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SOME RESULTS ON KRONECKER, DIRICHLET AND HELSON SETS

by Thomas William KÖRNER

1. Introduction.

We start with a short general discussion. The reader seeking more information should read [8], [9] and ([13] Chap. 5). Moreover anyone acquainted with the field should simply skim through this section which contains only definitions and well known consequences. We work on the circle $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ (where $\mathbf{R}$ is the additive group of real numbers, $\mathbf{Z}$ the subgroup of integers). As usual we define the *characters* χ_m by $\chi_m(x) = e^{imx}$ for $x \in \mathbf{T}$. $\mathrm{C}(\mathbf{T})$ is the set of continuous functions $f\colon \mathbf{T} \to \mathbf{C}$ (where $\mathbf{C}$ is the set of complex numbers). What we shall be concerned with is the possibility of approximating members of the set $\mathrm{S} = \{f \in \mathrm{C}(\mathbf{T}) : |f(x)| = 1 \text{ for all } x \in \mathbf{T}\}$ by characters. Clearly this is only possible on « thin » subsets of $\mathbf{T}$ and it is these subsets we shall study.

A closed set $\mathrm{E} \subseteq \mathbf{T}$ is called *Kronecker* if for every $g \in \mathrm{S}$, $\varepsilon > 0$ we can find an n such that $|g(x) - \chi_n(x)| \leqslant \varepsilon$ for all $x \in \mathrm{E}$. A closed set $\mathrm{E} \subseteq \mathbf{T}$ is called *Dirichlet* if we can find an increasing sequence $n(j)$ such that

$$\sup_{x \in \mathrm{E}} |1 - \chi_{n(j)}(x)| \to 0$$

as $j \to \infty$. A set $\mathrm{E} \subseteq \mathbf{T}$ (respectively $\mathrm{E} \subseteq \mathbf{R}$) is called *independent* if given $x_1, \ldots, x_r \in \mathrm{E}$ distinct, $\sum_{q=1}^{r} m_q x_q = 0$

with $m_1, m_2, \ldots, m_r \in \mathbf{Z}$ implies $m_1 = m_2 = \cdots = m_r = 0$. The relations between these 3 concepts are very close and we shall use the following well known facts without comment.

LEMMA 1.1. — *(i) Kronecker's Theorem: If $x_1, \ldots, x_r \in \mathbf{T}$ are independent, then given any $\lambda_1, \ldots, \lambda_r \in \mathbf{C}$ with $|\lambda_1| = \cdots = |\lambda_r| = 1$, $\mathrm{N} \in \mathbf{Z}$, $\varepsilon > 0$ we can find an $n \geqslant \mathrm{N}$ such that $\sup\limits_{1 \leqslant q \leqslant r} |\chi_n(x_q) - \lambda_q| \leqslant \varepsilon$. In particular every finite independent set is Kronecker.*

(ii) Dirichlet's Theorem: Any finite set is Dirichlet.

Proof. — These results are classical ([3] Chap. 3, § 2 and Chap. 1, § 5).

LEMMA 1.2. — *(i) Every Kronecker set is independent.*

(ii) Every Kronecker set is Dirichlet.

(iii) If E is a Kronecker set, $f \in \mathrm{S}$, $\varepsilon > 0$, $n_0 \in \mathbf{Z}$ we can find an $n \geqslant n_0$ such that $\sup\limits_{x \in \mathrm{E}} |\chi_n(x) - f(x)| \leqslant \varepsilon$.

Proof. — (i) Suppose $\sum\limits_{q=1}^{r} m_q x_q = 0$ and

$$\sup_{1 \leqslant q \leqslant r} |\chi_{n(j)}(x_q) - f(x_q)| \to 0$$

as $j \to \infty$. Then

$$1 = \chi_{n(j)}(0) = \chi_{n(j)}\left(\sum_{q=1}^{r} m_q x_q\right) = \prod_{q=1}^{r} [\chi_{n(j)}(x_q)]^{m_q} \to \prod_{q=1}^{r} [f(x_q)]^{m_q}$$

and so $1 = \prod\limits_{q=1}^{r} [f(x_q)]^{m_q}$.

(ii) Take $c_r = \chi_1(2^{-(r+2)}\pi)$. If E is Kronecker, then for each $r \geqslant 1$ we can find an $n(r)$ such that

$$\sup_{x \in \mathrm{E}} |\chi_{n(r)}(x) - c_r| \leqslant 1/2|c_r - c_{r-1}|.$$

Clearly $\chi_{n(1)}, \chi_{n(2)}, \ldots$ are distinct and $\sup\limits_{x \in \mathrm{E}} |\chi_{n(r)}(x) - 1| \to 0$ as $r \to \infty$. We have thus $|n(r)| \to \infty$ and

$$\sup_{x \in \mathrm{E}} |\chi_{|n(r)|}(x) - 1| = \sup_{x \in \mathrm{E}} |\chi_{n(r)}(x) - 1| \to 0$$

as $r \to \infty$.

(*iii*) This follows from (*ii*).

We shall use the following trivial Lemma repeatedly.

LEMMA 1.3. — *If* $R = \{x_m : m \in \mathbf{Z}\}$ *is independent*, $E \subseteq \mathbf{T}$ (*respectively* $E \subseteq \mathbf{R}$) *uncountable, then there exists a* $y \in E \setminus R$ *such that* $\{y\} \cup R$ *is independent.*

Proof. — We prove the result for $E \subseteq \mathbf{T}$. The proof for $E \subseteq \mathbf{R}$ is similar. Let $T = \left\{ \sum_{-\infty}^{\infty} q_m x_m : q_m \neq 0 \right.$ only finitely often, $\left. q_m \in 2\pi\mathbf{Q} \right\}$ (where $\mathbf{Q}$ is the set of rationals). Then T is countable. In particular $E \setminus T \neq \varnothing$ and we may choose for y any $z \in E \setminus T$.

In stating our results we shall be mainly concerned with countable and perfect sets. The reader may therefore find it useful to recall that,

LEMMA 1.4. — *The following conditions on* $E \subseteq \mathbf{T}$ *are equivalent:*

(*i*) E *is perfect and totally disconnected*;

(*ii*) E *with the subspace topology is homeomorphic to* $D_\infty = \prod_1^\infty D_2$ (*where* D_2 *is the group of* 2 *elements with the discrete topology*);

(*iii*) *There exist* $\mathfrak{T}_1, \mathfrak{T}_2, \ldots$ *finite collections of disjoint (closed) intervals such that setting* $P_m = \cup \{I : I \in \mathfrak{T}_m\}$ *we have* $P_m \supseteq P_{m+1}$ *for* $m \geqslant 1$, $\max\{\operatorname{diam} I : I \in \mathfrak{T}_m\} \to 0$ *as* $m \to \infty$, *card*$\{I \in \mathfrak{T}_m : I \subseteq J\} \to \infty$ *as* $m \to \infty$ *for all* $J \in \mathfrak{T}_n$, *and* $E = \bigcup_{m=1}^{\infty} P_m$.

Proof. — Standard. ([7] § 9.15 prove a more general result).

Any set E satisfying the conditions of *Lemma* 1.4 is called a *Cantor set.* It seems worth remarking that

LEMMA 1.5. — *If* E *is a closed independent set, then* E *is the union of a countable set and a Cantor set.*

Proof. — $E \cap 2\pi\mathbf{Q} = \varnothing$ so E is totally disconnected. Since by the Theorem of Cantor-Bendixson ([6] § 27) every closed set is the union of a perfect and a countable set, the result follows.

Corresponding to the strong notions of Kronecker and Dirichlet sets there exist weak ones. Let $M(\mathbf{T})$ be the set of measures on $\mathbf{T}$ and $M^+(\mathbf{T})$ the subset of positive measures. We say that a closed set $E \subseteq \mathbf{T}$ is *weak Kronecker* (respectively *weak Dirichlet*) if for all $\mu \in M^+(\mathbf{T})$, $\varepsilon > 0$, $\eta > 0$ given $f \in S$ we can find an n such that $\mu\{x \in E : |\chi_n(x) - f(x)| \geqslant \varepsilon\} \leqslant \eta$ (respectively given n_0 we can find an $n \geqslant n_0$ such that $\mu\{x \in E : |\chi_n(x) - 1| \geqslant \varepsilon\} \leqslant \eta$). It should be stressed at this point that Wik [15] uses entirely different definitions. We have, however, followed Kahane [8] and Varopoulos [17].

At the risk of stating the obvious, we remark

Lemma 1.6. — *(i) If μ is a measure supported on a closed countable set E then for any $\varepsilon > 0$ we can find F a finite subset of E such that $W \subseteq E \setminus F$ implies $|\mu(W)| \leqslant \varepsilon$.*

(ii) If E is closed independent and countable, then E is weak Kronecker.

(iii) If E is weak Kronecker, then E is independent.

(iv) Every weak Kronecker set is weak Dirichlet.

(v) If E is a Kronecker set $\mu \in M^+(\mathbf{T})$, $f \in S$, $\varepsilon > 0$, $\eta > 0$, $n_0 \in \mathbf{Z}$ we can find an $n \geqslant n_0$ such that

$$\mu\{x \in E : \sup_{x \in E} |\chi_n(x) - f(x)| \geqslant \varepsilon\} \leqslant \eta.$$

Proof. — (i) We have $\sum_{x \in E} |\mu(e)| = \|\mu\| < \infty$. Selecting $e_1, \ldots, e_r \in E$ distinct such that $\sum_{q=1}^{r} |\mu(e_q)| \geqslant \|\mu\| - \varepsilon$ and setting $F = \{e_1, \ldots, e_r\}$ we have the required result.

(ii) now follows from *Lemma* 1.1.

(iii) Suppose $e_1, \ldots, e_r$ distinct points of E with associated δ-measures $\delta_1, \ldots, \delta_r$ (so $\int g\,d\delta_q = g(e_q)$ for $g \in C(\mathbf{T})$, $1 \leqslant q \leqslant r$). Let $F = \{e_1, \ldots, e_r\}$ and $\mu = \sum_{q=1}^{r} \delta_q$. We have that $\mu \in M^+(\mathbf{T})$ and μ is supported by $F \subseteq E$. Thus given $\varepsilon > 0$, $f \in S$ we can find an n such that

$$\mu\{x \in E : |\chi_n(x) - f(x)| \geqslant \varepsilon\} \leqslant 1/2,$$

i.e. $\sup_{x \in E} |\chi_n(x) - f(x)| \leqslant \varepsilon$. Thus F is Kronecker and so independent.

(*iv*) The proof proceeds as in *Lemma* 1.2 (*ii*).

(*v*) We can find a closed $E_1 \subseteq E$, $n_1 \in \mathbf{Z}$ such that $\mu(E \setminus E_1) \leqslant \eta/2$ and $\sup_{x \in E_1} |\chi_{n_1}(x) - f(x)| \leqslant \varepsilon/2$. We can find a closed $E_2 \subseteq E$, $n_2 \geqslant n_0 - n_1$ such that $\mu(E \setminus E_2) \leqslant \eta/2$ and $\sup_{x \in E_2} |\chi_{n_2}(x) - 1| \leqslant \varepsilon/2$. Now $\mu(E \setminus (E_1 \cap E_2)) \leqslant \eta$ and for $x \in E_1 \cap E_2$, $n = n_1 + n_2$,

$$\begin{aligned} |\chi_n(x) - f(x)| &\leqslant |\chi_n(x) - \chi_{n_1}(x)| + |\chi_{n_1}(x) - f(x)| \\ &= |\chi_{n_2}(x) - 1| + |\chi_{n_1}(x) - f(x)| \\ &\leqslant \varepsilon \end{aligned}$$

whilst $n \geqslant n_0$. This is the required result.

The concepts of Dirichlet and weak Dirichlet sets are in a certain sense modifications of the older concepts of an N set and of an R set. A closed set E is called an *N set* if we can find $a_n \geqslant 0$ such that $\sum_{n=1}^{\infty} a_n$ diverges, yet $\sum_{n=1}^{\infty} a_n \sin nx$ converges absolutely on E. Salem (B. 6 [14]) defines a type of set which has similar properties. A closed set E is called an *R set* if we can find $a_n \geqslant 0$, $\xi_n \in \mathbf{R}$ such that $\limsup a_n > 0$ but $\sum_{n=1}^{\infty} a_n \cos (nx - \xi_n)$ converges pointwise on E. (N sets are so named in honour of Nemytzkii, R sets in honour of Rajchman.) We shall also use the concept of an N_0 set. A closed set E is called an *N_0 set* if there exists an infinite subset Y of $\mathbf{Z}$ with $\sum_{n \in Y} |\sin nx|$ pointwise convergent on E. Clearly an N_0 set is both an N and an R set. The reader will observe in e.g. *Theorem* 3 that R and N_0 sets appear much more amenable to our methods than N sets. A full discussion of N, N_0 and R sets may be found in ([2] Chapters XII and XIII). Usually the condition E closed is dropped, but it will be found that our results are valid in this case also.

We require the following results of which the most important are due to Salem.

LEMMA 1.7. — (*i*) *Every Dirichlet set is an* N_0 *set and so an* N *and an* R *set;*

(*ii*) *Every* R *set is weak Dirichlet* (*and so every* N_0 *set is*);

(*iii*) *Every* N *set is weak Dirichlet;*

(*iv*) *Every closed countable set* E *is* N_0 *(and so weak Dirichlet).*

Proof. — (*i*) Suppose E is a Dirichlet set. Then we can find $0 < n(1) < n(2) < \ldots$ such that $\sup_{x\in E} |\chi_{n(r)}(x) - 1| \leqslant 2^{-r}$. Setting $Y = \{n(r) : r \geqslant 1\}$ we have the result.

(*ii*) Let E be an R set. Then we can find $a_n \geqslant 0, \xi_n$ such that $\Sigma a_n \cos(nx + \xi_n)$ converges on E but we can find $0 < m(1) < m(2) < \ldots$ and $a > 0$ such that $a_{n(r)} \geqslant a$. Automatically $\cos(m(r)x + \xi_{m(r)}) \to 0$ pointwise on E. Choose any $x_0 \in E$ and set $y = x - x_0$, $\zeta_n = \xi_n + nx_0$. Then we have $\cos(m(r)y + \zeta_{m(r)}) \to 0$ as $r \to \infty$ for all $x \in E$. In particular taking $x = x_0$ we have $\cos \zeta_{m(r)} \to 0$ and so $\sin m(r)y \sin \zeta_{m(r)} = \cos m(r)y \cos \zeta_{m(r)} - \cos(m(r)y + \zeta_{m(r)}) \to 0$ whilst $\sin m(r)y \cos \zeta_{m(r)} \to 0$ as $r \to \infty$. Since

$$|\sin \zeta_{m(r)}| + |\cos \zeta_{m(r)}| \geqslant 1$$

it follows that $\sin m(r)y \to 0$ and so $\sin^2 m(r)y \to 0$ as $r \to \infty$ for all $x \in E$. Now $|\sin^2 m(r)y| \leqslant 1$ for all r and all x so by Lebesgue's theorem on dominated convergence $\int_{x\in E} \sin^2 m(r)y \, d\nu(x) \to 0$ for all $\nu \in M^+(\mathbf{T})$ whence $\int_E \sin^2 m(r)x \, d\mu(x) \to 0$ as $r \to \infty$ for all $\mu \in M^+(\mathbf{T})$. Thus taking $\varepsilon > 0$, $\eta > 0$ we can find an m such that

$$\mu\{x \in E : \sin^2 mx \geqslant \varepsilon^2/8\} \leqslant \eta$$

and so

$$\mu\{x \in E : |\chi_m(x) - 1| \geqslant \varepsilon\} \leqslant \eta.$$

E is thus weak Dirichlet.

(*iii*) The proof is quite similar to (*ii*). Suppose E is an N set. Then we can find $a_n \geqslant 0$ with $\sum_{n=1}^{\infty} a_n$ divergent, whilst $\sum_{n=1}^{\infty} a_n |\sin nx|$ and thus $\sum_{n=1}^{\infty} a_n \sin^2 nx$ are convergent on E. Setting $f_p(x) = \left(\sum_{n=1}^{p} a_n \sin^2 nx\right) / \left(\sum_{n=1}^{p} a_n\right)$ for $p \geqslant 1$ we have $0 \leqslant f_p(x) \leqslant 1$ and $f_p(x) \to 0$ as $p \to \infty$ for all $x \in E$. Hence by Lebesgue's theorem $\int_E f_p(x) \, d\mu(x) \to 0$ as $p \to \infty$.

But $\liminf_{p\to\infty} \int_E f_p(x)\, d\mu(x) \geq \liminf_{m\to\infty} \int_E \sin^2 mx \;\; d\mu(x) \geq 0$ so $\liminf_{m\to\infty} \int_E \sin^2 mx\, d\mu(x) = 0$ for all $\mu \in M^+(\mathbf{T})$ and as in (*ii*) we see that E is weak Dirichlet.

(*iv*) Let $E = \{x_1, x_2, \ldots\}$. By Dirichlet's theorem we can find an $n(1) > 0$ such that $|\chi_{n(1)}(x_1) - 1| \leq 2^{-1}$, an $n(2) > n(1)$ such that $\sup_{1\leq i\leq 2} |\chi_{n(i)}(x_i) - 1| \leq 2^{-2}$ and in general an $n(r) > n(r-1)$ such that $\sup_{1\leq i\leq r} |\chi_{n(r)}(x_i) - 1| \leq 2^{-r}$ $[r \geq 2]$. Clearly $|\sin n(r)x_s| \leq 2^{-r}$ for all $r \geq s \geq 1$ so $\sum_{r=1}^{\infty} |\sin n(r)x_s|$ converges for all $s \geq 1$, i.e. E is an N_0 set.

Finally we require the following definitions. If $\mu \in M(\mathbf{T})$ we write $\hat{\mu}(n) = \int \chi_{-n}\, d\mu$. Suppose $E \subseteq \mathbf{T}$ is a closed set. We write $M(E) = \{\mu \in M(\mathbf{T}) : \text{supp}\, \mu \subseteq E\}$. Suppose $0 \leq u \leq 1$. We say that E is *at most an* H_u *set* (a *Helson* u *set*) if for every $v > u$ we can find a $\mu \in M(E)$ with $\|\mu\| = 1$ and $\limsup_{|m|\to\infty} |\hat{\mu}(m)| \leq v$. We say that E is *at least an* H_u *set* if, for all $\mu \in M(E)$, $\limsup_{|m|\to\infty} |\hat{\mu}(m)| \geq u$. E is said to be an H_u *set* if both conditions are satisfied. E is said to be *Helson* (or *Carleson-Helson*) if it is not H_0, i.e. if there exists a $\delta > 0$ such that $\limsup_{|m|\to\infty} \left|\int \chi_m\, d\mu\right| \geq \delta \int |d\mu|$ for all measures μ with support contained in E. The reader who has not already done so, is strongly advised to look at ([9] Chapter XI) where the concept of a Helson set is put in its natural setting, but this definition is all we shall require. We note the following well known results.

LEMMA 1.8 : (*i*) *Every weak Kronecker set is* H_1.

(*ii*) *In particular every closed countable independent set is* H_1.

Proof. — (*i*) It is a standard result (e.g. following directly from § 14.12 [7]) that if $\mu \in M(\mathbf{T})$ there exists $h_r \in S$ with $\int h_r\, d\mu \to \|\mu\|$. The lemma is thus a consequence of *Lemma* 1.6 (*v*).

(*ii*) By *Lemma* 1.6 (*ii*) every closed countable independent set is weak Kronecker.

I have attempted in the introduction and elsewhere to make this paper self contained. But clearly it can only be so in the narrowest sense. In particular, well known results are not traced back to their sources which must be sought in the references (especially [2], [9] and [13]).

2. Results.

2 questions arise at once. The first asks what the relations are between the concepts defined above. It would, for example, be very interesting to know whether every weak Dirichlet set is necessarily an N set. (Yves, Björk) Often, to prevent triviality (since e.g. a Kronecker set must be independent, yet $\{\pi/4\}$ is Dirichlet), only independent sets are considered. In *Theorem* 4 we answer a question of Kahane [8] by constructing an independent Dirichlet set which is not Kronecker. Kahane has asked further in conversation whether there exist independent Dirichlet sets which are not even Helson. We conclude the paper by constructing such a set (*Theorem* 9). In *Lemma* 3.1 we construct a set which is N_0 but not Dirichlet.

In *Theorem* 3 we construct a weak Kronecker (and so weak Dirichlet) perfect set which is not an R (and so not an N_0) set. According to Bary ([2] Chapter 12, § 10) it was an open question whether a set could be weak Dirichlet and yet not an R set. But she also reports in detail work of Arbault [1] in which he shows that there exist N sets (which by *Lemma* 1.7 (*ii*) are automatically weak Dirichlet) which are not R sets. I do not know whether the fact that Arbault's construction also answers the question stated above has been generally overlooked or not. In any case we tackle the proof by entirely different methods, obtaining a stronger result (since our set is weak Kronecker and so independent). We then vary our construction to obtain, in *Lemma* 4.8, a weak Kronecker (so independent) N set which is not an R set. That the addition of independence to the conditions of the Theorem and Lemma is not entirely a trivial generalisation is best seen by noting that Arbault's proof of his result depends on the lack of independence in the set constructed.

Wik has shown, effectively, that there exist weak Kronecker sets which are not Dirichlet ([15] Theorem 2). In *Theorem* 2 we construct a perfect non Dirichlet set every proper closed subset of which is Kronecker. Wik's result appears as a consequence (*Corollary* 2.2). In *Theorem* 5 we construct a perfect Dirichlet non Kronecker set every proper closed subset of which is Kronecker. In *Corollary* 5.2 we see that there exist weak Kronecker sets which are Dirichlet but not Kronecker. Although the 2 proofs of Wik's result spring from different ideas (his being a modification of a theorem on the Hausdorff measures of Kronecker sets, ours of a theorem on the union of 2 Kronecker sets) the constructions used turn out to be fairly similar. On the other hand our method also gives the result of *Corollary* 5.2 which does not seem as accessible by his methods. At the beginning of Section 6 we give an alternative proof of Wik's result, obtaining it as a consequence of the existence of countable independent closed sets which are not Kronecker.

The second question concerns the union problem. What can we say about the finite union or the countable union of particular types of set? In particular, is it a set of the same type? Again, to prevent triviality, we often add the condition that the sets be disjoint and their union independent. (As an example of the difference this may make, recall that a closed countable independent set is necessarily Helson (*Lemma* 1.8 (*ii*)) but that a closed countable set need not be (direct consequence of Theorem VIII of [9] Chapter XII)). The most important unanswered question in this direction asks whether the union of 2 Helson sets is necessarily Helson (Yes, Drury and Varopoulos).

The 2 most powerful techniques for solving these problems turn out to be probabilistic ([9] Chapter VIII and elsewhere) and functional analytic (this technique owes a great deal to an idea of Kaufman [11]). Typically the results obtained tell us that « with probability 1 or « quasi-always » certain sets have a required property. In contrast to these non constructive proofs we shall obtain our results by the direct construction of suitable examples.

Bernard and Varopoulos [16] have shown by functional analytic methods that the independent union of 2 disjoint

Kronecker sets is « quasi-always » not Dirichlet and « quasi-always » not H_1. (Strictly speaking this statement is meaningless, since I have not indicated the space, let alone the metric with respect to which « quasi-always » is defined. Since a full discussion would take us too far afield, I have included the word « quasi-always » simply to give the flavour of the results.) In *Theorem* 1 we construct 2 countable disjoint Kronecker sets whose union is independent but not Dirichlet. A modification of this construction gives *Theorem* 2 which in turn implies that the independent union of 2 disjoint perfect Kronecker sets may be independent but not Dirichlet (*Corollary* 2.1). As another example of what can occur if independence conditions are dropped we give in *Lemma* 3.4 a constructive proof of a result obtained by functional analytic methods by Varopoulos [17], which shows that the sum of 2 disjoint Kronecker sets may be the whole of **T**. In *Theorem* 6 we construct 2 disjoint perfect Kronecker sets whose union is independent but not even weak Dirichlet. Examination of the proof and consideration of the methods of Bernard and Varopoulos indicate that the novelty here lies in the exhibition of an independent set which is not weak Dirichlet. Since (as we saw in Section 1) all Kronecker sets are Dirichlet, weak Kronecker, weak Dirichlet and N sets and all Kronecker, Dirichlet, weak Kronecker and N sets are weak Dirichlet, *Theorem* 6 provides a complete negative answer to the union problem for these types of set. For example it improves on the standard result (due to Marcinkiewicz) that the union of two N sets need not be an N set ([9] Chapter VII, § 5) by exhibiting an independent union of two N sets which is not an N set (the standard result again depends crucially on the lack of independence of the given union).

In *Theorem* 7 we show that the independent union of q disjoint Kronecker sets may be such that it is at most $H_{1/q}$. We very briefly discuss a result of Varopoulos [17] which states that the independent union of q disjoint Kronecker sets is at least $H_{1/q}$ and so shows our result to be best possible. A simple modification of the proof of *Theorem* 7 gives *Theorem* 8 : the closed independent union of a countable collection of disjoint Kronecker sets need not be Helson. Rudin has shown by probabilistic means that independent non Helson sets

(Rudin sets) exist ([9] Chapter VII). As a by-product of *Theorems* 7 and 8 we obtain in *Lemma* 7.4 and *Lemma* 7.8 by direct construction examples of independent perfect non Helson sets which both do and do not carry non zero measures σ with $\lim_{|n|\to\infty} \hat{\sigma}(n) = 0$. We conclude Section 7 by using our techniques together with the result of Varopoulos just mentioned to prove some minor results of which the most interesting is that the independent union of an H_s and an H_t set can be an $H_{st/t+s}$ set $[0 < s, t \leqslant 1]$.

The paper proceeds at a leisurely pace. My excuse is that whilst the impatient reader may in any case omit those results and proofs which he finds uninteresting, I wish to help those who want to construct such sets themselves by exhibiting as varied as selection of constructions as possible. For example there is a considerably longer discussion of independence in Section 6 than the paper needs. In Section 7 we discuss discrete Kronecker sets in **R** because the techniques used seem to be simpler. Finally several results on H_r sets are included only because it seems to me that any results in this field are worth having.

Essentially the proofs merely consist of repeated applications of Kronecker's theorem and the trivial observation that for any interval I we can find an m_0 such that for $m > m_0$ the wave length of χ_m is very much smaller than the length of I. We apply this to situations of gradually increasing complexity. Thus although the proofs of our theorems (except in the case of *Theorems* 2 and 5 and *Theorems* 7 and 8) are independent, it may be helpful to absorb the ideas of the earlier theorems before proceeding to the later ones. However, anyone well versed in the subject will find that reading the heuristic preceding *Theorem* 1 and the proof of *Theorem* 7 gives a good idea both of the methods used and the ideas behind them. Others may wish to omit the proof of *Theorem* 5 as adding nothing to the ideas of *Theorems* 2 and 4, and may feel that they do not need *Theorem* 6 as a stepping stone to *Theorem* 7 but having read the latter may feel the former easily obtainable.

Unfortunately our constructions are necessarily inductive and the proofs thus obscure the simplicity (or triviality) of the ideas. Thus once the reader has grasped the idea, he may prefer to construct his own proof. Except in the case of

Theorems 4 and 9 where some verification is required, once the idea is recognised the result is obvious. To encourage this process we have included a certain amount of heuristic. The reader may also find it helpful to follow the induction through a couple of cycles drawing a diagram of the state of the system at each step (the results were obtained in this manner). When doing this it becomes natural to think of the construction as proceeding in time : at time m we examine the system with respect to χ_m and make suitable modifications. We use this metaphor in our heuristics.

Finally we establish some notation. Since $C(\mathbf{T})$ is separable, so is S and there exist $g_i \in S$ such that $\{g_1, g_2, \dots\}$ is dense (under the uniform norm) in S (for convenience we take $g_1 = 1$). Write $f_{1+2+\cdots+r+s-1} = g_s$ for $1 \leqslant s \leqslant r$. Then trivially E is Kronecker if and only if $\inf_{s\geqslant 1} \sup_{x\in E} |f_r(x) - \chi_s(x)| \to 0$ as $r \to \infty$ and weak Kronecker if and only if we can find $\varepsilon_r \to 0$ such that for every $\mu \in M^+(\mathbf{T})$

$$\inf_{s\geqslant 1} \mu\{x \in E : |f_r(x) - \chi_s(x)| > \varepsilon_r\} \to 0 \quad \text{as} \quad r \to \infty.$$

The reader unfamiliar with the convention should note that we write $N(y, \varepsilon) = \{x \in \mathbf{T} : |x - y| \leqslant \varepsilon\}$ $[y \in \mathbf{T}, \pi \geqslant \varepsilon > 0]$.

3. First Results.

Suppose we construct successively $x_1, x_2, \dots$ and set $E = \{x_1, x_2, \dots\}$. Then E independent does not imply $\overline{E}$ independent. Thus in what follows we first select a limit point α_0 and then construct $\alpha_1, \alpha_2, \dots, \alpha_r \to \alpha_0$ such that, for each s, $\{\alpha_1, \dots, \alpha_s, \alpha_0\}$ is independent. Then $E = \{\alpha_0, \alpha_1, \alpha_2, \dots\}$ is closed. Suppose $y_1, y_2, \dots, y_n \in E$ distinct. Then, for some s, $\{y_1, \dots, y_n\} \subseteq \{\alpha_1, \dots, \alpha_s, \alpha_0\}$ so $y_1, y_2, \dots, y_n$ are independent. Thus E is independent.

We use this idea in the proof that follows. Here the reader will note that at time m we introduce a new point α_m to prevent the set E from being well behaved with respect to χ_m.

LEMMA 3.1. — *There exists a closed set* E *which is not Dirichlet yet independent and countable* (*and so weak Kronecker and* N_0).

Proof. — We construct such an E. Select $\alpha_0 \in \mathbf{T}$ independent (i.e. $\alpha_0 \notin 2\pi\mathbf{Q}$). The central inductive step in the construction runs as follows. Suppose we have constructed $\alpha_0, \alpha_1, \ldots, \alpha_m$ independent. By *Lemma* 1.3 we can find α_{m+1} such that $|\chi_{m+1}(\alpha_{m+1}) - 1| \geqslant 1, |\alpha_{m+1} - \alpha_0| \leqslant 2\pi/(m+1)$, and $\alpha_0, \alpha_1, \ldots, \alpha_{(m+1)}$ are independent. The induction now proceeds.

We set $E = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$. By the remarks above E is independent and closed. On the other hand $|\chi_r(\alpha_r) - 1| \geqslant 1$ for all $r \geqslant 1$ so E is not Dirichlet.

The reader will notice that the early stages of the construction do not really affect the final result. This is true for all the constructions in this paper. In fact the first few steps are often atypical. Because of this and also because we may wish to impose conditions that complicate the early parts of the construction but not the later ones (e.g. that the set lie in an interval I), we shall be deliberately vague as to how the inductions are started. Inserting these details is a genuinely trivial matter but would still further obscure the ideas of the proof.

We now prove *Theorem* 1. The idea here is crudely as follows. Consider the construction proceeding at times $r = 1, 2, \ldots$. At time $r = M$ we have a great deal of latitude in constructing one set A, we use this latitude at times $r = M, M+1, \ldots$ to ensure that A is badly behaved with respect to $\chi_M, \chi_{M+1}, \ldots$. By Kronecker's theorem there will come a time P when what we have constructed of B is well behaved with respect to χ_P. By the continuity of $\chi_1, \ldots, \chi_P$ there is a certain small latitude allowed us in adding to our construction of B while retaining its desirable characteristics in relation to them. As r continues to increase, the wave length associated with χ_r decreases and there will come a time M′ when the latitude allowed in constructing B is large compared with the wave length of $\chi_{M'}$. We now reverse the rôles of A and B and start again. This construction ensures that A and B are each well behaved infinitely often but $A \cup B$ never is.

THEOREM 1. — *There exist* A, B *disjoint countable Kronecker sets such that* $A \cup B$ *is independent but not Dirichlet.*

Proof. — We construct such an A and B. Select α_0, β_0 independent with $|\alpha_0 - \beta_0| > 2/5$ say. We give the central inductive step in the construction of A and B. Suppose we have $\alpha_0, \alpha_1, \ldots, \alpha_{M(n,B)}, \beta_0, \beta_1, \ldots, \beta_{M(n,B)}$ independent and $1/10 \geqslant \varepsilon(n, A) \geqslant \varepsilon(n, B) > 0$ such that $M(n, B)\varepsilon(n, A) \geqslant 2\pi$. Since $\alpha_0, \alpha_1, \ldots, \alpha_{M(n,B)}$ are independent, there exists a $P(n+1, A) > M(n, B)$ such that $|\chi_{P(n+1,A)}(\alpha_s) - f_{n+1}(\alpha_s)| \leqslant 2^{-(n+2)}$ for $0 \leqslant s \leqslant M(n, B)$. Since f_{n+1} and $\chi_{P(n+1,A)}$ are continuous there exists an $0 < \varepsilon(n+1, A) \leqslant 1/2\ \varepsilon(n, B)$ such that $|\chi_{P(n+1,A)}(x) - f_{n+1}(x)| \leqslant 2^{-(n+1)}$ for all $x \in N(\alpha_0, \varepsilon(n+1, A))$. There exists an $M(n+1, A) > P(n+1, A)$ such that $M(n+1, A)\varepsilon(n+1, B) \geqslant 2\pi$. Select (using *Lemma* 1.3) $\alpha_{M(n,B)+1}, \ldots, \alpha_{M(n+1,A)} \in N(\alpha_0, \varepsilon(n+1, A))$ such that $\alpha_0, \ldots, \alpha_{M(n+1,A)}, \beta_0, \ldots, \beta_{M(n,B)}$ are independent. Because $M(n, B)\varepsilon(n, A) \geqslant 2\pi$ we can find (by *Lemma* 1.3) $\beta_{M(n,B)+1}, \ldots, \beta_{M(n+1,A)} \in N(\beta_0, \varepsilon(n, A))$ such that $|\chi_s(\beta_s) - 1| \geqslant 1$ for $M(n+1, A) \geqslant s \geqslant M(n, B) + 1$ and $\alpha_0, \ldots, \alpha_{M(n+1,A)}, \beta_0, \ldots, \beta_{M(n+1,A)}$ are independent.

Repeating the work mutatis mutandis we obtain $\alpha_0, \ldots, \alpha_{M(n+1,B)}, \beta_0, \ldots, \beta_{M(n+1,B)}$ independent points and $\varepsilon(n+1, B), P(n+1, B), M(n+1, B)$ with the following properties :

$$\varepsilon(n+1, A) \geqslant 1/2\ \varepsilon(n+1, B) > 0,$$
$$M(n+1, B)\varepsilon(n+1, A) \geqslant 2\pi,$$
$$M(n+1, B) > P(n+1, B) > M(n+1, A)$$
and
$$\sup_{0 \leqslant s \leqslant M(n+1,B)} |\chi_{P(n+1,B)}(\beta_s) - f_{n+1}(\beta_s)| \leqslant 2^{-(n+1)},$$
$$\sup_{x \in N(\beta_0, \varepsilon(n+1,B))} |\chi_{P(n+1,B)}(x) - f_{n+1}(x)| \leqslant 2^{-(n+1)},$$

whilst $|\chi_s(\alpha_s) - 1| \geqslant 1$ for
$$M(n+1, B) \geqslant s \geqslant M(n+1, A) + 1.$$

The induction proceeds.

Let $A = \{\alpha_r : r \geqslant 0\}$, $B = \{\beta_r : r \geqslant 0\}$. Since $\alpha_r \to \alpha_0$, $\beta_r \to \beta_0$ as $r \to \infty$, A and B are closed. Trivially A and B are countable and, provided the induction is started suitably, disjoint. Suppose $y_1, y_2, \ldots, y_n \in A \cup B$ distinct. Then, for some s, $\{y_1, \ldots, y_n\} \subseteq \{\alpha_0, \ldots, \alpha_s, \beta_0, \ldots, \beta_s\}$ so $y_1, y_2, \ldots, y_n$ are independent. Thus, as before $A \cup B$ is independent. By construction $\sup_{s \geqslant 0} |\chi_{P(n,A)}(\alpha_s) - f_n(\alpha_s)| \leqslant 2^{-n}$ so A and similarly B are Kronecker. But $\max(|\chi_s(\alpha_s) - 1|, |\chi_s(\beta_s) - 1|) \geqslant 1$ for s large enough (depending on how the induction is started), so $A \cup B$ cannot be Dirichlet.

The reader will note that (in heuristic terms) there is a great deal of « slack » in our constructions, both here and later. Having ensured the good behaviour of part of our set at time r (i.e. for the character χ_r) we are content not to tamper with that part so as to try to ensure good behaviour until much later. This is, I think, unavoidable to some degree, since a Kronecker (or similar) set has a deep structure which does not appear explicitly in our construction. For example, on the face of it, our methods should enable us to construct a Kronecker set K such that $2K = \{2x : x \in K\}$ is not Kronecker (i.e. K is Kronecker but not all members of S can be approximated by characters of the form χ_{2n}). The following easy Lemma shows that this is not possible. The reader may derive some benefit by considering where a proposed construction for a counter example breaks down.

LEMMA 3.2. — *If* K *is a Kronecker set and* $q \in \mathbf{Q}\setminus\{0\}$ *then* $qK = \{qx : x \in K\}$ *is Kronecker.*

Proof. — Clearly qK is closed. If $m, n \in \mathbf{Z}\setminus\{0\}$ then $\chi_{rn}(mx/n) = \chi_{rm}(x)$. The result is thus equivalent to showing that if $n \in \mathbf{Z}^+\setminus\{0\}$ every $f \in S$ can be uniformly approximated arbitrarily well by characters of the form χ_{rn}. Since K is closed and $K \neq \mathbf{T}$ there is an open interval $I \subseteq \mathbf{T}$ such that $I \cap K = \varnothing$. Thus if $f \in S$ there is a $g \in S$ such that $g|K = f|K$ and $[\arg g(t)]_0^{2\pi}$ is a multiple of $2\pi n$. In particular there exists an $h \in S$ such that $h^n = g$. Suppose $\varepsilon > 0$ is given. Then we can find an r such that $|\chi_r(x) - h(x)| \leqslant \varepsilon/n$ for $x \in E$ and so

$$\begin{aligned} |\chi_{rn}(x) - f(x)| &= |\chi_{rn}(x) - h^n(x)| \\ &= |\chi_r^n(x) - h^n(x)| \\ &= |\chi_r(x) - h(x)| \left| \sum_{s=0}^{n-1} \chi_r^s(x) h^{n-1-s}(x) \right| \\ &\leqslant |\chi_r(x) - h(x)| \sum_{s=0}^{n-1} |\chi_r^s(x) h^{n-1-s}(x)| \\ &= |\chi_r(x) - h(x)| n \leqslant \varepsilon \quad \text{for all} \quad x \in E. \end{aligned}$$

This is the required result.

