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A CLASS FORMULA FOR L-SERIES IN POSITIVE CHARACTERISTIC

by Florent DEMESLAY

Abstract. — We prove a formula for special L-values of Anderson modules, analogue in positive characteristic of the class number formula. We apply this result to two kinds of L-series.

Résumé. — Nous prouvons une formule pour les valeurs spéciales des séries L associées aux modules d'Anderson, cette formule étant un analogue de la formule analytique du nombre de classes. Nous appliquons nos résultats à deux types de fonctions L.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements and θ an indeterminate over $\mathbb{F}_q$. We denote by A the polynomial ring $\mathbb{F}_q[\theta]$ and by K the fraction field of A. For a A-module M having a finite number of elements, we denote by $[M]_A$ the monic generator of the Fitting ideal of M. The *Carlitz zeta value* at a positive integer n is defined as

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty := \mathbb{F}_q((\theta^{-1})),$$

where A_+ is the set of monic polynomials of A.

The *Carlitz module* C is the functor that associates to an A-algebra B the A-module $C(B)$ whose underlying $\mathbb{F}_q$-vector space is B and whose A-module structure is given by the homomorphism of $\mathbb{F}_q$-algebras

$$\begin{aligned} \varphi_C \colon A &\longrightarrow \mathrm{End}_{\mathbb{F}_q}(B) \\ \theta &\longmapsto \theta + \tau, \end{aligned}$$

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where τ is the Frobenius endomorphism $b \mapsto b^q$. Similarly, we denote by $\mathrm{Lie}(C)$ the functor where the A-module structure is given by scalar multiplication. For P a prime of A (i.e. a monic irreducible polynomial), one can show (see [8, Theorem 3.6.3]) that $[C(A/PA)]_A = P - 1$. Thus

$$\zeta_A(1) = \prod_{P \text{ prime}} \left(1 - \frac{1}{P}\right)^{-1} = \prod_{P \text{ prime}} \frac{[\mathrm{Lie}(C)(A/PA)]_A}{[C(A/PA)]_A}. \tag{1.1}$$

Recently, Taelman [11] associates, to a Drinfeld module ϕ over the ring of integers R of a finite extension of K, a finite A-module called the *class module* $H(\phi/R)$ and an L-series value $L(\phi/R)$. In particular, if ϕ is the Carlitz module and R is A, thanks to (1.1), we have

$$L(C/A) = \zeta_A(1).$$

These objects are related by a *class formula*: $L(\phi/R)$ is equal to the product of $[H(\phi/R)]_A$ times a regulator (see [11, Theorem 1]).

This class formula was generalized by Fang [7], using the theory of shtukas and ideas of Vincent Lafforgue, to Anderson modules over A, which are n-dimensional analogues of Drinfeld modules. In particular, for $\mathrm{C}^{\otimes n}$, the n^{th} tensor power of the Carlitz module, introduced by Anderson and Thakur [2], we have

$$L(\mathrm{C}^{\otimes n}/A) = \zeta_A(n)$$

and this is related to a class module and a regulator as in the work of Taelman.

On the other hand, Pellarin [9] introduced a new class of L-series. Let $t_1, \dots, t_s$ be indeterminates over $\mathbb{C}_\infty$, the completion of a fixed algebraic closure of K_∞. For each $1 \leqslant i \leqslant s$, let $\chi_{t_i} : A \to \mathbb{F}_q[t_1, \dots, t_s]$ be the $\mathbb{F}_q$-linear ring homomorphism defined by $\chi_{t_i}(\theta) = t_i$. Then, *Pellarin's L-value* at a positive integer n is defined as

$$L(\chi_{t_1} \cdots \chi_{t_s}, n) := \sum_{a \in A_+} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{a^n} \in \mathbb{F}_q[t_1, \dots, t_s] \otimes_{\mathbb{F}_q} K_\infty.$$

In this paper, we prove that these series are naturally attached to some Anderson module (see Section 4.2) and that a class formula (Theorem 2.9) links these series to a class module *à la Taelman* [11]. Let us describe briefly our main result (Theorem 2.9).

Let L be a finite extension of K and $L_{s,\infty} := L \otimes_K \mathbb{F}_q(t_1, \dots, t_s)((\theta^{-1}))$. Let τ be the continuous $\mathbb{F}_q(t_1, \dots, t_s)$-endomorphism such that $\tau(x) = x^q$ for all $x \in L \otimes_K \mathbb{F}_q((\theta^{-1}))$. For all $n \geqslant 1$, we naturally extend τ in a $\mathbb{F}_q(t_1,\ldots,t_s)$-algebra endomorphism of $M_n(L_{s,\infty}) : \tau((a_{i,j})_{1\leqslant i,j\leqslant n}) := (\tau(a_{i,j})_{1\leqslant i,j\leqslant n}), a_{i,j} \in L_{s,\infty}$. We set

$$R_s := \mathbb{F}_q(t_1,\ldots,t_s)[\theta] \simeq A \otimes_{\mathbb{F}_q} \mathbb{F}_q(t_1,\ldots,t_s),$$

and let $R_{L,s}$ be the integral closure of R_s in $L(t_1,\ldots t_s)$ ($R_{L,s} \simeq \mathcal{O}_L \otimes_{\mathbb{F}_q} \mathbb{F}_q(t_1,\ldots,t_s)$ where $\mathcal{O}_L$ is the integral closure of A in L).

We recall that an Anderson t-module ψ is in particular a morphism of $\mathbb{F}_q$-algebras $A \to M_n(F)\{\tau\}$ where F is a $\mathbb{F}_q$-algebra equipped with a structure of A-module and where $\forall\ x \in F, \tau(x) = x^q$. In the case where $F = L$ is a finite extension of K and $\psi : A \to M_n(\mathcal{O}_L)\{\tau\}$, Taelman ([11]) and Fang ([7]) proved an "analytic class number formula" for its associated L-series. In this article, we will replace A by R_s, $\mathcal{O}_L$ by $R_{L,s}$, and we will be interested by a variant of Anderson modules and their associated L-series in this context. More precisely, let ϕ be an "Anderson module" defined on $R_{L,s}$, i.e. a morphism of $\mathbb{F}_q(t_1,\ldots,t_s)$-algebras $\phi : R_s \to M_n(R_{L,s})\{\tau\}$ for a certain integer n such that

$$\phi(\theta) \equiv \theta I_n + N_\phi \mod \tau, \text{ with } N_\phi \in M_n(R_{L,s}) \text{ verifying } N_\phi^n = 0.$$

If B is an $R_{L,s}$-algebra, we denote by $\phi(B)$ the $\mathbb{F}_q(t_1,\ldots,t_s)$-vector space B^n of column vectors with coefficients in B equipped with the R_s-module structure induced by ϕ. We also define $\mathrm{Lie}(\phi)(B)$ as the $\mathbb{F}_q(t_1,\ldots,t_s)$-vector space B^n whose R_s-module structure is given by

$$\theta \cdot b = (\theta I_n + N_\phi)b \quad \text{for all } b \in \mathrm{Lie}(\phi)(B).$$

According to the work of Taelman [11], we can associate to this object the infinite product

$$L(\phi/R_{L,s}) := \prod_{\mathfrak{m}} \frac{[\mathrm{Lie}(\phi)(R_{L,s}/\mathfrak{m}R_{L,s})]_{R_s}}{[\phi(R_{L,s}/\mathfrak{m}R_{L,s})]_{R_s}}$$

where $\mathfrak{m}$ runs through maximal ideals of $\mathcal{O}_L$, the integral closure of A in L and, if M is a finitely generated and torsion R_s-module, $[M]_{R_s}$ is the monic generator of the Fitting ideal of the R_s module M. This product converges to an element of $1 + \theta^{-1}\mathbb{F}_q(t_1,\ldots,t_s)((\theta^{-1}))$ (see Proposition 3.5).

For example, if $L = K$ and $\phi_\theta = \theta + (t_1 - \theta)\cdots(t_s - \theta)\tau$, we have (see Propositions 4.10 and 4.13)

$$L(\phi/R_s) = \sum_{a\in A_+} \frac{\chi_{t_1}(a)\cdots\chi_{t_s}(a)}{a^n}.$$

Thus, we recover L-series introduced by Pellarin in [9] and we have an equality in the manner of (1.1). The interest of these series, as they are in the Tate algebra in s indeterminates $t_1,\ldots,t_s$ with coefficients in $\mathbb{F}_q((\theta^{-1}))$, is that we can evaluate them specializing $t_1, \dots, t_s$ in elements of the algebraic closure of $\mathbb{F}_q$. Such specializations give us special values of Dirichlet–Goss L-series (see for example [4]).

Let us return to the general case and let ϕ be an Anderson module over $R_{L,s}$. There exists a unique series $\exp_\phi \in M_n(L(t_1, \dots, t_s))\{\{\tau\}\}$ such that

$$\exp_\phi(\theta I_n + N_\phi) = \phi(\theta) \exp_\phi .$$

Moreover, $\exp_\phi$ converges on $\mathrm{Lie}(\phi)(L_{s,\infty})$ (Proposition 2.5). Then, we set

$$U(\phi/R_{L,s}) := \left\{x \in \mathrm{Lie}(\phi)(L_{s,\infty}),\ \exp_\phi(x) \in \mathrm{Lie}(\phi)(R_{L,s})\right\}$$

$$\text{and} \quad H(\phi/R_{L,s}) := \frac{\mathrm{Lie}(\phi)(L_{s,\infty})}{\mathrm{Lie}(\phi)(R_{L,s}) + \exp_\phi\left(\mathrm{Lie}(\phi)(L_{s,\infty})\right)}.$$

We show that $U(\phi/R_{L,s})$ is an R_s-lattice in $L_{s,\infty}$ and that $H(\phi/R_{L,s})$ is a finitely generated R_s-module and a torsion R_s-module (Proposition 2.8). If $s = 0$, these objects coincide with unit module and class module introduced by Taelman in [11]. As $U(\phi/R_{L,s})$ and $R_{L,s}$ are two R_s-lattices in $L_{s,\infty}$, we can define a "regulator" (see Section 2.3)

$$[R_{L,s} : U(\phi/R_{L,s})]_{R_s} \in \mathbb{F}_q(t_1, \dots, t_s)((\theta^{-1}))^\times.$$

Inspired by ideas developed by Taelman in [11], we prove that we have the class formula

$$L(\phi/R_{L,s}) = [R_{L,s} : U(\phi/R_{L,s})]_{R_s}[H(\phi/R_{L,s})]_{R_s}.$$

In particular, for $s = 0$, we recover Theorem 1.10 of [7]. Note also that a weak version of this class formula play a significant role in [4]. We mention that one could work with a $\mathbb{F}_q$-algebra k instead of $\mathbb{F}_q(t_1, \dots, t_s)$, in that case one should replace R_s by $A \otimes_{\mathbb{F}_q} k$, $R_{L,s}$ by $\mathcal{O}_L \otimes_{\mathbb{F}_q} k$, $L_{s,\infty}$ by $(L \otimes_{\mathbb{F}_q} \mathbb{F}_q((\theta^{-1}))) \otimes_{\mathbb{F}_q} k$ and $\tau : L \otimes_{\mathbb{F}_q} \mathbb{F}_q((\theta^{-1})) \to L \otimes_{\mathbb{F}_q} \mathbb{F}_q((\theta^{-1})), x \mapsto x^q$ by $\tau \otimes 1$. However, for the arithmetic applications we had in mind, we have focused on the case $k = \mathbb{F}_q(t_1, \dots, t_s)$.

Finally, let $a \in A_+$ be squarefree and L be the cyclotomic field associated with a, i.e. the finite extension of K generated by the a-torsion of the Carlitz module. It is a Galois extension with group $\Delta_a \simeq (A/aA)^\times$. Let $\chi\colon (A/aA)^\times \to F^*$ be a homomorphism where F is a finite extension of $\mathbb{F}_q$. The special value at a positive integer n of Goss L-series associated to χ is defined as

$$L(n, \chi) := \sum_{b \in A_+} \frac{\chi(\bar{b})}{b^n} \in F \otimes_{\mathbb{F}_q} K_\infty,$$

where $\bar{b}$ is the image of b in $(A/aA)^\times$. Combining the techniques used to prove Theorem 2.9 and ideas developped in [5, Section 8], we give some new information on the arithmetic of the special values of these Dirichlet–Goss L-series $L(n,\chi)$. We can group all the $L(n,\chi)$ together in one equivariant L-value $L(n,\Delta_a)$. Then, we prove an equivariant class formula for these L-values (see Theorem 4.16), generalizing that of Anglès and Taelman [5] in the case $n=1$.

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2. Anderson modules and class formula

Let $\mathbb{F}_q$ be the finite field with q elements and θ an indeterminate over $\mathbb{F}_q$. We denote by A the polynomial ring $\mathbb{F}_q[\theta]$ and by K the fraction field of A. Let ∞ be the unique place of K which is a pole of θ and v_∞ the discrete valuation of K corresponding to this place with the normalization $v_\infty(\theta)=-1$. The completion of K at ∞ is denoted by K_∞. We have $K_\infty=\mathbb{F}_q((\theta^{-1}))$. We denote by $\mathbb{C}_\infty$ a fixed completion of an algebraic closure of K_∞. The valuation on $\mathbb{C}_\infty$ that extends v_∞ is still denoted by v_∞.

