|_{\alpha,p;\mathfrak{a}} < \varepsilon.$$

Then, by (33),

$$\|f\tau_r\|_{\alpha,p;\mathfrak{a}} \leq \text{const.}\,\rho_{\mathfrak{a}}(r)\,\varepsilon + \|(e^{i\xi} - 1)^{|\alpha|+1}\phi.\{(e^{i\xi}-1)\tau_r\}\|_{\alpha,p;\mathfrak{a}}.$$

Since $(e^{i\xi} - 1)^{|\alpha|+1}\phi$ belongs to $I_{\alpha,p}(0)$, substitution of estimate (31) in theorem (3.2) clearly gives (28) provided $\int_0^1 \rho_{\mathfrak{a}}(-r)dr < \infty$ and $\rho_{\mathfrak{a}}(r) = O(1)$, $r \to 1-$.

(3.13) *Remark.* — It can be shown that

$$\|\tau_r\|_{\alpha,p;\mathfrak{a}} = O((1-r)^{-\alpha}) \quad , \quad r \to 1-,$$

is the best possible estimate for $\{\tau_r\}$. Hence $\{\tau_r\}_{0<r<1}$ is unbounded in the multiplier norm on $\Lambda(\alpha, p; \mathfrak{A})$ but bounded in the multiplier norm on $I_{\alpha,p}(0)$, $\alpha > 0$.

Some important examples illustrate the general theory of this section. (For formal definitions and extended discussion see [5, pp. 226-232], [23, chap. VIII]). Denote by $W_q^m(T)$, $1 \leqslant q \leqslant \infty$, $m \geqslant 0$, the Sobolev spaces :

$$W_q^m(T) = \{f : f, Df, \ldots, D^m f \in L^q(T)\} \quad , \quad 1 \leqslant q < \infty,$$

$$W_\infty^m(T) = \{f : f, Df, \ldots, D^m f \in \mathcal{C}(T)\} \quad , \quad q = \infty.$$

The interpolation spaces

$$\left.\begin{array}{lll} B_q^{\alpha p}(T) = (L^q(T) \, , \, W_q^m(T))_{\theta, p} & , & q < \infty \\ B_\infty^{\alpha p}(T) = (\mathcal{C}(T) \, , \, W_\infty^m(T))_{\theta, p} & , & q = \infty \end{array}\right\} \theta = \alpha/m,$$

are just the classical periodic Besov spaces (cf. [2]). Appendix I of [8] and stability indicate why we have chosen to define $B_\infty^{\alpha p}$ via the space $\mathcal{C}(T)$ rather than via $L^\infty(T)$ which at first sight might seem more natural. It is known that

$$B_q^{\alpha 1} \subset \mathcal{C}(T) \quad , \quad \alpha = 1/q,$$

(a simple proof for the non-periodic case appears in [8, p. 240], cf. also [11, p. 289)]. For convenience of notation we set

$$\mathcal{B}^q = B_q^{1/q,1}(T) \quad , \quad 1 \leq q \leq \infty \quad ; \quad \mathcal{B}^\infty = \mathcal{C}(T) \quad , \quad q = \infty.$$

Then, by the stability theorem,

$$\Lambda(\alpha \, , p \, ; \mathcal{B}^q) = B_q^{\beta p} \quad , \quad \beta = \alpha + 1/q \quad , \quad \alpha > 0,$$

Hence (cf. also [19, ex. 2.3] for an entirely different proof) :

(3.14) THEOREM. – *The Besov* spaces $B_q^{\alpha p}(T)$ *are Banach algebras for*

$$\alpha \geq 1/q \qquad p = 1 \; ; \alpha > 1/q \quad , \quad 1 \leq p \leq \infty,$$

-the so-called SCHAUDER ALGEBRAS. Furthermore, $\mathcal{B}^q$,

$$\Lambda(\alpha \, , p \, ; \mathcal{B}^q) \quad , \quad p \neq \infty \qquad \text{and} \qquad \lambda(\alpha \, , p \, ; \mathcal{B}^q)$$

are homogeneous algebras on T.

