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On the Hartogs extension theorem for unbounded domains in $\mathbb{C}^n$

ON THE HARTOGS EXTENSION THEOREM FOR UNBOUNDED DOMAINS IN $\mathbb{C}^n$

by Al BOGGESS, Roman DWILEWICZ[†] & Egmont PORTEN

To the memory of Roman Dwilewicz

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ABSTRACT. — Let $\Omega \subset \mathbb{C}^n$, $n \geqslant 2$, be a domain with smooth connected boundary. If Ω is relatively compact, the Hartogs–Bochner theorem ensures that every CR distribution on $\partial\Omega$ has a holomorphic extension to Ω. For unbounded domains this extension property may fail, for example if Ω contains a complex hypersurface. The main result in this paper tells that the extension property holds if and only if the envelope of holomorphy of $\mathbb{C}^n \setminus \overline{\Omega}$ is $\mathbb{C}^n$. It seems that it is the first result in the literature which gives a geometric characterization of unbounded domains in $\mathbb{C}^n$ for which the Hartogs phenomenon holds.

Comparing this to earlier work by the first two authors and Z. Słodkowski, one observes that the extension problem changes in character if one restricts to CR functions of higher regularity.

RÉSUMÉ. — Soit $\Omega \subset \mathbb{C}^n$, $n \geqslant 2$, un domaine à bord lisse et connexe. Si Ω est relativement compact, le théorème de Hartogs et Bochner assure que toute distribution CR définie sur $\partial\Omega$ admet une extension holomorphe à Ω. Cela ne se généralise pas forcément aux domaines non-bornés, par exemple si Ω contient une hypersurface complexe. Le résultat principal de l'article dit que la propriété d'extension holomorphe a lieu si et seulement si l'enveloppe d'holomorphie de $\mathbb{C}^n \setminus \overline{\Omega}$ est $\mathbb{C}^n$. Il apparaît que cela est le premier résultat connu à donner une caractérisation géométrique des domaines non-bornés pour lesquels le phénomène de Hartogs est valide.

En se référant à des résultats antérieurs des deux premiers auteurs et Z. Słodkowski, on constate que le problème d'extension change de nature si on se restreint à des fonctions CR plus régulières.

1. Introduction

Throughout this article we consider a domain $\Omega = \Omega^- \subset \mathbb{C}^n$, $n \geqslant 2$, with $\mathcal{C}^\infty$-smooth connected boundary M. If Ω is relatively compact in $\mathbb{C}^n$,

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the classical Hartogs–Bochner theorem tells that every CR function on M admits holomorphic extension to Ω. Via a convenient notion of weak boundary values, this result naturally generalizes to CR distributions.

The classical Hartogs extension theorem made an important influence not only on Complex Analysis, but also on other areas of mathematics, like Algebraic Geometry or Partial Differential Equations. The theorem still inspires researchers and there is a renewed interest in recent years: Harz–Shcherbina–Tomassini [15, 16], Øvrelid–Vassiliadou [28], Damiano–Struppa–A.Vajiac–M.Vajiac [9], Palamodov [29], Ohsawa [27], Coltoiu–Ruppenthal [8], Lewandowski [21], and papers by the authors with other colleagues [3, 4, 5, 6], [24, 25].

A good deal of the mentioned contributions consider extension from boundaries of unbounded domains. Easy examples show that the Hartogs–Bochner theorem may fail for unbounded domains, leading to the problem to understand the precise nature of the obstacles. The essence of the present article is a geometric characterization of the Hartogs extension property for CR distributions.

Let S be a smooth real hypersurface of $\mathbb{C}^n$ and $\omega \subset \mathbb{C}^n$ a domain such that $\omega \backslash S$ has two connected components ω^- and ω^+. A function $f \in \mathcal{O}(\omega^-)$ is said to have *polynomial growth at* $p \in S \cap \omega$, if there are $k \geqslant 0$ and $\epsilon > 0$ such that

$$(1.1) \qquad\qquad |f(z)| \leqslant C \operatorname{dist}(z, S)^{-k}$$

holds for $z \in \omega^- \cap B_\epsilon(p)$. We say that f has *polynomial growth towards* S if it has polynomial growth at every $p \in S \cap \omega$. It is well-known that such functions have unique weak boundary values in $\mathcal{D}'_{CR}(S)$, the space of CR distributions on S, see [1, Chapter VII] and also Section 2.

DEFINITION 1.1. — *We say that* Hartogs extension *holds for* Ω, *if every* $u \in \mathcal{D}'_{CR}(M)$ *is the boundary value of some* $f \in \mathcal{O}(\Omega)$ *with polynomial growth along* M.

The most straightforward examples for domains without Hartogs extension are domains containing a complex hypersurface, but these are very far from exhausting all possible obstructions. Despite of considerable recent activity, see [9, 25, 27, 28, 31], or the older [22], to mention a few, a satisfying understanding of Hartogs extension for unbounded domains seems still to be missing, even in the case that M is strictly pseudoconvex at every point. The main result of the present note is a geometric characterization that establishes a close link to envelopes of holomorphy.

THEOREM 1.2 (Main Theorem). — *Let $\Omega \subset \mathbb{C}^n$, $n \geqslant 2$, be a domain with connected smooth boundary M. Then Hartogs extension holds for Ω if and only if the envelope of holomorphy of the outer domain $\Omega^+ = \mathbb{C}^n \setminus \overline{\Omega}$ is $\mathbb{C}^n$.*

Actually, Theorem 1.2 is the global version of the more general Theorem 4.1, where Ω^+ is replaced by arbitrary outer collars attached to M. Moreover, Theorem 1.2 straightforwardly generalizes to domains in Stein manifolds.

THEOREM 1.3 (cf. Theorem 6.1). — *Let X be a Stein manifold with $\dim_{\mathbb{C}} X \geqslant 2$, and let $\Omega \subset X$ be a domain with connected smooth boundary M. Then Hartogs extension holds for Ω if and only if the envelope of holomorphy of the outer domain $\Omega^+ = X \setminus \overline{\Omega}$ is X.*

Note also that Theorem 1.2 easily implies the classical Hartogs–Bochner theorem for bounded domains, since holomorphic extension from the complement of a closed round ball to $\mathbb{C}^n$ (containing $\overline{\Omega}$) can be proved by combining the one-dimensional Cauchy formula along parallel slices.

Since pseudoconvex domains coincide with their envelope of holomorphy, we immediately obtain

COROLLARY 1.4. — *If Ω and M are as in Theorem 1.2 and Ω^+ is contained in a pseudoconvex* proper *subdomain of $\mathbb{C}^n$, then Hartogs extension fails. This holds in particular if $\overline{\Omega}$ contains a closed complex subvariety of $\mathbb{C}^n$ of dimension $n-1$.*

If M is unbounded the assumptions of Theorem 1.2 are symmetric with respect to the sides $\Omega^{\pm}$. In view of Corollary 1.4, the reader may wonder about similarities with the theorem of Trépreau [35] on local extension of CR functions from real hypersurfaces. We will elaborate on the relation between the two results in Section 7.