Let $s\geqslant 0$ be an integer and $t_1,\dots,t_s$ indeterminates over $\mathbb{C}_\infty$. We set $k_s:=\mathbb{F}_q(t_1,\dots,t_s)$, $R_s:=k_s[\theta]$, $K_s:=k_s(\theta)$ and $K_{s,\infty}:=k_s((\theta^{-1}))$. For $f\in\mathbb{C}_\infty[t_1,\dots,t_s]$ a polynomial expanded as a finite sum

$$f=\sum_{i_1,\dots,i_s\in\mathbb{N}}\alpha_{i_1,\dots,i_s}t_1^{i_1}\cdots t_s^{i_s},$$

with $\alpha_{i_1,\dots,i_s}\in\mathbb{C}_\infty$, we set

$$v_\infty(f):=\inf\bigl\{v_\infty(\alpha_{i_1,\dots,i_s})\bigm| i_1,\dots,i_s\in\mathbb{N}\bigr\}.$$

For $f\in\mathbb{C}_\infty(t_1,\dots,t_s)$, there exists g and h in $\mathbb{C}_\infty[t_1,\dots,t_s]$ such that $f=g/h$, then we define $v_\infty(f):=v_\infty(g)-v_\infty(h)$. We easily check that v_∞ is a valuation, trivial on k_s, called the *Gauss valuation*. For $f\in\mathbb{C}_\infty[t_1,\dots,t_s]$, we set $\|f\|_\infty:=q^{-v_\infty(f)}$ if $f\neq 0$ and $\|0\|_\infty=0$. The function $\|\cdot\|_\infty$ is called the *Gauss norm*.

We denote by $\mathbb{C}_{s,\infty}$ the completion of $\mathbb{C}_\infty(t_1,\dots,t_s)$ with respect to v_∞.

2.1. Lattices

Let k be a field of characteristic q and θ be an indeterminate over k. We set $R := k[\theta]$ and $F := k((\theta^{-1}))$. We equipped R with the discrete valuation v trivial on k and normalized such that $v(\theta) = -1$. This valuation extends naturally to F and, for $f \in F$, we set $|f| = q^{-v(f)}$ if $f \neq 0$ and $|0| = 0$.

Let V be a finite dimensional k-vector space and $\|\cdot\|$ be a norm on V compatible with $|\cdot|$ on F, i.e. : $\forall\, v \in V$, $\forall\, f \in F$, $\|fv\| = |f|\|v\|$. For $r > 0$, we denote by $B(0, r) := \{v \in V \mid \|v\| < r\}$ the open ball of radius r, which is a k-subspace of V.

Definition 2.1. — *A sub-R-module M of V is an R-*lattice *of V if it is free of rank n and the F-vector space spanned by M is V.*

We can characterize these lattices.

Lemma 2.2. — *Let V be a F-vector space of dimension $n \geqslant 1$ and M be a sub-R-module of V. The following assertions are equivalent:*

(1) *M is an R-lattice of V;*
(2) *M is discrete in V and every open subspace of the k-vector space V/M is of finite co-dimension.*

Proof. — Let us suppose that M is an R-lattice of V, i.e. there exists a family $(e_1, \dots, e_n)$ of elements of M such that

$$M = \bigoplus_{i=1}^{n} Re_i \quad \text{and} \quad V = \bigoplus_{i=1}^{n} Fe_i.$$

Any element v of V can be uniquely written as $v = \sum_{i=1}^{n} v_i e_i$ with $v_i \in F$. Then, we set $\|v\| := \max\{|v_i| \mid i = 1, \dots, n\}$. Since R is discrete in F, this implies that M is discrete in V. Now, let $m \geqslant 0$ be an integer. We have

$$B\left(0, q^{-m}\right) = \bigoplus_{i=1}^{n} \theta^{-m-1} k[\![\theta^{-1}]\!] e_i.$$

In particular, we have $V = M \oplus B(0, 1)$ and

$$\dim_k \frac{B(0, q^{-m})}{B(0, q^{-m-1})} = n.$$

This implies that every open k-subspace of V/M is of finite co-dimension.

Reciprocally, let us suppose that M is discrete in V and every open subspace of the k-vector space V/M is of finite co-dimension. Let W be the F-subspace of V generated by M and m be its dimension. There exist $e_1, \dots, e_m$ in M such that

$$W = \bigoplus_{i=1}^{m} F e_i.$$

Set

$$N = \bigoplus_{i=1}^{m} R e_i.$$

This is a sub-R-module of M and an R-lattice of W. In particular, M/N is discrete in W/N. Since any open k-subspace of W/N is of finite co-dimension, we deduce that M/N is a finite dimensional k-vector space. This implies that M is a finitely generated R-module, and therefore, since R is a principal ideal domain, we conclude that M is a free R-module of rank m. Finally, observe that, if $m < n$, V/M can not satisfy the co-dimensional property and thus $W = V$. □

In Section 2.3, we will introduce some R_s-lattices needed for the statement of the class formula.

2.2. Anderson modules and exponential map

Let L be a finite extension of K, $L \subseteq \mathbb{C}_\infty$. We define $R_{L,s}$ to be the subring of $L_s := L(t_1, \dots, t_s)$ generated by k_s and $\mathcal{O}_L$, where $\mathcal{O}_L$ is the integral closure of A in L. We set $L_{s,\infty} := L \otimes_K K_{s,\infty}$. This is a finite dimensional $K_{s,\infty}$-vector space. We denote by $S_\infty(L)$ the set of places of L above ∞. For a place $\nu \in S_\infty(L)$, we denote by L_ν the completion of L with respect to ν. Let π_ν be a uniformizer of L_ν and $\mathbb{F}_\nu$ be the residue field of L_ν. Then, we define $L_{s,\nu} := \mathbb{F}_\nu(t_1, \dots, t_s)((\pi_\nu))$ viewed as a subfield of $\mathbb{C}_{s,\infty}$. Let's observe that $L_{s,\nu}$ is the completion of L_s for the Gauss norm attached to ν. We have an isomorphism of $K_{s,\infty}$-algebras

$$L_{s,\infty} \simeq \prod_{\nu \in S_\infty(L)} L_{s,\nu}.$$

Observe that $R_{L,s}$ is an R_s-lattice in the $K_{s,\infty}$-vector space $L_{s,\infty}$.

Let $\tau \colon \mathbb{C}_{s,\infty} \to \mathbb{C}_{s,\infty}$ be the morphism of k_s-algebras given by the q-power map on $\mathbb{C}_\infty$.

Lemma 2.3. — *The elements of $\mathbb{C}_{s,\infty}$ fixed by τ are those of k_s.*

Proof. — Obviously, $k_s \subseteq \mathbb{C}_{s,\infty}^{\tau=1}$. Reciprocally, observe that $\mathbb{C}_{s,\infty}^{\tau=1} \subseteq \{f \in \mathbb{C}_{s,\infty} \mid v_\infty(f) = 0\}$. But we have the direct sum of $\mathbb{F}_q[\tau]$-modules

$$\{f \in \mathbb{C}_{s,\infty} \,|\, v_\infty(f) \geqslant 0\} = \overline{\mathbb{F}_q}(t_1, \dots, t_s) \oplus \{f \in \mathbb{C}_{s,\infty} \,|\, v_\infty(f) > 0\}.$$

Since $\overline{\mathbb{F}_q}(t_1, \dots, t_s)^{\tau=1} = k_s$, we get the result. □

The action of τ on $L_{s,\infty} = L \otimes_K K_{s,\infty}$ is the diagonal one $\tau \otimes \tau$.

As $R_s = k_s[\theta]$, a morphism of k_s-algebras is entirely defined by the image of θ.

Definition 2.4. — *Let r be a positive integer. An* Anderson module E *over $R_{L,s}$ is a morphism of k_s-algebras*

$$\phi_E \colon R_s \longrightarrow M_n(R_{L,s})\{\tau\}$$
$$\theta \longmapsto \sum_{j=0}^{r} A_j \tau^j$$

for some $A_0, \dots, A_r \in M_n(R_{L,s})$ such that $(A_0 - \theta I_n)^n = 0$.

These objects are usually called *abelian t-modules* as in the terminology of [1] but, to avoid confusion between t and the indeterminates $t_1, \dots, t_s$, we prefer called them Anderson modules. Note also that Drinfeld modules are Anderson modules with $n = 1$.

For a matrix $A=(a_{ij}) \in M_n(\mathbb{C}_{s,\infty})$, we set $v_\infty(A) := \min_{1 \leqslant i,j \leqslant n}\{v_\infty(a_{ij})\}$ and $\tau(A) := (\tau(a_{ij})) \in M_n(\mathbb{C}_{s,\infty})$.

Proposition 2.5. — *There exists a unique skew power series* $\exp_E := \sum_{j \geqslant 0} e_j \tau^j$ *with coefficients in $M_n(L_s)$ such that*

(1) $e_0 = I_n$;
(2) $\exp_E A_0 = \phi_E(\theta) \exp_E$ *in* $M_n(L_s)\{\{\tau\}\}$;
(3) $\lim_{j \to \infty} \frac{v_\infty(e_j)}{q^j} = +\infty$.

Proof. — See [1, Proposition 2.1.4]. □

Observe that $\exp_E$ is locally isometric. Indeed, by the third point,

$$c := \sup_{j \geqslant 1} \left(\frac{-v_\infty(e_j)}{q^j - 1} \right)$$

is finite. Then, for any $x \in L_{s,\infty}^n$ such that $v_\infty(x) > c$, we have

$$v_\infty \left(\sum_{j \geqslant 0} e_j \tau^j(x) - x \right) \geqslant \min_{j \geqslant 1} \big(v_\infty(e_j) + q^j v_\infty(x) \big) > v_\infty(x).$$

If B is an $R_{L,s}$-algebra together with a $\mathbb{F}_q(t_1, \dots, t_s)$-linear endomorphism $\tau_B : B \to B$ such that $\tau_B(rb) = \tau(r)\tau_B(b)$ for all $r \in R_{L,s}$ and $b \in B$.

We denote by $E(B)$ the k_s-vector space B^n equipped with the structure of R_s-module induced by ϕ_E. For example, if $n=1$ and $\phi(\theta)=\theta+\tau$, then the action of θ on B is given by $\theta.b=\theta b+\tau_B(b)$.

We can also consider the tangent space $\mathrm{Lie}(E)(B)$ which is the k_s-vector space B^n whose R_s-module structure is given by the morphism of k_s-algebras

$$\begin{aligned}\partial\colon R_s &\longrightarrow M_n(R_{L,s})\\ \theta &\longmapsto A_0.\end{aligned}$$

In particular, by the previous proposition, we get a continuous R_s-linear map

$$\exp_E\colon \mathrm{Lie}(E)(L_{s,\infty})\longrightarrow E(L_{s,\infty}).$$

2.3. The class formula

In this section, we define a class module and two lattices in order to state the main result.

LEMMA 2.6.

(1) $A_0^{q^n}=\theta^{q^n}I_n$;
(2) $\inf_{j\in\mathbb{Z}}\big(v_\infty(A_0^j)+j\big)$ *is finite.*

Proof. — See [7, Lemma 1.4]. □

By the second point, for any $a_j\in k_s$ and $m\in\mathbb{Z}$, the series $\sum_{j\geqslant m}a_jA_0^{-j}$ converges in $M_n(L_{s,\infty})$. Thus, ∂ can be uniquely extended to a morphism of k_s-algebras by

$$\begin{aligned}\partial\colon\qquad K_{s,\infty} &\longrightarrow M_n(L_{s,\infty})\\ \sum_{j\geqslant m}a_j\frac{1}{\theta^j} &\longmapsto \sum_{j\geqslant m}a_jA_0^{-j},\end{aligned}$$

where $a_j\in k_s$ and $m\in\mathbb{Z}$. Then, $\mathrm{Lie}(E)(L_{s,\infty})$ inherits a $K_{s,\infty}$-vector space structure. Observe, by the first point of the lemma, that, for any $f\in k_s((\theta^{-q^n}))$, we have $\partial(f)=fI_n$, i.e. the action is the scalar multiplication for these elements. In particular, we get an isomorphism $\mathrm{Lie}(E)(L_{s,\infty})\simeq L_{s,\infty}^n$ as $k_s((\theta^{-q^n}))$-modules. We deduce that $\mathrm{Lie}(E)(L_{s,\infty})$ is a $k_s((\theta^{-q^n}))$-vector space of dimension nq^n, so of dimension n over $K_{s,\infty}$.

PROPOSITION 2.7. — *The R_s-module* $\mathrm{Lie}(E)(R_{L,s})$ *is an R_s-lattice of* $\mathrm{Lie}(E)(L_{s,\infty})$.

