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MULTIPLICATIVE FUNCTIONALS OF DUAL PROCESSES

by Ronald K. GETOOR (*)

1. Introduction.

Let X and $\hat{X}$ be a pair of standard processes in duality relative to a Radon measure ξ, that is X and $\hat{X}$ satisfy the conditions on page 259 of [1]. We refer the reader to [1] for all terminology and notation not explicitly defined here. One of the most important properties of such dual processes is (VI-1.16) — all such references are to [1] — which states that if B is a Borel set, then for all $\alpha \geqslant 0$ and x, y in the state space E

$$(1.1) \qquad P_B^\alpha u^\alpha(x, y) = u^\alpha \hat{P}_B^\alpha(x, y).$$

This result which is due to Hunt [4] may be viewed as stating that the process X killed when it first hits B and the process $\hat{X}$ killed when it first hits B are in duality. Define

$$Q_t f(x) = E^x\{f(X_t); t < T_B\} \text{ and } f\hat{Q}_t(x) = \hat{E}^x\{f(X_t); t < T_B\}.$$

For typographical convenience we will omit the hat « ^ » in those places where it is obviously required. Thus

$$\hat{E}^x\{f(X_t);\ t < T_B\}$$

is short for $\hat{E}^x\{f(\hat{X}_t); t < \hat{T}_B\}$. It is easy to see that (1.1) is equivalent to

$$(1.2) \qquad (f\hat{Q}_t,\ g) = (f,\ Q_t g)$$

for all $t \geqslant 0$ and for all f, g in $\mathbf{C}_K$. Here $\mathbf{C}_K$ denotes the

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real valued continuous functions with compact support, and $(\varphi, \psi) = \int \varphi(x)\psi(x)\,dx$ with $\xi\,(dx) = dx$.

The purpose of this paper is to extend (1.1) and (1.2) to a more general class of multiplicative functionals than those of the form $M_t = I_{(0,T_B)}(t)$. Our basic result is that if M is an exact MF (multiplicative functional) of X, then there exists a unique exact MF, $\hat{M}$, of $\hat{X}$ such that (1.2) holds where (Q_t) and $(\hat{Q}_t)$ are the semigroups generated by M and $\hat{M}$ respectively. In addition an appropriate analogue of (1.1) also holds. Actually the existence of $\hat{M}$ is an easy consequence of a result of Meyer [6] and undoubtedly is known to many people. Our main contribution is the fact that this correspondence is multiplicative, that is, $\widehat{MN} = \hat{M}\hat{N}$, and it is this property that turns the above correspondence into a useful tool.

This paper represents an extension of work begun by Hunt in Sections 17 and 21 of [4]. In particular it provides an answer to the rather cryptic comment at the top of page 304 in [1]. Some of these results were announced in [3].

We now will outline the contents of this paper. The basic results and some consequences are established in Sections 3 and 4. In particular Section 3 treats the existence of $\hat{M}$ and Section 4 the multiplicative property of the correspondence between M and $\hat{M}$. Some examples of this correspondence are discussed in Section 5, while Section 6 contains an extension of these results to non-exact multiplicative functionals. In Section 7 we associate a measure μ_M with each natural multiplicative functional M of X, and in Section 8 we show that $\mu_M = \mu_{\hat{M}}$. These results are then used to show, roughly speaking, that M is natural if and only if $\hat{M}$ is natural and that M is continuous if and only if $\hat{M}$ is continuous. See Theorem 8.6 for the precise statements. Finally Section 9 contains a few elementary applications of the above results to additive functionals. We intend to devote a future paper to some deeper applications in this same direction.

The notation used in this paper is that of [1]. However, for convenience of the reader we collect here some of the less standard notation. The state space E for our processes is a locally compact space with a countable base. $E_\Delta = E \cup \{\Delta\}$ where Δ is adjoined to E as the point at infinity if E is not compact or as an isolated point if E is compact. All numerical functions f on E are automatically extended to E_Δ by setting $f(\Delta) = 0$ unless explicitly stated otherwise. For any such f, $\| f \| = \sup \{| f(x)| : x \in E\}$. Of course, $\| f \|$ may be infinite. $\mathcal{E}(\mathcal{E}^*)$ denotes the σ-algebra of Borel (universally measurable) subsets of E. We will write $f \in \mathcal{E}(\mathcal{E}^*)$ to indicate that the numerical function f is Borel (universally) measurable. If $\mathcal{H}$ is any collection of numerical functions, $b\mathcal{H}$ denotes the bounded functions in $\mathcal{H}$ and $\mathcal{H}^+$ (or $\mathcal{H}_+$) denotes the nonnegative elements of $\mathcal{H}$. For example, $f \in b\mathcal{E}^+$ means that f is a bounded nonnegative Borel function. $\mathbf{C_K}$ denotes the real valued continuous functions with compact support on E. By $\int_a^b$ we will always mean the integral over $(a, b]$ unless explicitly stated otherwise. Finally we often will omit the qualifying phrase « almost surely ».

2. Preliminaries.

We fix once and for all a pair of standard processes $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ and $\hat{X} = (\hat{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \hat{X}_t, \hat{\theta}_t, \hat{P}^x)$ on the same state space E which are in duality relative to a fixed Radon measure ξ. We often write simply dx for $\xi(dx)$ and (f, g) will always stand for $\int fg\, d\xi = \int f(x)g(x)\, dx$ whenever the integral makes sense. Naturally not all that follows depends on the existence of the dual process $\hat{X}$. We will use the results of Section VI-1 of [1] without special mention. Note, however, that we are assuming here that the basic measure ξ is a Radon measure whereas in [1] it is only assumed to be σ-finite. This is merely for convenience. On the other hand we make no regularity assumptions on the resolvents of X or $\hat{X}$ such as those made in Sections VI-2 and VI-4 of [1].

Let $M = (M_t)$ be a multiplicative functional (MF) of X. Throughout this paper all MF'*s* are assumed to be right continuous, decreasing, and to satisfy $0 \leqslant M_t \leqslant 1$. Also *equality between* MF'*s always means equivalence*; that is, $M = N$ provided that (almost surely) $t \to M_t$ and $t \to N_t$

are identical functions on $[0, \zeta)$. See (III-1.6). For $f \in \mathcal{E}_+^*$ define

(2.1) $$Q_t f(x) = E^x\{f(X_t)M_t\}, \quad t \geqslant 0$$
$$V^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t) M_t \, dt, \quad \alpha \geqslant 0$$

so that (Q_t) and (V^α) denote the semigroup and resolvent generated by M respectively. Since $f(\Delta) = 0$ we have

$$Q_t f(x) = E^x\{f(X_t)M_t;\ t < \zeta\}$$

and $$V^\alpha f(x) = E^x \int_0^\zeta e^{-\alpha t} f(X_t) M_t \, dt.$$

We will use the results and terminology of Chapter III of [1] without special mention. In particular, recall that $E_M = \{x:\ P^x(M_0 = 1) = 1\}$ is the set of permanent points for M, and in the present situation $t \to M_t$ is identically zero almost surely P^x if x is not in E_M.

We now introduce an operator associated with M that will play a fundamental role in the sequel. For $f \in \mathcal{E}_+^*$ and $\alpha \geqslant 0$ define

(2.2) $$\begin{aligned} P_M^\alpha f(x) &= -E^x \int_0^\infty e^{-\alpha t} f(X_t)\, dM_t \qquad &\text{if} \quad x \in E_M \\ &= f(x) &\text{if} \quad x \notin E_M. \end{aligned}$$

Observe that the integration in (2.2) extends only over the interval $(0, \zeta)$ by our convention on f. Of course, P_M^α is given by a kernel and as usual we write

$$P_M^\alpha f(x) = \int P_M^\alpha(x, dy)\, f(y).$$

Note that if T is a terminal time and $M_t = I_{[0,T)}(t)$, then $P_M^\alpha f = P_T^\alpha f$. Thus P_M^α extends the notion of « hitting operator » to a general MF [1]. Obviously $P_M^\alpha 1 \leqslant 1$ for all $\alpha \geqslant 0$. The following relationship is well-known but we will include a proof for completeness.

(2.3) Proposition. — *If* $U^\alpha f$ *is finite, then*

$$U^\alpha f - V^\alpha f = P_M^\alpha U^\alpha f.$$

Of course, if $\alpha > 0$ *and* f *bounded, then* $U^\alpha f$ *is bounded.*

[1] It is well-known that on a possibly larger Ω space one can introduce a stopping time R such that $P_M^\alpha = P_R^\alpha$.

Proof. — If $x \notin E_M$, then $V^\alpha f(x) = 0$ and so the result is obvious in this case. The simplest way to prove the desired identity for $x \in E_M$ is to make use of the following lemma which is due to Meyer. See [5, Chap. VII, Th. 15]. This lemma will be used several times in the sequel.

(2.4) LEMMA. — *Let $(a_t;\ t \geqslant 0)$ be a right continuous increasing process adapted to $(\mathcal{F}_t)$ and such that $a(0) = 0$. Let (y_t) and (z_t) be two nonnegative (measurable with respect to $\mathcal{F}$) processes such that for all stopping times* T *and some fixed initial measure* μ *one has* $E^\mu(y_T) = E^\mu(z_T)$ *where we set* $y_\infty = z_\infty = 0$. *Then*

$$E^\mu \int_0^\infty y_t\, da_t = E^\mu \int_0^\infty z_t\, da_t.$$

The proof of (2.3) now goes as follows. It suffices to consider $f \geqslant 0$ and $x \in E_M$. Let $a_t = M_0 - M_t$, $y_t = e^{-\alpha t} U^\alpha f(X_t)$, and $z_t = \int_t^\infty e^{-\alpha s} f(X_s)\, ds$. The hypotheses of (2.4) are satisfied and one obtains

$$\begin{aligned} P_M^\alpha U^\alpha f(x) &= E^x \int_0^\infty e^{-\alpha t} U^\alpha f(X_t)\, da_t \\ &= E^x \int_0^\infty \left(\int_t^\infty e^{-\alpha s} f(X_s)\, ds\right) da_t \\ &= E^x \int_0^\infty e^{-\alpha s} f(X_s) a_s\, ds \\ &= U^\alpha f(x) - V^\alpha f(x), \end{aligned}$$

since $M_0 = 1$ almost surely P^x.

The following result is essentially known. See Meyer [6]. Once again we will give the proof for completeness. For the first time we make use of the dual process $\hat{X}$. Our notation follows the pattern of [1] for the most part. For example, $P_M^\alpha u^\alpha(x, y) = \int P_M^\alpha(x, dz) u^\alpha(z, y)$.

(2.5) THEOREM. — *Let* M *be an exact* MF *of* X. *Then for each* $\alpha \geqslant 0$ *there exists a function* $v^\alpha(x, y) \geqslant 0$ *such that*

(i) $$V^\alpha f(x) = \int v^\alpha(x, y) f(y)\, dy$$

(ii) $$u^\alpha(x, y) = v^\alpha(x, y) + P_M^\alpha u^\alpha(x, y).$$

Let $f\hat{V}^{\alpha}(x) = \int dy\, f(y) v^{\alpha}(y, x)$. *Then* $(\hat{V}^{\alpha})$ *is a resolvent exactly subordinate to* $(\hat{U}^{\alpha})$. *Finally* $x \to v^{\alpha}(x, y)$ *is* $\alpha - (X, M)$ *excessive for each* y, *and* $y \to v^{\alpha}(x, y)$ *is* $\alpha - \hat{V}$ *excessive for each* x.

Proof. — First of all we suppose that $\alpha > 0$. If $f \geqslant 0$ is bounded, then $P_M^{\alpha} U^{\alpha} f = U^{\alpha} f - V^{\alpha} f$ is α-excessive by definition of an exact MF. Hence $P_M^{\alpha} f \in \mathfrak{S}^{\alpha}$ and $P_M^{\alpha} f \leqslant f$ whenever $f \in \mathfrak{S}^{\alpha}$. Here $\mathfrak{S}^{\alpha}$ denotes the set of α-excessive functions. Also if $\alpha < \beta$ then using (2.3)

$$\begin{aligned} P_M^{\alpha} U^{\alpha} f(x) &= E^x \int_0^{\infty} e^{-\alpha t}(1 - M_t) f(X_t)\, dt \\ &\geqslant E^x \int_0^{\infty} e^{-\beta t}(1 - M_t) f(X_t)\, dt = P_M^{\beta} U^{\beta} f(x). \end{aligned}$$

In particular for each y, $x \to P_M^{\alpha} u^{\alpha}(x, y)$ is α-excessive and $P_M^{\beta} u^{\beta} \leqslant P_M^{\alpha} u^{\alpha} \leqslant u^{\alpha}$ if $\beta > \alpha$. Also for each x, $y \to P_M^{\alpha} u^{\alpha}(x, y)$ is in $\hat{\mathfrak{S}}^{\alpha}$ (the α-coexcessive functions) because it is the α-copotential of the measure $P_M^{\alpha}(x, .)$.

