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p-ADIC L-FUNCTIONS AND THE GEOMETRY OF HIDA FAMILIES

by Joe KRAMER-MILLER

Abstract. — A major theme in the theory of p-adic deformations of automorphic forms is how p-adic L-functions over eigenvarieties relate to the geometry of these eigenvarieties. In this article we prove results in this vein for the ordinary part of the eigencurve (i.e. Hida families). We show that the crossing of two Hida families is determined by the local behavior of p-adic L-functions on those Hida families. In addition, we prove that the local behavior of p-adic L-functions determines when a Hida family ramifies over the weight space. Our methods involve proving a converse to a result of Vatsal relating congruences between eigenforms to their algebraic special L-values and then p-adically interpolating congruences.

Résumé. — Une question centrale dans la théorie de la déformations p-adique des formes automorphes est la suivante: quelle est la relation entre une L-fonction p-adic sur une variété propre et la géométrie de cette variété propre ? Dans l'article nous montrons certains résultats dans cet esprit pour la partie ordinaire de la courbe propre (i.e., les familles d'Hida). Plus précisément, nous montrons que l'intersection de deux familles d'Hida est déterminée par le comportement local des L-fonctions p-adic des familles. Nous prouvons aussi que le comportement local des L-fonctions p-adique nous dit quand une famille d'Hida est ramifiée sur l'espace de poids. La technique principale consiste à prouver une réciproque d'un théorème de Vatsal et puis à interpoler p-adiquement certaines congruences.

1. Introduction

1.1. Congruences between Cusp Forms and Special Values

The relationship between special values of L-functions and congruent eigenforms was first observed by Mazur. The underlying principle is that congruent forms should have congruent special values. Vatsal proved general results in this direction in [31] with applications to nonvanishing theorems in mind. It is natural to ask if the converse is true: if the special

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values of two eigenforms are congruent, are the eigenforms themselves congruent? One may then consider p-adic families of eigenforms and p-adic L-functions. The parallel question becomes: can one determine intersections of Hida families and ramification over the weight space by looking at p-adic L-functions? In this article we answer both questions affirmatively.

Let us first recall Vatsal's result. Let $p > 3$ be a prime and let K be a finite extension of $\mathbb{Q}_p$. Let $N_0 > 3$ be relatively prime to p and $k > 1$. For $r > 0$ we will consider the space of eigenforms $S_k(\Gamma, \mathcal{O}_K)$, where Γ is the congruence subgroup $\Gamma_1(N_0 p^r)$. We will assume that K is large enough to contain all the eigenvalues of the Hecke operators acting on $S_k(\Gamma, \mathcal{O}_K)$. We define the Hecke algebra $\mathbb{T}_{N_0 p^r, k}$ as the $\mathcal{O}_K$-algebra generated by the operators T_l for $l \nmid N_0 p$ and U_l for $l | N_0 p$ acting on $S_k(\Gamma, \mathcal{O}_K)$. A maximal ideal $\mathfrak{m}$ of $\mathbb{T}_{N_0 p^r, k}$ then corresponds to a residual Galois representation $\bar{\rho}$ up to semisimplification. We define $\mathbb{T}_{N_0 p^r, k, \mathfrak{m}}$ to be the localization of $\mathbb{T}_{N_0 p^r, k}$ at $\mathfrak{m}$. Denote by $L_{k-2}(\mathcal{O}_K)$ the $\mathcal{O}_K[\Gamma]$-module of homogeneous polynomials in two variables of degree $k-2$ with the standard action of Γ. For the moment, assume there exists an isomorphism

$$H_1(\mathbb{H}/\Gamma, \mathcal{L}_{k-2}(\mathcal{O}_K))^{\pm}_{\mathfrak{m}} \cong \mathbb{T}_{N_0 p^r, k, \mathfrak{m}}$$

as $\mathbb{T}_{N_0 p^r, k}$-modules, where $\mathcal{L}_{k-2}(\mathcal{O}_K)$ is the local system associated to $L_{k-2}(\mathcal{O}_K)$ on $\mathbb{H}/\Gamma$ and the $\pm$ means we only consider the $+$ or $-$ eigenspace with respect to complex conjugation (see Subsection 2.2). This isomorphism is unique up to an element in $\mathcal{O}_K^{\times}$ and a choice of isomorphism corresponds to choosing a period. If f and g are two eigenforms with residual representation $\bar{\rho}$, then we have two $\mathcal{O}_K$-algebra homomorphisms, δ_f and δ_g, from $\mathbb{T}_{N_0 p^r, k, \mathfrak{m}}$ to $\mathcal{O}_K$. Any congruence satisfied between f and g is necessarily satisfied between δ_f and δ_g. Evaluating δ_* on the appropriate cycle (maybe extending scalars to include the necessary roots of unity) yields special L-values. These special values will satisfy any congruence between f and g.

In Theorem 3.8 we prove that the converse is true under the assumption that both f and g are p-ordinary. We show that if periods $\Omega_f^{\pm}$ and $\Omega_g^{\pm}$ can be chosen so that the algebraic special values of both eigenforms are congruent mod π_K^r, then we have $f \equiv g \bmod \pi_K^r$. In fact, we only need to consider a subset $\mathcal{A}_\epsilon$ of Dirichlet characters, which is defined in 3.1. To prove this result, we use the theory of modular symbols introduced by Manin [24] and generalized further by Ash and Stevens [1]. We show that a p-ordinary modular symbol is completely determined by its special values. This will allow us to construct a congruence module and use a standard congruence module argument (see [15] or [27]).

THEOREM 3.8. — *Let f and g be p-ordinary eigenforms in $S_k(\Gamma, \mathcal{O}_K)$. If there exists periods $\Omega_f^{\pm}$ and $\Omega_g^{\pm}$ (these are defined canonically up to p-adic unit) that satisfy*

$$\frac{L(f,\chi,1)}{2\pi i\Omega_f^{\pm}} \equiv \frac{L(g,\chi,1)}{2\pi i\Omega_g^{\pm}} \bmod \pi_K^r,$$

for all $\chi \in \mathcal{A}_\epsilon$, then $f \equiv g \bmod \pi_K^r$.

This theorem can be viewed as a generalization of a result of Stevens ([30, Theorem 2.1]), who proves a similar result when $k = 2$. More precisely, Stevens proves that if a modular symbol with constant coefficients vanishes at a certain set of special values, then the modular symbol must be trivial. One can then deduce Theorem 3.8 for weight 2 eigenforms using a standard congruence module argument. The proof of Theorem 3.8 relies essentially on the ordinary assumption. Without this assumption, it would be necessary for us to require congruences between $\frac{L(f,\chi,m)}{2\pi i\Omega_f^{\pm}}$ and $\frac{L(g,\chi,m)}{2\pi i\Omega_g^{\pm}}$ for $m \geqslant 2$.

After proving Theorem 3.8 we briefly describe the p-adic L-functions $L_p^{\pm}(f,\chi) \in \mathcal{O}_K[\![T]\!]$ associated to a p-ordinary eigenform as described in [26]. An immediate consequence of the interpolation properties is:

THEOREM. — *Assume that for all Dirichlet characters χ in $\mathcal{A}_\epsilon$ we have*

$$L_p^{\pm}(f,\chi) \equiv L_p^{\pm}(g,\chi) \bmod \pi_K^s \mathcal{O}_K[\![T]\!].$$

Then $f \equiv g \bmod \pi_K^s$.

1.2. Crossing components in Hida families

In the second half of this article we prove a geometric analogue to the results of the first half. Let us first give a geometric interpretation of Theorem 3.8 as motivation. Consider the space $X = \mathrm{Spec}(\mathbb{T}_{N_0p^r,k})$. The points of codimension zero in X correspond to cuspidal eigenforms of level N_0p^r and the points of co-dimension one correspond to residual representations. Let x_f and x_g be points of X corresponding to eigenforms f and g. Let $\overline{x_f}$ and $\overline{x_g}$ denote the Zariski closure of x_f and x_g. Then x_f and x_g specialize to the same co-dimension one point x_ρ if and only if $f \equiv g \bmod \pi_K$. We then define the intersection multiplicity of the components $\overline{x_f}$ and $\overline{x_g}$ at x_ρ to be

$$\dim_{\mathcal{O}_K/\pi_K} \mathbb{T}_{N_0p^r,k,\mathfrak{m}}/(\mathfrak{p}_f + \mathfrak{p}_g)$$

where $\mathfrak{p}_*$ is the prime corresponding to x_*. This definition agrees with the algebraic definition provided in [11]. One can check that the largest power of

π_K for which f and g are congruent is equal to this intersection multiplicity. Theorem 3.8 can therefore be reformulated to relate congruences between special L-values and intersection multiplicities.

This view of Theorem 3.8 suggests that Hida families that cross should have p-adic L-functions that satisfy certain congruences. Let $\mathbb{T}_{N_0,\mathfrak{m}}$ be a localized Hecke–Hida algebra (see Section 4.1 for a precise definition and for the conditions we assume on the residual Galois representation) and let

$$\pi : \operatorname{Spec}(\mathbb{T}_{N_0,\mathfrak{m}}) \to \operatorname{Spec}(\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!])$$

be the map onto the weight space (cf. Section 4.1). Under certain conditions on $\bar{\rho}$, there exists an isomorphism of $\mathbb{T}_{N_0,\mathfrak{m}}$-modules

$$H_1(N_0p^\infty, \mathcal{O}_K)_{\mathfrak{m}}^{\pm\mathrm{ord}} \cong \mathbb{T}_{N_0,\mathfrak{m}}, \tag{1.1}$$

which is defined up to multiplication by $\mathbb{T}_{N_0,\mathfrak{m}}^\times$. Fix such an isomorphism for the remainder of this introduction. For any primitive Dirichlet character χ, the isomorphism (1.1) determines p-adic valued function $L_p^\pm(\mathbb{T}_{N_0,\mathfrak{m}}, \chi)$ on $\operatorname{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$ that has the following interpolation property:

$$L_p^\pm(\mathbb{T}_{N_0,\mathfrak{m}}, \chi)(x) = \frac{L(f_x, \chi, 1)}{2\pi i \Omega_{f_x}^\pm} u,$$

where $x \in \operatorname{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$ corresponds to a classical eigenform f_x and u is a p-adic unit. The period $\Omega_{f_x}^\pm$ at each point is determined by the isomorphism (1.1). In particular, we may think of a choice of isomorphism (1.1) as choosing a family of p-adically compatible periods.

By a Hida family we mean an irreducible component of $\operatorname{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$. The weight space $\operatorname{Spec}(\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!])$ is equal to $p-1$ disjoint copies $\operatorname{Spec}(\Lambda)$, where $\Lambda = \mathcal{O}_K[\![T]\!]$ (see Section 3.4). We may think of $\operatorname{Spec}(\Lambda)$ as the open unit ball in $\mathcal{O}_K$. If C is a Hida family then $\pi(C)$ is equal to one copy of $\operatorname{Spec}(\Lambda)$, and we may regard π as a map from C to $\operatorname{Spec}(\Lambda)$. By restricting $L_p^\pm(\mathbb{T}_{N_0,\mathfrak{m}}, \chi)$ to C we obtain a p-adic L-function $L_p^\pm(C, \chi)$ on C that interpolates the special values of eigenforms lying on C. We prove that the local behavior of these p-adic L-functions around a point x determine when two Hida families intersect. Let $\widehat{\mathcal{O}}_{C,x}$ be the completion of the germ of analytic functions defined at x. When x is a smooth $\mathcal{O}_K$-point we may noncanonically identify $\widehat{\mathcal{O}}_{C,x}$ with $K[\![T_0]\!]$, where T_0 is a formal variable. The image of $L_p^\pm(C, \chi)$ in $K[\![T_0]\!]$ is just the Taylor expansion of $L_p^\pm(C, \chi)$ around the point x in terms of a parameter T_0. We may now state our main result.

THEOREM 1.1. — *Let C_1 and C_2 be two Hida families in $\mathbb{T}_{N_0,\mathfrak{m}}$. Let $x_i \in C_i$ be an $\mathcal{O}_K$-point such that $\kappa = \pi(x_1) = \pi(x_2)$ and π is étale at x_i. We assume that κ is a p-adic limit of classical weights. There are canonical isomorphisms induced by π:*

$$\widehat{\mathcal{O}}_{C_1,x_1} \cong \widehat{\mathcal{O}}_{\mathrm{Spec}(\Lambda),\kappa} \cong \widehat{\mathcal{O}}_{C_2,x_2}.$$

Let $\mathfrak{m}_\kappa$ be the maximal ideal of $\widehat{\mathcal{O}}_{\mathrm{Spec}(\Lambda),\kappa}$. Then x_1 and x_2 are the same point in $\mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$ (in particular, the two components cross over κ) if and only if there exists $u \in \Lambda^\times$ such that

$$L_p^\pm(C_1,\chi) \equiv L_p^\pm(C_2,\chi)u \bmod \mathfrak{m}_\kappa,$$

for all $\chi \in \mathcal{A}_\epsilon$. Furthermore, the intersection multiplicity is at least d if and only if we can pick u satisfying

$$L_p^\pm(C_1,\chi) \equiv L_p^\pm(C_2,\chi)u \bmod \mathfrak{m}_\kappa^d.$$

Roughly speaking, Theorem 1.1 states that two Hida families must cross in $\mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$ if they have the same special values at some point. We can further determine how these two components cross by asking how our p-adic L-functions behave on infinitesimal deformations of the point along the two families. In terms of the parameter T_0, Theorem 1.1 states that two Hida families cross with multiplicity at least d if and only if the Taylor expansions of $L_p^\pm(C_1,\chi)$ and $L_p^\pm(C_2,\chi)$ agree for the first d terms.

The fact that crossing Hida families have congruent p-adic L-functions follows directly from the construction of the p-adic L-functions. The proof of the converse proceeds in several steps. First, we reduce the problem to the situation where both components look *almost* like $\mathrm{Spec}(\Lambda)$. This involves choosing small affinoid neighborhoods of x in the rigid fiber and choosing an appropriate integral model. This integral model is chosen in a way that allows us to remember information about congruences between cusp forms. We develop the necessary geometry in Section 5. When both components look like $\mathrm{Spec}(\Lambda)$ we may employ the p-adic Weierstrass preparation theorem to the p-adic L-functions. Finally, we will apply Theorem 3.9 to a limit of classical weights approaching $\pi(x)$.

1.3. Ramification over the weight space

In the final section we describe the behavior of the p-adic L-functions at points on Hida families that are ramified above the weight space. Informally our result says that a component is étale over the weight space if and only if no poles are introduced when we differentiate each L-function along the weight space. Let C be a Hida family and let T be any parameter for our weight space $\mathrm{Spec}(\Lambda)$. Our parameter defines a derivation on the function field of $\mathrm{Spec}(\Lambda)$ denoted $\frac{\mathrm{d}}{\mathrm{d}T}$. This derivation extends to the function field K of C. If C is étale over $\mathrm{Spec}(\Lambda)$, then $\frac{\mathrm{d}}{\mathrm{d}T}$ will give a derivation on the global functions A of C. If for some $x \in C$ there exists $f \in A$ such that $\frac{\mathrm{d}f}{\mathrm{d}T}$ has a pole at x, then x must be ramified over the weight space. Our main result is that it is enough to check if there exists a Dirichlet character such that $\frac{\mathrm{d}}{\mathrm{d}T} L_p^{\pm}(C,\chi)$ has poles. Furthermore, the order of pole determines the ramification index.

Theorem 1.2. — *A regular $\mathcal{O}_K$-point $x \in C$ is ramified over $\pi(x) \in \mathrm{Spec}(\Lambda)$ if and only if there exists a Dirichlet character $\chi \in \mathcal{A}_\epsilon$ such that $\frac{\mathrm{d}}{\mathrm{d}T} L_p^{\pm}(C,\chi)$ has a pole at x, where T is a parameter of the weight space. The ramification index e of x over $\pi(x)$ is equal to one more than the largest order pole occurring.*

The proof of this theorem is similar to the proof of Theorem 1.1. We first take a small affinoid neighborhood around x which comes naturally equipped with a formal model that is isomorphic to $\mathrm{Spec}(\mathcal{O}_K\langle Y\rangle)$. This allows us to apply the p-adic Weierstrauss preparation theorem and Theorem 3.9 repeatedly. We then deduce by carefully keeping track of congruences as we approach x.

Assume that C is ramified over $\mathrm{Spec}(\Lambda)$ with ramification index e and that $L_p^{\pm}(C,\chi)$ is nonzero at a x for some $\chi \in \mathcal{A}_\epsilon$. By Theorem 3.8 such a χ exists. We may find $g \in \mathbb{T}_{N_0,\mathfrak{m}}^{\times}$ such that $\frac{\mathrm{d}}{\mathrm{d}T} g$ has a pole of order $e-1$. Then if $\frac{\mathrm{d}}{\mathrm{d}T} L_p^{\pm}(C,\chi)$ does not have a pole of order $e-1$, we know by the Leibnitz rule that $\frac{\mathrm{d}}{\mathrm{d}T} g L_p^{\pm}(C,\chi)$ has a pole of order $e-1$. Thus we may choose the isomorphism (1.1) so that Theorem 1.2 holds. The content of Theorem 1.2 is that, regardless of our choice of isomorphism, there exists $L_p^{\pm}(C,\chi)$ that detects the ramification at x. Together, Theorem 1.1 and Theorem 1.2 suggest that $\mathrm{Spec}(\mathbb{T}_{N_0,m})$ can be understood entirely from finitely many p-adic L-functions $L_p^{\pm}(C,\chi_1),\dots,L_p^{\pm}(C,\chi_n)$. In other words, there should be an embedding $\mathrm{Spec}(\mathbb{T}_{N_0,m}) \to \mathbb{A}^{n+1}$ defined by

$$x \in \mathrm{Spec}(\mathbb{T}_{N_0}) \to (\pi(x), L_p(\mathbb{T}_{N_0},\chi_1)(x), \dots L_p(\mathbb{T}_{N_0},\chi_n)(x)).$$

Such an embedding would mean that the p-adic L-functions detect all “tangent” geometric data of $\mathrm{Spec}(\mathbb{T}_{N_0,m})$. Theorem 1.1 and Theorem 1.2 imply that the p-adic L-functions detect tangent data at non-smooth points over the weight space.

