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UNIFORM PERFECTNESS FOR INTERVAL EXCHANGE TRANSFORMATIONS WITH OR WITHOUT FLIPS

by Nancy GUELMAN & Isabelle LIOUSSE (*)

ABSTRACT. — Let $\mathcal{G}$ be the group of all Interval Exchange Transformations. Results of Arnoux–Fathi, Sah and Vorobets state that $\mathcal{G}_0$ the subgroup of $\mathcal{G}$ generated by its commutators is simple. Arnoux proved that the group $\overline{\mathcal{G}}$ of all Interval Exchange Transformations with flips is simple.

We establish that the commutator length is at most 6 for any element of $\overline{\mathcal{G}}$. Moreover, we give conditions on $\mathcal{G}$ that guarantee that the commutator lengths of the elements of $\mathcal{G}_0$ are uniformly bounded, and in this case for any $g \in \mathcal{G}_0$ this length is at most 5.

RÉSUMÉ. — Soit $\mathcal{G}$ le groupe des échanges d'intervalles. Des résultats d'Arnoux–Fathi, Sah et Vorobets indiquent que $\mathcal{G}_0$ le sous-groupe de $\mathcal{G}$ engendré par ses commutateurs est simple. Arnoux prouve que le groupe $\overline{\mathcal{G}}$ des échanges d'intervalles avec flips est simple.

Nous établissons que tout élément de $\overline{\mathcal{G}}$ a une longueur des commutateur inférieure ou égale à 6. De plus, nous exhibons des conditions sur $\mathcal{G}$ qui garantissent que les longueurs des commutateurs des éléments de $\mathcal{G}_0$ sont uniformément bornées et dans ce cas pour tout $g \in \mathcal{G}_0$ nous montrons que cette longueur est au plus 5.

1. Introduction

Let $J = [a, b)$ be a half-open interval.

An *interval exchange transformation (IET)* of J is a right continuous bijective map $f : J \to J$ defined by a finite partition of J into half-open

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subintervals I_i and a reordering of these intervals by translations. We denote by $\mathcal{G}_J$ the group consisting in all IET of J.

An *interval exchange transformations with flips (FIET)* on J is a bijection $f : J \to J$ for which there exists a subdivision $a = a_1 < \cdots < a_m < a_{m+1} = b$ such that $f|_{(a_i,a_{i+1})}$ is a continuous isometry. Note that f is not necessarily right continuous since the orientation of some intervals can be reversed, and there exists a *flip-vector* $U(f) = (u_1, \ldots, u_m) \in \{-1, 1\}^n$ such that $f|_{(a_i,a_{i+1})}$ is a direct isometry if $u_i = 1$ or an indirect one if $u_i = -1$.

It is worth noting that several authors consider FIET as piecewise isometries of J with a finite number of discontinuity points reversing at least one of the intervals of continuity. This is not our case, so an IET is an FIET.

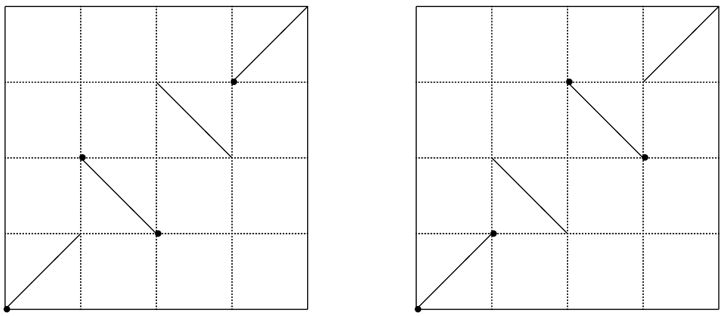


Figure 1.1. Two equivalent FIET.

We say that two FIET on J, f and g are equivalent if the set $\{x \in J : f(x) \neq g(x)\}$ is finite (see Figure 1.1). We denote by $\overline{\mathcal{G}}_J$ the corresponding quotient set. Note that $\overline{\mathcal{G}}_J$ is the quotient group of the FIET group by its normal subgroup consisting of elements which are trivial except possibly at finitely many points. By abuse of terminology, elements of $\overline{\mathcal{G}}_J$ are also called FIET. The map U is still well defined on $\overline{\mathcal{G}}_J$ and the set of all elements of $\overline{\mathcal{G}}_J$ such that $U(f) \subset \{1\}^m$ is identified with $\mathcal{G}_J$.

Let f be an IET or an FIET on J. The *continuity intervals* of f are the maximal connected subsets of J on which f is continuous and they are denoted by $I_1, \ldots, I_m$. The *associated permutation* $\pi = \pi(f) \in \mathcal{S}_m$ is defined by $f(I_i) = J\pi(i)$, where the J_j's are the ordered images of the I_i's. By convention, f is not continuous at the left endpoint of J and we define $BP(f)$ to be the set of the discontinuity points of f. Note that $BP(f^{-1}) = f(BP(f))$ and $BP(f \circ g) \subset BP(g) \cup g^{-1}(BP(f))$.

Let m be a positive integer, we denote by $\mathcal{G}_m$ (resp. $\mathcal{G}_{m,\pi}$) the set of all elements of $\mathcal{G}$ having at most m discontinuity points (resp. whose associated permutation is $\pi \in \mathcal{S}_m$).

From now on, without mention of the defining interval J, an IET or an FIET is defined on $I = [0,1)$ and $\mathcal{G}_I$ and $\overline{\mathcal{G}}_I$ will be denoted by $\mathcal{G}$ and $\overline{\mathcal{G}}$ respectively.

Remark 1.1. — The group $\mathcal{G}_J$ (resp. $\overline{\mathcal{G}}_J$) is conjugated by the direct homothecy that sends J to I to the group $\mathcal{G}$ (resp. $\overline{\mathcal{G}}$). The subgroup of $\mathcal{G}$ (resp. $\overline{\mathcal{G}}$) consisting of elements with support in J can be identified, by taking restriction, with $\mathcal{G}_J$ (resp. $\overline{\mathcal{G}}_J$).

Since the late seventies, the dynamics and the ergodic properties of a single interval exchange transformation were intensively studied (see e.g. Viana survey [36]). A natural extension is to consider the dynamics in terms of group actions. The most famous problem was raised by Katok: does $\mathcal{G}$ contain copies of $\mathbb{F}_2$, the free group of rank 2? Dahmani, Fujiwara and Guirardel established that such subgroups are rare ([7, Theorem 5.2]).

More generally, one can ask for a description of possible subgroups of $\mathcal{G}$. According to Novak *there is no distortion in* $\mathcal{G}$ ([27, Theorem 1.3]) and as a standard consequence *any finitely generated nilpotent subgroup of* $\mathcal{G}$ *is virtually abelian* (see e.g. [13]).

Among many things, Dahmani, Fujiwara and Guirardel proved that *any finitely generated subgroup of* $\mathcal{G}$ *is residually finite* ([7, Theorem 7.1]), $\mathcal{G}$ *contains no infinite Kazhdan groups* ([7, Theorem 6.2]), *any finitely generated torsion free solvable subgroup of* $\mathcal{G}$ *is virtually abelian* ([8, Theorem 3]) and provide examples of non virtually abelian solvable subgroups of $\mathcal{G}$ ([8, Theorem 6]). Thus, finding torsion free finitely generated non virtually abelian subgroups of $\mathcal{G}$ seems very difficult, especially as works of Juchenko, Monod ([18]) suggest that $\mathcal{G}$ could be amenable as it is conjectured by Cornulier.

The group $\mathcal{G}$ shares many of the properties of the group of piecewise affine increasing homeomorphisms of the unit interval, $\mathrm{PL}^+(I)$. For instance these two groups are not simple but have simple derivated subgroups (see [10] for the PL case). As noted in Remark 1.2 of [28], they satisfy no law (i.e. there does not exist $\omega \in \mathbb{F}_2 \setminus \{e\}$ such that $\phi(\omega) = \mathrm{Id}$ for every homomorphism $\phi : \mathbb{F}_2 \to G$), their nilpotent subgroups are virtually abelian (see [11] for the PL case) and the main result of [7] can be seen has a generic version of the Brin and Squier theorem [5] which asserts that $\mathrm{PL}^+(I)$ does not contain non abelian free subgroups. It's however still unknown whether $\mathrm{PL}^+(I)$ is amenable.

Considerably less is known about $\overline{\mathcal{G}}$. However, due to their connections with non oriented measured foliations on surfaces and billiards, the dynamics and ergodic properties of a single FIET were firstly explored by Gutierrez ([15]) and Arnoux ([2]).