Let $\Gamma_{\alpha} = \{(x, y) : u^{\alpha}(x, y) < \infty\}$ and define

$$\begin{aligned} w^{\alpha}(x, y) &= u^{\alpha}(x, y) - P_M^{\alpha} u^{\alpha}(x, y) &\quad \text{if} \quad (x, y) \in \Gamma_{\alpha} \\ &= +\infty &\quad \text{if} \quad (x, y) \notin \Gamma_{\alpha}. \end{aligned}$$

Let $\Gamma_{\alpha}^x = \{y : (x, y) \in \Gamma_{\alpha}\}$. Then for each x, $E - \Gamma_{\alpha}^x$ is of measure zero and hence polar. Consequently (2.3) implies that for bounded f

$$(2.6) \qquad V^{\alpha} f(x) = \int w^{\alpha}(x, y) f(y)\, dy.$$

But $\Gamma_{\alpha} \subset \Gamma_{\alpha+\beta}$ since $u^{\alpha+\beta} \leqslant u^{\alpha}$. Therefore using the resolvent equation for (V^{γ}) and (2.6) we see that for each x and $\beta \geqslant 0$

$$(2.7) \qquad \beta V^{\alpha+\beta} w^{\alpha}(x, y) + w^{\alpha+\beta}(x, y) = w^{\alpha}(x, y)$$

almost everywhere. Now Γ_{α}^x is cofinely open and $y \to w^{\gamma}(x, y)$ is cofinely continuous on Γ_{α}^x provided $\gamma \geqslant \alpha$. If $\mu(.) = \beta V^{\alpha+\beta}(x, .)$ and $\nu(.) = \beta V^{\alpha+\beta} P_M^{\alpha}(x, .)$, then for $y \in \Gamma_{\alpha}^x$, $\beta V^{\alpha+\beta} w^{\alpha}(x, y) = \mu U^{\alpha}(y) - \nu U^{\alpha}(y)$ which is cofinely continuous on Γ_{α}^x. As a result (2.7) holds everywhere on Γ_{α}^x. Consequently $\beta V^{\alpha+\beta} w^{\alpha}(x, y) \leqslant w^{\alpha}(x, y)$ everywhere, that is, for each y, $x \to w^{\alpha}(x, y)$ is $\alpha - V$ supermedian (III-4.5).

Therefore $\beta V^{\alpha+\beta} w^{\alpha}(x, y)$ increases with β and we define $v^{\alpha}(x, y) = \lim_{\beta \to \infty} \beta V^{\alpha+\beta} w^{\alpha}(x, y)$. Then $v^{\alpha} \leqslant w^{\alpha}$ and for each y, $x \to v^{\alpha}(x, y)$ is $\alpha - (X, M)$ excessive. Moreover if f is in $b\mathcal{E}_{+}^{*}$, $\|V^{\alpha+\beta} f\| \leqslant (\alpha + \beta)^{-1} \| f \|$, and so (2.6), (2.7), and the monotone convergence theorem imply that

$$V^{\alpha} f(x) = \int v^{\alpha}(x, y) f(y) \, dy,$$

establishing (2.5 i).

Before establishing (2.5 ii) we need to make a few preliminary observations. First of all in light of (VI-1.5) it is clear that $u^{\alpha+\beta}(x, y)$ decreases to zero as $\beta \to \infty$ for $(x, y) \in \Gamma_{\alpha}$. But $P_M^{\alpha+\beta} u^{\alpha+\beta}(x, y) \leqslant u^{\alpha+\beta}(x, y)$ and so $P_M^{\alpha+\beta} u^{\alpha+\beta}$ decreases to zero as $\beta \to \infty$ on Γ_{α}. Now fix $(x, y) \in \Gamma_{\alpha}$. Then

$$\begin{aligned} \beta V^{\alpha+\beta} w^{\alpha}(x, y) &= w^{\alpha}(x, y) - w^{\alpha+\beta}(x, y) \\ &= w^{\alpha}(x, y) - u^{\alpha+\beta}(x, y) + P_M^{\alpha+\beta} u^{\alpha+\beta}(x, y), \end{aligned}$$

and letting $\beta \to \infty$ we find that $v^{\alpha} = w^{\alpha}$ on Γ_{α}. Therefore for $(x, y) \in \Gamma_{\alpha}$.

$$u^{\alpha}(x, y) = v^{\alpha}(x, y) + P_M^{\alpha} u^{\alpha}(x, y). \tag{2.8}$$

Now fix y. Then $\{x : (x, y) \notin \Gamma_{\alpha}\}$ is of measure zero and so (2.8) holds almost everywhere. But E_M is finely open and $x \to v^{\alpha}(x, y)$ is finely continuous on E_M since M is exact; see (III-5.8). Therefore (2.8) holds if x is in E_M. But if x is not in E_M, $v^{\alpha}(x, y) = 0$ and (2.8) holds by the definition of P_M^{α}. Since y is arbitrary this establishes (2.5 ii).

For $f \in b\mathcal{E}^{*}$ and $\alpha > 0$ define $f\hat{V}^{\alpha}(y) = \int v^{\alpha}(x, y) f(x) \, dx$. Then for $f, g \in C_K$ we have $(f\hat{V}^{\alpha}, g) = (f, V^{\alpha} g)$. Note that $f\hat{V}^{\alpha} \leqslant f\hat{U}^{\alpha}$ if $f \geqslant 0$ and so $f\hat{V}^{\alpha}$ is bounded if $f \in b\mathcal{E}^{*}$. Recall that we write the action of $\hat{U}^{\alpha}$ on a function f as $f\hat{U}^{\alpha}(x) = \int f(y) \hat{U}^{\alpha}(dy, x) = \int f(y) u^{\alpha}(y, x) \, dy$. Consequently for $f \in C_K$ and $\alpha, \beta > 0$

$$f\hat{V}^{\alpha} - f\hat{V}^{\beta} - (\beta - \alpha) f \hat{V}^{\alpha}\hat{V}^{\beta} = 0 \tag{2.9}$$

almost everywhere. If $g \in b\mathcal{E}_{+}^{*}$ and $\gamma > 0$, then $g\hat{V}^{\gamma} = g\hat{U}^{\gamma} - \mu\hat{U}^{\gamma}$ where $\mu(.) = \int dx \, g(x) P_M^{\gamma}(x, \; .)$, and since all terms are bounded it follows that $g\hat{V}^{\gamma}$ is cofinely continuous. As a result the resolvent equation (2.9) holds identically. Thus $(\hat{V}^\alpha)$ is a resolvent subordinate to $(\hat{U}^\alpha)$. Moreover the subordination is exact because $f\hat{U}^\alpha - f\hat{V}^\alpha = \mu\hat{U}^\alpha$ is α-coexcessive for $f \in b\mathcal{E}^*_+$. Here $\mu(.) = \int dx\, f(x) P^\alpha_M(x, .)$. See (III-4.8).

It remains to show that $y \to v^\alpha(x, y)$ is $\alpha - \hat{V}$ excessive for each x in order to complete the proof of Theorem 2.5. First observe that the resolvent equation for $(\hat{V}^\gamma)$ implies that if $\beta > \alpha > 0$ and y is fixed, then

$$(2.10) \quad \int v^\alpha(x, z) v^\beta(z, y)\, dz = \int v^\beta(x, z) v^\alpha(z, y)\, dz$$

almost everywhere in x. But as functions of x both integrals are finely continuous on E_M and vanish off E_M. Hence (2.10) holds identically in x and y. Now fix x and let $u(y) = v^\alpha(x, y)$. Then

$$\begin{aligned} \beta u\hat{V}^{\alpha+\beta}(y) &= \beta \int v^\alpha(x, z) v^{\alpha+\beta}(z, y)\, dz \\ &= \beta \int v^{\alpha+\beta}(x, z) v^\alpha(z, y)\, dz \\ &= \beta V^{\alpha+\beta} v^\alpha(x, y) \uparrow v^\alpha(x, y) = u(y) \end{aligned}$$

as $\beta \to \infty$ since $z \to v^\alpha(z, y)$ is $\alpha - (X, M)$ excessive for each y. Therefore u is $\alpha - \hat{V}$ excessive, completing the proof of Theorem 2.5 when $\alpha > 0$.

It is easy to see that $v^\alpha(x, y) \geq v^\beta(x, y)$ if $0 < \alpha < \beta$. Consequently defining

$$v(x, y) = v^0(x, y) = \lim_{\alpha \to 0} v^\alpha(x, y),$$

one easily checks that v has the desired properties, i.e., (2.5 i) and (2.5 ii) hold when $\alpha = 0$ and $v(x, y)$ is $(X - M)$ excessive as a function of x and $\hat{V}$ excessive as a function of y.

(2.11) *Remarks.* — Clearly v^α is the unique function satisfying (2.5 ii) and such that $x \to v^\alpha(x, y)$ is $\alpha - (X, M)$ excessive for each y. In particular $v^\alpha(x, y) = 0$ if x is not in E_M, and $y \to v^\alpha(x, y)$ vanishes almost everywhere on $E - E_M$. This last statement will be made more precise in the next section. See (3.4).

3. Dual multiplicative functionals.

In this section we will associate with each exact MF of X an exact MF of $\hat{X}$ in such a manner that (1.2) and an appropriate generalization of (1.1) hold. We begin by establishing some notation. If $\hat{M} = (\hat{M}_t)$ is a MF of $\hat{X}$, we write $(\hat{Q}_t)$ and $(\hat{V}^\alpha)$ for the semigroup and resolvent generated by $\hat{M}$. In keeping with the pattern of notation established in Section VI-1 of [1] we will write the action of these operators as follows :

$$f\hat{Q}_t(x) = \int f(y)\hat{Q}_t(dy, x) = \hat{E}^x\{f(X_t)M_t\}$$
$$f\hat{V}^\alpha(x) = \int f(y)\hat{V}^\alpha(dy, x) = \hat{E}^x \int_0^\infty e^{-\alpha t} f(X_t)M_t\, dt.$$

For notational convenience we will write $\hat{E}_M$ in place of $E_{\hat{M}}$ for the set of permanent points of $\hat{M}$. Similarly we will write $\hat{P}^\alpha_M$ in place of $\hat{P}^\alpha_{\hat{M}}$ for the operator defined in (2.2) relative to $\hat{M}$. In accordance with the above we will write the action of $\hat{P}^\alpha_M$ on a function f as

$$f\hat{P}^\alpha_M(x) = \int f(y)\hat{P}^\alpha_M(dy, x).$$

With these conventions (2.3) becomes

$$f\hat{U}^\alpha - f\hat{V}^\alpha = f\hat{U}^\alpha\hat{P}^\alpha_M \tag{3.1}$$

provided $f\hat{U}^\alpha$ is finite.

We now are prepared to state the main result of this section. Recall that $(f, g) = \int f(x)g(x)\, dx$ provided the integral exists.

(3.2) THEOREM. — *Let* M *be an exact* MF *of* X. *Then there exists a unique exact* MF, $\hat{M}$, *of* $\hat{X}$ *satisfying*

$$P^\alpha_M u^\alpha(x, y) = u^\alpha \hat{P}^\alpha_M(x, y). \tag{3.3}$$

For each $\alpha > 0$, (3.3) *is equivalent to* $(f\hat{V}^\alpha, g) = (f, V^\alpha g)$ *for all* $f, g \in C^+_K$. *Moreover* $E_M \Delta \hat{E}_M$ *is semipolar. Finally let* $F = E_M - \hat{E}_M$. *Then* $M_{T_F} = 0$ *almost surely on* $\{T_F < \zeta\}$. *In particular if* M *doesn't vanish on* $[0, \zeta)$, *then* $E_M = E$ *and* $E - \hat{E}_M$ *is polar.*

Proof. — Let $(\hat{V}^\alpha)$ be the resolvent constructed in Theorem 2.5. Then $(\hat{V}^\alpha)$ is exactly subordinate to $(\hat{U}^\alpha)$. Now by a result of Meyer [6, Th. 1.1] there exists a unique semigroup $(\hat{Q}_t)$ subordinate to $(\hat{P}_t)$ having $(\hat{V}^\alpha)$ as its resolvent and such that $1\hat{Q}_t \to 1\hat{Q}_0$ as $t \to 0$. Meyer assumes that $(\hat{U}^\alpha)$ maps continuous functions into continuous functions and this is used at one point of his proof. However, it suffices for this point to note that if f is a bounded continuous function, then $\alpha f\hat{U}^\alpha$ converges pointwise *boundedly* to f as $\alpha \to \infty$, and so this assumption on $(\hat{U}^\alpha)$ is not necessary. By another theorem of Meyer, see (III-2.3), there exists a MF, $\hat{M}$, of $\hat{X}$ which generates $(\hat{Q}_t)$. Clearly the resolvent corresponding to $\hat{M}$ is $(\hat{V}^\alpha)$. Consequently $\hat{M}$ is exact. Now (3.1) implies that for each fixed y

$$u^\alpha(x, y) = v^\alpha(x, y) + u^\alpha\hat{P}^\alpha_M(x, y)$$

almost everywhere in x. Combining this with (2.5 ii) we see that (3.3) holds almost everywhere on $\{x : u^\alpha(x, y) < \infty\}$ and hence almost everywhere on E if $\alpha > 0$. But both sides of (3.3) are α-excessive as functions of x and so (3.3) holds if $\alpha > 0$. The case $\alpha = 0$ is obtained by a passage to the limit. The fact that (3.3) is equivalent to $(f\hat{V}^\alpha, g) = (f, V^\alpha g)$ for all $f, g \in \mathbf{C}_K^+$ follows readily from (2.3) and (3.1).

Next let $\hat{N}$ be another exact MF of $\hat{X}$ such that $P^\alpha_M u^\alpha(x, y) = u^\alpha\hat{P}^\alpha_N(x, y)$ for all $\alpha > 0$. But using (3.1) we see that $\hat{M}$ and $\hat{N}$ generate the same resolvent, and hence the same semigroup. Therefore $\hat{M}$ and $\hat{N}$ are equivalent (III-1.9). Thus we have established the existence and uniqueness assertions in (3.2).