We note that if $y \in K$ then $2\pi y^{-1}K$ is not independent and so not Kronecker.

In what follows we shall use repeatedly the following simple fact:

LEMMA 3.3. — *(i) Suppose* $I_1, \ldots, I_s$ *are disjoint closed intervals in* $\mathbf{T}$ *and* $f \in S$, $r(1), r(2), \ldots, r(s) \geqslant 1$, $n_0 \geqslant 1$, $\varepsilon > 0$ *and* $\delta > 0$ *given. Then we can find an* $n > n_0$, *disjoint closed intervals* $I_{pq} \subseteq \operatorname{int} I_p$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$ *with diam* $I_{pq} \leqslant \delta$ *such that* $|f(x) - \chi_n(x)| \leqslant \varepsilon$ *for all* $x \in \bigcup_{p=1}^{s} \bigcup_{q=1}^{r(p)} I_{pq}$.

(ii) Under the hypotheses of (i) we can find an $n_1 > n_0$ *such that for any* $n > n_1$ *we can find disjoint closed intervals* $I_{pq} \subseteq int\ I_p$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$ *with diam* $I_{pq} \leqslant \delta$ *such that* $|f(x) - \chi_n(x)| \leqslant \varepsilon$ *for all* $x \in \bigcup_{p=1}^{s} \bigcup_{q=1}^{r(p)} I_{pq}$.

Proof. — Although *(ii)* implies *(i)* we choose to prove the 2 results separately. We do this because our proof of *(ii)* leans much more heavily on the special properties of $\mathbf{T}$, and because we need this stronger and « less natural » version only once for the non essential *Lemma* 6.1.

(i) By *Lemma* 1.3 we can find distinct independent $\alpha_{pq} \in int\ I_p$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$. By Kronecker's theorem we can find an $n > n_0$ such that $|f(\alpha_{pq}) - \chi_n(\alpha_{pq})| \leqslant \varepsilon/2$ and since f and χ_n are continuous, we can find disjoint intervals I_{pq} with diam $I_{pq} \leqslant \delta$, $\alpha_{pq} \in \operatorname{int} I_{pq} \subseteq I_{pq} \subseteq \operatorname{int} I_p$ such that $|f(x) - \chi_n(x)| \leqslant \varepsilon$ for $x \in I_{pq}$ $[1 \leqslant q \leqslant r(p),$ $1 \leqslant p \leqslant s]$ as required.

(ii) Since f is continuous and $\mathbf{T}$ compact, there exists a $0 < \eta < \min_{1 \leqslant p \leqslant s} \operatorname{diam} I_p$ such that $\sup_{|x-y| \leqslant \eta} |f(x) - f(y)| \leqslant \varepsilon/2$. Select (closed) intervals J_p and points y_p such that $y_p \in \operatorname{int} J_p \subseteq J_p \subseteq \operatorname{int} I_p$ and diam $J_p \leqslant \eta$ $[1 \leqslant p \leqslant s]$. Choose an $n_1 > n_0$ such that $n_1 \min_{1 \leqslant p \leqslant s} J_p \geqslant 4\pi$. Then if $n > n_1$ we have that $\chi_n(t)$ has period $2\pi/n$ (in t) and so there exists $z_p \in \operatorname{int} J_p$ with $\chi_n(z_p) = f(y_p)$ and so

$$\sup_{x \in J_p} |f(x) - \chi_n(z_p)| \leqslant \varepsilon/2.$$

By the continuity of χ_n we can find closed intervals $I_{pq} \subseteq \operatorname{int} J_p$ with diam $I_{pq} < \delta$ such that $|\chi_n(y) - \chi_n(z_p)| \leqslant \varepsilon/2$ for all $y \in J_{pq}$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$. Clearly the J_{pq} have the required properties.

Using this, we can push the method of *Theorem* 1 a little further to obtain *Theorem* 2. The idea here is that the bad behaviour of E with respect to χ_m can be ensured by bad behaviour in a small region (within a small closed interval F_r^* say) outside of which we can make E well behaved. As r increases with m we allow F_r^* to move round and round the circle growing ever smaller. In this way we ensure that although E is badly behaved, the removal of that (non empty) part of E lying within a small interval renders E well behaved infinitely often.

THEOREM 2. — *There exists a perfect non Dirichlet set* E *such that every proper closed subset of* E *is Kronecker.*

Proof. — We construct inductively $\mathfrak{J}_1, \mathfrak{J}_2, \ldots$ where $\mathfrak{J}_r$ is a finite collection of (closed) intervals and setting $E_r = \cup\{F : F \in \mathfrak{J}_r\}$ we have $0 \notin F_r$ and $F_r \supseteq F_{r+1}$. We then set $E = \bigcap_{r=1}^{\infty} F_r$ (cf. *Lemma* 1.4). The central inductive step runs as follows. Suppose we have $\mathfrak{J}_r$, $F_r^* \in \mathfrak{J}_r$, $1/10 \geqslant \varepsilon(r) \geqslant \delta(r) > 0$, $M(r) \geqslant N(r) \geqslant 1$ defined in such a way that $M(r)\,\varepsilon(r) \geqslant 2\pi$, diam $F \geqslant \delta(r)$ for all $F \in \mathfrak{J}_r$ and diam $F_r^* \geqslant \varepsilon(r)$. Let the sets of $\mathfrak{J}_r$ be numbered in the direction θ from 0 to 2π as $E_0, E_1, \ldots, E_q$ say (here as elsewhere in the proof the notation is obviously temporary to be maintained only in this step of the induction; if we needed to be more specific, we would talk of $E_0(r), E_1(r), \ldots, E_{q(r)}(r)$) with $E_s = F_r^*$. Select $\alpha_k \in \operatorname{int} E_k$ $[0 \leqslant k \leqslant q,\ k \neq s]$ independent. By *Lemma* 3.3 there exists a $P(r) > M(r)$ and an $0 < \varepsilon(r+1) \leqslant 1/4\ \delta(r)$ such that setting

$$G_k = N(\alpha_k, 1/2\ \varepsilon(r+1))$$

we have $G_k \subseteq F_k$ and $|\chi_{P(r)}(x) - f_{N(r)}(x)| \leqslant 2^{-N(r)}$ for all $x \in G_k[0 \leqslant k \leqslant q,\ k \neq s]$. Choose an $M(r+1) > P(r)$ such that $M(r+1)\varepsilon(r+1) \geqslant 2\pi$. Since $M(r)$ diam $E_s \geqslant 2\pi$ we can, trivially, find distinct $\beta_{M(r)+1}, \beta_{M(r)+2}, \ldots, \beta_{M(r+1)} \in \operatorname{int} E_s$

such that $|\chi_n(\beta_n) - 1| \geqslant 3/2$ for $M(r) + 1 \leqslant n \leqslant M(r+1)$. By the continuity of $\chi_{M(r)+1}, \chi_{M(r)+2}, \ldots, \chi_{M(r+1)}$ we can find a $0 < \delta(r+1) \leqslant \varepsilon(r+1)$ such that setting

$$H_n = N(\beta_n, 1/2\delta(r+1))$$

(or more specifically $H_n(r) = N(\beta_n(r), 1/2\, \delta(r+1))$) we have $H_n \subseteq E_s$, $|\chi_n(x) - 1| \geqslant 1$ for all

$$x \in H_n \ [M(r)+1 \leqslant n \leqslant M(r+1)]$$

and
$$H_{M(r)+1}, H_{M(r)+2}, \ldots, H_{M(r+1)}$$

disjoint. Setting

$$\mathfrak{I}_{r+1} = \bigcup_{\substack{0 \leqslant k \leqslant q \\ k \neq s}} G_k \cup \bigcup_{M(r)+1 \leqslant n \leqslant M(r+1)} H_n,\ F^*_{r+1} = G_{s+1}$$

(where $G_{q+1} = G_0$) and $N(r+1) = N(r) + 1$ if $s = M(r)$, $N(r+1) = N(r)$ otherwise.

The induction now proceeds.

Since the argument now depends crucially on the behaviour of F^*_r as r increases the reader is again advised to draw a diagram and observe the behaviour of F^*_r through several inductive steps. We see that indeed $N(r) \to \infty$ (in more colourful language F^*_r describes complete rotations), and $\varepsilon(r), \delta(r) \to 0$, card $\mathfrak{I}_r \to \infty$ as $r \to \infty$. Hence E is perfect (*Lemma* 1.4). Moreover if $F \in \mathfrak{I}_r$ $[r \geqslant 1]$ then $F \cap E \neq \varnothing$. It follows that if n is large enough (depending on how the induction is started), then we have for some r that $M(r+1) \geqslant n \geqslant M(r) + 1$ and therefore, selecting a

$$\beta \in E \cap H_n(r) \quad \text{we have} \quad \beta \in E \quad \text{and} \quad |\chi_r(\beta) - 1| \geqslant 1.$$

Thus E is not Dirichlet. Now suppose that A is a proper closed subset of E. Then there exists an $x \in E$ and a $\delta > 0$ such that $N(x, \delta) \cap A = \varnothing$. But we can find a q_0 such that for all $q \geqslant q_0$ there exists an r with $N(r) = q$ and $F^*_r \subseteq N(x, \delta)$ (more colourfully: eventually F^*_r lies entirely within $N(x, \delta)$ at some time during each revolution). Thus

$$|\chi_{P(r)}(t) - f_q(t)| = |\chi_{P(r)}(t) - f_{N(r)}(t)| \leqslant 2^{-N(r)} = 2^{-q}$$

for all $t \in A$. Hence A is Kronecker. This completes the proof.

We note that every finite subset of E is closed, so Kronecker and thus independent. Hence E is independent. Since E is perfect it is the union of 2 non void disjoint closed sets. From considering any such 2 sets A, B we obtain

COROLLARY 2.1. — *There exist 2 disjoint perfect Kronecker sets* A *and* B *with* $A \cup B$ *independent but not Dirichlet.*

Moreover if we are given $\mu \in M^+(\mathbf{T})$, $\varepsilon > 0$, $\eta > 0$ we can find an open non void subset K of E such that $\mu(K) < \eta$. Let $L = E \setminus K$. Then L is Kronecker and given $f \in S$ we can find an n such that $\sup_{x \in L} |f(x) - \chi_n(x)| \leqslant 1/2\,\varepsilon$ and so $\mu\{x \in E : |\chi_n(x) - f(x)| \geqslant \varepsilon\} < \eta$. We thus obtain Wik's result.

COROLLARY 2.2. — *There exists a perfect weak Kronecker set which is not Dirichlet.*

We conclude this section with a lemma (due, as we said in Section 2, to Varopoulos [17]) which shows that, in a certain sense, Kronecker sets are quite thick. The proof follows that of *Theorem* 1 in that we balance the construction of K and L against each other so as to ensure that each of K and L is well behaved infinitely often, but K + L is not well behaved.

LEMMA 3.4. — *We can find* K, L *disjoint perfect Kronecker sets such that* $K + L = \mathbf{T}$.

Proof. — We construct inductively $\mathcal{K}_1, \mathcal{K}_2, \ldots$ and $\mathcal{L}_1, \mathcal{L}_2, \ldots$ finite collections of disjoint (closed) intervals such that setting $K_r = \cup\{F : F \in \mathcal{K}_r\}$, $L_r = \cup\{F : F \in \mathcal{L}_r\}$ we have $K_r \supseteq K_{r+1}$, $L_r \supseteq L_{r+1}$, $K_r \cap L_r = \varnothing$. We then set $K = \bigcap_{r=1}^{\infty} K_r$, $L = \bigcap_{r=1}^{\infty} L_r$ (cf. *Lemma* 1.4) and show that K and L (which are certainly disjoint) have the required properties. The central inductive step runs as follows. Suppose we have $P(r, K) \geqslant 10$ and K_r, L_r defined in such a way that $\operatorname{int} K_r + \operatorname{int} L_r = \mathbf{T}$. Then $\bigcup_{x \in \operatorname{int} L_r} (x + \operatorname{int} K_r) = \mathbf{T}$. Since $\mathbf{T}$ is compact, we can find distinct $x_{1r}, x_{2r}, \ldots, x_{n(r)r} \in \operatorname{int} L_r$ such that $\bigcup_{k=1}^{n(r)} (x_{kr} + \operatorname{int} K_r) = \mathbf{T}$ and further every $F \in \mathcal{K}_r$

contains at least 2 members of $x_{1r}, x_{2r}, \ldots, x_{n(r)r}$. Since $\operatorname{int} K_r$ is the union of a finite collection of open intervals, we see that there exist disjoint closed intervals I_{1r}, $I_{2r}, \ldots, I_{n(r)r}$ such that $x_{kr} \in \operatorname{int} I_{kr} \subseteq I_{kr} \subseteq \operatorname{int} L_r$ $[1 \leqslant k \leqslant n]$ and for all $z_1 \in I_1, \ldots, z_{n(r)} \in I_{n(r)r}$ we have

$$\bigcup_{k=1}^{n(r)} (z_k + \operatorname{int} K_r) = \mathbf{T}.$$

Select $y_{kr} \in \operatorname{int} I_{kr}$ such that $y_{1r}, y_{2r}, \ldots, y_{n(r)r}$ are independent. By *Lemma* 3.3 there exists a $P(r+1, L) > P(r, K)$ and intervals J_{kr} with $y_{kr} \in \operatorname{int} J_{kr} \subseteq J_{kr} \subseteq I_{kr}$ and $\operatorname{diam} J_{kr} \leqslant 1/2 \operatorname{diam} I_{kr}$, such that $x \in J_{kr}$ implies

$$|\chi_{P(r+1,L)}(x) - f_{r+1}(x)| \leqslant 2^{-r-1} \; [1 \leqslant k \leqslant n(r)].$$

Setting $\mathcal{L}_{r+1} = \{J_{kr}: 1 \leqslant k \leqslant n(r)\}$ we have $L_{r+1} \subseteq L_r$, $\sup\limits_{x \in L_{r+1}} |\chi_{P(r+1,L)}(x) - f_{r+1}(x)| \leqslant 2^{-r-1}$ and $\operatorname{int} L_{r+1} + \operatorname{int} K_r = \mathbf{T}$. Mutatis mutandis we can find $\mathcal{K}_{r+1}$ a finite collection of disjoint intervals and $P(r+1, K) > P(r+1, L)$ such that $K_{r+1} \subseteq K_r$, $\sup\limits_{x \in K_{r+1}} |\chi_{P(r+1,K)}(x) - f_{r+1}(x)| \leqslant 2^{-r-1}$ and

$$\operatorname{int} L_{r+1} + \operatorname{int} K_{r+1} = \mathbf{T}.$$

The induction now proceeds.

As usual there is a great deal of liberty allowed in starting the induction, but it may be as well to point out that setting $\mathcal{K}_0 = \{[-\pi/40, \pi/40], [19\pi/40, 21\pi/40], [29\pi/40, 31\pi/40]\}$, and

$$\mathcal{L}_0 = \{[2\pi/40, 18\pi/40], [22\pi/40, 28\pi/40], [32\pi/40, -2\pi/40]\}$$

we do indeed have $\operatorname{int} K_0 + \operatorname{int} L_0 = \mathbf{T}$. We now show that K and L have the required properties. By construction K and L are perfect. Further, since $L \subseteq L_{r+1}$, we have $\sup\limits_{x \in L} |\chi_{P(r+1,L)}(x) - f_{r+1}(x)| \leqslant 2^{-r-1}$ so that L and similarly K are Kronecker. Now suppose $\xi \in \mathbf{T}$. We can find $\varkappa_r \in K_r$, $\lambda_r \in L_r$ such that $\varkappa_r + \lambda_r = \xi$. By compactness we can find $m(1) < m(2) < \cdots$ such that $\varkappa_{m(j)} \to \varkappa \in \mathbf{T}$, $\lambda_{m(j)} \to \lambda \in \mathbf{T}$ and so $\varkappa + \lambda = \xi$. But by construction $\varkappa \in K$, $\lambda \in L$. Thus $K + L = \mathbf{T}$ as we set out to prove.

4. R Sets and Weak Kronecker Sets.

The method we have used is clearly a very easy way to construct weak Kronecker and weak Dirichletsets. However, there are limits to its power as we shall now see. The results which show this, though not very deep, in turn form the heuristic for *Theorem* 3. We make the following temporary definitions for use in what follows. We will call a closed set E *almost Kronecker* (respectively *almost Dirichlet*) if every proper closed subset of E is Kronecker (respectively Dirichlet).

We first remark that, not surprisingly, not every weak Kronecker set E is almost Dirichlet. If we do not demand E perfect the result is trivial.

LEMMA 4.1. — *There exists a countable independent (and so weak Kronecker) set* E *which is not almost Dirichlet.*

Proof. — Take E as in *Lemma* 3.1, $E\setminus\{\alpha_1\}$ is not Dirichlet.

For E perfect we obtain the result by a simple modification of the construction in *Theorem* 2. The idea here is to ensure that F_r^* does not make a complete revolution. We give the proof in full, but for most of its length it follows that in *Theorem* 2 word for word, and the reader need only pay attention to the divergences.

LEMMA 4.2. — *There exists a weak Kronecker set which is perfect but not almost Dirichlet.*

Proof. — Suppose we have $\mathfrak{J}_r$ a finite collection of closed intervals, $F_r^* \in \mathfrak{J}_r$, $1/10 \geqslant \varepsilon(r) \geqslant \delta(r) > 0$, $M(r) \in \mathbf{Z}$ with $M(r)\varepsilon(r) \geqslant 2\pi$, diam $F \geqslant \delta(r)$ for all $F \in \mathfrak{J}_r$ and diam $F_r^* \geqslant \varepsilon(r)$. Let the sets of $\mathfrak{J}_r$ be numbered in the direction θ increasing as $E_0, E_1, \ldots, E_q$ say with $E_s = F_r^*$. Select $\alpha_{2k}, \alpha_{2k+1} \in \operatorname{int} E_k[0 \leqslant k \leqslant q, k \neq s]$ independent. By *Lemma* 3.3 there exists a $P(r) > M(r)$ and an $0 < \varepsilon(r+1) \leqslant 1/4\,\delta(r)$ such that setting $G_l = N(\alpha_l, \varepsilon(r+1))$ we have G_{2k}, G_{2k+1} disjoint subsets of $F_k[0 \leqslant k \leqslant q,\ k \neq s]$ and

$$|\chi_{P(r)}(x) - f_r(x)| \leqslant 2^{-r}$$

for all $x \in G_l$ $[0 \leqslant l \leqslant 2q+1,\ l \neq 2s,\ 2s+1]$. Choose an

$M(r+1) > P(r)$ such that $M(r+1)\varepsilon(r+1) \geqslant 2\pi$. Since $M(r)$ diam $E_s \geqslant 2\pi$ we can, as in *Theorem* 2, find disjoint subintervals $H_{M(r)+1}, \ldots, H_{M(r+1)}$ of E_s with diameter $\delta(r+1)$ where $\varepsilon(r+1) \geqslant \delta(r+1) > 0$ such that $|\chi_n(x) - 1| \geqslant 1$ for all $x \in H_n$ $[M(r)+1 \leqslant n \leqslant M(r+1)]$. We now set $\mathfrak{J}_{r+1} = \bigcup_{\substack{0 \leqslant l \leqslant 2q+1 \\ l \neq 2s,\ 2s+1}} \{G_l\} \cup \bigcup_{M(r)+1 \leqslant n \leqslant M(r+1)} \{H_n\}$, $F^*_{r+1} = G_{s+1}$ and restart the induction.

Now $\varepsilon(r+1) \leqslant 1/2\ \varepsilon(r)$ so $\varepsilon(r) \to 0$ as $r \to \infty$, for any $I \in \mathfrak{J}_s$, card $\{I' \in \mathfrak{J}_r : I' \subseteq I\} \geqslant 2^{s-r-1} \to \infty$ as $r \to \infty$ and setting $F_r = \cup \{F : F \in \mathfrak{J}_r\}$ we have $F_r \supseteq F_{r+1}$. Thus $E = \bigcap_{r=1}^{\infty} F_r$ is perfect. The reader is invited to draw a diagram and consider the behaviour of F^*_r. We note at once that $F^*_n \subseteq E_{s(r)+1}(r)$ for all $n \geqslant r$ and thus by well known topological results there exists an $x_0 \in E$ such that if $x_r \in F^*_r$ then $x_r \to x_0$. Further we see that $x_0 \notin F^*_r$ for any r. With the aid of these observations we can show E weak Kronecker but not almost Kronecker. Suppose $\mu \in M^+(\mathbf{T})$, $\varepsilon > 0$, $\eta > 0$ given. Then we can find a $\delta > 0$ such that setting $L = \{x_0\} \cup (E \setminus N(x_0, \delta))$ we have $\mu(L) > \mu(E) - \eta$. For r large enough $L \cap F^*_r = \varnothing$ and so $\sup_{x \in L} |\chi_{P(r)}(x) - f_r(x)| \leqslant 2^{-r}$, i.e. L is Kronecker. Thus if $f \in S$ we can find an n such that $\sup_{x \in L} |f(x) - \chi_n(x)| \leqslant 1/2\ \varepsilon$ and so

$$\mu\{x \in E : |\chi_n(x) - f(x)| \geqslant \varepsilon\} < \eta.$$

Thus E is weak Kronecker. However, for any $\delta > 0$ setting $K(\delta) = E \cap N(x_0, \delta)$ we have that for r large enough $K \cap F^*_r \neq \varnothing$ and so for n large enough $\sup_{x \in K} |\chi_n(x) - 1| \geqslant 1$. Hence K is not Dirichlet. In particular taking $\delta = 1/4$ diam E so $K \neq E$, we see that E is not almost Dirichlet. This completes the proof.

More disappointingly we have

Lemma 4.3 — (*i*) *Every almost Dirichlet Cantor set* E *is an* N_0 *set* (*and so an* N *set and an* R *set*).

Proof. — Choose L_1, K_1 non void disjoint closed subsets of E whose union is E. Choose L_2, K_2 non void disjoint closed subsets of K_1 whose union is K_1; L_3, K_3 non void disjoint closed subsets of K_2 whose union is K_2, and so on. Setting $E_r = E \setminus L_r$ we have E_r a proper closed subset of E. Thus we can find $0 < m(1) < m(2) < \cdots < m(r) < \cdots$ such that $|\chi_{m(r)}(x) - 1| \leqslant 2^{-r}$ for all $x \in E_r$ and so $|\sin m(r)x| \leqslant 2^{-r}$ for all $x \in E_r$. Since $L_1, L_2, \ldots$ are disjoint, we thus have that for any $x \in E$, $\sup |\sin m(r)x| > 2^{-r}$ for at most 1 value of $r \geqslant 1$. Thus $\sum_{r=1}^{\infty} |\sin m(r)x|$ converges $\left(\text{indeed } \sum_{r=1}^{\infty} |\sin m(r)x| \leqslant 2\right)$ for all $x \in E$. This proves the lemma.

This result does, however, have the immediate corollary (using *Theorem* 2)

LEMMA 4.3 — (*ii*) *There exist weak Kronecker (and so independent) sets which are* N_0 *sets but not Dirichlet.*

In Section 6 we shall prove this in a different way, directly from *Lemma* 3.1.

A certain amount of thought shows that similar results to that of *Lemma* 4.3 (*i*) (proved in a similar way) hold for the set E of *Lemma* 4.2. Moreover by *Lemma* 1.7 (*iv*) every closed countable set E is an N_0 set. In seeking for a proof of *Theorem* 3 we therefore try to construct sets which imitate the « natural disorder » of a typical perfect weak Kronecker set. I was also guided in attempting to obtain such results as *Lemma* 4.3 and *Theorem* 3 by the following idea which the reader may or may not find helpful. (In describing it I have tried to follow the notation of [12] especially § 4.1, but I hope that recourse to the reference will not be necessary.)

Consider the following game. Player A draws an infinite rooted tree (a circuit free connected unidirectional graph with a selected point a_0, i.e. something that looks like a tree springing from a_0). This tree corresponds to the construction of a Cantor set in *Lemma* 1.4 (*iii*) and elsewhere. $\mathfrak{L}_n$ corresponds to the collection of vertices at the n^{th} level (i.e. to the collection of points a_n such that we can find a path $(a_0a_1)(a_1a_2)(a_2a_3) \ldots (a_{n-1}a_n)$). $\mathfrak{L}_{n+1}$ corresponds to that at the $n+1^{\text{th}}$ level. If $I_n \in \mathfrak{L}_n$, $I_{n+1} \in \mathfrak{L}_{n+1}$ then the statement $I_n \supseteq I_{n+1}$ corresponds to the statement that the vertices a_n, a_{n+1} representing I_n, I_{n+1} are joined by an edge. Player

A then assigns a value 0 or 1 to each vertex (the vertices labelled 0 correspond to those intervals for which we cannot ensure misbehaviour, the others to those for which we can). Certain restrictions may be placed on the freedom of player A both in drawing and labelling the tree, and these determine the character of the game. They correspond to restrictions on the manner of our construction of E (for example: since P is to be perfect, there must, for each a_r, exist $a'_{r+s} \neq a_{r+s}$ such that $(a_r, a_{r+1})(a_{r+1}, a_{r+2}) \ldots (a_{r+s-1}, a_{r+s})$ and $(a_r, a'_{r+1})(a'_{r+1}, a'_{r+2}) \ldots (a'_{r+s-1}, a'_{r+s})$ are paths; again, if E is to be almost Kronecker, then without loss of generality we may assume that at an infinite number of levels at most 1 point can be labelled 1.). Player B then chooses integers $m(1) < m(2) < \cdots$ corresponding to the $m(1)^{\text{th}}$, $m(2)^{\text{th}}$ ... levels. (In the example given he can ensure that if $a_{m(r)}$ at the $m(r)^{\text{th}}$ level is labelled 1 then for every path $(a_{m(r)}, a_{m(r)+1})(a_{m(r)+1}, a_{m(r)+2}) \ldots (a_{m(r+s)-1}, a_{m(r+s)})$ we have that $a_{m(r+s)}$ is labelled 0.) Player A now chooses a path $(a_0a_1)(a_1a_2)(a_2a_3)(a_3a_4) \ldots$. If $a_{m(r)}$ is labelled 1 then A scores 1 point. A wins if his path gains him an infinite number of points (i.e. if he has found $P_s \in \mathfrak{L}_s$ with $P_1 \supseteq P_2 \supseteq \cdots$ and $P_{m(r)}$ badly behaved infinitely often) and loses otherwise. (In the example given he loses and we have the germ of the idea behind *Lemma* 4.3 (*iii*)).

Examining this game more closely, we see for example that A can win if he can ensure that given a_s he can find a t such that for all $r \geqslant t$ there is a path

$$(a_sa_{s+1})\ (a_{s+1}a_{s+2})\ (a_{s+2}a_{s+3})\ \ldots\ (a_{r-1}a_r)$$

with a_r labelled 1. We shall achieve this in *Theorem* 3. The construction there is planned with this and the following trivial lemma in mind.

LEMMA 4.4. — *Suppose* $E_1, E_2, \ldots, E_n \subseteq \mathbf{T}$ *are disjoint closed sets,* $n > m \geqslant 0$ *and* $\mu \in M^+(\mathbf{T})$. *Then for some* $m + 1 \leqslant r \leqslant n$ *we have* $\mu(E_r) \leqslant 1/(n-m)\|\mu\|$.

Proof. — Suppose, if possible, $\mu(E_r) > 1/(n-m)\|\mu\|$ for all $m+1 \leqslant r \leqslant n$. Then $\|\mu\| > \sum_{r=1}^{n} \mu(E_r) = \mu\left(\bigcup_{r=1}^{n} E_r\right) = \|\mu\|$ which is absurd.

Thus if at different times $E_1, \ldots, E_{s-1}, E_{s+1}, \ldots, E_n$ are well behaved, but E_s is not $[m+1 \leqslant s \leqslant n]$ then at some time a set of measure $\|\mu\|(n-m+1)/(n-m)$ is well behaved. By increasing the number of sets $E_1, \ldots, E_n$ we shall obtain good behaviour at some time, except on a set of arbitrarily small measure.

THEOREM 3. — *There exists a perfect weak Kronecker set* E *such that for any* ξ_r *and any* $n(r) \to \infty$ *we can find a* $z \in E$ *with* $\sin(m(r)z + \xi_{m(r)}) \not\to 0$. *In particular* E *is not an* R *set (and so not an* N_0 *set).*

Proof. — We construct inductively $\mathfrak{J}_1, \mathfrak{J}_2, \ldots$ finite collections of (closed) intervals such that setting $E_r = \cup\{F : F \in \mathfrak{J}_r\}$ we have $E_r \supseteq E_{r+1}$. We then put $E = \bigcap_{r=1}^{\infty} E_r$ (cf. *Lemma* 1.4) and show that E has the required properties. The induction runs in cycles covering steps $K(t)$ to $K(t+1)-1$, $K(t+1)$ to $K(t+2)-1$ and so on, where $K(t+1) = K(t) + t$. As usual, we ignore the initial stages of the construction and give only the central inductive step with $t \geqslant 2$, and $K(t) \geqslant 10$.

Suppose $K(t) \leqslant r \leqslant K(t+1) - 1$. We write

$$s = r - K(t) + 1$$

and, to bring the notation into line with that of *Theorem* 2, $N(r) = t$. At the r^{th} step we have

$$\mathfrak{J}_r = \mathcal{A}_{r0} \cup \mathcal{A}_{r1} \cup \mathcal{A}_{r2} \cup \cdots \cup \mathcal{A}_{rN(r)}$$

where the $\mathcal{A}_{r0}, \mathcal{A}_{r1}, \ldots, \mathcal{A}_{rN(r)}$ are disjoint and non empty and $F \in \mathcal{A}_{rs}$ implies $M(r)$ diam $F \geqslant 2\pi$. We also have $\mathcal{B}_{r1}, \mathcal{B}_{r2}, \ldots, \mathcal{B}_{rN(r)+1}$ disjoint non empty with

$$\mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \cdots \cup \mathcal{B}_{r,N(r)+1} = \mathcal{A}_{r0} \cup \mathcal{A}_{r1} \cup \cdots \cup \mathcal{A}_{r,s-1}.$$

By *Lemma* 3.3 we can find a $P(r) > M(r)$ and for each $F \in \mathcal{A}_{rk}$ $[0 \leqslant k \leqslant N(r), k \neq s]$ a (closed) interval $H(F, r) \subseteq F$ with diam $H(F, r) \leqslant 1/2$ diam F such that

$$\sup_{x \in H(F,\, r)} |\chi_{P(r)}(x) - f_{N(r)}(x)| \leqslant 2^{-N(r)}.$$

There exists an $M(r+1) > P(r)$ such that

$$M(r+1) \text{ diam } H(F, r) \geqslant 2\pi$$

for all $F \in \mathcal{A}_{rk}[0 \leqslant k \leqslant N(r),\ k \neq s]$. Two cases now arise according as $s < N(r)$ or $s = N(r)$. First suppose $s < N(r)$. Since $M(r)$ diam $F \geqslant 2\pi$ for any $F \in \mathcal{A}_{rs}$ we can find $2(N(r)+1)\ (M(r+1)-M(r))$ distinct points

$$\alpha(F, l, n, r, s, \varphi) \in \operatorname{int} F \quad (\text{where} \quad \varphi = 0 \quad \text{or} \quad \varphi = 1)$$

such that $|\sin n\alpha(F, l, n, r, s, 0)| \geqslant 19/20$,

$$|\sin n\alpha(F, l, n, r, s, 1)| \leqslant 1/20\ [F \in \mathcal{A}_{rs},\ 1 \leqslant l \leqslant N(r)+1,\\ M(r)+1 \leqslant n \leqslant M(r+1)].$$

Thus by continuity we can find $2(N(r)+1)(M(r+1)-M(r))$ disjoint (closed) intervals $J(F, l, n, r, s, \varphi)$ with

$$\alpha(F, l, n, r, s, \varphi) \in \operatorname{int} J(F, l, n, r, s, \varphi) \subseteq J(F, l, n, r, s, \varphi) \subseteq F$$

such that

$|\sin nx| \geqslant 9/10$ for all $x \in J(F, l, n, r, s, 0)$ and $|\sin nx| \leqslant 1/10$

for all

$$x \in J(F, l, n, r, s, 1)\ [F \in \mathcal{A}_{rs},\ 1 \leqslant l \leqslant N(r)+1,\\ M(r)+1 \leqslant n \leqslant M(r+1),\quad \varphi = 0,1].$$

We set

$$\mathcal{A}_{r+1k} = \{H(F, r) : F \in \mathcal{A}_{rk}\}\ [0 \leqslant k \leqslant N(r),\ k \neq s],$$
$$\mathcal{A}_{r+1s} = \{J(F, l, n, r, s, \varphi) : F \in \mathcal{A}_{rs},\\ 1 \leqslant l \leqslant N(r)+1, M(r)+1 \leqslant n \leqslant M(r+1), \varphi = 0,1$$

and

$$\mathcal{B}_{r+1l} = \{H(F, r) : F \in \mathcal{B}_{rl}\} \cup \{J(F, l, n, r, s, \varphi) : F \in \mathcal{A}_{rs},\\ M(r)+1 \leqslant n \leqslant M(r+1),\ \varphi = 0,1$$

$[1 \leqslant l \leqslant N(r)+1]$.

Now suppose $s = N(r)$ (so that $r = K(t+1)-1$ and we are at the end of a cycle). Since $M(r)$ diam $F \geqslant 2\pi$ for any $F \in \mathcal{A}_{rN(r)}$ it follows by a similar argument to the one just given that we can find $2(N(r)+2)(M(r+1)-M(r))$ disjoint intervals $J(F, l, n, r, s, \varphi) \subseteq F$ such that $|\sin nx| \geqslant 9/10$ for all $x \in J(F, l, n, r, s, 0)$ and $|\sin nx| \leqslant 1/10$ for all

$$x \in J(F, l, n, r, s, 1)\ [F \in \mathcal{A}_{rs},\ 1 \leqslant l \leqslant N(r)+2,\\ M(r)+1 \leqslant n \leqslant M(r+1),\ \varphi = 0,1].$$

We now set

$$\mathcal{A}_{r+1\,0} = \{J(F, l, n, r, s, \nu) : F \in \mathcal{A}_{rs},\ 1 \leqslant l \leqslant N(r) + 2,\ M(r) + 1 \leqslant n \leqslant M(r+1),\ \nu = 0,1$$

$$\mathcal{A}_{r+1k} = \{H(F, r) : F \in \mathcal{B}_{rk}\}\ [1 \leqslant k \leqslant N(r) + 1]$$

and

$$\mathcal{B}_{r+1l} = \{J(F, l, n, r, s, \nu) : F \in \mathcal{A}_{rs},\ M(r) + 1 \leqslant n \leqslant M(r+1),\ \nu = 0,1$$

$[1 \leqslant l \leqslant N(r) + 2]$. We note that $N(r+1) = N(r) + 1$ and restart the induction.

Let us pause at this stage of the proof to examine in general terms what we are doing. During the t^{th} cycle we split the collection of intervals under consideration into $t + 1$ blocks $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_t$ say. At the s^{th} step of the cycle we alter the contents of $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_t$ so that $\mathcal{A}_s$ is badly behaved but $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{s-1}, \mathcal{A}_{s+1}, \ldots, \mathcal{A}_t$ are well behaved. Thus each in turn of $\mathcal{A}_1, \ldots, \mathcal{A}_t$ is badly behaved while the remainder of the blocks are well behaved. However, we do not attempt to ensure bad behaviour for $\mathcal{A}_0$. At the end of the cycle we re-partition the new collection of intervals into $t + 2$ blocks (and so into a larger number of blocks) $\mathcal{A}'_0, \mathcal{A}'_1, \ldots, \mathcal{A}'_{t+1}$ say in such a manner that intervals ensuring the bad behaviour of $\mathcal{A}_0$ (during the last part of $t - 1^{\text{th}}$ cycle) and $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{t+1}$ (during the parts of the t^{th} cycle just discussed) are assigned to each of $\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_{t+1}$. Unfortunately the inductive hypothesis which demands that the intervals of $\mathcal{A}'_1$ must be large (more exactly $M(K(t+1))\operatorname{diam} F \geqslant 2\pi$ for all $F \in \mathcal{A}_{K(t+1)1}$) means that we cannot assign typical members of $\mathcal{A}_t$ to $\mathcal{A}'_1$. We therefore put the members of $\mathcal{A}_t$ into a separate block $\mathcal{A}'_0$ whose members will, when at the end of the $t + 1^{\text{th}}$ cycle we form new blocks $\mathcal{A}''_0, \mathcal{A}''_1, \ldots, \mathcal{A}''_{t+2}$, say, be distributed among $\mathcal{A}''_1, \mathcal{A}''_2, \ldots, \mathcal{A}''_{t+2}$ as described below. The reader may find it useful to draw a tree representing the behaviour of E_r during 3 or 4 cycles of the induction.

Returning to the proof, we first show that E (which, since $\max_{F \in \mathfrak{I}_r} \operatorname{diam} F \leqslant 2^{-K(r)+11}\pi \to 0$ as $r \to \infty$, is perfect) is indeed weak Kronecker. Suppose $\mu \in M^+(\mathbf{T})$. Set $F_{rk} = E \cap \cup \{F : F \in \mathcal{A}_{rk}\}$ $[0 \leqslant k \leqslant N(r)]$, and observe that

F_{rk} depends only on $N(r)$ so that we may write $E_{tk} = F_{rk}$ for $t = N(r)$ $[0 \leqslant k \leqslant t]$. Now $E_{t0}, E_{t1}, \ldots, E_{tt}$ are disjoint and $\bigcup_{k=0}^{t} E_{tk} = E$ so by *Lemma* 4.4 we can find an $1 \leqslant s \leqslant t$ such that $\mu(E_{ts}) \leqslant \|\mu\|/t$. But $|\chi_{P(1+K(t)+s)}(x) - f_t(x)| \leqslant 2^{-t}$ for all $x \in \bigcup_{k \neq s, 1 \leqslant k \leqslant t} E_{tk}$. Thus

$$\mu\{x \in E : |\chi_{P(1+K(t)+s)}(x) - f_t(x)| \geqslant 2^{-t+1}\} \leqslant \|\mu\|/t \to 0$$

as $t \to \infty$ and E is weak Kronecker.