The picture changes significantly, if we restrict to extension of *smooth* CR functions from the boundary of Ω. Obviously extension still fails if Ω contains a complex hypersurface. The case where $\overline{\Omega}$ contains a complex hypersurface but Ω does not is more delicate: Examples constructed in [5] by the first two authors and Z. Słodkowski show that simultaneous extension of smooth CR functions to Ω may be valid or not, depending on a finer geometry of intersection of the complex hypersurface and the boundary. These examples are constructed as domains $\Omega \subset \mathbb{C}^2$ with variables $(z, w) \in \mathbb{C}^2$ which contain the complex line $L = \{(0, w) : w \in \mathbb{C}\}$ in the boundary of Ω. Note that the envelope $\mathsf{E}(\mathbb{C}^2 \setminus \overline{\Omega})$ is different from $\mathbb{C}^2$ and hence Hartogs

extension *does not hold* for such domains. However, for each $w \in \mathbb{C}$, let $\Omega_w = \{z \in \mathbb{C} : (z, w) \in \Omega\}$ be the "cross section" of Ω at $(0, w) \in \partial\Omega$. Depending on how much Ω_w "twists" around the origin in $\mathbb{C}$ as w varies, Ω may or may not have the property that smooth CR functions on $\partial\Omega$ extend continuously to holomorphic functions on Ω. Therefore combining Theorem 1.2 with these examples constructed in [5, Section 5] we get

COROLLARY 1.5. — *There are domains $\Omega \subset \mathbb{C}^2$ with smooth connected boundary such that Hartogs extension fails but every $\mathcal{C}^\infty$-smooth CR function on $\partial\Omega$ has a holomorphic extension to Ω (which is smooth up to $\partial\Omega$).*

For more results and questions on particular domains (bounded or unbounded) we refer to [3, 4, 5, 6, 11, 15, 16, 22, 32]. It may also be interesting to study domains obtained by intersecting smoothly bounded domains in $\mathbb{CP}^2$ with $\mathbb{C}^2$, for example with respect to extension from parts of the boundary, see [26] for domains in $\mathbb{C}^2$ and further references.

The paper is organised as follows: After some preliminaries collected in Section 2, we prove the easier direction in Theorem 1.2 in Section 3. More precisely, we show how properties of the envelope of holomorphy of Ω^+ imply Hartogs extension by using jump formulas and $\bar{\partial}$-methods. The converse direction is treated in the Sections 4 and 5. Section 4 contains topological preparations, which permit in particular, to localise to envelopes of thin collars of the domain. Section 5 completes the proof of Theorem 1.2. The main ingredient is the use of holomorphic functions with polynomial growth in order to construct nonextendible CR distributions. In Section 6, a generalisation of the main theorem to domains in Stein manifolds is given. The final section relates our result to the topic of removable singularities. More precisely, we analyse obstructions to extension confined to M and exhibit analogies to Trépreau's theorem.

Acknowledgments

This article is dedicated to the memory of our dear colleague and friend, Roman Dwilewicz, who passed away on July 29, 2016. He touched many lives through his mathematical collaborations, his conference organizational skills, his extensive travels, and his ability to befriend just about anyone he met. We miss him dearly.

2. Preliminaries

Riemann domains. First we recall basic material on Riemann domains and envelopes of holomorphy, referring to the monograph [18] for a thorough introduction. For a domain $D \subset \mathbb{C}^n$, we denote by $\pi_D : \mathsf{E}(D) \to \mathbb{C}^n$ its envelope of holomorphy. It is a Riemann domain (i.e. π_D is a local biholomorphism) and there is a canonical embedding $\iota_D : D \hookrightarrow \mathsf{E}(D)$ satisfying $\pi_D \circ \iota_D = \mathrm{id}_D$, allowing us to identify D with $\iota_D(D) \subset \mathsf{E}(D)$. A classical theorem based on the solution of the Levi problem tells that $\mathsf{E}(D)$ is Stein.

Following Grauert and Remmert [12], one may associate to every Riemann domain $\pi : X \to \mathbb{C}^n$ an abstract closure $\bar{\pi} : \overline{X} \to \mathbb{C}^n$ and an abstract boundary $bX = \overline{X} \backslash X$. Referring to [18, Section 1.5] for a careful treatment of the subtle construction, see also Section 6, we record that $\overline{X}$ is equipped with a natural topology which restricts to the standard topology on X. Moreover $\overline{X}$ is the closure of X in $\overline{X}$, and $\bar{\pi}$ is the continuous extension of π. For $\mathcal{C}^1$-*smoothly* bounded domains $D \subset \mathbb{C}^n$, the abstract closure coincides with the usual one, but for rough boundaries the abstract boundary bD may be multi-sheeted above the standard boundary ∂D.

Distributions. Recall some basic facts on distributions on a real manifold M. We consider a covering $\{\omega_j\}$ of M by coordinate neighborhoods ω_j, i.e. open sets equipped with diffeomorphisms $\kappa_j : \omega_j \to \widetilde{\omega}_j \subset \mathbb{R}^m$, $m = \dim M$. Following [17, Section 6.3] a *distribution on M* is given by such a covering together with distributions $u_j \in \mathcal{D}'(\widetilde{\omega}_j)$ satisfying

$$(2.1) \qquad u_i[\varphi] = u_j \left[|J_{\kappa_{ij}}| \varphi \circ \kappa_{ij} \right]$$

for every $\varphi \in \mathcal{D}(\kappa_i(\omega_i \cap \omega_j))$, where $\kappa_{ij} = \kappa_i \circ \kappa_j^{-1}$ and $J_{\kappa_{ij}}$ is the Jacobian determinant of the transition map. This definition of distributions on M is natural in so far that every function $g \in \mathcal{C}(M)$ identifies with the distribution $g_j : \varphi \mapsto \int (g \circ \kappa_j^{-1}) \varphi \, dx$, $\varphi \in \mathcal{D}(\widetilde{\omega}_j)$, because of the transformation formula. Note on the other hand that distributions do not canonically correspond to elements of the dual space of $\mathcal{D}(M)$. In the proof of Proposition 3.1, we shall see how to use metrics to this end.

CR distributions. For a $\mathcal{C}^\infty$-smooth hypersurface M in $\mathbb{C}^n$, we say that a distribution $u = \{u_j\}$ is a *CR distribution* if the u_j satisfy the tangential Cauchy–Riemann equations in the weak sense. To a function $f \in \mathcal{O}(\Omega)$ satisfying (1.1) we associate weak boundary values in the following way: Locally we can represent M as a graph

$$(2.2) \qquad y_n = h(z_1, \ldots, z_{n-1}, x_n) = h(z', x_n),$$

with $h \in \mathcal{C}^\infty(\widetilde{\omega})$ with $\widetilde{\omega}^{\mathrm{open}} \subset \mathbb{C}^{n-1}_{z'} \times \mathbb{R}_{x_n}$, so that Ω lies on the side $\{y_n > h\}$. Then the distributions $f_\epsilon \in \mathcal{D}'(\widetilde{\omega})$ defined by

$$f_\epsilon[\varphi] = \int f(z', x_n + i(h(z', x_n) + \epsilon))\, \varphi(z', x_n)\, \mathrm{d}x_1\, \mathrm{d}y_1 \ldots \mathrm{d}x_{n-1}\, \mathrm{d}y_{n-1}\, \mathrm{d}x_n$$

tend to a CR distribution $f^* \in \mathcal{D}'(\widetilde{\omega})$ for $\epsilon \downarrow 0$, see [1, Theorem 7.2.6]. We can select a cover of M by open sets ω_j graphed over $\widetilde{\omega}^{\mathrm{open}}_j \subset \mathbb{C}^{n-1} \times \mathbb{R}$ and obtain distributions $f_j^* \in \mathcal{D}'(\widetilde{\omega}_j)$ as above. By the Baouendi–Trèves approximation theorem we can locally approximate the f_j^* by (pullbacks of) restrictions of entire functions and derive that the f_j^* satisfy (2.1).