We turn now to the relationships between E_M and $\hat{E}_M$. Since $E_M = \{V^1 1 > 0\} = \bigcup_n \{V^1 1 \geqslant 1/n\}$, in order to show that $E_M - \hat{E}_M$ is semipolar it will suffice to show that $K = \{V^1 1 \geqslant a\} - \hat{E}_M$ is thin for each $a > 0$. As usual we write $\Phi^1_K = P^1_K 1$. Let $\{h_n\}$ be a sequence in $b\mathcal{E}_+$ such that $U^1 h_n \uparrow 1$ and let $\mu_n = h_n\xi$. Then

$$P^1_M\Phi^1_K = \lim_n P^1_M P^1_K U^1\mu_n = \lim_n U^1\hat{P}^1_M\hat{P}^1_K\mu_n.$$

But $\hat{P}^1_K\mu_n$ is a measure carried by $K \cup {}^rK$ which is contained in $E - \hat{E}_M$ since $K \subset E - \hat{E}_M$ and $\hat{E}_M$ is cofinely open. However $\hat{P}^1_M(.,\ x) = \varepsilon_x$ if x is in $E - \hat{E}_M$ and so $\hat{P}^1_M\hat{P}^1_K\mu_n = \hat{P}^1_K\mu_n$. Consequently

$$P^1_M\Phi^1_K = \lim_n U^1\hat{P}^1_K\mu_n = \lim_n P^1_K U^1\mu_n = \Phi^1_K.$$

For x in $K \subset E_M$.

$$P^1_M 1(x) = -\ E^x \int_0^\infty e^{-t}\, dM_t = 1 - V^1 1(x) \leqslant 1 - a,$$

and so $\Phi^1_K(x) \leqslant P^1_M 1(x) \leqslant 1 - a$ for all x in K. Therefore K is (totally) thin and hence $E_M - \hat{E}_M$ is semipolar. By duality $\hat{E}_M - E_M$ is semipolar and so the symmetric difference of E_M and $\hat{E}_M$ is semipolar.

Actually somewhat more is true. Let K be as in the preceding paragraph. Then $\Phi^1_K = P^1_M\Phi^1_K$. If x is in E_M and $B_t = M_0 - M_t$, then using (2.4) one obtains

$$P^1_M\Phi^1_K(x) = E^x \int_0^\infty e^{-t}E^{X(t)}(e^{-T_K})\, dB_t = E^x \int_0^\infty e^{-(t+T_K \circ \theta_t)}\, dB_t.$$

Now let $T_0 = 0$ and $T_{n+1} = T_n + T_K \circ \theta_{T_n}$ be the iterates of T_K. Since K is totally thin, $T_n \uparrow +\infty$ and clearly $t + T_K \circ \theta_t = T_{n+1}$ if $T_n \leqslant t < T_{n+1}$. For notational convenience let $M^*_t = M_{t-}$ if $t > 0$, $M^*_0 = M_0$. Then for x in E_M

$$\begin{aligned} P^1_M\Phi^1_K(x) &= E^x \sum_{n \geqslant 0} e^{-T_{n+1}}[M^*_{T_n} - M^*_{T_{n+1}}] \\ &= E^x(e^{-T_1}) - E^x \sum_{n \geqslant 1} M^*_{T_n}[e^{-T_n} - e^{-T_{n+1}}]. \end{aligned}$$

But $P^1_M\Phi^1_K(x) = \Phi^1_K(x) = E^x(e^{-T_K})$ and since $T_K = T_1$, this implies that $E^x\{M^*_{T_n}[e^{-T_n} - e^{-T_{n+1}}] = 0$ for all $n \geqslant 1$ since each term is nonnegative. In particular when $n = 1$

$$E^x\{M^*_{T_K} e^{-T_K}[1 - e^{-T_K \circ \theta_{T_K}}]\} = 0,$$

and because K is thin, this tells us that $M^*_{T_K} = 0$ almost surely P^x on $\{T_K < \zeta\}$. Let $F = E_M - \hat{E}_M$. Then F is the increasing union of such sets K_n, and so $T_F = \inf T_{K_n}$. Now M is right continuous and so $M_{T_F} = 0$ almost surely P^x on $\{T_F < \zeta\}$ for each $x \in E_M$. Of course, if x is not in E_M then almost surely P^x, $t \to M_t$ is identically zero. This completes the proof of Theorem 3.2.

(3.4) *Remarks.* — It is now evident that $v^{\alpha}(x, y) = 0$ if x is not in E_M or if y is not in $\hat{E}_M$. The fact that $M_{T_F} = 0$ almost surely on $\{T_F < \zeta\}$ implies that F is polar with respect to the canonical subprocess (X, M) corresponding to X and M. The example at the end of Section 8 shows that $E_M = E$ does *not* imply that $E - \hat{E}_M$ is polar, while the example at the end of Section 9 shows that $E - \hat{E}_M$ need not be *empty* when M doesn't vanish on $[0, \zeta)$. Finally it is evident in view of the complete duality between X and $\hat{X}$ that the map $M \to \hat{M}$ is bijective from the class of exact MF's of X to the class of exact MF's of $\hat{X}$. *We will write* $M \longleftrightarrow \hat{M}$ *for this correspondence*, and we will say that M and $\hat{M}$ are *dual* functionals.

(3.5) Corollary. — *Let* M *be an exact* MF *of* X *and* $\hat{M}$ *be an exact* MF *of* $\hat{X}$. *Then* M *and* $\hat{M}$ *are dual functionals if and only if* $(f\hat{Q}_t, g) = (f, Q_t g)$ *for all* $t > 0$ *and* $f, g \in C_K^+$.

Proof. — If $(f\hat{Q}_t, g) = (f, Q_t g)$ we obtain $(f\hat{V}^{\alpha}, g) = (f, V^{\alpha} g)$ by taking Laplace transforms. This yields $M \longleftrightarrow \hat{M}$. Conversely if $M \longleftrightarrow \hat{M}$, the uniqueness theorem for Laplace transforms yields the desired equality almost everywhere (Lebesgue) in t, and since $t \to (f\hat{Q}_t, g)$ and $t \to (f, Q_t g)$ are right continuous for $f, g \in C_K^+$ the proof of (3.5) is complete.

4. The multiplicative property.

We come now to the fundamental property of the correspondence set up in Section 3. Recall (III-5.20) that the product of two exact MF's is again exact. We denote the product of two MF's M and N by MN, that is, $MN = (M_t N_t)$.

(4.1) Theorem. — *The correspondence* $M \longleftrightarrow \hat{M}$ *is multiplicative, that is, if* $M \longleftrightarrow \hat{M}$ *and* $N \longleftrightarrow \hat{N}$, *then* $MN \longleftrightarrow \hat{M}\hat{N}$.

We will break up the proof of this theorem into a series of lemmas. Let us fix M and $\hat{M}$ with $M \longleftrightarrow \hat{M}$. We will use the notation established in Section 3 without special mention. In particular $\{Q_t\}$ and $\{\hat{Q}_t\}$ are semigroups subordinate to

$\{P_t\}$ and $\{\hat{P}_t\}$ respectively, and so we can choose $q_t(x, y)$ and $\hat{q}_t(x, y)$ to be jointly universally measurable in (x, y), lying between zero and one, and such that

$$Q_t(x, dy) = q_t(x, y)P_t(x, dy)$$
$$\hat{Q}_t(dy, x) = \hat{q}_t(x, y)\hat{P}_t(dy, x).$$

(4.2) Lemma. — *Let* $G(x, y)$ *be nonnegative and jointly measurable* (i.e., $G \in (\mathcal{E}^\star \times \mathcal{E}^\star)_+$). *Then*

$$\int \left[\int P_t(x, dy)G(x, y)\right] dx = \int \left[\int G(x, y)\hat{P}_t(dx, y)\right] dy,$$

and a similar formula holds if P_t and $\hat{P}_t$ are replaced by Q_t and $\hat{Q}_t$ respectively.

Proof. — It suffices to prove this when $G(x, y) = g(x)h(y)$. But then the desired equality reduces to $(g, P_t h) = (g\hat{P}_t, h)$. Similarly $(g, Q_t h) = (g\hat{Q}_t, h)$ implies the formula involving Q_t and $\hat{Q}_t$.

(4.3) Lemma. — *Let* $g \in b\mathcal{E}^*_+$. *Then for almost all* y

$$\int g(x)q_t(x, y)\hat{P}_t(dx, y) = \int g(x)\hat{q}_t(y, x)\hat{P}_t(dx, y).$$

Proof. — Let $f \in C^*_K$. Then using (4.2)

$$(g, Q_t f) = \int \left[\int P_t(x, dy)q_t(x, y) f(y)\right] g(x)\, dx$$
$$= \int \left[\int \hat{P}_t(dx, y)q_t(x, y)g(x)\right] f(y)\, dy.$$

But $(g, Q_t f) = (g\hat{Q}_t, f)$ and the result follows since

$$(g\hat{Q}_t, f) = \int \left[\int \hat{P}_t(dx, y)\hat{q}_t(y, x)g(x)\right] f(y)\, dy.$$

(4.4) Lemma. — *Let* φ *and* ψ *be in* $b\mathcal{E}^*_+$ *with* $\varphi = \psi$ *almost everywhere. If* f *is bounded and integrable, then* $(\varphi\hat{P}_t, f) = (\psi\hat{P}_t, f)$ *and* $(\varphi\hat{Q}_t, f) = (\psi\hat{Q}_t, f)$ *for each* t. *Moreover*

$$\int \varphi(x)q_t(x, y)\hat{P}_t(dx, y) = \int \psi(x)\hat{q}_t(y, x)\hat{P}_t(dx, y) \tag{4.5}$$

almost everywhere in y.

Proof. — It suffices to consider the case $f \in C_K^+$ and φ and ψ integrable. Then $\mu(dx) = f(x)\,dx$ is a finite measure. Let $D = \{\varphi \neq \psi\}$. Then D is of a potential zero and so $\hat{P}^\mu(X_s \in D) = 0$ almost everywhere (Lebesgue) in s. Now fix t. Then there exists a sequence $\{t_n\}$ decreasing to t such that for each n

$$0 = \hat{P}^\mu[X_{t_n} \in D] = \int \hat{P}_{t_n}(D,\ x)\, f(x)\, dx.$$

Therefore $(\varphi\hat{P}_{t_n}, f) = (\psi\hat{P}_{t_n}, f)$. But

$$(\varphi\hat{P}_{t_n}, f) = (\varphi,\ P_{t_n}f) \to (\varphi,\ P_t f) = (\varphi\hat{P}_t,\ f)$$

since $f \in C_K^+$, and similarly $(\psi\hat{P}_{t_n}, f) \to (\psi\hat{P}_t, f)$. Hence $(\varphi\hat{P}_t, f) = (\psi\hat{P}_t, f)$. A similar argument yields

$$(\varphi\hat{Q}_t,\ f) = (\psi\hat{Q}_t,\ f).$$

Coming to (4.5) we again may assume that φ and ψ are integrable. If $f \in C_K^+$ then using (4.2)

$$(\varphi,\ Q_t f) = \int\Big[\int q_t(x,\ y)\varphi(x)\hat{P}_t(dx,\ y)\Big] f(y)\, dy.$$

But by the first assertion in (4.4),

$$(\varphi,\ Qf_t) = (\varphi\hat{Q}_t,\ f) = (\psi\hat{Q}_t,\ f),$$

and this yields (4.5) because

$$(\psi\hat{Q}_t, f) = \int\Big[\int \psi(x)\hat{q}_t(y,\ x)\hat{P}_t(dx,\ y)\Big] f(y)\, dy.$$

(4.6) LEMMA. — *Let* $F(x_0,\ x_1,\ \ldots,\ x_n)$ *be nonnegative, bounded, and measurable and let* $0 < t_1 < \cdots < t_n$. *Then for almost all* x_0

$$\int \cdots \int \hat{P}_{t_1}(dx_1,\ x_0)q_{t_1}(x_1,\ x_0)\ \ldots\ \hat{P}_{t_n - t_{n-1}}(dx_n,\ x_{n-1})\, q_{t_n - t_{n-1}}(x_n,\ x_{n-1})F(x_0,\ \ldots,\ x_n)$$
$$= \int \cdots \int \hat{P}_{t_1}(dx_1,\ x_0)\hat{q}_{t_1}(x_0,\ x_1)\ \ldots\ \hat{P}_{t_n - t_{n-1}}(dx_n,\ x_{n-1})\, \hat{q}_{t_n - t_{n-1}}(x_{n-1},\ x_n)F(x_0,\ \ldots\ x_n).$$

Proof. — It suffices to consider $F(x_0,\ \ldots,\ x_n) = \prod_{j=0}^{n} f_j(x_j)$ where each f_j is in C_K^+. In this case $f_0(x_0)$ plays no role and so we will drop it from our notation. The proof proceeds by induction on n. If $n = 1$ this reduces to Lemma 4.3. For the induction step let $s_1 = t_2 - t_1$, $s_2 = t_3 - t_2$, ..., $s_{n-1} = t_n - t_{n-1}$. Then the induction hypothesis implies that

$$\varphi(x_1) = \int \cdots \int \hat{P}_{t_2-t_1}(dx_2, x_1) q_{t_2-t_1}(x_2, x_1) \dots \hat{P}_{t_n-t_{n-1}}(dx_n, x_{n-1}) q_{t_n-t_{n-1}}(x_n, x_{n-1}) f_2(x_2) \dots f_n(x_n)$$

and

$$\psi(x_1) = \int \cdots \int \hat{P}_{t_2-t_1}(dx_2, x_1) \hat{q}_{t_2-t_1}(x_1, x_2) \dots \hat{P}_{t_n-t_{n-1}}(dx_n, x_{n-1}) \hat{q}_{t_n-t_{n-1}}(x_{n-1}, x_n) f_2(x_2) \dots f_n(x_n)$$

agree almost everywhere. Multiplying by $f_1(x_1)$ and using (4.5) we obtain Lemma 4.6.