Proof. – Apply corollary (2.3). That T is the maximal ideal space of $\mathcal{B}^q$, hence of $\Lambda(\alpha \, , p \, ; \mathcal{B}^q)$, $p \neq \infty$, and $\lambda(\alpha \, , p \, ; \mathcal{B}^q)$ is clear.

We now estimate $\rho_q(r)$ where

$$\|\Phi_r((1+\cos\xi)f)\|_{\mathcal{B}^q} \leqslant \rho_q(r)\,\|f\|_{\mathcal{B}^q}$$

For $q = \infty$, obviously

$$\rho_\infty(r) = 1 \quad , \quad -1 < r < 1.$$

Now, when $q < \infty$,

$$\int_0^{2\pi} |g(\phi_r(\xi))|^q\, d\xi = \int_0^{2\pi} |g(t)|^q\, \mathrm{P}_{-r}(t)\,dt$$

where $\mathrm{P}_r(t)$ is the Poisson kernel, $-1 < r < 1$. But

$$(1+\cos t)\ \mathrm{P}_{-r}(t) \leqslant 2\left(\frac{1-r}{1+r}\right) \quad , \quad -1 < r < 1$$

(notice again the importance of the factor (1 + cost)). Consequently,

$$\|\Phi_r((1+\cos\xi)f)\|_{L^q} \leq 2\left(\frac{1-r}{1+r}\right)^{1/q} \|f\|_{L^q}.$$

On the other hand, since

$$\int_0^{2\pi} |\mathrm{D}[g(\phi_r(\xi))]^q\, d\xi \leq (1-|r|)^{1-q} \int_0^{2\pi} |\mathrm{D}g(t)|^q\, dt,$$

we deduce that

$$\|\Phi_r((1+\cos\xi)f)\|_{\mathrm{W}_q^1} \leq \text{const.}\ (1+r)^{-1/a}\,(1-r)^{-1/q}\,\|f\|_{\mathrm{W}_q^1}$$

where $a = \min(q, q')$. The interpolation theorem thus gives :

(3.15) THEOREM. – *For the algebras* $\mathcal{B}^q(\mathrm{T})$,

$$\rho_q(r) \leq \text{const.}\ (1+r)^{-1/b} \quad , \quad -1 < r < 1,$$

where $b = \frac{1}{2}\, q\,(\min(q', 2))$. *In particular,* $\mathcal{B}^q(\mathrm{T})$ *satisfies condition* (R) *when* $q \geqslant 1$.

Consequently, the theory developed in this section applies to all the Schauder algebras $\mathcal{B}^q$, $(\mathcal{B}^q)^{(m)}$ and $\Lambda(\alpha, p; \mathcal{B}^q)$, provided $q > 1$.

(3.16) *Remarks.* – (i) The algebras

$$\lambda(\alpha, \infty; \mathcal{C}(T) \qquad \text{and} \qquad \Lambda(1, \infty; \mathcal{C}(T))$$

i.e. $\lambda(\alpha, \infty; \mathcal{B}^\infty)$ and $\Lambda(1, \infty; \mathcal{B}^\infty)$ in our notation, are of course the familiar Lipschitz algebras lip (T, d^α), Lip (T, d) respectively discussed extensively in [22].

(ii) The spaces $L^q(T)$ can be replaced with Lorentz L^{pq}-spaces or, more generally, rearrangement invariant Banach Function spaces on T (cf. [1]).

(iii) The Strong Ditkin condition has been established directly for the analogous spaces $\mathcal{B}^q(R^n)$ where the problem is very easy (cf. [11, p. 295]).