Boundary values of holomorphic functions. For detailed information on CR functions, we refer to [2, 24]. We will need the following fact: *Let D be a full neighborhood of M in $\mathbb{C}^n$ and let $f \in \mathcal{O}(D \setminus M)$ have polynomial growth towards M from both sides. If near every $z \in M$ the two local boundary values of f from opposite sides coincide, then f extends holomorphically through M.*

We sketch a proof based on the hypoanalytic wave front set $WF_{ha}(u)$, which is defined for CR distributions u on $\mathcal{C}^\infty$-smooth embedded CR manifolds, and refer to [36] for a thorough introduction and the basic structure theorems we use in the sequel. By definition $WF_{ha}(u)$ is a $\mathbb{R}_{>0}$-invariant subset of the pointed characteristic bundle. More precisely, the characteristic bundle is the real line bundle

$$H^0 M = \bigcup_{p \in M} \{\xi \in T_p^* M : \xi|_{H_p M} \equiv 0\} \subset T^* M,$$

and $WF_{ha}(u)$ is a subset of $H^0 M$ minus the zero section. Now the two local weak boundary values $f^- = f^+ = f^*$ coincide. The existence of each of the local extensions $f^\pm$ rules out one side of the zero section in $H^0 M$ from $WF_{ha}(f^*)$. Hence $WF_{ha}(f^*)$ is locally empty, whence f^* extends holomorphically to an ambient neighborhood.

3. Holomorphic extension

The following proposition yields sufficiency in Theorem 1.2.

PROPOSITION 3.1. — *Let $\Omega \subset \mathbb{C}^n$, $n \geqslant 2$, be a domain with smooth connected boundary M. If the envelope of holomorphy of $\mathbb{C}^n \setminus \overline{\Omega}$ is $\mathbb{C}^n$, Hartogs extension is valid for Ω.*

We will indicate how the proof can be pieced together from known techniques. Straightforward modifications yield versions for varying degrees of regularity, for example for continuous CR functions defined on a $\mathcal{C}^1$-smooth boundary.

Proof. — On M we select a smooth Riemannian metric μ and fix the orientation induced on M as the boundary of Ω. We may restrict to graph representations as in (2.2) such that

$$\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_{n-1} \wedge \mathrm{d}y_{n-1} \wedge \mathrm{d}x_n$$

is positive and write μ's volume form σ in the local coordinates κ_j as

$$\sigma = \sigma_j \, \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_{n-1} \wedge \mathrm{d}y_{n-1} \wedge \mathrm{d}x_n.$$

Since $\sigma_i = J_{\kappa_{ji}} \sigma_j$ by the transformation formula, the coordinate-wise defined products $\sigma_j u_j$ glue to an element of $u_\mu \in \mathcal{D}'(M)$, if $u = \{u_j\}$ is a distribution.

To every CR distribution $u = \{u_j\}$ on M, we may canonically associate a current T_u on $\mathbb{C}^n$ of bidegree $(0,1)$ in the following way: For a smooth compactly supported $(2n-1)$-form $\psi \in \mathcal{D}_{(2n-1)}(M)$ the function ψ/σ (defined as the unique function $\widetilde{\psi}$ satisfying $\psi = \widetilde{\psi}\sigma$) has compact support. Hence

$$T_{u,M}[\psi] = (u_\mu)[\psi/\sigma]$$

is a $(2n-1)$-dimensional current $T_{u,M}$ on M. Writing

$$\omega_{x,y'} = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_{n-1} \wedge \mathrm{d}y_{n-1} \wedge \mathrm{d}x_n,$$

the equality

$$(u_j)_{\mu_j} \left[\frac{\psi}{\sigma_j \omega_{x,y'}} \right] = u_j[\psi/\omega_{x,y'}]$$

holds locally. Thus $T_{u,M}$ is independent of μ. Finally we set

$$T_u[\varphi] = T_{u,M}[(\iota_M)^* \varphi]$$

for smooth $(n, n-1)$-forms $\varphi \in \mathcal{D}_{(n,n-1)}(\mathbb{C}^n)$. Here ι_M is the embedding of M and $(\iota_M)^* \varphi$ is the pullback of φ to M. Again T_u is μ-independent and $\bar{\partial}$-closed. For the last property, one may observe that $\bar{\partial}$-closedness is a local property (since M is properly embedded) and use the Baouendi–Trèves approximation theorem.

Since $H^1_{\bar{\partial}}(\mathbb{C}^n) = 0$, the Dolbeault isomorphism gives a distribution solution $f \in \mathcal{D}'(\mathbb{C}^n)$ of

$$\bar{\partial} f = T_u.$$

Since T_u has no mass outside M, f restricts to holomorphic functions f^- on $\Omega^- = \Omega$ and f^+ on $\Omega^+ = \mathbb{C}^n \setminus \bar{\Omega}$, by elliptic regularity. By [7, 20], u

is the jump from f^- to f^+ in the following sense: If $r \in \mathcal{C}^\infty(U)$ is a local defining function of $M \cap U$, $U \Subset \mathbb{C}^n$, then

$$\int_{r=\epsilon} f^+ \varphi - \int_{r=-\epsilon} f^- \varphi \longrightarrow T_u[\varphi], \text{ if } \epsilon \downarrow 0,$$

holds for all $\varphi \in \mathcal{C}^\infty_{(n,n-1)}(U)$ with supp $\varphi \subset U$. Now f^+ extends to a holomorphic function on $\mathbb{C}^n$ by assumption, and

$$\lim_{\epsilon \downarrow 0} \int_{r=\epsilon} f^+ \varphi = \lim_{\epsilon \downarrow 0} \int_{r=-\epsilon} f^+ \varphi = \int_M f^+ \varphi$$

holds by continuity. Hence $f^+ - f^-$ defines the desired extension of u to Ω. $\qquad \square$

4. Localization near M

A domain $C \subset \mathbb{C}^n \setminus \overline{\Omega}$ is called an *outer collar of M* if $C \cup M$ is a relative neighborhood of M in $\mathbb{C}^n \setminus \Omega = \Omega^+ \cup M$, see Figure 4.1. Of course $\Omega^+ = \mathbb{C}^n \setminus \overline{\Omega}$ itself is an outer collar. In this section we show that Ω^+ can be replaced by an arbitrary outer collar in the assumptions of Theorem 1.2. The most general version of our main result is
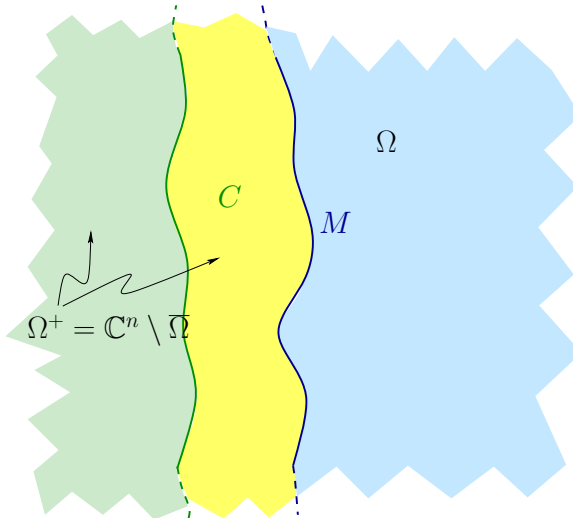

Figure 4.1. Outer collar C

THEOREM 4.1. — *For a domain $\Omega \subset \mathbb{C}^n$, $n \geqslant 2$, with connected smooth boundary M the following properties are equivalent:*

(a) *The envelope of holomorphy of Ω^+ is $\mathbb{C}^n$.*
(b) *For every outer collar C of M, the canonical embedding $\iota_C : C \hookrightarrow \mathsf{E}(C)$ extends to a (unique) lifting of $\Omega \cup M \cup C$ to $\mathsf{E}(C)$.*
(c) *There is an outer collar C of M such that ι_C extends as in* (b).
(d) *Every $u \in \mathcal{D}'_{CR}(M)$ has a holomorphic extension to Ω.*

Here we give the topological part of the proof, postponing the extension part (d) to the next section.