We now fix t. Let

$$\mathcal{U} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$$

be a finite subdivision of $[0, t]$, and define

$$M_t(\mathcal{U}) = q_{t_1}(X_0, X_{t_1}) q_{t_2-t_1}(X_{t_1}, X_{t_2}) \dots q_{t-t_{n-1}}(X_{t_{n-1}}, X_t).$$

Meyer has proved that if $\{\mathcal{U}_k\}$ is an increasing sequence of such subdivisions such that $\bigcup \mathcal{U}_k$ is dense in $[0, t]$, then $M_t(\mathcal{U}_k) \to M_t$ almost surely. See, for example, the proof of (III-2.3) in [1]. We also define for such a subdivision

$$\hat{M}_t(\mathcal{U}) = \hat{q}_{t_1}(\hat{X}_0, \hat{X}_{t_1}) \dots \hat{q}_{t-t_{n-1}}(\hat{X}_{t_{n-1}}, \hat{X}_t)$$
$$M_t^*(\mathcal{U}) = q_{t_1}(\hat{X}_{t_1}, \hat{X}_0) \dots q_{t-t_{n-1}}(\hat{X}_t, \hat{X}_{t_{n-1}}).$$

Of course, $\hat{M}_t(\mathcal{U}_k) \to \hat{M}_t$ almost surely, and we are going to study the relationship between $\hat{M}_t(\mathcal{U})$ and $\hat{M}_t^*(\mathcal{U})$.

(4.7) LEMMA. — *Let μ be an absolutely continuous initial measure on* E. *Then $\hat{M}_t(\mathcal{U}) = M_t^*(\mathcal{U})$ almost surely $\hat{P}^\mu$.*

Proof. — It suffices to prove that if $F(x_0, x_1, \dots, x_n)$ is a nonnegative bounded Borel function, then

$$\hat{E}^x\{F(\hat{X}_{t_0}, \dots, \hat{X}_t)\hat{M}_t(\mathcal{U})\} = \hat{E}^x\{F(\hat{X}_{t_0}, \dots, \hat{X}_t)M_t^*(\mathcal{U})\}$$

almost everywhere in x. But writing these expectations out in terms of the finite dimensional distributions of $\hat{X}$, this reduces to Lemma 4.6.

Combining Lemma 4.7 and the preceding discussion we see that $M_t^*(\mathcal{U}_k) \to \hat{M}_t$ almost surely $\hat{P}^\mu$ provided that μ is absolutely continuous.

We are now prepared to prove Theorem 4.1. Let M and N be exact MF's of X and let $M \longleftrightarrow \hat{M}$ and $N \longleftrightarrow \hat{N}$. Let (R_t) and $(\hat{R}_t)$ be the semigroups generated by N and $\hat{N}$ respectively and let $R_t(x, dy) = r_t(x, y)P_t(x, dy)$ and $\hat{R}_t(dy, x) = \hat{r}_t(x, y)\hat{P}_t(dy, x)$. If $\mathcal{U}$ is as before, we define $N_t(\mathcal{U})$, $\hat{N}_t(\mathcal{U})$, and $N_t^*(\mathcal{U})$ in the obvious fashion. By the preceding discussion $M_t(\mathcal{U}_k)N_t(\mathcal{U}_k) \to M_tN_t$ almost surely, $\hat{M}_t(\mathcal{U}_k)\hat{N}_t(\mathcal{U}_k) \to \hat{M}_t\hat{N}_t$ almost surely, and

$$M_t^*(\mathcal{U}_k)N_t^*(\mathcal{U}_k) \to \hat{M}_t\hat{N}_t$$

almost surely $\hat{P}^\mu$ provided μ is absolutely continuous. In these statements $\{\mathcal{U}_k\}$ is any increasing sequence of partitions of $[0, t]$ whose union is dense in $[0, t]$. It will be convenient to let $\mathcal{U}_k = \{0, t2^{-k}, 2t2^{-k}, \ldots, t\}$. Let $f, g \in b\mathcal{E}_+$. Then $\mu(dx) = f(x)\,dx$ and $\nu(dx) = g(x)\,dx$ are σ-finite measures on E. We claim that ($\mathcal{U} = \mathcal{U}_n$ for a fixed n)

$$(4.8) \qquad E^\mu\{M_t(\mathcal{U})N_t(\mathcal{U})g(X_t)\} = \hat{E}^\nu\{\hat{M}_t(\mathcal{U})\hat{N}_t(\mathcal{U})\, f(\hat{X}_t)\}.$$

Let us assume the truth of (4.8) for the moment and use it to complete the proof of Theorem 4.1. If we set $\mathcal{U} = \mathcal{U}_n$ in (4.8) and assume that $f, g \in \mathbf{C}_\mathbf{K}^+$, then letting $n \to \infty$ we obtain

$$(4.9) \qquad E^\mu\{M_tN_tg(X_t)\} = \hat{E}^\nu\{\hat{M}_t\hat{N}_tf(\hat{X}_t)\}.$$

Thus if we define (S_t) and $(\hat{S}_t)$ to be the semigroups generated by MN and $\hat{M}\hat{N}$ respectively, then (4.9) states that $(f, S_tg) = (f\hat{S}_t, g)$ for each t and all $f, g \in \mathbf{C}_\mathbf{K}^+$. Hence by (3.5), $MN \longleftrightarrow \hat{M}\hat{N}$.

Therefore to complete the proof of Theorem 4.1 we must establish (4.8). Using Lemma 4.7 we see that the right side of (4.8) reduces to $\hat{E}^\nu\{M_t^*(\mathcal{U})N_t^*(\mathcal{U})\, f(\hat{X}_t)\}$. Thus in order to establish (4.8) we must show that

$$(4.10) \qquad E^\mu\{M_t(\mathcal{U})N_t(\mathcal{U})g(X_t)\} = \hat{E}^\nu\{M_t^*(\mathcal{U})N_t^*(\mathcal{U})\, f(\hat{X}_t)\}.$$

We will prove (4.10) for any partition $\mathcal{U}$ of $[0, t]$ into *equally spaced* points by induction on the number of such points. If $\mathcal{U} = \{0, t\}$, (4.10) is an immediate consequence of Lemma 4.2. Now let $\mathcal{U} = \{0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t\}$ where $t_{j+1} - t_j = s$ for all j, and let $\mathcal{V} = \{0 = t_0 < t_1 < \cdots < t_n\}$. Define

$$\varphi(x_1) = \int \cdots \int \hat{P}_s(dx_2, x_1)q_s(x_2, x_1)r_s(x_2, x_1) \dots \hat{P}_s(dx_{n+1}, x_n)q_s(x_{n+1}, x_n)r_s(x_{n+1}, x_n) f(x_{n+1})$$
$$= \hat{E}^{x_1}\{M^*_{t_n}(\mathcal{V})N^*_{t_n}(\mathcal{V}) f(\hat{X}_{t_n})\}.$$

Then by Lemma 4.2 (or the case $n = 1$) the right side of (4.10) is

$$\int g(x_0)\, dx_0 \int \hat{P}_s(dx_1, x_0)q_s(x_1, x_0)r_s(x_1, x_0)\varphi(x_1)$$
$$= \int \varphi(x_1)\, dx_1 \int P_s(x_1, dx_0)q_s(x_1, x_0)r_s(x_1, x_0)g(x_0)$$
$$= \int \varphi(x_1)\psi(x_1)\, dx_1 = \hat{E}^{\lambda}\{M^*_{t_n}(\mathcal{V})N^*_{t_n}(\mathcal{V}) f(\hat{X}_{t_n})\}$$

where $\psi(x_1)$ denotes the integral over x_0 and

$$\lambda(dx_1) = \psi(x_1)\, dx_1.$$

But by the induction hypothesis this last expression becomes

$$E^{\mu}\{M_{t_n}(\mathcal{V})N_{t_n}(\mathcal{V})\psi(X_{t_n})\} = E^{\mu}\{M_t(\mathcal{U})N_t(\mathcal{U})g(X_t)\},$$

establishing (4.10). Thus finally the proof of Theorem 4.1 is complete.

Theorem 4.1 will have a number of important consequences in the following sections. Here is one which generalizes (1.1). If T is an exact terminal time of X, then $M_t = I_{[0, T)}(t)$ is an exact MF of X with $M^2 = M$. Consequently if $M \longleftrightarrow \hat{M}$, then according to (4.1), $\hat{M}^2 = \hat{M}$. Let $\hat{T} = \inf\{t: \hat{M}_t = 0\}$. Then $\hat{T}$ is a terminal time and $\hat{M}_t = I_{[0, \hat{T})}(t)$. Hence $\hat{T}$ is an exact terminal time of $\hat{X}$. Also $P^{\alpha}_{M} = P^{\alpha}_{T}$ and $\hat{P}^{\alpha}_{M} = \hat{P}^{\alpha}_{\hat{T}}$ ($= \hat{P}^{\alpha}_{T}$ for typographical convenience). If we agree to say that two exact terminal times are equivalent if the corresponding MF's are equal (i.e. equivalent as MF's), then we have proved the following corollary.

(4.11) Corollary. — *Let* T *be an exact terminal time of* X. *Then there exists a unique (up to equivalence) exact terminal time* $\hat{T}$ *of* $\hat{X}$ *such that* $P_T^\alpha u^\alpha(x, y) = u^\alpha \hat{P}_T^\alpha(x, y)$.

The discussion leading to (4.11) depended on the fact that $(M^2)^\wedge = (\hat{M})^2$. The next corollary is a useful extension of this fact. First observe that if $\lambda > 0$ and M is a MF of X, then $M^\lambda = (M_t^\lambda)$ is again a MF of X. Moreover (III-5.9) and (III-5.20) imply that M^λ is exact if M is.

(4.12) Corollary. — *Let* M *be an exact* MF *of* X *and let* $M \longleftrightarrow \hat{M}$. *Then for any* $\lambda > 0$, $M^\lambda \longleftrightarrow \hat{M}^\lambda$.

Proof. — If λ is a positive integer this is an immediate consequence of (4.1). Let $M^{1/2} \longleftrightarrow \hat{N}$. Then using (4.1), $\hat{N}^2 = \hat{M}$ and hence $\hat{N} = \hat{M}^{1/2}$. Consequently $M^\lambda \longleftrightarrow \hat{M}^\lambda$ whenever λ is a dyadic rational. Given $\lambda > 0$ let $\{\lambda_n\}$ be a sequence of dyadic rationales approaching λ. Then $(M_t)^{\lambda_n} \to (M_t)^\lambda$ and $(\hat{M}_t)^{\lambda_n} \to (\hat{M}_t)^\lambda$. Combining this with (3.5) yields (4.12).

We will close this section by discussing a generalization of the switching identity (1.1). To this end we fix an exact MF, M with dual $\hat{M}$, and as usual (Q_t) and (V^α) will denote the semigroup and resolvent generated by M. If N is another exact MF of X we define operators Q_N^α as follows:

$$(4.13)\qquad \begin{aligned} Q_N^\alpha f(x) &= -\,E^x \int_0^\infty e^{-\alpha t} M_t f(X_t)\, dN_t, \qquad && x \in E_N \\ &= f(x) I_{E_M}(x), && x \notin E_N. \end{aligned}$$

Observe that if T is an exact terminal time and $N_t = I_{[0,T)}(t)$, then

$$Q_N^\alpha f(x) = E^x\{e^{-\alpha T} M_T f(X_T)\} = Q_T^\alpha f(x).$$

Let (W^α) denote the resolvent corresponding to MN. Using (2.4) it is easy to check that if $f \in \mathcal{E}^+$ and $V^\alpha f$ is finite, then

$$(4.14)\qquad Q_N^\alpha V^\alpha f = V^\alpha f - W^\alpha f.$$

A straightforward computation now shows that (W^α) is exactly subordinate to (V^α) and that Q_N^α maps $\mathcal{S}^\alpha(M)$ into itself. Here $\mathcal{S}^\alpha(M)$ denotes the set of α — (X, M) excessive functions.

In the next proposition $\hat{Q}^\alpha_N$ is defined from $\hat{M}$ and $\hat{N}$ in the same way that Q^α_N is defined from M and N. Of course we write $f\hat{Q}^\alpha_N(x) = \int f(y)\hat{Q}^\alpha_N(dy, x)$ for the action of $\hat{Q}^\alpha_N$ on a function f.

(4.15) Proposition. — *For each* $\alpha \geqslant 0$,

$$Q^\alpha_N v^\alpha(x, y) = v^\alpha \hat{Q}^\alpha_N(x, y).$$

Proof. — It suffices to prove this when $\alpha > 0$. Let f and g be in $\mathbf{C}^+_K$. Using (4.14) and its dual along with (4.1) we have

$$(g, Q^\alpha_N V^\alpha f) = (g, (V^\alpha - W^\alpha)f) = (g(\hat{V}^\alpha - \hat{W}^\alpha), f) = (g\hat{V}^\alpha \hat{Q}^\alpha_N, f).$$

Consequently $Q^\alpha_N V^\alpha f(x) = \int v^\alpha \hat{Q}^\alpha_N(x, y) f(y)\, dy$ almost everywhere, and hence everywhere since both sides are in $\mathcal{S}^\alpha(M)$ as functions of x. This in turn implies that for a fixed x, $Q^\alpha_N v^\alpha(x, y) = v^\alpha \hat{Q}^\alpha_N(x, y)$ almost everywhere in y, and hence everywhere since both functions of y are in $\hat{\mathcal{S}}^\alpha(M)$. Thus (4.15) is established.

The most important special case of (4.15) is when $N_t = I_{[0,T_B)}(t)$ with B a Borel set. In this case (4.15) states

$$Q^\alpha_B v^\alpha(x, y) = v^\alpha \hat{Q}^\alpha_B(x, y). \tag{4.16}$$

5. Some examples.

In this section we will give some examples of dual multiplicative functionals.

Undoubtedly the most important example is given by the dual exact terminal times T_B and $\hat{T}_B$ where B is a Borel set. This example already was discussed in (1.1) and (1.2). At the opposite end of the spectrum from this example is the case of « classical » functionals which is treated in the next two propositions.