1.4. Further Remarks

Recently there has been good deal of work studying ramification over the weight space at points with weight one. For example, in [3], Bellaïche and Dimitrov give a criterion for étaleness. In the ramified situation, work of Betina in [4] proves that the ramification index is exactly two under certain conditions. Previous examples of the ramification index being exactly two were known due to Cho and Vatsal (see [6]). At these ramification points of degree two, it would be interesting to try and find p-adic L-functions that see this ramification, as predicted by Theorem 1.2.

It is also worth mentioning the relationship to the degree three adjoint L-function. In [15] and [16], Hida proves that congruences between cusp forms are determined by special values of the degree three adjoint L-function. This suggests that there should be a two variable p-adic adjoint L-function varying over a Hida family, whose zeros determine ramification over the weight space. This p-adic L-function was constructed by Walter Kim in his PhD thesis (see [23]), and it is proven that the zeros of this p-adic L-function are related to ramification over the weight space. The connection to Kim's work and Theorem 1.2 is unclear. The discussion at the end of 1.3 suggests that the p-adic L-functions $L_p^{\pm}(C, \chi)$ should determine the Hida family C. It seems unlikely that this is true for Kim's p-adic adjoint L-function. However, it is still interesting to ask if one can relate the L-functions $L_p^{\pm}(C, \chi)$ to Kim's p-adic adjoint L-function.

It would be interesting to extend these results to the positive slope part of the eigencurve. There is one technical difficulties that immediately come to mind. The construction of Coleman and Mazur [7] does not come with an integral model. The large Hecke–Hida algebra over the integers of a local field allows us to see congruences. Without an integral model that captures all congruences, our methods fail.

The results in this article should have generalizations to ordinary families of automorphic forms for larger algebraic groups. A several variable p-adic L-function was constructed by Dimitrov in [8] that varies over ordinary families of Hilbert modular forms. It seems likely that our geometric methods could be adapted to this context. Even more generally, it seems plausible that one could construct measures using compatible families of automorphic cycles living in Emerton's completed cohomology (see [9]) that detect ramification over the weight space and crossing.

The author is currently investigating the extension of the results in this paper to points of characteristic p. Following the philosophy of [20] we may view these points as the boundary of our Hida families. These points can be regarded as the ordinary part of the spectral halo conjectured by Coleman. In [20] a formal model is constructed for the part of the eigencurve living over the outer Halo of the weight space. It is plausible that p-adic L-functions can be constructed on the spectral halo and that this formal model could be used to imitate the techniques used in this paper.

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2. Modular symbols and the Eichler-Shimura isomorphism

In this chapter we summarize the theory of modular symbols developed by Manin and then generalized by Ash and Stevens (see [1] and [24]). We also give an overview of Eichler-Shimura theory.

2.1. Modular symbols and cohomology

Throughout this section we will fix $N > 3$ and $\Gamma = \Gamma_1(N)$. Let D_0 be the divisors of $\mathbb{P}(\mathbb{Q})$ of degree 0. Then $GL_2(\mathbb{Q})$, and therefore also Γ, acts on D_0 by linear fractional transformations. For any left $\mathbb{Z}[\Gamma]$-module E, we let $\Phi(E) = \mathrm{Hom}_\Gamma(D_0, E)$. These are *modular symbols* with values in E (see, for example, [1]). When the action on E extends to $GL_2(\mathbb{Q})$ (resp. $GL_2(\mathbb{Z})$) we may define a right action on $\mathrm{Hom}(D_0, E)$ (resp. $GL_2(\mathbb{Z})$). Explicitly, if $\alpha \in \Phi(E)$ and $g \in GL(\mathbb{Q})$ then $\alpha|_g$ sends $(r_1 - r_2)$ to $g^{-1}\alpha(g(r_1) - g(r_2))$. The Γ-invariant elements of $\mathrm{Hom}(D_0, E)$ are precisely $\Phi(E)$.

There is a locally constant sheaf $\widetilde{E}$ on $\mathbb{H}/\Gamma$ that is associated to E. The sections of $\widetilde{E}$ are sections of the E-torsor $s : E \times \mathbb{H}/\Gamma \to \mathbb{H}/\Gamma$. More precisely, for an open set $U \subset \mathbb{H}/\Gamma$ the sections $\Gamma(U, \widetilde{E})$ are continuous functions $f : U \to s^{-1}(U)$ such that $f \circ s$ is the identity (here we give E the discrete topology). If U is small enough to trivialize s (i.e. $s^{-1}(U) = U \times E$) then $\Gamma(U, \widetilde{E})$ is just isomorphic to E. It is known that $\Phi(E) \cong H^1_c(\mathbb{H}/\Gamma, \widetilde{E})$ (see [1, Proposition 4.2]). We define $H^1_!(\mathbb{H}/\Gamma, \widetilde{E})$ to be the image of $H^1_c(\mathbb{H}/\Gamma, \widetilde{E})$ in $H^1(\mathbb{H}/\Gamma, \widetilde{E})$. Explicitly, we may think of $H^1_!(\mathbb{H}/\Gamma, \widetilde{E})$ as the cohomology classes in $H^1(\mathbb{H}/\Gamma, \widetilde{E})$ that can be represented by a 1-form with compact support. Let $[c] \in H^1(\mathbb{H}/\Gamma, \widetilde{E})$ and let ω

be a 1-form representing $[c]$. Then for any $z_0 \in \mathbb{H}$ we may define a 1-cocycle on Γ with values in E:

$$g \to \int_{z_0}^{g(z_0)} \omega.$$

A different choice of ω or z_0 will result in a 1-cocycle that differs by a 1-coboundary. When $[c]$ is in $H^1_!(\mathbb{H}/\Gamma, \widetilde{E})$ we may take ω to have compact support. This allows us to choose $z_0 \in \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$. If $z_0 \in \mathbb{P}^1(\mathbb{Q})$ then the 1-cocycle is zero when restricted to the parabolic subgroup P_{z_0} that fixes z_0. Putting this together gives the following commutative diagram:

$$\begin{tikzcd} \Phi(E) \cong H^1_c(\mathbb{H}/\Gamma, \widetilde{E}) \arrow[r] & H^1_!(\mathbb{H}/\Gamma, \widetilde{E}) \arrow[r] \arrow[d] & H^1(\mathbb{H}/\Gamma, \widetilde{E}) \arrow[d] \\ & H^1_P(\Gamma, E) \arrow[r] & H^1(\Gamma, E). \end{tikzcd}$$

Here we define

$$H^1_P(\Gamma, E) := \ker(H^1(\Gamma, E) \to \prod_{z_0 \in \mathbb{P}^1(\mathbb{Q})} H^1(P_{z_0}, E)).$$

In general, these vertical maps are isomorphisms as long as Γ contains a torsion free subgroup of finite index that is coprime to the exponent of E. This condition is satisfied regardless of E, since we have taken Γ to be torsion free.

2.2. The complex conjugation involution

The involution σ of $\mathbb{H}$ given by $z \to -\bar{z}$ induces involutions on the cohomology groups discussed above. Consider the 1-cocycle β defined by a 1-form ω_β. Then β^σ is the 1-cocycle

$$g \to \int_i^{-\overline{g(i)}} \omega_\beta = \int_i^{g(i)} \sigma^*(\omega_\beta).$$

Thus β is sent to the 1-cocycle $g \to \beta(\xi g \xi^{-1})$, where $\xi = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. On de Rham cohomology the 1-form ω is send to its pullback $\sigma^*(\omega)$ under σ. In particular, holomorphic forms are sent to anti-holomorphic forms and vice versa. The involution σ sends a modular symbol $\alpha \in \Phi(E)$ to $\alpha|_\xi$.

If E is 2-divisible (i.e. E is a $\mathbb{Z}[\frac{1}{2}]$-module) then the cohomology groups considered in Subsection 2.1 decompose into eigenspaces of σ. For example, we have $H^1(\Gamma,E) = H^1(\Gamma,E)^+ \oplus H^1(\Gamma,E)^-$, where σ fixes the $H^1(\Gamma,E)^+$ and negates $H^1(\Gamma,E)^-$. This yields:

$$\begin{array}{ccccc} \Phi(E)^\pm \cong H^1_c(\mathbb{H}/\Gamma,\widetilde{E})^\pm & \longrightarrow & H^1_!(\mathbb{H}/\Gamma,\widetilde{E})^\pm & \longrightarrow & H^1(\mathbb{H}/\Gamma,\widetilde{E})^\pm \\ & & \downarrow & & \downarrow \\ & & H^1_P(\Gamma,E)^\pm & \longrightarrow & H^1(\Gamma,E)^\pm. \end{array}$$

2.3. The Eichler-Shimura isomorphism

For any ring A, we define $L_n(A)$ to be the space of degree n homogeneous polynomials in two variables with coefficients in A. Then $L_n(A)$ comes equipped with a left action of Γ. When $k \geqslant 2$ there is a map from $S_k(\Gamma,\mathbb{C})$, the weight k cusp forms on Γ with coefficients in $\mathbb{C}$, to the cohomology group $H^1(\mathbb{H}/\Gamma,\widetilde{L}_{k-2}(\mathbb{C}))$: the cusp form $f(z) \in S_k(\Gamma,\mathbb{C})$ is sent to the 1-form

$$\omega_f = f(z)(x-zy)^n \mathrm{d}z.$$

Since $f(z)$ vanishes at cusps $z_0 \in \mathbb{P}^1(\mathbb{Q})$ we may consider the 1-cocycle:

$$g \to \int_{z_0}^{g(z_0)} \omega_f.$$

This 1-cocycle vanishes on P_{z_0}, which lets us infer that

$$\omega_f \in H^1_!(\mathbb{H}/\Gamma,\widetilde{L}_{k-2}(\mathbb{C})) \cong H^1_P(\Gamma,L_{k-2}(\mathbb{C})).$$

By projecting onto the $\pm$ parts we obtain the Eichler-Shimura isomorphism (see [29, Chapter 8] for a full proof):

$$S_k(\Gamma,\mathbb{C}) \cong H^1_P(\Gamma,L_{k-2}(\mathbb{C}))^\pm.$$

2.4. Hecke operators and integral cohomology

We may define Hecke operators on the cohomology groups from Section 2.1 (see for example [29, Chapter 8.3] or [1, Section 2]). These operators are compatible with the Eichler-Shimura isomorphism. Let $f \in S_k(\Gamma,\mathbb{C})$ be a normalized eigenform and let $\omega_f^\pm$ be the projection of the 1-form ω_f onto the $\pm$ part. We define a modular symbol $\alpha_f^\pm$ by

$$\alpha_f^\pm(\{r_1\}-\{r_2\}) = \int_{r_1}^{r_2} \omega_f^\pm.$$

This gives a Hecke equivariant map $s : S_k(\Gamma, \mathbb{C}) \to \Phi(L_{k-2}(\mathbb{C}))^{\pm}$. By a theorem of Shimura (see [13, Theorem 4.8]) the subspace of $\Phi(L_{k-2}(\mathbb{C}))^{\pm}$ that has the same Hecke eigenvalues as f is one dimensional.

Fix an isomorphism $\mathbb{C}_p \cong \mathbb{C}$ and let $K \subset \mathbb{C}$ be a finite extension of $\mathbb{Q}_p$ that contains the Hecke eigenvalues of f. Let $\mathcal{O}_K$ be the ring of integers of K with uniformizer π_K. Since modular symbols commute with flat base change (cf. [2, Lemma III.1.2]) we have a Hecke equivariant isomorphism

$$\Phi(L_{k-2}(\mathcal{O}_K))^{\pm} \otimes_{\mathcal{O}_K} \mathbb{C} \cong \Phi(L_{k-2}(\mathbb{C}))^{\pm}.$$

As $\mathcal{O}_K$ contains the Hecke eigenvalues of f, the subspace of $\Phi(L_{k-2}(\mathcal{O}_K))$ that has the same Hecke eigenvalues as f is a free $\mathcal{O}_K$-module of rank one. From this we see that there exist periods $\Omega_f^{\pm}$ such that

$$\frac{\alpha_f^{\pm}}{\Omega_f^{\pm}} \in \Phi(L_{k-2}(\mathcal{O}_K))^{\pm}$$

and

$$\frac{\alpha_f^{\pm}}{\Omega_f^{\pm}} \notin \pi_K \Phi(L_{k-2}(\mathcal{O}_K))^{\pm}.$$

These periods are unique up to multiplication by a unit in $\mathcal{O}_K$.

Now consider the Hecke equivarient commutative diagram

$$\begin{array}{ccccc} \Phi(L_{k-2}(\mathcal{O}_K))^{\pm} & \xrightarrow{i} & H_P^1(\Gamma, L_{k-2}(\mathcal{O}_K))^{\pm} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \Phi(L_{k-2}(\mathbb{C}))^{\pm} & \xrightarrow{i} & H_P^1(\Gamma, L_{k-2}(\mathbb{C}))^{\pm} & \longrightarrow & 0 \end{array}$$

Note that $i(\alpha_f^{\pm}) = \omega_f^{\pm}$. In particular, we see that the subspace of $H_P^1(\Gamma, L_{k-2}(\mathcal{O}_K))^{\pm}$ that has the same Hecke eigenvalues as f is a free $\mathcal{O}_K$-module of rank one generated by $i(\frac{\alpha_f^{\pm}}{\Omega_f^{\pm}}) = \frac{\omega_f^{\pm}}{\Omega_f^{\pm}}$. Furthermore we see that

$$H^1(\Gamma, L_{k-2}(\mathcal{O}_K))^{\pm} \cap \mathbb{C}\omega_f = \mathcal{O}_K \frac{\omega_f^{\pm}}{\Omega_f^{\pm}}.$$

3. Congruences Between Cusp Forms and L-functions

The aim of this chapter is to prove that two cusp forms are congruent if and only if the "algebraic" special values of their L-functions admit congruences for all twists (see Theorem 3.8). The heart of the proof is Theorem 3.1, which specifies certain linear combinations of cycles on $\mathbb{H}/\Gamma$ that generate the ordinary part of $H_1(\mathbb{H}/\Gamma)$. This type of result was first observed by Glenn Stevens and in particular Theorem 3.1 was inspired by Theorem 2.1 in [30].

3.1. Special values of modular symbols

For this section we will take $\Gamma = \Gamma_1(N_0p^r)$, where N_0 is prime to p and $r \geqslant 1$. We let $N = N_0p^r$. Let $\mathcal{O}_K$ be the ring of integers of a finite extension K of $\mathbb{Q}_p$. Let π_K be a uniformizing element of $\mathcal{O}_K$. Let $s > 0$ and assume $\pi_K^s | p^r$ (if this is not the case we may replace Γ with a smaller congruence subgroup by increasing r). The purpose of this section is to prove a nonvanishing result for the special values of modular symbols with values in $L_n(\mathcal{O}_K/\pi_K^s)$. We let $[\frac{x}{y}]$ denote the degree zero divisor $\{\frac{x}{y}\} - \{\infty\}$. For a Dirichlet character χ of conductor m_χ we define

$$\Lambda(\chi) = \sum_{i=0}^{m_\chi - 1} \overline{\chi(i)} \left[\frac{i}{m_\chi}\right] \in D_0 \otimes \mathbb{Z}[\chi].$$

If α is the modular symbol associated to a cusp form then the first coordinate (i.e. the coefficient of X^n) of $\alpha(\Lambda(\chi))$ is a normalized special value (see Section 3.2). For $P(X,Y) \in L_n(\mathcal{O}_K/\pi_K^s)$ the coefficient of X^n is $P(1,0)$. Therefore it makes sense if we write $\alpha(\Lambda(\chi))(1,0)$ to denote the coefficient of X^n in $\alpha(\Lambda(\chi))$. The next theorem says that under certain conditions a modular symbol is completely determined by its special values. For $\epsilon > 0$ we define $\mathcal{P}_\epsilon$ to be the set of primes q larger than ϵ that satisfy the congruences

$$q \equiv -1 \bmod N.$$

We then define $\mathcal{A}'_\epsilon$ to be the set of all primitive character whose conductor lies in $\mathcal{P}_\epsilon$ and we define $\mathcal{A}_\epsilon = \mathcal{A}'_\epsilon \cup \{\chi_{\mathrm{triv}}\}$. Our main result of this section is

Theorem 3.1. — *Let $\alpha \in \Phi(L_n(\mathcal{O}_K/\pi_K^s))$. Assume the following conditions:*

1. *For every primitive Dirichlet character $\chi \in \mathcal{A}_\epsilon$ the special value $\alpha(\Lambda(\chi))(1,0)$ is zero.*
2. *The image of α in $H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^s))$ lies in the p-ordinary subspace $H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^s))^{\mathrm{ord}}$ (see, for example,* [19, Chapter 7]*).*
3. *The Nebentypus of α is a Dirichlet character ψ (i.e. for $\gamma \in \Gamma_0(N)$ we have $\alpha|_\gamma = \psi(d)\alpha$). The conductor of ψ is necessarily N.*

Then the image of α in $H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^s))$ is zero.

