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ABSOLUTELY CONVEX SETS IN BARRELLED SPACES (¹)

by Manuel VALDIVIA

We present here some new theorems on increasing sequences of absolutely convex sets in barrelled spaces. Some extensions of well known theorems are given :

a) *If* F *is a subspace, of finite codimension, of a barrelled space* E, *then* F *is barrelled* [*J. Dieudonné*, 3].

b) *If* F *is a subspace, of finite numerable codimension, of a metrizable barrelled space* E, *then* F *is barrelled.* [*I. Amemiya and Y. Komura*, 1].

Our theorem 3 contains the proof that if F is a infinite numerable codimensional subspace of a barrelled space E, then F is barrelled.

c) *If* E *is a metrizable barrelled space, then it is not the union of an increasing sequence of closed, nowhere dense and absolutely convex sets.* [*I. Amemiya and Y. Komura*, 1].

Our theorem 4 allows us to substitute in *c*) the condition that E is a metrizable barrelled space by the condition that E is barrelled and its completion $\hat{E}$ is a Baire space.

Finally we give a theorem on closed operators related with some results obtained by G. Köthe, [5].

The vector spaces used here are defined on the field K of the real or complex numbers.

The modification of a method used by G. Köthe to prove a

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property of Cauchy filters in a numerable inductive limit space [4, p. 162] allows us to give the following lemma.

LEMMA 1. — *Let* $\{A_n\}_{n=1}^{\infty}$ *be an increasing sequence of absolutely convex sets of the barrelled space* E, *such that* $A = \bigcup_{n=1}^{\infty} A_n$ *is absorbing. Let* $\mathcal{F}$ *be a Cauchy filter in* A *and* $\mathcal{G}$ *is the filter which basis is constituted by all sets of the form* $M + U$, *where* M *varies in* $\mathcal{F}$ *and* U *in the filter of the neighborhoods of the origin in* E, *then there exists a positive integer* n_0 *such that* $\mathcal{G}$ *induces a filter in* $2A_{n_0}$.

Proof. — If the lemma is false there exists a decreasing sequence $\{W_n\}_{n=1}^{\infty}$ of absolutely convex neighborhoods of the origin and a sequence $\{M_n\}_{n=1}^{\infty}$ of elements of $\mathcal{F}$, such that $M_n - M_n \subset W_n$ and $(M_n + W_n) \cap 2A_n = \emptyset$, $n = 1, 2, 3, \ldots$ If V is the convex hull of $\bigcup_{n=1}^{\infty} \frac{1}{2}(W_n \cap A_n)$, then V is absorbing, since for each $x \in E$ there exists a $\lambda \in K$, $\lambda \neq 0$ such that $\lambda x \in \frac{1}{2} A$, since A is an absorbing set. Therefore, there exists an integer n_1, such that $\lambda x \in \frac{1}{2} A_{n_1}$. Since there exists a real number μ in the open interval $]0, 1[$ such that $\mu(\lambda x) \in \frac{1}{2} W_{n_1}$, then $\mu\lambda x \in \frac{1}{2}(W_{n_1} \cap A_{n_1}) \subset V$. If V_n is the convex hull of

$$\frac{1}{2}(W_1 \cap A_1) \cup \frac{1}{2}(W_2 \cap A_2) \cup \cdots \cup \frac{1}{2}(W_{n-1} \cap A_{n-1}) \cup \frac{1}{2} W_n,$$

then V_n is a neighborhood of the origin and, clearly, $V \subset V_n$. If $\tilde{V}$ and $\tilde{V}_n$ are the closures of V and V_n, respectively, then $\frac{1}{2}\tilde{V} \subset \frac{1}{2}\tilde{V}_n \subset V_n$. Since V is absorbing and absolutely convex, $\tilde{V}$ is a barrel, and therefore, a neighborhood of the origin. Hence there exists an element $P \in \mathcal{F}$ such that $P - P \subset \frac{1}{2}\tilde{V}$.