4. Subalgebras of $\ell^1(Z)$.

Let $\mathfrak{a}(Z)$ be a convolution Banach subalgebra of $\ell^1(Z)$ with maximal ideal space T and Gelfand Transform the usual Fourier Transform.

$$\mathfrak{F} : \{f_n\} \to f(\xi) = \sum_{-\infty}^{\infty} f_n e^{in\xi} ;$$

throughout this section $\mathfrak{A}(T) = \mathfrak{F}(\mathfrak{a}(Z))$. We shall always assume that $\{\chi(t) : 0 < t < \infty\}$,

$$\chi(t) : \{f_n\} \to \{e^{int} f_n\} \quad , \quad \{f_n\} \in \mathfrak{a}(Z),$$

is a semi-group of contraction operators on $\mathfrak{A}(Z)$ of class $(\mathcal{C}_0)$. Thus $\mathfrak{A}(T)$ is a homogeneous algebra on T in the sense of Silov which we suppose satisfies also : $\mathcal{C}^\infty(T) \subset \mathfrak{A}(T)$. The infinitesimal generator of $\{\chi(t) : 0 < t < \infty\}$ is $M : \{f_n\} \to \{inf_n\}$. From corollary (2.4) we deduce that the interpolation space

$$\Lambda(\alpha, p\,; \mathfrak{a}(Z)) = (\mathfrak{a}(Z) \quad , \quad \mathfrak{a}^{(k)}(Z))_{\theta, p; K} \qquad \theta = \alpha/k,$$

is a convolution Banach subalgebra of $\mathfrak{A}(Z)$ for all $\alpha > 0$ and $1 < p \leqslant \infty$. Of course, with obvious notation,

$$\mathfrak{F}(\Lambda(\alpha, p\,; \mathfrak{a}(Z))) = \Lambda(\alpha, p\,; \mathfrak{A}(T))$$

and $\Lambda(\alpha, p\,; \mathfrak{A}(T))$ is a Banach subalgebra of the Wiener algebra $\mathfrak{F}(\ell^1(Z))$; in fact, $\Lambda(\alpha, p\,; \mathfrak{A}(T))$ has maximal ideal space T, while $\Lambda(\alpha, p\,; \mathfrak{A}(T))$, $p \neq \infty$, $\lambda(\alpha, \infty\,; \mathfrak{A}(T))$ and $\mathfrak{A}^{(m)}(T)$ are homogeneous algebras on T all containing $\mathcal{C}^\infty(T)$.

To overcome the fact that (very likely) the operators Φ_r do not map $\mathfrak{A}(T)$ into $\mathfrak{A}(T)$ (cf. [21, chap 4] for the example $\mathfrak{F}(\ell^1(Z))$) we introduce two conditions the second of which is of considerable technical interest. These conditions will be imposed as needed.

CONDITION (H – L). – *The Banach algebra* $\mathfrak{a}(Z)$ *is said to satisfy the Hardy-Littlewood condition if the operator* $F : \{f_n\} \to \{F_n\}$ *defined by*

$$F_n = \sum_{n+1}^{\infty} f_m \quad , \quad n \geq 0 \quad , \quad F_n = -\Big(\sum_{-\infty}^{n} f_m\Big) \quad , \quad n < 0$$

satisfies

$$\|\{F_n\}\|_{(k)} \leqslant \text{const.}\ \|\{f_n\}\|_{(k+1)} \quad , \quad \{f_n\} \in \mathfrak{a}^{(k+1)}(Z),$$

for each integer $k \geq 0$.

The significance of the operator F is clear : for by summation by parts, when $f(\xi) = \sum_{-\infty}^{\infty} f_n\, e^{in\xi}$,

$$(\Phi f)(\xi) = \frac{1}{\nu}\,(f(\xi) - f(0)) = \sum_{-\infty}^{\infty} F_n\, e^{in\xi} = \mathfrak{F}\{F_n\}. \tag{33}$$

CONDITION (B). – *The Banach algebra* $\mathfrak{A}(T)$ $(= F(\mathfrak{a}(Z)))$ *is said to satisfy the Bernstein condition if there is a homogeneous Banach algebra* $\mathfrak{B}(T)$ *on T satisfying condition* (R) *with* $\mathfrak{B}(T) \subset \mathfrak{A}(T)$.

The closed primary ideals $I_{\alpha,p}(\xi_0)$ and $I^{(k)}(\xi_0)$ in $\Lambda(\alpha, p; \mathfrak{A}(T))$, $\mathfrak{A}^{(k)}(T)$ respectively are defined exactly as in §2.