The proof of Theorem 3.1 will be broken up into several smaller lemmas.

Lemma 3.2. — *Let $\frac{c}{d}$ be a reduced fraction whose denominator is* $1 \bmod N$*. Then there exists $\gamma \in \Gamma_0(N)$ such that the denominators of $\gamma(\frac{c}{d})$ and $\gamma(0)$ are in $\mathcal{P}_\epsilon$.*

Proof. — Let l_1 be a prime number satisfying

$$l_1 \equiv -1 \bmod N.$$

We may take l_1 large enough to be contained in $\mathcal{P}_\epsilon$ and so that $l_1 \nmid c$. As l_1 and d are both coprime to Nc, it possible to choose a prime $z > l_1$ satisfying

$$z \equiv dl_1 \bmod Nc.$$

Then

$$z \equiv -1 \bmod N,$$

since

$$d \equiv 1 \bmod N \quad \text{and} \quad l_1 \equiv -1 \bmod N.$$

In particular $z \in \mathcal{P}_\epsilon$. We have $z = yNc + dl_1$ for some y and we set $l_2 = Ny$. Note that l_2 is not divisible by l_1: if $l_1 | l_2$ then we see that $l_1 | z$, which is impossible as z is a prime larger than l_1. Since $l_2 z$ and l_1 are relatively prime we may find t_2 and t_1 such that

$$l_1 t_2 - l_2 z t_1 = 1.$$

Thus the matrix

$$\gamma = \begin{pmatrix} t_2 & t_1 z \\ l_2 & l_1 \end{pmatrix}$$

is in $\Gamma_0(N)$. We compute

$$\begin{pmatrix} t_2 & t_1 z \\ l_2 & l_1 \end{pmatrix} \frac{c}{d} = \frac{t_2 c + t_1 z d}{z} \quad \text{and} \quad \begin{pmatrix} t_2 & t_1 z \\ l_2 & l_1 \end{pmatrix} 0 = \frac{t_1 z}{l_1}.$$

The fraction $\frac{t_1 z}{l_1}$ is reduced since l_1 is coprime to t_1 and z. Furthermore z is coprime to $t_2 c$ so that $\frac{t_2 c + t_1 z d}{z}$ is also reduced. This means γ satisfies the desired properties. □

Lemma 3.3. — *Let $m \in \mathcal{P}_\epsilon$. The coefficient of X^n in $\alpha([\frac{a}{m}])$ is zero if a is prime to m That is,*

$$\alpha\Big(\Big[\frac{a}{m}\Big]\Big)(1,0) = 0.$$

Proof. — Let $\mathcal{O}_{K_m}$ be the ring of integers of $K_m = \mathbb{Q}_p(\zeta_{m-1})$. Recall that D_0 is the group of degree zero divisors of $\mathbb{P}^1(\mathbb{Q})$. Let M be the free submodule of $D_0 \otimes_{\mathbb{Z}} \mathcal{O}_{K_m}$ generated by the elements $[\frac{1}{m}], \dots, [\frac{m-1}{m}]$ and let M' be the submodule of M spanned by each $\Lambda(\chi)$ for primitive χ of conductor m. We claim that $M = M' \oplus \mathcal{O}_{K_m}[\frac{1}{m}]$. To see this, consider the nonprimitive character 1_d defined by $1_d(a) = 1$ for $m \nmid a$ and $1_d(a) = 0$ otherwise. The index of $M' \oplus \mathcal{O}_{K_m}\Lambda(1_d)$ in M is given by the Vandermonde determinant

$$\prod_{0 \leqslant i < j \leqslant m-2} (\zeta_{m-1}^j - \zeta_{m-1}^i),$$

where ζ_{m-1} is an $m-1$th root of unity. We know that $p | m+1$ so that $p \nmid m-1$. Therefore $\zeta_{m-1}^j - \zeta_{m-1}^i$ is a p-adic unit whenver $j > i$ and in particular we find that $M' \oplus \mathcal{O}_{K_m}\Lambda(1_d) = M$. Now note that

$$\Lambda(1_d) = \sum_{\text{prim } \chi} \Lambda(\chi) - (m-2)\left[\frac{1}{m}\right].$$

From this we see $\Lambda(1_d)$ is contained in $M' \oplus \mathcal{O}_{K_m}[\frac{1}{m}]$ and thus

$$M = M' \oplus \mathcal{O}_{K_m}\left[\frac{1}{m}\right].$$

By the first hypothesis in Theorem 3.1 we know that $\alpha(t)(1,0)$ is zero for $t \in M'$. Therefore it will suffice to show $\alpha([\frac{1}{m}])(1,0)$ is zero. We have

$$\begin{aligned}\alpha\left(\left[\frac{1}{m}\right]\right)(1,0) &= \alpha\left(\left[\frac{1}{m}\right]\right)(1,0) - \alpha([0])(1,0)\\ &= \alpha\left(\left\{\frac{1}{m}\right\} - \{0\}\right)(1,0),\end{aligned}$$

where $\alpha([0])(1,0) = 0$ because $[0] = \Lambda(\chi_{\text{triv}})$. Let $\gamma_0 = \left(\begin{smallmatrix} 1 & 0 \\ -Nk & 1 \end{smallmatrix}\right)$, where $m = Nk - 1$. Then

$$\begin{aligned}\gamma_0\left(\left\{\frac{1}{m}\right\} - \{0\}\right) &= \{-1\} - \{0\}\\ &= [-1] - [0].\end{aligned}$$

There is an upper triangular matrix $\gamma_1 \in \Gamma$ such that $\gamma_1([-1]) = [0]$. The action of an upper triangular matrix on $L_n(\mathcal{O}_K/\pi_K^s)$ preserves the coefficient of X^n. In particular we see that $\alpha([-1])(1,0)$ is zero. Since γ_0 acts on $L_n(\mathcal{O}_K/\pi_K^s)$ as the identity, we find

$$\begin{aligned}\alpha\left(\left\{\frac{1}{m}\right\} - \{0\}\right)(1,0) &= \gamma_0\left(\alpha\left(\left\{\frac{1}{m}\right\} - \{0\}\right)\right)(1,0)\\ &= \alpha([-1] - [0])(1,0)\\ &= 0.\end{aligned}$$

The result follows. □

Lemma 3.4 (Hida). — *The map from $L_n(\mathcal{O}_K/\pi_K)$ to $L_0(\mathcal{O}_K/\pi_K)$ induced by projecting onto the coordinate of X^n induces an isomorphism*

$$H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K))^{\mathrm{ord}} \to H^1(\Gamma, \mathcal{O}_K/\pi_K)^{\mathrm{ord}}.$$

Proof. — See the proof of Theorem 2 in [19, Section 7.2] for the case of $\mathcal{O}_K = \mathbb{Z}_p$. To deduce the general case we first note that $\mathcal{O}_K/\pi_K$ is a free $\mathbb{Z}_p/p$-module. Therefore the natural map

$$H^1(\Gamma, L_n(\mathbb{Z}_p/p)) \otimes_{\mathbb{Z}_p/p} \mathcal{O}_K/\pi_K \to H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K))$$

is an isomorphism. This isomorphism commutes with the map induced by projecting onto the X^n-coordinate. It also commutes with the Hecke operators, which includes Hida's idempotent e. This gives a square of isomorphisms:

$$\begin{array}{ccc} H^1(\Gamma, L_n(\mathbb{Z}_p/p))^{\mathrm{ord}} \otimes_{\mathbb{Z}_p/p} \mathcal{O}_K/\pi_K & \longrightarrow & H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K))^{\mathrm{ord}} \\ \downarrow & & \downarrow \\ H^1(\Gamma, \mathbb{Z}_p/p)^{\mathrm{ord}} \otimes_{\mathbb{Z}_p/p} \mathcal{O}_K/\pi_K & \longrightarrow & H^1(\Gamma, \mathcal{O}_K/\pi_K)^{\mathrm{ord}} \end{array} \qquad \square$$

Lemma 3.5. — *Let $m' > m$. Then*

$$H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^m)) = H^1(\Gamma, L_n(\mathcal{O}_K)) \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi_K^m,$$

and

$$H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^m)) = H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^{m'})) \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi_K^m.$$

Furthermore, these isomorphisms commute with the Hecke operators and thus give isomorphisms of the ordinary subspaces.

Proof. — We see from (1.10_a) and (1.11) of [15] that

$$H^1(\Gamma, L_n(\mathbb{Z}/p^m)) = H^1(\Gamma, L_n(\mathbb{Z})) \otimes \mathbb{Z}/p^m.$$

Now consider the exact sequence

$$0 \longrightarrow L_n(\mathbb{Z}_p) \xrightarrow{\times p} L_n(\mathbb{Z}_p) \longrightarrow L_n(\mathbb{Z}_p/p) \longrightarrow 0.$$

The long exact sequence of cohomology groups gives the short exact sequence

$$0 \to H^1(\Gamma, L_n(\mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p \to H^1(\Gamma, L_n(\mathbb{Z}_p/p)) \to H^2(\Gamma, L_n(\mathbb{Z}_p))[p] \to 0,$$

where $H^2(\Gamma, L_n(\mathbb{Z}_p))[p]$ is the subgroup of p-torsion elements in $H^2(\Gamma, L_n(\mathbb{Z}_p))$. However we already know that the first map is an isomorphism, which means $H^2(\Gamma, L_n(\mathbb{Z}_p))[p] = 0$. Since $\mathcal{O}_K$ is a free $\mathbb{Z}_p$-module we see that

$$H^2(\Gamma, L_n(\mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathcal{O}_K = H^2(\Gamma, L_n(\mathcal{O}_K)).$$

In particular, $H^2(\Gamma, L_n(\mathcal{O}_K))$ has no p-torsion and consequently no π_K-torsion. Now consider the sequence

$$0 \longrightarrow L_n(\mathcal{O}_K) \xrightarrow{\times \pi_K^m} L_n(\mathcal{O}_K) \longrightarrow L_n(\mathcal{O}_K/\pi_K^m) \longrightarrow 0,$$

which gives rise to the sequence of cohomology groups

$$\begin{aligned} 0 \to H^1(\Gamma, L_n(\mathcal{O}_K)) \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi_K^m & \\ \to H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^m)) &\to H^2(\Gamma, L_n(\mathcal{O}_K))[\pi_K^m]. \end{aligned}$$

Since $H^2(\Gamma, L_n(\mathcal{O}_K))$ has no π_K-torsion we conclude

$$H^1(\Gamma, L_n(\mathcal{O}_K)) \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi_K^m = H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^m)).$$

The second isomorphism follows from the first together with that fact that if M is any $\mathcal{O}_K$-module then

$$M \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi_K^m = (M \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi_K^{m'}) \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi_K^m.$$

The map

$$H^1(\Gamma, L_n(\mathcal{O}_K)) \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi_K^m \to H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^m))$$

commutes with any double coset action, which gives the Hecke equivariance. □

LEMMA 3.6. — *Let $m>0$ such that $\pi_K^m | p^r$. Let $[c] \in H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^m))^{\mathrm{ord}}$ be a cohomology class represented by a 1-cocycle c. Assume that for all $\gamma \in \Gamma$ we have $c(\gamma)(1,0) = 0$ (i.e. the coefficient of X^n in $c(\gamma)$ is zero). Then $[c] = 0$.*

Proof. — We will prove this lemma by induction on m. When $m = 1$ this follows immediately from Lemma 3.4. Now let $m > 1$. The image of $[c]$ in $H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K))^{\mathrm{ord}}$ is 0 by Lemma 3.4. Then by Lemma 3.5 we know that $[c] = \pi_K[c_0]$ for some $[c_0]$ in $H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^m))^{\mathrm{ord}}$. Thus $\pi_K c_0 - c$ is a 1-coboundary b. There exists $p(X,Y) \in L_n(\mathcal{O}_K/\pi_K^m)$ such that $b(\gamma) = (1-\gamma)p(X,Y)$. Since π_K^m divides p^r, we know that γ reduces to an upper triangular matrix modulo π_K^m. This means that $b(\gamma)(1,0) = 0$, as applying γ to $p(X,Y)$ does not change the coefficient of X^n. Since $\pi_K c_0 = b + c$ we find

$$\begin{aligned} \pi_K c_0(\gamma)(1,0) &= b(\gamma)(1,0) + c(\gamma)(1,0) \\ &= 0. \end{aligned}$$

Therefore $c_0(\gamma)(1,0)$ is divisible by π_K^{m-1}. We now use our induction hypothesis to see that the cohomology class $[c_0] \in H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^{m-1}))$ is zero. Since

$$H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^m))^{\mathrm{ord}} \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi_K^{m-1} = H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^{m-1}))^{\mathrm{ord}},$$

This implies $[c_0]$ is divisible by π_K^{m-1} in $H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^m))^{\mathrm{ord}}$. It follows that $\pi_K[c_0] = [c] = 0$. $\square$

Proof of Theorem 3.1. — The image of α in $H^1(\Gamma, L_n(\mathcal{O}_K/\pi_K^s))$ can be represented by the 1-cocycle that sends $\gamma \in \Gamma$ to $\alpha(\{0\} - \{\gamma(0)\})$. Let x be $\gamma(0)$. The denominator of x is 1 modulo N, so we may apply Lemma 3.2. That is, we may find $g \in \Gamma_0(N)$ such that $g(\{x\})$ and $g(\{0\})$ have denominators in $\mathcal{P}_\epsilon$. In particular, we know that $\alpha(g(\{x\}) - g(\{0\}))(1,0) = 0$ by Lemma 3.3. Let c be the lower right entry of g. Then

$$\begin{aligned}\alpha(\{x\} - \{0\})(1,0) &= \psi(c)^{-1}\alpha|_g(\{x\} - \{0\})(1,0) \\ &= \psi(c)^{-1}g^{-1}\alpha(g(\{x\}) - g(\{0\}))(1,0) \\ &= 0\end{aligned}$$

Applying Lemma 3.6 proves the result. $\square$

3.2. Special values of L-functions

Let $f(z) \in S_k(\Gamma, \mathcal{O}_K)$ with $k \geqslant 2$ be a normalized eigenform whose eigenvalues are contained in K. We may write $f(z) = \Sigma a_n q^n$ where $q = e^{2i\pi z}$ and $a_n \in \mathcal{O}_K$. Then $L(f,s)$ is defined to be $\Sigma a_n n^{-s}$. Using the Mellin transform we may write the special values of $L(f,s)$ as an integral (see [1, Section 4]):

$$\int_0^{i\infty} f(z) z^j \mathrm{d}z = \frac{j! L(f, j+1)}{(-2\pi i)^{j+1}}.$$

More generally, if χ is a Dirichlet character of conductor m_χ we define $L(f,\chi,s)$ as $\Sigma a_n \chi(n) n^{-s}$. This can be written as the integral (see [26, Equation (8.6)]):

$$\sum_{a=0}^{m-1} \overline{\chi(a)} \int_{\frac{a}{m}}^{i\infty} f(z)(m_\chi z + a)^j \mathrm{d}z = \tau(\overline{\chi}) m_\chi^{j+1} j! \frac{L(f,\chi,j+1)}{(-2\pi i)^{j+1}}.$$

In particular, if $\alpha_f^{\pm}$ is the modular symbol defined in Section 2.4, then the coefficient of X^{k-2} in $\frac{\alpha_f^{\pm}(\Lambda(\chi))}{\Omega_f^{\pm}}$ is $\tau(\overline{\chi}) m_\chi \frac{L(f,\chi,1)}{-2\pi i \Omega_f^{\pm}}$ when the parity of α_f^* matches the parity of χ (when the parities are mixed the modular symbol evaluates to zero).

Theorem 3.7. — *Let f and g be two p-ordinary cuspidal eigenforms in $S_k(\Gamma, \mathcal{O}_K)$ with the same nebentypus. Let $s > 0$ such that $\pi_K^s | p^r$. If we can choose periods $\Omega_f^{\pm}$ and $\Omega_g^{\pm}$ (as in Section 2.4) such that*

$$\frac{L(f,\chi,1)}{2\pi i \Omega_f^{\pm}} \equiv \frac{L(g,\chi,1)}{2\pi i \Omega_g^{\pm}} \bmod \pi_K^s \mathcal{O}_K[\chi]$$

for all Dirichlet characters $\chi \in \mathcal{A}_\epsilon$, then the image of $\frac{\alpha_f^{\pm}}{\Omega_f^{\pm}} - \frac{\alpha_g^{\pm}}{\Omega_g^{\pm}}$ under the map

$$\Phi(L_{k-2}(\mathcal{O}_K)) \to H^1(\Gamma, L_{k-2}(\mathcal{O}_K))$$

from is contained in $\pi_K^s H^1(\Gamma, L_{k-2}(\mathcal{O}_K))$.