Dealing with irreducible permutations, Keane ([19]) proved that almost all IET are minimal and Masur ([23]) and Veech ([35]) that almost all are uniquely ergodic. For FIET that reverse orientation in at least one interval, Nogueira ([25]) proved that almost all have periodic points so are nonergodic. He also exhibited an example of a minimal uniquely ergodic one (see also [16] and [22]). Recently, Skripchenko and Troubetzkoy gave bounds for the Hausdorff dimension of the set of minimal maps ([33]) and Hubert and Paris-Romaskevich described all the minimal maps having 4 continuity intervals (see [17, Theorem 6 p. 39–40]).

The decomposition into minimal and periodic components was first studied for measured surfaces flows by Mayer ([24]) and restated for IET by Arnoux ([1]) and Keane ([19]). For FIET that reverse orientation in at least one interval of the m continuity intervals, Nogueira, Pires and Troubetzkoy proved that the sum of number of periodic components and twice the number of minimal components is bounded by m ([26]).

As mentioned by Paris-Romaskevich in [30], *one can interest ourselves in the dynamics of FIET from the point of view of geometrical group theory*: describe the possible groups that can be realized as groups of FIET or establish algebraic properties of the whole group.

Here we shall be concerned solely with the structure of the whole groups $\mathcal{G}$ and $\overline{\mathcal{G}}$. This is also motivated by the algebraic study of other transformation groups, particularly groups of homeomorphisms of low dimensional manifolds, that was initiated by Schreier and Ulam in 1934 ([32]) who were interested in the simplicity of such groups.

We recall that, given G a group,

- a *commutator* in G is an element of G of the form $[f,g] = fgf^{-1}g^{-1}$ with $f, g \in G$.
- G is *perfect* if $G = [G,G]$ the subgroup of G generated by its commutators.
- G is *simple* if any normal subgroup of G is either G or trivial.

In the seventies, lots of homeomorphisms or diffeomorphisms groups were studied by Epstein, Herman, Thurston, Mather, Banyaga, and proved to be simple; these works are survey in the books [3] or [4]. For interval exchange transformations, it has been shown by Arnoux ([2, III §2.4]), Sah ([31]) and Vorobets ([37]) that the subgroup $\mathcal{G}_0$ of $\mathcal{G}$ generated by its commutators is simple. In [2, III §1.4], Arnoux proved that $\overline{\mathcal{G}}$ is simple, this unpublished result has been recently recovered by Lacourte ([21]). In order to sharpen this property, it is convenient to give

Definition 1.2. — *Let G be a group and $g \in [G,G]$, the* commutator length *of g, denoted by $c(g)$, is the least number c such that g is a product of c commutators. We set $c(G) = \sup\{c(g), g \in [G,G]\}$ and we say that G is* uniformly perfect *if $c(G)$ is finite.*

The main theorems of this paper are

Theorem 1.3. — $c(\overline{\mathcal{G}}) \leqslant 6$.

For the group $\mathcal{G}$ we are not able to decide if $c(\mathcal{G})$ is finite. However, in the affirmative case, we give an explicit bound in the following

Theorem 1.4. — *If $\mathcal{G}$ is uniformly perfect then $c(\mathcal{G}) \leqslant 5$.*

In Section 6, we will prove stronger results, Theorems 6.1 and 6.5, which only require that commutator lengths are bounded when prescribing the number of discontinuity points or the arithmetic, that is when the elements considered belong to $\Gamma_\alpha := \{g \in \mathcal{G} : BP(g) \subset \Delta_\alpha\}$, where Δ_α is the abelian subgroup of $\mathbb{R}$ generated by p real numbers $\alpha_1, \dots, \alpha_p$ and 1.

Our proofs are based on an adaptation of a result of Dennis and Vaserstein giving a criterion for uniform perfectness ([9]). This is explained in Section 4.

It is plain that a simple group G is generated by any subset S which is invariant under conjugation. In particular, S can be the set consisting of commutators, involutions or finite order elements. Group invariants are therefore provided by considering S-lengths that is the least number l_S such that any element can be written as product of l_S elements of S. These lengths can be simultaneously considered by using the following

Definition 1.5. — *A group G is* uniformly simple *if there exists a positive integer N such that for any $f, g \in G \setminus \{\mathrm{Id}\}$, the element g can be written as a product of at most N conjugates of f or f^{-1}.*

Ulam and Von Neumann ([34]) showed that the group of homeomorphisms of $\mathbb{S}^1$ is uniformly simple. Burago and Ivanov ([6]) obtained implicitely the same conclusion for $\mathrm{PL}^+(\mathbb{S}^1)$ and $[\mathrm{PL}^+(I), \mathrm{PL}^+(I)]$ (see also [12, Theorem 1.1]). The question of the uniform simplicity of $[\mathcal{G},\mathcal{G}]$ and $\overline{\mathcal{G}}$ is formulated in [30]. However, Cornulier communicated us that $[\mathcal{G},\mathcal{G}]$ and $\overline{\mathcal{G}}$ are not uniformly simple. Indeed, if the support of an IET or FIET f has length less than $\frac{1}{N}$ then any product of N conjugates of f or f^{-1} can not have full support.

In a forthcoming paper we will prove the uniform simplicity of $\mathcal{A}$, the group of affine interval exchange transformations of I, i.e. bijections $I \to I$ defined by a finite partition of I into half-open subintervals such that the restriction to each of these intervals is a direct affine map. More generally, we will give conditions on groups of piecewise continuous bijective maps on I that ensure uniform simplicity. This study and the work of [12] suggest that most simple transformation groups are uniformly simple. The group $\overline{\mathcal{G}}$ provides an example of a non uniformly simple group with bounded commutator length. As far as we know it is an open problem to determine whether $\overline{\mathcal{G}}$ has bounded involution length. Recently, O. Lacourte ([20]) defined the analogues of Γ_α in $\overline{\mathcal{G}}$, namely $\overline{\Gamma_\alpha}$. He proved that $[\overline{\Gamma_\alpha}, \overline{\Gamma_\alpha}]$ are simple and it would be relevant to study their uniform perfectness.

Acknowledgements

We thank E. Ghys for communicating us, a long time ago, how Proposition 1 of [9] allows to conclude that certain groups of homeomorphisms are uniformly perfect. We thank Y. Cornulier and O. Lacourte for fruitful discussions. We also thank the anonymous referee for several comments which improved the clarity of the paper.

2. Preliminaries

The aim of this section is to fix notations and terminology, to collect a few results and to prove some basic results to be used in the sequel.

2.1. Restricted rotations and periodic IET

Definition 2.1. — *An IET with two continuity intervals is called a* rotation *and it is denoted by R_a, where a is the image of* 0.

An IET g whose support, $\mathrm{supp}(g) = \{x \in I : g(x) \neq x\}$, *is $J = [a, b) \subset [0, 1)$ is a* restricted rotation *if the direct homothecy that sends J to $[0, 1)$ conjugates $g_{|J}$ to a rotation. We denote it by $R_{\alpha,J}$ where α is given by $R_{\alpha,J}(x) = x + \alpha \pmod{|b-a|}$ for $x \in J$.*

An element g of $\mathcal{G}$ (resp. $\overline{\mathcal{G}}$) is periodic *if every g-orbit is finite.*

By [2, III p. 3], [27, Lemma 6.5] or [37, Lemma 2.1], any interval exchange transformation is a product of restricted rotations (see also our Lemma 3.3 for a proof). For periodic IET, Novak showed a sharper statement.

Lemma 2.2 ([27, Proof of Corollary 5.6]). — *Any periodic element g of $\mathcal{G}$ is conjugated in $\mathcal{G}$ to a product of finite order restricted rotations with disjoint supports. In particular, any periodic IET has finite order.*

2.2. Basic properties on commutators

Definition 2.3. — *Let G be a group. An element $a \in G$ is* reversible *in G if there exists $h \in G$ such that $a = ha^{-1}h^{-1}$.*

Properties 2.4. — *Let G be a group and let a, b, a', b' and h be elements of G.*

(1) *If a and b commute with both a' and b' then $[a,b][a',b'] = [aa', bb']$.*
(2) *If $a' = ha^{-1}h^{-1}$ then $aa' = [a,h]$.*
(3) *If a is reversible in G then a^2 is a commutator.*
(4) $h[a,b]h^{-1} = [hah^{-1}, hbh^{-1}]$.

Proof.