We now show that E satisfies the remaining condition of our claim. First observe that $\sup_{0 \leqslant v \leqslant 1} \inf_{x \in J(F, l, n, r, s, v)} |\sin (nx + \xi_n)| \geqslant 1/10$ for all ξ_n

$$[F \in \mathfrak{A}_{rs}, 1 \leqslant l \leqslant N(r) + 2, M(r) + 1 \leqslant n \leqslant M(r+1), r \geqslant 10].$$

We are now in a position to proceed with the argument prefigured in our discussion of trees. Suppose ξ_1, ξ_2 given. Observe that if $F \in \mathfrak{A}_{rl}$ where $K(t) \leqslant r \leqslant K(t+1) - 1$, $0 \leqslant l \leqslant t$, then provided $q \geqslant K(t+2)$ we can find for any $0 \leqslant u \leqslant N(q)$ a $G \in A_{qu}$ such that $G \subseteq F$. In particular if $F \in \mathfrak{J}_r$ then for p large enough (in fact for $p \geqslant P(K(t+2))$) there exists a $q > r$ (in fact such a q is given by $M(q) + 1 \leqslant p \leqslant M(q+1)$) and a $G \in \mathfrak{J}_q$ such that $|\sin (px + \xi_p)| \geqslant 1/10$ for $x \in G$ and $F \supseteq G$. Now suppose $m(1) \leqslant m(2) \leqslant \cdots$ given. Set $X = \{m(r) : r \geqslant 1\}$. We can find a $p(1) \in X$, an $r(1) \in \mathbf{Z}$ and an $F_1 \in \mathfrak{J}_{q(1)}$ with

$$|\sin (p(1)x + \xi_{p(1)})| \geqslant 1/10$$

for all $x \in F_1$. But by the result just given we can find a $p(2) \in X$ with $p(2) > p(1)$ (simply take a large enough member of X), a $q(2) > q(1)$ and an $F_2 \in \mathfrak{J}_{q(2)}$ with $|\sin (p(2)x + \xi_{p(2)})| \geqslant 1/10$ for all $x \in F_2$. Continuing in this manner we obtain $p(1) < p(2) < p(3) < \cdots$ with $p(r) \in X$, $q(1) < q(2) < q(3) < \cdots$ and $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ with $F_r \in \mathfrak{J}_{q(r)}$ and $|\sin (p(r)x + \xi_{p(r)})| \geqslant 1/10$ for all $x \in F_r [r \geqslant 1]$. Since F_r is closed and $\operatorname{diam} F_r \to 0$ as $r \to \infty$ we have by the Second Intersection Theorem ([6] § 26) that $\bigcap_{r=1}^{\infty} F_r = \{z\}$

for some $z \in \mathrm{E}$. We have however shown that

$$\sin (m(r)z + \xi_{m(r)}) \not\to 0$$

and this completes the proof.

(Incidentally *Theorem* 3 and *Lemma* 4.3 (i) give an alternative proof of *Lemma* 4.2 whilst both Theorem 3 and *Lemma* 4.2 have Wik's result as a corollary.)

It is natural to ask whether that part of the result which concerns N_0 sets can be improved to deal with more complex situations involving absolute convergence. We now give a simple modification of *Theorem* 3 which shows that we can obtain similar results for cases in which Σa_n does not diverge too slowly (and thus shows extremely clearly that the chief difficulties involved in dealing with N sets, at least by the methods of this paper, concern sums which diverge arbitrarily slowly).

We need the following simple fact:

Lemma 4.5. — *Given* $t \geqslant 1$, $m > n \geqslant 1$ *and* I *a closed interval of* **T** *such that* n diam $\mathrm{I} \geqslant 2\pi$ *we can find a* $w \geqslant 1$ *and* tw *disjoint closed subintervals* $\mathrm{J}_{pq}[1 \leqslant p \leqslant t, 1 \leqslant q \leqslant w]$ *such that for any* $a_n, a_{n+1}, \ldots, a_m \geqslant 0$ *there exists a* $1 \leqslant v \leqslant w$ *for which* $\sum_{r=n}^{m} a_r\, |\sin\, rx| \geqslant 1/10 \sum_{r=n}^{m} a_r$ *whenever* $x \in \bigcup_{p=1}^{t} \mathrm{J}_{pv}$.

Proof. — Since we can always take subintervals, it is sufficient to prove the result for $t = 1$. We note first that $|\sin rx| \geqslant \sin^2 rx$ for all r and x so that

$$\sum_{r=n}^{m} a_r\, |\sin\, rx| \geqslant \sum_{r=n}^{m} a_r \sin^2 rx.$$

By the uniform continuity of $\sin^2 ny, \sin^2 (n+1)y, \ldots, \sin^2 my$ (or directly from the theorem of Heine Borel) we can find a $w \geqslant 1$ and disjoint points $y_1, y_2, \ldots, y_w \in \operatorname{int} \mathrm{I}$ such that given any $x \in \mathrm{I}$ there exists a $1 \leqslant v \leqslant w$ such that $\sup_{n \leqslant r \leqslant m} |\sin^2 rx - \sin^2 ry_v| \leqslant 1/40$. Also by the continuity of $\sin^2 ny, \sin^2 (n+1)y, \ldots, \sin^2 my$ we can find disjoint intervals J_q with $y_q \in \operatorname{int} \mathrm{J}_q \subseteq \mathrm{J}_q \subseteq \operatorname{int} \mathrm{I}$ such that $y \in \mathrm{J}_q$ implies $\sup_{n \leqslant r \leqslant m} |\sin^2 ry_q - \sin^2 ry| \leqslant 1/40$ $[1 \leqslant q \leqslant w]$. But by the definition of $y_1, \ldots, y_w$ we see that if $a_n, a_{n+1}, \ldots, a_m \geqslant 0$

$$\begin{aligned}\max_{1\leqslant q\leqslant w} \sum_{r=n}^{m} a_r \sin^2 ry_q &\geqslant \sup_{y\in \mathrm{I}} \sum_{r=n}^{m} a_r (\sin^2 ry - 1/40)\\ &\geqslant \sup_{y\in \mathrm{I}} \sum_{r=n}^{m} a_r \sin^2 ry - 1/40 \sum_{r=n}^{m} a_r\end{aligned}$$

whence

$$\max_{1\leqslant q\leqslant w} \inf_{x\in \mathrm{J}_q} \sum_{r=n}^{m} a_r \sin^2 rx \geqslant \sup_{y\in \mathrm{I}} \sum_{r=n}^{m} a_r \sin^2 ry - 1/20 \sum_{r=n}^{m} a_r.$$

Now

$$\begin{aligned}\sup_{y\in \mathrm{I}} \sum_{r=n}^{m} a_r \sin^2 ry &\geqslant \frac{1}{\operatorname{diam} \mathrm{I}} \int_{\mathrm{I}} \sum_{r=n}^{m} a_r \sin^2 rt\, dt\\ &= \frac{1}{\operatorname{diam} \mathrm{I}} \sum_{r=n}^{m} a_r \int_{\mathrm{I}} \sin^2 rt\, dt\\ &= \frac{1}{\operatorname{diam} \mathrm{I}} \sum_{r=n}^{m} a_r \int_{\mathrm{I}} 1/2(1 - \cos 2rt)\, dt\\ &\geqslant \frac{1}{\operatorname{diam} \mathrm{I}} \sum_{r=n}^{m} a_r (1/2 \operatorname{diam} \mathrm{I} - 1/(2\, r))\\ &\geqslant \sum_{r=n}^{m} a_r (1/2 - 1/(2\, n \operatorname{diam} \mathrm{I}))\\ &\geqslant \sum_{r=n}^{m} a_r (1/2 - 1/(4\pi)).\end{aligned}$$

Thus

$$\max_{1<q\leqslant w} \inf_{x\in \mathrm{J}_q} \sum_{r=n}^{m} a_r |\sin rx| \geqslant (1/2 - 1/(4\pi) - 1/20) \sum_{r=n}^{m} a_r \geqslant 1/10 \sum_{r=n}^{m} a_r$$

as required.

It is now clear what modifications to the proof of *Theorem* 3 are necessary to obtain the following result, whose statement should be read carefully, as it is not as powerful or, I hope, quite as weak as it may appear at first sight.

LEMMA 4.6. — *Given* $m(1) < m(2) < m(3) < \cdots$ *we can construct a perfect weak Kronecker set* E *such that if* $a_r \geqslant 0$ *we have that* $\limsup_{u\to\infty} \sum_{h=m(u)+1}^{m(u+1)} a_h > 0$ *implies that* $\sum_{p=1}^{\infty} a_p |\sin pz|$ *diverges for some* $z \in \mathrm{E}$.

Proof. — The proof proceeds in parallel to that of *Theorem* 3. We use the same notation as was established in the first two paragraphs of the proof of that result and give the central inductive step in the same form. The reader need only pay attention to the divergences. We let $X = \{m(u) : u \geq 1\}$.

By *Lemma* 3.3 we can find a $P(r) > M(r)$ and for each $F \in \mathcal{A}_{rk}$ $[0 \leq k \leq N(r), k \neq s]$ a (closed) interval $H(F, r) \subseteq F$ with $\operatorname{diam} H(F, r) \leq 1/2 \operatorname{diam} F$ such that

$$\sup_{x \in H(F, r)} |\chi_{P(r)}(x) - f_{N(r)}(x)| \leq 2^{-N(r)}.$$

There exists an $M(r+1) \in X$ such that $M(r+1) > P(r)$ and $M(r+1) \operatorname{diam} H(F, r) \geq 2\pi$ for all

$$F \in \mathcal{A}_{rk} \ [0 \leq k \leq N(r), k \neq s].$$

Two cases now arise according as $s < N(r)$ or $s = N(r)$. First suppose $s < N(r)$. By *Lemma* 4.5 since $M(r) \operatorname{diam} F \geq 2\pi$ we can find, for any $F \in \mathcal{A}_{rs}$ a $w(F, r)$ and $w(F, r)(N(r) + 1)$ disjoint intervals

$$J(F, l, n, r, s) \subseteq F[1 \leq l \leq N(r) + 1, \ 1 \leq n \leq w(F, r)]$$

with the property that for any $1 \leq l \leq N(r) + 1$ and any $a_{M(r)+1}, a_{M(r)+2}, \ldots, a_{M(r+1)} \geq 0$ we have

$$\max_{1 \leq n \leq w(F, r)} \inf_{x \in J(F, l, n, r, s)} \sum_{h=M(r)+1}^{M(r+1)} a_h |\sin hx| \geq 1/10 \sum_{h=M(r)+1}^{M(r+1)} a_h.$$

We set $\mathcal{A}_{r+1k} = \{H(F, r) : F \in \mathcal{A}_{rk}\}$ $[0 \leq k \leq N(r), k \neq s]$, $\mathcal{A}_{r+1s} = \{J(F, l, n, r, s) : 1 \leq l \leq N(r) + 1, 1 \leq n \leq w(F, r), F \in \mathcal{A}_{rs}\}$ and

$$\mathcal{B}_{r+1l} = \{H(F, r) : F \in \mathcal{B}_{rl}\} \cup \{J(F, l, n, r, s) : 1 \leq n \leq w(F, r), F \in \mathcal{A}_{rs}\}$$

$[1 \leq l \leq N(r) + 1]$.

Now suppose $s = N(r)$. By *Lemma* 4.5 since

$$M(r) \operatorname{diam} F \geq 2\pi$$

we can find for any $F \in \mathcal{A}_{rN(r)}$ a $w(F)$ and $w(F, r)(N(r) + 2)$ disjoint intervals

$$J(F, l, n, r, s) \subseteq F[1 \leq l \leq N(r) + 2, 1 \leq n \leq w(F, r)]$$

with the property that for any $1 \leq l \leq N(r) + 2$ and any

$a_{M(r)+1}, a_{M(r)+2}, \ldots, a_{M(r+1)} \geqslant 0$ we have

$$\max_{1 \leqslant n \leqslant w(F,r)} \inf_{x \in J(F,l,n,r,s)} \sum_{h=M(r)+1}^{M(r+1)} a_h |\sin hx| \geqslant 1/10 \sum_{h=M(r)+1}^{M(r+1)} a_h.$$

We set

$$\mathcal{A}_{r+1,0} = \{J(F,l,n,r,s) : 1 \leqslant l \leqslant N(r)+2,\ 1 \leqslant n \leqslant w(F, r),\ F \in \mathcal{A}_{rs}\}$$
$$\mathcal{A}_{r+1k} = \{H(F,r) : F \in \mathcal{B}_{rk}\}\ [1 \leqslant k \leqslant N(r)+1]$$

and

$$\mathcal{B}_{r+11} = \{J(F,l,n,r,s) : 1 \leqslant n \leqslant w(F,r),\ F \in \mathcal{A}_{rs}\}\ [1 \leqslant l \leqslant N(r)+2].$$

We now restart the induction.

As before E is weak Kronecker. Suppose now

$$a_1, a_2, a_3, \ldots, a_h$$

given with $a_h \geqslant 0$. If $F \in \mathfrak{J}_r$ then for u large enough there exists a q and a $G \in \mathfrak{J}_q$ such that

$$\sum_{p=M(u)+1}^{M(u+1)} a_h |\sin hx| \geqslant 1/10 \sum_{h=M(u)+1}^{M(u+1)} a_h$$

for all $x \in G$ and $F \supseteq G$. Now suppose

$$\limsup_{u \to \infty} \sum_{h=m(u)+1}^{m(u+1)} a_h = \delta > 0.$$

Then

$$\limsup_{u\to\infty} \sum_{h=M(u)+1}^{M(u+1)} a_h \geqslant \delta. \text{ Let } Y = \left\{u : \sum_{h=M(u)+1}^{M(u+1)} a_h \geqslant \delta/2\right\}$$

so that Y is infinite. Then as in the proof of *Theorem* 3 we can obtain $u(1) < u(2) < u(3) < \cdots$ with $u(r) \in Y$, $q(1) < q(2) < q(3) < \cdots$ and $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ with $F_r \in \mathfrak{J}_{q(r)}$ and $\inf_{x \in F_r} \sum_{h=M(u(r))+1}^{M(u(r)+1)} a_h |\sin hx| \geqslant 1/10 \sum_{h=M(u(r))+1}^{M(u(r)+1)} a_h \geqslant \delta/20$. Let $\{z\} = \bigcap_{r=1}^{\infty} F_r$. Then $\sum_{h=1}^{\infty} a_h |\sin hz|$ diverges as was to be shown.

Arbault (Chapter III, § 1 [1]) has shown that there exist N sets which are not R sets and so not N_0 sets. (In fact he simply proves the existence of R sets which are not N_0, but, as Bary points out, the stronger result easily follows on examination of his construction (Chapter XII, § 7 [2])). His ingenious proof effectively depends on showing that a perfect subset E of the points of convergence of $\sum_{n=1}^{\infty} 1/n\,|\sin 2^n\pi x|$ exists such that for any $n(1) < n(2) < \cdots$

$$\limsup_{k\to\infty} |\sin n(k)y| > 0 \text{ for some } y \in E.$$

Since E is automatically an R set this gives the result.

We conclude this section by giving an alternative proof obtained by modifying the construction in *Theorem* 3. The set E we obtain is independent (since weak Kronecker) and this, as we said in Section 2, constitutes the novelty of the result. The nature of Arbault's proof does not seem to yield such an additional condition very easily. *Lemma* 4.7 (*ii*) forms an expected complement to *Lemma* 4.6. Before commencing the proof, we briefly state the idea behind it. This is to remark that $\sum_{n=1}^{\infty} 1/n\ a_n$ may converge if a_n is usually small but at increasingly rare intervals takes values near 1 (e.g. to take an extreme example: if $a_n = 1$ when $n \in \{m^4;\ m \in \mathbf{Z}\}$, $a_n = 0$ otherwise we have $\sum_{n=1}^{\infty} 1/n\ a_n = \sum_{n=1}^{\infty} n^{-4} < \infty\Big)$. We obtain this kind of situation in which for any particular $z \in E$, $\sin m(r)z$ is large very infrequently by allowing the number of blocks $\mathcal{A}_0, \ldots, \mathcal{A}_m$ which we have been considering to increase rapidly.

LEMMA 4.7 : (*i*) *There exists a perfect weak Kronecker set* E *which is an* N *set and yet such that for any* ξ_r *and any* $m(r) \to \infty$ *we can find a* $z \in E$ *with* $\sin(m(r)z + \xi_{m(r)}) \not\to 0$. *In particular* E *is not an* R *set* (*and so not an* N_0 *set*).

(*ii*) *Given* $m(1) < m(2) < m(3) < \cdots$ *we can construct a perfect weak Kronecker set* E *such that, if* $a_r \geq 0$,

$$\limsup_{u\to\infty} \sum_{h=m(u)+1}^{m(u+1)} a_h > 0$$

implies $\sum_{h=1}^{\infty} a_h\ |\sin hz|$ *divergent for some* $z \in E$, *but* E *is an* R *set.*

Proof. — We adopt the notation established in the first paragraph of the proof of *Theorem* 3 but with

$K(t+1) = K(t) + t^4$, and attempt, as far as possible, the mimic the remainder of that proof. As usual the reader should simply note the divergences.

Suppose $K(t) \leqslant r \leqslant K(t+1) - 1$. We write

$$s = r - K(t) + 1,\ L(r, q) = (t+q)^4 \text{ and } N(r) = t.$$

At the r^{th} step we have $\mathcal{J}_r = \mathcal{A}_{r0} \cup \mathcal{A}_{r1} \cup \mathcal{A}_{r2} \cup \cdots \cup \mathcal{A}_{rL(r,1)}$ where the $\mathcal{A}_{r0}, \mathcal{A}_{r1}, \ldots, \mathcal{A}_{rL(r,1)}$ are disjoint and non empty and $F \in \mathcal{A}_{rs}$ implies $M(r)$ diam $F \geqslant 2\pi$. We also have $\mathcal{B}_{r1}, \mathcal{B}_{r2}, \ldots, \mathcal{B}_{rL(r,1)}$ disjoint non empty with

$$\mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \cdots \cup \mathcal{B}_{rL(r,1)} = \mathcal{A}_{r0} \cup \mathcal{A}_{r1} \cup \cdots \cup \mathcal{A}_{r,s-1}.$$

By *Lemma* 3.3 applied twice we can find a $P(r) > Q(r) > M(r)$ and for each $F \in \mathcal{A}_{rk}[0 \leqslant k \leqslant L(r,0), k \neq s]$ a closed interval $H(F, r) \subseteq F$ with diam $H(F, r) \leqslant 1/2$ diam F such that $\sup_{x \in H(F,r)} |\sin Q(r)x| \leqslant \sup_{x \in H(F,r)} |\chi_{Q(r)}(x) - 1| \leqslant 1/r$ and

$$\sup_{x \in H(F,r)} |\chi_{P(r)}(x) - f_{N(r)}(x)| \leqslant 2^{-N(r)}.$$

There exists an $M(r+1) > P(r)$ such that

$$M(r+1) \text{ diam } H(F, r) \geqslant 2\pi$$

for all $F \in \mathcal{A}_{rk}[0 \leqslant k \leqslant N(r), k \neq s]$. Two cases now arise according as $s < N(r)$ or $s = N(r)$. First suppose $s < N(r)$. As in *Theorem* 3 we can find $2\,L(r, 1)(M(r+1) - M(r))$ disjoint (closed) intervals $J(F, l, n, r, s, \nu) \subseteq F$ such that $|\sin nx| \geqslant 9/10$ for all $x \in J(F, l, n, r, s, 0)$ and $|\sin nx| \leqslant 1/10$ for all $x \in J(F, l, n, r, s, 1)$

$[F \in \mathcal{A}_{rs}, 1 \leqslant l \leqslant L(r, 1), M(r) + 1 \leqslant n \leqslant M(r+1), \nu = 0,1]$.

We set $\mathcal{A}_{r+1k} = \{H(F, r) : \mathcal{A}_{rk}\}$ $[0 \leqslant k \leqslant L(r, 0), k \neq s]$

$$\mathcal{A}_{r+1s} = \{J(F, l, n, r, s, \nu) : F \in \mathcal{A}_{rs}, 1 \leqslant l \leqslant L(r, 1),\\ M(r) + m \leqslant n \leqslant M(r+1), 0 \leqslant \nu \leqslant 1\}$$

and

$$\mathcal{B}_{r+1l} = \{H(F, r) : F \in \mathcal{B}_{rl}\} \cup \{J(F, l, n, r, s, \nu) : F \in \mathcal{A}_{rs},\\ M(r) + 1 \leqslant n \leqslant M(r+1),\ 0 \leqslant \nu \leqslant 1\}$$

$[1 \leqslant l \leqslant L(r, 1)]$.

Now suppose $s = N(r)$. As before, we can find

$$2N(r, 2)(M(r+1) - M(r))$$

disjoint intervals $J(F, l, n, r, s, \nu) \subseteq F$ such that

$$|\sin nx| \geqslant 9/10$$

for all $x \in J(F, l, n, r, s, 0)$ and $|\sin nx| \leqslant 1/10$ for all $x \in J(F, l, n, r, s, 1)$

$[F \in \mathcal{A}_{rs}, 1 \leqslant l \leqslant L(r, 2), 1 + M(r) \leqslant n \leqslant M(r+1), \nu = 0,1]$.

We now set

$$\begin{aligned} \mathcal{A}_{r+10} &= \{J(F, l, n, r, s, \nu) : F \in \mathcal{A}_{rs}, 1 \leqslant l \leqslant L(r, 2), \\ &\qquad M(r) + 1 \leqslant n \leqslant M(r+1),\ 0 \leqslant \nu \leqslant 1\}, \\ \mathcal{A}_{r+1k} &= \{H(F, r) : F \in \mathcal{A}_{rk}\}\ [1 \leqslant k \leqslant L(r, 1)] \text{ and} \\ \mathcal{B}_{r+1l} &= \{J(F, l, n, r, s, \nu) : F \in \mathcal{A}_{rs}, \\ &\qquad 1 + M(r) \leqslant n \leqslant M(r+1),\ 0 \leqslant \nu \leqslant 1\} \end{aligned}$$

$[1 \leqslant l \leqslant L(r, 2)]$. We note that $L(r+1, q) = L(r, q+1)$ and restart the induction.

As before we see that E is weak Kronecker and that for any ξ_r and any $m(r) \to \infty$ there exists a $z \in E$ with $\sin(m(r)z + \xi_{m(r)}) \not\to 0$. We now show that

$$\sum_{n=1}^{\infty} 1/n|\sin Q(n)x|$$

converges for all $x \in E$. Since $Q(n) \to \infty$ this will show that E is an N set and complete the proof. Suppose $y \in E$. Then, for each $r, y \in F(y, r) \in \mathcal{A}_{rs(y)}$ for some $s(y)$ and some $F(y, r)$. We observe that $s(y)$ is a function of $t = N(r)$ only and so we may write $s(y) = u(t)$. Now $1/r\ |\sin ry| \leqslant 1/r^2$ for $r \neq u(t)$ and $1/r\,|\sin ry| \leqslant 1/r \leqslant (t-1)^{-4}$ for $r = u(t)$ $[t \geqslant 3]$. Since $\sum_{r=1}^{\infty} 1/r^2$ and $\sum_{n=2}^{\infty} (t-1)^{-4}$ converge absolutely, both the convergence of $\sum_{n=1}^{\infty} 1/n\ |\sin Q(n)x|$ and the lemma follow.

Proof of (*ii*). — This is obtained by modifying the proof of *Lemma* 4.7 in the same way as we have just modified the proof of *Theorem* 3 to get a proof of (*i*). We leave this as a, very mildly, instructive exercise. It need hardly be remarked that we can construct an E satisfying the conditions of (*i*) and (*ii*) (and therefore of *Lemma* 4.7 and *Theorem* 3) simultaneously. This too is left as an exercise.

5. Kronecker and Dirichlet Sets.

The main result of this section is *Theorem* 4 that there exists a countable independent Dirichlet set which is not Kronecker. As usual we give a long heuristic preamble which may be skipped or not as the reader feels more useful.

We revert to the methods of the first part of Section 3 selecting a γ independent (i.e. $\gamma \notin 2\pi\mathbf{Q}$) and choosing distinct points $\alpha_1, \alpha_2, \alpha_3, \ldots$ successively such that $\alpha_r \to \gamma$ as $r \to \infty$ and $\{\gamma, \alpha_1, \ldots, \alpha_r\}$ is independent. It should be noted that (by choice rather than absolute necessity) γ is not treated on par with the α_r in this construction, but is given a less prominent role. We set $E = \{\gamma, \alpha_1, \alpha_2, \ldots\}$ and have E closed and independent. If we can ensure that for every $1 \leqslant w$ we can find an $r \geqslant 1$ such that $\chi_w(\alpha_r)$ is far from -1 we shall have obtained E non Kronecker (since -1 cannot be approximated by characters on E). Suppose we have so far succeeded that we have found $\alpha_1, \alpha_2, \ldots, \alpha_h$ independent such that for every $1 \leqslant w \leqslant N$ we can find an $h \geqslant r \geqslant 1$ for which $\chi_w(\alpha_r)$ is far from -1. By Kronecker's theorem we can now find a $P \geqslant 4N$, say, such that $\chi_P(\gamma)$ is close to 1 and $\chi_P(\alpha_r)$ is very close indeed to 1 for $1 \leqslant r \leqslant h$. Let us closely examine what this means. It means that, for $1 \leqslant r \leqslant h$, $\chi_w(\alpha_r)$ is « practically periodic » in w with « period » P, i.e. that the values of $\chi_v(\alpha_r)$, $\chi_{v+P}(\alpha_r)$, $\chi_{v+2P}(\alpha_r)$, $\ldots$, $\chi_{v+tP}(\alpha_r)$ are very close for t quite large (but not too large). In particular we have that for every $|w| < N$ there exists an $1 \leqslant r \leqslant h$ such that $\chi_w(\alpha_r)$ and thus $\chi_{w+tP}(\alpha_r)$ is far from -1 for t not too large (since

$$|\chi_w(x) + 1| = |\chi_{-w}(x)||1 + \chi_{-w}(x)| = |\chi_{-w}(x) + 1|,$$

$\chi_w(\alpha_r)$ far from -1 implies $\chi_{-w}(\alpha_r)$ far from -1). However, we have no control over the constructed points in the « gaps »

$$N < w < P - N, P + N < w < 2P - N,$$

and so on.

Clearly we must add more points to ensure that $\sup_{x \in E} |\chi_w(x) + 1|$ is large in these gaps. But here we recall that we wish E to be Dirichlet. It would therefore be appropriate, in view of our definition of P, to have $\sup_{x\in E}|\chi_P(x)-1|$ small. Ignoring for the moment the independence condition this suggests adding points k_q of the form $2q\pi/P$ for which certainly $\chi_P(k_q)=1$. This is however not the only complication, since we wish E to have a single limit point γ. We therefore confine our scrutiny to those k_q near γ. If among these, given any $N < w \leqslant P - N$, we can find a k_q such that $\chi_w(k_q)$ is far from -1 (and so since $\chi_w(k_q)$ is periodic in w with period P we have $\chi_{w+P}(k_q), \chi_{w+2P}(k_q), \ldots, \chi_{w+tP}(k_q)$ far from -1) then perturbing the k_q slightly to obtain $\alpha_{h+1}, \ldots, \alpha_{h'}$ with $\gamma, \alpha_1, \alpha_2, \ldots, \alpha_{h'}$ independent, we may hope to have for some large N′ that $\sup_{1\leqslant r\leqslant h'}|\chi_w(\alpha_r)+1|$ is large for all $1 \leqslant r \leqslant N'$ yet $\sup_{1\leqslant r\leqslant h'}|\chi_P(\alpha_r)-1|$ is small. We then restart the induction.

On the face of it this does not look a very promising program, since even if the k_q can be chosen to satisfy our conditions, it is by no means clear that the induction will not break down. Yet, surprisingly, this naive approach works. As we remarked in Section 2 this proof stands out from the others in this paper in that once their idea is grasped it is obvious that they work, whereas here the idea does not seem sufficiently powerful and only the full proof can provide the necessary verification.

First we have a, perhaps overcomplicated, proof of the following easy result:

LEMMA 5.1. — *Suppose* $a, b, N, P \in \mathbf{Z}$ *and*

$$(b-a)N > P > 4N > 0.$$

Let τ *be a constant with* $|\tau| = 1$ *and set* $k_q = 2q\pi/P$ *for* $a \leqslant q \leqslant b$. *Then for any* $N \leqslant r \leqslant P - N$ *we can find a* $a \leqslant p \leqslant b$ *such that* $|\chi_r(k_p)-\tau| \geqslant 1/2$.

Proof. — $|\chi_r(k_m)-\chi_r(k_n)| \leqslant |\chi_r(k_m)-\tau| + |\chi_r(k_n)-\tau|$ so that $\sup_{a\leqslant q\leqslant b}|\chi_r(k_q)-\tau| = 1/2 \sup_{a\leqslant n\leqslant m\leqslant b}|\chi_r(k_m)-\chi_r(k_n)|$. Now $|\chi_r(k_m)-\chi_r(k_n)| = |\chi_r(k_m-k_n)-1| = |\exp(2\pi iur/P)-1|$ where $u = m - n$. We therefore consider $\exp(2\pi iur/P)$ as u increases from 0 to $b-a$. Either $\pi/2 \leqslant 2\pi r/P \leqslant 3\pi/2$ in which case $|\exp(2\pi iu/P)-1| \geqslant 1$ for $u = 1$ or the unit vector representing $\exp(2\pi iu/\mathrm{P})$ in the complex plane rotates through an angle less than (in modulus) $\pi/2$ as u increases by 1. But $(b-a)\mathrm{N} > \mathrm{P}$ so the unit vector representing $\exp(2\pi iu/\mathrm{P})$ must rotate through over 2π (in modulus) as u increases from 0 to $b-a$, and in the second case it is clear that the unit vector must lie in the left hand half $\{z \in \mathbf{C} : \operatorname{Re} z \leqslant 0\}$ of the complex plane for some $0 \leqslant u \leqslant b-a$, i.e. $|\exp(2\pi iu/\mathrm{P}) - 1| \geqslant 1$ for some $0 \leqslant n \leqslant b-a$.

Thus

$$\sup_{a \leqslant q \leqslant b} |\chi_r(k_q) - \tau| \geqslant 1/2 \sup_{0 \leqslant u \leqslant b-a} |\exp(2\pi iur/\mathrm{P}) - 1| \geqslant 1/2$$

as required.

Using this we now prove

THEOREM 4. — *There exists a countable independent Dirichlet set* E *which is not Kronecker.*

Proof. — We construct such a set E. Select γ independent. We give the central inductive step in the construction of E. Suppose we have at the n^{th} stage $\gamma, \alpha_1, \alpha_2, \ldots, \alpha_{h(n)}$ independent, $\mathrm{N}(n) = 2^{n+7}\mathrm{P}(n)$, $\mathrm{P}(n) \geqslant 10$ such that

$$\sup_{1 \leqslant w \leqslant h(n)} |\chi_r(\alpha_w) + 1| = 1/8\ (1 + 2^{-n}) \quad \text{for all} \quad 1 \leqslant r \leqslant \mathrm{N}(n).$$

Now by Kronecker's theorem there exists a

$$\mathrm{P}(n+1) \geqslant 4\mathrm{N}(n)$$

such that $|\chi_{\mathrm{P}(n+1)}(\alpha_w) - 1| \leqslant 2^{-(2n+12)}$ for all $1 \leqslant w \leqslant h(n)$ and $|\chi_{\mathrm{P}(n+1)}(\gamma) - 1| \leqslant 2^{-(n+2)}$. Trivially

$$\begin{aligned} |\chi_{t\mathrm{P}(n+1)}(\alpha_w) - 1| &\leqslant \sum_{s=1}^{t} |\chi_{s\mathrm{P}(n+1)}(\alpha_w) - \chi_{(s-1)\mathrm{P}(n+1)}(\alpha_w)| \\ &\leqslant t|\chi_{\mathrm{P}(n+1)}(\alpha_w) - 1| \\ &\leqslant 2^{-(n+4)} \quad \text{for} \quad 1 \leqslant w \leqslant h(n),\ 1 \leqslant t \leqslant 2^{n+8}. \end{aligned}$$

Set $\mathrm{N}(n+1) = 2^{n+8}\mathrm{P}(n+1)$. Since

$$2^{-(n+4)} \frac{\mathrm{P}(n+1)\mathrm{N}(n)}{\mathrm{P}(n)} \geqslant \mathrm{P}(n+1)$$

we can find $a(n+1)$, $b(n+1) \in \mathbf{Z}$ such that

$$(*) \qquad \begin{gathered}[b(n+1)-a(n+1)]\mathrm{N}(n) \geqslant \mathrm{P}(n+1)\\ \frac{b(n+1)-a(n+1)}{\mathrm{P}(n+1)} \leqslant \frac{2^{-(n+4)}}{\mathrm{P}(n)}\end{gathered}$$

and

$$\gamma \in \left[\frac{2a(n+1)\pi}{\mathrm{P}(n+1)}, \frac{2b(n+1)\pi}{\mathrm{P}(n+1)}\right]$$

Set $k_{q,n+1} = \dfrac{2\,q\pi}{\mathrm{P}(n+1)}$ for $a(n+1) \leqslant q \leqslant b(n+1)$. Then trivially $\chi_{\mathrm{P}(n+1)}(k_{q,n+1}) = 1$. Also if

$$(t-1)\mathrm{P}(n+1) + \mathrm{N}(n) \leqslant r \leqslant t\mathrm{P}(n+1) - \mathrm{N}(n)$$

then by *Lemma* 5.1

$$\sup_{a(n+1)\leqslant q\leqslant b(n+1)} |\chi_r(k_{q,n+1}) + 1| \geqslant 1/2 \geqslant 1/4(1 + 2^{-(n+1)}),$$

whilst if $0 \leqslant v \leqslant \mathrm{N}(n)$, $0 \leqslant t \leqslant 2^{n+8}$ then

$$\begin{aligned}\sup_{1\leqslant w\leqslant h(n)} &|\chi_{t\mathrm{P}(n+1)+v}(\alpha_w) + 1|\\ &= \sup_{1\leqslant w\leqslant h(n)} |\chi_{t\mathrm{P}(n+1)}(\alpha_w) + \chi_{-v}(\alpha_w)|\\ &\geqslant \sup_{1\leqslant w\leqslant h(n)} \big||\chi_{-v}(\alpha_w) + 1| - |\chi_{t\mathrm{P}(n+1)}(\alpha_w) + 1|\big|\\ &= \sup_{1\leqslant w\leqslant h(n)} \big||1 + \chi_v(\alpha_w)| - |\chi_{t\mathrm{P}(n+1)}(\alpha_w) + 1|\big|\\ &\geqslant |1/8(1 + 2^{-n}) - 2^{-n-4}|\end{aligned}$$

and similarly

$$\sup_{1\leqslant w\leqslant h(n)} |\chi_{t\mathrm{P}(n+1)+v}(\alpha_w) + 1| \geqslant 1/8(1 + 2^{-(n+1)}).$$

Now writing

$$\begin{aligned}\mathrm{Y}(n+1) = \{t\mathrm{P}(n+1) + \mathrm{N}(n) \leqslant r\\ \leqslant (t+1)\mathrm{P}(n+1) - \mathrm{N}(n) :\ 0 \leqslant t \leqslant 2^{n+8} - 1\}\end{aligned}$$

we know that the characters χ_r with $r \in \mathrm{Y}(n+1)$ are continuous so by *Lemma* 1.3 we can find

$$\alpha_{h(n)+r+1} \in \left[\frac{2a(n+1)\pi}{\mathrm{P}(n+1)}, \frac{2b(n+1)\pi}{\mathrm{P}(n+1)}\right]$$

« near to $k_{h(n)+r,n+1}$ » $[0 \leqslant r \leqslant b(n+1) - a(n+1)]$ such that writing $h(n+1) = h(n) + 1 + b(n+1) - a(n+1)$ we have $\gamma, \alpha_1, \ldots, \alpha_{h(n+1)}$ independent,

$$\sup_{h(n)\leqslant w < h(n+1)} |\chi_{P(n+1)}(\alpha_w) - 1| \leqslant 2^{-(n+1)}$$

and $\sup\limits_{h(n)+1 < w \leqslant h(n+1)} |\chi_r(\alpha_w) + 1| \geqslant 1/8(1 + 2^{-(n+1)})$. This together with what we already know about $\alpha_1, \alpha_2, \ldots, \alpha_{h(n)}$ gives $\sup\limits_{1\leqslant w < h(n+1)} |\chi_r(\alpha_w) + 1| \geqslant 1/8 \ (1 + 2^{-(n+1)})$ for

$$1 \leqslant r \leqslant N(n+1), \quad \text{and} \quad |\chi_{P(n+1)}(\alpha_w) - 1| \leqslant 2^{-(n+1)}$$

for $1 \leqslant w \leqslant h(n+1)$. Moreover $|\gamma - x| \leqslant \dfrac{2^{-(n+5)}}{P(n+1)} \cdot 2\pi$ implies

$$\begin{aligned}|\chi_{P(n+1)}(x) - 1| &\leqslant |\chi_{P(n+1)}(\gamma) - 1| + |\chi_{P(n+1)}(x) - \chi_{P(n+1)}(\gamma)| \\ &= |\chi_{P(n+1)}(\gamma) - 1| + |\exp(i(\gamma - x)P(n+1)) - 1| \\ &\leqslant 2^{-(n+2)} + 2^{-(n+2)} = 2^{-(n+1)}\end{aligned}$$

and looking at (*) we see that all the points we construct in the later stages will indeed satisfy this condition. We now recommence the induction.