Proof. — By the first point of the previous lemma, $\mathrm{Lie}(E)(R_{L,s})$ and $R^n_{L,s}$ are isomorphic as $k_s[\theta^{q^n}]$-modules. Thus, $\mathrm{Lie}(E)(R_{L,s})$ is a finitely generated $k_s[\theta^{q^n}]$-module. On the other hand, the action of an element $a \in R_s$ is the left multiplication by $aI_n + N$ where N is a nilpotent matrix. Since $aI_n + N$ is an invertible matrix, $\mathrm{Lie}(E)(R_{L,s})$ is a torsion-free R_s-module. Moreover, the $k_s((\theta^{-q^n}))$-vector space generated by $\mathrm{Lie}(E)(R_{L,s})$ and $K_{s,\infty}$ is $L^n_{s,\infty} \simeq \mathrm{Lie}(E)(L_{s,\infty})$. Therefore, $\mathrm{Lie}(E)(R_{L,s})$ is a free R_s-module of finite rank. Looking at the dimension as K_s-vector space, the rank is necessarily n. □

Proposition 2.8.

(1) *Set*

$$H(E/R_{L,s}) := \frac{E(L_{s,\infty})}{\exp_E(\mathrm{Lie}(E)(L_{s,\infty})) + E(R_{L,s})}.$$

This is a finite dimensional k_s-vector space, thus a finitely generated R_s-module and a torsion R_s-module, called the class module.

(2) *The R_s-module* $\exp_E^{-1}(E(R_{L,s}))$ *is an R_s-lattice in* $\mathrm{Lie}(E)(R_{L,s})$.

Proof. — Let V be an open neighbourhood of 0 in $L^n_{s,\infty}$ on which $\exp_E$ acts as an isometry and such that $\exp_E(V) = V$. We have a natural surjection of k_s-vector spaces

$$\frac{L^n_{s,\infty}}{R^n_{L,s} + V} \twoheadrightarrow H(E/R_{L,s}).$$

By Proposition 2.7, the left hand side is a finite dimensional k_s-vector space, hence *a fortiori* $H(E/R_{L,s})$ is as well.

Now, let us prove that $\exp_E^{-1}(E(R_{L,s}))$ is an R_s-lattice in $\mathrm{Lie}(E)(L_{s,\infty})$. Since the kernel of $\exp_E$ and $\mathrm{Lie}(E)(R_{L,s})$ are discrete in $\mathrm{Lie}(E)(L_{s,\infty})$, so is $\exp_E^{-1}(E(R_{L,s}))$. Let V be an open neighbourhood of 0 on which $\exp_E$ is isometric and such that $\exp_E(V) = V$. The exponential map induces a short exact sequence of k_s-vector spaces

$$0 \longrightarrow \frac{\mathrm{Lie}(E)(L_{s,\infty})}{\exp_E^{-1}(E(R_{L,s})) + V} \xrightarrow{\exp_E} \frac{E(L_{s,\infty})}{E(R_{L,s}) + V} \longrightarrow H(E/R_{L,s}) \longrightarrow 0.$$

Since the last two k_s-vector spaces are of finite dimension, the first one is of finite dimension too; thus $\exp_E^{-1}(E(R_{L,s}))$ satisfies the co-dimensional property. □

An element $f \in K_{s,\infty}$ is *monic* if

$$f = \frac{1}{\theta^m} + \sum_{i>m} x_i \frac{1}{\theta^i},$$

where $m \in \mathbb{Z}$ and $x_i \in k_s$. For an R_s-module M which is a finite dimensional k_s-vector space, we denote by $[M]_{R_s}$ the monic generator of the Fitting ideal of M.

Let V be a finite dimensional $K_{s,\infty}$-vector space. Let M_1 and M_2 be two R_s-lattices in V. There exists $\sigma \in \mathrm{GL}(V)$ such that $\sigma(M_1) = M_2$. Then, we define $[M_1 : M_2]_{R_s}$ to be the unique monic representative of $k_s^\times \det \sigma$.

The aim of the next section is to prove a class formula *à la Taelman* for Anderson modules:

Theorem 2.9. — *Let E be an Anderson module over $R_{L,s}$. The infinite product*

$$L(E/R_{L,s}) := \prod_{\substack{\mathfrak{m} \text{ maximal} \\ \text{ideal of } \mathcal{O}_L}} \frac{[\mathrm{Lie}(E)(R_{L,s}/\mathfrak{m}R_{L,s})]_{R_s}}{[E(R_{L,s}/\mathfrak{m}R_{L,s})]_{R_s}}$$

converges in $K_{s,\infty}$. Furthermore, we have

$$L(E/R_{L,s}) = [\mathrm{Lie}(E)(R_{L,s}) : \exp_E^{-1}(E(R_{L,s}))]_{R_s}[H(E/R_{L,s})]_{R_s}.$$

3. Proof of the class formula

The proof is very close to ideas developed by Taelman in [11] so we will only recall some statements and point out differencies.

3.1. Nuclear operators and determinants

Let k be a field and V a k-vector space equipped with a non-archimedean norm $\|\cdot\|$. Let φ be a continuous endomorphism of V. We say that φ is *locally contracting* if there exist an non empty open subspace $U \subseteq V$ and a real number $0 < c < 1$ such that $\|\varphi(u)\| \leqslant c\|u\|$ for all $u \in U$. Any such open subspace U which moreover satisfies $\varphi(U) \subseteq U$ is called a *nucleus* for φ. Observe that any finite collection of locally contracting endomorphisms of V has a common nucleus. Furthermore if φ and ϕ are locally contracting, then so are the sum $\varphi + \psi$ and the composition $\varphi\psi$.

For every positive integer N, we denote by $V[\![Z]\!]/Z^N$ the $k[\![Z]\!]/Z^N$-module $V \otimes_k k[\![Z]\!]/Z^N$ and by $V[\![Z]\!]$ the $k[\![Z]\!]$-module $V[\![Z]\!] := \varprojlim V[\![Z]\!]/Z^N$ equipped with the limit topology. Observe that any continuous $k[\![Z]\!]$-linear endomorphism $\Phi\colon V[\![Z]\!] \to V[\![Z]\!]$ is of the form

$$\Phi = \sum_{n \geqslant 0} \varphi_n Z^n,$$

where the φ_n are continuous endomorphisms of V. Similarly, any continuous $k[\![Z]\!]/Z^n$-linear endomorphism of $V[\![Z]\!]/Z^N$ is of the form

$$\sum_{n=0}^{N-1} \varphi_n Z^n.$$

We say that the continuous $k[\![Z]\!]$-linear endomorphism Φ of $V[\![Z]\!]$ (resp. of $V[\![Z]\!]/Z^N$) is *nuclear* if for all n (resp. for all $n < N$), the endomorphism φ_n of V is locally contracting.

From now on, we assume that for any open subspace U of V, the k-vector space V/U is of finite dimension.

Let Φ be a nuclear endomorphism of $V[\![Z]\!]/Z^N$. Let U_1 and U_2 be common nuclei for the φ_n, $n < N$. Since Proposition 8 in [11] is valid in our context,

$$\det_{k[\![Z]\!]/Z^N}\left(1 + \Phi \mid V/U_i \otimes_k k[\![Z]\!]/Z^N\right) \in k[\![Z]\!]/Z^N$$

is independent of $i \in \{1, 2\}$. We denote this determinant by

$$\det_{k[\![Z]\!]/Z^N}(1 + \Phi \mid V).$$

If Φ is a nuclear endomorphism of $V[\![Z]\!]$, then we denote by $\det_{k[\![Z]\!]}(1 + \Phi \mid V)$ the unique power series that reduces to $\det_{k[\![Z]\!]/Z^N}(1 + \Phi \mid V)$ modulo Z^N for every N.

Note that Proposition 9, Proposition 10, Theorem 2 and Corollary 1 of [11] are also valid in our context. We recall the statements for the convenience of the reader.

Proposition 3.1.

(1) *Let Φ be a nuclear endomorphism of $V[\![Z]\!]$. Let $W \subseteq V$ be a closed subspace such that $\Phi(W[\![Z]\!]) \subseteq W[\![Z]\!]$. Then Φ is nuclear on $W[\![Z]\!]$ and $(V/W)[\![Z]\!]$, and*

$$\det_{k[\![Z]\!]}(1 + \Phi \mid V) = \det_{k[\![Z]\!]}(1 + \Phi \mid W) \det_{k[\![Z]\!]}(1 + \Phi \mid V/W).$$

(2) *Let Φ and Ψ be nuclear endomorphisms of $V[\![Z]\!]$. Then $(1 + \Phi)(1 + \Psi) - 1$ is nuclear, and*

$$\det_{k[\![Z]\!]}((1 + \Phi)(1 + \Psi) \mid V) = \det_{k[\![Z]\!]}(1 + \Phi \mid V) \det_{k[\![Z]\!]}(1 + \Psi \mid V).$$

Theorem 3.2.

(1) *Let φ and ψ be continuous k-linear endomorphisms of V such that φ, $\varphi\psi$ and $\psi\varphi$ are locally contracting. Then*

$$\det_{k[\![Z]\!]}(1 + \varphi\psi Z \mid V) = \det_{k[\![Z]\!]}(1 + \psi\varphi Z \mid V).$$

(2) *Let $N \geqslant 1$ be an integer. Let φ and ψ be continuous k-linear endomorphisms of V such that all compositions φ, $\varphi\psi$, $\psi\varphi$, φ^2, etc. in φ and ψ, containing at least one endomorphism φ and at most $N-1$ endomorphisms ψ, are locally contracting. Let $\Delta = \sum_{n=1}^{N-1}\gamma_n Z^n$ such that*
$$1+\Delta = \frac{1-(1+\varphi)\psi Z}{1-\psi(1+\varphi)Z} \mod Z^N.$$
Then Δ is a nuclear endomorphism of $V[\![Z]\!]$ and
$$\det_{k[\![Z]\!]}(1+\Delta \mid V) = 1 \mod Z^N.$$

3.2. Taelman's trace formula

Let L be a finite extension of K and E be the Anderson module given by
$$\phi\colon R_s \longrightarrow M_n(R_{L,s})\{\tau\}$$
$$\theta \longmapsto \sum_{j=0}^{r} A_j\tau^j$$
for some $A_0,\dots,A_r \in M_n(R_{L,s})$ such that $(A_0-\theta I_n)^n = 0$. Let $M_n(R_{L,s})\{\tau\}[\![Z]\!]$ be the ring of formal power series in Z with coefficients in $M_n(R_{L,s})\{\tau\}$, the variable Z being central.

We set
$$\Theta := \sum_{n\geqslant 1}(\partial_\theta-\phi_\theta)\partial_\theta^{n-1}Z^n \in M_n(R_{L,s})\{\tau\}[\![Z]\!].$$

Lemma 3.3. — *Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{O}_L$. In $K_{s,\infty}$, the following equality holds:*
$$\frac{[\mathrm{Lie}(E)(R_{L,s}/\mathfrak{m}R_{L,s})]_{R_s}}{[E(R_{L,s}/\mathfrak{m}R_{L,s})]_{R_s}} = \det_{k_s[\![Z]\!]}\left(1+\Theta \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\right)^{-1}|_{Z=\theta^{-1}}.$$

Proof. — We have:
$$\Theta = (1-\phi_\theta Z)\frac{1}{1-\partial_\theta Z} - 1.$$
Furthermore:
$$\begin{aligned}&[\mathrm{Lie}(E)(R_{L,s}/\mathfrak{m}R_{L,s})]_{R_s}\\&\qquad= \det_{k_s[Z^{-1}]}\left(Z^{-1}-\partial_\theta \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\otimes_{k_s} k_s[Z^{-1}]\right)|_{Z^{-1}=\theta},\end{aligned}$$
$$\begin{aligned}&[E(R_{L,s}/\mathfrak{m}R_{L,s})]_{R_s}\\&\qquad= \det_{k_s[Z^{-1}]}\left(Z^{-1}-\phi_\theta \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\otimes_{k_s} k_s[Z^{-1}]\right)|_{Z^{-1}=\theta}.\end{aligned}$$

Now:

$$\frac{\det_{k_s[Z^{-1}]}\left(Z^{-1}-\partial_\theta \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\otimes_{k_s}k_s[Z^{-1}]\right)}{\det_{k_s[Z^{-1}]}\left(Z^{-1}-\phi_\theta \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\otimes_{k_s}k_s[Z^{-1}]\right)} = \frac{\det_{k_s[Z]}\left(1-\partial_\theta Z \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\otimes_{k_s}k_s[Z]\right)}{\det_{k_s[Z]}\left(1-\phi_\theta Z \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\otimes_{k_s}k_s[Z]\right)},$$

and:

$$\frac{\det_{k_s[Z]}\left(1-\partial_\theta Z \mid ((R_{L,s}/\mathfrak{m}R_{L,s})^n\otimes_{k_s}k_s[Z]\right)}{\det_{k_s[Z]}\left(1-\phi_\theta Z \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\otimes_{k_s}k_s[Z]\right)} = \det_{k_s[\![Z]\!]}\left(1+\Theta \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\right)^{-1}.$$

Thus:

$$\frac{\det_{k_s[Z^{-1}]}\left(Z^{-1}-\partial_\theta \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\otimes_{k_s}k_s[Z^{-1}]\right)|_{Z^{-1}=\theta}}{\det_{k_s[Z^{-1}]}\left(Z^{-1}-\phi_\theta \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\otimes_{k_s}k_s[Z^{-1}]\right)|_{Z^{-1}=\theta}} = \det_{k_s[\![Z]\!]}\left(1+\Theta \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\right)^{-1}|_{Z=\theta^{-1}}. \quad \square$$

Let S be a finite set of places of L containing $S_\infty(L)$. Denote by $\mathcal{O}_S$ the ring of functions regular outside S. In particular $\mathcal{O}_L\subseteq\mathcal{O}_S$. Let $R_{S,s}$ be the subring of L_s generated by $\mathcal{O}_S$ and k_s. For example, if $S=S_\infty(L)$, we have $R_{S,s}=R_{L,s}$.

Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}_L$ which is not in S. The natural inclusion $\mathcal{O}_L\hookrightarrow\mathcal{O}_S$ induces an isomorphism $R_{L,s}/\mathfrak{p}R_{L,s}\xrightarrow{\sim}R_{S,s}/\mathfrak{p}R_{S,s}$. By the previous lemma, we obtain

$$\frac{[\mathrm{Lie}(E)(R_{L,s}/\mathfrak{p}R_{L,s})]_{R_s}}{[E(R_{L,s}/\mathfrak{p}R_{L,s})]_{R_s}} = \det_{k_s[\![Z]\!]}\left(1+\Theta\,\middle|\,(R_{S,s}/\mathfrak{p}R_{S,s})^n\right)^{-1}\Big|_{Z=\theta^{-1}}. \tag{3.1}$$

Let $v_\mathfrak{p}$ de notes the $\mathfrak{p}$-adic Gauss valuation on $L[t_1,\dots,t_s]$, *i.e.* :

$$v_\mathfrak{p}\left(\sum_{i_1,\dots,i_s\in\mathbb{N}}\alpha_{i_1,\dots,i_s}t_1^{i_1}\cdots t_s^{i_s}\right) := \inf_{i_1,\dots,i_s\in\mathbb{N}}\{v_\mathfrak{p}(\alpha_{i_1,\dots,i_s})\},$$

where $v_\mathfrak{p}$ is the normalized $\mathfrak{p}$-adic valuation on L. Then $v_\mathfrak{p}$ extends to a valuation on L_s and we denote by $L_{s,\mathfrak{p}}$ the completion of L_s for the $\mathfrak{p}$-adic valuation $v_\mathfrak{p}$. Denote by $\mathcal{O}_{s,\mathfrak{p}}$ the valuation ring of $L_{s,\mathfrak{p}}$. By the strong approximation theorem, for any $n>0$, there exists $\pi_n\in L$ such that $v_\mathfrak{p}(\pi_n)=-n$ and $v(\pi_n)\geqslant 0$ for all $v\notin S\cup\mathfrak{p}$. Thus, we have

$$L_{s,\mathfrak{p}}=\mathcal{O}_{s,\mathfrak{p}}+R_{S\cup\{\mathfrak{p}\},s} \quad\text{and}\quad R_{S,s}=\mathcal{O}_{s,\mathfrak{p}}\cap R_{S\cup\{\mathfrak{p}\},s}. \tag{3.2}$$

Finally, denote by $L_{s,S}$ the product of the completions of L_s with respect to places of S. For example, if $S=S_\infty(L)$, we have $L_{s,S}=L_{s,\infty}$.

Recall that $R_{S,s}$ is a Dedekind domain, discrete in $L_{s,S}$ and such that every open subspace of $L_{s,S}/R_{S,s}$ is of finite co-dimension. Observe also that any element of $M_n(R_{S,s})\{\tau\}$ induces a continuous k_s-linear endomorphism of $(L_{s,S}/R_{S,s})^n$ which is locally contracting. In particular, the endomorphism Θ is a nuclear operator of $(L_{s,S}/R_{S,s})^n[\![Z]\!]$.

Lemma 3.4. — *Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}_L$ which is not in S. Then*

$$\det_{k_s[\![Z]\!]}\left(1+\Theta \mid (R_{S,s}/\mathfrak{p}R_{S,s})^n\right) = \frac{\det_{k_s[\![Z]\!]}\left(1+\Theta \,\middle|\, \left(\frac{L_{s,S}\times L_{s,\mathfrak{p}}}{R_{S\cup\{\mathfrak{p}\},s}}\right)^n\right)}{\det_{k_s[\![Z]\!]}\left(1+\Theta \,\middle|\, \left(\frac{L_{s,S}}{R_{S,s}}\right)^n\right)}.$$

Proof. — The proof is the same as that of Lemma 1 of [11], using equalities (3.2). □

Proposition 3.5. — *The following equality holds in $K_{s,\infty}$:*

$$L(E/R_{L,s}) = \det_{k_s[\![Z]\!]}\left(1+\Theta \mid (L_{s,\infty}/R_{L,s})^n\right)|_{Z=\theta^{-1}}.$$

In particular, $L(E/R_{L,s})$ converges in $K_{s,\infty}$.

Proof. — By Lemma 3.3, we have

$$L(E/R_{L,s}) = \prod_{\mathfrak{m}} \det_{k_s[\![Z]\!]}\left(1+\Theta \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\right)^{-1}|_{Z=\theta^{-1}},$$

where the product runs through maximal ideals of $\mathcal{O}_L$. Fix $S \supseteq S_\infty(L)$ as above (the case $S = S_\infty(L)$ suffices). By equality (3.1), we have

$$\prod_{\mathfrak{m}} \det_{k_s[\![Z]\!]}\left(1+\Theta \mid (R_{L,s}/\mathfrak{m}R_{L,s})^n\right)^{-1} = \prod_{\mathfrak{m}} \det_{k_s[\![Z]\!]}\left(1+\Theta \mid (R_{S,s}/\mathfrak{m}R_{S,s})^n\right)^{-1},$$

where the products run through maximal ideals of $\mathcal{O}_L$ which are not in S.

Define $S_{D,N}$ as in [11]. It suffices to prove that for any $1+F \in S_{D,N}$, the infinite product

$$\prod_{\mathfrak{m}\notin S\setminus S_\infty(L)} \det_{k_s[\![Z]\!]/Z^N}\left(1+F \,\middle|\, \left(\frac{R_{S,s}}{\mathfrak{m}R_{S,s}}\right)^n\right)$$

converges to

$$\det_{k_s[\![Z]\!]/Z^N}\left(1+F \,\middle|\, \left(\frac{L_{s,S}}{R_{S,s}}\right)^n\right)^{-1}.$$

Let $\mathfrak{m}_1,\ldots,\mathfrak{m}_r$ be the maximal ideals of $\mathcal{O}_L$ which are not in S and such that $\mathfrak{m}_iR_{S,s}$ is a maximal ideal of $R_{S,s}$ verifying $\dim_{k_s} R_{S,s}/\mathfrak{m}_iR_{S,s} < D$.

Applying successively Lemma 3.4 to $R_{S,s}$, $R_{S\cup\{\mathfrak{m}_1\},s}$, $R_{S\cup\{\mathfrak{m}_1,\mathfrak{m}_2\},s}$, etc., we obtain the following equality:

$$\det_{k_s[\![Z]\!]}\left(1+F\,\middle|\,\left(\frac{L_{s,S}}{R_{S,s}}\right)^n\right)\prod_{\mathfrak{m}}\det_{k_s[\![Z]\!]}\left(1+F\,\middle|\,\left(\frac{R_{S,s}}{\mathfrak{m}R_{S,s}}\right)^n\right)$$
$$=\det_{k_s[\![Z]\!]}\left(1+F\,\middle|\,\left(\frac{L_{s,S}\times L_{s,\mathfrak{m}_1}\times\cdots\times L_{s,\mathfrak{m}_r}}{R_{S\cup\{\mathfrak{m}_1,\ldots,\mathfrak{m}_r\},s}}\right)^n\right)$$
$$\times\prod_{\mathfrak{m}\neq\mathfrak{m}_1,\ldots,\mathfrak{m}_r}\det_{k_s[\![Z]\!]}\left(1+F\,\middle|\,\left(\frac{R_{S,s}}{\mathfrak{m}R_{S,s}}\right)^n\right).$$

This allows us, replacing $R_{S,s}$ by $R_{S\cup\{\mathfrak{m}_1,\ldots,\mathfrak{m}_r\},s}$, to suppose that $R_{S,s}$ has not maximal ideal of the form $\mathfrak{m}R_{S,s}$ with $\mathfrak{m}$ maximal ideal of $\mathcal{O}_L$ which is not in S such that $\dim_{k_s} R_{S,s}/\mathfrak{m}R_{S,s} < D$. Then, we can finish the proof as in [11]. □

3.3. Ratio of co-volumes

Let V be a finite dimensional $K_{s,\infty}$-vector space and $\|\cdot\|$ be a norm on V compatible with $\|\cdot\|_\infty$ on $K_{s,\infty}$. Let M_1 and M_2 be two R_s-lattices in V and $N\in\mathbb{N}$. A continuous k_s-linear map $\gamma\colon V/M_1\to V/M_2$ is *N-tangent to the identity* on V if there exists an open k_s-subspace U of V such that

(1) $U\cap M_1 = U\cap M_2=\{0\}$;
(2) γ restricts to an isometry between the images of U;
(3) for any $u\in U$, we have $\|\gamma(u)-u\|\leqslant q^{-N}\|u\|$.

The map γ is *infinitely tangent to the identity* on V if it is N-tangent for every positive integer N.

Proposition 3.6. — *Let $\gamma\in M_n(L_s)\{\{\tau\}\}$ be a power series convergent on $L_{s,\infty}^n$ with constant term equal to 1 and such that $\gamma(M_1)\subseteq M_2$. Then γ is infinitely tangent to the identity on $L_{s,\infty}^n$.*

Proof. — See [11, Proposition 12]. □

For example, by Proposition 2.5, the map

$$\exp_E\colon \frac{\mathrm{Lie}(E)(L_{s,\infty})}{\exp_E^{-1}(E(R_{L,s}))}\longrightarrow\frac{E(L_{s,\infty})}{E(R_{L,s})}$$

is infinitely tangent to the identity on $L_{s,\infty}^n$.

Now, let H_1 and H_2 two finite dimensional k_s-vector spaces which are also R_s-modules and set $N_i := \frac{V}{M_i}\times H_i$ for $i=1,2$. A k_s-linear map

$\gamma\colon N_1 \to N_2$ is *N-tangent* (resp. *infinitely tangent*) to the identity on V if the composition

$$\frac{V}{M_1} \hookrightarrow N_1 \xrightarrow{\gamma} N_2 \twoheadrightarrow \frac{V}{M_2}$$

is so. For a k_s-linear isomorphism $\gamma\colon N_1 \to N_2$, we define an endomorphism

$$\Delta_\gamma := \frac{1-\gamma^{-1}\partial_\theta \gamma Z}{1-\partial_\theta Z} - 1 = \sum_{i\geqslant 1} (\partial_\theta - \gamma^{-1}\partial_\theta\gamma)\partial^{n-1}Z^n$$

of $N_1[\![Z]\!]$.

Proposition 3.7. — *If γ is infinitely tangent to the identity on V, then Δ_γ is nuclear and*

$$\det_{k_s[\![Z]\!]}(1+\Delta_\gamma \mid N_1)|_{Z=\theta^{-1}} = [M_1 : M_2]_{R_s}\frac{[H_2]_{R_s}}{[H_1]_{R_s}}.$$

Proof. — See [11, Theorem 4]. ☐

3.4. Proof of Theorem 2.9

By Theorem 3.5, $L(E/R_{L,s})$ converges in $K_{s,\infty}$ and

$$L(E/R_{L,s}) = \det_{k_s[\![Z]\!]}\left(1+\Theta \mid (L_{s,\infty}/R_{L,s})^n\right)|_{Z=\theta^{-1}}.$$

The exponential map $\exp_E$ induces a short exact sequence of R_s-modules

$$0 \longrightarrow \frac{\mathrm{Lie}(E)(L_{s,\infty})}{\exp_E^{-1}(E(R_{L,s}))} \longrightarrow \frac{E(L_{s,\infty})}{E(R_{L,s})} \longrightarrow H(E/R_{L,s}) \longrightarrow 0.$$

By Proposition 2.8, the k_s-vector space $H(E/R_{L,s})$ is of finite dimension. Moreover, since the R_s-module on the left is divisible and R_s is principal, the sequence splits. The choice of a section gives rise to an isomorphism of R_s-modules

$$\frac{\mathrm{Lie}(E)(L_{s,\infty})}{\exp_E^{-1}(E(R_{L,s}))} \times H(E/R_{L,s}) \simeq \frac{E(L_{s,\infty})}{E(R_{L,s})}.$$

This isomorphism can be restricted to an isomorphism of k_s-vector spaces

$$\gamma\colon \frac{\mathrm{Lie}(E)(L_{s,\infty})}{\exp_E^{-1}(E(R_{L,s}))} \times H(E/R_{L,s}) \xrightarrow{\sim} \left(\frac{L_{s,\infty}}{R_{L,s}}\right)^n.$$

Observe that γ corresponds with the map induced by $\exp_E$. By Proposition 3.6, the map γ is infinitely tangent to the identity on $L_{s,\infty}^n$. By the second point of Proposition 2.5, we have $\exp_E \partial_\theta \exp_E^{-1} = \phi_\theta$, hence the equality of $k_s[\![Z]\!]$-linear endomorphisms of $\left(\frac{L_{s,\infty}}{R_{L,s}}\right)^n[\![Z]\!]$:

$$1+\Theta = \frac{1-\gamma\partial_\theta\gamma^{-1}Z}{1-\partial_\theta Z}.$$

Thus, by Proposition 3.7, we obtain

$$\begin{aligned}\det\nolimits_{k_s[\![Z]\!]}&(1+\Theta \mid (L_{s,\infty}/R_{L,s})^n)|_{Z=\theta^{-1}}\\ &= [\mathrm{Lie}(E)(R_{L,s}) : \exp_E^{-1}(E(R_{L,s}))]_{R_s}[H(E/R_{L,s})]_{R_s}.\end{aligned}$$

This concludes the proof.