(1). — As a' commutes with a and b, we have

$$[a,b][a',b'] = aba^{-1}b^{-1}a'b'a'^{-1}b'^{-1} = aa'ba^{-1}b^{-1}b'a'^{-1}b'^{-1}.$$

Repeating this process with b' and then a'^{-1}, we get

$$[a,b][a',b'] = aa'bb'a^{-1}a'^{-1}b^{-1}b'^{-1} = aa'bb'a'^{-1}a^{-1}b'^{-1}b^{-1} = [aa', bb'].$$

(2). — $aa' = aha^{-1}h^{-1} = [a,h]$.

(3). — $a^2 = aha^{-1}h^{-1} = [a,h]$.

(4). — $[hah^{-1}, hbh^{-1}] = hah^{-1}hbh^{-1}ha^{-1}h^{-1}hb^{-1}h^{-1} = h[a,b]h^{-1}$. □

2.3. Periodic IET are commutators

In [14, Theorem 4], the authors proved that any periodic IET is reversible in $\mathcal{G}$. We recall this argument briefly. Let f be a periodic IET, by the Arnoux Decomposition Theorem (see [2, Proposition p. 20]), the interval $[0,1)$ can be written as the union of finitely many f-periodic components M_i, $i = 1, \ldots, n$ of period p_i. In particular, $M_i = \bigsqcup_{k=1}^{p_i} J_k$, where $J_k = f^{k-1}(J_1)$ are half-open intervals and f is continuous on J_k.

Eventually conjugating f by an IET, we can suppose that the J_k's are ordered consecutive intervals so the M_i's are intervals and $\pi = \pi(f|_{M_i}) = (1, 2, \ldots, p_i)$.

We consider the IET h, that is defined on each M_i by h is continuous on J_k and $h(J_k) = J_{\tau(k)}$, where $\tau \in \mathcal{S}_{p_i}$ satisfies $\tau^{-1}\pi\tau(k) = \pi^{-1}$ (such a permutation exists by Proposition 3.4 of [29]). One has that $h^{-1} \circ f \circ h$ is continuous on J_k and $h^{-1} \circ f \circ h(J_k) = J_{\tau^{-1}\pi\tau(k)} = J_{\pi^{-1}(k)}$. Therefore $h^{-1} \circ f \circ h = f^{-1}$, meaning that f is reversible in $\mathcal{G}$.

This implies

Proposition 2.5. — *Any periodic IET is a commutator in $\mathcal{G}$.*

Indeed, we claim that any periodic IET can be written as the square of another periodic element. To see this, it is enough to consider rotations, by Lemma 2.2. This is obvious since $R_\alpha = R^2_{\frac{\alpha}{2}}$, so any periodic IET is the square of a reversible IET. Finally, the result follows from Properties 2.4 (3).

3. Generalities on commutators in $\bar{\mathcal{G}}$ and $\mathcal{G}$

3.1. Commutators in $\bar{\mathcal{G}}$

3.1.1. Fundamental examples

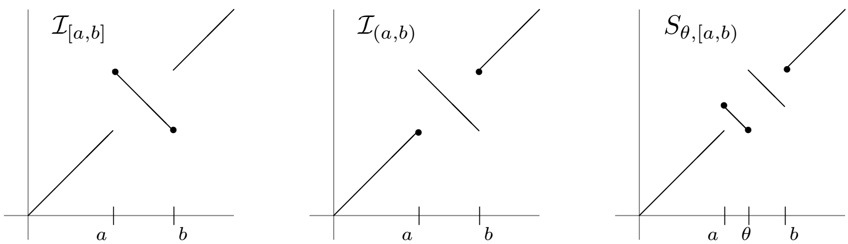
Let a, b satisfy that $0 \leqslant a < b \leqslant 1$.

We denote by $\mathcal{I}_{[a,b]}$ the symmetry of $[a, b]$, i.e. the FIET defined by:

$$\mathcal{I}_{[a,b]}(x) = x \text{ if } x \notin [a, b] \text{ and } \mathcal{I}_{[a,b]}(x) = a + b - x \text{ if } x \in [a, b].$$

Similarly, we denote by $\mathcal{I}_{(a,b)}$ the symmetry of (a, b), i.e. the FIET defined by:

$$\mathcal{I}_{(a,b)}(x) = \begin{cases} x & \text{if } x \notin (a, b) \\ a + b - x & \text{if } x \in (a, b). \end{cases}$$



Clearly, $\mathcal{I}_{[a,b]}$ and $\mathcal{I}_{(a,b)}$ are involutions and they represent the same element of $\overline{\mathcal{G}}$. Therefore, given J a subinterval of I, we define $\mathcal{I}_J$ to be the element of $\overline{\mathcal{G}}$ represented by $\mathcal{I}_{\overline{J}}$.

Let $\theta \in [a,b)$, we define another involution $S_{\theta,[a,b)}$ on $[a,b)$ by

$$S_{\theta,[a,b)} = \mathcal{I}_{[a,\theta]} \circ \mathcal{I}_{(\theta,b)}.$$

In particular, $S_{\theta,[0,1)} = \theta - x \pmod 1$ and it is denoted by S_θ.

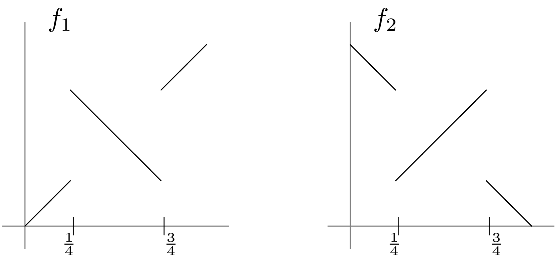
Property 3.1.

(1) $S_\theta \circ S_{\theta'} = R_{\theta-\theta'}$.
(2) $R_\alpha \circ S_\theta \circ R_\alpha^{-1} = S_{\theta+2\alpha}$.

Lemma 3.2 ([2, III p. 3]). — *The maps $\mathcal{I}_{(a,b)}$ and $R_{\alpha,[a,b)}$ are commutators in $\overline{\mathcal{G}}_{[a,b)}$ and then in $\overline{\mathcal{G}}$.*

Proof. — Taking restrictions and conjugating by a homothecy as in Remark 1.1, it is sufficient to prove that $\mathcal{I}_{(0,1)}$ et $R_{\alpha,[0,1)}$ are commutators.

It is easy to see that $\mathcal{I}_{(0,1)}$ is the product of the involutions f_1 and f_2 described as below:



As f_2 is conjugated to $f_1 = f_1^{-1}$ by $R_{\frac{1}{2}}$, Item (2) of Properties 2.4 implies that the map $\mathcal{I}_{(0,1)}$ is a commutator.

According to Property 3.1, any rotation is the product of 2 symmetries that are conjugated by a rotation; thus $R_{\alpha,[0,1)}$ is a commutator. □

3.1.2. Decomposition in involutions and restricted rotations

Lemma 3.3 ([2, III p. 3]).

(1) *Any $f \in \overline{\mathcal{G}}$ can be written as the product of an element of $\mathcal{G}$ and an involution that is a commutator.*
(2) *Any $g \in \mathcal{G}_m$ can be written as the product of $m-1$ restricted rotations.*

Proof.

(1). — Let $f \in \overline{\mathcal{G}}$, we denote by I_i its continuity intervals and by $U = (u_i)$ its flip-vector. It is easy to check that $f \circ \prod_{\{i|u_i=-1\}} \mathcal{I}_{I_i}$ belongs to $\mathcal{G}$. Moreover the $\mathcal{I}_{I_i}$'s have disjoint supports, so they commute and then $\prod_i \mathcal{I}_{I_i}$ is an involution and a commutator by Lemma 3.2 and Properties 2.4.

(2). — For clarity, given $J = [a, b)$ and $K = [b, c)$ two consecutive half-open intervals, we denote by $R_{J,K}$ the restricted rotation of support $J \sqcup K$ whose interior discontinuity point is b.

Let $g \in \mathcal{G}_{m,\pi}$ with continuity intervals $I_1, \dots I_m$ and let $g(I_i) = J_{\pi(i)}$. We consider $R_1 = R_{J,K}$, where $J = J_1 \cup \dots \cup J_{\pi(1)-1}$ and $K = J_{\pi(1)}$. One directly has that $R_1 \circ g|_{I_1} = \mathrm{Id}$ and $\#\mathrm{BP}(g_1) \leqslant m - 1$, where $g_1 = R_1 \circ g|_{I_2 \cup \dots \cup I_m}$.

Starting with g_1, we define similarly R_2 and we get that $R_2 \circ g_1|_{I_2} = \mathrm{Id}$ and $\#\mathrm{BP}(g_2) \leqslant m - 2$, where $g_2 = R_2 \circ g_1|_{I_3 \cup \dots \cup I_m}$.