Proof. — Let η denote the modular symbol $\frac{\alpha_f^{\pm}}{\Omega_f^{\pm}} - \frac{\alpha_g^{\pm}}{\Omega_g^{\pm}}$. By the discussion above we have

$$\begin{aligned}\eta(\Lambda(\chi))(1,0) &= \frac{\alpha_f^{\pm}(\Lambda(\chi))(1,0)}{\Omega_f^{\pm}} - \frac{\alpha_g^{\pm}(\Lambda(\chi))(1,0)}{\Omega_g^{\pm}} \\ &= \tau(\overline{\chi}) m_\chi \frac{L(f,\chi,1)}{-2\pi i \Omega_f^{\pm}} - \tau(\overline{\chi}) m_\chi \frac{L(g,\chi,1)}{-2\pi i \Omega_g^{\pm}},\end{aligned}$$

which is contained in $\tau(\overline{\chi}) m_\chi \pi_K^s \mathcal{O}_K$ by our hypothesis. Thus $\eta(\Lambda(\chi)) \in \pi_K^s \mathcal{O}_K$ for all $\chi \in \mathcal{A}_\epsilon$. Then by Theorem 3.1 it follows that the image of η in $H^1(\Gamma, L_{k-2}(\mathcal{O}_K/\pi_K^s))$ is zero. From Lemma 3.5 we deduce that η is contained in $\pi_K^s H^1(\Gamma, L_{k-2}(\mathcal{O}_K))$. □

3.3. Congruences between special values

We may now prove Theorem 3.8 by combining Theorem 3.7 with a standard congruence module argument (cf [27] and [15]). Let f and g be normalized p-ordinary cuspidal eigenforms of weight $k \geqslant 2$ for the congruence subgroup $\Gamma = \Gamma_1(N_0 p^r)$. Assume that $f(z) = \Sigma a_n q^n$ and $g(z) = \Sigma b_n q^n$ have Fourier coefficients in $\mathcal{O}_K$. Let $s > 0$ such that

$$\frac{L(f,\chi,1)}{2\pi i \Omega_f^{\pm}} \equiv \frac{L(g,\chi,1)}{2\pi i \Omega_g^{\pm}} \bmod \pi_K^s,$$

for all primitive Dirichlet characters χ whose conductor is prime to p. Since f and g are also eigenforms for $\Gamma_1(N_0 p^{r'})$ for $r' > r$, we may replace Γ with a smaller congruence subgroup and assume that π_K^s divides p^r. Let ω_f and ω_g be the differential forms associated to f and g. In particular, if i denotes the map from modular symbols to cohomology then $i(\alpha_*) = \omega_*$ (see Section 2.4). By abuse of notation, when we refer to $H^1(\Gamma, L_{k-2}(\mathcal{O}_K))^{\pm}$

we will actually mean the image of $H^1(\Gamma, L_{k-2}(\mathcal{O}_K))^\pm$ in $H^1(\Gamma, L_{k-2}(\mathbb{C}))^\pm$ (i.e. the torsion free part viewed as a "lattice" in the complex cohomology group).

Let $M^\pm$ be $H^1(\Gamma, L_{k-2}(\mathcal{O}_K))^\pm \cap (\mathbb{C}\omega_f^\pm \oplus \mathbb{C}\omega_g^\pm)$. Then by end of Section 2.4 we know that $M^\pm$ is a rank two free $\mathcal{O}_K$-submodule of $H^1(\Gamma, L_{k-2}(\mathcal{O}_K))^\pm$ fixed by the Hecke operators. Let $M_*^\pm = M^\pm \cap \mathbb{C}\omega_*^\pm$. This is the subspace of $H^1(\Gamma, L_{k-2}(\mathcal{O}_K))^\pm$ whose Hecke eigenvalues are the same as f. By Section 2.4 $M_*^\pm$ is generated by $\frac{\omega_*^\pm}{\Omega_*^\pm}$ as a $\mathcal{O}_K$-module. We also let $M^{*\pm}$ be the projection of $M^\pm$ onto $\mathbb{C}\omega_*^\pm$. Note that $M_*^\pm \subset M^{*\pm}$. By Theorem 3.7 we know that

$$\frac{i(\alpha_f^\pm)}{\Omega_f^\pm} - \frac{i(\alpha_g^\pm)}{\Omega_g^\pm} = \frac{\omega_f^\pm}{\Omega_f^\pm} - \frac{\omega_g^\pm}{\Omega_g^\pm} \in \pi_K^s H^1(\Gamma, L_{k-2}(\mathcal{O}_K))^\pm.$$

In particular, we may find $x \in H^1(\Gamma, L_{k-2}(\mathcal{O}_K))^\pm$ with $\pi_K^s x = \frac{\omega_f^\pm}{\Omega_f^\pm} - \frac{\omega_g^\pm}{\Omega_g^\pm}$. As x is in the $\mathcal{O}_K$-submodule spanned by $\frac{\omega_f^\pm}{\Omega_f^\pm}$ and $\frac{\omega_g^\pm}{\Omega_g^\pm}$ we see that $M^\pm$ contains x. Thus we have a map

$$\mathcal{O}_K/\pi_K^s\mathcal{O}_K \to \frac{M^\pm}{M_f^\pm \oplus M_g^\pm}, \text{ defined by}$$
$$1 \bmod \pi_K^s\mathcal{O}_K \to x \bmod M_f^\pm \oplus M_g^\pm.$$

In fact this map is an injection. This is true because $M_f^\pm \oplus M_g^\pm$ is a free rank two $\mathcal{O}_K$-module generated by $\frac{\omega_f^\pm}{\Omega_f^\pm}$ and $\frac{\omega_g^\pm}{\Omega_g^\pm}$. If the map had a kernel then

$$\pi_K^{s'} x = \pi_K^{s'-s}\frac{\omega_f^\pm}{\Omega_f^\pm} - \pi_K^{s'-s}\frac{\omega_g^\pm}{\Omega_g^\pm} \in M_f^\pm \oplus M_g^\pm$$

for some $s' < s$, which is impossible.

There is an equivalence of Hecke modules (see for example [12, Lemma 1]):

$$\frac{M^{f\pm}}{M_f^\pm} \cong \frac{M^{f\pm} \oplus M^{g\pm}}{M^\pm} \cong \frac{M^{g\pm}}{M_g^\pm}, \text{ and}$$
$$\frac{M^{f\pm} \oplus M^{g\pm}}{M^\pm} \cong \frac{M^\pm}{M_f^\pm \oplus M_g^\pm}.$$

In particular we find that

$$\frac{M^{f\pm}}{M_f^\pm} \otimes \mathcal{O}_K/\pi_K^s\mathcal{O}_K \cong \frac{M^{g\pm}}{M_g^\pm} \otimes \mathcal{O}_K/\pi_K^s\mathcal{O}_K,$$

as Hecke modules. Since $M_*^{\pm}$ is isomorphic to $\mathcal{O}_K$ we know that $\frac{M^{*\pm}}{M_*^{\pm}}$ is isomorphic to $\mathcal{O}_K/\pi_K^e\mathcal{O}_K$ for some $e \geqslant 0$. By the above isomorphisms there is an injection $\mathcal{O}_K/\pi_K^s \to \frac{M^{*\pm}}{M_*^{\pm}}$ and therefore $e \geqslant s$. Thus $\frac{M^{*\pm}}{M_*^{\pm}} \otimes \mathcal{O}_K/\pi_K^s\mathcal{O}_K$ is isomorphic to $\mathcal{O}_K/\pi_K^s\mathcal{O}_K$. The Hecke operator T_n acts on $\frac{M^{f\pm}}{M_f^{\pm}} \otimes \mathcal{O}_K/\pi_K^s\mathcal{O}_K$ (resp $\frac{M^{g\pm}}{M_g^{\pm}} \otimes \mathcal{O}_K/\pi_K^s\mathcal{O}_K$) through scalar multiplication by $a_n \bmod \pi_K^s\mathcal{O}_K$ (resp $b_n \bmod \pi_K^s\mathcal{O}_K$). The isomorphism of Hecke modules then implies

$$a_n \equiv b_n \bmod \pi_K^s\mathcal{O}_K.$$

Putting this together gives the following theorem:

Theorem 3.8. — *Let f and g be eigenforms as above. If there exist periods $\Omega_f^{\pm}$ and $\Omega_g^{\pm}$ as in Subsection 2.4 that satisfy*

$$\frac{L(f,\chi,1)}{2\pi i\Omega_f^{\pm}} \equiv \frac{L(g,\chi,1)}{2\pi i\Omega_g^{\pm}} \bmod \pi_K^s,$$

for all Dirichlet characters $\chi \in \mathcal{A}_\epsilon$, then

$$f \equiv g \bmod \pi_K^s.$$

3.4. The one variable cyclotomic p-adic L-function

We will now reinterpret Theorem 3.8 in terms of p-adic L-functions. Our construction roughly follows the cyclotomic p-adic L-function described in [10]. The only difference is that we consider p-adic L-functions that are twisted by a Dirichlet character of level prime to p. Let f be an p-ordinary eigenform in $S_k(\Gamma,\mathcal{O}_K)$. We define $\mathbb{T}_{N_0p^r,k}$ to be the Hecke algebra over $\mathcal{O}_K$ generated by T_l for $l \nmid N_0p$, U_l for $l|N_0p$, and the diamond operators $\langle a\rangle$ for $a \bmod N$. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}_{N_0p^r,k}$ corresponding to the residue of f modulo p and recall that $\mathbb{T}_{N_0p^r,k,\mathfrak{m}}$ is the localization of $\mathbb{T}_{N_0p^r,k}$ at $\mathfrak{m}$. We will assume that there is an isomorphism of Hecke modules:

$$\text{(A)} \qquad H_1(\mathbb{H}/\Gamma,\widetilde{L}_{k-2}(\mathcal{O}_K))_{\mathfrak{m}}^{\pm} \cong \mathbb{T}_{N_0p^r,k,\mathfrak{m}}.$$

For example, we say that f is *p-distinguished* if its residual representation has distinct Jordan–Holder factors when restricted to the decomposition group $\mathrm{Gal}(\mathbb{Q}_p^{alg}/\mathbb{Q}_p)$. Then by Proposition 3.1.1 in [10] this isomorphism holds for p-distinguished eigenforms.

There is a natural map δ_f from $\mathbb{T}_{N_0p^r,k,\mathfrak{m}}$ to $\mathbb{C}$ sending each Hecke operator to its eigenvalue on f. The image of this map lies in $\mathcal{O}_K$. The 1-form

$\omega_f^{\pm}$ also induces a map from $H_1(\mathbb{H}/\Gamma, \widetilde{L}_{k-2}(\mathcal{O}_K))_{\mathfrak{m}}^{\pm}$ to $\mathbb{C}$, which we will also refer by $\omega_f^{\pm}$, by integrating along each cycle. Then we have

$$\frac{\omega_f^{\pm}}{\Omega_f^{\pm}} = \delta_f,$$

where $\Omega_f^{\pm}$ is a period as in Section 2.4. Choosing a different isomorphism for A results in a different choice of period.

Let $\Lambda = \mathcal{O}_K[\![T]\!]$ be the standard Iwasawa algebra. For $M > 0$ prime to p we define Λ_M to be the group ring $\Lambda[\mathbb{Z}/pM\mathbb{Z}^{\times}]$. Then there is a non-canonical isomorphism

$$\Lambda[\mathbb{Z}/pM\mathbb{Z}^{\times}] \cong \varprojlim \mathcal{O}_K[\mathbb{Z}/Mp^s\mathbb{Z}^{\times}],$$

where $1+T$ goes to the topological generator $1+p$ of $1+p\mathbb{Z}_p$. Recall that for any $\mathcal{O}_K$-module A we may think of elements of $\Lambda_M \otimes_{\mathcal{O}_K} A$ as measures on $\mathbb{Z}_p^{\times} \oplus \mathbb{Z}/M\mathbb{Z}^{\times}$ with values in A (see [25, Section 7]). Thus we may define an element $L_{M,\mathfrak{m}}^{\pm}$ of $\Lambda_M \otimes_{\mathcal{O}_K} H_1(\mathbb{H}/\Gamma, \widetilde{L}_{k-2}(\mathcal{O}_K))_{\mathfrak{m}}^{\pm}$ as follows: the open set $(a + p^r\mathbb{Z}_p, a + M\mathbb{Z})$ in $\mathbb{Z}_p^{\times} \oplus \mathbb{Z}/M\mathbb{Z}^{\times}$ is sent to the homology class $U_p^{-r}\left[\frac{a}{p^rM}\right] \in H_1(\mathbb{H}/\Gamma, \widetilde{L}_{k-2}(\mathcal{O}_K))_{\mathfrak{m}}^{\pm}$ (we are thinking of U_p as acting on the homology class defined by the cycle $\left[\frac{a}{p^rM}\right]$). This gives a well defined measure on $\mathbb{Z}_p^{\times} \oplus \mathbb{Z}/M\mathbb{Z}^{\times}$ with values in $H_1(\mathbb{H}/\Gamma, \widetilde{L}_{k-2}(\mathcal{O}_K))_{\mathfrak{m}}^{\pm}$.

Let $L_{M,f}^{\pm}$ be the image of $L_{M,\mathfrak{m}}^{\pm}$ under the map $1 \otimes \delta_f$. Then $L_{M,f}^{\pm}$ is an element of Λ_M. Specializing at certain $\mathbb{C}_p$ points of $\mathrm{Spec}(\Lambda_M)$ will give us special values of $L(f,\chi,s)$. Let us explain this interpolation property in more detail. A $\mathbb{C}_p$-point of $\mathrm{Spec}(\Lambda_M)$ is defined by an element of $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^{\times} \oplus \mathbb{Z}/M\mathbb{Z}^{\times}, \mathbb{C}_p^{\times})$. In particular any primitive Dirichlet character χ of conductor Mp^r induces a continuous homomorphism $\chi : \mathbb{Z}_p^{\times} \oplus \mathbb{Z}/M\mathbb{Z}^{\times} \to \mathbb{C}_p^{\times}$ and thus defines a $\mathbb{C}_p$-point of $\mathrm{Spec}(\Lambda_M)$. Evaluating $L_{M,f}^{\pm}$ at the point defined by χ gives

$$\begin{aligned} L_{M,f}^{\pm}(\chi) &= U_p^{-r}\delta_f\left(\sum_{a=0}^{m_\chi}\chi(a)\left[\frac{a}{m_\chi}\right]\right) \\ &= U_p^{-r}\sum_{a=0}^{m-1}\chi(a)\int_{\frac{a}{m}}^{i\infty}\frac{f(z)}{\Omega_f^{\pm}}\mathrm{d}z \\ &= a_p^{-r}\tau(\chi)m_\chi\frac{L(f,\chi,1)}{2\pi i\Omega_f^{\pm}}, \end{aligned}$$

where a_p is the eigenvalue of U_p on f. Note that $\mathrm{Spec}(\Lambda_M)$ is equal to $\phi(pM)$ copies of the open unit p-adic ball: one copy for each character of $(\mathbb{Z}/pM\mathbb{Z})^{\times}$. For a primitive character χ of conductor pM we let $L_{\chi,f}$ denote the restriction of $L^{\pm}_{M,f}$ to the unit ball corresponding to χ.

Theorem 3.9. — *Let f and g be two p-ordinary eigenforms of weight $k \geqslant 2$ and level N_0p^r whose coefficients are in $\mathcal{O}_K$. Assume that both eigenforms have the same residual representation and*

$$H_1(\mathbb{H}/\Gamma, \widetilde{L}_{k-2}(\mathcal{O}_K))^{\pm}_{\mathfrak{m}} \cong \mathbb{T}_{N_0p^r,k,\mathfrak{m}},$$

where $\mathfrak{m}$ is the maximal ideal of $\mathbb{T}_{N_0p^r,k}$ corresponding to this residual representation. The following are equivalent

- *The forms f and g are congruent modulo π_K^s.*
- *The p-adic L-functions $L_{\chi,f}$ and $L_{\chi,g}$ are congruent modulo π_K^s for all Dirichlet characters $\chi \in \mathcal{A}_\epsilon$.*

Proof. — If f and g are congruence modulo π_K^s then we know that $\delta_f \equiv \delta_g \bmod \pi_K^s$. From this it's clear that $L_{\chi,f} \equiv L_{\chi,g} \bmod \pi_K^s$. This was originally proven by Vatsal in [31] using the p-adic Weierstrass preparation theorem. Conversely, if the p-adic L-functions are congruent then the special values are congruent. Then from Theorem 3.8 we know that $f \equiv g \bmod \pi_K^s$. □

4. p-adic L-functions on Hida families

4.1. Hida Theory

We will now summarize the main ideas of Hida theory, which was first introduced in [17] and [18]. For an accessible introduction to the theory with tame level 1 see [19] and for a general overview see [10]. Let $N_0 > 0$ be relatively prime to p and let $k \geqslant 2$. We define

$$S_k(N_0p^\infty, \mathcal{O}_K)^{\mathrm{ord}} = \varinjlim_{r>0} S_k(N_0p^r, \mathcal{O}_K)^{\mathrm{ord}},$$

which is the space of all p-ordinary cusp forms with tame level N_0. There is a natural action of $\mathbb{Z}/p^r\mathbb{Z}^\times$ on $S_k(N_0p^r, \mathcal{O}_K)^{\mathrm{ord}}$ given by the product of the Nebentypus action and the character $\gamma \to \gamma^k$. These actions are compatible with the inclusion of $S_k(N_0p^r, \mathcal{O}_K)^{\mathrm{ord}}$ in $S_k(N_0p^{r+s}, \mathcal{O}_K)^{\mathrm{ord}}$ for any $s > 0$. Thus we may take the action on the direct limit to get an action of $\mathbb{Z}_p^\times$ on $S_k(N_0p^\infty, \mathcal{O}_K)^{\mathrm{ord}}$. In particular $S_k(N_0p^\infty, \mathcal{O}_K)^{\mathrm{ord}}$ is an $\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!]$-module. Since $\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!] \cong \Lambda[\mathbb{Z}/p\mathbb{Z}^\times]$, as described in Section 3.4, we may view $S_k(N_0p^\infty, \mathcal{O}_K)^{\mathrm{ord}}$ as a $\Lambda[\mathbb{Z}/p\mathbb{Z}^\times]$-module.