Let $P_n \in \mathcal{F}$ be an element such that $P_n - P_n \subset V_n$, let us prove that $(P_n + V_n) \cap A_n = \varnothing$. If $x_0 \in P_n \cap M_n \in \mathcal{F}$ and $y \in P_n$, then $y - x_0 \in P_n - P_n \subset V_n$, which implies

$$y = x_0 + \sum_{p=1}^{n} a_p x_p, \quad \text{with} \quad a_p \geqslant 0,\ p = 1, 2, \ldots, n,\ \sum_{p=1}^{n} a_p = 1$$

$$x_p \in \frac{1}{2}(W_p \cap A_p),\ p = 1, 2, \ldots, n-1,\ x_n \in \frac{1}{2} W_n.$$

If $z \in P_n + V_n$ we can write

$$z = x_0 + \sum_{p=1}^{n} a_p x_p + \sum_{p=1}^{n} b_p y_p,$$

$$\text{with} \quad b_p \geqslant 0, \quad p = 1, 2, \ldots, n, \ \sum_{p=1}^{n} b_p = 1,$$

$$y_p \in \frac{1}{2}(W_p \cap A_p), \quad p = 1, 2, \ldots, n-1, \quad y_n \in \frac{1}{2} W_n.$$

Clearly

$$\sum_{p=1}^{n-1} a_p x_p \in \frac{1}{2} A_{n-1}, \qquad \sum_{p=1}^{n-1} b_p y_p \in \frac{1}{2} A_{n-1},$$

and hence

$$\sum_{p=1}^{n-1} a_p x_p + \sum_{p=1}^{n-1} b_p y_p = v \in \frac{1}{2} A_{n-1} + \frac{1}{2} A_{n-1} = A_{n-1} \subset A_n.$$

Furthermore $a_n x_n \in \frac{1}{2} W_n$, $b_n y_n \in \frac{1}{2} W_n$ and therefore

$$a_n x_n + b_n x_n \in \frac{1}{2} W_n + \frac{1}{2} W_n \subset W_n.$$

Since $x_0 \in M_n$ we obtain $x_0 + a_n x_n + b_n y_n \in M_n + W_n$, which not intersects $2A_n$, and therefore

$$(1) \qquad x_0 + a_n x_n + b_n y_n \notin 2A_n,$$

If $z \in A_n$ then $x_0 + a_n x_n + b_n y_n = z - v \in A_n + A_n = 2A_n$ which contradicts (1). Hence $(P_n + V_n) \cap A_n = \varnothing$.

If $z_0 \in P$, there exists a positive integer n_2 such that $z_0 \in A_{n_2}$. If w is an arbitrary element of P_{n_2} then $z_0 \notin w + V_{n_2}$, i.e. $z_0 - w \notin V_{n_2}$. Since $\frac{1}{2} \tilde{V} \subset V_{n_2}$ we obtain

$z_0 - w \notin \frac{1}{2}\tilde{V}$. Furthermore, $P - P \subset \frac{1}{2}\tilde{V}$ and $z_0 \in P$ and hence we deduce that $w \notin P$, i.e. $P \cap P_{n_2} = \emptyset$ which is clearly non true and the lemma has been proved.

THEOREM 1. — *If the barrelled space* E *has an increasing sequence* $\{W_n\}_{n=1}^{\infty}$ *of complete and absolutely convex sets, such that* $W = \bigcup_{n=1}^{\infty} W_n$ *is absorbing, then* E *is complete.*

Proof. — For each positive integer n, let $A_n = n.W_n$. Then the sequence $\{A_n\}_{n=1}^{\infty}$ is increasing and all elements are absolutely convex complete sets. Since $\bigcup_{n=1}^{\infty} A_n = E$ the conditions of the above given Lemma are satisfied and therefore if $\mathcal{F}$ is a Cauchy filter in E, there exists a positive integer n_0, such that the filter $\mathcal{G}$ described in this Lemma, induces in $2A_{n_0}$ a filter $\mathcal{G}'$. Obviously $\mathcal{G}'$ is a Cauchy filter and therefore there exists a $x \in 2A_{n_0}$ such that $\mathcal{G}'$ converges to x and also $\mathcal{F}$ converges to x.

THEOREM 2. — *Let* $\{W_n\}_{n=1}^{\infty}$ *be an increasing sequence of absolutely convex sets in the barrelled space* E, *such that* $E = \bigcup_{n=1}^{\infty} W_n$. *If* U *is an absorbing absolutely convex set, such that* $U \cap W_n$ *is closed in* W_n, $n = 1, 2, \ldots,$ *then* U *is a neighborhood of the origin in* E.