(5.1) Proposition. — *Let* h *be a bounded nonnegative Borel function. Then* $M_t = M_t(h) = \exp\left[-\int_0^t h(X_s)\,ds\right]$ *and* $\hat{M}_t = \hat{M}_t(h) = \exp\left[-\int_0^t h(\hat{X}_s)\,ds\right]$ *are dual functionals.*

Proof. — Both M and $\hat{M}$ are continuous and they are exact since $E_M = \hat{E}_M = E$. In order to show that $M \longleftrightarrow \hat{M}$ it suffices to prove that for $f, g \in C_K^+$ and for each fixed $t > 0$

$$(5.2) \qquad E^f\{M_t g(X_t)\} = \hat{E}^g\{\hat{M}_t f(\hat{X}_t)\}$$

where we have written $E^f\{.\}$ for $E^\mu\{.\}$ when $\mu(dx) = f(x)\,dx$ and similarly for $\hat{E}^g$. First of all we consider the case $h \in C_K^+$. Then

$$\int_0^t h(X_s)\,ds = \lim_n \frac{t}{n} \sum_{k=1}^n h\left[X\left(\frac{kt}{n}\right)\right] = \lim_n \frac{t}{n} \sum_{k=0}^n h\left[\left(X\,\frac{kt}{n}\right)\right]$$

since h is in C_K^+. Thus if $H_n(x) = \exp\left[-\frac{t}{n}h(x)\right]$ we have

$$M_t = \lim_{n\to\infty} \prod_{k=0}^n H_n\left[X\left(\frac{kt}{n}\right)\right] \quad \text{and} \quad \hat{M}_t = \lim_n \prod_{k=0}^n H_n\left[\hat{X}\left(\frac{kt}{n}\right)\right], \text{ and}$$

hence to establish (5.2) it will suffice to prove

$$(5.3) \quad E^f\left\{\prod_{k=0}^n H\left[X\left(\frac{kt}{n}\right)\right] g(X_t)\right\} = \hat{E}^g\left\{\prod_{k=0}^n H\left[\hat{X}\left(\frac{kt}{n}\right)\right] f(\hat{X}_t)\right\}.$$

for any $H \in b\mathcal{E}_+$.

If $n = 1$, (5.3) reduces to the identity

$$(fH,\ P_t(Hg)) = ((fH)\hat{P}_t,\ Hg).$$

It is easy to check by induction on n that (5.3) may be written in the form $(fH,\ (P_{t/n}H)^n g) = (f(H\hat{P}_{t/n})^n,\ Hg)$ (²). But for a fixed s the identity

$$(fH,\ (P_sH)^n g) = (f(H\hat{P}_s)^n,\ Hg)$$

is immediate, and this establishes (5.3). Thus we have proved (5.2) when $h \in C_K^+$.

Let $\mathcal{H}$ denote the class of all bounded nonnegative Borel functions h for which (5.2) holds. Clearly $\mathcal{H}$ is closed under bounded pointwise limits and we have just seen that $C_K^+ \subset \mathcal{H}$. Consequently $\mathcal{H}$ contains all bounded nonnegative Borel functions. Thus Proposition 5.1 is established.

We now are going to extend Proposition 5.1 to the case of an arbitrary nonnegative Borel function h. However, we must

(²) Here $(P_sH)^n$ denotes the n-th iterate of the operator P_sH which maps a bounded function g into the bounded function $P_s(Hg)$, and $(H\hat{P}_s)^n$ is defined similarly.

exercise a certain amount of care in this case since $\exp\left[-\int_0^t h(X_s)\,ds\right]$ need not be right continuous if h is unbounded. To overcome this, let

$$T = \inf\left\{t : \int_0^t h(X_s)\,ds = \infty\right\}.$$

Then T is a terminal time and we define

$$(5.4)\qquad M_t = M_t(h) = I_{[0,T)}(t)\exp\left[-\int_0^t h(X_s)\,ds\right].$$

Clearly M is a right continuous MF of X, and if h is bounded then $T = \infty$ so that this is consistent with our previous definition of $M_t(h)$. Of course, when h is unbounded E_M need not be all of E. However if x is not in E_M, then $P^x(T = 0) = 1$ and almost surely P^x, $\int_0^t h(X_s)\,ds = \infty$ for all $t > 0$. As a result, using (III-5.9), it is easy to check that M is exact. One more observation is needed. Namely if we define $M_t^* = \exp\left[-\int_0^t h(X_s)\,ds\right]$, then $t \to M_t$ and $t \to M_t^*$ agree for all values of t except possibly $t = T$, and so $V^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t) M_t^*\,dt$. Of course, $\hat{M}_t(h)$ is defined analogously.

(5.5) Proposition. — *If h is a nonnegative Borel function, then $M(h)$ and $\hat{M}(h)$ are dual functionals.*

Proof. — Let $h_n = h \wedge n$. Then $M_t(h_n) \to M_t^*(h)$ and $\hat{M}_t(h_n) \to \hat{M}_t^*(h)$ where the notation is that introduced above. Let (V_n^α) and $(\hat{V}_n^\alpha)$ denote the resolvents of $M_t(h_n)$ and $\hat{M}_t(h_n)$ respectively. Then by (5.1), $(f, V_n^\alpha g) = (f\hat{V}_n^\alpha, g)$ and letting $n \to \infty$ and using the remark preceding the statement of Proposition 5.5 we obtain $(f, V^\alpha g) = (f\hat{V}^\alpha, g)$. Hence $M \longleftrightarrow \hat{M}$ and (5.5) is proved.

Proposition 5.5 was proved by Hunt in [4]. Hunt used a different method. If we combine (4.1), (5.5), and the duality of T_B and $\hat{T}_B$ for a Borel set B, we obtain the full duality result proved by Hunt in [4].

6. Duality of non-exact multiplicative functionals.

It often is useful to extend the notion of duality to general (always right continuous, decreasing, and satisfying $0 \leqslant M_t \leqslant 1$) multiplicative functionals.

(6.1) Definition. — *Two* MF'*s*, M *and* $\hat{M}$ *of* X *and* $\hat{X}$ *respectively are said to be dual (or to be in duality) provided that* $(f, Q_t g) = (f\hat{Q}_t, g)$ *for all* $t \geqslant 0$ *and* $f, g \in C_K^+$. *This condition is equivalent to* $(f, V^\alpha g) = (f\hat{V}^\alpha, g)$ *for all* $\alpha > 0$ *and* $f, g \in C_K^+$.

It follows from (2.3) that if M and $\hat{M}$ are dual MF'*s*, then $P_M^\alpha u^\alpha(x, y) = u^\alpha \hat{P}_M^\alpha(x, y)$ almost everywhere with respect to $\xi \times \xi$ on $E \times E$. This will be an identity in x and y if and only if M and $\hat{M}$ are exact.

(6.2) Proposition. — *Let* M *be a* MF *of* X. *Then there exists a unique exact* MF, $\hat{M}$ *of* $\hat{X}$ *that is dual to* M.

Proof. — It is known that there exists a unique exact MF, $M^\star$ of X such that if (W^α) is the resolvent corresponding to $M^\star$, then $V^\alpha \leqslant W^\alpha$, $E_M \subset E_{M^*}$, $V^\alpha(x, .) = W^\alpha(x, .)$ if $x \in E_M$, and $t \to M_t$ and $t \to M_t^*$ are identical almost surely P^x on $[0, \zeta)$ if $x \in E_M$. In particular $M_t \leqslant M_t^*$ almost surely. Finally $E_{M^*} - E_M$ is of potential zero and hence of measure zero. See (III-4.9) and its proof and (III-4.25). We will call $M^\star$ the *regularization* of M. Since $V^\alpha g = W^\alpha g$ almost everywhere we have $(f, V^\alpha g) = (f, W^\alpha g)$ for all $f, g \in C_K^+$. Thus if $\hat{M} \longleftrightarrow M^\star$, then $\hat{M}$ and M are dual MF'*s*.

(6.3) *Remark.* — The proof of (6.2) shows that M and $\hat{M}$ are dual if and only if their regularizations are dual.

It follows from (III-4.25) that for each $t > 0$ we have almost surely $M_t^* = \lim_{s \downarrow 0} M_{t-s} \circ \theta_s$. Consequently if M and N are MF'*s*, then for each t almost surely $(MN)_t^* = M_t^* N_t^*$. But $(MN)^\star$ and $M^\star N^\star$ are right continuous and so $(MN)^\star = M^\star N^\star$. Similarly $(M^\lambda)^\star = (M^\star)^\lambda$ for any $\lambda > 0$. The following result now follows from (6.3), (4.1), and (4.12).

(6.4) COROLLARY. — *Let* M *and* $\hat{M}$ *and* N *and* $\hat{N}$ *be dual respectively. Let* $\lambda > 0$. *Then* MN *and* $\hat{M}\hat{N}$ *and* M^λ *and* $\hat{M}^\lambda$ *are dual respectively.*

We will say that two terminal times T and $\hat{T}$ of X and $\hat{X}$ respectively are dual provided the corresponding MF'*s* are dual. Observe that if T is a terminal time of X and $M_t = I_{[0,T)}(t)$, then $(M^\star)^2 = M^\star$ since $M^2 = M$. Consequently there exists an exact terminal time $T^\star$ such that $M_t^\star = I_{[0,T^\star)}(t)$. We call $T^\star$ the regularization of T. It is immediate that almost surely $T \leqslant T^\star$ and that if x is not regular for T then $T = T^\star$ almost surely P^x.

If M is a MF define $S = \inf\{t < \zeta : M_t = 0\}$ if the set in braces is not empty and $S = \zeta$ if it is empty. (Note that we are *not* assuming that $M_t = 0$ if $t \geqslant \zeta$.) Then S is a terminal time, but it need not be exact even when M is exact. (See the example on page 131 of [1].) The following result is sometimes of interest.

(6.5) PROPOSITION. — *Let* M *and* $\hat{M}$ *be dual* MF'*s. Then* S *and* $\hat{S}$ *are dual terminal times.*

Proof. — There is no loss of generality in assuming that $M_t = 0$ if $t \geqslant \zeta$ since (M_t) and $(I_{[0,\zeta)}(t)M_t)$ are equivalent MF'*s*. Similarly we may assume that $\hat{M}_t = 0$ if $t \geqslant \hat{\zeta}$. Let $N_t = \lim_{\lambda \downarrow 0} (M_t)^\lambda$. Then $N_t = 1$ if $t < S$ and $N_t = 0$ if $t > S$, that is, $N_t = I_{[0,S)}(t)$ for all t except possibly $t = S$. In particular if (W^α) is the resolvent corresponding to S, then $W^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} N_t f(X_t)\, dt = \lim_{\lambda \to 0} V_\lambda^\alpha f(x)$ for all bounded f and $\alpha > 0$ where (V_λ^α) is the resolvent of M^λ. Similarly $f\hat{W}^\alpha(x) = \lim_{\lambda \to 0} f\hat{V}^\alpha(x)$ where $(\hat{W}^\alpha)$ and $(\hat{V}_\lambda^\alpha)$ are the resolvents of $\hat{S}$ and $\hat{M}^\lambda$ respectively. As a result (6.4) implies that S and $\hat{S}$ are dual terminal times.

7. Measures associated with multiplicative functionals.

In this section we will associate with certain MF'*s* a measure in a natural and useful manner. The ideas and techniques of this section are due to Revuz [9]. We merely adapt his methods to cover the situation which interests us here. These results will be applied to the study of dual functionals in Section 8.

It will be convenient to single out a particular representative of a given MF. Recall that we are identifying equivalent MF's. Therefore in the remainder of this paper we will assume that each MF, M is *normalized* as follows: $M_t(\omega) = 1$ for all $t \geqslant 0$ if $X_0(\omega) = \Delta$, $M_t(\omega) = \lim_{u \uparrow \zeta(\omega)} M_u(\omega) = M_{\zeta-}(\omega)$ if $t \geqslant \zeta(\omega)$ and $X_0(\omega) \in E$. This is no loss of generality since each MF is equivalent to a normalized one. See (III-4.23). As in Section 6 we define S, or S_M if the dependence on M needs emphasis, as follows: $S = \inf\{t < \zeta : M_t = 0\}$ if the set in braces is not empty and $S = \zeta$ if it is empty.

We now fix a normalized exact MF, M *with dual* $\hat{M}$ *also assumed to be normalized.* Let $B_t = M_0 - M_t$. Then $B_{t+s} = B_t + M_t(B_s \circ \theta_t)$. That is B is an *additive functional* of (X, M) except that $t \to B_t$ need not be continuous at $t = S$. See (IV-1.1) for the definition. However $t \to B_t$ is constant on $[S, \infty]$ and is continuous at $t = \zeta$ in view of our normalization of M. Note that $U_B^\alpha f(x) = P_M^\alpha f(x)$ if x is in E_M, but that $U_B^\alpha f(x) = 0$ if x is not in E_M. Here U_B^α is the α-potential operator associated with B, that is,

$$U_B^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t)\, dB_t.$$

For the purposes of this paper the following definition is appropriate.

(7.1) Definition. — *A family* $A = (A_t : t \geqslant 0)$ *of nonnegative random variables is called an additive functional* (AF) *of* (X, M) *provided:*

(i) *for each* t, A_t is $\mathcal{F}_t$ *measurable*;

(ii) *for each* t *and* s, $A_{t+s} = A_t + M_t(A_s \circ \theta_t)$ *almost surely*,

(iii) *almost surely* $t \to A_t$ *is right continuous, increasing, constant on the interval* $[S, \infty]$, *continuous at* $t = \zeta$, *and* $A_0 = 0$.

As remarked above this differs slightly from (IV-1.1) in that it does not require A to be continuous at S when $S < \zeta$. (In [1] we assumed that $M_t = 0$ if $t \geqslant \zeta$ which implies that A is constant on $[S, \infty]$, and so it was not necessary to require this in (IV-1.1).) Let $\mathbf{A}(M)$ denote the class of all AF's of (X, M) which are finite on $[0, \zeta)$. In particular $B \in \mathbf{A}(M)$. For simplicity in what follows we will restrict our attention to $\mathbf{A}(M)$, although the results are valid somewhat more generally. This is justified because our main concern in this paper is B.