Repeating the previous argument $m - 1$ times leads to $\#\mathrm{BP}(g_{m-1}) \leqslant 1$ so $g_{m-1} = \mathrm{Id}$.

Extending the restricted rotations R_i to $[0, 1[$ by the identity map, we conclude that

$$R_{m-1} \circ \dots \circ R_1 \circ g = \mathrm{Id}\,. \qquad \square$$

A direct consequence of Lemmas 3.2 and 3.3 is

Proposition 3.4 ([2, III §1.4]). — *The group $\overline{\mathcal{G}}$ is perfect and any $g \in \mathcal{G}_m$ is the product of $m - 1$ commutators in $\overline{\mathcal{G}}$.*

3.2. Commutators in $\mathcal{G}$

In the introduction, we have indicated a few similarities between the groups $\mathcal{G}$ and $\mathrm{PL}^+(I)$. In particular, the simplicity of their derivative subgroups relies on a result of Epstein (see [10, 1.1.Theorem]). In the context of finding bounds for the commutator length, a substantial difference between these two groups is that an element f of $[\mathrm{PL}^+(I), \mathrm{PL}^+(I)]$ is a map whose support J satisfies $\overline{J} \subset (0, 1)$ and f can not be written as a product of commutators of maps with support in J. This contrasts with

Remark 3.5. — Let J be a half-open subinterval of I. If $g \in \mathcal{G}$ has support in J then $g \in [\mathcal{G}, \mathcal{G}]$ if and only if $g|_J \in [\mathcal{G}_J, \mathcal{G}_J]$.

Indeed, according to Theorem 1.1 of [37], there is a morphism $\mathrm{SAF}_J : \mathcal{G}_J \to \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$ such that $[\mathcal{G}_J, \mathcal{G}_J] = \mathrm{SAF}_J^{-1}(0)$. More precisely for $f \in \mathcal{G}_J$, $\mathrm{SAF}_J(f) = \sum \lambda_k \otimes \delta_k$, where the vectors (λ_k), (δ_k) encode the lengths of exchanged intervals and the corresponding translation constants respectively.

Let $g \in \mathcal{G}$ with support in J. From the previous definition, it is easy to check that $\mathrm{SAF}_I(g) = \mathrm{SAF}_J(g|_J)$. Therefore $g \in [\mathcal{G}, \mathcal{G}]$ if and only if $g|_J \in [\mathcal{G}_J, \mathcal{G}_J]$.

4. The adapted Dennis and Vaserstein argument

In this section, we first we recall Proposition 1 (c) of Dennis and Vaserstein ([9]).

4.1. The original criterion

Definition 4.1. — *Two subsets S_1 and S_2 of a group G are* commuting *if any $a \in S_1$ commutes with any $a' \in S_2$.*

Dennis and Vaserstein's criterion. — *If a group G contains two commuting subgroups H_1 and H_2 such that for each finite subset S of G there are elements $g_i \in G$, $i = 1, 2$, such that $g_i^{-1} S g_i \leqslant H_i$ for $i = 1, 2$, then $c(G) \leqslant 3$.*

As an illustrating example indicated by Ghys, the group $[\mathrm{PL}^+(I), \mathrm{PL}^+(I)]$ consists in all g of $\mathrm{PL}^+(I)$ such that $g'(0) = g'(1) = 1$. Thus, for any finite collection $\{g_i\}$ in $[\mathrm{PL}^+(I), \mathrm{PL}^+(I)]$ there exist $0 < a < b < 1$ such that (a, b) contains the support of all the g_i. The required groups H_1 and H_2 are obtained as groups of maps with disjoint supports by setting $H_1 = \langle g_i \rangle$ and $H_2 = h \langle g_i \rangle h^{-1}$ where h is an element of $[\mathrm{PL}^+(I), \mathrm{PL}^+(I)]$ that carries (a, b) into $(\frac{a}{2}, a)$.

Unfortunately, this argument doesn't apply immediately in $[\mathcal{G}, \mathcal{G}]$ and $\overline{\mathcal{G}}$. This is essentially due to the facts that both groups contain maps with full support and that if the length of the support of $g \in \overline{\mathcal{G}}$ exceeds $\frac{1}{2}$ then it is impossible to find a conjugate of g in $\overline{\mathcal{G}}$ with a disjoint support. To avoid these difficulties, we will compose with suitable periodic maps to obtain an IET with arbitrary small support (see Propositions 5.1 and 6.1) and then we will apply the following iterated version of Dennis and Vaserstein's criterion.

4.2. The iterated version

Let $n \in \mathbb{N}^*$, we denote by H_n (resp. $\overline{H}_n$) the subgroup of $\mathcal{G}$ (resp. $\bar{\mathcal{G}}$) consisting of elements whose support is included in $[1-\frac{1}{n},1)$. By Remark 1.1, H_n (resp. $\overline{H}_n$) is identified with $\mathcal{G}_{[1-\frac{1}{n},1)}$ (resp. $\bar{\mathcal{G}}_{[1-\frac{1}{n},1)}$). Moreover, Remark 1.1 and Proposition 3.4 imply that $\overline{H}_n$ is perfect and Remark 3.5 leads to $H_n \cap [\mathcal{G},\mathcal{G}] = [H_n, H_n]$.

Lemma 4.2.

(1) *If $g \in H_2 \cap [\mathcal{G},\mathcal{G}] = [H_2,H_2]$ then $c_{\mathcal{G}}(g) \leqslant \frac{1}{2} c_{H_2}(g) + \frac{3}{2}$.*
(2) *If $g \in \overline{H_2}$ then $c_{\bar{\mathcal{G}}}(g) \leqslant \frac{1}{2} c_{\overline{H_2}}(g) + \frac{3}{2}$.*

Proof. — Proofs of Items (1) and (2) are similar, changing $\mathcal{G}$ for $\bar{\mathcal{G}}$ and H_2 for $\overline{H}_2$, so we only prove Item (1).

Let $g \in H_2 \cap [\mathcal{G},\mathcal{G}]$. We write $c_{H_2}(g) = 2p - r$ with $p \in \mathbb{N}^*$ and $r = 0, 1$. Therefore

$$g = (c_1 \dots c_p)(c_{p+1} \dots c_{2p}),$$

where $c_i = [a_i, b_i]$ with a_i, b_i in H_2 and the last commutator c_{2p} is eventually trivial.

Let R be the rotation of angle $\frac{1}{2}$. We denote by $f' = R \circ f \circ R^{-1}$.

Note that if $f, k \in H_2$ then f and k' have disjoint supports and they commute. We write

$$\begin{aligned} g &= (c_1 \dots c_p)(c'_{p+1} \dots c'_{2p})(c'_{p+1} \dots c'_{2p})^{-1}(c_{p+1} \dots c_{2p}) \\ &= (c_1 c'_{p+1}) \dots (c_p c'_{2p})\, C, \end{aligned}$$

where $C = (c'_{p+1} \dots c'_{2p})^{-1}(c_{p+1} \dots c_{2p})$.

On one hand, by Properties 2.4 (1), we have that $c_i c'_{p+i}$, $i = 1, \dots, p$, are commutators.

On the other hand, by Properties 2.4 (2), it holds that

$$C = (c'_{p+1} \dots c'_{2p})^{-1}(c_{p+1} \dots c_{2p})$$

is a commutator since it is the product of $(c_{p+1} \dots c_{2p})$ and the conjugate by R of its inverse.

Finally, we have $c_{\mathcal{G}}(g) \leqslant p + 1$, thus

$$2c_{\mathcal{G}}(g) \leqslant 2p + 2 = c_{H_2}(g) + r + 2 \leqslant c_{H_2}(g) + 3. \qquad \square$$

Repeatedly applying Lemma 4.2, we get

Proposition 4.3. — *Let $t \in \mathbb{N}^*$.*

(1) *If $g \in H_{2^t} \cap [\mathcal{G},\mathcal{G}] = [H_{2^t}, H_{2^t}]$ then $c_{\mathcal{G}}(g) < \frac{1}{2^t} c_{H_{2^t}}(g) + 3$.*
(2) *If $g \in \overline{H_{2^t}}$ then $c_{\bar{\mathcal{G}}}(g) < \frac{1}{2^t} c_{\overline{H}_{2^t}}(g) + 3$.*

Proof. — As noted earlier, we only prove Item (1).