Proof. — Let us define $A_n = U \cap W_n$, $n = 1, 2, \ldots,$ and let x be an arbitrary element of $\tilde{U}$, closure of $U = \bigcup_{n=1}^{\infty} A_n$ in E. Clearly the sequence $\{A_n\}_{n=1}^{\infty}$ satisfies the conditions of Lemma 1. Hence if $\mathcal{F}$ is a Cauchy filter in U, converging toward x, it is possible to find a positive integer n_0 such that the filter $\mathcal{G}$ described in Lemma 1, induces a filter $\mathcal{G}'$ in $2A_{n_0}$. It is immediate that n_0 can be taken in such a way that $x \in 2W_{n_0}$. Since A_{n_0} is closed in W_{n_0}, then $2A_{n_0}$ is closed in $2W_{n_0}$, hence $x \in 2A_{n_0} \subset 2U$, i.e. $\tilde{U} \subset 2U$. Since $\tilde{U}$ is a barrel in E, then 2U is a neighborhood of the origin in E, and teh theorem is proved.

COROLLARY 1.2. — *Let* $\{E_n\}_{n=1}^{\infty}$ *be an increasing sequence of subspaces of the barrelled space* E, *such that* $E = \bigcup_{n=1}^{\infty} E_n$. *If* U *is a set such that* $U \cap E_n$, $n = 1, 2, \ldots$, *is a barrel in* E_n, *then* U *is a neighborhood of the origin in* E.

Proof. — This is a special case of Theorem 2, taking $E_n = W_n$, $n = 1, 2, \ldots$

M. de Wilde has proved [2] the following result:

d) *Let* E *be a locally convex Haussdorff space and* F *a subspace of* E *with finite codimension. If* T *is a barrel in* F, *there exists a barrel* T′ *in* E *such that* $T' \cap F = T$.

The proof given M. de Wilde of result *d*) is still true, with a small modification, in the case that E is not a Hausdorff space.

THEOREM 3. — *If* E_1 *is a subspace, with numerable infinite codimension, of the barrelled space* E, *then* E_1 *is barrelled.*

Proof. — Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a numerable infinite set of vectors, such that $E_1 \cup \{x_1, x_2, \ldots, x_n, \ldots\}$ generates E. Let us denote by E_n the space generated by

$$E_1 \cup \{x_1, x_2, \ldots, x_{n-1}\}, \quad n = 2, 3, \ldots$$

Let U_1 be a barrel in E_1. Since E_1 has finite codimension in E_2 there exists (result *d*)) a barrel U_2 in E_2, such that $U_2 \cap E_1 = U_1$. Following this method we can go on and if a barrel U_n in E_n is constructed we can find a barrel U_{n+1} in E_{n+1} such that $U_{n+1} \cap E_n = U_n$. If $U = \bigcup_{n=1}^{\infty} U_n$, then

$$U \cap E_n = \left(\bigcup_{p=1}^{\infty} U_p\right) \cap E_n = \left(\bigcup_{p=n}^{\infty} U_p\right) \cap E_n = U_n, \quad n = 1, 2, \ldots,.$$

Hence, using Corollary 1.2, U_1 is a neighborhood of the origin in E_1. q.e.d.

THEOREM 4. — *Let* E *be a barrelled space such that it has a completion* $\hat{E}$ *which is a Baire space. Let* $\{U_n\}_{n=1}^{\infty}$ *be an increasing sequence of absolutely convex closed sets in* E, *such that for any* n, U_n *has no inner point, then* $U = \bigcup_{n=1}^{\infty} U_n$ *is not absorbing.*