Taking $E = \{\gamma\} \cup \{\alpha_w : w \geqslant 1\}$ we have at once E closed. Also E is independent, since if $\gamma, \alpha_{j(1)}, \ldots, \alpha_{j(m)}$ are given, then $j(1), j(2), \ldots, j(m) \leqslant h(n)$ for some n, and so $\gamma, \alpha_{j(1)}, \alpha_{j(2)}, \ldots, \alpha_{j(m)}$ are independent. By construction $P(n) \to \infty$ and $\sup\limits_{x\in E} |\chi_{P(n)}(x) - 1| \leqslant 2^{-n} \to 0$ as $n \to \infty$ so E is Dirichlet. But $\sup\limits_{x\in E} |\chi_r(x) + 1| \geqslant 1/8$ for all $r \geqslant 0$ so E is not Kronecker.

Varopoulos [18] has shown that the independent union of a Kronecker set and a single point is still Kronecker. Thus $E(s) = \{\gamma\} \cup \{\alpha_r : r \geqslant s\}$ also satisfies the conditions of our theorem. However, by inspection $\inf\limits_{r\geqslant s} \sup\limits_{x\in E(s)} |\chi_r(x) + 1| \to 0$ as $s \to \infty$. Moreover a little thought shows that the convergence can be very rapid. (On the other hand, writing

$$F(s) = \{\gamma\} \cup \{\alpha_r : r \neq s \ r \geqslant 1\}$$

we have $\inf\limits_{r\geqslant 1} \sup\limits_{x\in F(s)} |\chi_r(x) + 1| \to \inf\limits_{r\geqslant 1} \sup\limits_{x\in E} |\chi_r(x) + 1|$ and indeed the removal of points constructed in the later stages need have very little effect.)

The following easy Lemma also shows how closely the behaviour of χ_n on a Dirichlet set E for large n is bound up with its behaviour for small n.

LEMMA 5.2. — *If* E *is weak Dirichlet and* $\mu \in M(E)$ *then* $\limsup_{|n|\to\infty} |\hat{\mu}(n)| = \sup_n |\hat{\mu}(n)|$. *In particular, if* $\mu \neq 0$ *then* $\limsup_{|n|\to\infty} |\hat{\mu}(n)| > 0$.

Proof. — Suppose $\varepsilon > 0$, m, $p \in \mathbf{Z}$ given. Then by definition we can find a $q > m - p$ such that

$$|\mu|\{x : |\chi_q(x) - 1| \geqslant \varepsilon/20\} \leqslant \varepsilon/4.$$

Setting $n = q + p$, we have $n > m$ and

$$\begin{aligned} |\hat{\mu}(-n)| &= \left|\int \chi_n \, d\mu\right| \\ &\geqslant \left|\int \chi_p \, d\mu\right| - \left|\int (\chi_n - \chi_p) \, d\mu\right| \\ &\geqslant |\hat{\mu}(p)| - \int |\chi_n - \chi_p| \, d\mu \\ &= |\hat{\mu}(p)| - \int |\chi_q - 1| \, d\mu \\ &\geqslant |\hat{\mu}(p)| - (2\pi\, \varepsilon/20 + 2\, \varepsilon/4) \\ &\geqslant |\hat{\mu}(p)| - \varepsilon. \end{aligned}$$

Thus $\limsup_{n\to-\infty} |\hat{\mu}(n)| \geqslant \sup |\hat{\mu}(p)|$ and similarly

$$\limsup_{n\to\infty} |\hat{\mu}(n)| \geqslant \sup |\hat{\mu}(p)|.$$

Combining the ideas of *Theorems* 2 and 4 we have *Theorem* 5. This is included chiefly for the sake of completeness, since the main result, that there exist perfect independent Dirichlet non Kronecker sets, can be obtained directly from *Theorem* 4 using *Lemma* 6.1 below, whilst in *Theorem* 9 we obtain a much stronger result.

THEOREM 5. — *There exists a perfect Dirichlet non Kronecker set* E *such that every proper closed subset of* E *is Kronecker.*

Proof. — We obtain E as the closure of a set F constructed by an induction of which the following is the n^{th} step. We have $A(n)$, $B(n)$ 2 finite sets of points ($A(n)$ consists of points constructed in this cycle, $B(n)$ of points constructed earlier) such that $C(n) = A(n) \cup B(n)$ is independent, and $b_n \in B(n)$ (b_n is a marker point just as F_r^* was a marker set in *Theorem* 2). We also have $N(n) = 2^{n+7} P(n)$, $P(n) \geqslant 10$ (playing the roles assigned to them in *Theorem* 4) and $M(n)$

(a counter, telling us that we are on the $M(n)^{\text{th}}$ cycle of the induction, i.e. playing the role of $N(r)$ in *Theorem* 2), such that $\sup_{x \in C(n)} |\chi_r(x) + 1| \geqslant 1/8(1 + 2^{-n})$ for all $1 \leqslant r \leqslant N(n)$. Now there exists a $Q(n+1) \geqslant 2N(n)$ such that

$$\sup_{x \in C(n)} |f_{M(n)}(x) - \chi_{Q(n+1)}(x)| \leqslant 2^{-(M(n)+1)}$$

and a $P(n+1) \geqslant 2Q(n+1)$ such that

$$|\chi_{P(n+1)}(x) - 1| \leqslant 2^{-(2n+12)} \quad \text{for all} \quad x \in C(n).$$

As in *Theorem* 4 we can find a finite set of points

$$D(n+1) \subseteq N\left(b_n, \frac{2^{-(n+4)}}{P(n-1)} 2\pi\right)$$

such that setting $C(n+1) = D(n+1) \cup A(n) \cup B(n)$ we have $C(n+1)$ independent,

$$\sup_{x \in C(n+1)} |\chi_r(x) + 1| \geqslant 1/8\,(1 + 2^{-(n+1)})$$

for

$$1 \leqslant r \leqslant N(n+1) = 2^{n+8}P(n+1)$$

and $|\chi_{P(n+1)}(x) - 1| \leqslant 2^{-(n+1)}$ for all $x \in C(n+1)$. Two cases now arise (according as we have or have not completed a cycle, i.e. a complete rotation of the marker point). Let the 2 points in $B(n) \cup \{0\}$ nearest to b_n in the direction θ increasing from 0 to 2π be c_n, d_n in that order. If $c_n \neq 0$ set $b_{n+1} = c_n$, $B(n+1) = B(n)$, $A(n+1) = A(n) \cup D(n+1)$ and $M(n+1) = M(n)$. If $c_n = 0$ set

$$b_{n+1} = d_n,\ B(n+1) = C(n+1),\ A(n+1) = \varnothing$$

and $M(n+1) = M(n) + 1$. We now restart the induction.

Let $F = \bigcup_{n=1}^{\infty} C(n)$ and set $E = \overline{F}$. By the arguments of *Theorem* 4 $\sup_{x \in E} |\chi_{P(n)}(x) - 1| = \sup_{x \in F} |\chi_{P(n)}(x) - 1| \leqslant 2^{-(n+1)} \to 0$. Clearly E is perfect. Suppose K is a proper closed subset of E. To show K Kronecker we argue much as in *Theorem* 2 and note that there exists a $y \in E/K$ and a $\delta > 0$ such that $N(y, \delta) \cap K = \varnothing$, and that there exists a $q_0 \geqslant 20$ such that for all $q \geqslant q_0$ we can find an n with $q = M(n)$ and

$N\left(b_n, \frac{2^{-(n+4)}}{P(n)} 2\pi\right) \subseteq N(y, \delta)$. The remainder of the argument (which, however, the reader may be prepared to take on trust) is slightly obscured by notational difficulties. Recall the definitions of g_i, f_i given at the end of Section 2. The result will be proved if given any g_i and $\varepsilon > 0$ we can find an l such that $\sup_{x \in F} |g_i(x) - \chi_l(x)| < \varepsilon$. To do this we first note that g_i is continuous and so there exists a $\eta > 0$ such that $\sup_{z, y \in T, |z-y|<\eta} |g_i(z) - g_i(y)| < \varepsilon/3$. By the definition of $f_1, f_2, f_3, \ldots$ we can find arbitrarily large q with $f_q = g_i$. In particular we can find n, q with $a = M(n)$, $f_q = g_i$, $N\left(b_n, \frac{2^{-(n+4)}}{P(n)} 2\pi\right) \subseteq N(y, \delta)$ and $2^{-n} \leqslant \eta$, $2^{-M(n)} \leqslant \varepsilon/3$. Then setting $\delta = \frac{2^{-(n+4)}}{P(n+1)} 2\pi$ so $\delta < \eta$, we have

$$
\begin{aligned}
&\sup_{x \in E} |\chi_{Q(n+1)}(x) - f_{M(n)}(x)| \\
&\qquad \leqslant \sup_{z \in C(n) \setminus \{b_n\}} |\chi_{Q(n+1)}(z) - f_{M(n)}(z)| \\
&\qquad + \sup_{y \in E} \inf_{z \in ((n)} |\chi_{Q(n+1)}(z) - \chi_{Q(n+1)}(y)| \\
&\qquad + \sup_{y \in E} \inf_{z \in C(n)} |f_{M(n)}(z) - \chi_{M(n)}(y)| \\
&\qquad \leqslant 2^{-(M(n)+2)} + \sup_{y \in E} \inf_{z \in C(n)} |\exp\,(Q(n+1)(z-y)) - 1| \\
&\qquad + \sup_{y \in E} \inf_{z \in C(n)} |f_q(z) - f_q(y)| \\
&\qquad \leqslant 2^{-(M(n)+2)} + \left| \exp\left(2^{-(n+4)} \frac{Q(n+1)}{P(n+1)} 2\pi\right) - 1 \right| \\
&\qquad + \sup_{z, y \in T\, |z-y|<\delta} |f_q(z) - f_q(y)| \\
&\qquad \leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}
$$

This completes the proof.

We obtain the following corollaries analogous to those of *Theorem* 2, again included mainly for completeness.

COROLLARY 5.1. — *There exist* K, L *disjoint perfect Kronecker sets such that* $K \cup L$ *is independent Dirichlet but not Kronecker.*

COROLLARY 5.2. — *There exists a perfect independent weak Kronecker set* E *which is Dirichlet but not Kronecker.*

In passing it should be noted that *Theorem* 2 renders any attempt to obtain an independent Dirichlet non Kronecker set directly by taking a Dirichlet subset of an independent non Kronecker set considerably more complicated than it appears at first sight.

Because of the interest attached to the classification of subgroups of S (and not because of any intrinsic interest) we remark that a trivial modification of *Theorem* 4 gives

LEMMA 5.3. — *If* $f \in S$ *and* $f \notin \{\chi_n : n \in \mathbf{Z}\}$ *then there exists a countable independent Dirichlet set* E *for which* $\inf_{n\in\mathbf{Z}} \sup_{x\in E} |\chi_n(x) - f(x)| \neq 0$.

Proof. — Choose γ independent. By the continuity of f (and the compactness of $\mathbf{T}$) we can find a $\eta > 0$ such that $x \in N(\gamma, \eta)$ implies $|f(x) - f(\gamma)| \leqslant 1/4$. Then with the notation of *Lemma* 5.1 if $[k_a, k_b] \subseteq N(x, \eta)$ we have

$$\sup_{a<q<b} |\chi_r(k_q) - f(k_q)| \geqslant 1/4. \quad \text{Now} \quad \{\chi_n : n \in \mathbf{Z}\}$$

is a closed subset of S so that $\inf_{n\in\mathbf{Z}} \sup_{x\in\mathbf{T}} |\chi_n(x) - f(x)| \geqslant \delta$ for some $1/16 > \delta > 0$. Using *Lemma* 1.3 (*i*) we can thus choose P(1) such that $\frac{2^{-3}}{P(1)} \leqslant \frac{\eta}{4}$, $N(1) = 2^8 P(1)$, and find $\alpha_1, \alpha_2, \ldots, \alpha_{h(1)}$ with $\gamma, \alpha_1, \ldots, \alpha_{h(1)}$ independent and $\sup_{1<w<h(1)} |\chi_r(\alpha_w) + 1| \geqslant 1/8\ \delta\ (1 + 2^{-1})$. An easy rewriting of the inductive argument of *Theorem* 4 (e.g. we might take $P(n+1)$ such that $\sup_{l\leqslant w<h(n)} |\chi_r(\alpha_w) + 1| \geqslant 1/8\ \delta\ (1 + 2^{-n})$) now gives the result.

Méla has raised in conversation the question of how far the results of *Theorem* 1 and *Theorem* 4 carry over from $\mathbf{T}$ to $\mathbf{R}$. The rest of this section will be devoted to this topic. We prove no deep results and will not use what we do prove later so that the reader may, if he wishes, simply skip this part of the paper.

First we establish some appropriate definitions. A closed set $E \subseteq \mathbf{R}$ is called *Dirichlet* if (setting $\chi_\lambda(t) = \exp i\lambda t$ for $t \in \mathbf{R}, \lambda \in \mathbf{R}$) we can find $\lambda(r) \to \infty$ such that

$$\sup_{x\in E} |\chi_{\lambda(r)}(x) - 1| \to 0 \quad \text{as} \quad r \to \infty.$$

It is more difficult to find a notion of a Kronecker set for $\mathbf{R}$ parallel to that for $\mathbf{T}$ and 2 alternative definitions have been proposed. We call a discrete set $E \subseteq \mathbf{R}$ (i.e. a set E for which $\inf_{x,\, y \in E, x \neq y} |x-y| > 0$) *discrete Kronecker* if given an $\varepsilon > 0$ and any $f: E \to \mathbf{C}$ with $|f(E)| = 1$ we can find a $\lambda \in \mathbf{R}$ with $\sup_{x \in E} |\chi_\lambda(x) - f(x)| \leqslant \varepsilon$. Varopoulos [19] has proposed that a closed set $E \subseteq \mathbf{R}$ be called *uniform Kronecker* if given an $\varepsilon > 0$ and any uniformly continuous $f \in C(\mathbf{R})$ with $f(0) = 1$ and $|f(E)| = 1$ we can find a $\lambda \in \mathbf{R}$ with $\sup_{x \in E} |\chi_\lambda(x) - f(x)| \leqslant \varepsilon$. We shall deal with discrete Kronecker sets. But if the reader bears in mind the obvious fact that for E discrete such that $0 \notin E$, E is discrete Kronecker if and only if E is uniform Kronecker, he will be able to extract similar results for unbounded uniform Kronecker sets as we obtain for discrete Kronecker sets.

It is at once obvious (using similar proofs to those of Section 1) that discrete Kronecker sets (and uniform Kronecker sets) are independent and Dirichlet. We show that there exist discrete independent Dirichlet sets which are not Kronecker and that the union of 2 discrete Kronecker sets may be independent and discrete yet not even Dirichlet. Our proofs depend essentially on the following 2 simple facts (the first is a very simple and well known version of a theorem of Hartman and Ryll-Nardzewski [5]).

LEMMA 5.4: *(i) — Suppose* $E = \{x_i : i \geqslant 1\}$, *with* $x_{i+1} > x_i > 0$ $[i \geqslant 1]$ *say, is an independent discrete set with* $x_{i+1}/x_i \to \infty$. *Then* E *is discrete Kronecker.*

(ii) Suppose $E = \{x_i : i \geqslant 1\}$ *with* $x_{i+1} > x_i > 0$ $[i \geqslant 1]$ *say. Then if* E *is discrete Kronecker* $\sum_{i=1}^{\infty} (x_{i+1} - x_i)^{-1}$ *converges.*

Proof (i). — Suppose $0 < y_1 < y_2 < \cdots < y_n$ and $y_1, y_2, \ldots, y_n$ independent. Suppose $y_{i+1}/y_i \geqslant A > 1$ for $i \geqslant n$. Set $F = \{y_i : i \geqslant 1\}$ and suppose $g: F \to \mathbf{C}$ such that $|g(F)| = 1$. By Kronecker's theorem there exists a $\lambda > 0$ such that $\sup_{1 \leqslant i \leqslant n} |\chi_\lambda(y_i) - f(y_i)| \leqslant \varepsilon$. Now since

$\exp it$ has period 2π there exists a $\delta(1)$ with $|\delta(1)| \leqslant y_{n+1}/\pi$ such that $\chi_{\lambda+\delta(1)}(y_{n+1}) = f(y_{n+1})$. Similarly there exist $\delta(2)$, $\delta(3)$, ..., $\delta(r)$, ... with $|\delta(r)| \leqslant y_{n+r}/\pi$ and

$$\chi_{\lambda+\delta(1)+\delta(2)+\cdots+\delta(r)}(y_{n+r}) = f(y_{n+r}).$$

Since $y_{n+r} \geqslant A^r y_n$, $|\delta(r)| \leqslant y_n^{-1}A^{-r}$ and $\lambda + \sum_1^\infty \delta(r)$ converges to η say. Now

$$\begin{aligned}|\chi_\eta(y_{n+r}) - f(y_{n+r})| &= |\chi_\eta(y_{n+r}) - \chi_{\lambda+\delta(1)+\cdots+\delta(r)}(y_{n+r})|\\ &= \left|\exp i\left(y_{n+r}\sum_{s=r+1}^{\infty}\delta(s)\right) - 1\right|\\ &\leqslant \left|y_{n+r}\sum_{s=r+1}^{\infty}\delta(s)\right|\\ &\leqslant \sum_{s=r+1}^{\infty} y_{n+r}|\delta(s)|\\ &\leqslant \sum_{u=1}^{\infty} A^{-u} = A^{-1}/(1-A^{-1})\end{aligned}$$

Similarly, if $1 \leqslant i \leqslant n$ $|\chi_\eta(y_i) - f(y_i)| \leqslant \varepsilon + A^{-1}/(1-A^{-1})$. Since $A^{-1}/(1-A^{-1}) \to 0$ as $A \to \infty$ this proves the result.

(*ii*) Write $E_n = \{x_i : n \geqslant i \geqslant 1\}$. We define inductively functions f_n: $E_n \to \mathbf{C}$ with $|f_n(E_n)| = 1$ such that $f_n(x_r) = f_m(x_r)$ for all $r \leqslant m \leqslant n$. Eventually we shall define f: $E \to \mathbf{C}$ with $|f(E)| = 1$ by $f(x_r) = f_r(x_r)$. Setting

$$\begin{aligned}\lambda(n) &= \lambda(x_1, x_2, \ldots, x_n, f_n)\\ &= \inf\{\lambda \geqslant 0 : |\chi_\lambda(x_r) - f_n(x_r)| \leqslant 1/100 \quad \text{for} \quad 1 \leqslant r \leqslant n\}\end{aligned}$$

we see that $0 \leqslant \lambda(1) \leqslant \lambda(2) \leqslant \ldots \leqslant \lambda(n)$.

The central inductive step runs as follows. Suppose f_n and so $\lambda(n)$ defined. Set $f_{n+1}(x_r) = f_n(x_r)$ for $1 \leqslant r \leqslant n$ and $f_{n+1}(x_{n+1}) = -\chi_{\lambda(n)}(x_{n+1})$. Then if $\lambda \geqslant 0$ and

$$|\chi_\lambda(x_r) - f_{n+1}(x_r)| \leqslant 1/100 \quad \text{for} \quad 1 \leqslant r \leqslant n+1$$

we have $\lambda \geqslant \lambda(n)$ and $|\chi_\lambda(x_{n+1}) + \chi_{\lambda(n)}(x_{n+1})| \leqslant 1/100$, whilst

$$\begin{aligned}|\chi_\lambda(x_n) - \chi_{\lambda(n)}(x_n)| \leqslant |\chi_\lambda(x_n) &- f_n(x_n)|\\ &+ |\chi_{\lambda(n)}(x_n) - f_n(x_n)| \leqslant 2/100.\end{aligned}$$

Thus $|\chi_{\lambda-\lambda(n)}(x_{n+1}) + 1| \leqslant 1/100$ whilst

$$|\chi_{\lambda-\lambda(n)}(x_n) - 1| \leqslant 2/100,$$

so adding, $|\chi_{\lambda-\lambda(n)}(x_{n+1}) + \chi_{\lambda-\lambda(n)}(x_n)| \leqslant 3/100$ whence

$$|\chi_{\lambda-\lambda(n)}(x_{n+1} - x_n) + 1| \leqslant 3/100.$$

Thus

$$|\lambda - \lambda(n)||x_{n+1} - x_n| \geqslant \pi/2$$

and so $\lambda \geqslant \lambda(n) + \pi/(2(x_{n+1} - x_n))$. Hence

$$\lambda(n+1) \geqslant \lambda(n) + \pi/(2(x_{n+1} - x_n)).$$

If $\sum_{m=1}^{\infty} (x_{n+1} - x_n)^{-1}$ diverges, we have $\lambda(n) \to \infty$ as $n \to \infty$. There then exists no $\lambda \geqslant 0$ such that

$$\sup_{r \geqslant 1} |\chi_\lambda(x_r) - f(x_r)| \leqslant 1/100$$

and so (either by repeating the argument this time for $\lambda \leqslant 1$ or by using a result of the type established in *Lemma* 1.2 (*iii*)) E cannot be discrete Kronecker.

We also need

LEMMA 5.5. — *There exists an independent Dirichlet set* $E = \{x_r : r \geqslant 1\}$ *with* $|x_r - 2r\pi| \leqslant 1/10r$.

Proof. — We construct $x_1, x_2, \ldots, x_n, \ldots$ inductively. At the n^{th} step we have $x_1, x_2, \ldots, x_n$ independent, an $N(n) \in \mathbf{Z}$ and a $0 < \delta(n) \leqslant 1/(10(n+1))$. By *Lemma* 1.3 we can find $x_{n+1} \in N(2(n+1)\pi, \delta(n))$ with $x_1, x_2, \ldots, x_{n+1}$ independent. By Kronecker's theorem there exists an $N(n+1) > N(n)$ such that $|\chi_{N(n+1)}(x_r) - 1| \leqslant 2^{-n}$ for $1 \leqslant r \leqslant n$. By the continuity of $\chi_{N(1)}, \chi_{N(2)}, \ldots, \chi_{N(n+1)}$ and the fact that

$$\chi_{N(r)}(2(n+2)\pi) = 1$$

there exists a $0 < \delta(n+1) \leqslant 1/(10(n+2))$ such that

$$|\chi_{N(r)}(x) - 1| \leqslant 2^{-r}$$

for all $x \in N(2(n+2)\pi, \delta(n+1))$ and $1 \leqslant r \leqslant n+1$. We now restart the induction.

Setting $E = \{x_r : r \geqslant 1\}$ we have by construction $|x_r - 2r\pi| \leqslant 1/(10r)$ and $\sup_{x \in E} |\chi_{N(n)}(x) - 1| \leqslant 2^{-n}$. Since $N(n) \to \infty$ this shows E Dirichlet. Suppose $x_{r(1)}, \ldots, x_{r(s)} \in E$.

Then setting $\max_{1 \leqslant k \leqslant s} r(k) = n$ we have that

$$\{x_{r(1)}, x_{r(2)}, \ldots, x_{r(s)}\} \subseteq \{x_1, x_2, \ldots, x_n\}$$

and so is independent. Thus E is independent.

Dirichlet sets need not have the simple form given here. A little thought shows that the construction above can be modified to give e.g. an unbounded independent perfect set which is Dirichlet. Varopoulos [19] has also found uniform Kronecker and thus Dirichlet sets of a similarly complex type. However all we need here is the simple construction of *Lemma* 5.5 and the results of *Lemma* 5.4 to obtain

LEMMA 5.6. — *(i) There exists an independent discrete Dirichlet set which is not discrete Kronecker.*

(ii) There exist 2 disjoint discrete Kronecker sets whose union is discrete independent and Dirichlet but not a discrete Kronecker set.

Proof. — Take $y_r = x_{m(r)}$, $z_r = x_{m(r)+r}$ where $x_1, x_2, x_3, \ldots$ and E are as constructed in *Lemma* 5.5 and $m(r) = 2^{2^r}$. Set $K = \{y_r : r \geqslant 1\}$, $L = \{z_r : r \geqslant 1\}$ and $M = K \cup L$. Now $y_1 < z_1 < y_2 < z_2 < \cdots$ and $\sum_{p=1}^{\infty} (y_r - z_r)^{-1} = \sum_{r=1}^{\infty} r^{-1}$ diverges, so, by *Lemma* 5.4 *(ii)*, M is not Kronecker. But $M \subseteq E$ so M is discrete, independent and Dirichlet. This proves *(i)*. On the other hand $y_{r+1}/y_r \to \infty$, $z_{r+1}/z_r \to \infty$ so, by *Lemma* 5.4 *(i)* K and L are discrete Kronecker. This proves *(ii)*.

It is worth noting though we did not explicitly demand it in the statement of the Lemma that in these counter examples $|z_r - y_r| \to \infty$.

We have also promised to give an example of a union which is not even Dirichlet. This is obtained by a simple modification of the proof of *Lemma* 5.4 *(ii)* and the reader may well wish to skip this.

LEMMA 5.7. — *There exist 2 disjoint Kronecker sets whose union is discrete and independent but not Dirichlet.*

Proof. — We construct inductively $x_1, y_1, x_2, y_2, \ldots$. Suppose $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ have been constructed independent with $1 < x_1 < y_1 < x_2 < \cdots < x_n < y_n$. We write

$$\lambda^*(n) = \inf \{\lambda \geqslant 100 : |\chi_\lambda(x_r) - 1| \leqslant 1/100, \\ |\chi_\lambda(y_r) - 1| \leqslant 1/100, \ 1 \leqslant r \leqslant n\}.$$

By *Lemma* 1.3 we can find an $x_{n+1} \geqslant (n+1)y_n$ with $x_1, y_1, \ldots, x_n, y_n, x_{n+1}$ independent. Since $\chi_{\lambda^*(n)}(t)$ has period $2\pi/\lambda^*(n) \leqslant \pi/50$ (as a function of t) we can find again by *Lemma* 1.3 using the continuity of $\chi_{\lambda^*(n)}$ a y_{n+1} with

$$x_{n+1} + n - 1/10 \leqslant y_{n+1} \leqslant x_{n+1} + n + 1/10, \\ |\chi^*_{\lambda(n)}(y_{n+1}) + 1| \leqslant 1/100$$

and $x_1, y_1, \ldots, x_n, y_n, x_{n+1}, y_{n+1}$ independent. We write

$$\lambda(n) = \inf \{\lambda \geqslant 100 : |\chi_\lambda(x_r) - 1| \leqslant 1/100, \\ |\chi_\lambda(y_s) - 1| \leqslant 1/100, 1 \leqslant r \leqslant n+1, 1 \leqslant s \leqslant n\}.$$

Now suppose $\lambda \geqslant 100$ and

$$|\chi_\lambda(x_r) - 1| \leqslant 1/100, |\chi_\lambda(y_r) - 1| \leqslant 1/100 \text{ for } 1 \leqslant r \leqslant n+1.$$

Then as in *Lemma* 5.4 (*ii*) we see that $\lambda \geqslant \lambda(n) \geqslant \lambda^*(n)$ and

$$\begin{aligned} |\chi_{\lambda-\lambda(n)}(x_{n+1} - y_{n+1}) + 1| & \\ &= |\chi_{\lambda-\lambda(n)}(x_{n+1}) + \chi_{\lambda-\lambda(n)}(y_{n+1})| \\ &\leqslant |\chi_{\lambda-\lambda(n)}(y_{n+1}) + 1| + |\chi_{\lambda-\lambda(n)}(x_{n+1}) - 1| \\ &= |\chi_\lambda(y_{n+1}) + \chi_{\lambda(n)}(y_{n+1})| + |\chi_\lambda(x_{n+1}) - \chi_{\lambda(n)}(x_{n+1})| \\ &\leqslant |\chi_\lambda(y_{n+1}) - 1| + |\chi_{\lambda(n)}(y_{n+1}) + 1| \\ &\quad + |\chi_\lambda(x_{n+1}) - 1| + |\chi_{\lambda(n)}(x_{n+1}) - 1| \leqslant 4/100. \end{aligned}$$

Thus $|\lambda - \lambda(n)||y_{n+1} - x_{n+1}| \geqslant \pi/2$ and so

$$\lambda \geqslant \lambda(n) + \pi/(2(y_{n+1} - x_{n+1})) \geqslant \lambda^*(n) + \pi/2n,$$

whence $\lambda^*(n+1) \geqslant \lambda^*(n) + \pi/2n$.

Setting $K = \{x_r : r \geqslant 1\}, L = \{y_r : r \geqslant 1\}$ and $M = K \cup L$ we have, by the usual argument, M discrete and independent. Since $\lambda^*(n) \to \infty$ there does not exist a $\lambda \geqslant 100$ for which $\sup_{x \in M} |\chi_\lambda(x) - 1| \leqslant 1/100$ and so M is not Dirichlet. On the other hand $x_{n+1}/x_n \to \infty$, $y_{n+1}/y_n \to \infty$ so that K and L are discrete Kronecker by *Lemma* 5.4 (*i*). This completes the proof.

Our answers for $\mathbf{R}$ are thus considerably simpler than those for $\mathbf{T}$. This may be because in the case of $\mathbf{R}$ we have not asked the right questions.

6. Independence and Measure.

In this section we lay the foundations for Section 7. In the first part we prove a series of simple lemmas. Some of these, like e.g. *Lemma* 6.1, clear up points from earlier sections, some are included, like the necessity part of *Lemma* 6.3, for interest and to convey the drift of our argument, but some are basic to the understanding of what follows. Assuming the reader already knows *Lemma* 6.2 and the sufficiency part of *Lemma* 6.3, they are *Lemma* 6.4, *Lemma* 6.11 (*i*), and *Lemma* 6.11 (*ii*). Though easy, they must be fully understood in order to follow the arguments of Section 7. In the second part we use them to prove *Theorem* 6. In a certain sense this provides a « dress rehearsal » for Section 7.

The first lemma provides an alternative method for extending some of our results on countable closed sets to cover perfect sets.

LEMMA 6.1. — *Suppose* E *is a countable closed set*

(*i*) *We can find an* N_0 *(and so weak Dirichlet) perfect set* $P \supseteq E$;

(*ii*) *If* E *is independent we can find a weak Kronecker (and so independent)* N_0 *perfect set* $P \supseteq E$;

(*iii*) *If* E *is a Kronecker set we can find a perfect Kronecker set* $P \supseteq E$;

(*iv*) *If* E *is a Dirichlet set we can find a perfect Dirichlet set* $P \supseteq E$;

(*v*) *If* E *is an independent Dirichlet set we can find a weak Kronecker (and so independent) perfect Dirichlet set* $P \supseteq E$.

Proof. — It should be remarked that the proofs for E of a simple form (with 1 or 2 limit points, say) are considerably shorter. Since this is all we require in the discussion that follows, the reader may wish to prove these results for himself in the simple form. The proofs of (*i*) and (*ii*) and of (*iii*) and (*iv*) are similar and the reader having read one may simply note the dissimilarities in the proof of the other.

Take $E = \{x_1, x_2, x_3, \ldots\}$ where the x_i are distinct. We proceed in each case by constructing inductively $\mathcal{F}_1$, $\mathcal{F}_2, \ldots$ finite collections of disjoint closed sets with $\max_{F\in\mathcal{F}_n} \operatorname{diam} F \to 0$ as $n \to \infty$. We set $P_n = \cup\{F : F \in \mathcal{F}_n\}$ and ensure that $\mathcal{F}_n$ satisfies the conditions of *Lemma* 1.4 (*iii*) *whilst* $P_n \supseteq E$ $[n \geqslant 1]$. Setting $P = \lim_{n\to\infty} P_n$ we have P perfect and $P \supseteq E$. We give the central inductive step and then examine P so constructed. Note that since $P_n \supseteq E$, $\mathcal{F}_n$ is the union of 2 disjoint sets $\mathcal{K}_n$, $\mathcal{L}_n$ such that setting $K_n = \cup\{F : F \in \mathcal{K}_n\}$, $L_n = \cup\{F : F \in \mathcal{L}_n\}$ we have $K_n \cap E = E$ (so $L_n \cap E = \varnothing$) and such that $F \in \mathcal{K}_n$ implies $F \cap E \neq \varnothing$. (*i*) *and* (*ii*). The construction is similar for (*i*) and (*ii*). In (*i*) we set $h_n = 1$, in (*ii*) $h_n = f_n$. Suppose we have $M(n)$ and $\mathcal{F}_n = \mathcal{K}_n \cup \mathcal{L}_n$. By *Lemma* 3.3 (*ii*) (in (*i*) we use the weak form with $f = 1$, in (*ii*) the strong form with f not constant) we can find an $R(n+1) > M(n)$ such that given $\nu \geqslant R(n+1)$ we can find for each $F \in \mathcal{L}_n$ disjoint subintervals F', F'' with $\operatorname{diam} F', \operatorname{diam} F'' \leqslant 1/3 \operatorname{diam} F$ and $|\chi_\nu(x) - h_{n+1}(x)| \leqslant 2^{-(n+1)}$ for all $x \in F' \cup F''$. Now (by Dirichlet's theorem in (*i*), by Kronecker's theorem in (*ii*)) we can find $M(n+1) \geqslant R(n+1)$ such that

$$\sup_{1\leqslant r\leqslant n+1} |\chi_{M(n+1)}(x_r) - h_{n+1}(x_r)| \leqslant 2^{-(n+2)}.$$

By the continuity of $\chi_{M(n+1)}$ and h_{n+1} there exists a $\delta(n+1) > 0$ such that $x \in \bigcup_{r=1}^{n+1} N(x_r, \delta(n+1))$ implies $|\chi_{M(n+1)}(x) - h_{n+1}(x)| \leqslant 2^{-(n+1)}$. Thus we can find $\mathcal{F}_{n+1}$ a collection of disjoint closed intervals such that $E \subseteq P_{n+1} \subseteq P_n$ and $\max_{G\in\mathcal{F}_{n+1}} \operatorname{diam} G \leqslant 1/3 \max_{F\in\mathcal{F}_n} \operatorname{diam} F$ with the following properties. If $F \in \mathcal{L}_n$ we can find $F', F'' \in \mathcal{F}_{n+1}$ disjoint with $F', F'' \subseteq F$, moreover $|\chi_{M(n+1)}(x) - h_{n+1}(x)| \leqslant 2^{-(n+1)}$ for all $x \in G \in \mathcal{F}_{n+1}$ with $G \subseteq F$. No 2 of $x_1, \ldots, x_{n+1}$ belong to the same member G of $\mathcal{F}_{n+1}$. If

$$x_r \in G \in \mathcal{F}_{n+1} \; [1 \leqslant r \leqslant n+1]$$

then $\sup_{x\in G} |\chi_{M(n+1)}(x) - h_{n+1}(x)| \leqslant 2^{-(n+1)}$. If $F \in \mathcal{K}_n$ then there exists an $F' \in \mathcal{L}_{n+1}$ with $F' \subseteq F$ and $F \cap E = \varnothing$ (so there exists an $F'' \in \mathcal{L}_{n+1}$ with $F'' \subseteq F$ and $F'' \cap E \neq \varnothing$). The induction now restarts.

We now consider the P so constructed in the 2 cases. First we take the construction for (i) with $h_n = 1$. Consider $x \in P$. Suppose $x \notin L_n$ for all n. Then $x \in \cap K_n$ and so there exists a $y_n \in E$ such that if $x \in F_n \in \mathcal{L}_n$ then $y_n \in F_n [n \geqslant 1]$. Since $\operatorname{diam} F_n \to 0$ as $n \to \infty$, $y_n \to x$ and since E is closed $x \in E$. If $x \in L_r$ for some r then

$$|\chi_{M(s)}(x) - 1| \leqslant 2^{-(s+1)}$$

for all $s \geqslant r + 1$. If $x \in E$ then $x = x_r$ for some r and $|\chi_{M(s)}(x) - 1| \leqslant 2^{-(s+1)}$ for all $s \geqslant r$. In either case (and so for all $x \in P$) $\sum_{n=1}^{\infty} |\sin M(n)x|$ converges. Thus P is an N_0 set and (i) is proved.

Next we consider the case (ii). As above we see that $\sum_{n=1}^{\infty} \left|\sin M_{\frac{1}{2}n(n+1)+1}\, x\right|$ converges for all $x \in P$ (recall that $f_{\frac{1}{2}n(n+1)+1} = 1$) and so P is an N_0 set. It only remains to show that P is a weak Kronecker set. Suppose $\mu \in M^+(P)$, $\eta > 0$ are given. For each r we can find a $\rho_r > 0$ such that $\mu(N(x_r, 2\rho_r)\setminus\{x_r\}) \leqslant 2^{-(r+2)}\eta$. Now the family $\{N(x_r, \rho_r)\}_{r=1}^{\infty}$ form a covering by intervals of the compact set E so that we can find a finite subcover $\{N(x_{r(s)}, \rho_{r(s)})\}_{s=1}^{k}$ say. Set

$$J = P \cap \bigcup_{s=1}^{k} N(x_{r(s)}, 2\rho_{r(s)}),\ H = P\setminus J.$$

Then

$$\begin{aligned}
\mu(H) &= \mu(P) - \mu(J)\\
&\geqslant \mu(P) - \sum_{s=1}^{k} \mu(N(x_{r(s)}, \rho_{r(s)}))\\
&= \mu(P) - \sum_{s=1}^{k} [\mu(N(x_{r(s)}, \rho_{r(s)})\setminus\{x_{r(s)}\}) + \mu(\{x_{r(s)}\})]\\
&\geqslant \mu(P) - \sum_{r=1}^{\infty} [\mu(N(x_r, \rho_r)\setminus\{x_r\}) + \mu(\{x_r\})]\\
&\geqslant \mu(P) - \eta/2 - \mu(E).
\end{aligned}$$

Now let $\rho = \min\limits_{1 \leqslant s \leqslant k} \rho_{r(s)}$. If $x \in \mathrm{H}$ then $|x - y| \geqslant \rho > 0$ for all $y \in \mathrm{E}$. But $\max\limits_{\mathrm{F} \in \mathcal{L}_n} \operatorname{diam} \mathrm{F} \to 0$ as $n \to \infty$ so for some n_1 we have that $n \geqslant n_1$ implies $\operatorname{diam} \mathrm{F} \leqslant 1/2\ \rho$ for all $\mathrm{F} \in \mathcal{L}_n$. In particular $\mathrm{K}_n \cap \mathrm{H} = \emptyset$ for all $n \geqslant n_1 + 1$. Thus $\sup\limits_{x \in \mathrm{H}} |\chi_{\mathrm{M}(n)}(x) - f_n(x)| \leqslant 2^{-n}$ for all $n \geqslant n_1 + 2$.