Proof that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). — Since the implications (b) $\Rightarrow$ (a) $\Rightarrow$ (c) are tautological, it suffices to show (c) $\Rightarrow$ (b). We let C_1 be a collar as granted by (c) and have to show the lifting property for an arbitrary collar C_2. $\qquad\square$

LEMMA 4.2. — *If the lifting property holds for some subcollar $C_2' \subset C_2$, it also holds for C_2.*

Proof. — The lifting property is equivalent to the fact that all $f \in \mathcal{O}(C_2')$ extend to $\Omega \cup M \cup C_2'$. Applying this extension property to restrictions $g|_{C_2'}$, $g \in \mathcal{O}(C_2)$, we get the extension property and thereby the lifting property for C_2. $\qquad\square$

Hence it suffices to prove the lifting property for an appropriate subcollar of C_2, allowing us to assume that $C_2 \subset C_1$.

LEMMA 4.3. — *Let M' be a smooth hypersurface obtained by isotoping M into C_2. Then $M_1' = \iota_{C_1}(M')$ disconnects $\mathsf{E}(C_1)$, and $M_2' = \iota_{C_2}(M')$ disconnects $\mathsf{E}(C_2)$ (see Figure 4.2).*

Proof. — The argument is the same for M_1' and M_2'. Since M_1' is connected, $\mathsf{E}(C_1) \setminus M_1'$ has at most two connected components. If there is only one, there is a smoothly embedded loop $\gamma \subset \mathsf{E}(C_1)$, which has exactly one transverse intersection point with M_1' (take a small arc transverse to M_1' and link the endpoints by another arc that does not intersect M_1'). By a result of Kerner [19], see also [33], ι_{C_1} induces a surjective homomorphism $(\iota_{C_1})_* : \pi(C_1) \hookrightarrow \pi(\mathsf{E}(C_1))$ between the fundamental groups. Hence there is a loop $\widetilde{\gamma} \subset C_1$ such that $\iota_{C_1}(\widetilde{\gamma})$ and γ are homotopic within $\mathsf{E}(C_1)$.

We use intersection numbers of oriented loops $\lambda \subset \mathbb{C}^n$ with M', which can be defined as follows: Let Ω' be the domain in $\mathbb{C}^n$ bounded by M' and containing Ω. For λ transverse to M', we compute the intersection number by subtracting the number of points where λ enters Ω' from the number of

points where λ leaves Ω'. The definition extends to general λ because the intersection number is homotopy invariant, see [13] for details.

Since $M \cap C_1 = \emptyset$ the intersection number of $\widetilde{\gamma}$ and M is zero and the same holds for the intersection of $\widetilde{\gamma}$ and M' (which is isotopic to M). Pushing forwards by ι_{C_1}, we get zero intersection number between $\iota_{C_1}(\widetilde{\gamma})$ and M'_1 (the intersection number is calculated locally at the intersection points). This contradicts the stability of intersection numbers under homotopy and the fact that the intersection number of γ and M'_1 is ± 1. $\qquad\square$
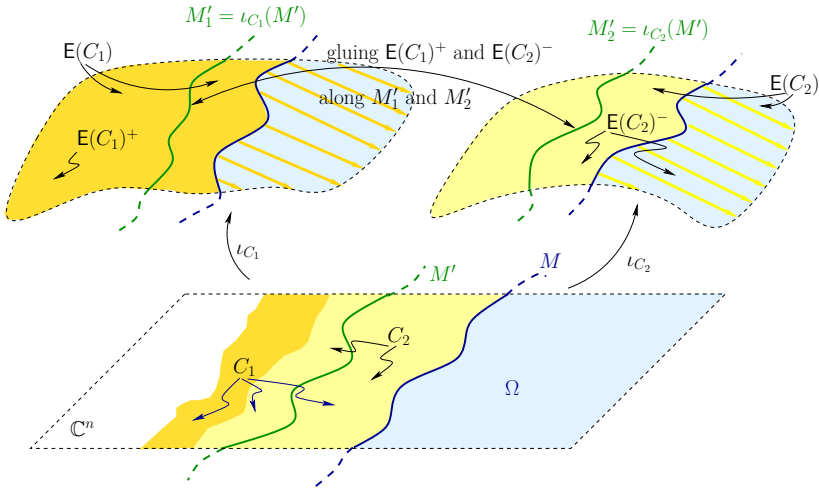

Figure 4.2. Two collars and corresponding envelopes

Continuation of the proof (c) $\Rightarrow$ (b). — Denote by $\mathsf{E}(C_2)^-$ the connected component of $\mathsf{E}(C_2) \setminus M'_2$ which lies on the side of Ω, see Figure 4.2. More precisely, M can be identified, via the canonic embedding $C_2 \hookrightarrow \mathsf{E}(C_2)$, with a subset of the abstract closure of $\mathsf{E}(C_2)$, and $\mathsf{E}(C_2)^-$ is the connected component containing the lifting of M in its closure. It suffices to show that the domain Ω' considered in the proof of Lemma 4.3 lifts to $\mathsf{E}(C_2)^-$ biholomorphically.

To this end, we construct a new Riemann domain $\pi_{X'} : X' \to \mathbb{C}^n$ by gluing $\mathsf{E}(C_2)^-$ with $\mathsf{E}(C_1)^+$ along M'. More precisely, $\mathsf{E}(C_1)^+$ is the connected component of $\mathsf{E}(C_1) \setminus M'_1$ on the side opposite to M, and the gluing identifies M'_2 with M'_1. Denote by $\widetilde{M}$ the corresponding hypersurface of X'. Note that there is a natural embedding ι' of C_1 into X', which coincides with ι_{C_1} along M'.

Since pseudoconvexity is a local property at points of the abstract boundary and X' is obtained by gluing two pseudoconvex Riemann domains along a set in the interior, X' is pseudoconvex by [10]. Treating the sides of $\widetilde{M}$ separately, we see that every $f \in \mathcal{O}(C_1)$ extends to X'. Hence X' is an extension of C_1 in the terminology of [18, Section 1.4], and by the pseudoconvexity of X', this extension is maximal. Hence X' is equivalent to the envelope of holomorphy $\mathsf{E}(C_1)$ as a Riemann domain over $\mathbb{C}^n$, meaning that Ω' lifts to X'. Since the lifting has image in the side of $\widetilde{M}$ which is equivalent with $\mathsf{E}(C_2)^-$, we have proved the existence of the desired lifting for C_2.