(4.1) THEOREM. – *Suppose* $\mathfrak{A}(Z)$ *satisfies the condition* (H – L) *and* $\mathfrak{A}(T) = \mathfrak{F}(\mathfrak{a}(Z))$. *Then, provided* $\alpha \neq [\alpha]$.

$$\|f\sigma\|_{\alpha,p;\mathfrak{A}} \leq \text{const.}\ \|f\|_{\alpha,p;\mathfrak{A}} \left(\sum_{k=0}^{[\alpha]+1} \|\nu^k D^k \sigma\|_{\mathfrak{A}} \right) \tag{34}$$

for all f in the ideal $I_{\alpha,p}(0)$ *in* $\Lambda(\alpha, p; \mathfrak{A}(T))$ *and* σ *in* $\mathcal{C}^\infty(T)$.

Proof. – In view of (34) and the Hardy-Littlewood condition, inequalities (18) and (19) hold with

$$M_0(\sigma) = \sum_{k=0}^{[\alpha]} \|\nu^k D^k \sigma\|_{\mathfrak{A}} \quad , \quad M_1(\sigma) = \sum_{k+1}^{[\alpha]+1} \|\nu^k D^k \sigma\|_{\mathfrak{A}}$$

Inequality (34) follows by interpolation.

Estimate (34) thus shows that

$$\|f\sigma.(e^{i\xi}+1)^{[\alpha]+1}\|_{\alpha,p;\mathfrak{A}} \leq \text{const.}\ \|f\|_{\alpha,p;\mathfrak{A}} \left(\sum_{k=0}^{[\alpha]+1} \|\sin^k \xi D^k \sigma\|_{\mathfrak{A}} \right),$$

for all f in $I_{\alpha,p}(0)$ and σ in $\mathcal{C}^\infty(T)$. If $\mathfrak{A}(T)$ satisfies also the Bernstein condition, say $\mathfrak{B}(T) \subset \mathfrak{A}(T)$ where $\mathfrak{B}(T)$ satisfies condition (R), then

$$\|f\sigma.(e^{i\xi}+1)^{[\alpha]+1}\|_{\alpha,p;\mathfrak{A}} \leq \text{const.}\ \|f\|_{\alpha,p;\mathfrak{A}} \left(\sum_{k=0}^{[\alpha]+1} \|\sin^k \xi D^k \sigma\|_{\mathfrak{B}} \right) \tag{35}$$

for all f in $I_{\alpha,p}(0)$ and σ in $\mathcal{C}^\infty(T)$. Choose any σ in $\mathcal{C}^\infty(T)$ satisfying (27) and set

$$\sigma_r = \Phi_r(\sigma) \quad , \quad \tau_r(\xi) = [(e^{i\xi}+1)^{[\alpha]+1}]\,\sigma_r \quad , \quad 0 < r < 1.$$

(4.2) THEOREM. – *Suppose* $\mathfrak{a}(Z)$ *satisfies the Hardy-Littlewood condition and* $\mathfrak{A}(T)$ $(= \mathfrak{F}(\mathfrak{a}(Z)))$ *the Bernstein condition for some Banach algebra* $\mathfrak{B}(T)$. *If* $\rho_{\mathfrak{B}}(r) = O(1)$, $r \to 1-$, *then,*

$$\lim_{r \to 1-} \|f\tau_r\|_{\alpha,p;\mathfrak{A}} = 0 \quad , \quad \alpha \neq [\alpha],$$

for all f in the closed ideal $\mathrm{I}_{\alpha,p}(0)$ *in* $\Lambda(\alpha, p; \mathcal{A}(\mathrm{T}))$, $p \neq \infty$, *or in* $\lambda(\alpha, \infty; \mathcal{A}(\mathrm{T}))$.