Proof. — If U is absorbing, then if we define $A_n = nU_n$, the increasing sequence $\{A_n\}_{n=1}^{\infty}$ is such that $E = \bigcup_{n=1}^{\infty} A_n$. Let x be an arbitrary element of $\hat{E}$ and $\mathcal{F}$ a filter in E converging to x. Since $\mathcal{F}$ is a Cauchy filter there exists a positive integer n_0, such that the filter $\mathcal{G}$ described in Lemma 1, induces a Cauchy filter $\mathcal{G}'$ in $2A_{n_0}$, converging to x, as can be proved easily. Hence if $\overline{A_{n_0}^*}$ is the closure of A_{n_0} in $\hat{E}$, then $x \in \overline{2A_{n_0}^*}$ and therefore $\hat{E} = \bigcup_{n=1}^{\infty} \overline{2A_n^*}$. Since E is a Baire space, there exists a positive number n_1 such that $\overline{2A_{n_1}^*}$ has an inner point in $\hat{E}$, hence $2A_{n_1} \cap E = 2_{n_1}U_{n_1}$ has an inner point in E, which contradicts our starting hypothesis. q.e.d.

Corollary 1.4. — *Let* E *be a barrelled space with a completion* $\hat{E}$ *which is a Baire space. Let us assume that there exists a numerable sequence* $\{B_n\}_{n=1}^{\infty}$ *of bounded sets, such that* $\bigcup_{n=1}^{\infty} B_n = E$, *then* E *is a seminormed space.*

Proof. — Let us denote by A_n the absolutely convex and closed hull in E of $B_1 \cup B_2 \cup \cdots \cup B_n$, $n = 1, 2, \ldots$ The increasing sequence of bounded sets $\{A_n\}_{n=1}^{\infty}$ is such that $\bigcup_{n=1}^{\infty} A_n = E$. Using Theorem 4, there exists a positive integer n_0 such that A_{n_0} has an inner point, hence E is seminormed.

Corollary 2.4. — *Let* E *be a barrelled Hausdorff space, which completion* $\hat{E}$ *is a Baire space. If* E_1 *is a closed subspace of* E *with codimension at most numerable, then this codimension is finite.*

Proof. — If the codimension of E_1 is infinite there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of linearly independent vectors such that the space generated by them is the algebraic complementary of E_1. Let E_n be the space generated by

$$E_1 \cup \{x_1, x_2, \ldots, x_{n-1}\},\ n = 2, 3, \ldots$$

Since E_1 is closed, E_2 is also closed and in general E_n, $n = 1, 2, \ldots$ are closed subspaces. Each of the sets of the sequence $\{E_n\}_{n=1}^{\infty}$ is closed, absolutely convex and without inner point. Theorem 4 contradicts the fact that $\bigcup_{n=1}^{\infty} E_n = E$.

THEOREM 5. — *Let* $\{U_n\}_{n=1}^{\infty}$ *be an increasing sequence of absolutely convex sets in the barrelled space* E, *such that* $E = \bigcup_{n=1}^{\infty} U_n$. *If* V *is an absolutely convex set such that, for each positive integer* n, $V \cap U_n$ *is a neighborhood of* 0 *in* U_n, *with the topology induced by* E, *then* V *is a neighborhood of the origin in* E.

Proof. — Given a positive n it is possible to find a neighborhood of the origin in E, W_n, absolutely convex, such that $W_n \cap U_n \subset V \cap U_n$.

If $(V \cap U_n)^{\star}$ denote the closure of $V \cap U_n$ in U_n and x is an arbitrary element of $(V \cap U_n)^{\star}$, then

$$\varnothing \neq [(x + W_n) \cap U_n] \cap (V \cap U_n).$$

Hence an element z exists in the last set. Since $x \in U_n$ and $z \in U_n$, then $z - x \in 2U_n$. Furthermore $z \in x + W_n$, hence $z - x \in W_n \subset 2W_n$ and therefore

$$z - x \in (2W_n) \cap (2U_n) = 2(W_n \cap U_n) \subset 2(V \cap U_n).$$

Since $z \in V \cap U_n$ then $x \in 3(V \cap U_n)$, i.e. $(V \cap U_n)^{\star} \subset 3(V \cap U_n)$, (2).