This finishes the proof that the first three properties are equivalent. The link to (d) will be completed in the subsequent section. □

5. Obstructions to Hartogs extension

In this section, we will prove the harder direction in Theorem 1.2.

Geometry of $\mathsf{E}(\Omega^+)$. Recall that we consider a domain $\Omega = \Omega^-$ with smooth connected boundary M, and assume that the envelope of holomorphy $\pi_{\Omega^+} : \mathsf{E}(\Omega^+) \to \mathbb{C}^n$ of $\Omega^+ = \mathbb{C}^n \setminus \overline{\Omega}$ differs from $\mathbb{C}^n$. For notational simplicity we write X^+ instead of $\mathsf{E}(\Omega^+)$, π instead of π_{Ω^+}, $\bar{\pi}$ instead of the continuous extension $\overline{(\pi_{\Omega^+})} : \overline{X^+} \to \mathbb{C}^n$, and $\iota : \Omega^+ \hookrightarrow X^+$ instead of ι_{Ω^+}.

Let M_1 be a smooth hypersurface obtained by slightly deforming M into Ω^+. Let Ω_1 be the domain that is bounded by M_1 and contains Ω. Then the intersection $C = \Omega_1 \cap \Omega^+$ is a one-sided collar of M, lying opposite to Ω. As in the proof of Theorem 4.1, we see that $\iota(M_1)$ disconnects X^+ into two domains. Let X_1^- be the connected component of $X^+ \setminus \iota(M_1)$ that contains $\iota(C)$. Obviously $X_1^- = \iota(C) \cup (X^+ \setminus \iota(\Omega^+))$. The arguments of Section 4 actually allow us to identify X_1^- with a subdomain of the envelope of C, but we will not need this here. Note that $\pi(X_1^-)$ need not be contained in Ω_1 and that there may be multi-sheetedness over both sides of M, as indicated in Figure 5.1.

We distinguish two cases:

Case 1: X^+ is univalent. — In this case we may identify both X^+ and X_1^- with domains in $\mathbb{C}^n$, and the mappings ι and π are inclusions. Since Ω_1 is one of the sides of M_1 in $\mathbb{C}^n$ and X_1^- contains C, X_1^- is contained in Ω_1. It has to be a *proper* subset, for X^+ would be biholomorphic to $\mathbb{C}^n$ otherwise. It follows that the abstract boundary bX_1^- is the disjoint union of M_1 with a *nonvoid* set S satisfying $\bar{\pi}(S) \subset \overline{\Omega}$. Note that $\bar{\pi}$ may
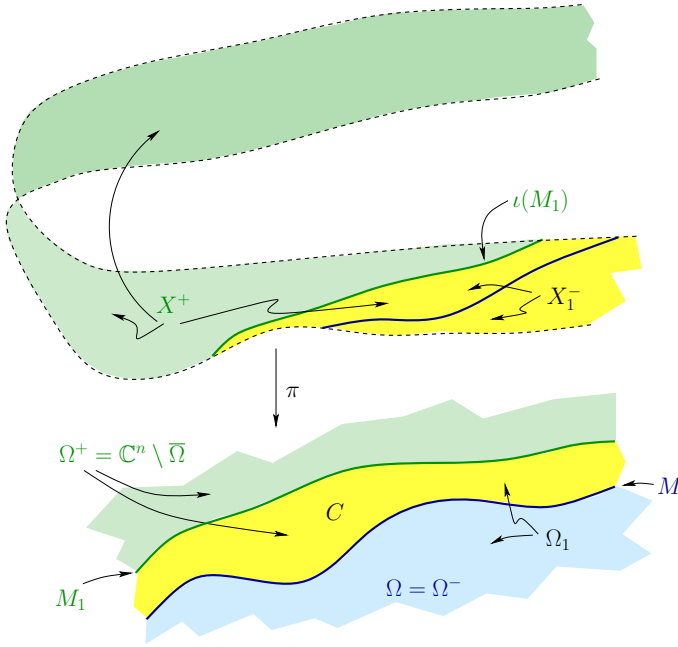
Figure 5.1. Multi-sheetedness

become multi-sheeted on S, which therefore cannot be identified with a subset of $\mathbb{C}^n$.

Case 2: X^+ is multi-sheeted. — Obviously, this can only happen if $\iota(\Omega^+)$ is a proper subset of X^+. For later use, we will only need that there is a $p_0 \in X^+ \setminus \iota(\Omega^+)$ such that the fiber $\pi^{-1}(\pi(p_0))$ contains at least two elements.

Construction of CR distributions. Lifting the Euclidean distance, we get a Riemannian metric on any Riemann domain $\pi : X \to \mathbb{C}^n$ and thereby the distance[1] $\mathrm{dist}(p, bX)$ between a point $p \in X$ and the abstract boundary bX. For a nonnegative integer k we consider the Banach space

$$\mathcal{O}^{(k)}(X) = \{f \in \mathcal{O}(X) : f(p)\, \delta_X^k(p) \text{ is bounded on } X\},$$

where

$$\delta_X(p) = \min\left(\mathrm{dist}(p, bX), \frac{1}{\sqrt{1 + |\pi(p)|^2}} \right),$$

---

[1] $\mathrm{dist}(p, bX)$ is the supremum of all $r > 0$ such that $B_r(\pi(p))$ can be lifted to X so that $\pi(p)$ is mapped to p.

see [18, Section 2.5] for detailed information. As observed in [26], see also [30], these spaces are useful for constructing CR distributions with prescribed singularities.

LEMMA 5.1. — Let $f \in \mathcal{O}^{(k)}(X^+)$ be given.

(a) The restriction $f^+ = f|_{\iota(\Omega^+)}$ has a unique CR distribution $f^* \in \mathcal{D}'_{CR}(M)$ of order $k+1$ as weak boundary values on M.

(b) If f^* has a holomorphic extension $f^- \in \mathcal{O}(\Omega)$ then f^- and f^+ glue to an entire function. In particular f^* is smooth.

Proof. — For $z \in \Omega^+$ close to M, we have

$$|f_{\Omega^+}(z)| \leqslant C \operatorname{dist}^{-k}(z, bX^+) \leqslant C \operatorname{dist}^{-k}(z, M).$$

Thus $f|_{\Omega^+}$ has at most polynomial growth towards M, and (a) follows from classical results on boundary values of holomorphic functions, see [1, Ch. VII]. If there is also an extension f^- to the other side, the hypoanalytic wave front set of f^* is empty, meaning that f^* is locally a restriction of a holomorphic function, and (b) follows from the uniqueness of holomorphic extension.   □

From [26] we recall the following

LEMMA 5.2. — Let $\pi_Y : Y \to \mathbb{C}^n$ be a pseudoconvex Riemann domain and q a point on the abstract boundary bY. Then there is a sequence $Y \ni p_j \to q$ and a function $f \in \mathcal{O}^{(2n+1)}(Y)$ such that $|f(p_j)| \to \infty$.

Now we are ready to construct a nonextendable CR distribution in the two cases from our previous discussion of the geometry. In Case 1 there is a point $q \in bX^+ \cap \pi_{X^+}^{-1}(\overline{\Omega})$, a sequence $X^+ \ni p_j \to q$ (convergence with respect to topology of the abstract closure of X^+, which may be finer than the subspace topology coming from $\mathbb{C}^n$) and a function $f \in \mathcal{O}^{(2n+1)}(X^+)$ as in Lemma 5.2.