4. Applications

4.1. The n^{th} tensor power of the Carlitz module

Let α be a non-zero element of R_s. Let E_α be the Anderson module defined by the morphism of k_s-algebras $\phi\colon R_s \to M_n(R_s)\{\tau\}$ given by

$$\phi_\theta = \partial_\theta + N_\alpha\tau,$$

where

$$\partial_\theta = \begin{pmatrix} \theta & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \theta \end{pmatrix} \quad \text{and} \quad N_\alpha = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & \vdots \\ \alpha & 0 & \cdots & 0 \end{pmatrix}.$$

In other words, if ${}^t(x_1,\dots,x_n) \in \mathbb{C}_{s,\infty}^n$, we have

$$\phi_\theta \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \theta x_1 + x_2 \\ \vdots \\ \theta x_{n-1} + x_n \\ \theta x_n + \alpha\tau(x_1) \end{pmatrix}.$$

The case $\alpha = 1$ is denoted by $\mathrm{C}^{\otimes n}$, the n^{th} tensor power of Carlitz module, introduced in [2]. In this section, we show that the exponential map associated to $\mathrm{C}^{\otimes n}$ is surjective on $\mathbb{C}_{s,\infty}^n$ and we recall its kernel.

4.1.1. Surjectivity and kernel of $\exp_{\mathrm{C}^{\otimes n}}$

By Proposition 2.5, there exists a unique exponential map $\exp_{\mathrm{C}^{\otimes n}}$ associated with $\mathrm{C}^{\otimes n}$ and by [2, Section 2], there exists a unique formal power series

$$\log_{\mathrm{C}^{\otimes n}} = \sum_{i \geqslant 0} P_i \tau^i \in M_n(\mathbb{C}_{s,\infty})\{\{\tau\}\}$$

such that $P_0 = I_n$ and $\log_{\mathrm{C}^{\otimes n}} \mathrm{C}_\theta^{\otimes n} = \partial_\theta \log_{\mathrm{C}^{\otimes n}}$. These two maps are inverses of each other, i.e. we have the equality of formal power series

$$\log_{\mathrm{C}^{\otimes n}} \exp_{\mathrm{C}^{\otimes n}} = \exp_{\mathrm{C}^{\otimes n}} \log_{\mathrm{C}^{\otimes n}} = I_n.$$

Furthermore, by [2, Proposition 2.4.2 and 2.4.3], the series $\exp_{\mathrm{C}^{\otimes n}}(f)$ converges for all $f \in \mathbb{C}_{s,\infty}^n$ and $\log_{\mathrm{C}^{\otimes n}}(f)$ for all $f = (f_1, \dots, f_n) \in \mathbb{C}_{s,\infty}^n$ such that $v_\infty(f_i) > n - i - \frac{nq}{q-1}$ for $1 \leqslant i \leqslant n$.

For an n-tuple $(r_1, \dots, r_n)$ of real numbers, we denote by $D_n(r_i, i = 1, \dots, n)$ the polydisc

$$\left\{ f \in \mathbb{C}_{s,\infty}^n \mid v_\infty(f_i) > r_i,\ i = 1, \dots, n \right\}.$$

Proposition 4.1. — *The exponential map $\exp_{\mathrm{C}^{\otimes n}}$ is surjective on $\mathbb{C}_{s,\infty}^n$.*

To prove this, we reduce to the one dimensional case.

Lemma 4.2. — *The following assertions are equivalent:*

(1) *$\exp_{\mathrm{C}^{\otimes n}}$ is surjective on $\mathbb{C}_{s,\infty}^n$;*
(2) *$\mathrm{C}_\theta^{\otimes n}$ is surjective on $\mathbb{C}_{s,\infty}^n$;*
(3) *$\tau - 1$ is surjective on $\mathbb{C}_{s,\infty}$.*

Proof. — It is easy to show that (1) implies (2). Indeed, let $y \in \mathbb{C}_{s,\infty}^n$. By hypothesis, there exists $x \in \mathbb{C}_{s,\infty}^n$ such that $\exp_{\mathrm{C}^{\otimes n}}(x) = y$. Hence we have

$$\mathrm{C}_\theta^{\otimes n} \exp_{\mathrm{C}^{\otimes n}}(\partial_\theta^{-1} x) = \exp_{\mathrm{C}^{\otimes n}}(x) = y.$$

Next we prove that (2) implies (3). Since $\mathrm{C}_\theta^{\otimes n}$ is supposed to be surjective on $\mathbb{C}_{s,\infty}^n$, for any $y = (y_1, \dots, y_n) \in \mathbb{C}_{s,\infty}^n$, there exists $x = (x_1, \dots, x_n) \in \mathbb{C}_{s,\infty}^n$ such that

$$\begin{cases} \theta x_1 + x_2 = y_1 \\ \qquad \vdots \\ \theta x_{n-1} + x_n = y_{n-1} \\ \theta x_n + \tau(x_1) = y_n. \end{cases}$$

In particular, we get

$$\tag{4.1} \tau(x_1) - (-\theta)^n x_1 = \sum_{i=1}^{n} (-\theta)^{n-i} y_i.$$

Thus $\tau - (-\theta)^n$ is surjective on $\mathbb{C}_{s,\infty}$. But we have

$$\tau\left((-\theta)^{\frac{n}{q-1}}\right) = (-\theta)^n (-\theta)^{\frac{n}{q-1}},$$

hence $\tau - 1$ is also surjective on $\mathbb{C}_{s,\infty}$.

In fact, it is also easy to check that (3) implies (2). As in the previous case, the surjectivity of $\tau - (-\theta)^n$ is deduced from the surjectivity of $\tau - 1$. Hence, for a fixed $y = (y_1, \dots, y_n) \in \mathbb{C}_{s,\infty}^n$, there exists $x_1 \in \mathbb{C}_{s,\infty}$ verifying equation (4.1). Then, by back-substitution, we find successively $x_2, \dots, x_n \in \mathbb{C}_{s,\infty}$ such that $x = (x_1, \dots, x_n)$ satisfies $\mathrm{C}_\theta^{\otimes n}(x) = y$.

We finally prove that (2) implies (1). Since $\log_{\mathrm{C}^{\otimes n}}$ converges on the polydisc $D_n(n - i - \frac{nq}{q-1}, i = 1, \dots, n)$ and $\exp_{\mathrm{C}^{\otimes n}} \log_{\mathrm{C}^{\otimes n}}$ is the identity map on it, this polydisc is included in the image of the exponential. We will "grow" this polydisc to show that $\exp_{\mathrm{C}^{\otimes n}}$ is surjective. For $i = 1, \dots, n$, we define

$$r_{0,i} := n - i - \frac{nq}{q-1} = -i - \frac{n}{q-1},$$

and for $k \geqslant 1$,

$$r_{k+1,i} = \begin{cases} r_{k,i+1} & \text{if } 1 \leqslant i \leqslant n-1 \\ q r_{k,1} & \text{if } i = n. \end{cases}$$

By induction, we prove that for any integer $k \geqslant 0$ and any $1 \leqslant i \leqslant n-1$,

$$\tag{4.2} r_{k,i+1} \leqslant r_{k,i} - 1.$$

We also prove that for any integer $k \geqslant 0$ and $i \in \{1, \dots, n\}$, we have $r_{k,i} \leqslant r_{0,i} - k$. In particular, for any $1 \leqslant i \leqslant n$, the sequence $(r_{k,i})$ tends to $-\infty$, i.e. the polydiscs $D_n(r_{k,i}, i = 1, \dots, n)$ cover $\mathbb{C}_{s,\infty}^n$. Thus, it suffices to show that $D_n(r_{k,i}, i = 1, \dots, n) \subseteq \operatorname{Im} \exp_{\mathrm{C}^{\otimes n}}$ for any integer $k \geqslant 0$.

The case $k = 0$, corresponding to the convergence domain of $\log_{\mathrm{C}^{\otimes n}}$, is already known. Let us suppose that $D_n(r_{k,i}, i = 1, \dots, n)$ is included in the image of $\exp_{\mathrm{C}^{\otimes n}}$ for an integer $k \geqslant 0$. Let y be an element of $D_n(r_{k+1,i}, i = 1, \dots, n) \setminus D_n(r_{k,i}, i = 1, \dots, n)$.

We claim that there exists $x \in D_n(r_{k,i}, i = 1, \dots, n)$ such that $\mathrm{C}_\theta^{\otimes n}(x) = y$.

Assume temporally this. Since $D_n(r_{k,i}, i = 1, \dots, n) \subseteq \operatorname{Im} \exp_{\mathrm{C}^{\otimes n}}$, there exists $z \in \mathbb{C}_{s,\infty}^n$ such that $\exp_{\mathrm{C}^{\otimes n}}(z) = x$. Thus

$$\exp_{\mathrm{C}^{\otimes n}}(\partial_\theta z) = \mathrm{C}_\theta^{\otimes n} \exp_{\mathrm{C}^{\otimes n}}(z) = \mathrm{C}_\theta^{\otimes n}(x) = y.$$

In particular y is in the image of the exponential as expected.

It only remains to prove the claim. By hypothesis, there exists $x = (x_1, \dots, x_n) \in \mathbb{C}_{s,\infty}^n$ such that

$$\begin{cases} x_2 = y_1 - \theta x_1 \\ \vdots \\ x_n = y_{n-1} - \theta x_n \\ \tau(x_1) - (-\theta)^n x_1 = \sum_{i=1}^{n} (-\theta)^{n-i} y_i. \end{cases}$$

We need to show that x is in $D_n(r_{k,i}, i = 1, \dots, n)$. Let begin by showing $v_\infty(x_1) > r_{k,1}$. If $v_\infty(x_1) = \frac{-n}{q-1}$, then $v_\infty(x_1) > r_{0,1} > r_{k,1}$. So we may suppose that $v_\infty(x_1) \neq \frac{-n}{q-1}$. Then

$$v_\infty(\tau(x_1) - (-\theta)^n x_1) = \min(q v_\infty(x_1) \ ; \ v_\infty(x_1) - n).$$

In particular,

$$\begin{aligned} q v_\infty(x_1) &\geqslant v_\infty\left(\sum_{i=1}^{n} (-\theta)^{n-i} y_i\right) \\ &\geqslant \min_{1 \leqslant i \leqslant n} (v_\infty(y_i) - n + i) > \min_{1 \leqslant i \leqslant n} (r_{k+1,i} - n + i), \end{aligned}$$

where the last inequality comes from the fact that y is in $D_n(r_{k+1,i}, i = 1, \dots, n)$. But, by the inequality (4.2), we have

$$r_{k+1,n} \leqslant r_{k+1,n-1} - 1 \leqslant \cdots \leqslant r_{k+1,1} - n + 1.$$

Hence we get

$$q v_\infty(x_1) > r_{k+1,n} = q r_{k,1},$$

as desired.