If u is an arbitrary element of E there exists a positive integer n_1, such that $u \in U_{n_1}$. Furthermore there exists a real number $\lambda \in]0, 1[$, such that $\lambda u \in W_{n_1}$, hence $\lambda u \in W_{n_1} \cap U_{n_1} \subset V \cap U_1 \subset V$, i.e. V is absorbing. If $A_n = \frac{1}{2}(V \cap U_n)$, then $\bigcup_{n=1}^{\infty} A_n = \frac{1}{2} V$ and, therefore, the sequence $\{A_n\}_{n=1}^{\infty}$ satisfies the conditions of Lemma 1. If v is arbitrary element of $\frac{1}{2}\tilde{V}$, closure of $\frac{1}{2}V$, there exists a filter $\mathcal{F}$ in $\frac{1}{2}V$ converging to v, and therefore, there exists a positive integer n_0 such that the filter $\mathcal{G}$ described in Lemma 1, induces a filter $\mathcal{G}'$ in $2A_{n_0}$ converging to v. It is clear that n_0 can be taken in such a way that $v \in U_{n_0}$. Then, taking into account relation (2), we obtain

$$v \in (2A_{n_0})^* = (V \cap U_{n_0})^* \subset 3(V \cap U_{n_0}),$$

hence $\frac{1}{2}\tilde{V} \subset \bigcup_{n=1}^{\infty} 3(V \cap U_n) = 3V$. Since $\frac{1}{6}\tilde{V}$ is a barrel, V is a neighborhood of the origin in E. q.e.d.

The two next corollaries are immediate consequences of Theorem 5.

Corollary 1.5. — *Let $\{E_n\}_{n=1}^{\infty}$ be an increasing sequence of subspaces of the barrelled space $E = \bigcup_{n=1}^{\infty} E_n$ then E is the strict inductive limit of this sequence.*

Corollary 2.5. — *Let $\{W_n\}_{n=1}^{\infty}$ be an increasing sequence of absolutely convex sets of the barrelled space E, such that $\bigcup_{n=1}^{\infty} W_n = E$. If g is a linear mapping of E in the locally convex space F, then g is continuous if and only if its restriction to W_n, $n = 1, 2, \ldots$ is continuous.*

Theorem 6. — *Let $\{U_n\}_{n=1}^{\infty}$ be an increasing sequence of closed absolutely convex sets in the barrelled space E, such that $\bigcup_{n=1}^{\infty} U_n = E$. Let $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of non-zero real numbers, such that $\lim_n \lambda_n = \infty$. Then, if B is an arbitrarely given bounded set, there exists a positive integer n_0 such that $B \subset \lambda_{n_0} U_{n_0}$.*

Proof. — For each positive integer n, if B is not contained in $\lambda_n U_n$, it will be possible to find a $x_n \in B$ such that $x_n \notin \lambda_n U_n$ and hence $y_n = \frac{1}{\lambda_n} x_n \notin U_n$. Since U_n is closed it is possible to find and absolutely convex neighborhood of the origin such that $(y_n + V_n) \cap U_n = \emptyset$, hence

$$\left(y_n + \frac{1}{2} V_n\right) \cap \left(U_n + \frac{1}{2} V_n\right) = \emptyset,$$

which implies $y_n \notin U_n + \frac{1}{2} V_n$. The set $U_n + \frac{1}{2} V_n$ is an absolutely convex neighborhood of the origin and therefore $U = \bigcap_{n=1}^{\infty} \left(U_n + \frac{1}{2} V_n\right)$ is absolutely convex. The sequence $\{y_n\}_{n=1}^{\infty}$ converges to 0 and all their terms are not contained in U. If we prove that U is a neighborhood of the origin we have a contradiction and the theorem is proved. Let p be an arbitrary positive integer, then

$$U_p \cap U = U_p \cap \left[\bigcup_{n=1}^{\infty}\left(U_n + \frac{1}{2}\right)V_n\right] = U_n \cap \left[\bigcap_{n=1}^{p=1}\left(U_n + \frac{1}{2} V_n\right)\right].$$

Since $\bigcap_{n=1}^{p=1} \left(U_n + \frac{1}{2} V_n\right)$ is a neighborhood of the origin, then $U_p \cap U$ is a neighborhood of 0 in U_p and applying Theorem 5 we can prove that U is a neighborhood of the origin in E.

q.e.d.

COROLLARY 1.6. — *Let* $\{U_n\}_{n=1}^{\infty}$ *be an increasing sequence of closed absolutely convex sets in the barrelled space* E, *such that* $\bigcup_{n=1}^{\infty} U_n = E$, *and let* E_n *be the subspace generated by* U_n, $n = 1, 2, \ldots$ *Then, if* B *is an arbitrary bounded set, there exists a positive integer* n_0, *such that* $B \subset E_{n_0}$.