We claim that the CR distribution f^* associated to f by Lemma 5.1 does not extend holomorphically to Ω. Otherwise Lemma 5.1 (b) yields an entire function F. The identity principle shows that F is an extension of f (with X^+ considered as a subset of $\mathbb{C}^n$), which is impossible since $|f(p_j)| \to \infty$.

In Case 2 we get a function $f \in \mathcal{O}^{(6n+1)}(X)$ which separates the points in the fiber $\pi^{-1}(\pi(p_0))$ from [18, Proposition 2.5.5]. Actually it suffices that f attains different values at p_0 and a second point $p_1 \in \pi^{-1}(\pi(p_0))$. Again we claim that the induced CR distribution f^* does not extend to Ω. Otherwise the extension and $f|_{\Omega^+}$ glue along M to an entire function F. The identity principle yields $f = F \circ \pi$, and therefore $f(p_0) = f(p_1)$, in contradiction to the choice of f.

Remark 5.3.

- (a) In some cases (for example if M is strictly pseudoconvex at every point and Ω is the pseudoconvex side) the CR distributions we find to be obstructions to Hartogs extension are smooth on M. In general this cannot always be achieved because of examples constructed in [5], see Corollary 1.5.
- (b) It may happen that each CR distribution u on M possesses a weaker kind of holomorphic extension to Ω which attains u as weak boundary values only along an open subset of M. We look at this in Section 7.

Conclusion of the proof of Theorem 1.2. — Sufficiency was shown in Section 3.1. To derive necessity, we argue by contraposition, assuming that the envelope of Ω^+ differs from $\mathbb{C}^n$. In each of the occuring cases, we have constructed a CR function without extension to Ω, which completes the proof of Theorem 1.2 and Theorem 4.1.   □

6. Generalisation of the main result to domains in Stein manifolds

THEOREM 6.1. — *Theorem 4.1 is still valid if $\mathbb{C}^n$ is replaced by a Stein manifold Y of complex dimension $n \geqslant 2$.*

Most of the proof of Theorem 4.1 is easily generalised to Stein manifolds. The only ingredient which is specifically related to domains over $\mathbb{C}^n$ are the spaces $\mathcal{O}^{(k)}(X)$. We shall give an extension to Stein manifolds, which is less precise than the original results but still sufficient for our needs.

Let $\pi : X \to Y$ be a Riemann domain over a complex manifold Y. Recall that an abstract boundary point $q \in bX$ can be specified by associating to every open neighborhood $U \subset Y$ of $p = \pi(q)$ the connected component V of $\pi^{-1}(U)$ containing q in its closure. If U is relatively compact in Y and $z_1, \ldots, z_n$ are local holomorphic coordinates defined in a neighborhood of $\overline{U}$, we may view V as a Riemann domain over $\mathbb{C}^n$. Then we call V *adapted neighborhood* and $z_1, \ldots, z_n$ *adapted coordinates*. We say that a function $f \in \mathcal{O}(X)$ has *polynomial growth of degree k at $q \in bX$*, if there is an adapted neighborhood V and $k \in \mathbb{N}_0$ such that $f|_V$ is an element of $\mathcal{O}^{(k)}(V)$ with respect to the corresponding adapted coordinates. It is elementary to verify that this property does not depend on the choice of adapted coordinates. Define $\mathcal{O}^{\mathrm{pol}}(X)$ as the algebra of *holomorphic functions of*

polynomial growth, i.e. of all $f \in \mathcal{O}(X)$ which have polynomial growth at every $q \in bX$.

In contrast to the Banach algebras $\mathcal{O}^{(k)}(X)$, defined for X spread over $\mathbb{C}^n$, $\mathcal{O}^{\mathrm{pol}}(Y)$ is only Fréchet in general. However, the following theorem is enough for constructing CR distributions and permits to extend the proof of Theorem 4.1 to Theorem 6.1.

THEOREM 6.2. — *Let Y be a Stein manifold of dimension n and $\pi :$ $X \to Y$ a pseudoconvex Riemann domain over Y. Then the following hold:*

(a) *For every $q_0 \in bX$, there is a sequence $x_j \in X$ with $x_j \to q_0$ and a function $f \in \mathcal{O}^{\mathrm{pol}}(X)$ with $|f(x_j)| \to \infty$.*

(b) *For every pair $q_1, q_2 \in X$ with $\pi(q_1) = \pi(q_2)$ and $q_1 \neq q_2$, there is $f \in \mathcal{O}^{\mathrm{pol}}(X)$ with $f(q_1) \neq f(q_2)$.*

Proof. — By the Bishop–Narasimhan–Remmert embedding theorem (see [14, Chapter VII, Section C and Notes on p. 233]), we may assume that Y is a properly embedded complex submanifold of $\mathbb{C}^m$ for some $m \geqslant n$. The normal bundle $\pi_N : N \to Y$ of Y in $\mathbb{C}^m$ (i.e. the bundle with fibers $T_y\mathbb{C}^m/T_yY$, $y \in Y$) has a natural holomorphic structure, with respect to which it is a Stein manifold. Let $\mathsf{o} : Y \hookrightarrow N$ be the canonical embedding of Y as zero section. By [10, Satz 3], there is a holomorphic mapping $\Phi : N \to \mathbb{C}^m$ which is a locally biholomorphic outside some purely $(m-1)$-dimensional complex subvariety A of N that does not intersect $\mathsf{o}(Y)$. Thus $\widetilde{\Phi} = \Phi|_{N \setminus A} : N \setminus A \to \mathbb{C}^m$ is a pseudoconvex Riemann domain.

Consider the Riemann domain $\widetilde{\pi} : \widetilde{X} \to N \setminus A$ that is obtained by gluing at every point $x \in X$ the corresponding fiber $\widetilde{\Phi}^{-1}(\pi(x)) \setminus A$, or formally

$$\widetilde{X} = \{(x, v) \in X \times (N \setminus A) : \pi(x) = \widetilde{\Phi}(v)\}, \quad \widetilde{\pi}(x, v) = v.$$

It is straightforward to see that $\widetilde{X}$ is Stein and that $\alpha : x \mapsto (x, \mathsf{o} \circ \pi(x))$ embeds X into $\widetilde{X}$ in such a way that the image consists of the points lying above the zero section of N. Moreover, $\widetilde{\Phi} \circ \widetilde{\pi}$ turns $\widetilde{X}$ into a Riemann domain over $\mathbb{C}^m$, and we get the commutative diagram

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\ \widetilde{\pi}\ } & N \setminus A \\
{\scriptstyle \alpha}\big\uparrow & & \big\downarrow{\scriptstyle \widetilde{\Phi}} \\
X & \xrightarrow{\ \pi\ } & \mathbb{C}^m
\end{array}
$$

where π is viewed as a mapping with values in $\mathbb{C}^m$.

Hence we may apply results from [18] in the same way as in the proof of [26, Lemma 2.2], in order to obtain functions $f \in \mathcal{O}^{(6m+1)}(\widetilde{X})$ which

explode at a given $\widetilde{q}_0 \in b\widetilde{X}$ or separate two points $\widetilde{q}_1 \neq \widetilde{q}_2$ lying in the same fiber of $\widetilde{\Phi}\circ\widetilde{\pi}$. Here $\mathcal{O}^{(6m+1)}(\widetilde{X})$ is defined with respect to the standard structure of $\mathbb{C}^m$.