Let $\mathbb{T}_{N_0}$ be the p-adic completion of the $\mathcal{O}_K$-algebra generated by the Hecke operators and diamond operators acting on $S_k(Np^\infty, \mathcal{O}_K)^{\mathrm{ord}}$. Since $\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!]$ acts on $S_k(Np^\infty, \mathcal{O}_K)^{\mathrm{ord}}$ via the diamond operators, we may view $\mathbb{T}_{N_0}$ as a $\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!]$-algebra (and thus also a $\Lambda[\mathbb{Z}/p\mathbb{Z}^\times]$-algebra). In particular there is a map $\pi : \mathrm{Spec}(\mathbb{T}_{N_0}) \to \mathrm{Spec}(\Lambda[\mathbb{Z}/p\mathbb{Z}^\times])$. For a prime $\mathfrak{p}$ of $\Lambda[\mathbb{Z}/p\mathbb{Z}^\times]$ of height one we write $\mathcal{O}(\mathfrak{p})$ to denote $\Lambda[\mathbb{Z}/p\mathbb{Z}^\times]/\mathfrak{p}$. We say that $\mathfrak{p}$ is classical of weight k if the residue has characteristic zero and if the induced homomorphism $\kappa_\mathfrak{p} : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ equals the k-th power map on a small enough open subgroup of $\mathbb{Z}_p^\times$. Just as before, we set $\mathcal{O}(\mathfrak{p}) := \mathbb{T}_{N_0}/\mathfrak{p}$ for a height one prime $\mathfrak{p}$ of $\mathbb{T}_{N_0}$ and define $\kappa_\mathfrak{p}$ to be the homomorphism $\mathbb{Z}_p^\times \to \mathbb{C}_p$ induced by the ring homomorphism

$$\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!] \to \mathbb{T}_{N_0} \to \mathcal{O}(\mathfrak{p}),$$

when $\mathcal{O}(\mathfrak{p})$ has characteristic zero. We say that $\mathfrak{p}$ is classical of weight k if the character $\kappa_\mathfrak{p}$ corresponds to a classical weight k point of $\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!]$. Equivalently, $\mathfrak{p}$ is classical of weight k if $\pi(\mathfrak{p}) \in \mathrm{Spec}(\Lambda[\mathbb{Z}/p\mathbb{Z}^\times])$ is classical of weight k. The fibers of π above classical weight k primes of Λ recover the Hecke algebra acting on weight k cusp forms of tame level N_0. More specifically, Hida proved the following theorem in [18].

Theorem 4.1. — *Using the above notation we have:*

- *The map $\pi : \mathrm{Spec}(\mathbb{T}_{N_0}) \to \mathrm{Spec}(\Lambda[\mathbb{Z}/p\mathbb{Z}^\times])$ is a finite morphism.*
- *Let $\mathfrak{p} \in \mathrm{Spec}(\Lambda[\mathbb{Z}/p\mathbb{Z}^\times])$ be a classical height one prime of weight k. Then*

  $$\pi^{-1}(\mathfrak{p}) = \mathrm{Spec}(\mathbb{T}_{N_0} \otimes_{\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!]} \mathcal{O}(\mathfrak{p}))$$

  is equal to the full Hecke algebra acting on $S_k(Np^\infty, \mathcal{O}(\mathfrak{p}))^{\mathrm{ord}}[\kappa_\mathfrak{p}]$, the subspace of $S_k(Np^\infty, \mathcal{O}(\mathfrak{p}))^{\mathrm{ord}}$ where $\mathbb{Z}_p^\times$ acts via $\kappa_\mathfrak{p}$.

The Hecke algebra $\mathbb{T}_{N_0}$ is a semi-local ring. Its maximal ideals correspond to residual Galois representations (up to semi-simplification) that are associated to cusp forms of tame level N_0. By Theorem 4.1 the $\mathrm{Spec}(\mathbb{C}_p)$-points of $\mathrm{Spec}(\mathbb{T}_{N_0})$ that map to classical points in $\mathrm{Spec}(\Lambda[\mathbb{Z}/p\mathbb{Z}^\times])$ are in one-to-one correspondence with p-ordinary eigenforms of tame level N_0. We may think of $\mathrm{Spec}(\mathbb{T}_{N_0})$ as a geometric object whose points interpolate eigenforms in the following way: consider the formal power series

$$f(q) = \Sigma T_n q^n \in \mathbb{T}_{N_0}[\![q]\!].$$

If $\mathfrak{p} \in \mathrm{Spec}(\mathbb{T}_{N_0})$ is classical of weight k then the image of $f(q)$ in $\mathcal{O}(\mathfrak{p})[\![q]\!]$ gives the q-expansion of a weight k eigenform. Two prime ideals correspond to congruent eigenforms if and only if they are contained in the same maximal ideal. The minimal primes of $\mathbb{T}_{N_0}$ then correspond to maximal irreducible families of eigenforms. We are primarily interested in the interaction between different irreducible families.

Definition 4.2. — *A Hida family is an irreducible component of* $\mathrm{Spec}(\mathbb{T}_{N_0})$.

Let C be a Hida family and recall that $\mathrm{Spec}(\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!])$ is isomorphic to $p-1$ copies of $\mathrm{Spec}(\Lambda)$. The image of $\pi|_C$ is equal to one of these copies. This lets us regard π as a map from C to $\mathrm{Spec}(\Lambda)$. It is know that $\pi|_C$ is a finite morphism that is étale at all classical points (see [18]).

4.2. p-adic L-functions

We will now construct a p-adic L-function that varies over a Hida family. Our construction is based on the p-adic L-functions described in [10, Section 3.1]. Let $\mathfrak{m}$ be a maximal prime of $\mathbb{T}_{N_0}$ corresponding to the semi-simplified residual Galois representation $\bar{\rho}_{\mathfrak{m}}^{ss}$ and let $\mathbb{T}_{N_0,\mathfrak{m}}$ be the localization of $\mathbb{T}_{N_0}$ at $\mathfrak{m}$. The classical height one primes of $\mathbb{T}_{N_0,\mathfrak{m}}$ correspond to congruent eigenforms whose semi-simplified residual Galois representation is $\bar{\rho}_{\mathfrak{m}}^{ss}$. From now on we will assume that $\bar{\rho}_{\mathfrak{m}}^{ss}$ is p-distinguished (see Section 3.4) and irreducible. Then we have the following multiplicity one result:

Theorem 4.3. — *Let $\mathfrak{p}$ be a maximal prime of $\mathbb{T}_{N_0p^r,k,\mathfrak{p}}$ whose residual representation is irreducible and p-distinguished. Then*

$$H_1(\mathbb{H}/\Gamma_1(N_0p^r), \widetilde{L}_{k-2}(\mathcal{O}_K))_{\mathfrak{p}}^{\pm\mathrm{ord}} \cong \mathbb{T}_{N_0p^r,k,\mathfrak{p}}$$

as $\mathbb{T}_{N_0p^r,k,\mathfrak{p}}$-modules.

Proof. — This is [10, Proposition 3.1.1]. □

We define

$$H_1(N_0p^\infty, \mathcal{O}_K)_{\mathfrak{m}}^{\mathrm{ord}} := \varprojlim H_1(\mathbb{H}/\Gamma_1(N_0p^r), \mathcal{O}_K)_{\mathfrak{m}}^{\mathrm{ord}}.$$

By Proposition 3.3.1 in [10], our assumptions on $\bar{\rho}_{\mathfrak{m}}^{ss}$ imply that $H_1(N_0p^\infty, \mathcal{O}_K)_{\mathfrak{m}}^{\pm\mathrm{ord}}$ is a rank one free $\mathbb{T}_{N_0,\mathfrak{m}}$-module. Fix an isomorphism

$$H_1(N_0p^\infty, \mathcal{O}_K)_{\mathfrak{m}}^{\pm\mathrm{ord}} \cong \mathbb{T}_{N_0,\mathfrak{m}}. \tag{B}$$

Similar to the p-adic L-function of Section 3.4, our p-adic L-functions will be given by a measure on $\mathbb{Z}_p^\times \oplus \mathbb{Z}/M\mathbb{Z}$ with values in $H_1(N_0p^\infty, \mathcal{O}_K)_{\mathfrak{m}}^{\pm\mathrm{ord}}$. More precisely, we consider the measure sending the open set $(a + p^r\mathbb{Z}_p,$

$a + M\mathbb{Z})$ in $\mathbb{Z}_p^\times \oplus \mathbb{Z}/M\mathbb{Z}$ to $U_p^{-r}\{\frac{a}{p^r M}, \infty\} \in H_1(N_0 p^\infty, \mathcal{O}_K)^{\pm \mathrm{ord}}_{\mathfrak{m}}$. This defines an element

$$L_p^\pm(\mathbb{T}_{N_0,\mathfrak{m}}, M) \in H_1(N_0 p^\infty, \mathcal{O}_K)^{\pm \mathrm{ord}}_{\mathfrak{m}} \otimes_{\mathcal{O}_K} \Lambda[\mathbb{Z}/Mp\mathbb{Z}^\times].$$

Using our fixed isomorphism (B), we may view $L_p^\pm(\mathbb{T}_{N_0,\mathfrak{m}}, M)$ as an element of $\mathbb{T}_{N_0,\mathfrak{m}} \otimes_{\mathcal{O}_K} \Lambda[\mathbb{Z}/Mp\mathbb{Z}^\times]$. We may view $L_p^\pm(\mathbb{T}_{N_0,\mathfrak{m}}, M)$ as a two variable p-adic L-function that varies over the Hida family (i.e. $\mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$) and the cyclotomic variable. By specializing to a classical weight one prime of $\mathbb{T}_{N_0,\mathfrak{m}}$ corresponding to an eigenform f we recover the p-adic L-function $L^\pm_{M,f}$ from Section 3.4. In particular, if $\mathfrak{p}$ is the classical height one prime corresponding to f then the image of $L_p^\pm(\mathbb{T}_{N_0,\mathfrak{m}}, M)$ in $\mathcal{O}(\mathfrak{p}) \otimes_{\mathcal{O}_K} \Lambda[\mathbb{Z}/Mp\mathbb{Z}^\times]$ is $L^\pm_{M,f} u$ where u is in $\mathcal{O}(\mathfrak{p})^\times$.

When we specialize the cyclotomic variable at a character we obtain a p-adic L-function that interpolates the special values of the eigenforms in a Hida family. More precisely, let χ be a primitive Dirichlet character whose tame conductor is M. Then χ induces a ring homomorphism:

$$\kappa_\chi : \Lambda[\mathbb{Z}/Mp\mathbb{Z}^\times] \to \mathcal{O}_K[\chi].$$

Let $L_p^\pm(\mathbb{T}_{N_0,\mathfrak{m}}, \chi)$ be the image of $L_p^\pm(\mathbb{T}_{N_0,\mathfrak{m}}, M)$ under the map $1 \otimes \kappa_\chi$ so that $L_p^\pm(\mathbb{T}_{N_0,\mathfrak{m}}, \chi)$ is an element of $\mathbb{T}_{N_0,\mathfrak{m}} \otimes_{\mathcal{O}_K} \mathcal{O}_K[\chi] = \mathbb{T}_{N_0,\mathfrak{m}}[\chi]$. Now let $\mathfrak{p}$ be a height one prime corresponding to a classical eigenform f. We have the following interpolation property:

$$L_p^\pm(\mathbb{T}_{N_0,\mathfrak{m}}, \chi) \equiv \frac{L(f,\chi,1)}{2\pi i \Omega_f^\pm} \bmod \mathfrak{p},$$

where $\Omega_f^\pm$ is a period independent of χ.

Remark. — The choice of the isomorphism in equation (B) will determines the periods $\Omega_f^\pm$ in a p-adically compatible way. Changing this isomorphism amounts to scaling the periods in a p-adic analytic compatible manner.

5. Some Geometric Preliminaries

Let C_1 and C_2 be irreducible components of $\mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$ that cross at a point x. It is often the case that the structure maps π from C_i to $\mathrm{Spec}(\Lambda)$ are isomorphisms. This is an ideal situation, as functions on $\mathrm{Spec}(\Lambda)$ are understood through the Weierstrass preparation theorem. However, there are many examples of components whose structure maps are not isomorphisms (e.g. families of CM forms where the class group of the imaginary quadratic field is divisible by some power of p). The solution is to take small enough affinoid neighborhoods around x so that π becomes isomorphic. We then must choose integral models of these smaller neighborhoods in a way so that they still carry information about congruences between cusp forms. The purpose of this section is to prove some geometric theorems that allow us to make this type of geometric simplification. In the first subsection we prove a specific form of inverse function theorem. This allows us to choose formal models for these small affinoids in a precise manner. In the second subsection we introduce an auxiliary p-adic metric on the set of $\mathcal{O}_K$-points of a schemes over $\mathcal{O}_K$. The purpose of this metric is to help keep track of congruences between eigenforms. Finally, in the third subsection, we give our precise definition of intersection multiplicity and explain its basic properties.

5.1. The inverse function theorem for formal models

Throughout this section we let $\pi : X \to \mathrm{Spec}(\Lambda)$ be a finite morphism. We will assume X is a reduced affine scheme with a local coordinate ring A. By an irreducible component of X we mean a closed subscheme $C = \mathrm{Spec}(A/\mathfrak{a})$ of X where $\mathfrak{a}$ is a minimal prime ideal of A. There is a rigid analytic space $\mathcal{X} = X^{rig}$ associated to $X = \mathrm{Spec}(A)$. If $f_1, \dots, f_n \in \Gamma(\mathcal{O}_{\mathcal{X}}, \mathcal{X})$ are global functions on $\mathcal{X}$ we define $\mathcal{X}(f_1, \dots, f_n)$ to be the subdomain whose points are

$$\{x \in \mathcal{X} \ ; \ |f_i(x)|_p \leqslant 1 \ \text{ for all i}\}.$$

Whenever R is a $\mathcal{O}_K$-algebra, we will let R' denote the quotient of R by the ideal generated by π_K power torsion.

Fix an irreducible component C of X and let $x \in C$ be an $\mathcal{O}_K$-point at which π is étale. Since C is finite over $\mathrm{Spec}(\Lambda)$, we have $C = \mathrm{Spec}(A)$ where $A = \mathcal{O}_K[\![T]\!][T_1, \dots, T_d]/I$ and π is given by projecting onto the T-coordinate. After a change of variables, we may assume that x corresponds to the point $T = T_i = 0$. We will write $\mathcal{C}$ for the rigid space associated to C and $\mathcal{W}$ for the rigid space associated to $\mathrm{Spec}(\Lambda)$. The coordinate ring of $\mathcal{C}$ is $A_K := A \otimes_{\mathcal{O}_K} K$.

LEMMA 5.1. — *There is an affinoid neighborhood U of x in $\mathcal{C}$ and V of $\pi(x)$ in $\mathcal{W}$ such that U and V are isomorphic as rigid varieties. For any m sufficiently large we may find $N \geqslant m$ such that*

$$\pi : \mathcal{C}(p^{-N}T, p^{-m}T_i) \to \mathcal{W}(p^{-N}T)$$

is an isomorphism. Furthermore, we may choose N so that

$$\pi : \mathrm{Spec}(A\langle p^{-N}T, p^{-m}T_1, \dots, p^{-m}T_d\rangle') \to \mathrm{Spec}(\mathcal{O}_K\langle p^{-N}T\rangle)$$

is an isomorphism.

Proof. — Being étale at x just means that the Jacobian matrix, which only has one element, is invertible. Thus the existence of U and V is just a rigid analytic inverse function theorem. A proof can be found in [28, Chapter III]. It remains to prove that U and V can be written as above. We may find n large enough so that

$$U' = \mathcal{C}(p^{-n}T, p^{-n}T_i)$$

is contained in U. By the universal property of affinoid subdomains (see [5, Definition 9]) we see that U' is an affinoid subdomain of U. Then $V' = \pi(U')$ is an affinoid subdomain of $\mathcal{W}$ that is of the form

$$\mathcal{W}(p^{-n_0}T) \cong \mathrm{Sp}(K\langle p^{-n_0}T\rangle)$$

for some $n_0 \geqslant n$. With this in mind, we can rewrite U' as

$$U' = \mathcal{C}(p^{-n_0}T, p^{-n}T_i).$$

Since $U' \cong V'$, we see that $T_i = f_i(p^{-n_0}T)$ where $f_i(p^{-n_0}T) \in K\langle p^{-n_0}T\rangle$. Moreover, we know that each f_i has no constant term since all of the coordinates of x are zero.

Let $m > n$. We may find N sufficiently large so that for all i we have

$$|f_i(a)|_p < p^{-m},$$

whenever $a \in \mathcal{O}_{\mathbb{C}_p}$ satisfies $|a|_p < p^{-N}$. This is possible because $f_i(0) = 0$. Then for $a \in \mathrm{Sp}(K\langle p^{-N}\rangle)$ the T_i coordinate of $\pi|_{U'}^{-1}(a)$ has p-adic absolute value less than p^{-m}. Thus

$$\pi_{U'}^{-1}(\mathrm{Sp}(K\langle p^{-N}T\rangle)) = \mathcal{C}(p^{-N}T, p^{-m}T_i),$$

from which the first claimed isomorphism follows.