The proof of (4.2) follows from the multiplier inequality (35) and estimates (29), (31) applied to $\mathcal{B}(\mathrm{T})$. The analogues of theorem (3.11) and corollary (3.12) for $\Lambda(\alpha, p; \mathcal{A}(\mathrm{T}))$ hold because

$$\Lambda(\alpha, p; \mathcal{B}(\mathrm{T})) \subset \Lambda(\alpha, p; \mathcal{A}(\mathrm{T})) \subset \Lambda(\alpha, p; \mathcal{C}(\mathrm{T}))$$

(using the Lipschitz characterization (5) or interpolation, for instance). We omit the details.

Exactly as in corollaries (3.7) and (3.8) of theorem (3.6) we deduce from theorem (4.2) the following important results.

(4.3) COROLLARY. – *Under the hypotheses of theorem (4.2) the Banach algebras* $\Lambda(\alpha, p; \mathcal{A}(\mathrm{T}))$, $p \neq \infty$, *and* $\lambda(\alpha, p; \mathcal{A}(\mathrm{T})$ *satisfy the Strong Ditkin and Strong Analytic Ditkin conditions whenever* $\alpha \neq [\alpha]$.

Finally, we discuss some examples of algebras satisfying the Hardy-Littlewood and Bernstein conditions. On $\ell^q(\mathrm{Z})$, $1 \leqslant q < \infty$, and $c_0(\mathrm{Z})$ the family $\{\chi(t) : 0 < t < \infty\}$ is a semi-group of contraction operators of class $(\mathcal{C}_0)$. For convenience of notation we define $\ell^q_\alpha(\mathrm{Z})$ by

$$\ell^q_\alpha(\mathrm{Z}) = \left\{ \{f_n\} : \left(\sum_{-\infty}^{\infty} [(1 + |n|)^\alpha \, |f_n|]^q \right)^{1/q} < \infty \right\}$$

with obvious modifications when $q = \infty$. Then $\ell^q_k(\mathrm{Z})$, $k = 1, 2, \ldots$, is the domain of definition of M^k. The Beurling space $b^{\alpha p}_q(\mathrm{Z})$ is the interpolation space

$$b^{\alpha p}_q(\mathrm{Z}) = \Lambda(\alpha, p; \ell^q(\mathrm{Z})) = (\ell^q(\mathrm{Z}) \;,\; \ell^q_k(\mathrm{Z}))_{\theta, p; \mathrm{K}} \qquad \theta = \alpha/k,$$

and similarly for $q = \infty$. It is known that $b^{\alpha p}_q(\mathrm{Z}) \subseteq \ell^1(\mathrm{Z})$ whenever

$$\alpha \geq 1 - 1/q \;,\; p = 1 \;;\; \alpha > 1 - 1/q \;,\; 1 \leq p \leq \infty,$$

(essentially this is contained in proposition 2.1 of [11], however, see [7] for alternative proof). Set

$$b^q(Z) = b_q^{\beta 1}(Z) \quad , \quad \beta = 1 - 1/q \quad ; \quad b^1(Z) = \ell^1(Z) \ ;$$

then by stability

$$\Lambda(\alpha, p\, ; b^q(Z)) = b_q^{\gamma p}(Z) \quad , \quad \gamma = \alpha + (1 - 1/q),$$

and by corollary (2.3) we have :

(4.4) THEOREM. – *The Beurling spaces* $b_q^{\alpha p}(Z)$ *are convolution Banach subalgebras of* $\ell^1(Z)$ *whenever*

$$\alpha \geq 1 - 1/q \quad , \quad p = 1 \quad ; \quad \alpha > 1 - 1/q \quad , \quad 1 \leq p \leq \infty,$$

-the so-called BEURLING ALGEBRAS. Furthermore,

$$\mathfrak{F}(b^q) \quad , \quad \mathfrak{F}(\Lambda(\alpha, p\, ; b^q)) \quad , \quad p \neq \infty \quad \textit{and} \quad \mathfrak{F}(\lambda(\alpha, \infty\, ; b^q))$$

are homogeneous algebras on T.

Proof. – That T is the maximal ideal space of b^q, hence of $\Lambda(\alpha, p\, ; b^q)$, $p \neq \infty$, and of $\lambda(\alpha, \infty\, ; b^q)$, is easy to establish as are the remaining homogeneity properties.