To prove (a) and (b), we choose $\widetilde{q}_j = \alpha(q_j)$. Since $\widetilde{X}$ is a product near every point of $\widetilde{\pi}^{-1}(Y)$, the restriction f to $\alpha(X)$ defines an element of $\mathcal{O}^{\mathrm{pol}}(X)$. The proofs of Theorem 6.2 and of Theorem 6.1 are complete. $\square$

7. CR orbits and removable singularities

In this section we will slightly change our viewpoint by treating the sides of M on equal footing. Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface and $z \in M$. Consider ambient neighborhoods ω of z such that $\omega \setminus M$ has exactly two connected components $\omega^\pm$. Then z is called *local obstruction point of M* if there is a neighborhood basis of z by neighborhoods ω as above and functions $f^\pm \in \mathcal{O}(\omega^\pm)$ which do not extend holomorphically to any neighborhood of z. Trépreau's theorem [35] says that z is a local obstruction point of M if and only if there is a local holomorphic hypersurface Z satisfying $z \in Z \subset M$. In the literature, it is customary to call M *minimal at z* iff z is no local obstruction point.

For M as in Theorem 1.2, with sides $\Omega^- = \Omega$ and $\Omega^+ = \mathbb{C}^n \setminus \overline{\Omega}$, let $\pi_\pm : X^\pm \to \mathbb{C}^n$ denote the envelopes of holomorphy of $\Omega^\pm$, $bX^\pm$ their abstract boundaries, $\overline{\pi}_\pm$ the continuous extensions of $\pi_\pm$ to the abstract closures $X^\pm \cup bX^\pm$, and $\iota_\pm : \Omega^\pm \hookrightarrow X^\pm$ the canonical embeddings. A point $z \in M$ is called a *global obstruction point of M* if there are liftings $z^- \in bX^-$ and $z^+ \in bX^+$ satisfying

$$(7.1) \qquad z^\pm \in bX^\pm \cap \overline{\iota_\pm(\Omega^\pm)} \cap \overline{\pi}_\pm^{-1}(z).$$

Observe that (7.1) is equivalent to the existence of functions $f^\pm \in \mathcal{O}(\Omega^\pm)$ without holomorphic extension across z. We study the *global obstruction set $M_{\mathsf{obs}} \subset M$* of all global obstruction points of M.

To investigate the geometry of M_{obs}, we recall the notion of CR orbits. Two points of M lie in the same *CR orbit* if they can be linked by a piecewise smooth CR curve, i.e. a curve whose velocity vectors are contained in the complex tangent bundle $HM = \bigcup_{p \in M}(T_p M \cap J_p T_p M)$. Obviously the CR orbits form a disjoint decomposition of M. A fundamental result of Sussmann [34] tells that every orbit is an injectively immersed smooth manifold with real dimension at least equal to the rank $2n - 2$ of HM. From basic facts on ordinary differential equations, it follows that orbits are either open subsets of M or injectively immersed complex manifolds of

complex dimension $n - 1$, see [24, Section 3.1], and that the union M_{hol} of the lower-dimensional orbits is closed in M.

The following theorem shows that the global obstruction set is completely determined by the CR geometry of M.

THEOREM 7.1. — *In the situation of Theorem 1.2, the sets M_{obs} and M_{hol} coincide. In particular, $M_{\mathsf{obs}} \subset M$ is either empty or unbounded and of positive $(2n - 2)$-dimensional Hausdorff measure.*

Proof. — First we claim that M_{hol} is contained in M_{obs}. To see this, we fix $z \in M_{\mathsf{hol}}$. Since $\mathbb{C}^n \setminus M_{\mathsf{hol}}$ is Stein, there are functions $f^\pm \in \mathcal{O}^{(6n+1)}(\mathbb{C}^n \setminus M_{\mathsf{hol}})$ and sequences $\{z_j^\pm\} \subset \Omega^\pm$ approaching z from the $\Omega^\pm$-side, respectively, such that $|f(z_j^\pm)| \to \infty$. In $\overline{X^\pm}$, the lifted sequences $\{\iota_\pm(z_j^\pm)\}$ converge to elements $z^\pm \in bX^\pm$ above z, and the claim follows.

Observe that M_{obs} is closed as the intersection of the two closed sets

$$\pi_\pm \left(bX^\pm \cap \overline{(\iota_\pm(\Omega^\pm))} \right).$$

Fix $z_0 \in M_{\mathsf{obs}}$. For any $0 < \epsilon \ll 1$ the open set $B_\epsilon(z_0) \setminus M$ has two connected components $\omega^\pm \subset \Omega^\pm$. The envelope $\mathsf{E}(\omega^\pm)$ is also a Riemann domain over $X^\pm$ and it is readily verified that

$$z_0 \in \overline{\pi}_{\omega^+} \left(b\mathsf{E}(\omega^+) \cap \overline{\iota_{\omega^+}(\omega^+)} \right) \cap \overline{\pi}_{\omega^-} \left(b\mathsf{E}(\omega^-) \cap \overline{\iota_{\omega^-}(\omega^-)} \right).$$

Hence Trépreau's theorem implies that there is a local complex hypersurface $Z \subset M$ passing through z_0. Since Z is the CR orbit through z_0 with respect to a sufficiently small neighborhood of z_0, Z is smooth and tangent to HM.

We claim that a neighborhood of z_0 in Z is contained in M_{obs}. Otherwise there is a point $z_1 \in Z$ such that all functions in $\mathcal{O}(\Omega^*)$, where $*$ is one of the signs $+$ or $-$, extend to a uniform ambient neighborhood of z_1. Now we get a contradiction to (7.1) from the general theorem about propagation of extension to full neighborhoods along complex submanifolds of M. Below we provide some details on how to apply propagation arguments to envelopes of holomorphy.

Consider the CR orbit $\mathcal{O}(z_0, M)$ of z_0 in M. The proof of Theorem 7.1 will be complete, as soon as we have shown that $\mathcal{O}(z_0, M)$ is a lower-dimensional orbit and satisfies

$$(7.2) \qquad\qquad \mathcal{O}(z_0, M) \subset M_{\mathsf{obs}}.$$

Let us first show (7.2) in case that $\mathcal{O}(z_0, M)$ is lower-dimensional. Then $\mathcal{O}(z_0, M)$ can be parametrized by an injective holomorphic immersion $\alpha : Z \hookrightarrow \mathcal{O}(z_0, M)$ of a connected $(n - 1)$-dimensional complex manifold Z.

Note that the manifold topology of $\mathcal{O}(z_0, M)$, i.e. the pushforward of the topology of Z under α, may be finer than the topology induced from ambient space. However, the above arguments imply that $M_{\mathsf{obs}} \cap \mathcal{O}(z_0, M)$ is both open and closed in $\mathcal{O}(z_0, M)$ with respect to the manifold topology. This proves (7.2) for $\mathcal{O}(z_0, M)$ lower-dimensional.

It remains to rule out the case that $\mathcal{O}(z_0, M)$ is open in M. Then M is minimal at some point $z_1 \in \mathcal{O}(z_0, M)$ (otherwise $\mathcal{O}(z_0, M)$ were foliated by complex hypersurfaces), and Trépreau's theorem implies that CR functions locally extend to one side of M. Since this property propagates along CR orbits, CR functions extend to one side at every point of $\mathcal{O}(z_0, M)$, in particular at z_0. Below we will outline how the information on extension of CR functions yields that z_0 is contained in the envelope of at least one of the domains $\Omega^\pm$. This contradicts $z_0 \in M_{\mathsf{obs}}$, and completes the proof of (7.2).