For the second isomorphism, let $g_i \in K\langle p^{-N}T\rangle$ such that $p^{-m}T_i = g_i(p^{-N}T)$. We may assume that $g_i \in \mathcal{O}_K\langle p^{-N}T\rangle$ by sufficiently increasing N. Consider the commutative diagram of coordinate rings:

$$\begin{array}{ccc} \mathcal{O}_K\langle p^{-N}T\rangle & \longrightarrow & A\langle p^{-N}T, p^{-m}T_1, \dots, p^{-m}T_d\rangle' \\ \downarrow & & \downarrow \\ K\langle p^{-N}T\rangle & \longrightarrow & A_K\langle p^{-N}T, p^{-m}T_1, \dots, p^{-m}T_d\rangle. \end{array}$$

The left vertical map and the bottom row map are injective so that the top row is also injective. Since $p^{-m}T_i - g_i(p^{-N}T)$ is zero in

$A_K\langle p^{-N}T, p^{-m}T_1, \dots, p^{-m}T_d\rangle$ we know that it is killed by a power of p in $A\langle p^{-N}T, p^{-m}T_1, \dots, p^{-m}T_d\rangle$. In particular $p^{-m}T_i - g_i(p^{-N}T)$ is zero in $A\langle p^{-N}T, p^{-m}T_1, \dots, p^{-m}T_d\rangle'$. Thus the image of the morphism on the top of the diagram contains $p^{-m}T_i$. It follows that the top arrow is surjective and therefore an isomorphism. □

5.2. p-adic distances and congruences

We'll start by giving an informal description: consider an open n-dimensional ball B in $\mathcal{O}_K$ centered around the origin of radius one. We think of $\mathcal{O}_K[\![T_1, \dots, T_n]\!]$ as the ring of bounded analytic functions on B with integral values. The $\mathcal{O}_K$-points of $\operatorname{Spec}(\mathcal{O}_K[\![T_1, \dots, T_n]\!])$ correspond to the points in B. If $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are two $\mathcal{O}_K$-points in B, a natural choice for the distance between them is

$$\max_i |x_i - y_i|_p.$$

It would be great to translate this definition into something more intrinsic and algebraic. In particular, we want a definition that will work for all quotients of $\mathcal{O}_K[\![T_1, \dots, T_n]\!]$ and $\mathcal{O}_K\langle T_1, \dots, T_n\rangle$.

Definition 5.2. — *Let R be a ring that is a quotient of either $\mathcal{O}_K[\![T_1, \dots, T_n]\!]$ or $\mathcal{O}_K\langle T_1, \dots, T_n\rangle$. Let I be an ideal of R and let $\mathfrak{p}$ be a prime ideal of R corresponding to an $\mathcal{O}_K$-point. We define $I(\mathfrak{p})$ be the ideal $I + \mathfrak{p} \bmod \mathfrak{p}$ in $R/\mathfrak{p}$. Let $|I(\mathfrak{p})|_p$ be the largest absolute value occurring in $I(\mathfrak{p})$.*

Definition 5.3. — *Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be prime ideals of R corresponding to $\mathcal{O}_K$-points. We define $d(\mathfrak{p}_1, \mathfrak{p}_2)$ to be $|\mathfrak{p}_1(\mathfrak{p}_2)|_p$.*

Lemma 5.4. — *The following properties of $d(\,\cdot\,,\cdot\,)$ hold:*

1. $d(\mathfrak{p}_1, \mathfrak{p}_2) = d(\mathfrak{p}_2, \mathfrak{p}_1)$
2. $d(\mathfrak{p}_1, \mathfrak{p}_1) = 0$
3. *Let $(f_1, \dots, f_r)$ be any set of generators of $\mathfrak{p}_1$. Then $d(\mathfrak{p}_1, \mathfrak{p}_2) = \max |f_i \bmod \mathfrak{p}_2|_p$.*
4. *Suppose R is $\mathcal{O}_K[\![T_1, \dots, T_n]\!]$ or $\mathcal{O}_K\langle T_1, \dots, T_n\rangle$. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}(R)$ be $\mathcal{O}_K$-points corresponding to the coordinates $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$. Then*

$$d(\mathfrak{p}_1, \mathfrak{p}_2) = \max_i |x_i - y_i|_p.$$

(5) *Let R and S be rings over $\mathcal{O}_K$ as in the Definition 5.2. Let $f : \operatorname{Spec}(R) \to \operatorname{Spec}(S)$ be a closed embedding. Then $d(\mathfrak{p}_1, \mathfrak{p}_2) = d(f(\mathfrak{p}_1), f(\mathfrak{p}_2))$.*

(6) *The non-Archemedian triangle inequality holds. That is $d(\mathfrak{p}_1, \mathfrak{p}_3) \leqslant \max(d(\mathfrak{p}_1, \mathfrak{p}_2), d(\mathfrak{p}_2, \mathfrak{p}_3))$.*

Proof. — Statements (2), (3), and (5) are immediate. To prove (4) note that $\mathfrak{p}_1 = (T_1 - x_1, \ldots, T_n - x_n)$ and $\mathfrak{p}_2 = (T_1 - y_1, \ldots, T_n - y_n)$. The image of $\mathfrak{p}_2$ in $R/\mathfrak{p}_1$ is the ideal $(x_1 - y_1, \ldots, x_n - y_n)$. It follows that $d(\mathfrak{p}_1, \mathfrak{p}_2) = \max |x_i - y_i|_p$. Statements (1) and (6) follow from (4). □

Lemma 5.5. — *Let R be a ring over $\mathcal{O}_K$ as in Definition 5.2. Let r be an integer with $p^{\frac{-r}{e}} = d(\mathfrak{p}_1, \mathfrak{p}_2)$, where e is the ramification index of $\mathcal{O}_K$ over $\mathbb{Z}_p$. Then r is the largest integer making the following diagram commute:*

$$\begin{tikzcd}
 & R/\mathfrak{p}_1 \arrow[dr] & \\
R \arrow[ur] \arrow[dr] & & \mathcal{O}_K/\pi_K^r. \\
 & A/\mathfrak{p}_2 \arrow[ur] &
\end{tikzcd}$$

Proof. — Any such map $R \to \mathcal{O}_K/\pi_K^{r'}$ that factors through $R/\mathfrak{p}_1$ and $R/\mathfrak{p}_2$ must also factor through $R/(\mathfrak{p}_1 + \mathfrak{p}_2)$. Since

$$R/(\mathfrak{p}_1 + \mathfrak{p}_2) \cong \frac{R/\mathfrak{p}_1}{\mathfrak{p}_2 A/\mathfrak{p}_1} \cong \mathcal{O}_K/\pi_K^r,$$

we know that $\mathcal{O}_K/\pi_K^{r'}$ is a quotient of $\mathcal{O}_K/\pi_K^r$. The Lemma follows. □

Now consider the Hecke algebra $\mathbb{T}_{N_0}$ described in Section 4.1. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_{N_0}$. The ring $\mathbb{T}_{N_0,\mathfrak{m}}$ is local, reduced, finite over Λ and $\mathfrak{m}$-adically complete. Let $r_1, \ldots, r_n$ generate $\mathbb{T}_{N_0,\mathfrak{m}}$ as a Λ-algebra. If r_i is a unit then $r_i \equiv a_i \bmod \mathfrak{m}$ for some $a_i \in \mathcal{O}_K^\times$. By replacing r_i with $r_i - a_i$ we may assume that each r_i is in $\mathfrak{m}$ so the r_i are topologically nilpotent. This lets us define a surjection $\mathcal{O}_K[\![T_1, \ldots, T_n]\!] \to \mathbb{T}_{N_0,\mathfrak{m}}$ by sending T_i to r_i (i.e. we have an embedding of $\operatorname{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$ into the n-dimensional open unit ball over $\mathcal{O}_K$).

Lemma 5.6. — *The local big Hecke algebra $\mathbb{T}_{N_0,\mathfrak{m}}$ embeds into an n-dimensional open unit ball. This mapping is "isometric" with respect to the natural p-adic metric on the p-adic ball and the metric $d(\,\cdot\,,\cdot\,)$ defined above. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$ be classical $\mathcal{O}_K$-points corresponding to normalized eigenforms f_1 and f_2. If $d(\mathfrak{p}_1, \mathfrak{p}_2) = p^{\frac{-r}{e}}$ then*

$$f_1 \equiv f_2 \bmod \pi_K^r,$$

and

$$f_1 \not\equiv f_2 \bmod \pi_K^{r+1}.$$

Informally, this means that the distance between two points in $\mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$ *tells us exactly how congruent the corresponding eigenforms are.*

Proof. — The map $\mathbb{T}_{N_0,\mathfrak{m}} \to \mathbb{T}_{N_0,\mathfrak{m}}/\mathfrak{p}_i \cong \mathcal{O}_K$ sends T_l to the l-th Fourier coefficient of f_i. This means that $f_1 \equiv f_2 \bmod \pi_K^s$ if and only if the following diagram commutes:

$$\begin{tikzcd} & \mathbb{T}_{N_0,\mathfrak{m}}/\mathfrak{p}_1 \arrow[dr] & \\ \mathbb{T}_{N_0,\mathfrak{m}} \arrow[ur] \arrow[dr] & & \mathcal{O}_K/\pi_K^s. \\ & \mathbb{T}_{N_0,\mathfrak{m}}/\mathfrak{p}_2 \arrow[ur] & \end{tikzcd}$$

Then the Lemma is a consequence of Lemma 5.5. □

5.3. Intersection Multiplicities

In this section we will define the intersection multiplicity of two crossing components on a curve over $\mathcal{O}_K$. We let X be a scheme that is finite over Λ or $\mathcal{O}_K\langle T\rangle$ whose rigid analytic fiber $\mathcal{X}$ has dimension one. Let C_1 and C_2 be irreducible components of X with rigid fibers $\mathcal{C}_1$ and $\mathcal{C}_2$. Let x be an $\mathcal{O}_K$-point of X. There is a natural inclusion map $j_i : C_i \to X$. We define $\mathcal{I}_i$ to be the $\mathcal{O}_X$-sheaf of ideals that define the component C_i.

Definition 5.7. — *The intersection multiplicty $I(X, C_1, C_2, x)$ of C_1 and C_2 at x is the K-dimension of*

$$(\mathcal{O}_{C_1}/j_1^*(\mathcal{I}_2))_x = (\mathcal{O}_X/(\mathcal{I}_1 + \mathcal{I}_2))_x = (\mathcal{O}_{C_2}/j_2^*(\mathcal{I}_1))_x.$$

Remark. — The intersection multiplicity is nonzero if and only if both C_1 and C_2 contain x.

Since this definition is Zariski local we may take an affine neighborhood $U = \mathrm{Spec}(A)$ of x. Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be the minimal prime ideals of A defining the components C_1 and C_2. Let $\mathfrak{p}_x$ be the prime corresponding to x. Then

$$I(X, C_1, C_2, x) = \dim_K(A_{\mathfrak{p}_x}/(\mathfrak{a}_1 + \mathfrak{a}_2)).$$

LEMMA 5.8. — *The following properties of intersection multiplicities are true.*

(1) *Let X' be another scheme of finite type over Λ or $\mathcal{O}_K\langle T\rangle$ whose rigid fiber is equal to $\mathcal{X}$. Let C_1' and C_2' be connected components corresponding to $\mathcal{C}_1$ and $\mathcal{C}_2$. These components cross at x' whose image in X^{rig} is x. Then $I(X, C_1, C_2, x) = I(X', C_1', C_2', x')$. In other words, the intersection multiplicity only depends on the rigid fiber.*

(2) *Recall how stalks are defined for a sheaf on a rigid analytic varieties*

$$\mathcal{F}_{X^{rig},x} = \varinjlim_{\substack{x\in U \\ U \text{ is affinoid}}} \mathcal{F}(U).$$

Then $I(X, C_1, C_2, x) = \dim_K(\mathcal{O}_{X^{rig}}/(\mathcal{I}_1^{rig} + \mathcal{I}_2^{rig}))_x$. That is, we can use the rigid analytic stalks or the Zariski stalks to compute intersection numbers.

(3) *Let $\mathcal{U}$ be an affinoid neighborhood of x. Let U be a integral model of $\mathcal{U}$ and let D_i be the component of U whose rigid fiber is $\mathcal{U} \cap \mathcal{C}_i$. Then $I(X, C_1, C_2, x) = I(U, D_1, D_2, x)$.*

(4) *Let $i : Y \to X$ be a closed subscheme such that $i(Y)$ contains the generic points of C_1 and C_2. Then $I(X, C_1, C_2, x)$ is the same as $I(Y, Y \times_X C_1, Y \times_X C_2, i^{-1}(x))$.*

Proof. — The first and third statement are immediate consequences of the second statement. To prove the second statement, pick an affine neighbodhood of x and use the fact that $\widehat{\mathcal{O}_{X,x}} \cong \widehat{\mathcal{O}_{X^{rig},x}}$, where $\widehat{A}$ denotes the completion of a local ring A along its maximal ideal (see [5, Section 1.7]). The last statement is easily checked by picking an affine neighborhood of x. $\square$

In the simple situation of two rational curves crossing in $\mathcal{O}_K\langle T, T_1\rangle$, we can come up with a precise formula for the intersection number. Let X be finite over $\mathrm{Spec}(\mathcal{O}_K\langle T\rangle)$ and let $X \to \mathcal{O}_K\langle T, T_1\rangle$ be a closed embedding. Let C_1 and C_2 be two connected components of X that are isomorphic both $\mathcal{O}_K\langle T\rangle$. Then we have embeddings $C_i \to \mathrm{Spec}(\mathcal{O}_K\langle T, T_1\rangle)$ that factor through X and we see that $C_i = \mathrm{Spec}(\mathcal{O}_K\langle T, T_1\rangle/(T_1 - f_i(T)))$ with $f_i(T) \in \mathcal{O}_K\langle T\rangle$. Then we can compute the intersection number:

LEMMA 5.9. — *Let x be the point of X corresponding to $T = 0$ and $T_1 = 0$. The intersection $I(X, C_1, C_2, x)$ is equal to the largest power of T dividing $f_1(T) - f_2(T)$.*

Proof. — We have to compute the dimension of

$$\mathcal{O}_K\langle T, T_1\rangle/(T_1 - f_1(T)), T_1 - f_2(T)) \cong \mathcal{O}_K\langle T\rangle/(f_1(T) - f_2(T)),$$

after localizing at x. This is the largest power of T dividing $f_1(T) - f_2(T)$. □

6. Proof of Theorem 1.1

We are now ready to prove Theorem 1.1. Recall our definition of $\mathbb{T}_{N_0,\mathfrak{m}}$ from Section 4.1 as a local component of the Hecke algebra that acts on the space of all cusp forms of tame level N_0. There is a map

$$\pi : \mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}}) \to \mathrm{Spec}(\Lambda) \cong \mathrm{Spec}(\mathcal{O}_K[\![T]\!]).$$

We will assume that $\mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$ has at least two distinct components C_1 and C_2. Let κ be an $\mathcal{O}_K$-point of $\mathrm{Spec}(\Lambda)$ that is the p-adic limit of classical weights and let x_1 (resp x_2) be an $\mathcal{O}_K$-point of C_1 (resp. C_2) of weight κ. After a change of variables we may take κ to be the point $T = 0$. We will assume that the restriction of π to C_1 (resp C_2) is étale at x_1 (resp x_2). If C_1 and C_2 cross above κ then it is possible to choose x_1 and x_2 to be the same point when viewed as points of $\mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$.

6.1. Reducing to the simplest geometric situation

We may write $C_i = \mathrm{Spec}(A_i)$ where A_i is generated as a Λ-algebra by $T_{i,1}, \dots, T_{i,n}$ and x_i corresponds to the origin in these coordinates. By applying Lemma 5.1 to C_1 and C_2 simultaneously we know that there exists $N > m > 0$ such that

$$B_i := A_i\langle p^{-N}T, p^{-m}T_{i,1}, \dots, p^{-m}T_{i,n}\rangle' \cong \mathcal{O}_K\langle p^{-N}T\rangle.$$

Let $Y = p^{-N}T$. Note that $\mathrm{Spec}(B_i)$ is a connected component of $\mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}}\langle Y\rangle)$. To see this we consider the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(B_i) \arrow[r] & \mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}}\langle Y\rangle) \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_K\langle Y\rangle) \arrow[u, "\pi|^{-1}_{\mathrm{Spec}(B_i)}"] \arrow[r, equal] & \mathrm{Spec}(\mathcal{O}_K\langle Y\rangle) \end{tikzcd}$$

From this diagram we see that the generic point of $\mathrm{Spec}(B_i)$ must be sent to a minimal prime in $\mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}}\langle Y\rangle)$ and that the map $\mathbb{T}_{N_0,\mathfrak{m}}\langle Y\rangle \to B_i$ is surjective.

Let Z be the scheme-theoretic union of $\mathrm{Spec}(B_1)$ and $\mathrm{Spec}(B_2)$ inside of $\mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}}\langle Y\rangle)$. This means that Z satisfies the following universal property: if $X \to \mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}}\langle Y\rangle)$ is a closed embedding such that both maps $\mathrm{Spec}(B_i) \to \mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}}\langle Y\rangle)$ factor through X then $Z \to \mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}}\langle Y\rangle)$ factors through X. Then Z comes naturally equipped with a map to $\mathrm{Spec}(\mathcal{O}_K\langle Y\rangle)$, which we call π by abuse of notation. Let B be the coordinate ring of Z, which is a finitely generated $\mathcal{O}_K\langle Y\rangle$-module. By the structure theorem for modules over $\mathcal{O}_K\langle Y\rangle$ we know that if B has no Λ-torsion then B is free (see [32, Theorem 13.12]). Let $b \in B$ be killed by a nonzero element $f \in \mathcal{O}_K\langle Y\rangle$. Since B_i has no torsion as an $\mathcal{O}_K\langle Y\rangle$-module we know that the image of b in B_i is zero. Thus the map $B \to B_i$ factors through $B/(b)$. The universal property satisfied by B then tells us that b is zero, so B has no torsion. In particular B is a free $\mathcal{O}_K\langle Y\rangle$-module of rank two and we may write

$$B = \mathcal{O}_K\langle Y\rangle[X]/f(X,Y),$$

where X is some indeterminate and f is monic of degree two in X. As π admits two sections (one for each $\mathrm{Spec}(B_i)$) the polynomial f factors into linear terms, i.e.

$$f(X,Y) = (X - g_1(Y))(X - g_2(Y)). \tag{C}$$

Here we have $g_i \in \mathcal{O}_K\langle Y\rangle$ and g_i corresponds to the closed subscheme $\mathrm{Spec}(B_i)$. The points x_1 and x_2 are the same if and only if g_1 and g_2 have the same constant term.