Let $t \in \mathbb{N}^*$ and $g \in H_{2^t} \cap [\mathcal{G}, \mathcal{G}]$. From Remark 1.1 and Lemma 4.2, we obtain

$$c_{H_{2^{t-1}}}(g) \leqslant \frac{1}{2} c_{H_{2^t}}(g) + \frac{3}{2}.$$

It is easy to check by induction that for $s \in \{1, \ldots, t\}$, we have

$$(E_s) c_{H_{2^{t-s}}}(g) \leqslant \frac{1}{2^s} c_{H_{2^t}}(g) + 3 \sum_{j=1}^{s} \frac{1}{2^j}.$$

Indeed, for $s = 1$, (E_1) is the first identity.

Fix $s \in \{1, \ldots, t-1\}$ and suppose that (E_s) holds. Then according to Remark 1.1 and Lemma 4.2

$$C_{H_{2^{t-(s+1)}}}(g) \leqslant \frac{1}{2} C_{H_{2^{t-s}}}(g) + \frac{3}{2}.$$

Thus, by induction hypothesis

$$C_{H_{2^{t-(s+1)}}}(g) \leqslant \frac{1}{2} \left(\frac{1}{2^s} C_{H_{2^t}}(g) + 3 \sum_{j=1}^{s} \frac{1}{2^j} \right) + \frac{3}{2}.$$

Therefore

$$C_{H_{2^{t-(s+1)}}}(g) \leqslant \left(\frac{1}{2^{s+1}} C_{H_{2^t}}(g) + 3 \sum_{j=1}^{s} \frac{1}{2^{j+1}} \right) + \frac{3}{2},$$

which leads immediately to (E_{s+1}).

Finally, noting that $H_1 = \mathcal{G}$ and $\sum_{j=1}^{t} \frac{1}{2^j} = 1 - \frac{1}{2^t} < 1$, the identity (E_t) leads to

$$c_{\mathcal{G}}(g) < \frac{1}{2^t} c_{H_{2^t}}(g) + 3. \qquad \square$$

5. The group $\overline{\mathcal{G}}$ is uniformly perfect

The aim of this section is to prove Theorem 1.3.

5.1. Background material

Let $g \in \mathcal{G}_m$. The *combinatorial description* of g is $(\lambda(g), \pi(g))$, where $\lambda(g)$ is an m-dimensional vector whose coordinates are the lengths of $I_1, \ldots, I_m$, the continuity intervals of g and $\pi(g) \in \mathcal{S}_m$ is the permutation on $\{1, \ldots, m\}$ that tells how the intervals I_i are rearranged by g.

We denote by $a_i(g)$ the discontinuity points of g. If g is continuous on a half-open interval J, we define $\delta_{\mathbf{J}}(g) := g(x) - x$, for $x \in J$.

The translations of g are $\delta_i(g) := \delta_{I_i}(g)$, $i = 1, \dots, m$.

Note that $a_i(g)$ and $\delta_i(g)$ are related to $(\lambda(g), \pi(g))$ by

$$(*) \quad a_i(g) = \sum_{k=1}^{i-1} \lambda_k(g) \quad \text{and} \quad \delta_i(g) = -\sum_{k=1}^{i-1} \lambda_k(g) + \sum_{k=1}^{\pi(i)-1} \lambda_{\pi^{-1}(k)}(g).$$

The map g is said to be *rational* if all its discontinuity points are rational. It is easy to see that rational IET are periodic.

Definition 5.1. — *Let m be a positive integer and $\pi \in \mathcal{S}_m$, we define a* metric *on $\mathcal{G}_{m,\pi}$ by*

$$d(f,g) = \sum_{i=1}^{m} |\lambda_i(f) - \lambda_i(g)|.$$

Properties 5.2. — *Let f and g be elements of $\mathcal{G}_{m,\pi}$. Then*

(1) $d(f^{-1}, g^{-1}) = d(f,g)$,
(2) $|a_i(f) - a_i(g)| \leqslant d(f,g)$,
(3) $|\delta_i(f) - \delta_i(g)| \leqslant 2d(f,g)$.

Proof.

(1). — The first item is due to the fact that $\lambda_{\pi(i)}(f^{-1}) = \lambda_i(f)$.

We deduce the remaining items from $(*)$, indeed

(2). — $|a_i(f) - a_i(g)| = |\sum_{k=1}^{i-1} \lambda_k(f) - \lambda_k(g)| \leqslant d(f,g)$

(3). — $|\delta_i(f) - \delta_i(g)| = |\big(-\sum_{k=1}^{i-1} \lambda_k(f) + \sum_{k=1}^{\pi(i)-1} \lambda_{\pi^{-1}(k)}(f)\big) - \big(-\sum_{k=1}^{i-1} \lambda_k(g) + \sum_{k=1}^{\pi(i)-1} \lambda_{\pi^{-1}(k)}(g)\big)| \leqslant 2d(f,g)$. □

Lemma 5.3. — *Let $g \in \mathcal{G}_m$ and let $l = |\mathrm{Fix}(g)|$ be the Lebesgue measure of the fixed point set of g. Then, there exists $h \in \mathcal{G}_m$ such that*

$$\mathrm{Fix}(h \circ g \circ h^{-1}) = [0, l).$$

In particular $\#\mathrm{BP}(h \circ g \circ h^{-1}) \leqslant 3m$.

Proof. — Denote by $F_1, F_3, \dots, F_{2p-1}$ the p ordered connected components of $I \setminus \mathrm{Fix}(g)$. We write $F_i = [\alpha_i, \alpha_{i+1})$, for $i = 2k-1$, $k = 1, \dots, p$. Note that $\alpha_i \in BP(g)$. Hence the connected components of $\mathrm{Fix}(g)$ are the possibly empty intervals $F_0 = [0, \alpha_1)$, $F_{2p} = [\alpha_{2p}, 1)$ and

$$F_{2k} = [\alpha_{2k}, \alpha_{2k+1}), \quad \text{for } k = 1, \dots, p-1.$$

The required map h is the IET whose combinatorial description is (λ, π)

$$\text{with } \begin{cases} \lambda_i = |F_i|, i = 0, \dots, 2p \quad \text{and} \quad \pi \in \mathcal{S}(\{0, \dots, 2p\}), \\ \pi(0) = 0, \quad \pi(2k) = k \text{ and } \pi(2k-1) = k + p, \quad k = 1, \dots, p. \end{cases}$$

Finally we note that $h \in \mathcal{G}_m$ since $BP(h) \subset \{\alpha_i\} \subset BP(g)$. □

5.2. Proof of Theorem 1.3

For proving Theorem 1.3, we need the following

PROPOSITION 5.4. — *Let n be a positive integer and let $f \in \mathcal{G}_m$. Then there exist two periodic elements $p, p' \in \mathcal{G}$ such that*

$$|\mathrm{supp}(p \circ f \circ p')| \leqslant \frac{1}{n} \quad \textit{and} \quad \#\mathrm{BP}(p \circ f \circ p') \leqslant 5m.$$

Proof. — Let n be a positive integer and $f \in \mathcal{G}_{m,\pi}$. We set $BP(f) = \{a_i, i = 1 \dots m\}$, $I_i = [a_i, a_{i+1})$ and $BP(f^{-1}) = \{b_i, i = 1, \dots, m\}$. Fix $0 < \epsilon < \frac{1}{2n}$ small enough ($\epsilon \ll |I_i|$). We consider $p \in \mathcal{G}_{m,\pi^{-1}}$ a rational IET such that $d(f^{-1}, p) \leqslant \frac{\epsilon}{2m}$ and $BP(p) = \{b'_i, i = 1, \dots, m\}$ satisfies $b_i - \frac{\epsilon}{2m} < b'_i \leqslant b_i$. This map p is periodic.

CLAIM 5.5. — *By construction, $f_\epsilon = p \circ f$ satisfies $\#\mathrm{BP}(f_\epsilon) \leqslant 2m$, it is continuous on $[a_i, a_{i+1} - \frac{\epsilon}{2m})$ and $\partial_i := \delta_{[a_i, a_{i+1} - \frac{\epsilon}{2m})}(p \circ f)$ satisfies $|\partial_i| \leqslant \frac{\epsilon}{m}$.*

Indeed, obviously $\#\mathrm{BP}(f_\epsilon) \leqslant \#\mathrm{BP}(f) + \#\mathrm{BP}(p) \leqslant 2m$.