Let us sketch the link between extension of CR functions from M and the envelopes of $\Omega^\pm$. We will use the method of analytic discs, see [1, 2, 24] for detailed information. The tools necessary to realise the following outline are explained in [24, Chapters 4 and 5], see also [23] for more on deformation of discs. An analytic disc is a mapping $A : \overline{\mathbb{D}} \to \mathbb{C}^n$ which is holomorphic in $\mathbb{D}$ and has some smoothness up to the boundary $\mathbb{T} = \partial\mathbb{D}$ (for our needs $\mathcal{C}^{2,\alpha}$ with $0 < \alpha < 1$ is enough). One works with discs attached to M (i.e. $A(\mathbb{T}) \subset M$) or with boundaries close to M. In our case, one starts from a chain of discs A_j, $j = 1, \ldots, m$, attached to M and linking z_1 and z_0 in the sense that $A_1(-1) = z_1$, $A_j(1) = A_{j+1}(-1)$, $j = 1, \ldots, m-1$, and $A_m(1) = z_0$. These discs are small in the sense that they are attached to subsets of M which can be represented as graphs and that the local solution theory of the Bishop equation can be used to deform discs. Since z_1 is a minimal point, we can sweep out one local side of M at z_1 by images of a 1-parameter family of discs. Then one uses this open set U_0 attached to M at z_1 in order to deform A_1 and produce a nearby disc $\widetilde{A}_1$, whose image, viewed as a parametrised surface, is transverse to M at $\widetilde{A}_1(1) = A_1(1)$. Sliding the $\widetilde{A}_1$ in the directions transverse to $\frac{\partial \widetilde{A}_1}{\partial\theta}$ (where $\mathbb{T} = \{e^{i\theta}, \theta \in \mathbb{R}\}$) yields a family that sweeps out a one-sided neighborhood U_1 attached to M at $A_1(1)$. Note that U_0 and U_1 may lie on opposite sides of M.

Iterating this procedure, we finally obtain a one-sided neighborhood U_m attached at $A_m(1)$. The continuity principle applied to the underlying families of discs shows that holomorphic functions defined in an *arbitrarily thin* ambient neighborhood of (a sufficiently large subset of) M extend to the open sets U_j. By construction U_m intersects one of the sides of M. To

fix ideas, we assume that this side is Ω^+. Let M_t, $t \in [0,1]$, be a smooth 1-parameter deformation of $M = M_0$ such that $M_t \subset \Omega^-$, $0 < t \leqslant 1$. Together the deformations of discs constructed above depend on finitely many parameters, the dependence being $\mathcal{C}^{2,\beta}$-smooth for some $\beta \in (0, \alpha)$. Inspection of the Bishop equation shows that we may extend these deformations to the parameter t for $0 \leqslant t \ll 1$. More precisely, we locally write the M_t as families of graphs and obtain the families for $t > 0$ by using the same data as for $t = 0$. Since the resulting discs depend $\mathcal{C}^{2,\beta}$-smoothly on all parameters including t, we get slightly deformed open sets $U_{m,t}$ attached to M_t such that functions holomorphic near M_t extend to $U_{m,t}$. If t is sufficiently close to 0, we get $z_0 \in U_{m,t}$, and hence that holomorphic functions extend from Ω^- to a uniform neighborhood of z_0, the desired contradiction. The proof of Theorem 7.1 is complete. $\square$

As an application, we revisit a special case of Theorem 1.2.

PROPOSITION 7.2. — *Let $\Omega \subset \mathbb{C}^n$, $n \geqslant 2$, be a domain with connected smooth boundary M. Assume that $X^+ = \mathsf{E}(\Omega^+)$ is univalent and that $\mathbb{C}^n \setminus \overline{X^+}$ is contained in M. Then we have $\mathbb{C}^n \setminus \overline{X^+} = M_{\mathsf{hol}}$. Moreover, M_{hol} is removable in the following sense: For every CR distribution $u \in \mathcal{D}'_{CR}(M \setminus M_{\mathsf{hol}})$, there is a function $\widetilde{u} \in \mathcal{O}(\Omega)$ which attains u as weak boundary value along $M \setminus M_{\mathsf{hol}}$.*

Proof. — Clearly M_{hol} is a proper subset of M, since otherwise Ω^+ would be Stein and coincide with X^+. Theorem 7.1 directly implies that M_{hol} and X^+ are disjoint. If $z \in M \setminus M_{\mathsf{hol}}$ we see like in the proof of Theorem 7.1 that holomorphic functions extend through z at least from one of the sides $\Omega^\pm$. Since X^+ is Stein and by assumption contains both sides, we conclude $z \in X^+$, and the first part of the proposition follows.

As for removability, we argue similarly as in the proof of Proposition 3.1: The complement of $M \setminus M_{\mathsf{hol}}$ in $X^+ = \mathbb{C}^n \setminus M_{\mathsf{hol}}$ has two connected components $\Omega^\pm$. Solving a suitable $\overline{\partial}$ equation on the pseudoconvex domain X^+, we find $f^\pm \in \mathcal{O}(\Omega^\pm)$ so that u is the jump between f^- and f^+ along $M \setminus M_{\mathsf{hol}}$, in the sense of weak boundary values. By assumption f^+ admits an extension $\widetilde{f}^+ \in \mathcal{O}(X^+)$, and $\widetilde{u} = \widetilde{f}^+|_\Omega - f^-$ is the desired extension of u. $\square$

We do not get more even if u is a CR distribution on M that is the global weak boundary value of a holomorphic function on Ω^+, as shown by

Example 7.3. — Let $G = \{\zeta \in \mathbb{C} : \rho(\zeta) < 0\} \Subset \mathbb{C}^1$ be a smoothly bounded disc such that $0 \in \partial G$ and $T_0 \partial G = \{\eta = \mathrm{Re}(\zeta) = 0\}$. In addition, we assume that G is strictly concave at 0 and that $\nabla \rho(0)$ is proportional

to $-\frac{\partial}{\partial\eta}$, meaning that all $\zeta \in \partial G \setminus \{0\}$ close to 0 are contained in $\{\eta < 0\}$. The unbounded domain

$$\Omega = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \rho\left(z_1 \exp\left(1 + |z_2|^2\right)\right) < 0 \right\}$$

has smooth connected boundary M homeomorphic to the cylinder $S^1 \times \mathbb{C}$. It is routine to verify that M decomposes into two CR orbits, the z_2-axis A and the open orbit $M \setminus A$. Applying the continuity principle to families of complex lines parallel to A shows that $\mathbb{C}^2 \setminus A$ is the envelope of $\Omega^+ = \mathbb{C}^2 \setminus \overline{\Omega}$. Thus Proposition 7.2 implies that every CR distribution defined on $M \setminus A$ has a holomorphic extension to Ω. The function

$$g(z) = \exp(1/z_1)|_{\Omega^+}$$

is holomorphic and locally bounded along $\partial\Omega^+$. In fact, g is continuous near $\partial\Omega^+ \setminus A$ and $|g| < 1$ holds near every $z \in A$. Hence its weak boundary value g^* is a CR function in $L^\infty_{loc}(M)$. Obviously the extension to Ω is $g_\Omega = \exp(1/z_1)|_\Omega$. Note that g_Ω does not have polynomial growth along A, meaning that g^* is not the weak boundary value of g_Ω along A.

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