The advantage of working with Z is that both components are isomorphic to a p-adic ball. We still need to understand what happens to the p-adic distances introduced in Section 5.2. Let $y_1, y_2 \in Z$ be $\mathcal{O}_K$-points of equal weight (i.e. $\pi(y_1) = \pi(y_2)$). Consider the closed embedding

$$s : Z \to \mathbb{T}_{N_0,\mathfrak{m}}\langle Y\rangle.$$

By Lemma 5.4 closed embeddings are compatible with our p-adic distances, so we know that $d(y_1,y_2) = d(s(y_1), s(y_2))$. Now consider the map $t : \mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}}\langle Y\rangle) \to \mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$. Let $T, X_1, \dots, X_d$ be topological generators of $\mathbb{T}_{N_0,\mathfrak{m}}$ as an $\mathcal{O}_K$-algebra. This means $Y = p^{-N}T, X_1, \dots, X_d$ are topological generators for $\mathbb{T}_{N_0,\mathfrak{m}}\langle Y\rangle$. In terms of these coordinates the map t is given by

$$(y, x_1, \dots, x_d) \to (p^N y, x_1, \dots, x_d).$$

In particular, write $s(y_1) = (a, a_1, \dots, a_d)$ and $s(y_2) = (b, b_1, \dots, b_d)$ in terms of these coordinates. Then $a = b$ since both points have the same weight. Thus by Lemma 5.4 we have

$$d(s(y_1), s(y_2)) = \max(|a_i - b_i|_p),$$

which does not change after applying t.

LEMMA 6.1. — *Assume that the images of y_1 and y_2 in* $\mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$ *correspond to eigenforms f_{y_1} and f_{y_2}. Then $d(y_1, y_2) \leqslant p^{\frac{m}{e}}$ if and only if*

$$f_{y_1} \equiv f_{y_2} \bmod \pi_K^m.$$

Informally this states that the distance between y_1 and y_2 tells us exactly how congruent the corresponding forms are.

Proof. — This follows from the above discussion and from Lemma 5.6. □

6.2. An ideal of differences of L-values

Recall that for a Dirichlet character χ with conductor prime to p there is a one variable p-adic L-function $L_p^{\pm}(\mathbb{T}_{N_0,\mathfrak{m}}, \chi) \in \mathbb{T}_{N_0,\mathfrak{m}}[\chi]$. Let $L_p^{\pm}(B_i, \chi)$ be the image of this function induced by the map $\mathrm{Spec}(B_i[\chi]) \to \mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}}[\chi])$. Then for $x \in \mathrm{Spec}(B_i)$ corresponding to a classical modular form f_x we see that $L_p^{\pm}(B_i, \chi)$ evaluated at x is equal to the algebraic part of $L(f_x, \chi, 1)$. The isomorphism $\pi : \mathrm{Spec}(B_i[\chi]) \to \mathrm{Spec}(\mathcal{O}_K[\chi]\langle Y\rangle)$ allows us to regard $L_p^{\pm}(B_i, \chi)$ as a power series in $\mathcal{O}_K[\chi]\langle Y\rangle$. In particular, let $\kappa \in \mathrm{Spec}(\mathcal{O}_K\langle Y\rangle)$ be an $\mathcal{O}_K$-point corresponding to an element in $\pi_K\mathcal{O}_K$ (which we refer to as κ by abuse of notation). Let $y_i = \pi^{-1}(\kappa) \in \mathrm{Spec}(B_i[\chi])$. Then the evaluation of this power series at κ is equal to the evaluation of $L_p^{\pm}(B_i, \chi)$ at y_i. Informally we write $L_p^{\pm}(B_i, \chi)(\kappa) = L_p^{\pm}(B_i, \chi)(y_i)$.

DEFINITION 6.2. — *Let $u(Y)$ be a unit in $\mathcal{O}_K\langle Y\rangle$ and let $\mathcal{O}_K^{nr}$ be the ring of integers of the maximal unramified extension of K. Then we define I_u to be the ideal of $\mathcal{O}_K^{nr}\langle Y\rangle$ generated by the elements $L_p^{\pm}(B_1, \chi) - u(Y)L_p^{\pm}(B_2, \chi)$ for all Dirichlet characters $\chi \in \mathcal{A}_\epsilon$.*

LEMMA 6.3. — *Suppose that $\kappa \in \mathrm{Spec}(\mathcal{O}_K\langle Y\rangle)$ corresponds to a classical weight. Then $|I_u(\kappa))|_p$ is greater than or equal to $|g_1(\kappa) - g_2(\kappa)|_p$ (recall that $|I_u(\kappa))|_p$ is the supremum of the absolute values of the elements in I_u evaluated at κ).*

Proof. — Let y_i be the point of $\mathrm{Spec}(B_i)$ lying above κ. From (C) we may view y_i as being in the XY-plane with coordinates $(g_i(\kappa), \kappa)$. In particular we find that $d(y_1, y_2) = |g_1(\kappa) - g_2(\kappa)|_p$. Assume $|I_u(\kappa)|_p = p^{\frac{-n}{e}}$. This means

$$L_p^{\pm}(B_1, \chi)(\kappa) \equiv L_p^{\pm}(B_2, \chi)(\kappa) u(\kappa) \bmod \pi_K^n$$

for $\chi \in \mathcal{A}_\epsilon$. Let f_{y_i} be the eigenform corresponding to y_i. The interpolation property of our p-adic L-function gives

$$\frac{L(f_{y_1}, \chi, 1)}{2\pi i \Omega^{\pm}_{f_{y_1}}} \equiv \frac{L(f_{y_2}, \chi, 1)}{2\pi i \Omega^{\pm}_{f_{y_2}}} u(\kappa) \bmod \pi_K^n.$$

Since $u(\kappa)$ is a unit in $\mathcal{O}_K$ we know by Theorem 3.8 that $f_{y_1} \equiv f_{y_2} \bmod \pi_K^n$. From Lemma 6.1 we conclude

$$|g_1(\kappa) - g_2(\kappa)|_p = d(y_1, y_2) \leqslant p^{\frac{-n}{e}} = |I_u(\kappa)|_p. \qquad \square$$

6.3. Proof of Theorem 1.1

We are now ready to prove Theorem 1.1 by comparing $g_1(Y) - g_2(Y)$ with the ideal I_u. The proof involves using Lemma 6.3 to see how congruences behave as we approach the crossing point at classical weights.

Lemma 6.4. — *The largest power of Y dividing the ideal I_u is at most $I(\mathbb{T}_{N_0}, C_1, C_2, x_1)$.*

Proof. — Let $n = I(\mathbb{T}_{N_0}, C_1, C_2, x)$ and let m be the largest power of Y dividing I_u. Since $\mathcal{O}_{K^{nr}}\langle Y \rangle$ is Noetherian we may find $\chi_1, \ldots, \chi_d$ such that

$$I_u = (L_p^{\pm}(B_1, \chi_1) - u(Y) L_p^{\pm}(B_2, \chi_1), \ldots, L_p^{\pm}(B_1, \chi_d) - u(Y) L_p^{\pm}(B_2, \chi_d)).$$

The Weierstrass preparation theorem allows us to write

$$L_p^{\pm}(B_1, \chi_i) - u(Y) L_p^{\pm}(B_2, \chi_i) = Y^{m_{\chi_i}} P_{\chi_i}(Y) u_{\chi_i}(Y) \pi_K^{r_{\chi_i}},$$

where $P_{\chi_i}(Y)$ is a polynomial not divisible by Y and $u_{\chi_i}(Y)$ is a unit in $\mathcal{O}_{K^{nr}}\langle Y \rangle$. Similarly, by Lemma 5.9 we may write

$$g_1(Y) - g_2(Y) = Y^n P(Y) u(Y) \pi_K^r,$$

where $P(Y)$ is a polynomial not divisible by Y and $u(Y)$ is a unit in $\mathcal{O}_K\langle Y \rangle$.

Pick a sequence t_k of points in $\mathrm{Spec}(\mathcal{O}_K\langle Y \rangle)$ that converge to zero p-adically such that each t_k corresponds to a classical weight. By picking the t_k close enough to zero we may assume that $c = v_p(P(t_k))$ is constant. By Lemma 6.3 we know that

$$\min_i v_p(L_p^{\pm}(B_1, \chi_i)(t_k) - u(t_k) L_p^{\pm}(B_2, \chi_i)(t_k)) \leqslant v_p(g_1(t_k) - g_2(t_k)).$$

Then since $\min(m_{\chi_i}) = m$ we find

$$\begin{aligned} mv_p(t_k) &\leqslant \min_i v_p(L_p^{\pm}(B_1,\chi_i)(t_k) - u(t_k)L_p^{\pm}(B_2,\chi_i)(t_k)) \\ &\leqslant v_p(g_1(t_k) - g_2(t_k)) \\ &= nv_p(t_k) + \frac{r}{e} + c. \end{aligned}$$

Letting k go to infinity (so $v_p(t_k)$ gets large) we see that $m \leqslant n$. □

Proof of Theorem 1.1. — Let us first assume that $x_1 \in C_1$ and $x_2 \in C_2$ are equal to the same point x in $\mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}})$ and that the two components cross at x with multiplicity d. Let $\mathfrak{a}_i$ be the minimal prime ideal of B such that $B/\mathfrak{a}_i = B_i$. Consider the commutative diagrams obtained by localizing and completing the rings $B, B_i, B/(\mathfrak{a}_1+\mathfrak{a}_2)$ and $\mathcal{O}_K\langle Y\rangle$ at x.

$$\begin{tikzcd} & \widehat{B}_{2,x_2} \arrow[dr] & \\ \widehat{\mathbb{T}_{N_0,\mathfrak{m},x}} \arrow[ur] \arrow[dr] & K[\![Y]\!] \arrow[u] \arrow[d] \arrow[r] & \widehat{B}_x/(\mathfrak{a}_1+\mathfrak{a}_2). \\ & \widehat{B}_{1,x_1} \arrow[ur] & \end{tikzcd}$$

Here the vertical arrows are isomorphisms. The K-dimension of $\widehat{B}_x/(\mathfrak{a}_1 + \mathfrak{a}_2)$ is d by our definition of intersection multiplicity (see Section 5.7). Furthermore, since the arrows into $\widehat{B}_x/(\mathfrak{a}_1+\mathfrak{a}_2)$ are surjective, we see that $\widehat{B}_x/(\mathfrak{a}_1+\mathfrak{a}_2) = K[Y]/Y^d$. Following $L_p^{\pm}(\mathbb{T}_{N_0,\mathfrak{m}},\chi)$ along the top and bottom of the diagram shows us that

$$L_p^{\pm}(B_1,\chi) \equiv L_p^{\pm}(B_2,\chi) \bmod Y^d,$$

when we view both L-functions as elements of $K[\![Y]\!]$. Since Y generates the maximal ideal of $K[\![Y]\!]$, we have proven one direction of the theorem.

Conversely, assume that there exists $u(Y) \in \mathcal{O}_K\langle Y\rangle^{\times}$ such that

$$L_p^{\pm}(B_1,\chi) \equiv u(Y)L_p^{\pm}(B_2,\chi) \bmod Y^d,$$

for $\chi \in \mathcal{A}_{\epsilon}$. This means the ideal I_u is divisible by Y^d and then Lemma 6.4 tells us that the two components intersect with multiplicity at least d. □

7. Some examples

In this section we look at two Hida families of different levels with the same residual representation. Under a suitable hypothesis on the levels, we can determine if the two families will cross in a higher level by looking at the L-functions on each family modified by appropriate Euler factors. These modified L-functions were introduced in [10] for trivial tame character using results from [33].

7.1. Components of different level

In this subsection we apply our results on crossing components to the situation described in [10, Section 2.6]. We will briefly summarize the set up. For details and references see [10]. Let $\bar{\rho}$ be a modular residual Galois representation and let $\overline{V}$ be the $\mathbb{F}_q$ vector space on which $G_{\mathbb{Q}}$ acts. We will assume that $\bar{\rho}$ is odd, irreducible, p-ordinary, and p-distinguished. Fix a p-stabilization of $\bar{\rho}$. Let $N(\bar{\rho})$ be the conductor of $\bar{\rho}$. For a prime $l \neq p$ let n_l be the dimension of the I_l-coinvariant of $\overline{V}$ (i.e. the largest quotient of $\overline{V}$ that is invariant under the inertial group I_l). Let Σ be a finite set of primes not containing p. Define

$$N(\Sigma) := N(\bar{\rho}) \prod_{l \in \Sigma} l^{n_l}.$$

For any tame level N_0 we let $\mathbb{T}'_{N_0}$ be the Hecke algebra acting on $S(N_0 p^\infty, \mathcal{O}_K)$ generated by U_p and T_l for all $l \nmid N_0 p$ (explicitly, we are just leaving out the Atkin Lehner operators).

We let $\mathbb{T}^{\mathrm{new}}_{N_0}$ denote the Hecke algebra generated by T_l for primes $l \nmid N_0 p$ and U_l for $l | N_0 p$ acting on the subspace of $S(N_0 p^\infty, \mathcal{O}_K)$ consisting of all newforms. Then we have a natural map of Λ-algebras

$$\mathbb{T}'_{N(\Sigma)} \to \Pi_{M | N(\Sigma)} \mathbb{T}^{\mathrm{new}}_M.$$

This map becomes an isomorphism after tensoring over Λ with its fraction field $\mathcal{L}$. As described by Hida [19] there is a Galois representation $\rho'_M : G_{\mathbb{Q}} \to GL_2(\mathbb{T}^{\mathrm{new}}_M \otimes \mathcal{L})$ for any M. This gives a Galois representation $\rho' : G_{\mathbb{Q}} \to GL_2(\mathbb{T}'_{N(\Sigma)} \otimes \mathcal{L})$. We have the following two theorems

Theorem 7.1. — *There exists a unique maximal prime $\mathfrak{m}$ of $\mathbb{T}'_{N(\Sigma)}$ such that the residual representation of the composition*

$$G_{\mathbb{Q}} \to GL_2(\mathbb{T}'_{N(\Sigma)}) \to GL_2(\mathbb{T}_{N(\Sigma)}/\mathfrak{m})$$

is $\bar{\rho}$. Furthermore, there is a unique maximal prime $\mathfrak{n}$ of $\mathbb{T}_{N(\Sigma)}$ such that the two local Hecke algebras are isomorphism:

$$\mathbb{T}_{N(\Sigma),\mathfrak{n}} \cong \mathbb{T}'_{N(\Sigma),\mathfrak{m}}.$$

Proof. — This is [10, Theorem 2.1.2]. □

Theorem 7.2. — *Let $\mathfrak{a}$ be a minimal prime ideal of $\mathbb{T}_{N(\Sigma),\mathfrak{n}}$. There is an integer $N(\mathfrak{a})$, depending on $\mathfrak{a}$, that divides $N(\Sigma)$ and a minimal prime ideal $\mathfrak{a}'$ of $\mathbb{T}^{\mathrm{new}}_{N(\mathfrak{a})}$ that makes the following diagram commute.*

$$\begin{tikzcd} \mathbb{T}_{N(\Sigma),\mathfrak{n}} \cong \mathbb{T}'_{N(\Sigma),\mathfrak{m}} \arrow[r] \arrow[d] & \Pi_{M|N(\Sigma)}\mathbb{T}^{\mathrm{new}}_{M} \arrow[d] \\ \mathbb{T}_{N(\Sigma),\mathfrak{n}}/\mathfrak{a} \arrow[r] & \mathbb{T}^{\mathrm{new}}_{N(\mathfrak{a})}/\mathfrak{a}' \end{tikzcd}$$

Proof. — This follows from 7.1 and the isomorphism

$$\mathbb{T}'_{N(\Sigma)} \otimes \mathcal{L} \to \Pi_{M|N(\Sigma)}\mathbb{T}^{\mathrm{new}}_{M} \otimes \mathcal{L}.$$

See [10, Proposition 2.5.2] for more details. □

Remark. — We may think of $\mathrm{Spec}(\mathbb{T}_{N(\Sigma),\mathfrak{n}}/\mathfrak{a})$ as a family of old forms of level $N(\Sigma)$ and $\mathrm{Spec}(\mathbb{T}^{\mathrm{new}}_{N(\mathfrak{a})}/\mathfrak{a}')$ as a family of new forms of level $N(\mathfrak{a})$. If $x \in \mathrm{Spec}(\mathbb{T}_{N(\Sigma),\mathfrak{n}})$ corresponds to the classical old form f_x, then there is a corresponding $x' \in \mathrm{Spec}(\mathbb{T}^{\mathrm{new}}_{N(\mathfrak{a})}/\mathfrak{a}')$ that is sent to x under the map $\mathrm{Spec}(\mathbb{T}^{\mathrm{new}}_{N(\mathfrak{a})}/\mathfrak{a}') \to \mathrm{Spec}(\mathbb{T}_{N(\Sigma),\mathfrak{n}}/\mathfrak{a})$. The point x' corresponds to a newform $f_{x'}$ of level $N(\mathfrak{a})$. The Fourier coefficients of f_x and $f_{x'}$ agree away from the primes dividing the level $N(\Sigma)$.