Now consider $\mu|\mathrm{E}$ (μ restricted to E). Since E is countable, we can find an m such that

$$\mu(\{x_1, x_2, \ldots, x_m\}) \geqslant \mu(\mathrm{E}) - \eta.$$

Set $n_2 = \max\ (m, n) + 4$. Then setting

$$\mathrm{H}_m = \mathrm{H} \cup \{x_1, x_2, \ldots, x_m\}$$

we have $|\chi_{\mathrm{M}(n)}(x) - f_n(x)| \leqslant 2^{-n}$ for all $n \geqslant n_2$ and $\mu(\mathrm{H}_m) \geqslant \mu(\mathrm{P}) - \eta$. Thus

$$\mu\{x \in \mathrm{P} : |\chi_{\mathrm{M}(n)}(x) - f_n(x)| \geqslant 2^{-n+1}\} \leqslant \eta$$

for all $n \geqslant n_2$ and P is weak Kronecker.

We note that using this method it is as easy in (*i*) to construct P an N_0 set as it is to construct P a weak Dirichlet set.

(*iii*) *and* (*iv*) Again the construction is similar for (*iii*) and (*iv*). In (*iii*) we set $h_n = f_n$, in (*iv*) $h_n = 1$. Suppose we have $\mathrm{M}(n)$ and $\mathcal{L}_n = \mathcal{K}_n \cup \mathcal{L}_n$. By *Lemma* 3.3 (*ii*) (in (*iv*) we use the weak form with $f = 1$, in (*iii*) the strong form with f non constant) we can find an $\mathrm{R}(n+1) > \mathrm{M}(n)$ such that given $v \geqslant \mathrm{R}(n+1)$ we can find for each $\mathrm{F} \in \mathcal{L}_n$ disjoint subintervals F', F'' with diam F', diam $\mathrm{F}'' \leqslant 1/3$ diam F and $|\chi_v(x) - h_{n+1}(x)| \leqslant 2^{-(n+1)}$ for all $x \in \mathrm{F}' \cup \mathrm{F}''$. Now by the definition of E we can find a $\mathrm{M}(n+1) \geqslant \mathrm{R}(n+1)$ such that $\sup\limits_{y \in \mathrm{E}} |\chi_{\mathrm{M}(n+1)}(y) - h_{n+1}(y)| \leqslant 2^{-(n+2)}$. By the continuity of $\chi_{\mathrm{M}(n+1)}$ and h_{n+1} (and the compactness of $\mathbf{T}$) there exists a $\delta(n+1) > 0$ such that for all $y \in \mathrm{E}$, $z \in \mathrm{N}(y, \delta(n+1))$ implies $|\chi_{\mathrm{M}(n+1)}(z) - h_{n+1}(z)| \leqslant 2^{-(n+2)}$. Now the family $\{\operatorname{int} \mathrm{N}(y, \delta(n+1))\}_{y \in \mathrm{E}}$ form an open covering of the compact set E. We can therefore find a subcovering

$$\{\operatorname{int} \mathrm{N}(y_{n+1s}, \delta(n+1))\}_{s=1}^{k}.$$

Observing that

$$\{\text{int}\,(\mathrm{N}(y_{n+1s}, \delta(n+1)) \cap \mathrm{F}) : \mathrm{F} \in \mathcal{K}_n,\ 1 \leqslant s \leqslant k\}$$

is then itself a set of open intervals (together possibly with $\varnothing$) covering E we see that we can find $\mathcal{F}_{n+1}$ a collection of disjoint closed intervals with the following properties. As demanded in the introduction $\mathrm{P}_n \supseteq \mathrm{P}_{n+1} \supseteq \mathrm{E}$ and

$$\max_{\mathrm{G} \in \mathcal{F}_{n+1}} \operatorname{diam} \mathrm{G} \leqslant 1/3 \max_{\mathrm{F} \in \mathcal{F}_n} \operatorname{diam} \mathrm{F}.$$

If $\mathrm{F} \in \mathcal{F}_n$ we can find $\mathrm{F}', \mathrm{F}'' \in \mathcal{F}_{n+1}$ disjoint with $\mathrm{F}', \mathrm{F}'' \subseteq \mathrm{F}$, moreover $|\chi_{\mathrm{M}(n+1)}(x) - h_{n+1}(x)| \leqslant 2^{-(n+1)}$ for all $x \in \mathrm{G} \in \mathcal{F}_{n+1}$ with $\mathrm{G} \subseteq \mathrm{F}$. We now restart the induction.

We now consider the P constructed in case (*iii*). Since $\mathrm{P} \subseteq \mathrm{P}_n$ we have $|\chi_{\mathrm{M}(n)}(x) - h_n(x)| \leqslant 2^{-n}$ for all $x \in \mathrm{P}$ and P is Kronecker as required. A similar proof gives (*iv*).

(*v*) It is clear that we need only alternate the inductive steps of (*ii*) and (*iv*) (i.e. first proceed as in (*ii*), then as in (*iv*), then as in (*ii*), then as in (*iv*), and so on) to obtain a suitable P.

As promised *Lemma* 6.1 (*v*) and *Theorem* 4 give at once an alternative proof of

Corollary 5.2. — *There exists a perfect weak Kronecker (and so independent) Dirichlet non Kronecker set.*

Proof. — Let E be as in *Theorem* 4, form P as in *Lemma* 6.1 (*v*). P is perfect weak Kronecker and Dirichlet. But $\mathrm{E} \subseteq \mathrm{P}$ and E is not Kronecker so P can not be.

Similarly Wik's result (our *Corollary* 2.2) that there exist weak Kronecker non Dirichlet sets becomes a consequence of the fact that there exist closed countable independent sets E which are not Dirichlet (*Lemma* 3.1). This is perhaps the neatest of the several proofs of Wik's result in our paper. Again *Lemma* 3.1 combined with *Lemma* 6.1 (*ii*) gives the existence of weak Kronecker (and so independent) perfect N_0 sets which are not Dirichlet (*Lemma* 4.3 (*ii*)). The results of *Lemma* 6.1 also enable us to obtain *Corollary* 2.1 (the independent union of 2 disjoint Kronecker perfect sets need not be Dirichlet) directly from *Theorem* 1.

Up to now we have proved sets independent by showing them weak Kronecker. But in Section 7 we shall want to construct independent closed sets which are not H_1 (and so in particular not weak Kronecker). We therefore develop methods for constructing independent sets in a series of lemmas stated and proved on the lines of ([8] Chapter I, § 11) (from where in particular *Lemma* 6.2 and the sufficiency part of *Lemma* 6.3 are taken directly). We start with a definition. Suppose $I_1, I_2, \ldots, I_n$ are disjoint (closed) sets, we say that $I_1, I_2, \ldots, I_n$ are *M-independent* if whenever

$$x_i \in I_i \ [1 \leqslant i \leqslant n],$$

it follows that $0 < \sum_{j=1}^{n} |m_j| \leqslant M$ implies $\sum_{j=1}^{n} m_j x_j \neq 0$ $[M \geqslant 1]$. Similarly we call $x_1, x_2, \ldots, x_n$ M-independent if

$$0 < \sum_{j=1}^{n} |m_j| \leqslant M \quad \text{implies} \quad \sum_{j=1}^{n} m_j x_j \neq 0.$$

For example we have

LEMMA 6.2: (*i*) *In* $\mathbf{R}$ *given* $I_1, I_2, \ldots, I_n$ *disjoint closed intervals and* $x_1, x_2, \ldots, x_n$ *M-independent (so in particular given* $x_1, x_2, \ldots, x_n$ *independent) with* $x_i \in \operatorname{int} I_i$ $[1 \leqslant i \leqslant n]$ *we can find* $J_1, J_2, \ldots, J_n$ *closed intervals such that*

$$x_i \in \operatorname{int} J_i \subseteq I_i \ [1 \leqslant i \leqslant n] \quad \textit{and} \quad J_1, J_2, \ldots, J_n$$

are M-independent.

Thus by *Lemma* 1.3 given $I_1, I_2, \ldots, I_n$ disjoint closed intervals we can find $J_1, J_2, \ldots, J_n$ closed subintervals which are M-independent.

(*ii*) *The result of* (*i*) *holds in* $\mathbf{T}$.

Proof. — In $\mathbf{R}^n$ the point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ does not lie in any of the closed hyperplanes

$$\Pi_{m_1, m_2, \ldots, m_n} = \left\{ \mathbf{z} : \sum_{j=1}^{n} m_j z_j = 0 \right\} \left[M \geqslant \sum_{j=1}^{n} |m_j| > 0 \right]$$

and so there exists a closed hypercube

$$J_1 \times J_2 \times \cdots \times J_n \subseteq I_1 \times I_2 \times \cdots \times I_n$$

such that $\mathbf{x} \in \operatorname{int}(J_1 \times J_2 \times \cdots \times J_n)$ and

$$(J_1 \times J_2 \times \cdots \times J_n) \cap \bigcup_{M \geqslant \Sigma |m_i| > 0} \Pi_{m_1, m_2, \ldots, m_n} = \varnothing.$$

This proves (i); (ii) follows as a corollary or by a similar method.

Suppose E is a Cantor set constructed after the manner of *Lemma* 1.4 (iii). We show that there is one and (in a certain very limited sense) only one method of obtaining E independent.

LEMMA 6.3. — *With the notation of Lemma* 1.4 (iii) E *is independent if and only if given* $\delta > 0$, $m_1, m_2, \ldots, m_n \in \mathbf{Z} \setminus \{0\}$ *there exists an* i_0 *such that* $x_1, x_2, \ldots, x_n \in P_i$, $|x_k - x_j| \geqslant \delta$ *for* $k \neq j$ *together imply* $\sum_1^n m_j x_j \neq 0$, *for all* $i \geqslant i_0$.

Proof. — *Sufficiency.* Suppose $x_1, x_2, \ldots, x_n \in E$ are distinct. Then we can find a $\delta \geqslant 0$ such that $|x_k - x_j| \geqslant \delta$ for $k \neq j$. Given $m_1, m_2, \ldots, m_n \neq 0$ we can therefore find an i_0 as above. But $x_1, x_2, \ldots, x_n \in P_{i_0}$ (since $P_{i_0} \supseteq E$) and so $\sum_{j=1}^n m_j x_j \neq 0$. Hence E is independent as required.

Necessity. Suppose the condition fails. Then there exist $\delta > 0$, $m_1, m_2, \ldots, m_n \in \mathbf{Z} \setminus \{0\}$ such that for infinitely many i there exist $x_{1i}, x_{2i}, \ldots, x_{ni} \in P_i$ such that $|x_{ki} - x_{ji}| \geqslant \delta$ for $k \neq j$ $[1 \leqslant k, j \leqslant n]$ and $\sum_{r=1}^n m_r x_{ri} = 0$. Without loss of generality we may assume $\max_{A \in \mathcal{L}_1} \operatorname{diam} A \leqslant \delta/2$. Since $\mathcal{L}_1$ is finite there exist $A_1^1, A_2^1, \ldots, A_n^1 \in \mathcal{L}_1$ distinct such that for infinitely many $i \geqslant 2$ there exist

$$x_{1i} \in A_1^1 \cap P_i,\ x_{2i} \in A_2^1 \cap P_i,\ \ldots,\ x_{ni} \in A_n^1 \cap P_i$$

with $\sum_{r=1}^n m_r x_{ri} = 0$. Since $\mathcal{L}_2$ is finite, there exist

$$A_1^2, A_2^2, \ldots, A_n^2 \in \mathcal{L}_n \quad \text{with} \quad A_s^2 \subseteq A_s^1 \ [1 \leqslant s \leqslant n]$$

such that for infinitely many $i \geqslant 3$ there exist

$$x_{1i} \in A_1^2 \cap P_i,\ x_{2i} \in A_2^2 \cap P_i,\ \ldots,\ x_{ni} \in A_n^2 \cap P_i \quad \text{with} \quad \sum_{r=1}^n m_r x_{ri} = 0.$$

We continue inductively. Let $\{y_s\} = \bigcap_{t=1}^{\infty} A_s^t$ $[1 \leqslant s \leqslant n]$ (that $\bigcap_{t=1}^{\infty} A_s^t$ contains one and only one point results from the Second Intersection Theorem ([6] § 26), cf. our proof of *Theorem* 3). Select $y_{su} \in A_s^u$ $[1 \leqslant s \leqslant n]$ such that $\sum_{r=1}^{n} m_r y_{ru} = 0$ $[u = 1, 2, 3, \ldots]$ (if $x_{is} \in A_s^u \cap P_i$ for some $i \geqslant u$ then $x_{is} \in A_s^u$). Now $y_{su} \to y_s$ as $u \to \infty$ so $\sum_{r=1}^{n} m_r y_r = 0$. But $y_s \in P$ $[1 \leqslant s \leqslant n]$ and (since $y_s \in A_s^1$) $y_1, y_2, \ldots, y_n$ are distinct so P is not independent.

One simple way to obtain the condition above is to demand that for any M we can find arbitrarily large i such that the intervals making up $\mathfrak{L}_i$ are M-independent. Speaking very roughly this is what occurs when we construct E Kronecker. Again we might demand that all the intervals making up $\mathfrak{L}_i$ except I_i say together with $\{x_0\}$ are M-independent. Then if $I_i \to \{x_0\}$ as $i \to \infty$ we have E independent. This corresponds to the construction in *Lemma* 4.2 of a weak Kronecker set. As a heuristic principle we may say that wherever these kinds of method are used to construct independent perfect sets with certain properties, we can construct Kronecker or weak Kronecker sets with the same properties.

We need in fact a slightly more subtle approach to deal with constructions in which we are « only allowed to tamper with a small bit of the set at a time ». The result is one which may be thought rather trivial to be announced with such a fanfare, but must nevertheless be fully absorbed.

LEMMA 6.4. — *Suppose*

$$\mathfrak{L}_1, \mathfrak{L}_2, \ldots, \mathfrak{L}_N \quad (\textit{where} \quad N = \tbinom{n}{M}) \; [M \leqslant n]$$

are collections of disjoint (closed) intervals such that setting $P_i = \cup \{F : F \in \mathfrak{L}_i\}$ $[1 \leqslant i \leqslant N]$ *we have* $P_1 \supseteq P_2 \supseteq \cdots \supseteq P_N \supseteq E$ *where* E *is a closed set. Suppose* $\mathfrak{L}_i = \bigcup_{k=1}^{n} \mathfrak{A}_{ik}$ *where* $\mathfrak{A}_{ij}$ *are disjoint and non empty* $[1 \leqslant j \leqslant n, \; 1 \leqslant i \leqslant N]$ *and setting* $A_{ij} = \cup \{F : F \in \mathfrak{A}_{ij}\}$ *we have* $A_{ij} = P_i \cap A_{1j}$. *(Thus* *we have divided our sets into* n *blocks.) Put*

$$E_j = E \cap A_{1j} \quad [1 \leqslant j \leqslant n].$$

Let σ *be a bijective map* $\sigma : \{1, \ldots, N\} \to \{\theta : \theta$ *injective map* $\theta : \{1, \ldots, M\} \to \{1, \ldots, n\}\}$. *Then if the sets* $A_{i(\sigma(i))(p)}$ $[1 \leqslant p \leqslant M]$ *are* M-*independent (and so in particular if the intervals of* $\bigcup_{p=1}^{M} A_{i(\sigma(i))(p)}$ *are M-independent* $[1 \leqslant i \leqslant N]$ *we have that* $E_1, E_2, \ldots, E_n$ *are M-independent.*

Proof. — Suppose $x_j \in E_j$ and $0 < \sum_{j=1}^{n} |m_j| \leqslant M$. Then at most M of $m_1, m_2, \ldots, m_n$ can be non zero. In particular we can take an $1 \leqslant i \leqslant N$ such that $j \notin \{(\sigma(i))r : M \geqslant r \geqslant 1\}$ implies $m_j = 0$. Now $x_{(\sigma(i))(r)} \in A_{i(\sigma(i))(r)}$ $[1 \leqslant r \leqslant M]$ and $A_{i(\sigma(i))(1)}, A_{i(\sigma(i))(2)}, \ldots A_{i(\sigma(i))(M)}$ are M-independent. Thus $\sum_{j=1}^{n} m_j x_j = \sum_{r=1}^{M} m_{(\sigma(i))(r)} x_{(\sigma(i))(r)} \neq 0$. Hence $E_1, E_2, \ldots, E_n$ are M-independent.

A more colourful way of stating the theorem is that if (in non rigorous terms) we can ensure that every combination of M blocks from $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is M-independent, then the blocks $\mathcal{A}_1, \ldots, \mathcal{A}_n$ will become M-independent.

In the next part of this section we diverge from the mainstream of the paper to consider some easy technical results on independence which might be found useful by those wishing to develop the methods of this paper and some of which form a background for *Theorem* 9. This constitutes easily the dullest part of the paper and those readers uninterested in *Theorem* 9 will probably prefer to resume reading at the end of *Lemma* 6.10.

The first 3 lemmas discuss how badly sets must be changed to ensure certain types of independence, and constitute an improvement on *Lemma* 6.2. In particular they show that in our standard construction some intervals may be left « large ».

LEMMA 6.5. — *Suppose* $n_1, n_2, \ldots, n_t, m_1, m_2, \ldots, m_s$ *given with* $\sum_{i=1}^{t} |n_i| > 0$. *Suppose* $I_1, I_2, \ldots, I_t, J_1, J_2, \ldots, J_s$ *are disjoint closed intervals in* $\mathbf{R}$. *Then we can find closed* *intervals* $I'_i \subseteq I_i$, $J'_j \subseteq J_j$ $[1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant s]$ *such that* diam $I'_i = 1/4$ diam I_i; *yet* $x_i \in I'_i$, $y_j \in J'_j$ *imply*

$$n_1x_1 + n_2x_2 + \cdots + n_tx_t + m_1y_1 + m_2y_2 + \cdots + m_sy_s \neq 0.$$

The result also holds with

$$\{m_1, m_2, \ldots, m_s\} = \emptyset,\ \{J_1, J_2, \ldots, J_s\} = \emptyset.$$

Proof. — Select $y'_j \in \operatorname{int} J_j$ $[1 \leqslant j \leqslant s]$ and consider in $\mathbf{R}^t$ the hyperplane $\Pi = \left\{ z : \sum_{i=1}^{t} n_iz_i = \gamma \right\}$ where $\gamma = -\sum_{1}^{s} m_jy'_j$, and the hypercuboid $\Gamma = \{z : z_i \in I_i,\ 1 \leqslant i \leqslant t\}$. It is clear that we can find a hypercuboid Γ'' of side at least $1/2$ that of Γ lying within Γ'' such that int Γ'' does not intersect Π. In other words we can find $I''_i \subseteq I_i$ a closed interval with diam $I''_i \geqslant 1/2$ diam I_i $[1 \leqslant i \leqslant t]$ such that setting $\Gamma'' = \{z : z_i \in I''_i,\ 1 \leqslant i \leqslant t\}$ we have $\operatorname{int}\Gamma'' \cap \Pi = \emptyset$. Thus we can find $\varepsilon > 0$, $I'_i \subseteq I''_i$ a closed interval with diam $I'_i \geqslant 1/4$ diam I_i $[1 \leqslant i \leqslant t]$ such that if $\Gamma' = \{z :\ z_i \in I'_i,\ 1 \leqslant i \leqslant t\}$, $\Pi' = \{z :\ \gamma - \varepsilon \leqslant \sum_{i=1}^{t} n_iz_i \leqslant \gamma + \varepsilon\}$ we have $\Pi' \cap \Gamma' = \emptyset$. Select δ such that $\delta \sum_{j=1}^{s} |m_j| < \varepsilon$ and $N(y'_j, \delta) \subseteq J_j$. Setting $J'_j = N(y'_j, \delta)$, the lemma follows at once.

We obtain as a corollary or, by repeating the proof,

LEMMA 6.6. — *There exists a* $\lambda = \lambda(m_1, m_2, \ldots, m_s) > 0$ *dependent only on* $m_1, m_2, \ldots, m_s \left[\sum_{j=1}^{s} |m_j| > 0\right]$ *for which the following is true. Suppose* $n_1, n_2, \ldots, n_t$ *given with* $\sum_{i=1}^{t} |n_i| > 0$. *Suppose* $I_1, I_2, \ldots, I_t, J_1, J_2, \ldots, J_s$ *are disjoint closed intervals in* $\mathbf{T}$ *such that* $x_i \in I_i$ $[1 \leqslant i \leqslant t]$ *implies* $\sum_{i=1}^{t} n_ix_i \neq 0$. *Then we can find closed intervals*

$$I'_i \subseteq I_i,\ J'_j \subseteq J_j\ [1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant s]$$

such that diam $I'_i \geqslant \lambda(m_1, m_2, \ldots, m_s)$ diam I_i; yet $x_i \in I'_i$, $y_j \in J'_j$ *yield*

$$n_1x_1 + n_2x_2 + \cdots + n_tx_t + m_1y_1 + m_2y_2 + \cdots + m_sy_s \neq 0.$$

This result holds also with

$$\{m_1, m_2, \ldots, m_s\} = \varnothing,\ \{J_1, J_2, \ldots, J_s\} = \varnothing.$$

As a trivial consequence we have by induction

LEMMA 6.7. — *Suppose* $I_1, I_2, \ldots, I_t, J_1, J_2, \ldots, J_s$ *are disjoint closed intervals in* $\mathbf{T}$ *such that* $I_1, I_2, \ldots, I_t$ *are M-independent. Suppose* $J_{pq} \subseteq J_p$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$ *and* $J_{p1}, J_{p2}, \ldots, J_{pr(p)}$ *are disjoint closed intervals. Then there exits a* $\lambda = \lambda_M(r(1), r(2), \ldots, r(s)) > 0$, *depending only on* M *and* $r(1), r(2), \ldots, r(s)$, *such that we can find closed intervals* $I'_i \subseteq I_i$, $J'_{pq} \subseteq J_{pq}$ $[1 \leqslant i \leqslant t,\ 1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$ *with diam* $I'_i \geqslant \lambda$ *diam* I_i *for wich* $x_i \in I'_i$, $y_p \in \bigcup_{q=1}^{r(p)} J_{pq}$ *imply* $\sum_{i=1}^{t} n_i x_i + \sum_{p=1}^{s} m_p y_p \neq 0$ *whenever*

$$M \geqslant \sum_{i=1}^{t} |n_i|,\ \sum_{p=1}^{s} |m_p| > 0.$$

On the face of it this is a stronger result than *Lemma* 6.2, but it turns out that (at least in the work that follows) the latter will suffice. We note that the values of λ can easily be calculated (though naturally they become rather complicated) and might be used to obtain numerical bounds.

In a similar spirit we ask what happens to certain of our basic tools (in particular Kronecker's Theorem and *Lemma* 1.3) if we relax the demand for independence merely asking for M-independence.

LEMMA 6.8. — (*i*) *If* $R = \{x_m : m \in \mathbf{Z}\}$ *is* M-*independent*, $E \subseteq \mathbf{T}$ (*respectively* $E \subseteq \mathbf{R}$) *uncountable, then there exists a* $y \in E \setminus R$ *such that* $\{y\} \cup R$ *is* M-*independent.*

(*ii*) *If* $x_1, x_2, \ldots, x_n$ *are* M-*independent*, $E \subseteq \mathbf{T}$ (*respectively* $E \subseteq \mathbf{R}$) *infinite, then there exists a* $y \in E \setminus \{x_1, x_2, \ldots, x_n\}$ *such that* $y, x_1, \ldots, x_n$ *are* M-*independent.*

Proof. — As for *Lemma* 1.3.

More interestingly (though not unexpectedly) we have the following form of Kronecker's Theorem.

LEMMA 6.9. — (*i*) *For fixed* n *there exists a*

$$\tau(M) = \tau(n, M) \to 0 \quad as \quad M \to \infty$$

such that if $x_1, x_2, \ldots, x_n$ *are M-independent and*

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n| = 1$$

we have $\inf_{r\in\mathbf{Z}} \sup_{1\leqslant s<n} |\chi_r(x_s) - \lambda_s| \leqslant \tau(\mathrm{M})$. *Moreover if*

$$x_1, x_2, \ldots, x_n \in 2\pi\mathbf{Q}$$

then the n-*tuple* $(\chi_m(x_1), \chi_m(x_2), \ldots, \chi_m(x_n))$ *is periodic in* m.

Proof. — This is a result of number theory known as the Quantitative Kronecker Theorem ([3] Chapter V, § 8). Values of $\tau(\mathrm{M})$ can be calculated.

For the purposes we shall suggest, however, the following trivial result will suffice.

LEMMA 6.9. — *(ii) If* $p_1, p_2, \ldots, p_h$ *are distinct primes and* $r_1, r_2, \ldots, r_n$ *are such that* $0 < r_i < p_i$ $[1 \leqslant i \leqslant h]$ *then setting* $x_i = 2\pi r_i/p_i, q = \min_{1\leqslant s<h} p_s$ *and* $\mathrm{P} = p_1p_2\ldots p_h$ *we have for any* $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_h| = 1$ *that*

$$\inf_{1\leqslant m\leqslant \mathrm{P}} \sup_{1\leqslant s\leqslant h} |\chi_m(x_i) - \lambda_i| \leqslant \pi/q.$$

Moreover the h-*tuple* $(\chi_m(x_1), \chi_m(x_2), \ldots, \chi_m(x_h))$ *has period* P *in* m.

We note that given $\delta, \varepsilon > 0$, $y_1, y_2, \ldots, y_h \in \mathbf{T}$ we can find $p_1, p_2, \ldots, p_h$ and $r_1, r_2, \ldots, r_h$ satisfying the conditions of the lemma such that $\pi/q \leqslant \varepsilon$ and $\sup_{1<s<h} |y_s - x_s| \leqslant \delta$.

These results provide a method of extending to the theorems so far discussed the technique illustrated in the following alternative proof of

THEOREM 4. — *There exists a countable independent Dirichlet set* E *which is not Kronecker.*

Proof. — We construct such a set E. Suppose we have at the n^{th} stage $\gamma_n, \alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{h(n)n} \in 2\pi\mathbf{Q}$ n-independent, $\mathrm{N}(n) = 2^{n+7}\mathrm{P}(n)$, $\mathrm{P}(n) \geqslant 10$ such that

$$\sup_{1<w<h(n)} |\chi_r(\alpha_{wn}) + 1| \geqslant 1/8\,(1 + 2^{-n})$$

for all $1 \leqslant r \leqslant \mathrm{N}(n)$ and $\delta(n) > 0$. Now by *Lemma* 6.2 *(i)* we can find a $1/4\ \delta(n) > \delta(n+1) > 0$ such that

$N(\gamma_n,\ \delta(n+1))$, $N(\alpha_{1n},\ \delta(n+1))$, $N(\alpha_{2n},\ \delta(n+1))$, $\ldots$, $N(\alpha_{h(n)n},\ \delta(n+1))$ are n-independent. Now

$$(\chi_r(\alpha_{1n}), \chi_r(\alpha_{2n}), \ldots, \chi_r(\alpha_{h(n)n}))$$

has period $P_0(n)$. Let $P(n+1)$ be a multiple of $P_0(n)$ such that $P(n+1) \geqslant 4N(n)$ and set

$$N(n+1) = 2^{n+8}P(n+1).$$

Since $\chi_{tP(n+1)+v}(\alpha_{wn}) = \chi_v(\alpha_{wn})$ we have

$$\sup_{1 \leqslant w \leqslant h(n)} |\chi_{tP(n+1)+v}(\alpha_{wn}) + 1| \geqslant 1/8(1 + 2^{-n}) \quad \text{for} \quad 0 \leqslant v \leqslant N(n)$$

or $P(n+1) - N(n) \leqslant v \leqslant P(n+1)$. Taking

$$a(n+1),\ b(n+1) \in \mathbf{Z}$$

such that

$$[b(n+1) - a(n+1)]N(n) \geqslant P(n+1)$$
$$\frac{b(n+1) - a(n+1)}{P(n+1)} \leqslant \frac{2^{-(n+4)}}{P(n)}$$

and

$$\gamma_n \in \left[\frac{2a(n+1)}{P(n+1)}\pi,\ \frac{2b(n+1)}{P(n+1)}\pi\right]$$

we set $k_{q,n+1} = \dfrac{2\pi q}{P(n+1)}$ for $a(n+1) \leqslant q \leqslant b(n+1)$. Trivially $\chi_{P(n+1)}(k_{q,n+1}) = 1$. Also

$$\sup_{a(n+1) \leqslant q \leqslant b(n+1)} |\chi_{tP(n+1)+v}(k_{q,n+1}) + 1| \geqslant 1/2 \geqslant 1/8(1 + 2^{-n})$$

for $N(n) \leqslant v \leqslant P(n+1) - N(n)$.

By the continuity of $\chi_1, \chi_2, \ldots, \chi_{N(n+1)}$ we can, using *Lemma* 6.8 (*ii*), find $\gamma_{n+1}, \alpha_{1n+1}, \alpha_{2n+1}, \ldots, \alpha_{h(n+1),n+1}$ $n+1$ — independent such that

$$|\gamma_{n+1} - \gamma_n|, |\alpha_{1n+1} - \alpha_{1n}|, |\alpha_{2n+1} - \alpha_{2n}|, \ldots,$$
$$|\alpha_{h(n)n+1} - \alpha_{h(n)n}| \leqslant \delta(n+1)/4$$

whilst

$$|\chi_r(\gamma_n) - \chi_r(\gamma_{n+1})|,\ \sup_{1 \leqslant w \leqslant h(n)} |\chi_r(\alpha_{wn}) - \chi_r(\alpha_{w,n+1})|,$$
$$\sup_{h(n)+1 \leqslant w \leqslant h(n+1)} |\chi_r(k_{a(n+1)+w-h(n)-1,n+1}) - \chi_r(\alpha_{w,n+1})| \leqslant 1/8\ \ 2^{-(n+1)}$$

for $1 \leqslant r \leqslant N(n+1)$ where

$$h(n+1) = h(n) + 1 + b(n+1) - a(n+1).$$

Thus $\sup\limits_{1 \leqslant w \leqslant h(n+1)} |\chi_r(\alpha_{w,n+1}) + 1| \geqslant 1/8\ (1 + 2^{-(n+1)})$,

$$\sup_{1 \leqslant w \leqslant h(n+1)} |\chi_{P(n+1)}(\alpha_{w,n+1}) - 1| \leqslant 1/8\ 2^{-(n+1)}$$

and for $1 \leqslant u \leqslant n$

$$\sup_{1 \leqslant w \leqslant h(n)} |\chi_{P(u)}(\alpha_{w,n+1}) - 1| \leqslant \sup_{1 \leqslant w \leqslant h(n)} |\chi_{P(u)}(\alpha_{wn}) - 1| + 1/8\ 2^{-(n+1)},$$

whilst $|\chi_{P(u)}(\gamma_{n+1}) - 1| \leqslant |\chi_{P(u)}(\gamma_n) - 1| + 1/8\ 2^{-(n+1)}$ and

$$\sup_{h(n)+1 \leqslant w \leqslant h(n+1),} |\chi_{P(u)}(\alpha_{w,n+1}) - 1| \leqslant 2^{-n}.$$

Now $\gamma_n \to \gamma$ say and $\alpha_{wn} \to \alpha_w$ say. By construction $E = \{\gamma, \alpha_1, \alpha_2, \ldots\}$ is closed with γ as limit point. Further

$$\sup_{x \in E} |\chi_{P(n)}(x) - 1| \leqslant 2^{-n+1} \qquad \text{and} \qquad \inf_{r \geqslant 0} \sup_{x \in E} |\chi_r(x) + 1| \geqslant 1/8.$$

Finally consider a finite subset $\{x_1, x_2, \ldots, x_k\}$ of E say. We have $\{x_1, x_2, \ldots, x_k\} \subseteq \{\gamma, \alpha_1, \ldots, \alpha_m\}$ for some m. But given any n large enough (so that $h(n) \geqslant m + 1$) we have $|\gamma - \gamma_n| \leqslant 1/2\ \delta(n+1)$ and $|\alpha_{wn} - \alpha_w| \leqslant 1/2\ \delta(n+1)$ for $1 \leqslant w \leqslant m$ and so $\gamma, \alpha_1, \ldots, \alpha_m$ are n-independent. Since we can take n arbitrarily large, this shows $\gamma, \alpha_1, \ldots, \alpha_m$ and so $\{x_1, \ldots, x_k\}$ independent. This gives E independent and completes the proof.

Although the 2 versions of the proof of *Theorem* 4 are very similar and require the same amount of work, I think that the second just given is easier to extend (and I will take it as a model in my proof of *Theorem* 9).

The method used above to construct for example γ independent as the limit of γ_n n-independent rationals is strongly reminiscent of Liouville's construction of transcendental numbers ([4] Chapter II, § 7). Let us call a set $E \subseteq \mathbf{T}$ (respectively $E \subseteq \mathbf{R}$) *algebraically independent* if given $x_1, x_2, \ldots, x_n$ distinct, $\sum\limits_{q=1}^{n} l_q x_q = 0$ with $l_1, l_2, \ldots, l_n$ algebraic implies $l_1 = l_2 = \cdots = l_n = 0$. Set $W_M = \Big\{ t \in \mathbf{R} : \sum\limits_{q=0}^{M} m_q t^q = 0$ in

$\mathbf{R}$ for some $m_0, m_1, \ldots, m_{\mathrm{M}} \in \mathbf{Z}$ with $0 < \sum_{q=0}^{\mathrm{M}} |m_q| \leqslant \mathrm{M}\Big\}$.
Call a collection of disjoint (closed) sets $\mathrm{I}_1, \ldots, \mathrm{I}_n$ *algebraically* M-*independent* if whenever $x_i \in \mathrm{I}_i$, $l_i \in \mathrm{W}_{\mathrm{M}}$ $[1 \leqslant i \leqslant n]$

$$\sum_{j=1}^{n} l_j x_j = 0$$

implies $l_1 = l_2 = \cdots = l_n = 0$. We then have the following extensions of previous results (*Lemma* 6.2 and *Lemma* 6.8 (*ii*)) :

LEMMA 6.10 : (*i*) In $\mathbf{R}$ (*respectively* $\mathbf{T}$) *given* $\mathrm{I}_1, \mathrm{I}_2, \ldots, \mathrm{I}_n$ *disjoint closed intervals and* $x_1, x_2, \ldots, x_n$ *algebraically* M-*independent with* $x_i \in \operatorname{int} \mathrm{I}_i$ $[1 \leqslant i \leqslant n]$ *we can find* $\mathrm{J}_1, \mathrm{J}_2, \ldots, \mathrm{J}_n$ *closed intervals such that*

$$x_i \in \operatorname{int} \mathrm{J}_i \subseteq \mathrm{I}_i \; [1 \leqslant i \leqslant n]$$

and $\mathrm{J}_1, \mathrm{J}_2, \ldots, \mathrm{J}_n$ *are algebraically* M-*independent.*

(*ii*) *If* $x_1, x_2, \ldots, x_n$ *are algebraically* M-*independent* $\mathrm{E} \subseteq \mathbf{T}$ (*respectively* $\mathrm{E} \subseteq \mathbf{R}$) *infinite, then there exists a* $y \in \mathrm{E} \setminus \{x_1, x_2, \ldots, x_n\}$ *such that* $y, x_1, \ldots, x_n$ *are algebraically* M-*independent.*

(*iii*) *Transcendental numbers exist.*

Proof. — (*i*) As for *Lemma* 6.2.

(*ii*) As for *Lemma* 6.8 (*ii*).

(*iii*) Suppose we have constructed x_n algebraically n-independent and $\delta_n > 0$. We can find a δ_{n+1} with

$$1/4\, \delta_n > \delta_{n+1} > 0 \quad \text{such that} \quad y \in \mathrm{N}(x_n, \delta_{n+1})$$

implies y algebraically n-independent. By (*ii*) we can find an $x_{n+1} \in \mathrm{N}(x_n, \delta_{n+1}/4)$ which is algebraically $n+1$-independent. Now $x_n \to x$ say where $x \in \mathrm{N}(x_n, \delta_{n+1})$ for all n. Thus $x/2\pi$ is algebraically independent i.e. transcendental. (Strictly speaking any $y \in \mathbf{R}$ belonging to the equivalence class of $x/2\pi \in \mathbf{T}$ is transcendental.)

The concept of algebraic independence does not seem to be very deep. For example $\{\sqrt{2\pi}\}$ is Kronecker in $\mathbf{T}$ but not algebraically independent. However, if the reader wishes, it is an easy if lengthy process, using mainly *Lemma* 6.10 (*i*), to substitute algebraic M-independent for M-independent, algebraically independent for independent, and algebraically independent Kronecker for Kronecker in the results of this paper.

We now return to the central argument of this section. In *Theorem* 6 and Section 7 we shall want to construct not merely an independent perfect set E but a measure μ supported on E which is badly behaved. The hint as to how to do this is provided by the observation that, for example, if $\mu(\mathbf{T}) = 1$ then $\int 1\, d\mu = 1$ whatever the finer structure of μ. Again if $f(x) = -1$ for $x \in [0, \pi)$, $f(x) = 1$ for $x \in [\pi, 2\pi)$ then provided $\mu([0, \pi)) = 1/2$, $\mu([\pi, 2\pi)) = 1/2$ we have $\int f d\mu = 0$ whatever the finer structure. We develop this idea in two lemmas, the first of which, *Lemma* 6.12 (*i*), is used in our proof of *Theorem* 6, and the second, *Lemma* 6.12 (*ii*), in Section 7. We make the following definitions. Given $I_1, I_2, \ldots, I_n \subseteq \mathbf{T}$ disjoint closed intervals and σ a measure with $\operatorname{supp} \sigma \subseteq \bigcup_{i=1}^{n} I_i$ we call σ' a *descendant measure* of σ (with respect to $I_1, I_2, \ldots, I_n$) if $\operatorname{supp} \sigma' \subseteq \bigcup_{i=1}^{n} I_i$ and $\sigma'(I_j) = \sigma(I_j)$ $[1 \leqslant j \leqslant n]$. If $I_1, I_2, \ldots, I_n \subseteq \mathbf{T}$ and σ are as above and further $\sigma | I_j = A_j \mu | I_j$ where A_j is a constant $[1 \leqslant j \leqslant n]$ and μ is Haar measure we call σ a *distributed measure* on $I_1, I_2, \ldots, I_n$. To avoid complications in the statement and proofs of results, we shall use both definitions in the mildly abusive style of the following remarks, the last 2 of which justify the nomenclature « descendant ».