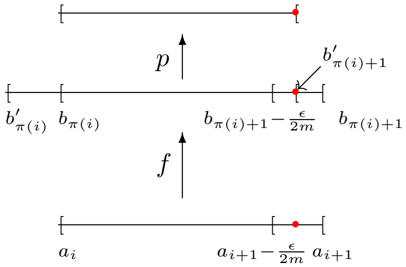
For every $x \in [a_i, a_{i+1} - \frac{\epsilon}{2m})$, one has $f(x) = x + \delta_{I_i}(f)$ and

$$f(x) \in \left[b_{\pi(i)}, b_{\pi(i)+1} - \frac{\epsilon}{2m}\right) \subset [b'_{\pi(i)}, b'_{\pi(i)+1}),$$
$$\text{then } p \circ f(x) = x + \delta_{I_i}(f) + \delta_{[b'_{\pi(i)}, b'_{\pi(i)+1})}(p).$$

Since $d(f^{-1}, p) \leqslant \frac{\epsilon}{2m}$, one has:

$$\frac{\epsilon}{m} \geqslant |\delta_{[b'_{\pi(i)}, b'_{\pi(i)+1})}(p) - \delta_{[b_{\pi(i)}, b_{\pi(i)+1})}(f^{-1})| = |\delta_{[b'_{\pi(i)}, b'_{\pi(i)+1})}(p) + \delta_{I_i}(f)|,$$
$$\text{therefore } |\partial_i| = |p \circ f(x) - x| = |\delta_{I_i}(f) + \delta_{[b'_{\pi(i)}), b'_{\pi(i)+1})}(p)| \leqslant \frac{\epsilon}{m}.$$

This ends the proof of the claim which is summarized by the following picture.



We turn now on to the proof of Proposition 5.4. Let $i \in \{1, \dots, m\}$.

If $\partial_i = 0$, we set $R_i = \mathrm{Id}$.

In the case that $\partial_i > 0$, we define R_i to be the finite order restricted rotation of support $[a_i, a_i + r_i\partial_i)$ and of angle ∂_i, where r_i is the greatest integer such that $a_i + r_i\partial_i \leqslant \min\{a_{i+1} - (\frac{\epsilon}{2m} - \partial_i), a_{i+1}\}$.

By definition R_i and f_ϵ coincide on the interval $[a_i, a_i + (r_i - 1)\partial_i)$ and $|[a_i + r_i\partial_i, a_{i+1})| \leqslant \frac{\epsilon}{m}$. Indeed, f_ϵ is continuous on $[a_i, a_i + (r_i - 1)\partial_i)$, since

$$a_i + (r_i - 1)\partial_i = a_i + r_i\partial_i - \partial_i \leqslant a_{i+1} - \left(\frac{\epsilon}{2m} - \partial_i\right) - \partial_i = a_{i+1} - \frac{\epsilon}{2m}.$$

In addition, by the maximality of r_i, either $a_i + (r_i + 1)\partial_i$ is greater

- than $a_{i+1} - (\frac{\epsilon}{2m} - \partial_i)$ and it follows that
$$|[a_i + r_i\partial_i, a_{i+1})| = a_{i+1} - (a_i + r_i\partial_i) < \partial_i + \left(\frac{\epsilon}{2m} - \partial_i\right) = \frac{\epsilon}{2m}$$
- or than a_{i+1} and then $|[a_i+r_i\partial_i, a_{i+1})| = a_{i+1}-(a_i+r_i\partial_i) < \partial_i \leqslant \frac{\epsilon}{m}$.

The same argument remains valid for negative ∂_i by using non positive integers r_i.

Finally, the map $g := f_\epsilon \circ \prod_1^m {R_i}^{-1}$ satisfies $\#\mathrm{BP}(g) \leqslant 5m$ because $\#\mathrm{BP}(R_i) \leqslant 3$.

Since $\mathrm{supp}(R_i) \subset [a_i, a_{i+1})$, the supports of the R_i's are disjoints and $p' = (\prod_1^m R_i)^{-1}$ is periodic and it is also a commutator in $\mathcal{G}$, according to Proposition 2.5.

But $g_{|R_i([a_i\ ,\ a_i+(r_i-1)\partial_i))} = \mathrm{Id}$ and therefore

$$\begin{aligned}|\mathrm{supp}(g)| &\leqslant 1 - \sum_{i=1}^m |[a_i, a_i + (r_i - 1)\partial_i)| \\ &\leqslant 1 - \sum_{i=1}^m \left(|[a_i, a_{i+1})| - \left(\partial_i + \frac{\epsilon}{m}\right)\right) \leqslant 2\epsilon \leqslant \frac{1}{n}.\end{aligned}$$ □

We turn now on to the proof of Theorem 1.3.

We first consider an IET $f \in \mathcal{G}_{m,\pi}$ viewed as an element of $\overline{\mathcal{G}}$. Let $t \in \mathbb{N}^*$.

Applying Proposition 5.4 to f and $n = 2^t$, we get that there exist two periodic elements $p, p' \in \mathcal{G}$ such that the support of $g = p \circ f \circ p' \in \mathcal{G}_{5m}$ has measure less than or equal to $\frac{1}{2^t}$.

By Lemma 5.3, the map g is conjugated to an element g' of H_{2^t} for which $\#\mathrm{BP}(g') \leqslant 15m$. Since p and p' are periodic and g and g' are conjugated, we have

$$c_{\overline{\mathcal{G}}}(f) \leqslant c_{\overline{\mathcal{G}}}(g) + 2 = c_{\overline{\mathcal{G}}}(g') + 2.$$

Then by Proposition 4.3 (2),

$$c_{\overline{\mathcal{G}}}(f) < \frac{1}{2^t} c_{\overline{H}_{2^t}}(g') + 5.$$

As $\#\mathrm{BP}(g'|_{[1-\frac{1}{2^t},1)}) \leqslant \#\mathrm{BP}(g')$, Remark 1.1 and Proposition 3.4 imply that

$$c_{\overline{H}_{2^t}}(g') \leqslant 15m - 1.$$

Finally, for any $t \in \mathbb{N}^*$ one has

$$c_{\overline{\mathcal{G}}}(f) < \frac{15m-1}{2^t} + 5$$

and choosing t large enough, we obtain

$$c_{\overline{\mathcal{G}}}(f) \leqslant 5.$$

Thus we get $c_{\overline{\mathcal{G}}}(f) \leqslant 5$, for any $f \in \mathcal{G}$.

For the general case, we consider $F \in \overline{\mathcal{G}}$. According to Lemma 3.3, the map F can be decomposed as the product of an involution that is a commutator and an element of $\mathcal{G}$. Therefore, we have proved the required inequality $c_{\overline{\mathcal{G}}}(F) \leqslant 1 + 5 = 6$, for any $F \in \overline{\mathcal{G}}$. □

6. Conditions for uniform perfectness of $\mathcal{G}$

In this section we give two sufficient conditions for $\mathcal{G}$ to be uniformly perfect.

6.1. The commutator length is bounded when fixing the number of discontinuity points

We prove the following statement that directly implies Theorem 1.4.

Theorem 6.1. — *If for any positive integer m, it holds that $C_m(\mathcal{G}) := \mathrm{supp}\{c_{\mathcal{G}}(g), g \in [\mathcal{G},\mathcal{G}] \cap \mathcal{G}_m\}$ is finite, then $\mathcal{G}$ is uniformly perfect and $c(\mathcal{G}) \leqslant 5$.*

Proof. — Let $f \in [\mathcal{G},\mathcal{G}] \cap \mathcal{G}_m$ and $t \in \mathbb{N}$. Proposition 5.4 and Lemma 5.3 with $n = 2^t$ show that there exist two periodic elements $p', p \in \mathcal{G}$ such that $g = p \circ f \circ p' \in \mathcal{G}_{5m}$ is conjugated to an element g' of $H_{2^t} \cap \mathcal{G}_{15m}$. By Proposition 2.5, p and p' are commutators then $g \in [\mathcal{G},\mathcal{G}]$. Moreover, $[\mathcal{G},\mathcal{G}]$ is normal so $g' \in H_{2^t} \cap [\mathcal{G},\mathcal{G}]$.

Therefore, according to Proposition 4.3

$$c_{\mathcal{G}}(f) \leqslant c_{\mathcal{G}}(g) + 2 = c_{\mathcal{G}}(g') + 2 < \frac{1}{2^t} c_{H_{2^t}}(g') + 5.$$

As $c_{H_{2^t}}(g') \leqslant C_{15m}(\mathcal{G})$, one has for any $t \in \mathbb{N}^*$

$$c_{\mathcal{G}}(f) < \frac{C_{15m}(\mathcal{G})}{2^t} + 5.$$

Choosing t large enough, we get $c_{\mathcal{G}}(f) \leqslant 5$. □

6.2. The commutator length is bounded when prescribing the arithmetic

Let $p \in \mathbb{N}^*$ and $\alpha = (\alpha_1, \dots, \alpha_p) \in [0,1)^p$ such that $\alpha_1 \notin \mathbb{Q}$.