Following Revuz [9], we define for $f \in \mathcal{E}_+$ and $A \in \mathbf{A}(M)$

$$\nu_A(f) = \sup_{t>0} t^{-1} E^{\xi} \int_0^t f(X_s)\, dA_s. \tag{7.2}$$

This definition does not depend on the existence of the dual process $\hat{X}$. One need only assume that X has a reference measure and that ξ is a fixed *excessive* reference measure. One says that A is *integrable* if $\nu_A(1) < \infty$. If $A = \Sigma A^n$ where each A^n is integrable, then A is said to be σ-integrable.

(7.3) *Remark.* — Our definition of σ-integrability differs from that of Revuz who requires that $E = \bigcup E_n$ where $\nu_A(I_{E_n}) < \infty$ for each n. Since we are *assuming* that $A \in \mathbf{A}(M)$, if such E_n exist then $A_t^n = \int_0^t I_{E_n}(X_s)\, dA_s$ is in $\mathbf{A}(M)$ for each n. Clearly $A = \Sigma A^n$ and each A^n is integrable. Thus our definition is somewhat more general than Revuz's and is the appropriate one for us. If one considers additive functionals which are *not* in $\mathbf{A}(M)$, then A^n defined above need not be an additive functional and one is forced to adopt Revuz's definition.

Exactly as in [9] one establishes the following proposition.

(7.4) Proposition. — *Let* $A \in \mathbf{A}(M)$. *If* $f \in b\mathcal{E}_+$ *then*

$$\nu_A(f) = \lim_{t \downarrow 0} t^{-1} E^{\xi} \int_0^t f(X_s)\, dA_s = \lim_{\alpha \to \infty} \alpha(1, U_A^{\alpha} f),$$

and this last limit is increasing. If A *is* σ-*integrable* ν_A *defines a measure which is a countable sum of finite measures; it is finite if and only if* A *is integrable. Clearly* ν_A *does not charge polar sets and if* A *is continuous* ν_A *does not charge semipolar sets.*

Proposition 7.4 is relatively elementary and is valid under the assumption that ξ is an excessive reference measure for X.

The next result is much deeper and is the key to our later applications. We do not strive for the utmost generality in its statement. Recall that A is *natural* provided that almost surely $t \to A_t$ and $t \to X_t$ have no common discontinuities. Furthermore we will say that a Borel set D *carries* A provided that almost surely $t \to A_t$ and $t \to \int_0^t I_D(X_s)\, dA_s$ agree on $[0, \zeta)$. Evidently this implies that $U_A^\alpha f = U_A^\alpha(f I_D)$ for all $\alpha \geqslant 0$ and $f \in \mathcal{E}_+$.

(7.5) THEOREM. — *Let* $A \in \mathbf{A}(M)$ *and assume that* A *is natural,* σ*-integrable, and* $u_A^\alpha = U_A^\alpha 1$ *is finite. If* A *is carried by* $\hat{E}_M$*, then* $u_A^\alpha = V^\alpha \nu$ *where* $\nu = \nu_A$*, that is,*

$$E^x \int_0^\infty e^{-\alpha t}\, dA_t = \int v^\alpha(x, y)\nu(dy).$$

Proof. — This is essentially Proposition V.1 of [9]. As in [9] it suffices to prove it when $\alpha > 0$ and A is integrable. Let $f \in b\mathcal{E}_+$ be integrable (with respect to ξ). Then since $\hat{M}$ is exact $f\hat{V}^\alpha = f\hat{U}^\alpha - (f\hat{U}^\alpha - f\hat{V}^\alpha)$ is the difference of two bounded α-coexcessive functions, and so $t \to f\hat{V}^\alpha(X_{t-})$ is left continuous on $(0, \zeta)$ by Weil's theorem [10]. Now arguing exactly as in [9] one shows that $V^\alpha\nu \leqslant u_A^\alpha$. Again just as in [9] one finds that

$$\int (u_A^\alpha - V^\alpha\nu)\, d\xi \leqslant \frac{1}{\alpha} \lim_{\beta \to \infty} [(1, \beta u_A^{\alpha+\beta}) - (1, \beta V^{\alpha+\beta}\nu)].$$

But $(1, \beta u_A^{\alpha+\beta}) \to \nu(1)$ by definition, while $(1, \beta V^{\alpha+\beta}\nu) \to \nu(\hat{E}_M)$ because $\beta(1\hat{V}^{\alpha+\beta})$ increases to the indicator function of $\hat{E}_M$ as $\beta \to \infty$. The fact that A is carried by $\hat{E}_M$ obviously implies that ν also is carried by $\hat{E}_M$. Consequently $u_A^\alpha = V^\alpha\nu$ almost everywhere, and hence everywhere since both functions are $\alpha - (X, M)$ excessive. This establishes Theorem 7.5.

Remark. — Under the assumptions of (7.5) the measure ν_A must, in fact, be σ-finite. Indeed if μ is any measure carried by $\hat{E}_M$ and $V^\alpha\mu$ is finite almost everywhere, then one can find a strictly positive f such that $(f, V^\alpha\mu) < \infty$. But $f\hat{V}^\alpha$ is strictly positive on $\hat{E}_M$ and $\int (f\hat{V}^\alpha_\cdot)\, d\mu = (f, V^\alpha\mu) < \infty$. Consequently μ is σ-finite.

Next the following uniqueness theorem is proved in the same manner as (VI-1.15).

(7.6) Proposition. — *Let* μ *be a measure. If* $V^\alpha\mu$ *is finite almost everywhere, then* $V^\alpha\mu$ *determines the restriction of* μ *to* $\hat{E}_M$.

Clearly this is the most that one could expect since for each x, $v^\alpha(x, y) = 0$ if $y \notin \hat{E}_M$ and so $V^\alpha\mu(x) = \int_{\hat{E}_M} v^\alpha(x, y)\mu(dy)$. Thus under the assumptions of Theorem 7.5, ν_A is the *unique* measure carried by $\hat{E}_M$ such that $u^\alpha_A = V^\alpha\nu_A$. Using (4.16) the proof of (VI-3.1) can easily be modified to obtain the following result.

(7.7) Proposition. — *Let* A *satisfy the hypotheses of* (7.5). *Then for any* $f \in \mathcal{E}^+$, $U^\alpha_A f = V^\alpha(f\nu_A)$.

Now we turn our attention to sufficient conditions that the hypotheses of (7.5) hold. Once again we follow Revuz [9]. The next lemma is the analogue of Lemma II.2 of [9] and the proof is exactly the same.

(7.8) Lemma. — *Suppose that* A *and* D *are in* $\mathbf{A}(M)$ *and that* D *is integrable. If* $f \in b\mathcal{E}^+$ *is such that for some* $\alpha \geq 0$, $U^\alpha_A f \leq u^\alpha_D < \infty$ *then* $\nu_A(f) < \infty$.

(7.9) Lemma. — *Let* $A \in \mathbf{A}(M)$. *If* A *is continuous at* S, *then* A *is carried by* E_M.

Proof. — Let R be the hitting time of E^c_M. Recall from (III-5.3) that almost surely $S \leq R$. Since A is continuous at S

$$A_t = \int_0^{S\wedge t} I_{E_M}(X_u)\, dA_u = \int_0^t I_{E_M}(X_u)\, dA_u,$$

and hence A is carried by E_M.

The criteria for σ-integrability contained in the next two propositions will suffice for our purposes. However, we readily admit that Proposition 7.11 is not particularly satisfying as it stands.

(7.10) PROPOSITION. — *Let* $A \in \mathbf{A}(M)$. *If* A *is continuous, then* A *is* σ*-integrable.*

Proof. — Let $C_t = \int_0^t (M_u)^{-1}\, dA_u$. Note that the integration extends over $(0, t \wedge S)$ since A is continuous and constant on $[S, \infty)$. Let g be a bounded strictly positive integrable function and define

$$\varphi(x) = E^x \int_0^\infty e^{-t} M_t \exp[-C_t] g(X_t)\, dt.$$

Clearly $\varphi > 0$ on E_M. A straightforward computation shows that

$$U_A^1 \varphi(x) = E^x \int_0^\infty e^{-t} M_t (1 - e^{-C_t}) g(X_t)\, dt \leqslant V^1 g(x).$$

If $E_n = \{\varphi \geqslant 1/n\}$, then $I_{E_n} \leqslant n\varphi$ and so

$$U_A^1 I_{E_n} \leqslant n U_A^1 \varphi \leqslant n V^1 g.$$

If $D_t = \int_0^t M_u g(X_u) du$ then $D \in \mathbf{A}(M)$ and $u_D^\alpha = V^\alpha g$. But for any $\alpha > 0$, $\alpha(1, V^\alpha g) = \alpha(1\hat{V}^\alpha, g) \leqslant \int g\, d\xi < \infty$ and hence D is integrable. It now follows from (7.8) that $\nu_A(E_n) < \infty$ for each n. Since $E_M = \bigcup E_n$ Lemma 7.9 implies that A is σ-integrable, completing the proof of (7.10).

(7.11) PROPOSITION. — *Let* $A \in \mathbf{A}(M)$ *and assume that for some* $\alpha \geqslant 0$, u_A^α *is finite. If, in addition,* $A(S) - A(S-)$ *is bounded by a constant, then* $L_t = \int_0^t I_{F_M}(X_u)\, dA_u$ *is* σ*-integrable.*

Proof. — Let $\{a_n : -\infty < n < \infty\}$ be a two-sided sequence of positive numbers such that $a_n < a_m$ if $n < m$ and $\lim_{n\to-\infty} a_n = 0$, $\lim_{n\to+\infty} a_n = +\infty$. For each n define

$$T_n = \inf\{t < S : a_n M_t \leqslant A_t - A_{t-} < a_{n+1} M_t\}.$$

Then each T_n is a complete terminal time (IV-4.6) and using standard techniques we can write

$$A_t = A_t^c + A_t^* + \sum_{n=-\infty}^{\infty} A_t^n \tag{7.12}$$

where A^c is continuous, A^* is constant except possibly for a jump at S, and each A^n is a pure jump element of $\mathbf{A}(M)$ which is continuous at S and satisfies

$$a_n M_t \leqslant A^n_t - A^n_{t-} < a_{n+1} M_t$$

at its (discrete) points of discontinuity. According to (7.9), A^c and each A_n is carried by E_M. Let $L^*_t = \int_0^t I_{F_M}(X_u) dA^*_u$. Then in light of (7.10) in order to establish (7.11) it will suffice to show that L^* and each A_n are σ-integrable.

Now fix n and let $D = A_n$. Then $D \in \mathbf{A}(M)$, u^α_D is finite, D is carried by E_M, and $D_t - D_{t-} \leqslant a M_t$ where $a = a_{n+1}$. We now use an argument of Revuz to show that D is σ-integrable. We may assume $\alpha > 0$ since u^α_D decreases as α increases. Let $f \in b\mathcal{E}_+$ be strictly positive and integrable. Then $V^\alpha f$ is bounded and strictly positive on E_M. Define $F_k = \{u^\alpha_D \leqslant k V^\alpha f\} \cap \{V^\alpha f \geqslant k^{-1}\}$. Clearly $\bigcup F_k = E_M$. Fix k and let $T = T_{F_k}$, $g = I_{F_k}$. Then

$$\begin{aligned} U^\alpha_D g(x) &= E^x \int_{[T,\infty)} e^{-\alpha t} g(X_t) dD_t \\ &= E^x\{e^{-\alpha T} g(X_T)[D_T - D_{T-}]\} + Q^\alpha_{F_K} U^\alpha_D g(x). \end{aligned}$$

But $U^\alpha_D g \leqslant u^\alpha_D \leqslant k V^\alpha f$ on F_K and so $Q^\alpha_{F_K} U^\alpha_D g \leqslant k Q^\alpha_{F_K} V^\alpha f$. On the otherhand the first term is dominated by

$$a E^x\{e^{-\alpha T} g(X_T) M_T\} \leqslant ak Q^\alpha_{F_K} V^\alpha f(x).$$

Thus $U^\alpha_D g \leqslant k(a+1) V^\alpha f$. Consequently by (7.8), $\nu_D(F_K) < \infty$ and hence D is σ-integrable since it is carried by E_M.

Finally it remains to show that L^* is σ-integrable. Unfortunately we have been unable to find a simple proof of this fact and the argument that we are now going to give is a bit involved. It is essentially due to Meyer [7]. As we have remarked several times S need not be exact. Therefore let T be the regularization of S. (See the discussion following (6.4).) Thus T is an exact terminal time $S \leqslant T$ and $S = T$ almost surely P^x if $x \in E_M$. We will omit the phrase « almost surely » in the remainder of the proof of (7.11) in those places where it obviously applies. Since $S \leqslant \zeta$ we may assume that $T \leqslant \zeta$.

We define the iterates of T in the usual manner: $T_0 = 0$,

$T_{n+1} = T_n + T \circ \theta_{T_n}$ for $n \geqslant 0$, and let $R = \lim T_n$. Clearly each $T_n \leqslant \zeta$ and so $R \leqslant \zeta$. Since T is exact, T is a complete terminal time, that is, for each $k \geqslant 0$, $n \geqslant 1$, and stopping time Q, $T_{n+k} = Q + T_n \circ \theta_Q$ on $\{T_k \leqslant Q < T_{k+1}\}$. See (IV-4.6) and (IV-4.36), and also [8]. One checks easily that $T_{n-1} < T_n$ on $\{T_n < R\}$ and that R is a strong terminal time. If $E_R = \{x : P^x(R > 0) = 1\}$, then $E_M \subset E_R$. (In fact $E_R = E_T = \{x : P^x(T > 0) = 1\}$.) Let $a_0 = 0$ and $a_n = a$ for $n \geqslant 1$ where $0 < a < 1$. Define

$$N_t = \prod_{n, T_n \leqslant t} (1 - a_n).$$

Then N_t is right continuous and $N_t > 0$ if $t < R$, $N_t = 0$ if $t \geqslant R$. Using the fact that T is a complete terminal time one can easily check that $N = (N_t)$ is a MF. Next define

$$\begin{aligned} D_t &= -\int_{[0,t]} \frac{dN_u}{N_{u-}} \qquad t < R \\ &= \lim_{u \uparrow R} D_u \qquad t \geqslant R, \end{aligned}$$

and observe that D is an additive functional of (X, R). In fact $D_t = na$ if $T_n \leqslant t < T_{n+1}$. Next let g be a bounded strictly positive integrable Borel function and define

$$\varphi(x) = E^x \int_0^\infty e^{-t} N_t g(X_t)\, dt.$$

It is evident that $\varphi(x) > 0$ if and only if $x \in E_R$ and that $\varphi \in b\mathcal{E}_+^*$. Using the integration by parts formula [5] one finds

$$0 = d\left(N_t \cdot \frac{1}{N_t}\right) = \frac{dN_t}{N_{t-}} + N_t d\left(\frac{1}{N_t}\right)$$

on [0, R), and so $dD_t = N_t d\left(\frac{1}{N_t}\right)$. Making use of this a familiar calculation using (2.4) yields

$$U_D^1 \varphi(x) = E^x \int_0^R e^{-t}(1 - N_t) g(X_t)\, dt \leqslant W^1 g(x),$$

where (W^α) is the resolvent corresponding to R.