Particular examples of the algebras b^q, $\Lambda(\alpha, p\, ; b^q)$ have been widely studied in the literature with a completely different definition. For instance :

(i) $\Lambda(\alpha, 1\, ; b^1)$ is the classical Beurling algebra $\ell^1_\alpha(Z)$ (cf. [17, 18] and [20, p. 137]).

(ii) $\Lambda(1 - 1/q, 1\, ; \ell^q)$, i.e. the Banach algebra b^q, is precisely the discrete analogue of the algebra $\mathfrak{A}^q(R^n)$ introduced by Beurling ([3, p. 10] ; cf. also [11], [18, theorem (2.2)]).

We shall prove that the algebras b^q satisfy the Hardy-Littlewood condition and $\mathfrak{F}(b^q)$ the Bernstein condition so that the harmonic analysis of these particular examples can be developed within the framework of this paper.

(4.5) THEOREM. – *The algebras* $b^q(Z)$ *and* $\Lambda(\alpha, p\, ; b^q)$, $1 \leqslant q < \infty$, *all satisfy the Hardy-Littlewood condition.*

Proof. – Certainly F is bounded

$$F : \ell_{kq+1}(Z) \to \ell^q_k(Z) \quad , \quad 1 \leq q < \infty \quad , \quad k = 0, 1, \dots ,$$

(cf. [9, p. 143]). Hence, by interpolation, F is bounded

$$F : \Lambda(\alpha + 1, p ; \ell^q) \to \Lambda(\alpha, p ; \ell^q).$$

The theorem follows.

(4.6) THEOREM. – *For each algebra* $\mathfrak{F}(b^q)$, $1 \leqslant q \leqslant \infty$, *there is a homogeneous Banach algebra* $\mathcal{B}$ *on* T *satisfying condition* (R) *together with*

$$\text{(i)} \quad \mathcal{B} \subset \mathfrak{F}(b^q) \qquad \text{(ii)} \quad \rho_{\mathcal{B}}(r) = O(1) \quad , \quad r \to 1-. \quad (36)$$

In particular, $\mathfrak{F}(b^q)$ *satisfies the Bernstein condition,* $1 \leq q \leq \infty$.

Proof. – Taking Fourier Transforms we see that $\mathfrak{F}(b^2) = \mathcal{B}^2$. That

$$\mathcal{B}^2 \subset \mathfrak{F}(\ell^1(Z)) = \mathfrak{F}(b^1)$$

is Bernstein's theorem (or least a weak variant of it). Now it is known that $b^{q_1} \subseteq b^{q_0}$ whenever $q_1 \geq q_0$ (essentially this is [11, proposition 2.1 (iii)]). Hence

$$\mathcal{B}^2 = \mathfrak{F}(b^2) \subset \mathfrak{F}(b^q) \subset \mathfrak{F}(b^1) \quad , \quad 1 \leq q \leq 2.$$

On the other hand, by the Hausdorff-Young theorem and the Lipschitz characterization (5),

$$\mathfrak{F}^{-1}(\mathcal{B}^{q'}) \subset b^q \quad , \quad 2 \leq q \leq \infty \quad , \quad \frac{1}{q} + \frac{1}{q'} = 1$$

Hence, in all cases, there exists a Besov space $\mathcal{B}^s$ such that $\mathcal{B}^s \subseteq \mathfrak{F}(b^q)$. Theorem (3.15) shows that part (ii) of (36) holds also.

(4.7) *Remarks.* – (i) All the theory developed in this section applies to b^q, $\Lambda(\alpha, p ; b^q)$ as well as to Banach algebras obtained by replacing ℓ^q with, say, a rearrangement invariant Banach Function spacc on Z.

(ii) The Ditkin condition for $\ell^1_\alpha(Z)$ $(= \Lambda(\alpha, 1; \ell^1))$ has been established by Reiter ([20, p. 137]) by entirely different methods.

(iii) Entirely analogous results hold for the Banach algebras $\mathfrak{a}^{(k)}(Z)$ or the special examples $(b^q)^{(k)}$, $k = 1, 2, \ldots$.

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