LEMMA 6.11 : (*i*) *If* σ *is a measure and* $I_1, I_2, \ldots, I_n$ *disjoint closed intervals with* $\operatorname{supp} \sigma \subseteq \bigcup_{i=1}^{n} I_i$ *then if* $I'_j \subseteq I_j$ *are closed intervals* $[1 \leqslant j \leqslant n]$, *then there exists a unique descendant distributed measure on* $I'_1, I'_2, \ldots, I'_n$.

(*ii*) *If* σ *is a distributed measure on* $I_1, I_2, \ldots, I_n$ *disjoint intervals and* $\delta, \varepsilon > 0$ *are given, we can find*

$$I_{pq} \subseteq I_p \; [1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant n]$$

disjoint (closed) intervals and σ' *a descendant distributed measure such that* $\operatorname{diam} I_{pq} \leqslant \delta$, $\sigma(I_{pq}) \leqslant \varepsilon$.

(*iii*) *Suppose* $\mathfrak{L}_1, \mathfrak{L}_2, \ldots$ *collections of disjoint intervals such that setting* $P_r = \cup \{F : F \in \mathfrak{L}_r\}$ *we have* $P_{r+1} \subseteq P_r$ $[r \geqslant 1]$. *Suppose* $\sigma_1, \sigma_2, \ldots$ *measures with* $\operatorname{supp} \sigma_r \subseteq P_r$ *such that* σ_{r+1} *is a descendant measure of* σ_r, *with respect to* $\mathfrak{L}_r$ $[r \geqslant 1]$. *Then* σ_s *is a descendant measure of* σ_r *with respect to* $\mathfrak{L}_r$ *for all* $1 \leqslant r \leqslant s$.

(*iv*) *Under the conditions of* (*iii*), *if* $\sigma_n \to \sigma$ *in the weak star topology as* $n \to \infty$ *then* σ *is a descendant measure of* σ_r *with respect to* $\mathfrak{L}_r$ *for all* $r \geqslant 1$.

We now come to 2 key lemmas. They are obvious, but must be fully digested. If the reader considers them together with *Lemma* 6.1 (*iii*) and (*iv*) above, he will see how we hope to proceed.

LEMMA 6.12: (*i*) *Suppose* $I_1, I_2, \ldots, I_n$ *closed* (*but not necessarily disjoint*) *intervals and* σ *is any absolutely continuous* (*with respect to Haar measure*) *measure with* $\operatorname{supp} \sigma \subseteq \bigcup_{i=1}^{n} I_i$. *Then we can find closed disjoint intervals* $J_1, J_2, \ldots, J_m$ *with* $\bigcup_{j=1}^{m} J_j \subseteq \bigcup_{i=1}^{n} I_i$ *and a measure* σ' *distributed on* $J_1, J_2, \ldots, J_m$ *such that* if σ'' *is any descendant measure of* σ' (*with respect to* $J_1, J_2, \ldots, J_m$) *we have* $\sigma''(I_k) = \sigma(I_k)$ $[1 \leqslant k \leqslant n]$.

Proof. — Consider the end points of $I_1, I_2, \ldots, I_n$. Let them be $a_1, a_2, \ldots, a_u$ where $0 \leqslant a_1 < a_2 < a_3 < \cdots < a_u < 2\pi$. Define σ' as follows. Let

$$[a_v + (a_{v+1} - a_v)/4, a_{v+1} - (a_{v+1} - a_v)/4] = K_v$$

and $\sigma'|K_v = (2\sigma([a_v, a_{v+1}])/(a_{v+1} - a_v))\mu|K_v$ whilst

$$\sigma'\left|\left(\Pi \setminus \bigcup_{v=1}^{u-1} K_v\right)\right| = 0. \quad \text{Let} \quad J_1, J_2, \ldots, J_m$$

be disjoint intervals making up the support of σ'. Suppose now σ'' is a descendant measure of σ' (with respect to $J_1, J_2, \ldots, J_m$). Then for any $1 \leqslant k \leqslant n$ we can find

$X \subseteq \{1, 2, \ldots, u\}$ such that $K_v \subseteq I_k$ if $v \in X$ and $I_k \cap K_v = \emptyset$ otherwise. We then have

$$\sigma''(I_k) = \sum_{v \in X} \sigma''(K_v) = \sum_{v \in X} \sigma'(K_v) = \sigma(I_k).$$

LEMMA 6.12: (*ii*) *Suppose* I *is a closed interval and* σ *is a distributed measure on* I. *Suppose further* $h_1, h_2, \ldots, h_n$ *are continuous functions and* $\varepsilon > 0$ *are given. Then we can find disjoint closed subintervals* $J_1, J_2, \ldots, J_m$ *and a measure* σ' *distributed on* $J_1, J_2, \ldots, J_m$ *such that if* σ'' *is any descendant measure of* σ' *(with respect to* $J_1, J_2, \ldots, J_m$*) we have* $\left|\int h_i \, d\sigma'' - \int h_i \, d\sigma\right| \leqslant \varepsilon$ $[1 \leqslant i \leqslant n]$.

Proof. — Without loss of generality take h_i and σ real. Let $I = [a, b]$ where $0 \leqslant a \leqslant b \leqslant 2\pi$. Recall that for h a real continuous function (so that the Lebesgue and Riemann definition of integral coincide) there exists a δ such that for all dissections D:

$$a = x_0 < x_1 < x_2 < \cdots < x_l = b$$

with $\delta \geqslant \max_{l \geqslant r \geqslant 1} |x_r - x_{r-1}|$ we have setting $k = \sigma(I)/(b-a)$ and writing

$$S(D, h) = k \sum_{r=1}^{l} (x_r - x_{r-1}) M_r(h)$$

where

$$M_r(h) = \sup \{h(x) : x \in [x_{r-1}, x_r]\}$$

$$s(D, h) = k \sum_{r=1}^{l} (x_r - x_{r-1}) m_r(h)$$

where

$$m_r(h) = \inf \{h(x) : x \in [x_{r-1}, x_r]\}$$

that

$$S(D, h) \geqslant \int h \, d\sigma \geqslant s(D, h) \quad \text{and} \quad |S(D, h) - s(D, h)| \leqslant \varepsilon.$$

In particular there exists a dissection D_0:

$$a = y_0 < y_1 < \cdots < y_m = b$$

such that $S(D_0, h_i) \geqslant \int h_i \, d\sigma \geqslant s(D_0, h_i)$ and

$$|S(D_0, h_i) - s(D_0, h_i)| \leqslant \varepsilon \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

Let $J_s = [y_{s-1} + (y_s - y_{s-1})/4, \ y_s - (y_s - y_{s-1})/4]$ and $\sigma'|J_s = 2k\mu|J_s \ [1 \leqslant s \leqslant n]$ whilst $\sigma'|\left(\Pi \setminus \bigcup_{t=1}^{m} J_t\right) = 0$. Then if σ'' is a descendant measure of σ' (with respect to $J_1, J_2, \ldots, J_m$) we have

$$S(D_0, h_i) = \sum_{s=1}^{m} M_s(h_i)\sigma''(J_s) \geqslant \int f \, d\sigma'' \geqslant \sum_{s=1}^{m} m_s(h_i)\sigma''(J_s) = s(D_0, h_i)$$

and so $\left|\int h_i \, d\sigma - \int h_i \, d\sigma''\right| \leqslant \varepsilon$ for $1 \leqslant i \leqslant n$ as required.

These lemmas are, of course, very closely related and, indeed, each can be deduced from the other. They are capable of considerable generalisation but the restriction h_i continuous in *Lemma* 6.12 (*ii*) cannot be removed without altering the character of the result (consider σ'' the finite sum of point masses). We add the probably unnecessary caveat that whatever the character of σ we have

$$|\sigma|(E) \inf_{x \in E} |f(x)| \leqslant \int f \, d\sigma \leqslant |\sigma|(E) \sup_{x \in E} |f(x)|$$

for all f continuous, E closed.

We now employ the machinery set up in this section to prove

THEOREM 6. — *There exist* 2 *disjoint perfect Kronecker sets* L *and* M *such that* L ∪ M *is independent but not weak Dirichlet.*

Description of Proof. — We proceed as in *Theorem* 1 by balancing the construction of one set against the other so that at any time either L or M is badly behaved. To simplify matters we split the proof into steps.

The first 2 lemmas are purely manipulative and enable us to « prepare » the sets. The first enables us to « switch » from L to M, the second to « divide » the sets so far constructed sufficiently finely to proceed.

LEMMA 6 A. — *Suppose we are given* $\eta, \delta > 0$, $I_1, I_2, \ldots, I_t$, $J_1, J_2, \ldots, J_s$ *disjoint closed intervals such that* n diam $J_p \geqslant 2\pi$ $[1 \leqslant p \leqslant s]$, $N > n$ *and* σ *a positive measure distributed on* $I_1, I_2, \ldots, I_t, J_1, J_2, \ldots, J_s$. *Then we can find an* $m > N$, $J_{pq} \subseteq J_p$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$ *disjoint closed intervals and* σ' *a descendant measure of* σ *distributed on*

$$J_{11}, J_{12}, \ldots, J_{1r(1)}, \ldots, J_{pr(p)}, I_1, \ldots, I_t$$

with the following properties. If σ'' *is a descendant measure of* σ' *then for* $m \geqslant l \geqslant n$ *we have*

$$\sigma''\{x : |\chi_l(x) - 1| \geqslant 1\} \geqslant 1/4\, \sigma\left(\bigcup_{p=1}^{s} J_p\right).$$

Further m diam $I_i \geqslant 2\pi$ $[1 \leqslant i \leqslant t]$, diam $J_{pq} \leqslant \eta$ *and* $\sigma' J_{pq} \leqslant \delta$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$.

Proof. — Select $m > n$ such that

$$m \text{ diam } I_i \geqslant 2\pi\ [1 \leqslant i \leqslant t]$$

and remark that since n diam $J_p \geqslant 2\pi$ $[1 \leqslant p \leqslant s]$ we have for $m \geqslant l \geqslant n$ setting $E_l = \left\{x \in \bigcup_{p=1}^{s} J_p : |\chi_l(x) - 1| \geqslant 1\right\}$ that $\sigma''' E_l \geqslant 1/4\, \sigma\left(\bigcup_{p=1}^{s} J_p\right)$. Now use *Lemma* 6.12 (*i*) (and less crucially *Lemmas* 6.11 (*ii*) and (*iii*)).

LEMMA 6 B. — *Under the hypotheses of Lemma* 6A *we can find an* $m > N$, $J_{pq} \subseteq J_p$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$, $I_{k1} \subseteq I_k$ $[1 \leqslant l \leqslant u(k),\ 1 \leqslant k \leqslant t]$ *disjoint closed intervals and* σ' *a descendant measure of* σ *distributed on the* J_{pq}, I_{k1} *with the following properties. If* σ'' *is a descendant measure of* σ' *then for* $m \geqslant l \geqslant n$

$$\sigma''\ \{x : |\chi_l(x) - 1| \geqslant 1\} \geqslant 1/4 \min\left(\sigma\left(\bigcup_{k=1}^{t} I_k\right), \sigma\left(\bigcup_{p=1}^{s} J_p\right)\right).$$

Further m diam $J_{pq} \geqslant 2\pi$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$ *whilst* diam J_{pq}, diam $I_{kl} \leqslant \eta$ *and*

$$\sigma' J_{pq}, \sigma' I_{k1} \leqslant \delta\ [1 \leqslant l \leqslant u(k),\ 1 \leqslant k \leqslant t].$$

Proof. — Apply *Lemma* 6A twice, interchanging the role of the « I intervals » and « J intervals » on the second occasion.

Next we have a lemma which enables us to proceed along the lines of the proof of *Theorem* 1.

LEMMA 6 C. — *Suppose we are given* $f \in S$, $\delta > 0$ *and* $I_1, I_2, \ldots, I_t, J_1, J_2, \ldots, J_s$ *disjoint closed intervals such that* n diam $J_p \geqslant 2\pi$ *together with* σ *a positive measure distributed on* $I_1, I_2, \ldots I_t, J_1, J_2, \ldots, J_s$. *Then we can find* $m > N > n$, *disjoint closed intervals* $I'_i \subseteq I_i$ $[1 \leqslant i \leqslant t]$, $J_{pq} \subseteq J_p$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$ *and* σ' *a descendant measure of* σ *distributed on the* I'_i, J_{pq} *with the following properties. If* $x \in \bigcup_{i=1}^{t} I'_i$ *then* $|\chi_N(x) - 1| \leqslant \delta$ *and if* σ'' *is a descendant measure of* σ' *then*

$$\sigma'' \{x : |\chi_l(x) - 1| \geqslant 1\} \geqslant 1/4\, \sigma\left(\bigcup_{p=1}^{s} J_p\right) \quad \textit{for} \quad m \geqslant l \geqslant n.$$

In addition m diam $I'_i \geqslant 2\pi$.

Proof. — By *Lemma* 3.3 (i) we can find an $N > n$ and $I'_i \subseteq I_i$ closed intervals such that $|\chi_n(x) - f(x)| \leqslant \delta$ for all $x \in \bigcup_{i=1}^{t} I'_i$. Let σ''' be the unique descendant distributed measure of σ on $I'_1, I'_2, \ldots, I'_t, J_1, J_2, \ldots, J_s$ and apply *Lemma* 6 *A* to $I'_1, I'_2, \ldots, I'_t, J_1, J_2, \ldots, J_s, \sigma'''$, n and N.

The next 2 lemmas cover the other part of the construction, that ensuring independence. The reader should reread *Lemma* 6.4 to see what is going on.

LEMMA 6 D. — *Suppose we have* $I_1, I_2, \ldots, I_k, J_1, J_2, \ldots, J_v, J_{v+1}, \ldots, J_{v+s}$ *disjoint closed intervals. Suppose* $J_{pq} \subseteq J_p$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant \nu + s]$ *disjoint closed intervals and* σ *a positive distributed measure on the* I_i, J_{pq}. *Suppose* $\varepsilon > 0$, M *and* n *given with* n diam $J_{pq} \geqslant 2\pi$. *Then we can find closed intervals*

$$I'_j \subseteq I_j\ [1 \leqslant j \leqslant k],\ J'_{pq} \subseteq J_{pq}\ [1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant \nu],$$

$$J_{pqw} \subseteq J_{pq}\ [1 \leqslant w \leqslant h(p, q), 1 \leqslant q \leqslant r(p), \nu + 1 \leqslant p \leqslant \nu + s]$$

an $m > n$ *and* σ' *a descendant measure of* σ *(with respect to* I_j, J_{pq}*) distributed on the*

$$I'_j,\ J'_{pq}\ [1 \leqslant p \leqslant \nu],\ J_{pqw}\ [\nu + 1 \leqslant p \leqslant \nu + s]$$

such that the following is true. The I'_j, J'_{pq} $[1 \leqslant p \leqslant \nu]$ *form an* M*-independent set,* m diam $I'_j \geqslant 2\pi$ *and if* σ'' *is a descendant measure of* σ *then*

$$\sigma''\ \{x : |\chi_r(x) - 1| \geqslant 1\} \geqslant 1/4\ \sigma\left(\bigcup_{p=\nu+1}^{\nu+s} J_p\right) \quad \textit{for } m \geqslant r \geqslant n.$$

Proof. — By *Lemma* 6.2 we can find $I'_j \subseteq I_j$ $[1 \leqslant j \leqslant k]$, $J'_{pq} \subseteq J_{pq}$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant \nu]$ closed intervals which are M-independent. Now we apply *Lemma* 6 A, taking the I'_j, J'_{pq} $[1 \leqslant p \leqslant \nu]$ as « I intervals », the J_{pq} $[\nu + 1 \leqslant p \leqslant s]$ as « J intervals » and the unique descendant distributed measure σ''' of σ on the

$$I'_j,\ J'_{pq}\ [1 \leqslant p \leqslant \nu],\ J_{pq}\ [\nu + 1 \leqslant p \leqslant s]$$

as our starting measure.

In the heuristic terminology of *Lemma* 6.4 we have obtained block-wise M-independence of $I_1, I_2, \ldots, I_k, J_1, J_2, \ldots, J_v$. (We shall call the $J_1, J_2, \ldots, J_v$ in the above construction « free intervals »). Following the obvious line of attack we have

Lemma 6 E. — *Suppose we have* $I_1, I_2, \ldots, I_t, J_1, J_2, \ldots, J_s$ *disjoint closed intervals and* σ *a positive distributed measure on them. Suppose*

$$\sigma(J_p) \leqslant 1/2\,M\,\sigma\left(\bigcup_{w=1}^{s} J_w\right) \left(\text{so if } \sigma\left(\bigcup_{w=1}^{s} J_w\right) > 0\right.$$

it follows automatically that $s \geqslant 2M$, *we shall take* $s \geqslant 2M$ *in any case) and* n diam $J_p \geqslant 2\pi$ *but* diam $J_p \leqslant \delta$ $[1 \leqslant p \leqslant s]$ *and* diam $I_i \leqslant \delta$ $[1 \leqslant i \leqslant t]$. *Then we can find an* $m > n$ *and*

$$I_{ij} \subseteq I_i\ [1 \leqslant j \leqslant u(i),\ 1 \leqslant i \leqslant t],\ J_{pq} \subseteq J_p\ [1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$$

disjoint closed intervals together with σ' *a descendant measure of* σ *distributed on them having the following properties. If*

σ'' is a descendant measure of σ' then $m \geqslant r \geqslant n$ implies

$$\sigma'' \{x : |\chi_r(x) - 1| \geqslant 1\} \geqslant 1/8\, \sigma\left(\bigcup_{p=1}^{s} \mathrm{J}_p\right) \text{ whilst}$$

$$m \operatorname{diam} \mathrm{I}_{ij} \geqslant 2\pi \ [1 \leqslant j \leqslant u(i),\ 1 \leqslant i \leqslant t].$$

If we set $\mathrm{P} = \bigcup_{i=1}^{t} \bigcup_{j=1}^{u(i)} \mathrm{I}_{ij} \cup \bigcup_{p=1}^{s} \bigcup_{q=1}^{r(p)} \mathrm{J}_{pq}$ *then*

$$z_1, z_2, \ldots, z_{\mathrm{M}} \in \mathrm{P}, \ |z_k - z_l| > \delta \ \text{ for } \ \mathrm{M} \geqslant k > l \geqslant 1$$

and $\mathrm{M} \geqslant \sum_{k=1}^{\mathrm{M}} |m_k| > 0$ *together imply* $\sum_{k=1}^{\mathrm{M}} m_k z_k \neq 0.$

Proof. — We apply *Lemma* 6 D followed by *Lemma* 6 A repeatedly, taking every possible combination of M members of $\{\mathrm{J}_1, \mathrm{J}_2, \ldots, \mathrm{J}_s\}$ as free intervals. The result now follows as in *Lemma* 6.4 (or indeed directly from it).

We can now combine these results to obtain the

Proof of Theorem 6. — We construct inductively $\mathcal{L}_n$, $\mathcal{M}_n$ 2 finite collections of disjoint closed intervals and σ_n a positive distributed measure on $\mathcal{L}_n \cup \mathcal{M}_n$ with the following properties. Setting $\mathrm{L}_n = \cup\{\mathrm{F} : \mathrm{F} \in \mathcal{L}_n\}$, $\mathrm{M}_n = \cup\{\mathrm{F} : \mathrm{F} \in \mathcal{M}_n\}$ we have $\mathrm{L}_n \supseteq \mathrm{L}_{n+1}$, $\mathrm{M}_n \supseteq \mathrm{M}_{n+1}$, $\sigma_n(\mathrm{M}_n) = \sigma_n(\mathrm{L}_n) = 1/2$ and σ_{n+1} a descendant measure of σ_n with respect to $\mathcal{L}_n \cup \mathcal{M}_n$. Also there exist $m(1) < m(2) < m(3) < \cdots$ (to be chosen consistent with our demands in the next paragraph) such that for any descendant measure σ' of σ_n we have

$$\sigma' \{x : |\chi_r(x) - 1| \geqslant 1\} \geqslant 1/16 \quad \text{for} \quad m(n) \leqslant r \leqslant m(n+1).$$

Let $p = 1, 2, 3, \ldots,$. We can further arrange, using *Lemmas* 6 B and 6 C, that for some $h(\mathrm{L}, p)$ and some $n = k(\mathrm{L}, p)$ say (and so for all $n > k(\mathrm{L}, p)$) $|\chi_{h(\mathrm{L},p)}(x) - f_p(x)| \leqslant 2^{-p}$ for all $x \in \mathrm{L}_n$ and similarly for some $h(\mathrm{M}, p)$ and some $n = k(\mathrm{M}, p)$ say (and so for all $n > k(\mathrm{M}, p)$) $|\chi_{h(\mathrm{M},p)}(x) - f_p(x)| \leqslant 2^{-p}$ for all $x \in \mathrm{M}_n$. By *Lemma* 6 B we can arrange that for some $n = n_0(p)$ say (and so for all $n \geqslant n_0(p)$)

$$\max \operatorname{diam} \{\mathrm{F} : \mathrm{F} \in \mathcal{L}_n \cup \mathcal{M}_n\}, \ \max \ \{\sigma_n \mathrm{I} : \ \mathrm{I} \in \mathcal{L}_n \cup \mathcal{M}_n\} \leqslant 1/p.$$

Thus by *Lemma* 6 E we can ensure that for some $n_1(p)$ (and so for all $n \geqslant n_1(p))$

$$x_1, x_2, \ldots, x_p \in \mathrm{L}_n \cup \mathrm{M}_n,$$

$$\min_{p \geqslant u > v \geqslant 1} |x_u - x_v| \geqslant 1/p,\ p \geqslant \sum_{u=1}^{p} |m_u| > 0$$

together imply $\sum_{u=1}^{p} m_u x_u \neq 0$.

We now set $\mathrm{L} = \bigcap_{n=1}^{\infty} \mathrm{L}_n$, $\mathrm{M} = \bigcap_{n=1}^{\infty} \mathrm{M}_n$ and note that σ_n tends (in the weak star topology) to a positive σ with $\operatorname{supp} \sigma \subseteq \mathrm{L} \cup \mathrm{M}$ and $\sigma(\mathrm{L} \cup \mathrm{M}) = 1$. By the usual arguments L and M are perfect Kronecker sets and $\mathrm{L} \cup \mathrm{M}$ is independent (*Lemma* 6.3). But $\sigma\{x : |\chi_r(x) - 1| \geqslant 1\} \geqslant 1/16$ for $r \geqslant m(1)$ (*Lemma* 6.11) so $\mathrm{L} \cup \mathrm{M}$ is not weak Dirichlet.

The reader will no doubt have remarked that with a little care *Lemmas* 6 A and 6 B can be dispensed with (after all, eventually $\mathcal{L}_n$, $\mathcal{M}_n$ must split as finely as required), but I feel that to do this (or indeed to employ *Lemma* 6.7 which can be used to effect 2 parts of the construction at once), only shortens, but does not simplify the proof. He will also remark that we have not used the best constants (for example in *Lemma* 6 C). In fact with a little work we could obtain

LEMMA 6.13. — *There exist* 2 *disjoint perfect Kronecker sets* L *and* M *such that* $\mathrm{L} \cup \mathrm{M}$ *is independent, but there exists a positive measure* σ *on* $\mathrm{L} \cup \mathrm{M}$ *with* $\sigma(\mathrm{L} \cup \mathrm{M}) = 1$ *and*

$$\begin{aligned}\liminf_{r \to \infty} \sigma\{x : |\chi_r(x) - \lambda| \geqslant c\} &\geqslant 1/2\ \mu\ \{y \in \mathbf{T} : |\exp iy - \lambda| \geqslant c\} \\ &= 1/2\ \mu\ \{y \in \mathbf{T} : |\exp iy - 1| \geqslant c\}\end{aligned}$$

for all $|\lambda| = 1$ *and* $1 \geqslant c \geqslant -1$ (μ *Haar measure*).

The reader who wishes to prove it would perhaps be well advised to defer this until after reading Section 7, though it is possible to do it with the tools at our disposal. The idea is firstly to let the total mass carried by the free intervals of *Lemmas* 6 D and 6 E become small in the later stages. Secondly we modify all of *Lemmas* 6 B to 6 E in accordance with the following modification of *Lemma* 6 A.

LEMMA 6.14. — *Suppose we are given*

$$\eta, \delta > 0,\ \mathrm{P}, \mathrm{Q}, \mathrm{R} \in \mathbf{Z}^+ = \{r \geqslant 1 : r \in \mathbf{Z}\}$$

(*in particular we may have* $2R > Q$), $I_1, I_2, \ldots, I_t, J_1, J_2, \ldots, J_s$ *disjoint closed intervals such that* n diam $J_p \geqslant 2Q\pi$ $[1 \leqslant p \leqslant s]$, $N > n$ *and* σ *a positive measure distributed on* $I_1, I_2, \ldots, I_t$, $J_1, J_2, \ldots, J_s$. *Then we can find an* $m > N$,

$$J_{pq} \subseteq J_p \; [1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$$

disjoint closed intervals and σ' *a descendant measure of* σ *distributed on* $J_{11}, J_{12}, \ldots, J_{1r(1)}, J_{21}, \ldots, J_{p,r(p)}, I_1, I_2, \ldots, I_t$ *with the following properties. If* σ'' *is a descendant measure of* σ' *then for* $m \geqslant l \geqslant n$ *and* $2^P \geqslant k$, $q \geqslant 2^{-P}$ *we have*

$$\begin{aligned}\sigma''\ \{x : |\chi_l(x) - \exp 2i\pi k/2^P| \geqslant q/2^P\}&\\ \leqslant ((2Q-1)/4Q)\mu\ \{y \in \mathbf{T} :\ |\exp iy - 1| \geqslant q/2^P\}.&\end{aligned}$$

Further m diam $I_i \geqslant 2\pi R$ $[1 \leqslant i \leqslant t]$, diam $J_{pq} \leqslant \eta$ *and* $\sigma' J_{pq} \leqslant \delta$ $[1 \leqslant q \leqslant r(p),\ 1 \leqslant p \leqslant s]$.

Proof. — Note that setting

$$E_{lkq} = \left\{ x \in \bigcup_{p=1}^{s} J_p : |\chi_l(x) - \exp 2i\pi k/2^P| \geqslant q/2^P \right\}$$

we have $\sigma E_{lkq} \geqslant ((2Q-1)/4Q)\mu\ \{y \in \mathbf{T} : |\exp iy - 1| \geqslant q/2^P\}$ and proceed as in *Lemma* 6 A.

We can now repeat the construction in the proof of *Theorem* 6 to obtain L, M disjoint Kronecker perfect sets with $L \cup M$ independent and a positive measure σ on $L \cup M$ with $\sigma(L \cup M) = 1$ and

$$\liminf_{r \to \infty} \sigma\{x : |\chi_r(x) - \lambda| \geqslant c\} \geqslant 1/2\, \mu\ \{y \in \mathbf{T} : |\exp iy - 1| \geqslant c\}$$

where c and $\arg \lambda/2$ are dyadic fractions (i.e. have the form $k/2^P$), $1 \geqslant c \geqslant -1$ and $|\lambda| = 1$. The full result follows by continuity.

For the sake of consistency we note that in *Theorems* 1 and 2 (and where appropriate, as e.g. in *Lemma* 2.1) we can by a similar argument replace the statement E non Dirichlet by

$$\liminf_{r \to \infty} \sup_{x \in E} |\chi_r(x) - \lambda| > 0 \quad \text{for all} \quad |\lambda| = 1.$$

In Section 7 we shall use slightly more delicate methods than we used in *Theorem* 6. These can be adapted to prove the results we have just obtained (and to give in *Lemma* 7.5 an interesting alternative construction to those used elsewhere in Section 7). In order to emphasize the parallelism, we essentially restate *Lemma* 6.14 as

LEMMA 6.15. — (*i*) *Suppose* K *is a closed interval,* τ_r *a positive distributed measure, of mass* 1, *on it and* $B_r, N_r \geqslant 1$, $\varepsilon_r > 0$ *given. Then we can find an* $N_{r+1} > N_r$ *such that setting* $Q_r = 2^{B_r}$ *we have*

$$\tau_r \{x : |\chi_m(x) - \exp 2\pi ik/Q_r| \geqslant q/Q_r\} \\ \geqslant (1 - \varepsilon_r/2)\mu \{y \in \mathbf{T} : |\exp iy - 1| \geqslant q/Q_r\}$$

for $|m| \geqslant N_{r+1}$, $Q_r \geqslant |k|$, $q \geqslant 0$.

(*ii*) *Further if* $\varepsilon_r > 0$, N_{r+1}, B_r, K, τ_r *satisfy the conclusion of* (*i*) *and* $M_{n+1} > M_{r+1}$ *and* $\eta_{n+1} > 0$ *are given, we can find closed disjoint intervals* $I_{K1}, I_{K2}, \ldots, I_{Ks} \subseteq K$ *with*

$$\operatorname{diam} I_{Kj} \leqslant \eta_{n+1}$$

and τ_{n+1} *a descendant measure of* τ_n *distributed on the* I_{Kj} $[1 \leqslant j \leqslant s]$ *such that*

$$\tau'_{n+1} \{x : |\chi_m(x) - \exp 2\pi ik/Q_r| \geqslant q/Q_r\} \\ \geqslant (1 - \varepsilon_r)\mu\{y \in \mathbf{T} : |\exp iy - 1| \geqslant q/Q_r\}$$

for $M_{r+1} \leqslant |m| \leqslant M_{n+1}$, $Q_r \geqslant |k|$, $q \geqslant 0$ *and* τ'_{n+1} *any descendant measure of* τ_{n+1}.

7. Kronecker and Helson Sets.

We shall need

LEMMA 7.1. — (*i*) *Suppose* K *is a closed interval,* τ_r *a positive distributed measure on it, and* $N_r \geqslant 1$, $\varepsilon_r > 0$ *given. Then we can find an* $N_{r+1} > N_r$ *such that* $\left|\int \chi_m \, d\tau_r\right| \leqslant \varepsilon_r/2$ *for* $|m| \geqslant N_{r+1}$.

(*ii*) *Further, if* $\varepsilon_r > 0$, N_{r+1}, K, τ_r *satisfy the conclusion of* (*i*) *and* $N_{n+1} > N_{r+1}$ *and* $\eta_{n+1} > 0$ *are given, we can find disjoint closed intervals* $I_{K1}, I_{K2}, \ldots, I_{Ks} \subseteq K$ *with*

$$\operatorname{diam} I_{Kj} \leqslant \eta_{n+1}$$

and τ_{n+1} *a descendant measure of* τ_n *distributed on* *the* I_{Kj} $[1 \leqslant j \leqslant s]$ *such that* $\left|\int \chi_m \, d\tau'_{n+1}\right| \leqslant \varepsilon_r$ *for* $N_{r+1} \leqslant |m| \leqslant N_{n+1}$ *and* τ'_{n+1} *any descendant measure of* τ_{n+1}.

Proof. — (*i*) This is the Riemann-Lebesgue Lemma ([10] § 2.8)

(*ii*) Follows as a direct consequence of (*i*) and *Lemma* 6.12 (*ii*).

The construction we are about to use is perhaps easier to follow (especially in the light of our earlier work) than to describe. The reader may thus be well advised to read the construction first (as far as *Lemma* 7.2 say) and then read the heuristic. But in this case I do not think that it should be skipped altogether. Essentially we refine our earlier methods (used e.g. in *Theorem* 6) by tampering only with very small parts (by mass) of our construction at a time. In *Theorem* 6, we tampered, roughly speaking, with 1/2 at a time. Let us examine a little more closely what will happen to a single interval K. Associated with K is an integer N_{r+1} and an $\varepsilon_r > 0$. There is also a measure $\sigma_u[u = r+1, r+2, \ldots]$ such that $\sigma_u|K$ is a constant distributed measure. At some time N_n later than N_{r+1} we decide to act. In the manner of *Lemma* 7.1 we find disjoint closed subintervals

$$I_{K1}, I_{K2}, \ldots, I_{Ks}$$

and define σ_{n+1} so that $\sigma_{n+1}|K$ is the distributed descendant measure of $\sigma_n|K$ and such that for any descendant measure σ'_{n+1} of σ_n, $\left|\int_K \chi_m \, d\sigma'_{n+1}\right| \leqslant \varepsilon_r$ for $N_{r+1} \leqslant |m| \leqslant N_n$. We then find an N_{n+1} such that (among other properties)

$$\left|\int_{I_{Kj}} \chi_m \, d\sigma_{n+1}\right| \leqslant \varepsilon_{n+1} \quad \text{for} \quad u \geqslant N_{n+1}$$

(where ε_{n+1} is chosen in advance). The N_{n+1} is then associated with each I_{Kj}.

In naive terms the I_{Kj} are « out of control » for the period $N_n < m < N_{n+1}$ (ignoring modulus signs). They return to our control from N_{n+1} onwards. But immediately we assert this control in the manner of *Lemma* 7.1, at the same time making further adjustments (to ensure block-wise independence say) at N_t say, the new subintervals pass from our control until N_{t+1}. By only asserting our control (in order to ensure independence say) on a small collection of intervals (i.e. a collection carrying only a small mass) we ensure that only a small portion is out of control in the period following our intervention. In this way we can ensure that an increasing part of the set is under increasingly strict control.

This may seem a rather simple minded way of looking at the matter, but none the less the results were obtained in this way. Moreover it emphasizes certain limitations of the method which seem to put a natural barrier on certain improvements. For example very heavy (and to a great extent unknown) constraints are laid on the I_{Kj} (they must be « well distributed » over K). The reader should contrast this with the construction in *Theorem* 3 where rigid control is exercised throughout. The constructions of this section are built up from the following

Central Inductive Step. — At the n^{th} stage we have a positive measure σ_n, integers

$$0 < N_1 < N_2 < \cdots < N_n < N_{n+1},$$

$\mathcal{E}_{1n}, \mathcal{E}_{2n}, \ldots, \mathcal{E}_{nn}$ disjoint finite collections of disjoint (closed) intervals and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0$ with the following properties. Setting $\mathcal{E}_n = \bigsqcup_{i=1}^{n} \mathcal{E}_{in}$ and $P_n = \cup \{F \in \mathcal{E}_n : 1 \leqslant i \leqslant n\}$ we have σ_n distributed on $\mathcal{E}_n$ (so $\operatorname{supp} \sigma_n \subseteq P_n$). Moreover if $F \in \mathcal{E}_{rn}$ then $\left|\int_F \chi_m \, d\sigma_n\right| \leqslant \frac{\varepsilon_r}{2} \sigma_n(F)$ for $|m| \geqslant N_{r+1}$ $[1 \leqslant r \leqslant n]$.

Suppose we now select $\varepsilon_{n+1}, \eta_{n+1} > 0$ and $\mathcal{K}_n \subseteq \mathcal{E}_n$. (This is the first point $*$ say at which we can exercise choice.) Set $L_n = \cup \{F : F \in \mathcal{K}_n\}$ and $\sigma_n(L_n) = A_n$. We now commence our construction. For each $K \in \mathcal{K}_n$ we have $K \in \mathcal{E}_{rn}$ for some $n \geqslant r \geqslant 1$. By our inductive hypothesis

$$\left|\int_K \chi_m \, d\mu_n\right| \leqslant \frac{\varepsilon_r}{2} \sigma_n(K) \quad \text{for} \quad N_{n+1} \geqslant m \geqslant N_{r+1}.$$

As in *Lemma* 7.1 we can find $\eta_{n+1} > \delta_{K1}, \delta_{K2}, \ldots, \delta_{Ks(K)} > 0$ and closed disjoint intervals

$$I_{Kj} \subseteq K \quad \text{with} \quad \operatorname{diam} I_{Kj} \leqslant \eta_{n+1} \; [1 \leqslant j \leqslant s(K)]$$

such that $\sum_{j=1}^{s(K)} \delta_{Kj} = \sigma_n(K)$ and if ψ is any measure with $\psi(I_{Kj}) = \delta_{Kj}$ and supp $\psi \subseteq \bigcup_{j=1}^{s(K)} I_{Kj}$ then $\left|\int_K \chi_m \, d\psi\right| \leqslant \varepsilon_r \sigma_n(K)$ for $N_{r+1} \leqslant |m| \leqslant N_{n+1}$. Now select

$$J_{Kj} \subseteq I_{Kj} \; [1 \leqslant j \leqslant s(K), \; K \in \mathcal{K}_n]$$

in any desired way (this is the second point $**$ say at which we can exercise choice). Let

$$\mathcal{E}_{n+1\; n+1} = \{J_{Kj} : 1 \leqslant j \leqslant s(K), \; K \in \mathcal{K}\} \quad \text{and} \quad \mathcal{E}_{r\; n+1} = \mathcal{E}_{rn} \setminus \mathcal{K}_n.$$

Construct σ_{n+1} with supp $\sigma_{n+1} \subseteq P_{n+1}$ such that $\sigma_{n+1} = \sigma_n$ on $E_n \setminus L_n$ and $\mu_{n+1}|J_{Kj}$ is proportional to Lebesgue measure on J_{Kj} and $\mu_{n+1}(J_{Kj}) = \delta_{Kj}$. Finally we note as in *Lemma* 7.1 (*i*) that by the Riemann-Lebesgue Lemma there exists an $N_{n+2} > N_{n+1}$ such that $\left|\int_I \chi_m \, d\sigma_{n+1}\right| = \frac{\varepsilon_{n+1}}{2} \sigma_{n+1}(I)$ for all $|m| \geqslant N_{n+2}$ and $I \in \mathcal{E}_{n+1\; n+1}$. We can now restart the induction.