We claim that $\alpha(1, U_D^\alpha \varphi) \leqslant (1, g)$ for all $\alpha > 1$. This is the key step in the proof of (7.8), but we will give the argument for completeness. By the resolvent equation and its analogue for (U_D^α), (IV-2.3)

$$W^\alpha g - U_D^\alpha \varphi = W^1 g + (1-\alpha)W^\alpha W^1 g - [U_D^1\varphi + (1-\alpha)W^\alpha U_D^1 \varphi],$$

and since $\alpha 1\hat{W}^\alpha \leqslant 1$ this yields for $\alpha > 1$

$$\alpha(1,\ W^\alpha g - U_D^\alpha \varphi) \geqslant (1,\ W^1 g - U_D^1 \varphi) \geqslant 0.$$

Consequently $\alpha(1, U_D^\alpha\varphi) \leqslant \alpha(1, W^\alpha g) = \alpha(1\hat{W}^\alpha, g) \leqslant (1, g) < \infty$ because g is integrable.

Now returning to A^*, if b is a bound for $A(S) - A(S-)$ we have for each $x \in E_M$

$$U_{A^*}^\alpha \varphi(x) \leqslant b E^x\{e^{-\alpha S}\varphi(X_S)\} = bE^x\{e^{-\alpha T}\varphi(X_T)\} \leqslant \frac{b}{a} U_D^\alpha \varphi(x).$$

But $U_{A^*}^\alpha \varphi = 0$ off E_M and so

$$\alpha(1,\ U_{A^*}^\alpha\varphi) \leqslant \frac{\alpha b}{a}(1,\ U_D^\alpha \varphi) \leqslant \frac{b}{a}(1,\ g) < \infty.$$

Finally since $\varphi > 0$ on $E_R \supset E_M$ we see that

$$L_t^* = \int_0^t I_{E_M}(X_u)\, dA_u^*$$

is σ-integrable, completing the proof of (7.11).

(7.12) *Remark.* — In checking the hypotheses of (7.5) the following observation is helpful. Suppose that $A \in \mathbf{A}(M)$ is carried by E_M and that $A(S) - A(S-) \leqslant bM(S-)$ on $\{S < \infty\}$ where b is a positive constant. Then if u_A^α is finite, A is carried by $\hat{E}_M$. Indeed since $U_A^\alpha(x, .)$ is a finite measure it suffices to show that $U_A^\alpha I_K(x) = 0$ for each fixed $x \in E_M$ where $K = \{V^1 1 \geqslant a\} - \hat{E}_M$ with $a > 0$ because $E_M - \hat{E}_M$ is a countable union of such sets. But in the proof of (3.2) we showed that $M_{T_K-} = 0$ almost surely P^x on $\{T_K < \zeta\}$ for each x in E_M. Fix such an x. The following statements are understood to hold almost surely P^x. Clearly $S \leqslant T_K$ and so

$$\begin{aligned} U_A^\alpha I_K(x) &= E^x \int_{[T_K,\zeta)} e^{-\alpha t} I_K(X_t)\, dA_t \\ &\leqslant bE^x\{e^{-\alpha S} M_{S-};\ S = T_K < \zeta\} = 0, \end{aligned}$$

and so A is carried by $\hat{E}_M$.

Recall that $B_t = M_0 - M_t$. Suppose that

$$A_t = \int_0^t I_{E_M}(X_u)\, dB_u = -\int_0^t I_{F_M}(X_u)\, dM_u$$

is natural. Then combining (7.11), (7.12), and (7.7), we see that there exists a unique σ-finite measure $\mu = \nu_A$ carried by $E_M \cap \hat{E}_M$ such that $U_A^\alpha f = V^\alpha(f\mu)$ for all $f \in \mathcal{E}^+$. *We will call* μ *the measure associated with* M. In particular for each $x \in E_M$, $f \in \mathcal{E}^+$, and $\alpha \geqslant 0$

$$(7.13)\quad P_M^\alpha(f\, I_{E_M})(x) = U_A^\alpha f(x) = \int v^\alpha(x, y)\, f(y)\mu(dy).$$

8. Dual functionals and measures.

In this section we will make use of the results of Section 7 to study the relationship between M and $\hat{M}$. The next theorem is the key to what follows. We fix an exact normalized MF, M and let $B_t = M_0 - M_t$ and $A_t = \int_0^t I_{F_M}(X_u)\, dB_u$. Obviously B and A have bounded potentials. *We assume that* A *is natural and let* μ *be the measure associated with* M.

(8.1) THEOREM. — *Using the above notation, for each* $y \in \hat{E}_M$, $\alpha \geqslant 0$, $f \in \mathcal{E}^+$ *which vanishes off* $\hat{E}_M$,

$$f\hat{P}_M^\alpha(y) = \int f(x) v^\alpha(x, y)\mu(dx),$$

that is, the restriction of $\hat{P}_M^\alpha(dx, y)$ *to* $\hat{E}_M$ *is given by* $v^\alpha(x, y)\mu(dx)$.

Proof. — It suffices to prove this when $\alpha > 0$. Let R be the hitting time of E_M^c. Since $S \leqslant R$, $B_t - A_t$ is constant on $[0, R)$ and has a jump of magnitude M_{R-} at $t = R < \zeta$. Of course, $M_{R-} = 0$ unless $S = R < \zeta$. Therefore for if $x \in E_M$ and $g \in \mathcal{E}^+$ we have

$$P_M^\alpha g(x) = U_B^\alpha g(x) = U_A^\alpha g(x) + E^x\{e^{-\alpha R} g(X_R) M_{R-};\ R < \zeta\}.$$

Now fix $x \in E_M$ and $y \in \hat{E}_M$. Then

$$(8.2)\quad u^\alpha \hat{P}_M^\alpha(x, y) = P_M^\alpha u^\alpha(x, y) = \int v^\alpha(x, z)\mu(dz) u^\alpha(z, y) + E^x\{e^{-\alpha R} u^\alpha(X_R, y) M_{R-};\ R < \zeta\}.$$

Denote the expectation term in (8.2) by $q(x, y)$. Substituting $u^\alpha = v^\alpha + P^\alpha_M u^\alpha$ into the left side of (8.2) and $u^\alpha = v^\alpha + u^\alpha \hat{P}^\alpha_M$ into the right side of (8.2) we obtain

$$(8.3) \quad \int v^\alpha(x, z)\hat{P}^\alpha_M(dz, y) + \int P^\alpha_M u^\alpha(x, z)\hat{P}^\alpha_M(dz, y) \\ = \int v^\alpha(x, z)\mu(dz)v^\alpha(z, y) \\ + \int v^\alpha(x, z)\mu(dz)u^\alpha\hat{P}^\alpha_M(z, y) + q(x, y).$$

But

$$(8.4) \quad \int P^\alpha_M u^\alpha(x, z)\hat{P}^\alpha_M(dz, y) = \int P^\alpha_M(x, dz)u^\alpha\hat{P}^\alpha_M(z, y) \\ = \int v^\alpha(x, z)\mu(dz)u^\alpha\hat{P}^\alpha_M(z, y) \\ + E^x\{e^{-\alpha R}u^\alpha\hat{P}^\alpha_M(X_R, y)M_{R-};\ R < \zeta\}.$$

However if $R < \zeta$ then $X_R \in E - E_M$ and if $x_0 \notin E_M$, $u^\alpha(x_0, y) = u^\alpha\hat{P}^\alpha_M(x_0, y)$ because $v^\alpha(x_0, y) = 0$. Consequently the expectation in (8.4) is just $q(x, y)$. On the other hand the left side of (8.4) is dominated by $u^\alpha(x, y)$, and so combining (8.3) and (8.4) we see that for a fixed $y \in \hat{E}_M$

$$\int v^\alpha(x, z)\hat{P}^\alpha_M(dz, y) = \int v^\alpha(x, z)\mu(dz)v^\alpha(z, y)$$

almost everywhere in x on E_M, and hence everywhere on E since both sides are $\alpha - (X, M)$ excessive functions of x. Theorem 8.1 now follows from the uniqueness result (7.6) since $v^\alpha\hat{P}^\alpha_M \leq u^\alpha\hat{P}^\alpha_M \leq u^\alpha$.

If we set $\hat{B}_t = \hat{M}_0 - \hat{M}_t$ and $\hat{A}_t = \int_0^t I_{\hat{E}_M}(\hat{X}_u)\, d\hat{B}_u$ and if we write $\hat{U}^\alpha_A$ for the potential operators associated with $\hat{A}$, then following corollary is immediate.

(8.5) Corollary. — *Suppose that* A *is natural and that* μ *is the measure associated with* M. *Then* μ *is the unique measure carried by* $E_M \cap \hat{E}_M$ *such that for each* $\alpha \geq 0$

$$U^\alpha_A(x, dy) = v^\alpha(x, y)\mu(dy); \qquad \hat{U}^\alpha_A(dy, x) = v^\alpha(y, x)\mu(dy),$$

where we have written $\hat{U}^\alpha_A(dy, x)$ *for the measures associated with* $\hat{U}^\alpha_A$ *in keeping with our standard notational scheme. Moreover* μ *doesn't charge polar sets and if* A *is continuous* μ *doesn't charge semipolar sets.*

We come now to the main result of this section. In addition to the notation developed above we set $A_t^* = A_t$ if $t < S$ and $A_t^* = A_{S-}$ if $t \geqslant S$. Thus A^* is an additive functional of (X, M) that is continuous at S and $A_t^* = A_t = M_0 - M_t$ if $t < S$. We define $\hat{A}^*$ in a similar manner.

(8.6) THEOREM. — *Let* M *be an exact* MF *and* $\hat{M}$ *be its dual :*

(i) A *is natural if and only if* $\hat{A}$ *is natural.*

(ii) *If* A *is continuous, then* $\hat{A}^*$ *is continuous.*

(iii) *If* $\hat{X}$ *is special standard and* A *is continuous, then* $\hat{A}$ *is continuous.*

Proof. — If $C \in \mathbf{A}(M)$ and F is a Borel set, then it is easy to check that for any $\alpha \geqslant 0$

$$(8.7) \quad U_C^\alpha I_F(x) = E^x\{e^{-\alpha T_F} I_F(X_{T_F})[C(T_F) - C(T_F -)]\} + Q_F^\alpha U_C^\alpha I_F(x),$$

and as in (IV-2.4) it follows from (8.7) that $Q_G^\alpha U_C^\alpha I_G = U_C^\alpha I_G$ for all open sets G if C is natural. See (4.13) and the paragraphs which follow it for the definition of the operators Q_F^α. Now the proof of (IV-2.5) is easily modified by taking account of the possible jump at S to yield the fact that if C has a finite α-potential then C is natural if and only if $Q_G^\alpha U_C^\alpha I_G = U_C^\alpha I_G$ for all open sets G. We will make use of this fact in proving (i).

Assume that $\hat{A}$ is natural. Then according to the dual of (8.5), $U_A(x, dy) = v(x, y)\mu(dy)$ where μ is the measure associated with $\hat{A}$. Now let G be an open set. Then using (4.16) we obtain

$$Q_G U_A I_G(x) = \int Q_G(x, dz) \int_G v(z, y)\mu(dy) = \int v(x, z) \int_G \hat{Q}_G(dz, y)\mu(dy).$$

But μ is carried by $\hat{E}_M$ and for any point $y \in G \cap \hat{E}_M$, $\hat{Q}_G(\cdot, y) = \varepsilon_y$. Thus for any Borel set Γ,

$$\int_G \hat{Q}_G(\Gamma, y)\mu(dy) = \mu(\Gamma \cap G),$$

and so

$$Q_G U_A I_G(x) = \int_G \varphi(x, z)\mu(dz) = U_A I_G(x).$$

Therefore A is natural and (8.6 i) is established because of the complete duality between A and $\hat{A}$.

Next we will prove (8.6 iii). However for notational convenience we will actually prove the dual statement. That is we will assume that X is special standard and $\hat{A}$ is continuous and conclude that A is continuous. First observe that the proof of (IV-4.30) carries over to the present situation (the process X in (IV-4.30) is assumed to be special standard) and so if $C \in \mathbf{A}(M)$ has a finite potential then C is continuous if and only if $Q_K U_C I_K = U_C I_K$ for all compact K. Now by the dual of (8.5), $U_A(x, dy) = \varphi(x, y)\mu(dy)$ where μ is carried by $E_M \cap \hat{E}_M$ and doesn't charge semipolar sets. As in the proof of (i)

$$Q_K U_A I_K(x) = \int \varphi(x, z) \int_K \hat{Q}_K(dz, y)\mu(dy).$$

But $(K - {}^rK) \cap \hat{E}_M$ is semipolar and so this last displayed expression is just $\int_K \varphi(x, z)\mu(dz) = U_A I_K(x)$. Therefore A is continuous and (8.6 iii) is established.