6.2.1. Background material

We denote by Δ_α the abelian subgroup of $\mathbb{R}$ generated by $\alpha_1, \dots, \alpha_p$ and 1. Note that the condition $\alpha_1 \notin \mathbb{Q}$ insures that Δ_α is dense in $[0,1)$.

Let J be a half-open interval with endpoints in Δ_α.

Definition 6.2. — *We set* $\Gamma_\alpha := \{g \in \mathcal{G} : BP(g) \subset \Delta_\alpha\}$ *and* $\Gamma_\alpha(J) := \{g \in \mathcal{G}_J : BP(g) \subset \Delta_\alpha\}$.

It is plain that any $g \in \mathcal{G}$ is either periodic or belongs to some Γ_α. Indeed, if g is not periodic then its length vector λ has at least one irrational coordinate and α is obtained from λ by permutation.

Note that $\Gamma_\alpha(J)$ is the set of all maps $g \in \mathcal{G}_J$ whose extensions to I by the identity map belong to Γ_α. But $\Gamma_\alpha(J)$ does not coincide with the set obtained by conjugating Γ_α through the homothecy that carries J into I. For $J = [c,d)$, the last set is $\{g \in \mathcal{G}_J : BP(g) \subset c + \frac{\Delta_\alpha}{d-c}\}$. For this reason, the properties of $\Gamma_\alpha(J)$ are not direct consequences of the ones of Γ_α.

Properties 6.3. — *Let* $g \in \Gamma_\alpha(J)$ *and* I_i *be its continuity intervals.*

1. *The lengths of the* I_i *and the translations of* g *belong to* Δ_α.
2. $\Gamma_\alpha(J)$ *is a subgroup of* $\mathcal{G}_J$.

(3) *The endpoints of the connected components of* $\mathrm{Fix}(g)$ *are in* Δ_α.

Proof.

(1). — The endpoints of the I_i are the discontinuity points of g and the left endpoint of J. Therefore if $g \in \Gamma_\alpha(J)$ any length $\lambda_i = |I_i|$ belongs to Δ_α. The translations of g also belong to Δ_α, as linear combinations of the λ_i's with coefficients in $\{-1, 0, 1\}$.

(2). — According to (1), any $f \in \Gamma_\alpha(J)$ preserves Δ_α. Therefore the relations $BP(f^{-1}) = f(BP(f))$ and $BP(f_1 \circ f_2) \subset BP(f_2) \cup f_2^{-1}(BP(f_1))$ imply that $\Gamma_\alpha(J)$ is stable by taking inverse and composite.

(3). — From the definition of $\mathrm{Fix}(g)$, it follows that every endpoint of a connected component of $\mathrm{Fix}(g)$ is a discontinuity point of g. □

Before stating our last theorem, we give

Definition 6.4. — *We say that* $\mathcal{G}$ *has* partial uniform perfectness *if for any* $p \in \mathbb{N}^*$ *and* $\alpha \in [0,1)^p$ *it holds that* $C_\alpha(\mathcal{G}) := sup\{c_{\mathcal{G}}(g), g \in [\mathcal{G}, \mathcal{G}] \cap \Gamma_\alpha\}$ *is finite.*

Theorem 6.5. — *If* $\mathcal{G}$ *has partial uniform perfectness then* $\mathcal{G}$ *is uniformly perfect.*

A consequence of Theorems 6.1 and 6.5 is

Corollary 6.6. — *If* $\mathcal{G}$ *has partial uniform perfectness then* $c(\mathcal{G}) \leqslant 5$.

The main tool for the proof of Theorem 6.5 is

Proposition 6.7. — *Let* $f \in \Gamma_\alpha$. *Let* n *be a positive integer and set* $s_n = [\frac{\ln(n)}{\ln(1.25)}] + 1$.

Then there exist $g_n \in H_n \cap \Gamma_\alpha$, *a map* $h \in \Gamma_\alpha$ *and* s_n *involutions* $i_j \in \Gamma_\alpha$, $j = 1, 2, \ldots, s_n$ *such that* $f = i_1 \circ \cdots \circ i_{s_n} \circ (h \circ g_n \circ h^{-1})$.

For proving this proposition we use the following

Lemma 6.8. — *Let* $\epsilon \in (0,1)$ *and* J *be a half-open interval with endpoints in* Δ_α. *If* $f \in \Gamma_\alpha(J)$ *then there exists an involution* $i \in \Gamma_\alpha(J)$ *such that*

$$|\mathrm{Fix}(i \circ f)| \geqslant |\mathrm{Fix}(f)| + \frac{|J| - |\mathrm{Fix}\, f|}{5}(1 - \epsilon).$$

Proof. — Let $f \in \Gamma_\alpha(J)$, $BP(f) = \{0 = a_1, \ldots, a_m\}$ and set $a_{m+1} = 1$.

Case 1: $\mathrm{Fix}(f) = \emptyset$. — Fix $\delta \in \Delta_\alpha$ such that $0 < \delta < \min\{\frac{|J|\epsilon}{m}; |f(x) - x|, x \in I\}$.

For every $j \in \{1, \ldots, m-1\}$, we consider the unique integer n_j such that $(n_j - 1)\delta \leqslant |[a_j, a_{j+1})| < n_j \delta$. It holds that $[a_j, a_{j+1})$ is the union of

$(n_j - 1)$ intervals of length δ and an eventually empty interval F_j of length less than δ.

Therefore J can be decomposed as a finite union of pairwise disjoint half-open intervals $I_1, \dots, I_n$ and $F_1, \dots, F_m$ such that

- f is continuous on these intervals,
- $|I_j| = \delta$ for $j = 1, \dots, n$ and $|F_k| < \delta$ for $k = 1, \dots, m$.

It follows that $n\delta + \sum |F_k| = |J|$.

Since for any x it holds that $|f(x) - x| > \delta$, one has $f(I_j) \cap I_j = \emptyset$. Therefore there exists an involution i_1 of support $I_1 \cup f(I_1)$ such that $i_1|_{I_1} = f|_{I_1}$ and then $i_1 \circ f|_{I_1} = \mathrm{Id}|_{I_1}$. Now, we want to construct a similar involution i_2 on a second interval I_{p_2} so that i_1 and i_2 have disjoint supports. This can be done if and only if $(I_{p_2} \cup f(I_{p_2})) \cap (I_1 \cup f(I_1)) = \emptyset$. This means that

$$I_{p_2} \subset I \setminus (I_1 \cup f(I_1) \cup f^{-1}(I_1)).$$

Consequently, such an interval I_{p_2} and its corresponding involution i_2 with support $I_{p_2} \cup f(I_{p_2})$ exist provided that

$$I' \setminus (I_1 \cup D(f(I_1) \cup f^{-1}(I_1)) \neq \emptyset, \tag{6.1}$$

where $I' = I \setminus \cup F_k$ and $D(K) = \bigcup_{\{k | K \cap I_k \neq \emptyset\}} I_k$.

As any half-open interval of length δ meets at most two intervals I_k, the condition (6.1) means that $n > 5$.

By induction, we can define s involutions i_j with disjoint supports $I_{p_j} \cup f(I_{p_j})$ provided that

$$I' \setminus (I_1 \cup \dots I_{p_{s-1}} \cup D(f(I_1 \cup \dots I_{p_{s-1}}) \cup f^{-1}(I_1 \cup \dots I_{p_{s-1}})) \neq \emptyset.$$

That is $n > 5(s-1)$.

Let s be the largest integer such that $n > 5(s-1)$, we can construct the involutions i_j, $j = 1, \dots, s$ but $n \leqslant 5s$.

By the definition of i_j, the map $g = i_s \dots i_1 \circ f$ satisfies

$$g|_{I_1 \cup I_{p_2} \cup \dots \cup I_{p_s}} = \mathrm{Id}|_{I_1 \cup I_{p_2} \cup \dots \cup I_{p_s}},$$

then

$$\begin{aligned} |\mathrm{Fix}(g)| &\geqslant \sum_{j=1}^{s} |I_{p_j}| = s.\delta = \frac{s}{n}(|J| - \sum |F_k|) \\ &\geqslant \frac{1}{5}(|J| - \sum |F_k|) \\ &\geqslant \frac{1}{5}(|J| - m\delta) \geqslant \frac{1}{5}(|J| - |J|\epsilon). \end{aligned}$$

In conclusion, since i_j have disjoint supports, the map $i = i_s \dots i_1$ is an involution and $|\mathrm{Fix}(i \circ f)| \geqslant \frac{|J|}{5}(1-\epsilon)$, this is the desired conclusion for $|\mathrm{Fix}\, f| = 0$.