Starting the induction in any way we please, we consider $P = \bigcap_{n=1}^{\infty} P_n$ and σ the weak star limit of σ_n as $n \to \infty$. By construction P is closed and σ a positive measure. supported on it Further σ is a descendant measure of σ_{n+1} (with respect to $\mathcal{E}_{n+1}$) so that if $K \in \mathcal{K}_n \cap \mathcal{E}_{nr}$ we have

$$\left|\int_{K \cap P} \chi_m \, d\sigma\right| \leqslant \varepsilon_r \sigma(K \cap P) \quad \text{for} \quad N_{r+1} \leqslant |m| \leqslant N_{n+1}.$$

(If on the other hand $K \in \mathcal{E}_{rr}$ but $K \notin \mathcal{K}_n$ for any $n \geqslant r$ then $\left|\int_{K \cap P} \chi_m \, d\sigma\right| \leqslant \frac{\varepsilon_r}{2} \sigma(K \cap P) \leqslant \varepsilon_r \sigma(K \cap P)$ for all $|m| \geqslant N_{r+1}$ automatically.) Thus if $|m| \geqslant N_2$ so that $N_{n+2} \geqslant |m| \geqslant N_{n+1}$ for some $n \geqslant 1$ we have

$$\begin{aligned}
\left|\int_P \chi_m \, d\sigma\right| &\leqslant \left|\int_{P \cap L_n} \chi_m \, d\sigma\right| + \sum_{r=1}^{n} \sum_{K \in \xi_{nr} \setminus \mathcal{K}_n} \left|\int_{P \cap K} \chi_m \, d\sigma\right| \\
&\leqslant \sigma(P \cap L_n) + \sum_{r=1}^{n} \sum_{K \in \xi_{nr} \setminus \mathcal{K}_n} \varepsilon_r \sigma(P \cap K) \\
&\leqslant \sigma(P \cap L_n) + \sup_{r \geqslant 1} \varepsilon_r \sigma(P \setminus L_{n+1}) \\
&= A_n + \sup_{r \geqslant 1} \varepsilon_r (\sigma(P) - A_n).
\end{aligned}$$

Taking $A_0 = \sup_{n \geqslant 1} A_n$, $\varepsilon_0 = \sup_{r \geqslant 1} \varepsilon_r$ and (as we shall henceforth) $\sigma_1(P_1) = 1$ (so that $\sigma(P) = 1$) we have

$$\left| \int_P \chi_m \, d\sigma \right| \leqslant A_0 + \varepsilon_0.$$

Note. — In fact we can allow even more freedom in the construction of P by e.g. dropping the restriction σ_n positive (but recall the caveat after *Lemma* 6.12) (*ii*) or by taking, at $*\ *$, instead of 1 subinterval of I_{K_j} several disjoint ones. (In this manner all the sets we shall construct in this section can be obtained with arbitrary Hausdorff dimension.) We shall not, however, need these refinements except in *Lemma* 7.12.

We now begin to use the freedom allowed us in the choice of $\mathcal{K}_1, \mathcal{K}_2, \ldots$. Suppose (as we shall from now on) that $\eta_n \to 0$. As an inductive hypothesis suppose we know at the n^{th} stage that $\mathcal{E}_{1w}, \mathcal{E}_{2w}, \ldots, \mathcal{E}_{lw} = \emptyset$. Let

$$\mathcal{E}_{l+1w} = \{I_i : 1 \leqslant i \leqslant k\}. \quad \text{Set} \quad \mathcal{K}_{w+i} = \{I_i\} \ [1 \leqslant i \leqslant k].$$

Then $A_{w+1} \leqslant \eta_{l+1}$ and $\mathcal{E}_{l+1,\ w+k+1} = \emptyset$. In this way we can ensure that for any given r we have $\mathcal{E}_{rn} = \emptyset$ for large enough n. (For later reference let us call the set of integers $w + i$ used in ensuring this the set U. We can always take U such that $\mathbf{Z}^+ \setminus U$ is infinite, and in what follows we shall do this. Note that $A_u \to 0$ as $u \to \infty$ with $u \in U$.) If this is done, we have P perfect and by the formula

$$\left| \int_P \chi_m \, d\sigma \right| \leqslant A_0 + \varepsilon_0 \qquad \text{for} \qquad |m| \geqslant N_2$$

proved above we have

$$\left| \int_P \chi_m \, d\sigma \right| \leqslant \sup_{s \geqslant n+1} A_s + \sup_{s \geqslant k(m)} \varepsilon_s \ \text{for} \ |m| \geqslant N_{n+2}$$

where $k(n) = \inf \{h : \mathcal{E}_{hn} \neq \emptyset\} - 1 \to \infty$ as $n \to \infty$. In particular

$$\limsup_{|m| \to \infty} \left| \int_P \chi_m \, d\sigma \right| \leqslant A + \varepsilon$$

where $A = \limsup_{n \to \infty} A_n$, $\varepsilon = \limsup_{n \to \infty} \varepsilon_n$. In what follows all our P will be assumed constructed in this manner. Moreover we shall take $\varepsilon_n \to 0$ so that

$$\limsup_{|m|\to\infty} \left|\int_P \chi_m \, d\sigma\right| \leqslant A$$

and so P is at most an H_A set. (In particular if $A = 0$, then P is not a Helson set.)

Independence. — Suppose $1 \geqslant B \geqslant 0$, $M \geqslant 1$ and $w_0 \geqslant 1$ given. We can find an $N \geqslant 2M$ such that $1/N \leqslant B$. Since $\eta_m \to 0$ and $k(m) \to \infty$ as $m \to \infty$ we can find a $w \geqslant w_0 + 1$ such that for any $I \in \mathcal{E}_w$ we have diam $I \leqslant 1/2\,M$ and $\sigma_w I \leqslant 1/(4N^2)$. Let $\mathcal{E}_w = \{I_i : 1 \leqslant i \leqslant k\}$ and note that $k \geqslant 4N^2$ (since $\sigma(P_w) = 1$). We set $q = \binom{k}{M}$ and define successively $\mathcal{K}_{w+1}, \mathcal{K}_{w+2}, \ldots, \mathcal{K}_{w+q}$ by

$$\mathcal{K}_{w+i} = \{F \in \mathcal{E}_{w+i} : F \subseteq I_{p(1,i)} \cup I_{p(2,i)} \cup \cdots \cup I_{p(M,i)}\}$$

where $p(1, i) < p(2, i), < \cdots < p(M, i)$ range over all possible values with $1 \leqslant p(u) \leqslant M$ as i increases from 1 to q (cf. restatement of *Lemma* 6.4). We now use the fredom we have at the point $**$ by taking (as we may by *Lemma* 6.2) the

$$J_{Kj}\ [1 \leqslant j \leqslant s(K),\ K \in \mathcal{K}_{w+i}]$$

to be *M*-independent. By Lemma 6.4 we see that if

$$x_1, x_2, \ldots, x_m \in P_{w+q+1} \quad \text{and} \quad |x_l - x_t| \geqslant 1/M$$

for

$$M \geqslant l > t \geqslant 1 \quad \text{then} \quad M \geqslant \sum_{l=1}^{M} |m_l| \geqslant 1$$

implies

$$\sum_{l=1}^{M} m_l x_l \neq 0. \quad \text{Moreover} \quad A_{w+i} = \sum_{t=1}^{M} \sigma_w(I_{p(t,i)}) \leqslant B.$$

We can repeat this process for successively larger n_0 and N, and successively smaller B and δ (with values tending to 0 say). By *Lemma* 6.3. — P is then independent. Call the set of integers $w + i$ used in the process V (so $V \subseteq \mathbf{Z} \setminus U$). We note that $A_r \to 0$ as $r \to \infty$ for $r \in V$ (since the values of

B tend to zero). We can always take V such that $\mathbf{Z}^+\setminus(V \cup U)$ is infinite.

But first we consider the case in which we only require that $A_n \to 0$ as $n \to \infty$ (for example when we take

$$V \cup U = \{r \in \mathbf{Z} : r \geqslant 2\}$$

as we always can). Then $A = 0$ and we have shown by construction

Lemma 7.4. — *There exists a perfect independent non Helson set* P. (*Moreover there exists a measure* σ *supported on* P *such that* $\int_P \chi_m \, d\sigma \to 0$ *as* $|m| \to \infty$.)

We now digress very slightly to obtain the result by a different method.

Comparing *Lemma* 7.1 with *Lemma* 6.15 we see that the work of this section gives mutatis mutandis (and allowing $B_r \to \infty$ as $r \to \infty$):

Lemma 7.5. — *There exists a perfect independent set* P *and a positive measure* σ *of mass* 1 *supported on* P *such that*

$$\liminf_{|m|\to\infty} \sigma\,\{x \in P : |\chi_m(x) - \lambda| \geqslant c\} \geqslant \mu\,\{y \in \mathbf{T} : |\exp iy - \lambda| \geqslant c\}$$

for all $|\lambda| = 1,\ 1 \geqslant c \geqslant 0$, i.e.

$$\sigma\{x \in P : \ |\chi_m(x) - \lambda| \geqslant c\} \to \mu\{y \in \mathbf{T} : \ |\exp \ iy - \lambda| \geqslant c\}$$

as $|m| \to \infty$ *for all* $|\lambda| = 1,\ 1 \geqslant c \geqslant 0$ (μ *is Haar measure*). *In particular if* h *is a piece-wise continuous function* $h : \Pi \to \mathbf{C}$ (*where* $\Pi = \{z \in \mathbf{C} : |z| = 1\}$) *we have*

$$\int_P h \circ \chi_m \, d\sigma \to \int_T h \circ \chi_1 \, d\mu \qquad as \qquad |m| \to \infty.$$

We see at once that $\int_P 1 \, d\sigma \to \int_T 1 \, d\mu = 1$ as $|m| \to \infty$ so that $\sigma(P) = 1$. Again $\int_P \chi_m \, d\sigma \to \int_T \chi_1 \, d\mu = 0$ so that P, σ satisfy the conditions of *Lemma* 7.4. But more is true. We have for example $\int_P \operatorname{sgn}\,(\sin mx)\, d\sigma(x) \to \int_T \operatorname{sgn}\,(\sin x)\, d\sigma(x) = 0$ and $\int_P \sin^2 mx \, d\sigma(x) \to \int_T \sin^2 x \, d\sigma(x) = 1/2$.

That *Lemmas* 7.4 and 7.5 are equivalent may be seen by approximating h by polynomials.

As a particular case of *Lemma* 7.5 we have

LEMMA 7.6. — *There exists a perfect independent set* P *and a positive measure* σ *supported on* P *such that*

$$\liminf_{|m|\to\infty} \sigma\{x \in \mathrm{P} : \cos mx \leqslant 0\} \geqslant \pi.$$

However, π seems to be a natural constant resulting from the method. It would, therefore, be interesting to know whether this constant can be increased.

We now return to the main theme of this section by modifying our inductive construction to give

THEOREM 7. — *There exist* $\mathrm{L}_1, \mathrm{L}_2, \ldots, \mathrm{L}_q$ *disjoint perfect Kronecker sets such that* $\mathrm{L}_1 \cup \mathrm{L}_2 \cup \cdots \cup \mathrm{L}_q$ *is independent but at most* $\mathrm{H}_{1/q}$.

Proof. — Start the induction in such a way that for some k $\mathcal{E}_k = \bigcup_{r=1}^{q} \mathcal{M}_r$ where setting $\mathrm{M}_r = \cup\{\mathrm{F} : \mathrm{F} \in \mathcal{M}_r\}$ we have $\mathrm{M}_1, \mathrm{M}_2, \ldots, \mathrm{M}_q$ disjoint and $\sigma_k \mathrm{M}_r = 1/q$ $[1 \leqslant r \leqslant q]$. Suppose n and p are given. Set

$$\mathcal{K}_{n+r} = \{\mathrm{I} \in \mathcal{E}_n : \mathrm{I} \subseteq \mathrm{M}_r\} \; [1 \leqslant r \leqslant q].$$

We now use the freedom we have at the point $**$ by taking (as we may by *Lemma* 3.3 (*i*)) the $\mathrm{J}_{\mathrm{K}j}$ such that there exists an $\mathrm{Q}(p, r)$ with $|\chi_{\mathrm{Q}(p,r)}(x) - f_p(x)| \leqslant 2^{-p}$ for all $x \in \mathrm{J}_{\mathrm{K}j}$ $[1 \leqslant j \leqslant s(\mathrm{K}), \mathrm{K} \in \mathcal{K}_{w+r}]$.

We repeat this process for $p = 1, 2, 3, \ldots.$ Let $\mathrm{L}_r = \mathrm{M}_r \cap \mathrm{P}$. Then $\sup_{x \in \mathrm{L}_r} |\chi_{\mathrm{Q}(p,r)}(x) - f_p(x)| \leqslant 2^{-p}$ so L_r is Kronecker. $\mathrm{L}_1, \mathrm{L}_2, \ldots, \mathrm{L}_q$ are disjoint and perfect by construction. Call the set of integers $w + r$ used in the process D (so $\mathrm{D} \subseteq \mathbf{Z} \setminus (\mathrm{U} \cup \mathrm{V})$ taken infinite). We note that $\mathrm{A}_d = 1/q \to 1/q$ as $d \to \infty$ for $d \in \mathrm{D}$. It is possible to ensure then (for example by taking $\mathrm{D} \cup \mathrm{U} \cup \mathrm{V} = \{r \in \mathbf{Z} : r \geqslant 2\}$) that $\mathrm{A} = 1/q$ and so $\mathrm{P} = \mathrm{L}_1 \cup \mathrm{L}_2 \cup \cdots \cup \mathrm{L}_q$ is at most $\mathrm{H}_{1/q}$. This gives the result.

The modifications discussed for *Lemma* 7.5 now give (taking $q = 2$) *Lemma* 6.13 and so *Theorem* 6. We can, of course, obtain other results by taking $q \neq 2$, but this and the same task for *Theorem* 8 is left to the reader.

Varopoulos [18] has greatly increased the value of this result by showing that the independent union of q Kronecker sets is at least $H_{1/q}$. Our result is thus best possible. We will conclude this section by obtaining some further (rather simple) consequences of this theorem, but we first obtain a result dependent only on the methods so far developed.

THEOREM 8. — *There exist a countable collection*

$$L_0,\ L_1,\ L_2\ \ldots$$

of disjoint perfect Kronecker sets such that $P = \bigcup_{q=0}^{\infty} L_q$ *is a closed (indeed perfect) independent non Helson set.*

Proof. — Choose γ independent and set $L_0 = \{\gamma\}$. Automatically L_0 is Kronecker (*Lemma* 1.1 (*i*)). Let

$$F_q = [\gamma + 1/(3q),\ \gamma + 1/(3q+1)]$$

for $q = 1, 2, 3, \ldots.$ Set $\mathcal{E}_1^q = \{F_q\}$ and let σ_1^q be the distributed measure on F_q with $\sigma_1^q(F_q) = 1$. For each $\mathcal{E}_1^q$ we now commence the inductive construction established in this section choosing η_n^q, $\varepsilon_n^q > 0$, $\mathcal{K}_n^q$ and finding $\mathcal{E}_n^q$, σ_n^q, N_n^q, and A_n^q in the usual manner. (Note that N_n^q need not have the same value as N_n^{q+1}). It is clear that we can arrange to have ε_n^q, $\eta_n^q \to 0$ as $A_n^q \to 1/q$ as $n \to \infty$ while ensuring at the same time that the following conditions are fulfilled. Given any M we have for large enough n that if $x_1, x_2, \ldots, x_M$ $L_0 \cup \bigcup_{q=1}^{M} P_n^q$ and $|x_l - x_t| \geqslant 1/M$ $[M \geqslant l > t \geqslant 1]$ then $M = \sum_{l=1}^{M} |m_l| \geqslant 1$ implies $\sum_{l=1}^{M} m_l x_l \neq 0$. (This is done by ensuring block-wise independence as above.) The usual argument now shows $P = L_0 \cup \bigcup_{q=1}^{\infty} P^q$ independent.

Further we demand that P^q is the (independent) union of q disjoint Kronecker sets $L_{1+2\ldots+q+r}$ say such that

$$\limsup_{|m|\to\infty} \left| \int_{P^q} \chi_m \, d\sigma^q \right| \leqslant 1/q.$$

(This is done as in *Theorem* 7).

Summing up we see that P is the perfect (since γ is a limit point of P) independent union of the disjoint perfect Kronecker sets $L_0, L_1, L_2, \ldots$. Moreover we can find positive measures $\sigma^1, \sigma^2, \sigma^3, \ldots$ of mass 1 supported by P such that $\lim\sup_{|m|\to\infty} \left|\int_P \chi_m\, d\sigma^q\right| \leqslant 1/q$ as $q \to \infty$. Thus P is non Helson.

The interest of this result is twofold. Firstly any countable independent set is H. Secondly (and more importantly) well known results give

Lemma 7.7. — *If* P *is given as in Theorem* 8 *and* σ *is a measure supported by* P *with* $|\sigma|(P) > 0$ *then* $\lim\sup_{m\to\infty} |\sigma(m)| > 0$.

Proof. — Since $\sum_{q=0}^{\infty} |\sigma|(L_q) = |\sigma|\left(\bigcup_{q=0}^{\infty} L_q\right) > 0$ there exists a $v \geqslant 0$ such that $|\sigma|(L_v) > 0$. We can also find a closed set L'_v with int $L'_v \supseteq L_v$ and $|\sigma|(L'_v \setminus L_v) \leqslant 1/4|\sigma|(L_v)$. Choose a continuous function $\omega : \mathbf{T} \to [0, 1]$ with $\omega(L_v) = 1$ and $\omega(\mathbf{T}\setminus L'_v) = 0$. Since L_v is Kronecker and so by *Lemma* 1.7 (*i*) H_1 we have

$$\lim\sup_{m\to\infty}\left|\int_P \chi_{-m}\omega\, d\sigma\right| \geqslant \lim\sup_{m\to\infty}\left|\int_{L_v} \chi_{-m}\, d\sigma\right| - 1/4|\sigma|(L_v) = 3/4|\sigma|(L_v).$$

Now we can find a trigonometric polynomial T say with $\sup_{y\in\mathbf{T}} |T(y) - \omega(y)| \leqslant 1/4\, |\sigma|(L_v)$ (e.g. by Féjer's Theorem [10] § 3, 1). Then

$$\lim\sup_{m\to\infty}\left|\int_P T\chi_{-m}\, d\sigma\right| \geqslant \lim\sup_{m\to\infty}\left|\int_P \omega\chi_{-m}\, d\sigma\right| - 1/4\, |\sigma|(L_v) \geqslant 1/2\, |\sigma|(L_v) > 0.$$

But if $\lim\sup_{m\to\infty}\left|\int_P \chi_{-m}\, d\sigma\right| = 0$ then $\lim\sup_{m\to\infty}\left|\int T\chi_{-m}\, d\sigma\right| = 0$ which yields a contradiction. This proves the lemma.

Summing up we have proved by construction

Lemma 7.8. — *There exists an independent perfect non Helson set* P *which supports no non zero measure* τ *such that* $\hat{\tau}(m) \to 0$ *as* $|m| \to \infty$.

One weakness of our methods is that while we can construct (e.g. in *Theorem* 7) sets which are at most H_s, say, we cannot ensure that they are exactly H_s. We conclude by giving some (admittedly simple) examples of what can be achieved with the help of « converse » results, in this case those of Varopoulos, which we give in a form adapted to our particular purposes.

LEMMA 7.9. — *If* A *is a closed independent set with* $A = A_1 \cup A_2 \cup \cdots \cup A_n$ *where the* $A_1, A_2, \ldots, A_n$ *are disjoint and the union of any* m *of them is Kronecker, then* A *is at least* $H_{m/n}$.

This lemma is the only major result used in the paper which we shall not prove. The proof is based on totally different principles to those used here [18].

Using it or a simpler estimation we obtain, for example, the following 3 results. In each case we give a fairly detailed but not complete sketch of the proof.

LEMMA 7.10. — *There exists a perfect set* E *such that* $\sum_1^q E = \mathbf{T}$ *yet* E *is at least* $H_{1-2/q}$.

Proof. — We construct $E = E_1 \cup E_2 \cup \cdots \cup E_q$ where $E_1, E_2, \ldots, E_q$ are disjoint perfect sets such that every union of $q-1$ of them is Kronecker. The proof follows *Lemma* 3.4. Suppose at the $r = nq + m^{th}$ step $[1 \leqslant m \leqslant q-1]$ we have $\mathcal{L}_{1r}, \mathcal{L}_{2r}, \ldots, \mathcal{L}_{qr}$ disjoint collections of disjoint closed sets such that setting $L_{vr} = \cup \{F : F \in \mathcal{L}_{vr}\}$ we have

$$\sum_{v=1}^{q} \operatorname{int} L_{vr} = \mathbf{T}.$$

As in *Lemma* 3.4 we can find $\mathcal{L}_{1r+1}, \mathcal{L}_{2r+1}, \ldots, \mathcal{L}_{q,r+1}$ such that $\mathcal{L}_{m,r+1} \subseteq \mathcal{L}_{mr}$, $\sum_{v=1}^{q} \operatorname{int} L_{v,r+1} = \mathbf{T}$, if $I \in \mathcal{L}_{wr}$ then there exists $I_1, I_2 \in \mathcal{L}_{wr+1}$ disjoint with $I_1, I_2 \subseteq I$ $[w \neq m]$,

$$1/2 \max\{\operatorname{diam} I : I \in \mathcal{L}_{wr}, w \neq m\} \geqslant \max\{\operatorname{diam} J : J \in \mathcal{L}_{wr+1}, w \neq m\}$$

and further there exists an $N^{(r+1)}$ such that

$$|\chi_{N(r+1)}(x) - f(x)| \leqslant 2^{-n} \text{ for all } x \in \bigcup_{w \neq m} L_{w,r+1}.$$

Setting $E_v = \cap L_{vr}$ we have the required result.

LEMMA 7. 11. — *There exists a perfect independent set* E *which is at least* $H_{1-1/q}$ *but not weak Dirichlet.*

Proof. — We adapt the proof of *Theorem* 6 in the same way. At the n^{th} step we have $\mathcal{L}_{1n}, \mathcal{L}_{2n}, \ldots, \mathcal{L}_{qn}$ disjoint collections of disjoint closed sets such that

$$\max\{\operatorname{diam} I : I \in \mathcal{L}_{vn}\, q \geqslant v \geqslant 1\} \to 0$$

as $n \to \infty$ and a positive measure σ_n such that setting $L_{vn} = \cup\{F : F \in \mathcal{L}_{vn}\}$ we have $\sigma_n(L_{vn}) = 1/q$ $[1 \leqslant v \leqslant q]$. We can ensure that, given any n large enough, there exists a $v(n)$ such that

$$\sigma_n\{x \in L_{v(n)n} : |\chi_r(x) - 1| \geqslant 1\} \geqslant 1/(16q)\sigma_n(L_{v(n)n}) = 1/(16q),$$

but none the less the following is true. Given $1 \leqslant w \leqslant q$ and $p \geqslant 1$ we can find an $h(p, w)$ such that

$$|\chi_{h(p,w)}(x) - f_p(x)| \leqslant 2^{-p} \quad \text{for} \quad x \in \bigcup_{v \neq w} L_{vr}$$

for all r large enough. (We do this as in *Lemma* 6 C.) Moreover, we can ensure as in Lemma 6 D that, given an M, we have that if $x_1, x_2, \ldots, x_M \in \bigcup_{v=1}^{q} L_{vr}$ and $|x_l - x_j| \geqslant 1/M$ $[M \geqslant l > j \geqslant 1]$ then $M \geqslant \sum_{l=1}^{M} |m_l| \geqslant 1$ implies $\sum_{l=1}^{M} m_l x_l \neq 0$. Setting $E_v = \bigcap_{n=1}^{\infty} L_{vn}$ and σ to be the weak star limit of σ_n we have $E = \bigcup_{v=1}^{q} E_v$ is independent and at least $H_{1-1/q}$ (since the union of any $q - 1$ of the $E_1, E_2, \ldots, E_q$ is Kronecker), but $\operatorname{supp} \sigma \subseteq E$, $\sigma(E) = 1$ and $\sigma\{x : |\chi_m(x) - 1| \geqslant 1\} \geqslant 1/(16q)$ for m large enough, so E is not weak Dirichlet.

Finally we give the following (possibly deeper) result.

LEMMA 7.12. — *Given* $1 \geqslant s, t > 0$ *we can find* L *a perfect* H_s *set and* M *a perfect* H_t *set such that* L, M *are disjoint, but* $P = L \cup M$ *is an independent* $H_{st/(s+t)}$ *set.*

Proof. — To get the idea of the proof suppose $s = k/l$, $t = k/m$ (where k, l, m are positive integers). Consider $l + m$ blocks $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_l, \mathcal{A}_{l+1}, \ldots, \mathcal{A}_{l+m}$. Suppose we ensure that any k of them (but no more) are well behaved. Then at various times a proportion $k/l = s$ of the $\mathcal{A}_1, \ldots, \mathcal{A}_l$ are well behaved, a proportion $k/m = t$ of $\mathcal{A}_{l+1}, \ldots, \mathcal{A}_{l+m}$ are well behaved and a proportion $k/(l+m) = st/(t+s)$ of $\mathcal{A}_1, \ldots, \mathcal{A}_{l+m}$ are well behaved.

Select

$$1 = l(1) < l(2) < l(3) < \cdots, 1 = m(1) < m(2) < m(3) < \cdots$$

such that $l(\nu+1)$ is an integer multiple of $l(\nu)$, $m(\nu+1)$ of $m(\nu)$ $[\nu \geqslant 1]$ and $l(w)/(l(w)+m(w)) \to t/(s+t)$ as $w \to \infty$ (so that $m(w)/(l(w)+m(w)) \to s/(s+t)$). Now choose $k(1)$, $k(2), k(3), \ldots$ such that $k(w)/l(w) \to s$ (so that $k(w)/m(w) \to t$).

We now construct P in the inductive manner established in this section. Choose L_0, M_0 disjoint intervals. We start the induction in such a way that, for some $n(1)$, $\mathcal{E}_{n(1)} = \mathcal{L}_{11} \cup \mathcal{M}_{11}$ where setting $L_1 = \cup\{I : I \in \mathcal{L}_{11}\}$, $M_1 = \cup\{I : I \in \mathcal{M}_{11}\}$ we have $\sigma_{n(1)}(L_1) = t/(t+s)$, $\sigma_{n(1)}(M_1) = s/(t+s)$. Automatically $|\chi_0(x) - f_1(x)| = 0 \leqslant 2^{-1}$ for all $x \in L_1 \cup M_1$. Suppose that for some $n(p)$ we have $\mathcal{E}_{n(p)} = \bigcup_{u=1}^{l(p)} \mathcal{L}_{ap} \cup \bigcup_{b=1}^{m(p)} \mathcal{M}_{bp}$ where setting $L_{ap} = \cup\{I : I \in \mathcal{L}_{ap}\}$, $M_{bp} = \cup\{I : I \in \mathcal{M}_{bp}\}$ we have $L_{1p}, L_{2p}, \ldots, L_{l(p)p}$ disjoint subsets of L_0; $M_{1p}, M_{2p}, \ldots, M_{m(p)p}$ disjoint subsets of M_0 with

$$\sigma_{n(p)}(L_{1p}) = \sigma_{n(p)}(L_{2p}) = \cdots = \sigma_{n(p)}(L_{l(p)p})$$

and

$$\sigma_{n(p)}(M_{1p}) = \sigma_{n(p)}(M_{2p}) = \cdots = \sigma_{n(p)}(M_{m(p)p}).$$

Suppose we are given some $N > n(p)$. By using the freedom given us at $**$ (taking several subsets of the I_{K_j} if necessary) we can ensure that for some $n(p+1) \geqslant N$,

$$\mathcal{E}_{n(p+1)} = \bigcup_{a=1}^{l(p+1)} \mathcal{L}_{a,p+1} \cup \bigcup_{b=1}^{m(p+1)} \mathcal{M}_{b,p+1}$$

where, with the usual notation, each $L_{a\,p+1}$ is a subset of some L_{cp}, each $M_{n\,p+1}$ of some M_{dp} and

$$\sigma_{n(p+1)}(L_{1p+1}) = \sigma_{n(p+1)}(L_{2p+1}) = \cdots = \sigma_{n(p+1)}(L_{1(p+1),p+1}),$$
$$\sigma_{n(p+1)}(M_{1p+1}) = \sigma_{n(p+1)}(M_{2p+1}) = \cdots = \sigma_{n(p+1)}(M_{m(p+1),p+1}).$$

Next we choose $\mathfrak{J}_{n(p+1)+1}, \mathfrak{J}_{n(p+1)+2}, \ldots, \mathfrak{J}_{n(p+1)+h(p+1)}$ (where

$h(p+1) = \left(\binom{m(p+1)+l(p+1)}{k(p+1)}\right)$ such that then the $\mathfrak{I}_{n(p+1)+r}$ consist of every possible union of $k(p+1)$ distinct sets from the $\mathcal{L}_{1p+1}, \mathcal{L}_{2p+1}, \ldots, \mathcal{L}_{l(p+1)p+1}, \mathcal{M}_{1p+1}, \mathcal{M}_{2p+1}, \ldots, \mathcal{M}_{m(p+1),\,p+1}$. Setting

$$\mathfrak{K}_{n(p+1)+r-1} = \{\mathrm{K} \in \mathcal{E}_{n(p+1)+r} : \mathrm{K} \subseteq \mathrm{I} \in \mathfrak{I}_{n(p+1)+r}\}$$

$[1 \leqslant r \leqslant h(p+1)]$ we can ensure, as in *Theorem* 7, that for some $\mathrm{Q}(p+1,\ r)$ we have that $x \in \mathrm{J} \in \mathcal{E}_{n(p+1)r}$ where $\mathrm{J} \subseteq \mathrm{K} \in \mathfrak{K}_{n(p+1)+r}$ implies $|\chi_{\mathrm{Q}(p+1,\,r)}(x) - f_{p+1}(x)| \leqslant 2^{-(p+1)}$. In addition setting $\mathrm{F}_{n(p+1)+r} = \cup\,\{\mathrm{I} : \ \mathrm{I} \in \mathfrak{I}_{n(p+1)+r}\}$ we have

$$\begin{aligned}\mathrm{A}_{n(p+1)+r} = \sigma(\mathrm{F}_{n(p+1)+r}) & \\ &\leqslant k(p) \max\,(\sigma(\mathrm{L}_{1p+1}),\, \sigma(\mathrm{M}_{1p+1})) \\ &= k(p) \max\,(t/(l(p)(t+s)),\, s/(m(p)(t+s))) \\ &= \mathrm{B}_p\end{aligned}$$

where $\mathrm{B}_p = \max\,(tk(p)/l(p),\, sk(p)/m(p)/(t+s) \to ts/(t+s)$ as $p \to \infty$.

We repeat this process for $p = 2, 3, 4, \ldots$. By the usual method we can ensure P independent and still obtain $\mathrm{A} = \limsup_{r\to\infty} \mathrm{A}_r = \limsup_{p\to\infty} \mathrm{B}_p = st/(t+s)$. Thus P is an independent at most $\mathrm{H}_{st/(t+s)}$ set. Similar arguments show that we can, in addition, have $\mathrm{L} = \mathrm{L}_0 \cap \mathrm{P}$ an at most H_e set where

$$\begin{aligned} e &= (1/\mathrm{A}) \limsup_{p\to\infty} \Big(\sup_{h(p)\geqslant r\geqslant 1} \sigma(\mathrm{F}_{n(p)+r} \cap \mathrm{L}) \Big) \\ &= (s+t)/t \limsup_{p\to\infty} (tk(p)/(l(p)(s+t))) \\ &= ((s+t)/t)(ts/(s+t)) = s \end{aligned}$$

i.e. L is at most an H_s set. Similarly M is at most an H_t set. By construction P is the union of $l(p) + m(p)$ disjoint sets $\mathrm{L}_a^p = \mathrm{L}_{ap} \cap \mathrm{P}, \mathrm{M}_b^p = \mathrm{M}_{bp} \cap \mathrm{P}$ $[1 \leqslant a \leqslant l(p), 1 \leqslant b \leqslant m(p)]$ such that the union F say of any $k(p)$ of them satisfies $\sup_{x\in\mathrm{F}} |\chi_{\mathrm{Q}}(x) - f_p(x)| \leqslant 2^{-p}$ for some Q. Further the L_c^{p+1}, M_d^{p+1} form subsets of the L_a^p, M_b^p each L_a^p having the same number of subsets and each M_b^p having the same number of subsets. Since $k(p)/l(p) \to s$, $m(p)/l(p) \to t$ (and so $(k(p) + m(p)/l(p) \to st/(t+s))$ given any $\varepsilon > 0$ we can find a $p(\varepsilon)$ and $k(\varepsilon) \leqslant k(p(\varepsilon))$ such that every union of $k(\varepsilon)$

sets taken from the $L_a^{p(\varepsilon)}$, $M_b^{p(\varepsilon)}$ is Kronecker and

$$k(\varepsilon)/(l(p)+m(p)) \geqslant st/(t+s)-\varepsilon.$$

By the result of Varopoulos P is at least an $H_{st/(s+t)-\varepsilon}$ set. Since ε is arbitrary, P is at least an $H_{st/(s+t)}$ set. Similarly L is an H_s set, M an H_t set. This concludes the proof.

Incidentally we have now constructed independent H_t sets for all $1 \geqslant t \geqslant 0$. For various reasons, most of which become clear on reading the proof, there seems to be a natural barrier at $st/(t+s)$ when using our method. It would, therefore, be extremely interesting to have an improvement on this, i.e. to construct an H_s and an H_t set whose independent union was not at least $H_{st/(s+t)}$. (A proof that this was impossible would, of course, show that the independent union of 2 Helson sets was Helson.)

8. Dirichlet and Helson Sets.

The object of this final section is to prove

THEOREM 9. — *There exists a perfect independent Dirichlet set which is not Helson (and so in particular not weak Kronecker).*

The reader is advised to refresh his memory of *Lemmas* 5.1 and 5.2 together with the alternative proof of *Theorem* 4 given in Section 6.

In seeking a proof we naturally first try to obtain analogues of the results used in the proof of *Theorem* 4. For example a suitable analogue of *Lemma* 5.1 turns out to be

LEMMA 8.1. — *Suppose* N, K, P, p *are positive integers with* $P \geqslant 4N$ *and* $\dfrac{NK}{P} \geqslant 12\,800\,p^3$. *Then setting* $b_u = \dfrac{2u\pi}{P}$ *we have for* $P-N \geqslant r \geqslant N$ *that* $\left|\sum_{u=-K}^{K} \chi_r(b_u)\right| \leqslant K/(4p)$.

Proof. — Take a particular $P-K \geqslant r \geqslant K$. By *Lemma* 5.1 there exists a $\left|\dfrac{K}{6\,400\,p^3}\right. \geqslant q > 0$ such that

$$|\chi_r(b_q)-1| \geqslant 1/2.$$

Now consider $b_q, b_{2q}, \ldots, b_{800\,pq}$. These 800 p points lie on the circle $\Pi = \{\lambda : |\lambda| = 1\} \subseteq \mathbf{C}$. Thus at least one pair of them must be a distance apart of no more than $\pi/(400\,p)$

(and so less than $1/(100\,p)$ apart, possibly even coincident). We can therefore find $1 \leqslant s < t \leqslant 800\,p$ such that

$$|\chi_r(b_{(t-s)q}) - 1| = |\chi_r(b_{tq}) - \chi_r(b_{sq})| \leqslant 1/(100\,p),$$

and thus $1 \leqslant w \leqslant 800\,p$ with $|\chi_r(a_{qw}) - 1| \leqslant 1/(100\,p)$.

Taking any integers h, l we have

$$\begin{aligned}\left|\sum_{u=h}^{h+w-1} \chi_r(b_{qu+l})\right| &= \left|\sum_{u=0}^{w-1} \chi_r(b_{qu})\right| \\ &= \left|\frac{1-\chi_r(b_{qw})}{1-\chi_r(b_q)}\right| \\ &\leqslant \frac{1}{50\,p}.\end{aligned}$$

Thus setting $k = \left[\dfrac{2\mathrm{K}}{qw}\right] - 1$ we have

$$\begin{aligned}\left|\sum_{u=-\mathrm{K}}^{\mathrm{K}} \chi_r(a_u)\right| &\leqslant \sum_{h=0}^{k-1}\left|\sum_{l=0}^{qw-1} \chi_r(a_{hqw+l-\mathrm{K}})\right| + \sum_{u=\mathrm{K}-2qw}^{\mathrm{K}} |\chi_r(a_u)| \\ &\leqslant \frac{\mathrm{K}}{25p} + 2qw \\ &\leqslant \frac{\mathrm{K}}{25p} + 800\,p.\frac{\mathrm{K}}{6\,400\,p^3} \\ &\leqslant \frac{\mathrm{K}}{4p}.\end{aligned}$$

We also wish to know how much we can tamper with what we have already constructed.

LEMMA 8.2. — *Suppose* $\delta > 0$ *and* $h_1, h_2, \ldots, h_{\mathrm{N}} \in \mathrm{C}(\mathbf{T})$ *given. Then there exists an* $\varepsilon_0 = \varepsilon_0(\delta, h_1, h_2, \ldots, h_{\mathrm{N}}) > 0$ *with the following property. Suppose* $a_1, a_2, \ldots, a_n$ *distinct points of* $\mathbf{T}$ *and* σ *a measure on* $a_1, a_2, \ldots, a_n$ *with* $\|\sigma\| = 1$. *Then if* σ_i' *is a measure supported by* $\mathrm{N}(a_i, \delta)$ *with*

$$\sigma_i'(\mathrm{N}(a_i, \delta)) = \sigma(a_i)\ [1 \leqslant i \leqslant n]$$

we have, setting $\sigma' = \sum_{i=1}^{n} \sigma_i'$, *that* $\left|\int h_j\, d\sigma' - \int h_j\, d\sigma\right| \leqslant \delta$ *for all* $1 \leqslant j \leqslant \mathrm{N}$.

Proof. — Since $h_1, h_2, \ldots, h_{\mathrm{N}}$ are uniformly continuous (or directly from the Heine Borel theorem), we can find an $\varepsilon > 0$ such that $|h_j(x) - h_j(y)| \leqslant \delta/2$ for all $|x - y| \leqslant \varepsilon, 1 \leqslant j \leqslant \mathrm{N}$. Setting $\varepsilon_0 = \varepsilon/2$ we have the result.