Finally we turn to (8.6 ii). Once again for notational convenience we will prove the dual statement. It follows from the dual of (8.5) that $U_A^\alpha(x, dy) = \varphi^\alpha(x, y)\mu(dy)$ where μ doesn't charge semipolar sets. Fix $\alpha > 0$ and write $A = A^* + J$ where $J \in \mathbf{A}(M)$ is constant except possibly for a jump at S. By (8.6 i), A, and hence A^*, is natural. Let $u = u_{A^*}^\alpha$. We are going to show that u is a regular α-potential of (X, M), that is, if $\{T_n\}$ is an increasing sequence of stopping times with limit T, then $Q_T^\alpha u = \lim Q_{T_n}^\alpha u$. Let us assume this for the moment and complete the proof of (8.6 ii). By (IV-3.14) there exists a continuous additive function C of (X, M) such that $u_C^\alpha = u = u_{A^*}^\alpha$. But A^* is natural and continuous at S and so the uniqueness theorem (IV-2.13) implies that $A^* = C$. Hence A^* is continuous.

Thus to complete the proof of (8.6) we must show that $u = u_{A^*}^\alpha$ is a regular α-potential of (X, M). If $\{T_n\}$ is an increasing sequence of stopping times with limit T and if

$f \in \mathcal{S}^\alpha(M)$, then $Q^\alpha_{T_n} f$ decreases and always dominates $Q^\alpha_T f$. Thus it will suffice to show that $w = u^\alpha_A$ is a regular α-potential of (X, M) because $w = u + u^\alpha_J$ and all three functions are bounded elements of $\mathcal{S}^\alpha(M)$. But $w(x) = \int v^\alpha(x, y)\mu(dy)$ where μ doesn't charge semipolar sets. Now fix x, and let $\{T_n\}$ and T be as above. Then $Q^\alpha_{T_n} v^\alpha(x, y)$ decreases to a limit $q_x(y)$. Since $y \to Q^\alpha_{T_n} v^\alpha(x, y)$ is in $\hat{\mathcal{S}}^\alpha(M)$, q_x is $\alpha - (\hat{X}, \hat{M})$ super-mean-valued and by Doob's theorem it differs from its $\alpha - (\hat{X}, \hat{M})$ excessive regularization $\bar{q}_x$ on at most a semipolar set. See the discussion and footnote on page 198 of [1]. On the other hand if $f \in C^+_K$ then

$$\int Q^\alpha_{T_n} v^\alpha(x, y) f(y)\, dy = Q^\alpha_{T_n} V^\alpha f(x)$$
$$= E^x \int_{T_n}^\infty e^{-\alpha t} f(X_t) M_t\, dt \downarrow E^x \int_T^\infty e^{-\alpha t} f(X_t) M_t\, dt$$
$$= \int Q^\alpha_T v^\alpha(x, y) f(y)\, dy.$$

Consequently $q_x = Q^\alpha_T v^\alpha(x, .)$ almost everywhere and since $Q^\alpha_T v^\alpha(x, .) \in \hat{\mathcal{S}}^\alpha(M)$ it follows that $\bar{q}_x = Q^\alpha_T v^\alpha(x, .)$ everywhere. Hence for each fixed x, $Q^\alpha_{T_n} v^\alpha(x, y)$ decreases to $Q^\alpha_T v^\alpha(x, y)$ except on a semipolar set (in y). But μ doesn't charge semipolar sets and so

$$Q^\alpha_{T_n} w(x) = \int Q^\alpha_{T_n} v^\alpha(x, y)\mu(dy) \to \int Q^\alpha_T v^\alpha(x, y)\mu(dy) = Q^\alpha_T w(x).$$

Thus w is a regular α-potential of (X, M) and the proof of (8.6) is complete.

Remark. — Most likely (8.6 iii) is valid without the assumption that $\hat{X}$ is special standard. Indeed the proof (of the dual) of (8.6 ii) shows that u^α_A is a regular α-potential of (X, M). But $u^\alpha_A = u^\alpha_{A^*} + u^\alpha_J$ and so u^α_J is a regular α-potential of (X, M). Consequently there exists a continuous additive function C of (X, M) so that $u^\alpha_J = u^\alpha_C$. Making use of the relationship between J and A one can then show that $U^\alpha_J f = U^\alpha_C f$ for all bounded f. If one could conclude from this that $J = C$, then J would be zero and so A itself would be continuous. Unfortunately the uniqueness theorem (IV-2.12) does not apply because J charges S and we have been unable to overcome this difficulty.

The following corollary is an immediate consequence of Theorem 8.6.

(8.8) Corollary. — *Let* M *be an exact* MF *of* X. *If* M *is continuous, then* $t \to \hat{M}_t$ *is continuous on* $[0, \hat{S})$.

We close this section with an example to show that the conclusion of (8.8) can not be strengthened to assert that $\hat{M}$ is continuous. Let E be the real line. Let X be translation to the right at speed one and let $\hat{X}$ be translation to the left at speed one. Let $h(x) = |x|^{-1}$ if $x < 0$ and $h(x) = 1$ if $x \geqslant 0$. Define $M_t = \exp\left[-\int_0^t h(X_s)\, ds\right]$. An elementary calculation shows that

$$M_t = \begin{cases} e^{-t} & \text{if} \quad X_0 \geqslant 0 \\ \dfrac{x-t}{x} & \text{if} \quad 0 \leqslant t < -X_0 = x \\ 0 & \text{if} \quad t \geqslant -X(0) > 0, \end{cases}$$

and using (5.5)

$$\hat{M}_t = \begin{cases} \dfrac{x}{x+t} & \text{if} \quad \hat{X}_0 = -x < 0 \\ 0 & \text{if} \quad \hat{X}_0 = 0 \\ e^{-t} & \text{if} \quad 0 \leqslant t < \hat{X}_0 \\ 0 & \text{if} \quad t \geqslant \hat{X}_0 > 0. \end{cases}$$

Thus M is continuous and $E_M = E$. On the other hand $\hat{E}_M = E - \{0\}$ and $\hat{S} = \hat{D}_{\{0\}} = \inf\{t \geqslant 0: \hat{X}_t = 0\}$. Clearly $\hat{M}$ is continuous on $[0, \hat{S})$, but has a discontinuity at $\hat{S}$ if $\hat{X}_0 > 0$. Finally observe that $\hat{A}_t = -\int_0^t I(\hat{X}_{\hat{E}_M s})\, d\hat{M}_s$ is continuous as it should be according to (8.6 iii). However, in this example $\hat{A} = \hat{A}^*$.

9. Some applications.

In this section we give a few elementary applications of the results developed in the preceding sections to additive functionals of X. As before X and $\hat{X}$ are dual processes. Suppose M is a MF of X which doesn't vanish on $[0, \zeta)$, that is, $S = \zeta$. Then $E_M = E$ and M is exact. *Assume that* M *is natural.* Then by (7.13)

$$(9.1) \qquad P_M^\alpha(x, dy) = v^\alpha(x, y)\mu(dy)$$

for all $\alpha \geqslant 0$ and x in E where μ is a σ-finite measure carried by $\hat{E}_M$. According to (3.2), $E - \hat{E}_M$ is polar in this case, and, of course, μ doesn't charge polar sets. Now define

$$\begin{aligned} A_t &= -\int_0^t \frac{dM_u}{M_{u-}} \qquad \text{if} \qquad t < \zeta \\ &= A_{\zeta-} \qquad\qquad \text{if} \qquad t \geqslant \zeta. \end{aligned}$$

It is evident that A is a natural additive functional of X. *Do not confuse this additive functional with the* A *of sections* 7 *and* 8.

(9.2) Proposition. — *For each* $\alpha \geqslant 0$ *and* $f \in \mathcal{E}^+$

$$U_A^\alpha f(x) = \int u^\alpha(x, y)\, f(y)\mu\,(dy).$$

Proof. — It suffices to prove this when $\alpha > 0$. First note that $dA_t = -(M_{t-})^{-1}\, dM_t = M_t d(M_t)^{-1}$. If $f \in \mathcal{E}_+^*$ a standard calculation using (2.4) yields (see [7])

$$U_A^\alpha V^\alpha f = U^\alpha f - V^\alpha f = P_M^\alpha U^\alpha f.$$

Fix x. Then from the above and (8.1)

$$\begin{aligned} \int U_A^\alpha(x, dz)v^\alpha(z, y) = P_M^\alpha u^\alpha(x, y) &= u^\alpha \hat{P}_M^\alpha(x, y) \\ &= \int u^\alpha(x, z)\mu(dz)v^\alpha(z, y) \end{aligned}$$

almost everywhere on $\hat{E}_M$ and hence everywhere on E. Consequently the dual of the uniqueness theorem (7.6) implies that $U_A^\alpha(x, dz) = u^\alpha(x, z)\mu\,(dz)$, proving (9.2).

Using the above notation let us consider $\hat{M}$ the dual of M. Since $S = \zeta$ it follows from (6.5) that the regularization of $\hat{S}$ is $\hat{\zeta}$. In particular if $x \in \hat{E}_M$ then $\hat{S} = \hat{\zeta}$ almost surely $\hat{P}^x$. Since $E - \hat{E}_M$ is polar we may consider $\hat{X}$ as a process on

$\hat{E}_M$. However, for notational simplicity, let us assume that $\hat{E}_M = E$ rather than keep track of the inessential polar set $\hat{E} - E_M$. If we define $\hat{A}_t = -\int_0^t (\hat{M}_{u-})^{-1}\, d\hat{M}_u$ if $t < \hat{\zeta}$ and $\hat{A}_t = \hat{A}_{\hat{\zeta}-}$ if $t \geqslant \hat{S}$, then by (8.6 i), $\hat{A}$ is a NAF of $\hat{X}$. Evidently $\hat{U}_A^\alpha(dy, x) = \mu(dy)u^\alpha(y, x)$. Thus if $\langle f, g\rangle = \int fg d\mu$ we have

$$(9.3) \qquad \langle f, U_A^\alpha g\rangle = \langle f\hat{U}_A^\alpha, g\rangle$$

for $f, g \in \mathcal{E}_+$. We call A and $\hat{A}$ dual additive functionals.

Suppose we begin with a *continuous* additive functional A of X, and assume that $t \to A_t$ is finite on $[0, \zeta)$. Then $M_t = e^{-A_t}$ is a continuous MF of X which doesn't vanish on $[0, \zeta)$ and

$$A_t = -\int_0^t (M_{u-})^{-1}\, dM_u = -\int_0^t (M_u)^{-1}\, dM_u.$$

In this case $t \to \hat{M}_t$ is continuous on $[0, \hat{\zeta})$ by (8.6 ii) recall that we are assuming $\hat{E}_M = E$ so that $\hat{S} = \hat{\zeta}$. But then $\hat{A}$ is continuous and finite on $[0, \hat{\zeta})$. Clearly $\hat{A}_t = -\log \hat{M}_t$ in this situation. The following proposition summarizes this discussion. We write $\mathbf{A}_c(X)$ for the class of continuous additive functionals of X which are finite on $[0, \zeta)$.

(9.4) Proposition. — *Let $A \in \mathbf{A}^c(X)$. Then there exists a unique σ-finite measure μ not charging semipolar sets and a unique $\hat{A} \in \mathbf{A}^c(\hat{X})$ — more precisely $\hat{A}$ is in $\mathbf{A}^c(\hat{X}|_{\hat{E}_M})$ where $\hat{X}|_{\hat{E}_M}$ denotes the restriction of $\hat{X}$ to $\hat{E}_M$ — such that*

$$U_A^\alpha(x, dy) = u^\alpha(x, y)\mu(dy); \qquad \hat{U}_A^\alpha(dy, x) = \mu(dy)u^\alpha(y, x).$$

Proposition 9.4 has been obtained earlier by Revuz [9] using different methods. See [2] for earlier work. Revuz actually characterized the measures μ arising in (9.4).

We will end this section with an example to show that the exceptional polar set $E - \hat{E}_M$ can not be eliminated in general. This is the same example as in Section 8 of [2]. Unfortunately the conclusions drawn from this example in [2] are not necessarily valid because the description of the fine topology for X on page 151 of [2] is incorrect.

Let E be the real line and ξ be Lebesgue measure. Let X be the (increasing) stable subordinator of index 1/2 and $\hat{X}$ be the corresponding decreasing stable process of index 1/2. The potential kernel is given by

$$u(x, y) = \begin{cases} c(y - x)^{-1/2} & y > x \\ 0 & y \leqslant x \end{cases}$$

where c is a positive constant. Let $h(x) = 1$ if $x \geqslant 0$ and $h(x) = (-x)^{-1/2}$ if $x < 0$. Define $A_t = \int_0^t h(X_s)\, ds$. If T_b is the hitting time of $[b, \infty)$ and $x < b$, then

$$E^x\{A_{T_b}\} = \int_x^b u(x, y) h(y)\, dy < \infty,$$

and since $T_b \uparrow \infty$ as $b \to \infty$ it follows that A_t is finite. Thus $A \in \mathbf{A}^c(X)$ and obviously the measure μ in (9.4) is given by $\mu(dy) = h(y)\, dy$. Let $\hat{A}_t = \int_0^t h(\hat{X}_s)\, ds$ and $\hat{T} = \inf\{t : \hat{A}_t = \infty\}$. Then according to (5.5),

$$\hat{M}_t = I_{[0,\hat{T})}(t) e^{-\hat{A}_t}.$$

The same calculation as above shows that $\hat{P}^x(\hat{T} = \infty) = 1$ if $x \neq 0$, but $\hat{P}^0(\hat{T} = 0) = 1$. Consequently $E - \hat{E}_M = \{0\}$ and $\hat{A}$ is the dual of A. Therefore the exceptional polar set can not be eliminated in (9.4).

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