Finally, we show that $v_\infty(x_i) > r_{k,i}$ for all $2 \leqslant i \leqslant n$. Since $y \in D_n(r_{k+1,i}, i = 1, \dots, n)$, we have

$$v_\infty(x_2) \geqslant \min(v_\infty(y_1) \ ; \ v_\infty(x_1) - 1) > \min(r_{k+1,1} \ ; \ r_{k,1} - 1) = r_{k,2},$$

where the last equality comes from the definition of r_{k+1} and from inequality (4.2). On the same way, we obtain the others needed inequalities. $\square$

LEMMA 4.3. — *The application $\tau - 1 \colon \mathbb{C}_{s,\infty} \to \mathbb{C}_{s,\infty}$ is surjective.*

Proof. — Since $\sum_{i \geqslant 0} \tau^i(x)$ converges for $x \in \mathbb{C}_{s,\infty}$ such that $v_\infty(x) > 0$, we have

$$\{x \in \mathbb{C}_{s,\infty} \mid v_\infty(x) > 0\} \subseteq \mathrm{Im}(\tau - 1).$$

Thus, since $\mathbb{C}_\infty(t_1,\ldots,t_s)$ is dense in $\mathbb{C}_{s,\infty}$, it suffices to show that $\mathbb{C}_\infty(t_1,\ldots,t_s) \subseteq (\tau-1)(\mathbb{C}_{s,\infty})$. Observe that $(\tau-1)(\mathbb{C}_\infty[t_1,\ldots,t_s]) = \mathbb{C}_\infty[t_1,\ldots,t_s]$. Now let $f \in \mathbb{C}_\infty(t_1,\ldots,t_s)$. We can write

$$f = \frac{g}{h} \quad \text{with } g,h \in \mathbb{C}_\infty[t_1,\ldots,t_s] \text{ and } v_\infty(h) = 0.$$

Now write $h = \delta - z$ with $\delta \in \overline{\mathbb{F}_q}[t_1,\ldots,t_s] \setminus \{0\}$ and $z \in \mathbb{C}_\infty[t_1,\ldots,t_s]$ such that $v_\infty(z) > 0$. Then, in $\mathbb{C}_{s,\infty}$, we have

$$f = \frac{g}{h} = \sum_{k \geqslant 0} \frac{gz^k}{\delta^{k+1}}.$$

On the one hand, since the series converges, there exists $k_0 \in \mathbb{N}$ such that

$$v_\infty\left(\sum_{k \geqslant k_0} \frac{gz^k}{\delta^{k+1}}\right) > 0.$$

In particular, this sum is in the image of $\tau - 1$. On the other hand, we have

$$\sum_{k=0}^{k_0-1} \frac{gz^k}{\delta^{k+1}} \in \frac{1}{\delta^{k_0}} \mathbb{C}_\infty[t_1,\ldots,t_s].$$

But we can write $\frac{1}{\delta^{k_0}} = \frac{\beta}{\gamma}$ with $\beta \in \overline{\mathbb{F}_q}[t_1,\ldots,t_s]$ and $\gamma \in \mathbb{F}_q[t_1,\ldots,t_s] \setminus \{0\}$. Hence

$$\sum_{k=0}^{k_0-1} \frac{gz^k}{\delta^{k+1}} \in \frac{1}{\gamma}\mathbb{C}_\infty[t_1,\ldots,t_s] \subseteq (\tau-1)\left(\frac{1}{\gamma}\mathbb{C}_\infty[t_1,\ldots,t_s]\right).$$

Thus, by linearity of $\tau - 1$, we get $f \in (\tau-1)(\mathbb{C}_{s,\infty})$. □

Denote by Λ_n the kernel of the morphism of R_s-modules

$$\exp_{\mathrm{C}^{\otimes n}} \colon \operatorname{Lie}(\mathrm{C}^{\otimes n})(\mathbb{C}_{s,\infty}) \longrightarrow \mathrm{C}^{\otimes n}(\mathbb{C}_{s,\infty}).$$

Recall that the Carliz period $\widetilde{\pi}$ is defined as

$$\widetilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1} \in (-\theta)^{\frac{1}{q-1}} K_\infty,$$

where $(-\theta)^{\frac{1}{q-1}}$ is a choosen $(q-1)$-th root of $-\theta$.

Proposition 4.4. — *The R_s-module Λ_n is free of rank 1 and is generated by a vector with $\widetilde{\pi}^n$ as last coordinate.*

Proof. — See [2, Section 2.5]. □

4.1.2. Characterization of Anderson modules isomorphic to $\mathrm{C}^{\otimes n}$

We characterize Anderson modules which are isomorphic, in a sense described below, to the n^{th} tensor power of the Carlitz module. We obtain an n-dimensional analogue of Proposition 6.2 of [4].

DEFINITION 4.5. — *Two Anderson modules E and E' are* isomorphic *if there exists a matrix $P\in \mathrm{GL}_n(\mathbb{C}_{s,\infty})$ such that $E_\theta P = PE'_\theta$ in $M_n(\mathbb{C}_{s,\infty})\{\tau\}$.*

Let $\alpha \in R_s$. Denote by E_α the Anderson module defined at the beginning of Section 4.1. Note that E_α and $\mathrm{C}^{\otimes n}$ are isomorphic if and only if there exists a matrix $P \in \mathrm{GL}_n(\mathbb{C}_{s,\infty})$ such that

$$\partial_\theta P = P\partial_\theta \quad \text{and} \quad N_1\tau(P) = PN_\alpha. \tag{4.3}$$

Let us set

$$\mathcal{U}_s := \left\{\alpha \in \mathbb{C}_{s,\infty}^* \,\middle|\, \exists\, \beta \in \mathbb{C}_\infty^*, \gamma \in \overline{\mathbb{F}_q}(t_1,\dots,t_s), v_\infty\left(\alpha - \beta\frac{\tau(\gamma)}{\gamma}\right) > v_\infty(\alpha)\right\}.$$

LEMMA 4.6. — *The map which associates to any element x of $\mathbb{C}_{s,\infty}^*$ the element $\frac{\tau(x)}{x}$ of $\mathbb{C}_{s,\infty}^*$ induces a short exact sequence of multiplicative groups*

$$1 \longrightarrow k_s^* \longrightarrow \mathbb{C}_{s,\infty}^* \longrightarrow \mathcal{U}_s \longrightarrow 1.$$

Proof. — The kernel comes from Lemma 2.3.

Let $\alpha \in \mathbb{C}_{s,\infty}^*$ such that there exists $x \in \mathbb{C}_{s,\infty}^*$ verifying $\tau(x) = \alpha x$. Since $\mathbb{C}_\infty$ is an algebraically closed field, one can suppose that $v_\infty(\alpha) = 0$. We write $x = \gamma + m$ with $\gamma \in \overline{\mathbb{F}_q}(t_1,\dots,t_s)$ and $m \in \mathbb{C}_{s,\infty}^*$ such that $v_\infty(m) > 0$. Then, we have $v_\infty(\tau(\gamma) - \alpha\gamma) > 0$, i.e. $\alpha \in \mathcal{U}_s$.

Reciprocally, let $\alpha \in \mathcal{U}_s$ and $\beta \in \mathbb{C}_\infty^*$, $\gamma \in \overline{\mathbb{F}_q}(t_1,\dots,t_s)$ such that

$$v_\infty\left(\alpha - \beta\frac{\tau(\gamma)}{\gamma}\right) > v_\infty(\alpha).$$

We set $\delta := \beta\frac{\tau(\gamma)}{\gamma}$. Observe that $\prod_{i\geqslant 0} \frac{\tau^i(\delta)}{\tau^i(\alpha)}$ converges in $\mathbb{C}_{s,\infty}^*$. Now, since τ is k_s-linear, there exists $\varepsilon \in \mathbb{C}_\infty^*\overline{\mathbb{F}_q}(t_1,\dots,t_s)$ such that $\tau(\varepsilon) = \delta$. Then, we set

$$\omega_\alpha := \varepsilon \prod_{i\geqslant 0} \frac{\tau^i(\delta)}{\tau^i(\alpha)} \in \mathbb{C}_{s,\infty}^*. \tag{4.4}$$

Thus, we have $\tau(\omega_\alpha) = \alpha\omega_\alpha$. Observe that ω_α is defined up to a scalar factor in $\mathbb{F}_q^*$ whereas it depends *a priori* on the choices of β, γ and ε. $\square$

We are now able to characterize Anderson modules which are isomorphic to $\mathrm{C}^{\otimes n}$.

Proposition 4.7. — *Let $\alpha \in R_s$. The following assertions are equivalent:*

(1) *E_α is isomorphic to $C^{\otimes n}$,*
(2) *$\alpha \in \mathcal{U}_s$,*
(3) *$\exp_\alpha$ is surjective,*
(4) *$\ker \exp_\alpha$ is a free R_s-module of rank 1,*

where $\exp_\alpha$ is the exponential map associated with E_α by Proposition 2.5.

Proof. — Setting $P = \omega_\alpha I_n$ where ω_α is defined by (4.4), we see that (2) implies (1).

We prove that (1) implies (3). Let $P \in \mathrm{GL}_n(\mathbb{C}_{s,\infty})$ such that $C^{\otimes n}_\theta P = PE_\theta$. Using equalities (4.3), we check that

$$P^{-1} \exp_{C^{\otimes n}} P\partial_\theta = E_\theta P^{-1} \exp_{C^{\otimes n}} P.$$

Thus, by unicity in Proposition 2.5, we get $P^{-1} \exp_{C^{\otimes n}} P = \exp_\alpha$. In particular, by Proposition 4.1, we deduce that $\exp_\alpha$ is surjective.

Next, we prove that (3) implies (2). We can assume that $v_\infty(\alpha) = 0$. By Lemma 4.6, it suffices to show that $\ker(\alpha\tau - 1)$ is not trivial. Let us suppose the converse. As at the beginning of the proof of Lemma 4.2, we easily show that the surjectivity of $\exp_\alpha$ on $\mathbb{C}^n_{s,\infty}$ implies that of $\alpha\tau - 1$ on $\mathbb{C}_{s,\infty}$. Thus, $\alpha\tau - 1$ is an automorphism of the k_s-vector space $\mathbb{C}_{s,\infty}$. We verify that $v_\infty(f) = 0$ if and only if $v_\infty(\alpha\tau(f) - f) = 0$. Let $\bar{\alpha} \in \overline{\mathbb{F}_q}(t_1, \dots, t_s)$ such that $v_\infty(\alpha - \bar{\alpha}) > 0$. Then, $\bar{\alpha}\tau - 1$ is an automorphism of the k_s-vector space $\overline{\mathbb{F}_q}(t_1, \dots, t_s)$, which is obviously false.

It is easy to show that (1) implies (4). Indeed, since E_α is isomorphic to $C^{\otimes n}$, we have

$$\ker \exp_\alpha = \frac{1}{\omega_\alpha} \ker \exp_{C^{\otimes n}} .$$

Thus, by Proposition 4.4, $\ker \exp_\alpha$ is a free R_s-module of rank 1 generated by a vector with $\frac{\tilde{\pi}^n}{\omega_\alpha}$ as last coordinate.

Finally, we prove that (4) implies (2). Let f be a non zero element of $\ker \exp_\alpha$ such that $\partial_\theta^{-1} f \notin \ker \exp_\alpha$. Thus, the vector $g := \exp_\alpha(\partial_\theta^{-1} f) \in \mathbb{C}^n_{s,\infty}$ is non zero and $E_\theta(g) = 0$. Denote by $g_1, \dots, g_n$ its coordinates. We have

$$\begin{cases} \theta g_1 + g_2 = 0 \\ \vdots \\ \theta g_{n-1} + g_n = 0 \\ \theta g_n + \alpha\tau(g_1) = 0. \end{cases}$$

Since $g \neq 0$, we deduce that $g_i \neq 0$ for all $1 \leqslant i \leqslant n$. Summing, we obtain $\alpha\tau(g_1) - (-\theta)^n g_1 = 0$. Thus

$$\alpha\tau\left((-\theta)^{\frac{-n}{q-1}} g_1\right) = (-\theta)^{\frac{-n}{q-1}} g_1.$$

We conclude, by Lemma 4.6, that $\alpha \in \mathcal{U}_s$. □

Example. — Looking at the degree in t_1, we easily show that $t_1 \notin \mathcal{U}_s$. So E_{t_1} is not isomorphic to $\mathrm{C}^{\otimes n}$ and $\exp_{t_1}$ is not surjective.

4.2. Pellarin's L-functions

Let $\alpha \in R_s \backslash \{0\}$ and E_α be the Anderson module defined at the beginning of Section 4.1. By Theorem 2.9, we have a class formula for

$$L(E_\alpha/R_s) := \prod_{\substack{P \in A \\ \text{prime}}} \frac{[\mathrm{Lie}(E_\alpha)(R_s/PR_s)]_{R_s}}{[E_\alpha(R_s/PR_s)]_{R_s}}.$$

We compute the R_s-module structure of $\mathrm{Lie}(E_\alpha)(R_s/PR_s)$ and $E_\alpha(R_s/PR_s)$. Then, we show that we recover special values of Pellarin's L-functions if we take $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$.

4.2.1. Fitting ideal of $\mathrm{Lie}(E_\alpha)(R_s/PR_s)$

Let us recall some facts about hyperdifferential operators. For more details, we refer the reader to [6].