Unfortunately the conditions of the 2 lemmas cannot be satisfied simultaneously. On the one hand, in order to use *Lemma* 8.1 to control the behaviour of our construction with respect to χ_r, we must make alterations over a length large with respect to the wave length of χ_r. On the other hand, to preserve this good behaviour by the means suggested in *Lemma* 8.2, later alterations must only involve lengths small with respect to the wavelength of χ_r. Nor can we simply ignore this gap as, to a limited extent, we could in the inductive construction of Section 7. For examining *Lemma* 5.2 we see that if we allow $|\hat{\sigma}(n)|$ to be large for any n, then

$$\limsup_{|r|\to\infty} |\hat{\sigma}(r)| = \sup_{r\in\mathbf{Z}} |\hat{\sigma}(r)|$$

will be large.

Let us examine more closely how we propose to use *Lemma* 8.1. Suppose we have a finite set $A = C \cup D$ of points (where C and D are disjoint) and a measure σ on A with $\|\sigma\| = 1$. For each $a \in D$ we form $k(a, u) = \dfrac{2u\pi}{P} + a$ for $|u| \leqslant K$ with K and P as in the statement of *Lemma* 8.1. Setting

$$D' = \{k(a, u) : a \in D, |u| \leqslant K\}$$

$$\begin{aligned}
\sigma'(k(a, u)) &= \frac{1}{2K+1}\,\sigma(a) && \text{for} \quad a \in D, |u| \leqslant K \\
\sigma'(a) &= \sigma(a) && \text{for} \quad a \in C \\
\sigma'(E) &= 0 && \text{for} \quad E \cap (C \cup D') = \varnothing
\end{aligned}$$

we have $\|\sigma'\| = 1$ and $\operatorname{supp} \sigma' \subseteq C \cup D'$.

Clearly if $P - N \geqslant r \geqslant N$

$$\begin{aligned}
\left|\int_{D'} \chi_r\, d\sigma'\right| + |\sigma'|(C) &\leqslant \sum_{a\in D} \left|\int_{\{k(a,u):\,|u|\leqslant K\}} \chi_r d\sigma'\right| + |\sigma|(C) \\
&\leqslant \sum_{a\in D} \left|\sum_{u=-K}^{K} \chi_r(b_u)\right| \frac{\sigma(a)}{2K+1} + |\sigma|(C) \\
&\leqslant \frac{1}{8p} + |\sigma|(C).
\end{aligned}$$

Thus, if originally $\left|\int_D \chi_r\, d\sigma\right| \leqslant \dfrac{1}{8p} + |\sigma|(C)$, we can tamper with C as much as we like without making things worse (at least in the interval $P - N \geqslant r \geqslant N$).

There remains the question of what happens when

$0 \leqslant r \leqslant N$. We note first that if $C = \emptyset$ then although $\frac{1}{2K+1} \sum_{u=-K}^{K} \chi_r(b_u) = \gamma$ say need not be small, we do at least have $|\gamma| \leqslant 1$ and therefore

$$\left|\int_{D'} \chi_r \, d\sigma'\right| = |\gamma| \left|\int_D \chi_r \, d\sigma\right| \leqslant \left|\int_D \chi_r \, d\sigma\right|.$$

Of course, we wish to work with $C \neq \emptyset$ and $\sigma(C) > 0$. It can then be true that $|\gamma'|, |\gamma''| \leqslant 1$ yet not true that

$$\left|\gamma'' \int_C \chi_r \, d\sigma + \gamma' \int_D \chi_r\right| \leqslant \left|\int_A \chi_r \, d\sigma\right|.$$

However, if γ', γ'' remain close to γ then

$\left|\left|\gamma'' \int_C \chi_r \, d\sigma + \gamma' \int_D \chi_r \, d\sigma\right| - \left|\gamma \int_A \chi_r \, d\sigma\right|\right|$ remains small. This observation gives us the final piece of the jigsaw. Let us write

$$\gamma(r, \varepsilon) = \frac{\pi}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \chi_r \, d\mu \qquad \text{where } \mu \text{ is Haar measure}$$

$$\gamma(r, \varepsilon, Q) = \frac{1}{2L+1} \sum_{u=-L}^{L} \chi_r\left(\frac{2u\pi}{Q}\right) \quad \text{where} \quad L = \left[\frac{Q}{\varepsilon}\right].$$

We then have

LEMMA 8.3. — *For fixed* r, $\gamma(r, \varepsilon, Q) \to \gamma(r, \varepsilon)$ *as* $Q \to \infty$. *In particular given* $\delta > 0$ *and* N *a positive integer, we can find a* $Q(\delta, N)$ *such that*

$$|\gamma(r, \varepsilon) - \gamma(r, \varepsilon, Q)| \leqslant \delta \quad \textit{for all} \quad 0 \leqslant r \leqslant N.$$

We are now in a position to give the central inductive step. One of the main points to notice is how we avoid making circular definitions.

Central inductive step. — In what follows $h(n)$ and $m(n)$ will be positive integers which we shall choose so that

(1.1) $\quad m(n) \leqslant m(n+1) \leqslant m(n) + 1$

(1.2) $\quad h(n) \leqslant h(n+1) \leqslant h(n) + 1, \quad h(n) \leqslant n - 1$

and $\delta(r)$, $\eta(n)$ real numbers such that

(1.3) $\quad 0 < \eta(n+1) \leqslant \eta(n) \leqslant 1,$

(1.4) $\quad \eta(n) = \min_{0 \leqslant r \leqslant h(n)} \delta(r)$

(again we will choose $\delta(r)$, $\eta(n)$ as the induction proceeds).

Suppose that at the n^{th} stage of our construction we have:

(2.1) $A(n)$ a finite set of points

(2.2) $B(n, 1), B(n, 2), \ldots, B(n, k(n))$ disjoint sets with $A(n) = B(n, 1) \cup B(n, 2) \cup \cdots \cup B(n, k(n))$.

(2.3) $\sigma(n, 0), \sigma(n, 1), \ldots, \sigma(n, h(n))$ real measures supported by $A(n)$ with $\|\sigma(n, j)\| = 1$ $[0 \leqslant j \leqslant h(n)]$.

(2.4) $P(n) \geqslant 2^n$, $k(n) \geqslant k(n-1) > 0$ integers

(2.5) $$1/(4\, m(n) + 2),\ 2^{-n}/P(n) \geqslant \varepsilon(n) > 0$$

with the following properties

(2.6) If $a \in A$ then $\chi_{P(n)}(a) = 1$

(2.7) $$|\sigma(n, j)|B(n, l) \leqslant \frac{1}{m(n)}\delta(j)$$
$$[0 \leqslant j \leqslant h(n),\ 1 \leqslant l \leqslant k(n)]$$

(2.8) $$\operatorname{diam} B(n, l) = \max\{|x - y| : x, y \in B(n, l)\}$$
$$\leqslant \frac{4}{m(n)} - \varepsilon(n) \quad [1 \leqslant l \leqslant k(n)]$$

(2.9) $\operatorname{card} B(n, l) \geqslant 2^{n+4}$ $[1 \leqslant l \leqslant k(n)]$

(2.10) $\left|\int_{A(n)} \chi_r \, d\sigma(n, j)\right| \leqslant \left(1 + \frac{1}{2^j} - \frac{1}{2^n}\right)\delta(j)$ for all r $[0 \leqslant j \leqslant h(n)]$. (The « important » conditions are, of course, (2.6) and (2.10)).

From here the inductive step may proceed in 2 ways which we shall call case 1 and case 2. We set $A(n) = C(n) \cup D(n)$ where $C(n) \cap D(n) = \varnothing$ and $C(n) \cap B(n, l)$ has the value $\varnothing$ or $B(n, l)$ (i.e. each of $C(n)$, $D(n)$ is the union of blocks $B(n, l)$) $[1 \leqslant l \leqslant k(n)]$.

In case 2 we take $C(n) = \varnothing$.

In case 1 we let $C(n)$ be the union of selected $B(n, l)$ subject to the overriding condition

$$|\sigma(n, j)|C(n) \leqslant \delta(j)\ [0 \leqslant j \leqslant h(n)].$$

In particular we can and shall ensure that in case 1 $C(n)$ is the union of at least $m(n)$ selected $B(n, l)$ (cf. the methods for achieving blockwise independence in Section 6).

The reader is advised to keep the more complex case 1 in mind while studying the induction and afterwards consider case 2 as a simpler version.

We now define successively $\varepsilon'(n+1)$, $N(n+1)$, $Q(n+1)$, $L(n+1)$, $C''(n+1)$, $\tau(j, n+1)$, $\varepsilon'''(n+1)$, $C'(n+1)$, $\varepsilon''(n+1)$, $P(n+1)$, $K(n+1)$, $\varepsilon(n+1)$, $A(n+1)$ and $\sigma(n+1, j)$.

Set

$$(3.1) \quad \varepsilon'(n+1) = 1/8 \min (\varepsilon(n), 1/(4m(n)+8), \inf\{|x-y| : x, y \in A(n), x \neq y\}).$$

Choose an $N(n+1) > P(n)$ such that

$$(3.2) \quad N(n+1)\varepsilon'(n+1) \geqslant 12\,800\left(2^{n+5}\left(\left[\frac{1}{\eta(n)}\right]+1\right)\right)^3.$$

By *Lemma* 8.3 we can find a $Q(n+1)$ such that

$$(3.3) \quad |\gamma(r, \varepsilon'(n+1)) - \gamma(r, \varepsilon'(n+1), Q)| \leqslant 2^{-(n+5)}\eta(n)$$

whenever $Q \geqslant Q(n+1)$ and $0 \leqslant r \leqslant N(n)$ whilst $L(n+1) = \left[\dfrac{Q(n+1)}{\varepsilon'(n+1)}\right] \geqslant 2^{n+6}$.

Let $C'(n) = \{l(a, u) : a \in C(n)\ |u| \leqslant L(n+1)\}$ where $l(a, u) = a + \dfrac{2u\pi}{Q(n+1)}$ (we should more properly talk of $l(a, u, n+1)$). Set

$$\tau(n, j+1)(l(a, u)) = \frac{1}{2L(n+1)+1}\sigma(n, j)(a) \quad [a \in C(n), |u| \leqslant L(n+1)]$$

$$\tau(n, j+1)E = 0 \quad \text{if} \quad E \cap C'(n) = \varnothing$$

Then

$$\begin{aligned}&\left|\int_{C'()n}\chi_r\, d\tau(n, j+1) - \gamma(r, \varepsilon'(n+1))\int_{C(n)}\chi_r\, d\sigma(n, j)\right| \\ &\quad = |\gamma(r, \varepsilon'(n+1), Q(n+1)) - \gamma(r, \varepsilon'(n+1))|\left|\int_{C(n)}\chi_r\, d\sigma(n, j)\right| \\ &\quad \leqslant 2^{-(n+5)}\eta(n).\end{aligned}$$

(If $C(n) = \varnothing$ then we adopt the usual conventions concerning $\varnothing$.) By *Lemma* 8.2 we can find an $0 < \varepsilon'''(n+1) < 1/2\,\varepsilon'(n+1)$ such that

$$|l''(a, u) - l(a, u)| < \varepsilon'''(n+1) \quad [a \in C(n), |u| \leqslant L(n+1)]$$

implies (in obvious notation)

$$\left|\int \chi_r(l''(a, u))\, d\tau(n, j+1) - \int \chi_r(l(a, u))\, d\tau\,(n, j+1)\right| \leqslant 2^{-(n+5)}\eta(n)$$

$[0 \leqslant j \leqslant h(n)]$. By *Lemma* 6.8 (*ii*) we can find $l'(a, u) \in 2\pi\mathbf{Q}$ such that $|l'(a, u) - l(a, u)| < 1/2\ \varepsilon'''(n+1)$, and writing $C'(n) = \{l'(a, u) : a \in C(n),\ |u| \leqslant L(n+1)\}$ we have $C'(n)$ $m(n)$-independent. As a consequence of these conditions we have

(3.4) $C'(n) \subseteq 2\pi\mathbf{Q}$

(3.5) $$\left|\int_{C'(n)} \chi_r d\tau(n, j+1) - \gamma(r, \varepsilon'(n+1)) \int_{C(n)} \chi_r\, d\sigma(n, j)\right| \leqslant 2^{-(n+4)}\eta(n) \quad [0 \leqslant j \leqslant h(n)]$$

(3.6) $$|a - l'(a, u)| \leqslant 2\varepsilon'(n+1)$$

for all

$$a \in C(n), |u| \leqslant L(n+1).$$

Finally, by *Lemma* 6.2 we can find a

$$0 < \varepsilon''(n+1) < 1/4\ \varepsilon'''(n+1)$$

such that

(3.7) If

$$|b_r - c'_r| \leqslant \varepsilon''(n+1) \quad [1 \leqslant r \leqslant m(n)]$$

where $c'_1, c'_2, \ldots, c'_{m(n)} \in C'(n)$ are distinct, we have $b_1, b_2, \ldots, b_{m(n)}$ $m(n)$-independent.

We note (though it is not necessary in the proof of *Theorem* 9) that we could choose $C'(n)$ to satisfy (ignoring a certain amount of notational confusion) the conditions of *Lemma* 6.9 (*ii*) for $C'(n) = \{x_1, x_2, \ldots, x_h\}$ with $q \geqslant m(n)$.

Our next task is to choose $P(n+1)$, $K(n+1)$. This we do so that

(4.1) $$P(n+1) \geqslant Q(n+1),\ 2^{n+1}$$

(4.2) $P(n+1)c'$ is integral for all $c' \in C'(n)$

(4.3) $$K(n+1) = \frac{P(n+1)}{[\varepsilon'(n+1)]}$$

(4.4) $$K(n+1) \geqslant 2^{k(n)+4}.$$

(*Note.* Speaking in a mildly abusive manner, we see that (4.1) and (4.4) are both satisfied for all $P(n+1)$ large enough, whilst (4.2) is satisfied by all $P(n+1)$ with a certain divisor. (If $C(n) = \varnothing$ then this divisor can, of course, be taken to be 1). We can thus indeed satisfy (4.1), (4.2), (4.4) simultaneously. We further remark (though we shall not need this in the proof of the main theorem), that if $P(n+1)$ satisfies (4.1), (4.2) and (4.4), then so does $sP(n+1)$ $[s \geqslant 1, s \in \mathbf{Z}]$. As a particular instance, suppose we undertake t such constructions simultaneously $[t \geqslant 1, t \in \mathbf{Z}]$ forming $A^i(n)$, $\varepsilon^i(n)$, $P^i(n)$ and so on $[1 \leqslant i \leqslant t]$. With an obvious notation, suppose that $P^i(n+1)$ satisfies $(4.1)^{(i)}$, $(4.2)^{(i)}$ and $(4.4)^{(i)}$. Then so does $P(n+1) = \prod_{j=1}^{t} P^j(n+1)$. We may, therefore, always arrange our construction so as to have

$$P^1(n) = P^2(n) = \cdots = P^t(n)$$

for all sufficiently large n.)

We take

$$\varepsilon(n+1) = \frac{1}{4} \min\left(\frac{2^{-(n+5)}}{P(n+1)}, \varepsilon''(n+1)\right). \tag{4.5}$$

In the next part of the construction we define

$$A(n+1) \quad \text{and} \quad \sigma(n+1, j).$$

For each $a \in D(n)$ form points $b(a, u)$ (or, more accurately, $b(a, u, n+1)$), with $b(a, u) = \dfrac{2u\pi}{P(n+1)}$ $[|u| \leqslant K(n+1)]$. Let

$$D'(n) = \{b(a, u) : a \in D(n), \quad |u| \leqslant K(n+1)\}$$

and $\sigma(n+1, j)$ be the measure on $A(n+1) = C'(n) \cup D'(n)$ defined for $0 \leqslant j \leqslant h(n)$ by

$$\sigma(n+1, j)(l'(a, u)) = \frac{\sigma(n, j)(a)}{2L(n+1)+1}$$

for
$$a \in C(n), \quad |u| \leqslant L(n+1)$$

$$\sigma(n+1, j)(k(a, u)) = \frac{\sigma(n, j)(a)}{2K(n+1)+1}$$

for
$$a \in D(n), \quad |u| \leqslant K(n+1).$$

For notational purposes set

$$\mathrm{B}'(n, l) = \{l'(a, u) : a \in \mathrm{B}(n, l), |u| \leqslant \mathrm{L}(n+1)\}$$

if

$$\mathrm{B}(n, l) \subseteq \mathrm{C}(n)$$

$$\mathrm{B}'(n, l) = \{k(a, u) : a \in \mathrm{B}(n, l), |u| \leqslant \mathrm{K}(n+1)\}$$

if

$$\mathrm{B}(n, l) \subseteq \mathrm{D}(n)$$

$$\mathrm{F}(n, a) = \{l'(a, u) : |u| \leqslant \mathrm{L}(n+1)\} \quad \text{if} \quad a \in \mathrm{C}(n)$$

$$\mathrm{F}(n, a) = \{k(a, u) : |u| \leqslant \mathrm{K}(n+1)\} \quad \text{if} \quad a \in \mathrm{D}(n).$$

We now examine some of the consequences of our definitions. Recall that $\sigma(n, j)$ is real and so $\sigma(n+1, j)$ is. Thus in particular

$$\left|\int \chi_r \, d\sigma(n+1, j)\right| = \left|\int \chi_{-r} \, d\sigma(n+1, j)\right|\cdot$$

This, of course, is simply a technical convenience for, as the reader will easily see, the work of this section goes through with $\sigma(n, j)$ complex. But the choice of $\sigma(n, j)$ real does shorten the next piece of working.

(5.1) If

$$0 \leqslant r \leqslant \mathrm{N}(n+1) \quad \text{(and so if} \quad -\mathrm{N}(n+1) \leqslant r \leqslant 0)$$

we have

$$\begin{aligned}
&\left|\int \chi_r \, d\sigma(n+1, j)\right| \\
&= \left|\int_{\mathrm{C}'(n)} \chi_r \, d\sigma(n+1, j) + \int_{\mathrm{D}(n)'} \chi_r \, d\sigma(n+1, j)\right| \\
&= \left|\int_{\mathrm{C}'(n)} \chi_r \, d\tau(n, j+1)\right. \\
&\quad \left. + \gamma(r, \varepsilon'(n+1), \mathrm{P}(n+1)) \int_{\mathrm{D}(n)} \chi_r \, d\sigma(n, j)\right| \\
&\leqslant \left|\gamma(r, \varepsilon'(n+1)) \left\{\int_{\mathrm{C}(n)} \chi_r \, d\sigma(n, j) + \int_{\mathrm{D}(n)} \chi_r \, d\sigma(n, j)\right\}\right| \\
&\quad + \left|\int_{\mathrm{C}'(n)} \chi_r \, d\tau(n, j+1) - \gamma(r, \varepsilon'(n+1)) \int_{\mathrm{C}(n)} \chi_r \, d\sigma(n, j)\right| \\
&\quad + \left|(\gamma(r, \varepsilon'(n+1)) - \gamma(r, \varepsilon'(n+1), \mathrm{P}(n+1))) \int_{\mathrm{D}(n)} \chi_r \, d\sigma(n, j)\right| \\
&\leqslant \left|\int_{\mathrm{A}(n)} \chi_r \, d\sigma(n, j)\right| + 2^{-(n+2)}\eta(n) \\
&\leqslant \left(1 + \frac{1}{2^j} - \frac{1}{2^{n+1}}\right) \delta(j) \quad [0 \leqslant j \leqslant h(n)]
\end{aligned}$$

using (3.5), (4.1) and (3.3), the fact that $|\gamma(r, \varepsilon'(n+1))| \leqslant 1$ and our original inductive hypothesis (2.10), together with condition (1.4).

We also see that

(5.2) If $N(n+1) \leqslant r \leqslant P(n+1) - N(n+1)$, then

$$\begin{aligned}
&\left|\int \chi_r \, d\sigma(n+1, j)\right| \\
&\quad \leqslant |\sigma(n+1, j)|(C'(n)) + \left|\int_{D'(n)} \chi_r \, d\sigma(n+1, j)\right| \\
&\quad = |\sigma(n, j)|(C(n)) + |\gamma(r, \varepsilon'(n+1), P(n+1))||\sigma(n, j)|(D(n)) \\
&\quad \leqslant \delta(j) + 2^{-(n+5)}\eta(n) \\
&\quad \leqslant \left(1 + \frac{1}{2^j} - \frac{1}{2^{n+1}}\right)\delta(j) \quad [0 \leqslant j \leqslant h(n)]
\end{aligned}$$

using the definition of $C(n)$, (3.2) and *Lemma* 8.1, and conditions (1.4) and (1.2).

Moreover by construction

(5.3) $\qquad \chi_{P(n+1)}(a) = 1 \qquad$ for all $\qquad a \in A(n+1)$

and so, combining (5.1) and (5.2) we have

(5.4) $\qquad \left|\int \chi_r \, d\sigma(n+1, j)\right| \leqslant \left(1 + \frac{1}{2^j} - \frac{1}{2^{n+1}}\right)\delta(j)$

for all r $\qquad [0 \leqslant j \leqslant h(n)]$.

Tidying up we note

(5.5) $\|\sigma(n+1, j)\| = 1 \quad [0 \leqslant j \leqslant h(n)]$

(5.6) $\text{diam } B'(n, l) \leqslant \text{diam } B(n, l) - \varepsilon(n+1) + \varepsilon(n) \quad [1 \leqslant l \leqslant k(n)]$

(since $\varepsilon'(n+1) \leqslant 1/8\, \varepsilon(n)$, $\varepsilon'(n+1) + \varepsilon'''(n+1) \leqslant 1/4\, \varepsilon(n)$ and $\varepsilon(n+1) \leqslant 1/4\, \varepsilon''(n+1)$).

(5.7) $\text{diam } F(n, a) \leqslant 1/8 \max\limits_{1 \leqslant l \leqslant k(n)} \text{diam } B(n, l) \quad [a \in A(n)]$

(since $\varepsilon'(n+1) \leqslant 1/8 \max\{|x - y| : x, y \in A(n)\ x \neq y\}$ and by (2.9) card $B(n, l) \geqslant 16$).

(5.8) $|\sigma(n+1, j)|\ F(n, a)$
$\leqslant 1/16 \max\limits_{1 \leqslant l \leqslant k(n)} |\sigma(n, j)|(B(n, l)) \quad [0 \leqslant j \leqslant h(n),\ a \in A(n)]$

(since card $B(n, l) \geqslant 16$).

(5.9) card $F(n, a) \geqslant 2^{n+5} \quad [a \in A(n)]$.

In case 1 we set $k(n+1) = k(n)$, $B(n+1, l) = B'(n, l)$ $[1 \leqslant l \leqslant k(n)]$ and $m(n+1) = m(n)$. In case 2 we set

$k(n+1) = \text{card } \mathrm{A}(n)$ and, writing

$$\mathrm{A}(n) = \{a(n, l) : k(n+1) \geqslant l \geqslant 1\},$$

we put

$$\mathrm{B}(n+1, l) = \mathrm{F}(n, a(n, l)).$$

We allow (but do not insist that) $m(n+1) = m(n) + 1$.
In both cases we have

$$(5.10) \quad |\sigma(n+1, j)|\,(\mathrm{B}(n+1, l)) \leqslant \frac{\delta(j)}{m(n+1)} \quad [0 \leqslant j \leqslant h(n),\ 1 \leqslant l \leqslant k(n+1)]$$

$$(5.11) \quad \operatorname{diam} \mathrm{B}(n+1, l) \leqslant \frac{4}{m(n+1)} - \varepsilon(n+1) \quad [1 \leqslant l \leqslant k(n+1)]$$

(recalling (3.1)).

$$(5.12) \quad \text{card } \mathrm{B}(n+1, l) \geqslant 2^{n+5}.$$

Now suppose we can ensure that whenever

$$h(n+1) = h(n) + 1$$

we have a real measure $\sigma(n+1, h(n+1))$ on $\mathrm{A}(n+1)$ such that

$$(5.4') \quad \left|\int \chi_r \, d\sigma(n+1, h(n+1))\right| \leqslant \left(1 + \frac{1}{2^j} - \frac{1}{2^{n+1}}\right) \delta(h(n+1))$$

for all r,

$$(5.5') \quad \|\sigma(n+1, j)\| = 1.$$

$$(5.10') \quad |\sigma(n+1, j)|(\mathrm{B}(n+1, l)) \leqslant \frac{\delta(h(n+1))}{m(n+1)} \quad [1 \leqslant l \leqslant k(n+1)].$$

Then, comparing (2.3) with (5.10), (2.4) with (4.1), (2.5) with (3.1) and (4.5), (2.6) with (5.3), (2.7) with (5.10), (2.8) with (5.11), (2.9) with (5.12), and (2.10) with (5.4), we see that we can restart the induction. (We shall leave the non central question of how to define $h(n+1)$ and, when

$$h(n+1) = h(n) + 1,$$

$\sigma(n+1, h(n+1))$ till later.)

This completes the more difficult and important part of the construction. We now obtain some consequences.

Structure of the Limit Set. — In our construction there is a natural definition of a descendant point. Formally we say that if $a \in A(n)$ then the descendant points of a are the members of $F(n, a)$. We call a descendant of a descendant a descendant and so on inductively. If $b \in A(n)$ then by construction, if $c \in A(n+m)$ say is a descendant of b, we have, chiefly by (3.1),

$$\begin{aligned} |b - c| &\leqslant \sum_{r=1}^{m} \varepsilon'(n+r) + \varepsilon''(n+r) \\ &\leqslant 2 \sum_{r=1}^{\infty} \varepsilon'(n+r) \\ &\leqslant 2\varepsilon'(n+1) \sum_{r=0}^{\infty} (1/4)^r \\ &\leqslant 1/3 \min\ (\varepsilon(n),\ \max\{|x-y| : x,\ y \in A(n),\quad x \neq y\}). \end{aligned}$$

Thus, if we let E be the topological limit of $A(n)$, i.e. if we set $E = \{x : \exists x(n) \in A(n)$ with $x(n) \to x$ as $n \to \infty\}$ we easily see that E is perfect.

Moreover, if $x \in E$, then $|x - a| \leqslant 1/3\ \varepsilon(n)$ for some $a \in A(n)$ and so

$$|\chi_{P(n)}(x) - 1| \leqslant |\chi_{P(n)}(x) - \chi_{P(n)}(a)| \leqslant P(n)|x - a| \leqslant 2^{-n}.$$

Thus $\chi_{P(n)} \to 1$ uniforuly on E and by (2.4) $P(n) \to \infty$ as $n \to \infty$ giving E Dirichlet.

Now consider $\sigma(m, j)$ for some fixed m such that $j \leqslant h(m)$. We can find a continuous function f with $\|f\|_\infty = 1$ and $f(z) = \operatorname{sgn} \sigma(m, j)(a)$ for all $|z - a| \leqslant \varkappa(n)$ where

$$\varkappa(n) = \max\ \{|x - y| : x, y \in A(n), \quad x \neq y\}.$$

For $n \geqslant m$ we have, by the considerations above,

$$\int f\, d\sigma(n, j) = \sum_{a \in A(n)} f(a)(\sigma(n, j)(a)) = 1.$$

Thus $\sigma(n, j)$ has a weak star limit point $\sigma(j)$ with support in E (indeed, it isobvious that $\sigma(j)$ is the weak star limit) with norm 1. Since $\left|\int \chi_r\, d\sigma(n, j)\right| \leqslant \left(1 + \frac{1}{2^j} - \frac{1}{2^n}\right) \delta(j)$ for all r and all $n \geqslant m$ it follows that

$$\left|\int \chi_r \, d\sigma(j)\right| \leqslant \left(1 + \frac{1}{2^j}\right) \delta(j) \quad \text{for all } r.$$

Setting $\delta = \inf\limits_{j \geqslant 1} \left(\left(1 + \frac{1}{2^j}\right) \delta(j)\right)$ we see that E is an at most H_δ set.

Independence. — By repeating case 1 with every possible combination of $m(q)$ selected $B(q, l)$ at stages

$$q, q+1, \ldots, q+q'$$

say, we can ensure that if $x_1, x_2, \ldots, x_{m(q)}, y_1, y_2, \ldots, y_{m(q)}$ are such that $|x_i - y_i| \leqslant \varepsilon(q + q')$ and $y_1, y_2, \ldots, y_{m(q)}$ belong to disjoint $B(q, l)$, then $x_1, x_2, \ldots, x_{m(q)}$ are $m(q)$-independent. In this way we ensure that if $x_1, x_2, \ldots, x_{m(q)} \in E$ and $\inf\limits_{t \neq s} |x(t) - x(s)| \geqslant \frac{4}{m(q)}$ then $x_1, x_2, \ldots, x_{m(q)}$ are $m(q)$-independent. (Cf. (2.8) and our discussion of descendant points). At stage $q + q' + 1$ we repeat case 2 and put $m(q + q' + 1) = m(q + q') + 1$. Repeating this process infinitely often (but not necessarily succesively), we obtain in the usual manner (*Lemma* 6.3) E independent.

Introduction of New Measures. — The induction here has the rather pleasing property of being self-starting. (Strictly speaking we must define $A(1)$, $\sigma(1, 0)$, $\delta(0)$ etc., but there we need only take $\sigma(1, 0)$ as a « dummy » measure. For example, let $m(1) = 1$, $h(1) = 0, \delta(0) = \eta(0) = 1, k(0) = 2^5, B(1, 1) = A(1)$, $A(1)$ a collection of 2^5 points of $2\pi\mathbf{Q}$ lying in $[d, d+1]$, $P(1)$ such that $P(1)a$ is an integral multiple of 2π for all $a \in A(1)$ and $P(1) \geqslant 2$, $\varepsilon(1) = 1/4 \min (1/6, 1/P(n))$. The conditions (2.1)-(2.10) are then trivially satisfied for $n = 1$ for any $\sigma(1, 0)$ with support $A(1)$. But this is simply a technical trick and has nothing to do with the construction proper.)

Suppose we have at the beginning of the n^{th} stage $\sigma(n, 1)$, $\sigma(n, 2), \ldots, \sigma(n, h(n))$ and $\delta(h(n) + 1)$ defined. Repeat case 2 at stages $n, n+1, n+2, \ldots$ setting

$$m(n) = m(n+1) = m(n+2) = \cdots$$

Each of the $B(n + \nu, l)$ $[1 \leqslant l \leqslant k(n + \nu)]$ contains an arithmetic progression of at least 2^{n+v+4} terms. But it is well known ([9] Chapter XI, § 6) that

LEMMA 8.4. — *There exists a sequence* $Q(n) \to 0$ *such that if* $z_1, z_2, \ldots, z_n$ *distinct form an arithmetic progression, we can find a real measure* τ *with* $\|\tau\| = 1$ *and* $\sup |\hat{\tau}(m)| \leqslant Q(n)$.

It is now clear how we proceed. There exists a $v_0 \geqslant 1$ such that for all $v \geqslant v_0$ there exists a measure $\tau_{B(n+v,l)}$ on $B(n+v, l)$ with $\|\tau_{B(n+v,\,l)}\| = 1$ and $\sup |\hat{\tau}_{B(n+v,l)}(r)| \leqslant \eta(n)$. Now select points $y_1, y_2, \ldots, y_s \in A(n+v_0)$. (We may, for the sake of simplicity, take $\{y_1, y_2, \ldots, y_s\} = A(n+v_0)$, but this is not necessary). There exists a $v_1 \geqslant v_0$ such that writing $\mathcal{C}(t) = \{B(n+v_1+1, l) : B(n+v_1+1, l)$ consists of descendants of $y_t\}$ we have for $s \geqslant b \geqslant 1$,

$$\operatorname{card} \mathcal{C}(t) \geqslant 4\, m(n) \left(\left[\frac{1}{\delta(h(n)+1)}\right] + 1\right) + 8.$$

At the $n + v_1^{\text{th}}$ stage set $h(n+v_1+1) = h(n)+1$ (note that (1.2) remains satisfied) and let

$$\sigma(n+v_1+1, h(n)+1) = \frac{1}{s} \sum_{t=1}^{s} \frac{1}{\operatorname{card}\ \mathcal{C}(t)} \sum_{B(n+v_1+1,\,l) \in \mathcal{C}(t)} \tau_{B(n+v,+1,\,l)}.$$

A quick check shows that we have satisfied conditions (5.4)′, (5.5)′ and (5.10)′.

Provided that we increase $h(n)$ only under these circumstances this completes the full description of the induction promised in the remarks following the statement of (5.10)′. We repeat this process infinitely often. In this manner we obtain (since $\delta(r)$ decreases as $r \to \infty$)

$$\delta = \inf_{j \geqslant 1} \left(\left(1 + \frac{1}{2^j}\right) \delta(j)\right) = \lim_{n \to \infty} \delta(n).$$

Proof of Theorem 9. — Allowing $\delta(n) \to 0$ as $n \to \infty$ we obtain $\delta = 0$ and E a perfect independent Dirichlet non Helson set.

Note. — Together with *Lemma* 5.2 this gives an alternative proof of *Lemma* 7.8.

We conclude by adapting the methods above to prove extensions of *Theorems* 7 and 8.

LEMMA 8.5. — *There exist* $L_1, L_2, \ldots, L_q$ *disjoint Kronecker sets such that* $E = L_1 \cup L_2 \cup \cdots \cup L_q$ *is independent and Dirichlet, but at most (and so, by Varopoulos's result, exactly)* $H_{1/q}$.

Proof. — For later use we construct E in $[a, b]$ by taking $A(1) \subseteq [a+\varepsilon, b-\varepsilon]$ and $\varepsilon(1) = \varepsilon$. We ensure that $A(1)$ has at least q points so that we can put $A(1) = \bigcup_{t=1}^{q} L(1, t)$ where the $L(1, t)$ are disjoint and not empty. We can further ensure (by taking $\sigma(1, 0)$ as a dummy with $\delta(0) = 1$ if necessary) that $|\sigma(1, 0)|(L(1, t)) = 1/q$ $[1 \leqslant t \leqslant q]$.

Let $L(n, t)$ be composed of those points of $A(n)$ which are descendants of points in $L(1, t)$ $[1 \leqslant t \leqslant q]$. Then $A(n) = \bigcup_{t=1}^{q} L(n, t)$ and the $L(n, t)$ are disjoint and not empty. Let L_t be the (topological) limit of $L(n, t)$ as $n \to \infty$. Then by the arguments used when discussing descendants, we see that $L_1, L_2, \ldots, L_q$ are disjoint perfect sets with E as union. Moreover, by the remarks concerning the choice of $\{y_1, y_2, \ldots, y_s\}$ when we introduced new measures, it is clear that we can ensure $|\sigma(n, j)|(L(n, t)) = 1/q$ for all $h(n) \geqslant j \geqslant 0$. Taking $\delta(r) = 1/q$ $[r \geqslant 1]$, we obtain E as an at most $H_{1/q}$, independent Dirichlet set.

We still have to show how to construct $L_1, L_2, \ldots, L_q$ as Kronecker sets. This we do as follows. Since

$$|\sigma(n, j)|(L(n, t)) = 1/q$$

and $L(n, t)$ is the union of $B(n, l)$ (for large enough n), we can take $C(n) = L(n, t)$ and proceed as in case 1. We choose $C'(n)$ as suggested in the remark following (3.7). By *Lemma* 6.9 (*ii*) $\inf_{0 \leqslant r \leqslant P(n+1)} \sup_{a \in C'(n)} |\chi_r(a) - f(a)| \leqslant 2\pi/m(n)$ for all $f \in S$. Now if $x \in L_t$, then $|x - a| \leqslant \varepsilon(n+1)$ for some $a \in C'(n)$ where $\varepsilon(n+1)P(n+1) \leqslant 2^{-n}$. Thus

$$\inf_{0 \leqslant r \leqslant P(n+1)} \sup_{x \in L_t} |\chi_r(x) - f(x)| \leqslant 2\pi/m(n) + 2^{-n} + \sup_{|y-z| \leqslant 2^{-n}} |f(y) - f(z)|.$$

If we repeat this process infinitely often (though, of course, not in successive steps), we see that, since

$$2\pi/m(n),\ 2^{-n},\ \sup_{|y-z|\leqslant 2^{-n}} |f(y)-f(z)| \to 0 \quad \text{as} \quad n \to \infty$$

we obtain L_t Kronecker.

(We could, of course, obtain the same result more elegantly by using *Lemma* 6.9 (*i*)).

Note. — We remark that in *Theorem* 8 we found (speaking roughly) a σ with $\limsup_{|r|\to\infty} |\hat{\sigma}(r)| = 1/q$, here a sequence σ_j with $\lim_{j\to\infty} \limsup_{|r|\to\infty} |\hat{\sigma}_j(r)| = 1/q$.

Lemma 8.6. — *There exists a countable collection* $L_0, L_1, L_2, \ldots$ *of disjoint Kronecker sets such that* $E = \bigcup_{q=0}^{\infty} L_q$ *is a perfect independent Dirichlet non Helson set.*

Proof. — This is related to that of *Lemma* 8.5 in the same way that the proof of *Theorem* 8 is related to that of *Theorem* 7. (But note that — as in the alternative proof of *Theorem* 4 given in Section 6 — it is more convenient to take a sequence of rationals $\gamma(n)$ tending to some point γ than to start with γ fixed.) We ensure that E is Dirichlet by using the technique suggested in the last part of the note following (4.4).

It is possible, using these techniques, to prove an extension of *Lemma* 7.12.

Lemma 8.7. — *Given* $1 \geqslant s, t > 0$, *we can find* L *a perfect* H_s *set and* M *a perfect* H_t *set such that* L, M *are disjoint but* $P = L \cup M$ *is an independent Dirichlet* $H_{st/(t+s)}$ *set.*

However, since the proof simply involves combining the rather complicated proofs of *Lemma* 7.12 and *Lemma* 8.5 and needs no new ideas, we leave it as an exercise for the reader.

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APENDIX

We give in diagramatic form the main relations (known to me) between the sets discussed above. [I] refers to this paper, [II] to a long paper « Some Results on Kronecker, Dirichlet and Helson sets II » which will form part of my Cambridge thesis. Additionally the result (5) of Björk will apear in seminar notes of the Mit ag Leffler Institute. Drury and Varopoulos have now proved that the union of 2 Helson sets is Helson (see [17]).

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