By Theorem 1.1 we can determine when two components of $\mathbb{T}_{N(\Sigma)}$ cross by looking at p-adic L-functions on each component. It is then natural to ask if we can determine when a family of newforms of level $M_1|N(\Sigma)$ will cross a family of newforms of level $M_2|N(\Sigma)$ by looking at p-adic L-functions. To employ Theorem 1.1 it is necessary to relate our p-adic L-functions on $\mathrm{Spec}(\mathbb{T}_{N(\Sigma),\mathfrak{n}}/\mathfrak{a})$ to our p-adic L-functions on $\mathrm{Spec}(\mathbb{T}_{N(\mathfrak{a}),\mathfrak{n}}/\mathfrak{a}')$. The former interpolates special values of eigenforms for the Hecke algebra $\mathbb{T}_{N(\Sigma)}$ and the later interpolates special values of eigenforms for the Hecke algebra $\mathbb{T}_{N(\mathfrak{a})}$. As these two Hecke algebras only differ at $l|N(\Sigma)$, it is natural to suspect that the two L-functions will be the same after introducing some Euler factors for the primes $l|N(\Sigma)$.

Definition 7.3. — *Let $l \neq p$ be a prime and let χ be a Dirichlet character of level Mp^r. Define $E_{N(\mathfrak{a})}(\chi, l) \in \mathbb{T}_{N(\mathfrak{a})}$ as follows:*

$$E_{N(\mathfrak{a})}(\chi, l) := \begin{cases} 1 - \chi(l)T_l l^{-1} + \chi(l^2)\langle l\rangle l^{-3} & \textit{if } l \nmid N(\mathfrak{a}) \\ 1 - \chi(l)T_l l^{-1} & \textit{if } l|N(\mathfrak{a}) \end{cases}$$

We then define

$$E_\Sigma(\mathfrak{a}, \chi) := \Pi_{l\in\Sigma} E_{N(\mathfrak{a})}(\chi, l).$$

Remark. — This definition is similar to Definitions 2.7.1 and 3.6.1 in [10]. The Euler factor of [10] varies over a branch of the Hida family and the cyclotomic variable, while our definition only varies over the branch. If the conductor of χ is a power of p then our Euler factor is equal to a specialization of the Euler factor defined in [10].

Proposition 7.4. — *Let χ be a Dirichlet character. There exists a unit $u \in \mathbb{T}_{N(\mathfrak{a})}/\mathfrak{a}'$ independent of χ Dirichlet character such that*

$$L_p^{\pm}(\mathbb{T}_{N(\Sigma)}/\mathfrak{a}, \chi) = uE_{\Sigma}(\mathfrak{a}', \chi) L_p^{\pm}(\mathbb{T}_{N(\mathfrak{a})}/\mathfrak{a}', \chi).$$

Proof. — The proof follows from the computations in beginning of the proof of Theorem 3.6.2 in [10]. The only difference is that we are specializing in the cyclotomic variable and we allow a nontrivial tame conductor. $\square$

Theorem 7.5. — *Let M_1 and M_2 be two integers dividing $N(\Sigma)$. Let $\mathfrak{a}_i$ be a minimal prime ideal of $\mathbb{T}^{\mathrm{new}}_{M_i}$. Let C_1 and C_2 be the components of $\mathbb{T}_{N(\Sigma)}$ corresponding to $\mathfrak{a}_1$ and $\mathfrak{a}_2$. The following are equivalent:*

- *The components C_1 and C_2 cross at a point x. We assume that each component is étale at x over the weight space and the weight κ of x is the p-adic limit of classical weights.*
- *There exists a point x_i of $\mathrm{Spec}(\mathbb{T}^{\mathrm{new}}_{M_i})$ over κ and a unit u of Λ such that for all Dirichlet characters $\chi \in \mathcal{A}_\epsilon$ the value of $uL_p^{\pm}(\mathbb{T}^{\mathrm{new}}_{M_1}/\mathfrak{a}_1, \chi)E_{\Sigma}(\mathfrak{a}_1, \chi)$ evaluated at x_1 is the same as $L_p^{\pm}(\mathbb{T}^{\mathrm{new}}_{M_2}/\mathfrak{a}_2, \chi)E_{\Sigma}(\mathfrak{a}_2, \chi)$ evaluated at x_2.*

Proof. — This is a consequence of Proposition 7.4 and Theorem 1.1. $\square$

8. Ramification over the weigth space

In this section we describe how p-adic L-functions behave when a Hida family is ramified over the weight space. Let C be a Hida family contained in $\mathrm{Spec}(\mathbb{T}_{N_0})$. Then C is affine and we let A be the coordinate ring. For a Dirichlet character χ whose conductor is prime to p, we let $L_p^{\pm}(C, \chi)$ denote the restriction of $L_p^{\pm}(\mathbb{T}_{N_0,\mathfrak{m}}, \chi)$ to C. Informally, the main result of this section says that C has ramified points over $\mathrm{Spec}(\Lambda) = \mathrm{Spec}(\mathcal{O}_K[\![T]\!])$ if and only if there exists an L-function $L_p^{\pm}(C, \chi)$ that acquires singularities after being differentiated in the direction of the weight space. The singularities will be at ramified points.

We begin by making some geometric simplifications similar to those in Section 6.1. Let x be a regular $\mathcal{O}_K$-point of C that is the p-adic limit of classical points. Assume that π is ramified at x. The ring A_x is a discrete valuation ring since x is a regular point of codimension one. Let X be a uniformizing element of A_x. We may assume that X is in A by clearing any denominators. Since X is topologically nilpotent in A we have a map

$$g : C = \mathrm{Spec}(A) \to \mathrm{Spec}(\mathcal{O}_K[\![X]\!]),$$

which is étale at x. Let $Y_1, \ldots, Y_d$ be generators of A as an $\mathcal{O}_K[\![X]\!]$-algebra. We will assume that x is the origin in terms of these coordinates. Then Lemma 5.1 says that there exists m large enough and $N > m$ such that the map

$$g : \mathrm{Spec}(A\langle p^{-N}X, p^{-m}Y_1, \ldots, p^{-m}Y_d\rangle \to \mathrm{Spec}(\mathcal{O}_K\langle p^{-N}X\rangle)$$

is an isomorphism. If we let $X_0 = p^{-N}X$, we can write $T = f(X_0)$ where $f(X_0) \in \mathcal{O}_K\langle X_0\rangle$. Define B to be $\mathcal{O}_K\langle X_0, T\rangle/(T - f(X_0))$ and note that

$$B \cong A\langle p^{-N}X, p^{-m}Y_1, \ldots, p^{-m}Y_d\rangle.$$

We may think of $\mathrm{Spec}(B)$ as a small p-adic neighborhood of C embedded into the X_0T-plane. There is a natural map

$$h : \mathrm{Spec}(B) \to \mathrm{Spec}(\mathbb{T}_{N_0,\mathfrak{m}}),$$

and we let $L_p^{\pm}(B, \chi)$ denote the image of $L_p^{\pm}(\mathbb{T}_{N_0,\mathfrak{m}}, \chi)$ under this map. The following lemma relates distances in $\mathrm{Spec}(B)$ to our p-adic L-functions and may be thought of as a counterpart to Lemma 6.3

Lemma 8.1. — *Let $a_1, a_2 \in \mathrm{Spec}(B)$ be $\mathcal{O}_K$-points corresponding to classical eigenforms of equal weight (i.e. they have the same T-coordinate). Assume that*

$$\min_{\chi \in \mathcal{A}_\epsilon} v_p(L_p^{\pm}(B, \chi)(a_1) - L_p^{\pm}(B, \chi)(a_2)) = \frac{r}{e}.$$

Then

$$d(a_1, a_2) \leqslant p^{\frac{-r}{e} + N}.$$

Proof. — Let f_{a_1} and f_{a_2} be the eigenforms corresponding $h(a_1)$ and $h(a_2)$. The interpolation property of $L_p^{\pm}(B, \chi)$ gives

$$\frac{L(f_{a_1}, \chi, 1)}{2\pi i \Omega_{f_{a_1}}} \equiv \frac{L(f_{a_2}, \chi, 2)}{2\pi i \Omega_{f_{a_2}}} \bmod \pi_K^r.$$

Applying Theorem 3.8 and Lemma 5.6 gives

$$d(h(a_1), h(a_2)) \leqslant p^{\frac{-r}{e}}.$$

We need to relate $d(h(a_1), h(a_2))$ to $d(a_1, a_2)$. To do this, we note that C is a subscheme of $\mathrm{Spec}(\mathcal{O}_K\langle X, Y_1, \ldots, Y_d\rangle)$ and $\mathrm{Spec}(B)$ is a closed subscheme of $\mathrm{Spec}(\mathcal{O}_K\langle p^{-N}X, p^{-m}Y_1, \dots, p^{-m}Y_d\rangle)$. In terms of these coordinates, the map h is given by

$$(x, y_1, \dots, y_d) \to (p^N x, p^m y_1, \dots, p^m y_d).$$

By Lemma 5.4 we may compute distances with these coordinates. Since $N > m$ we see that $d(a_1, a_2) \leqslant p^N d(h(a_1), h(a_2))$. We conclude

$$d(a_1, a_2) \leqslant p^N d(h(a_1), h(a_2)) \leqslant p^{\frac{-r}{e}+N}. \qquad \square$$

Lemma 8.2. — *Let $s \in A$ be a function on C such that the image of s in $\mathcal{O}_K\langle X_0\rangle$ has a nonzero linear term. Let r be the ramification index of π at x. Then $\frac{\mathrm{d}}{\mathrm{d}T}(s)$ has a pole at x of order $r-1$.*

Proof. — Recall that $T = f(X_0)$ in B. The ramification index r is equal to the power of X_0 dividing $f(X_0)$. In particular $f(X_0) = p\,(X_0)X_0^r$, where $p(X_0)$ is not divisible by X_0. Differentiating the equation

$$T - p(X_0)X_0^r = 0$$

with respect to T yields

$$\frac{\mathrm{d}X_0}{\mathrm{d}T} = \frac{1}{X_0^{r-1}} \frac{1}{X_0 p'(X_0) + r p(X_0)}.$$

This shows that $\frac{\mathrm{d}X_0}{\mathrm{d}T}$ has a pole at $X_0 = 0$ of order $r-1$. By viewing s as a power series in $\mathcal{O}_K\langle X_0\rangle$ and differentiating term by term, we see that s has a pole of order $r-1$ if it has a nonzero linear term. $\square$

Proof of Theorem 1.2. — First let's assume that π is étale at x. Informally, this means that a small neighborhood of x looks just like part of $\mathrm{Spec}(\Lambda)$, so taking the derivative with respect to T should not introduce any poles. More precisely, let $\widehat{A_x}$ be the completion along the maximal ideal of the stalk of $\mathcal{O}_C$ at x. Define $\widehat{\Lambda_{K,\pi(x)}}$ to be the completion along the maximal ideal of the stalk of $\mathcal{O}_\Lambda$ at $\pi(x)$. The natural map from $\widehat{\Lambda_{K,\pi(x)}} \to \widehat{A_x}$ is an étale morphism of complete local rings with isomorphic residue fields. This means the two rings are isomorphic. This isomorphism commutes with the differential operator $\frac{\mathrm{d}}{\mathrm{d}T}$. In particular there is a map $A \to \widehat{\Lambda_{K,\pi(x)}}$ that commutes with $\frac{\mathrm{d}}{\mathrm{d}T}$ and the maximal ideal of $\widehat{\Lambda_{K,\pi(x)}}$ pulls back to x. It is then clear that for any $f \in A$ the function $\frac{\mathrm{d}}{\mathrm{d}T} f$ does not have a pole at x.

The converse is slightly more difficult. By Lemma 8.2 it suffices to show that there exists a primitive Dirichlet character $\chi \in \mathcal{A}_\epsilon$ such that $L_p^{\pm}(B, \chi)$ has a linear term when viewed as a power series in X_0. Assume the contrary. We may write

$$L_p^{\pm}(B, \chi) = \sum_{i=0}^{\infty} c_{i,\chi} X_0^i,$$

where $c_{i,\chi} \in \mathcal{O}_K$ and $c_{1,\chi}$ is zero. Now let $x_1 = (\alpha, a_1)$ and $x_2(\alpha, a_2)$ be two distinct $\mathcal{O}_K$-points in $\operatorname{Spec}(B)$, where α is the T-coordinate and a_i is the X_0-coordinate. We take both points to correspond to eigenforms, which have the same weight since they have the same T-coordinate. Since x, which is given by the coordinates $(0,0)$, is the limit of classical points, we may let α be arbitrarily close to zero. Then we may also take a_1 and a_2 to be close to zero, since a_1 and a_2 are roots of $\alpha - f(X_0)$. In particular, we may assume that $v_p(a_1)$ and $v_p(a_2)$ are both greater than N. This gives

$$v_p(a_1^j - a_2^j) = v_p(a_1 - a_2) + v_p(a_1^{j-1} + \cdots + a_2^{j-1}) > v_p(a_1 - a_2) + N,$$

whenever $j \geqslant 2$. We then apply Lemma 8.1 to get

$$\begin{aligned} v_p(a_1 - a_2) &= -\log_p d(a_1, a_2) \\ &\geqslant \min_{\chi} v_p(L_p^{\pm}(B,\chi)(a_1) - L_p^{\pm}(B,\chi)(a_2)) - N \\ &\geqslant \min_{\chi} v_p\left(\sum_{i=0}^{\infty} c_{i,\chi} a_1^i - \sum_{i=0}^{\infty} c_{i,\chi} a_2^i\right) - N \\ &= \min_{\chi} v_p\left(\sum_{i=2}^{\infty} c_{i,\chi}(a_1^i - a_2^i)\right) - N \\ &> v_p(a_1 - a_2), \end{aligned}$$

which gives a contradiction. □

For the previous result, we choose a parameter for the weight space. The result holds true for any parameter and it would be nice to have a statement that makes no reference to any choice of parameter. This can be achieved using the Gauss–Manin connection, which can be defined without choosing a basis. For an overview of the Gauss–Manin connection see [21] or [22]. More precisely, consider the relative 0-th de Rham cohomology group $H^0_{dR}(C/\operatorname{Spec}(\Lambda))$ (see for example [14]). We may identify $H^0_{dR}(C/\operatorname{Spec}(\Lambda))$ with $\pi_*(\mathcal{O}_C)$. Let U be an open subscheme of C such that $\pi|_U$ is étale. Then following [22] there is a Gauss–Manin connection

$$\nabla : H^0_{dR}(C/\operatorname{Spec}(\Lambda))|_U \to H^0_{dR}(C/\operatorname{Spec}(\Lambda)) \otimes \Omega_{\operatorname{Spec}(\Lambda)}|_U.$$

If $f \in \Gamma(\mathcal{O}_C, U)$ and T_0 is any parameter of the weight space then $\nabla(f) = \frac{\mathrm{d}}{\mathrm{d}T_0} f \mathrm{d}T_0$. The map ∇ makes sense on all of $\operatorname{Spec}(\Lambda)$ when we allow poles in the image.

Corollary 8.3. — *Let κ be a $\mathcal{O}_K$ point of Λ. The map π is étale at the points above κ if and only if for all $\chi \in \mathcal{A}_\epsilon$ whose conductor is prime to p we have*

$$\nabla(L_p^{\pm}(C,\chi)) \in \Gamma(\pi_*\mathcal{O}_C \otimes \Omega_{\operatorname{Spec}(\Lambda)}, V)$$

where V is some Zariski open containing κ.

Proof. — The "only if" direction follows from the existence of the Gauss–Manin connection for smooth maps. For the "if" let x be a point in $\pi^{-1}(\kappa)$. Let $\chi \in \mathcal{A}_\epsilon$. By our hypothesis $\nabla(L_p^\pm(C,\chi)) \in (\pi_*\mathcal{O}_C \otimes \Omega_{\mathrm{Spec}(\Lambda)})_\kappa$ Note that

$$(\pi_*\mathcal{O}_C \otimes \Omega_{\mathrm{Spec}(\Lambda)})_\kappa \cong A_\kappa \otimes_{\Lambda_{K,\kappa}} \Omega_{\mathrm{Spec}(\Lambda),\kappa}.$$

Choose a parameter T_0 of the weight space and let D be the map from $\Omega_{\mathrm{Spec}(\Lambda),\kappa} \to \mathcal{O}_{\mathrm{Spec}(\Lambda)}$ that sends dT_0 to 1. Then $D \circ \nabla$ is the map $A_\kappa \to A_\kappa$ given by differentiation with respect to T_0. There is a natural map $l : A_\kappa \to A_{(x)}$, the localization of A at x. Since $\nabla(L_p^\pm(C,\chi))$ is contained in $(\pi_*\mathcal{O}_C \otimes \Omega_{\mathrm{Spec}(\Lambda)})_\kappa$ we see that $l \circ D \circ \nabla(L_p^\pm(C,\chi))$ is contained in $A_{(x)}$. This means that $\frac{\mathrm{d}}{\mathrm{d}T_0} L_p^\pm(C,\chi)$ does not have a pole at x. The corollary then follows from Theorem 1.2 □

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