Let $j \geqslant 0$ be an integer. The j^{th} *hyperdifferential operator* D_j is the k_s-linear endomorphism of R_s given by $D_j(\theta^k) = \binom{k}{j}\theta^{k-j}$ for $k \geqslant 0$. For any $f, g \in R_s$, we have the Leibniz rule

$$D_j(fg) = \sum_{k=0}^{j} D_k(f) D_{j-k}(g).$$

Lemma 4.8. — *For any $a \in R_s$, we have*

$$\partial(a) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} D_{n-1}(a) \\ \vdots \\ D_1(a) \\ a \end{pmatrix}.$$

Proof. — By linearity, it suffices to prove the equality for $a = \theta^k$, $k \in \mathbb{N}$. The action of $\partial(\theta^k)$ is the left multiplication by

$$\begin{pmatrix} \theta & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \theta \end{pmatrix}^k = \left(\theta I_n + \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}\right)^k$$

$$= \sum_{i=0}^{k} \binom{k}{i} \theta^{k-i} \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}^i,$$

hence the result comes from the definition of hyperdifferential operators. □

Lemma 4.9. — *Let P be a prime of A and m a positive integer. Then $\partial(P^m)$ is zero modulo P if and only if m is greater than or equal to n.*

Proof. — By the previous lemma, it suffices to show that for any $k \geqslant 0$, the congruence $D_k(P^m) = 0 \mod P$ holds if and only if $m \geqslant k+1$. The case $k = 0$ being obvious, let us suppose the result for an integer k. By the Leibnitz rule, we have

$$\begin{aligned} D_{k+1}(P^m) &= \sum_{i+j=k+1} D_i(P^{m-1}) D_j(P) \\ &= P D_{k+1}(P^{m-1}) + D_1(P) D_k(P^{m-1}) + \cdots + D_{k+1}(P) P^{m-1}, \end{aligned}$$

which is zero modulo P if $m \geqslant k+2$. Reciprocally, observe that

$$\begin{aligned} D_{k+1}(P^{k+1}) &= P D_{k+1}(P^k) + D_1(P) D_k(P^k) \\ &\quad + D_2(P) D_{k-1}(P^k) + \cdots + D_{k+1}(P) P^k \\ &= D_1(P) D_k(P^k) \mod P \end{aligned}$$

which is non zero modulo P by hypothesis. □

Thanks to this lemma, we can compute the first Fitting ideal.

Proposition 4.10. — *Let P be a prime of A. The R_s-module $\mathrm{Lie}(E_\alpha)(R_s/PR_s)$ is isomorphic to $R_s/P^n R_s$ and is generated by the residue class of ${}^t(0, \ldots, 0, 1)$.*

Proof. — By definition, $\mathrm{Lie}(E_\alpha)(R_s/PR_s)$ is the k_s-vector space $(R_s/PR_s)^n$ equipped with the R_s-module structure given by ∂. This R_s-module is finitely generated and, since $\partial(P^{q^n}) = P^{q^n} I_n$ by Lemma 2.6, the polynomial P^{q^n} annihilates it. Since R_s is principal, by the structure theorem, there exists integers $e_1 \leqslant \cdots \leqslant e_m$ such that

$$\operatorname{Lie}(E_\alpha)(R_s/PR_s) \simeq \frac{R_s}{P^{e_1}R_s} \times \cdots \times \frac{R_s}{P^{e_m}R_s}.$$

Since $\operatorname{Lie}(E_\alpha)(R_s/PR_s)$ is a k_s-vector space of dimension $n \deg P$, we have $e_1 + \cdots + e_m = n$. But, by the previous lemma, the residue class of ${}^t(0,\ldots,0,1)$ is not annihilated by P^{n-1}, hence $e_m \geqslant n$. Thus, $\operatorname{Lie}(E_\alpha)(R_s/PR_s)$ is cyclic and generated by the residue class of this vector. □

4.2.2. Fitting ideal of $E_\alpha(R_s/PR_s)$

Let P be a prime of A and denote its degree by d. We consider $R := R_s/PR_s$ and $E_\alpha(R)$ the R_s-module R^n where the action of R_s is given by ϕ, as defined at the beginning of Section 4.1.

For $i = 1,\ldots,n$, we denote by $e_i\colon \mathbb{C}_{s,\infty}^n \to \mathbb{C}_{s,\infty}$ the projection on the i^{th} coordinate. By analogy with [2], we define the R_s-module

$$W_n(R) := \left\{ w \in R((t^{-1}))/R[t] \,\middle|\, \alpha\tau(w) = (t-\theta)^n w \mod R[t] \right\},$$

where $\tau(w) = \sum \tau(r_i)t^i$ if $w = \sum r_i t^i \in R((t^{-1}))$.

Proposition 4.11. — *The map*

$$\begin{aligned} \psi\colon E_\alpha(R) &\longrightarrow R((t^{-1}))/R[t] \\ c &\longmapsto -\sum_{i=1}^{\infty} e_1\phi_{\theta^{i-1}}(c)t^{-i} \end{aligned}$$

induces an isomorphism of R_s-modules between $E_\alpha(R)$ and $W_n(R)$.

Proof. — See [2, Proposition 1.5.1]. □

Observe that for any $c \in E_\alpha(R)$, we have $\psi(\phi_\theta(c)) = t\phi_\theta(c) \mod R[t]$. Moreover, since it is a k_s-vector space of dimension nd, $W_n(R)$ is a finitely generated and torsion $k_s[t]$-module.

For $w \in W_n(R)$, applying $d-1$ times $\alpha\tau$ to the relation $\alpha\tau(w) = (t-\theta)^n w$, we get

$$\alpha\tau(\alpha)\cdots\tau^{d-1}(\alpha)\tau^d(w) = \prod_{i=0}^{d-1}\left(t-\theta^{q^i}\right)^n w.$$

But $\tau^d(w) = w$ in $W_n(R)$ and $\prod_{i=0}^{d-1}(t-\theta^{q^i}) = P(t) \mod R[t]$ where $P(t)$ denotes the polynomial in t obtained substituting t form θ in P. Thus we obtain

$$P^n(t) - \alpha\tau(\alpha)\cdots\tau^{d-1}(\alpha) = 0 \text{ in } W_n(R). \tag{4.5}$$

Since we have the isomorphism

$$\frac{R_s}{PR_s} \simeq \frac{A}{PA} \otimes_{\mathbb{F}_q} k_s,$$

for any $x \in R_s$, there exists a unique $y \in k_s$ such that $x\tau(x)\cdots\tau^{d-1}(x) = y \mod PR_s$. We denote by $\rho_\alpha(P)$ the element of k_s such that $\rho_\alpha(P) = \alpha\tau(\alpha)\cdots\tau^{d-1}(\alpha) \bmod PR_s$. Note that, since P is prime, $\rho_\alpha(P) = 0 \bmod P$ if and only if P divides α in R_s. Then, by (4.5), we deduce that $W_n(R)$ is annihilated by $P^n(t) - \rho_\alpha(P)$, or equivalently

$$E_\alpha(R) \subseteq \ker \phi_{P^n - \rho_\alpha(P)} = \left\{x \in R^n \mid \phi_{P^n-\rho_\alpha(P)}(x) = 0\right\}. \tag{4.6}$$

Lemma 4.12. — *For any $a \in k_s[t]$ prime to $P(t) := P_{|\theta=t}$, the k_s-vector space $W_n(R)[a]$ of a-torsion points of $W_n(R)$ is of dimension at most $\deg_t a$.*

Proof. — By definition, we have

$$W_n(R)[a] = \left\{ w \in \frac{1}{a}R[t]/R[t] \,\middle|\, \alpha\tau(w) = (t-\theta)^n w \mod R[t] \right\}$$
$$\subseteq R((t^{-1}))/R[t].$$

Let $w \in W_n(R)[a]$. Since the t^i/a for $i \in \{0, \dots, \deg a - 1\}$ form an R-basis of $\frac{1}{a}R[t]/R[t]$, we can write

$$w = \sum_{i=0}^{\deg a - 1} \lambda_i \frac{t^i}{a},$$

where the λ_i are in R. Using the binomial formula and writing t^j/a for $j \geqslant \deg a$ in the above basis, the functional equation satisfied by w becomes

$$\sum_{i=0}^{\deg a-1} \alpha\tau(\lambda_i)\frac{t^i}{a} = \sum_{i=0}^{\deg a -1} \sum_j b_{i,j}\lambda_j \frac{t^i}{a},$$

where the $b_{i,j}$ are in R. Identifying the two sides, we obtain $\tau(\Lambda) = B\Lambda$ where Λ is the vector ${}^t(\lambda_0, \dots, \lambda_{\deg a - 1})$ and B is the matrix of $M_{\deg a}(R)$ with coefficients $b_{i,j}/\alpha$.

But the k_s-vector space $V := \left\{X \in R^{\deg a} \,\middle|\, \tau(X) = BX\right\}$ is of dimension at most $\deg a$. Indeed, observe that, if $v_1, \dots, v_m$ are vectors of $R^{\deg a}$ such that $\tau(v_i) = Bv_i$ for all $i \in \{1, \dots, m\}$, linearly independent over R, there are also linearly independent over $R^\tau = k_s$ (by induction on m, see [10, Lemma 1.7]). □

Proposition 4.13. — *Let P be a prime of A. We have the isomorphism of R_s-modules*

$$E_\alpha(R_s/PR_s) \simeq \frac{R_s}{(P^n - \rho_\alpha(P))R_s}.$$

Proof. — Recall that we denote R_s/PR_s by R. Observe that if P divides α, we have $\rho_\alpha(P) = 0$ and the isomorphism of R_s-modules $\mathrm{Lie}(E_\alpha)(R) \simeq E_\alpha(R)$. Then, the result is the same as in Proposition 4.10.

Hence, let us suppose that α and P are coprime. The k_s-vector space $E_\alpha(R)$ is of dimension nd. We deduce from Lemma 4.12 that $E_\alpha(R)$ is a cyclic R_s-module, i.e.

$$E_\alpha(R_s) \simeq \frac{R_s}{fR_s},$$

for some monic element f of R_s of degree nd. On the other hand, by the inclusion (4.6), $E_\alpha(R)$ is annihilated by $P^n - \rho_\alpha(P)$ thus f divides $P^n - \rho_\alpha(P)$. Since these two polynomials are monic and have the same degree, they are equal. □

4.2.3. L-values

Let a be a monic polynomial of A and $a = P_1^{e_1} \cdots P_r^{e_r}$ be its decomposition into a product of primes. Then, we define

$$\rho_\alpha(a) := \prod_{i=1}^{r} \rho_\alpha(P_i)^{e_i}.$$

By Propositions 4.10 and 4.13, we get

$$\begin{aligned} L(E_\alpha/R_s) = \prod_{\substack{P \in A \\ \text{prime}}} \frac{[\mathrm{Lie}(E_\alpha)(R_s/PR_s)]_{R_s}}{[E_\alpha(R_s/PR_s)]_{R_s}} &= \prod_{\substack{P \in A \\ \text{prime}}} \frac{P^n}{P^n - \rho_\alpha(P)} \\ &= \sum_{a \in A_+} \frac{\rho_\alpha(a)}{a^n} \in K_{s,\infty}. \end{aligned}$$

As in [4, Section 4.1], observe that for any prime P of A, $\rho_\alpha(P)$ is the resultant of P and α seen as polynomials in θ. In particular, if $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$, we obtain $\rho_\alpha(P) = P(t_1) \cdots P(t_s)$. Thus, by Theorem 2.9, we get a class formula for L-values introduced in [9]:

$$\begin{aligned} L(\chi_{t_1} \cdots \chi_{t_s}, n) &= \sum_{a \in A_+} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{a^n} \\ &= [\mathrm{Lie}(E_\alpha)(R_s) : \exp_E^{-1}(E_\alpha(R_s))]_{R_s} [H(E_\alpha/R_s)]_{R_s}, \end{aligned}$$

where $\chi_{t_i} \colon A \to \mathbb{F}_q[t_1, \ldots, t_s]$ are the ring homomorphisms defined respectively by $\chi_{t_i}(\theta) = t_i$.

4.3. Goss abelian L-series

This section is inspired by [5].

Let $a \in A_+$ be squarefree and L be the cyclotomic field associated with a, i.e. the finite extension of K generated by the a-torsion of the Carlitz module. We denote by Δ_a the Galois group of this extension, it is isomorphic to $(A/aA)^\times$.

Note that $A[\Delta_a] = \prod_i F_i[\theta]$ for some finite extensions F_i of $\mathbb{F}_q$. In particular, $A[\Delta_a]$ is a principal ideal domain and Fitting ideals are defined as usual. If M is a finite $A[\Delta_a]$-module, we denote by $[M]_{A[\Delta_a]}$ the unique generator f of $\mathrm{Fitt}_{A[\Delta_a]} M$ such that each component $f_i \in F_i[\theta]$ of f is monic.

We denote by $\widehat{\Delta}_a$ the group of characters of Δ_a, i.e. $\widehat{\Delta}_a = \hom(\Delta_a, \overline{\mathbb{F}_q}^\times)$. For $\chi \in \widehat{\Delta}_a$, we denote by $\mathbb{F}_q(\chi)$ the finite extension of $\mathbb{F}_q$ generated by the values of χ and we set

$$e_\chi := \frac{1}{\#\Delta_a} \sum_{\sigma \in \Delta_a} \chi^{-1}(\sigma)\sigma \in \mathbb{F}_q(\chi)[\Delta_a].$$

Then e_χ is idempotent and $\sigma e_\chi = \chi(\sigma) e_\chi$ for every $\sigma \in \Delta_a$.