It remains to prove that $i \in \Gamma_\alpha$. Since the a_i and δ are in Δ_α, the endpoints of I_i and $f(I_i)$ also belong to Δ_α. Combining this with the fact that the discontinuity points of the involutions i_j are endpoints of I_i or $f(I_i)$, we get that $BP(i_j) \subset \Delta_\alpha$, for $j = 1, \dots, s$. Therefore, by definition, the maps $i_j \in \Gamma_\alpha$ then so does i.

Case 2: $\mathrm{Fix}(f) \neq \emptyset$. — We set $J = [c, d)$.

As the endpoints of the connected components of $\mathrm{Fix}(f)$ belong to Δ_α, it holds that $a = |\mathrm{Fix}(f)| \in \Delta_\alpha$. Therefore a slight adaptation of Lemma 5.3 to $f \in \Gamma_\alpha(J)$, shows that there exists $h \in \Gamma_\alpha(J)$ such that $\mathrm{Fix}(h \circ f \circ h^{-1}) = [c, c+a)$.

Let $f_1 \in \Gamma_\alpha([c+a, d))$ be the restriction of $h \circ f \circ h^{-1}$ to $[c+a, d)$. By construction, $\mathrm{Fix}(f_1) = \emptyset$ hence Case 1 applies to f_1 and provides an involution $j_1 \in \Gamma_\alpha([c+a, d))$ such that

$$|\mathrm{Fix}(j_1 \circ f_1)| \geqslant \frac{|J| - a}{5}(1-\epsilon).$$

Let $j \in \Gamma_\alpha(J)$ be the involution of J defined by $j(x) = j_1(x)$ if $x \in [c+a, d)$ and $j(x) = x$ if $x \in [c, c+a)$. We have

$$|\mathrm{Fix}(j \circ h \circ f \circ h^{-1})| \geqslant |[c, c+a)| + |\mathrm{Fix}(j_1 \circ f_1)| \geqslant a + \frac{|J| - a}{5}(1-\epsilon).$$

In addition, as h preserves lengths, we have

$$|\mathrm{Fix}(j \circ h \circ f \circ h^{-1})| = |\mathrm{Fix}(h^{-1} \circ (j \circ h \circ f \circ h^{-1}) \circ h)| = |\mathrm{Fix}((h^{-1} \circ j \circ h) \circ f)|.$$

Setting $i = h^{-1} \circ j \circ h$, we get

$$|\mathrm{Fix}(i \circ f)| \geqslant a + \frac{|J| - a}{5}(1-\epsilon),$$

which completes the proof. □

We turn now on to the proof of Proposition 6.7.

Let $\epsilon \in (0, 1)$ small enough and such that $\frac{1}{5}(1-\epsilon) \in \Delta_\alpha$. Consider $f \in \Gamma_\alpha$ and set $L_0 = |\mathrm{Fix}(f)|$.

Applying Lemma 6.8 to f, there exists an involution $i_1 \in \Gamma_\alpha$ such that

$$|\mathrm{Fix}(i_1 \circ f)| \geqslant L_0 + \frac{1 - L_0}{5}(1-\epsilon) = \phi(L_0) := L_1,$$

where $\phi(x) := x + \frac{1-x}{5}(1-\epsilon) = \frac{4+\epsilon}{5}(x-1) + 1$ is a direct affine map whose fixed point is 1.

We now apply this argument again, with f replaced by $i_1 \circ f$, to obtain an involution $i_2 \in \Gamma_\alpha$ such that

$$\begin{aligned}|\mathrm{Fix}(i_2 \circ i_1 \circ f)| &\geqslant |\mathrm{Fix}(i_1 \circ f)| + \frac{1 - |\mathrm{Fix}(i_1 \circ f)|}{5}(1-\epsilon)\\ &\geqslant \phi(|\mathrm{Fix}(i_1 \circ f)|)\\ &\geqslant \phi(L_1) = \phi^2(L_0).\end{aligned}$$

Repeating this process s times, we get s involutions $i_k \in \Gamma_\alpha$ such that

$$|\mathrm{Fix}(i_s \circ \cdots \circ i_1 \circ f)| \geqslant \phi^s(L_0).$$

We now prove that $\phi^s(L_0) \geqslant 1 - \frac{1}{n}$ provided that $s \geqslant s_n = \left[\frac{\ln(n)}{\ln(1.25)}\right] + 1$. In order to get this inequality, we are looking for integers s such that

$$\phi^s(L_0) = \left(\frac{4+\epsilon}{5}\right)^s (L_0 - 1) + 1 \geqslant 1 - \frac{1}{n},$$

that is

$$-\left(\frac{4+\epsilon}{5}\right)^s (1 - L_0) \geqslant -\frac{1}{n}.$$

Using that $L_0 \geqslant 0$, it suffices to determine s satisfying

$$\begin{gathered}\left(\frac{4+\epsilon}{5}\right)^s \leqslant \frac{1}{n}\\ s \ln\left(\frac{4+\epsilon}{5}\right) \leqslant \ln\left(\frac{1}{n}\right) = -\ln(n)\\ s \geqslant \frac{\ln(n)}{\ln(\frac{5}{4+\epsilon})}\end{gathered}$$

Therefore, we can take $s = \left[\frac{\ln(n)}{\ln(\frac{5}{4+\epsilon})}\right] + 1$. In addition, as $\frac{\ln(n)}{\ln(\frac{5}{4})} \notin \mathbb{N}$, we have $\left[\frac{\ln(n)}{\ln(\frac{5}{4+\epsilon})}\right] = \left[\frac{\ln(n)}{\ln(\frac{5}{4})}\right]$ for $\epsilon > 0$ small enough.

Finally $i_s \circ \cdots \circ i_1 \circ f \in \Gamma_\alpha$ has a fixed point set of length at least $1 - \frac{1}{n}$ so it is conjugated to an element of H_n by some $h \in \Gamma_\alpha$, by a slight adaptation of Lemma 5.3 to $f \in \Gamma_\alpha$. □

6.2.2. Proof of Theorem 6.5

We consider $g_1 \in \Gamma_\alpha \cap [\mathcal{G}, \mathcal{G}]$ that realizes $C_\alpha(\mathcal{G})$. By Proposition 6.7 with $n = 2$ and thus $s_2 = \left[\frac{\ln(2)}{\ln(1.25)}\right] + 1 = 4$, there exist $g_2 \in H_2 \cap \Gamma_\alpha$, $h \in \Gamma_\alpha$ and four involutions $i_1, \ldots, i_4$ such that

$$g_1 = i_1 \circ i_2 \circ i_3 \circ i_4 \circ (h g_2 h^{-1}).$$

We can now estimate $C_\alpha(\mathcal{G})$. By Proposition 2.5 and the normality of $[\mathcal{G},\mathcal{G}]$,

$$C_\alpha(\mathcal{G}) = c_{\mathcal{G}}(g_1) \leqslant 4 + c_{\mathcal{G}}(g_2).$$

According to Lemma 4.2, we have

$$c_{\mathcal{G}}(g_2) \leqslant \frac{1}{2}c_{H_2}(g_2) + \frac{3}{2}.$$

Using Remark 1.1, the group H_2 inherits the partial uniform perfectness of $\mathcal{G}$ and this implies that for any $g \in [H_2,H_2] \cap \Gamma_{\frac{\alpha}{2}}$, one has $c_{H_2}(g) \leqslant C_\alpha(\mathcal{G})$. In particular since Γ_α is a subgroup of $\Gamma_{\frac{\alpha}{2}}$, we have $c_{H_2}(g_2) \leqslant C_\alpha(\mathcal{G})$. Hence,

$$\begin{aligned} C_\alpha(\mathcal{G}) &\leqslant 4 + \frac{1}{2}C_\alpha(\mathcal{G}) + \frac{3}{2} \\ \frac{1}{2}C_\alpha(\mathcal{G}) &\leqslant 4 + \frac{3}{2} = \frac{11}{2} \\ C_\alpha(\mathcal{G}) &\leqslant 11. \end{aligned}$$

Finally, as any IET g is either periodic or it belongs to some Γ_α, we get that $c_{\mathcal{G}}(g) \leqslant 11$ for all $g \in [\mathcal{G},\mathcal{G}]$. It means that $\mathcal{G}$ is uniformly perfect. ☐

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