Proof. — In the proof of the Theorem 6 consider $\lambda_n = n$, then is enough to notice that $nU_n \subset E_n$, $n = 1, 2, \ldots$ to prove this corollary.

COROLLARY 2.6. — *If the barrelled space* E *has a numerable family* $\{B_n\}_{n=1}$ *of bounded sets, such that* $E = \bigcup_{n=1}^{\infty} B_n$, *then* E *is a* (DF)*-space.*

Proof. — It is enough to prove that E has a fundamental numerable system of bounded sets. Up to this let us take an arbitrary positive integer n and let U_n be the closed absolutely convex hull of $B_1 \cup B_2 \cup \cdots \cup B_n$. Using Theorem 6 with $\lambda_n = n, n = 1, 2, \ldots$, we can prove that given a bounded set B there exists a positive integer n_0 such that $B \subset n_0 U_{n_0}$. This implies that the increasing sequence $\{nU_n\}_{n=1}^{\infty}$ of bounded sets is fundamental.

G. Köthe, [5], has proved the two following results:

e) Let E *a* B*-complete space,* F *a barrelled space of Hausdorff and* T *a closed lineal operator of a part of* E *into* F*, such that* TE *is of finite codimension. Then* T *is an open operator and* TE *is closed.*

f) Let E *be a* B*-complete space,* F *a metrizable barrelled space and* T *a closed operator of a part of* E *into* F*, such that* TE *is of codimension at most numerable. Then* T *is an open operator and* TE *is closed and of finite codimension.*

The result *e*) is still true if we substitute the condition that TE has finite codimension by the condition that TE has at most numerable codimension. The same proof of G. Köthe can be used taking into account not only the result *a*) of Dieudonné but also our Theorem 4.

The result *f*) is still true if the condition that F is a barrelled metrizable space is substituted by the condition that is a barrelled Hausdorff space which completion is a Baire space. The G. Köthe's proof must be changed since our Theorem 4 is needed in substitution of the results *c*) of I. Amemiya and Y. Komura.

An extension of result *e*) is the following.

THEOREM 7. — *Let* E *be the inductive limit of an increasing sequence* $\{E_n\}_{n=1}^{\infty}$ *of* B*-complete spaces and* F *a barrelled space of Hausdorff. Let* u *be a linear mapping of a subspace* L *of* E *into* F*, with closed graph in* $E \times F$*, and* $u(L \cap E_1)$ *a subspace of* F *with codimension at most numerable, then* u *is open and* $u(L)$ *is closed in* F.

Proof. — For all n positive integer $u(E_n \cap L)$ is of codimension in F at most numerable and therefore is a barrelled space. The same is true for $u(L)$.

If u_n is the restriction of u to $E_n \cap L$, its graph is closed in $E_n \times u_n(E_n \cap L)$ and since E_n is B-complete then u_n is open.

The u mapping is open since $u(L)$ is the strict inductive limit of the sequence $\{u(E_n \cap L)\}_{n=1}^{\infty}$, (Corollary 1.5).

The graph of u_n is closed in $E_n \times F$. Using the same type of arguments given by G. Köthe, [5], it is possible to prove that $u(E_n \cap L) = u_n(E_n \cap L)$ is closed in F.

Let us prove, now, that $u(L)$ is closed in F. Let $\mathcal{F}$ be a basis of filter in $u(L)$ converging toward an element $x \in F$. Such basis of filter is of Cauchy and therefore there exists, [4, p. 162], a positive integer n, such that if $\mathcal{G}$ is the basis of filter constructed with all sets $M + V$, where M varies on $\mathcal{F}$ and V in the filter of the neighborhoods of the origin in $u(L)$, $\mathcal{G}$ induces a basis of filter $\mathcal{H}$ in $u(E_n \cap L)$. Obviously $\mathcal{G}$ converges to x and $\mathcal{H}$ converges to x. Hence $x \in u(E_n \cap L) \subset u(L)$, i.e. $u(L)$ is closed in F. q.e.d.

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