Let F be the finite extension of $\mathbb{F}_q$ generated by the values of all characters, i.e. F is the compositum of all $\mathbb{F}_q(\chi)$ for $\chi \in \widehat{\Delta}_a$. If M is an $A[\Delta_a]$-module, we have the decomposition into χ-components

$$F \otimes_{\mathbb{F}_q} M = \bigoplus_{\chi \in \widehat{\Delta}_a} e_\chi \left(F \otimes_{\mathbb{F}_q} M\right).$$

Let V be a free $K_\infty[\Delta_a]$-module of rank n. A sub-$A[\Delta_a]$-module M of V is a *lattice* of V if M is free of rank one and $K_\infty[\Delta_a] \cdot M = V$. Let M be a lattice of V and $\chi \in \widehat{\Delta}_a$. Then $M(\chi) := e_\chi \left(\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} M\right)$ is a free $A(\chi)$-module of rank n, discrete in $V(\chi) := e_\chi \left(\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} V\right)$, where $A(\chi) := \mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} A$. Now let M_1 and M_2 be two lattices of V. For each $\chi \in \widehat{\Delta}_a$, there exists $\sigma_\chi \in \mathrm{GL}(V(\chi))$ such that $\sigma_\chi(M_1(\chi)) = M_2(\chi)$. Then, we define $[M_1(\chi) : M_2(\chi)]_{A(\chi)}$ to be the unique monic representative of $\det \sigma_\chi$ in $K_\infty(\chi) := \mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} K_\infty$. Finally, we set

$$[M_1 : M_2]_{A[\Delta_a]} := \sum_{\chi \in \widehat{\Delta}_a} [M_1(\chi) : M_2(\chi)]_{A(\chi)} e_\chi \in K_\infty[\Delta_a]^\times.$$

4.3.1. Gauss–Thakur sums

We review some basic facts on Gauss–Thakur sums, introduced in [12] and generalized in [3].

We begin with the case of only one prime. Let P be a prime of A of degree d and $\zeta_P \in \overline{\mathbb{F}_q}$ such that $P(\zeta_P) = 0$. We denote by Λ_P the P-torsion of the Carlitz module and let λ_P be a non zero element of Λ_P. We consider the cyclotomic extension $K_P := K(\Lambda_P) = K(\lambda_P)$ and we denote its Galois group by Δ_P. We have $\Delta_P \simeq (A/PA)^\times$. More precisely, if $b \in (A/PA)^\times$, the corresponding element $\sigma_b \in \Delta_P$ is uniquely determined by $\sigma_b(\lambda_P) = C_b(\lambda_P)$. We denote by $\mathcal{O}_{K_P}$ the integral closure of A in K_P. We have $\mathcal{O}_{K_P} = A[\lambda_P]$.

We define the *Teichmüller character*

$$\begin{aligned} \omega_P \colon \Delta_P &\longrightarrow \mathbb{F}_{q^d}^* \\ \sigma_b &\longmapsto b(\zeta_P), \end{aligned}$$

where σ_b is the unique element of Δ_P such that $\sigma_b(\lambda_P) = C_b(\lambda_P)$. Let $\chi \in \widehat{\Delta}_P$. Since the Teichmüller character generates $\widehat{\Delta}_P$, there exists $j \in \{0, \ldots, q^d - 2\}$ such that $\chi = \omega_P^j$. We expand $j = j_0 + j_1 q + \cdots + j_{d-1} q^{d-1}$ in base q ($j_0, \ldots, j_{d-1} \in \{0, \ldots, q-1\}$). Then, the *Gauss–Thakur sum* (see [12]) associated with χ is defined as

$$g(\chi) := \prod_{i=0}^{d-1} \left(- \sum_{\delta \in \Delta_P} \omega_P^{-q^i}(\delta)\delta(\lambda_P) \right)^{j_i} \in \mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} \mathcal{O}_{K_P}.$$

We compute the action of $\tau = 1 \otimes \tau$ on these Gauss–Thakur sums (see [12, proof of Theorem II]). Let $1 \leqslant j \leqslant d-1$. Since by the Carlitz action $\sigma_\theta \sigma_b(\lambda_P) = \theta\sigma_b(\lambda_P) + \tau(\sigma_b(\lambda_P))$, we have

$$\tau\left(g(\omega_P^{q^j})\right) = - \sum_{\sigma_b \in \Delta_P} \omega_P^{q^j}(\sigma_b) \left(\sigma_b\sigma_\theta(\lambda_P) - \theta\sigma_b(\lambda_P)\right).$$

Then, by substitution, we get

$$\tau\left(g(\omega_P^{q^j})\right) = \left(\zeta_P^{q^j} - \theta\right) g(\omega_P^{q^j}). \tag{4.7}$$

Now, we return to the general case. Since a is squarefree, we can write $a = P_1 \cdots P_r$ with $P_1, \ldots, P_r$ distinct primes of respective degrees $d_1, \ldots, d_r$. Since $\widehat{\Delta}_a \simeq \widehat{\Delta}_{P_1} \times \cdots \times \widehat{\Delta}_{P_r}$, for every character $\chi \in \widehat{\Delta}_a$, we have

$$\chi = \omega_{P_1}^{N_1} \cdots \omega_{P_r}^{N_r}, \tag{4.8}$$

for some integers $0 \leqslant N_i \leqslant q^{d_i} - 2$ and where ω_{P_i} is the Teichmüller character associated with P_i. The product $f_\chi := \prod_{N_i \neq 0} P_i$ is the *conductor* of χ. Then, the Gauss–Thakur sum (see [3, Section 2.3]) associated with χ is defined as

$$g(\chi) := \prod_{i=1}^{r} g(\omega_{P_i}^{N_i}) \in F \otimes_{\mathbb{F}_q} \mathcal{O}_L,$$

or equivalently

$$g(\chi) = \prod_{i=1}^{r} \prod_{j=0}^{d_i-1} g(\omega_{P_i}^{q^j})^{N_{i,j}},$$

where the $N_{i,j}$ are the q-adic digits of N_i. By equality (4.7), we obtain

$$\tau\left(g(\chi)\right) = \underbrace{\prod_{i=1}^{r} \prod_{j=0}^{d_i-1} \left(\zeta_{P_i}^{q^j} - \theta\right)^{N_{i,j}}}_{\alpha(\chi)} g(\chi). \tag{4.9}$$

Lemma 4.14. — *The ring $\mathcal{O}_L$ is a free $A[\Delta_a]$-module of rank one generated by $\eta_a := \sum_{\chi \in \widehat{\Delta}_a} g(\chi)$.*

Proof. — See [3, Lemma 16]. □

4.3.2. The Frobenius action on the χ-components

Recall that L is the extension of K generated by the a-torsion of the Carlitz module. Let $L_\infty := L \otimes_K K_\infty$ on which τ acts diagonally and Δ_a acts on L. As in Section 2.2, we have a morphism of $A[\Delta_a]$-modules

$$\exp_{\mathrm{C}^{\otimes n}} \colon \operatorname{Lie}(\mathrm{C}^{\otimes n})(L_\infty) \longrightarrow \mathrm{C}^{\otimes n}(L_\infty).$$

Let $\chi \in \widehat{\Delta}_a$. We get an induced map

$$\exp_{\mathrm{C}^{\otimes n}} \colon e_\chi \left(\operatorname{Lie}(\mathrm{C}^{\otimes n})(\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} L_\infty)\right) \longrightarrow \mathrm{C}^{\otimes n}\left(e_\chi(\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} L_\infty)\right),$$

where the action of τ on $\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} L_\infty$ is on the second component. But, by Lemma 4.14, we have

$$e_\chi(\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} L_\infty) = g(\chi) K_\infty(\chi),$$

where $K_\infty(\chi) := \mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} K_\infty$.

We have the obvious isomorphism of modules over $A(\chi) := \mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} A$

$$g(\chi) K_\infty(\chi) \xrightarrow{\sim} K_\infty(\chi),$$

where the action on the right hand side is denoted by $\widetilde{\tau}$ and given by $\widetilde{\tau}(f) = \alpha(\chi)(1 \otimes \tau)(f)$ for any $f \in K_\infty(\chi)$, where $\alpha(\chi)$ is defined by equality (4.9). In particular, this isomorphism maps $\mathrm{C}_\theta^{\otimes n}$ into $\partial_\theta + N_1 \widetilde{\tau} = \partial_\theta + N_{\alpha(\chi)}\tau$ with notation of Section 4.1 and $\exp_{\mathrm{C}^{\otimes n}}$ into $\exp_{\alpha(\chi)}$. Thus, by Lemma 4.14, we have the isomorphism of $A(\chi)$-modules

$$e_\chi \left(\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} H(\mathrm{C}^{\otimes n} / \mathcal{O}_L)\right) \simeq \frac{E_{\alpha(\chi)}(K_\infty(\chi))}{\exp_{\alpha(\chi)}\left(\operatorname{Lie}(E_{\alpha(\chi)})(K_\infty(\chi))\right) + E_{\alpha(\chi)}(A(\chi))}.$$

We denote the right hand side by $H\left(E_{\alpha(\chi)}/A(\chi)\right)$. Note that we have also

$$e_\chi\left(\mathbb{F}_q(\chi)\otimes_{\mathbb{F}_q}\exp^{-1}_{\mathrm{C}^{\otimes n}}(\mathrm{C}^{\otimes n}(\mathcal{O}_L))\right)=\exp^{-1}_{\alpha(\chi)}\left(E_{\alpha(\chi)}(A(\chi))\right).$$

4.3.3. L-values

Let $\chi\in\widehat{\Delta}_a$ and denote its conductor by f_χ. Recall that the special value at $n\geqslant 1$ of Goss L-series (see [8, Chapter 8]) associated with χ is defined by

$$L(n,\chi):=\sum_{b\in A_+}\frac{\chi(\sigma_b)}{b^n}\in K_\infty(\chi),$$

where the sum runs over the elements $b\in A_+$ relatively prime to f_χ. If $b\in A_+$ and f_χ are not coprime, we set $\chi(\sigma_b)=0$. Then, define the Goss abelian L-series

$$L(n,\Delta_a):=\sum_{\chi\in\widehat{\Delta}_a}L(n,\chi)e_\chi\in K_\infty[\Delta_a]^\times.$$

Lemma 4.15. — *The infinite product*

$$\prod_{\substack{P\in A\\ \text{prime}}}\frac{\left[\mathrm{Lie}(\mathrm{C}^{\otimes n})(\mathcal{O}_L/P\mathcal{O}_L)\right]_{A[\Delta_a]}}{\left[\mathrm{C}^{\otimes n}(\mathcal{O}_L/P\mathcal{O}_L)\right]_{A[\Delta_a]}}$$

converges in $K_\infty[\Delta_a]$ to $L(n,\Delta_a)$.

Proof. — On the one hand, for all $\chi\in\widehat{\Delta}_a$, we have

$$L(n,\chi)=\prod_{\substack{P\in A\\ \text{prime}}}\left(1-\frac{\chi(\sigma_P)}{P^n}\right)^{-1},$$

where $\chi(\sigma_P)=0$ if P divides f_χ. On the other hand, let $\chi\in\widehat{\Delta}_a$. We write $\chi=\omega_{P_1}^{N_1}\cdots\omega_{P_r}^{N_r}$ as in equality (4.8) and denote by $N_{i,j}$ the q-adic digits of N_i. Then, as in Section 4.2.2, we can prove that

$$\begin{aligned}\left[E_{\alpha(\chi)}(A(\chi)/PA(\chi))\right]_{A(\chi)}&=P^n-\prod_{i=1}^{r}\prod_{j=0}^{d_i-1}P\left(\zeta_{P_i}^{q^j}\right)^{N_{i,j}}\\&=P^n-\prod_{i=1}^{r}P(\zeta_{P_i})^{N_i}\\&=P^n-\chi(\sigma_P).\end{aligned}$$

Thus, we obtain

$$L(n,\chi)=\prod_{\substack{P\in A\\ \text{prime}}}\frac{\left[\mathrm{Lie}(E_{\alpha(\chi)})(A(\chi)/PA(\chi))\right]_{A(\chi)}}{\left[E_{\alpha(\chi)}(A(\chi)/PA(\chi))\right]_{A(\chi)}}.$$

Hence, we get the result by the discussion of Section 4.3.2 and definition of $L(n,\Delta_a)$. □

Finally, we obtain a generalization of Theorem A of [5]:

Theorem 4.16. — *Let $a\in A_+$ be squarefree and denote by L the extension of K generated by the a-torsion of the Carlitz module. In $K_\infty[\Delta_a]$, we have*

$$L(n,\Delta_a)=\left[\mathrm{Lie}(\mathrm{C}^{\otimes n})(\mathcal{O}_L):\exp_{\mathrm{C}^{\otimes n}}^{-1}(\mathrm{C}^{\otimes n}(\mathcal{O}_L))\right]_{A[\Delta_a]}\left[H(\mathrm{C}^{\otimes n}/\mathcal{O}_L)\right]_{A[\Delta_a]}.$$

Proof. — By the previous lemma, $L(n,\Delta_a)$ is expressed in terms of Anderson module and Fitting. Then, as in Proposition 3.5, we express $L(n,\Delta_a)$ as a determinant. The proof is similar but we deal with the χ-components $e_\chi(\mathbb{F}_q(\chi)\otimes_{\mathbb{F}_q}\mathcal{O}_L)$ for all $\chi\in\widehat{\Delta}_a$. Then, since $A[\Delta_a]$ is principal, we conclude as in Section 3.4. We refer to [5, Section 6.4] for more details. □

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