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AUTOMORPHISMS OF REAL DEL PEZZO SURFACES AND THE REAL PLANE CREMONA GROUP

by Egor YASINSKY (*)

ABSTRACT. — We study automorphism groups of real del Pezzo surfaces, concentrating on finite groups acting with invariant Picard number equal to one. As a result, we obtain a vast part of classification of finite subgroups in the real plane Cremona group.

RÉSUMÉ. — On étudie les groupes d'automorphismes des surfaces de del Pezzo réelles, en se concentrant sur les groupes finis qui agissent avec un nombre invariant de Picard égal à 1. En conséquence, on obtient une bonne part de la classification des sous-groupes finis du groupe de Cremona du plan réel.

1. Introduction

1.1. The classification problem

This paper is devoted to the study of finite automorphism groups of real del Pezzo surfaces. Our main motivation is the classification of finite subgroups of the real plane Cremona group; hence this paper may be viewed as a follow-up paper to [39]. Recall that the Cremona group $\mathrm{Cr}_n(\mathbb{k}) = \mathrm{Bir}(\mathbb{P}_{\mathbb{k}}^n)$ is the group of birational automorphisms of the n -dimensional projective space over a field \mathbb{k} . The finite subgroups of $\mathrm{Cr}_1(\mathbb{k}) \cong \mathrm{PGL}_2(\mathbb{k})$ have been known since Klein's time (see Lemma 2.5 and [2]). By contrast, the complete classification of finite subgroups of $\mathrm{Cr}_2(\mathbb{k})$ for $\mathbb{k} = \overline{\mathbb{k}}$ was obtained by I. Dolgachev and V. Iskovskikh only in 2009 and involves different hard techniques of modern birational geometry, such as Mori theory, equivariant

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resolution of singularities, etc. For the exposition of these results, as well as some historical notes, we refer the reader to the original papers [4] (case of abelian subgroups) and [13].

Much less is known for algebraically non-closed fields or $n \geq 3$. Classification of finite subgroups of $\text{Cr}_2(\mathbb{R})$ was initiated by the author in [39] where subgroups of odd order were classified up to conjugacy. The goal of this paper is to extend these results much further and to classify all finite groups acting minimally on real del Pezzo surfaces (see below). As will be explained below, this gives a vast part of classification of finite subgroups in $\text{Cr}_2(\mathbb{R})$.

As for the case $n \geq 3$, $\mathbb{k} = \mathbb{C}$, the classification seems out of reach at the moment. There are some partial results, see e.g. [24, 25]. Alternatively, one can try looking at things from a different point of view using the notion of *Jordan property* introduced in [23]. Recall that an abstract group Γ is called *Jordan* if there exists a positive integer m such that every finite subgroup $G \subset \Gamma$ contains a normal abelian subgroup $A \triangleleft G$ of index at most m . The minimal such m is called the *Jordan constant* of Γ and is denoted by $J(\Gamma)$. There is a remarkable result⁽¹⁾ [27]:

THEOREM 1.1 (Yu. Prokhorov, C. Shramov). — *Let $\text{char } \mathbb{k} = 0$. Then $\text{Cr}_n(\mathbb{k})$ is Jordan for each $n \geq 1$.*

This theorem allows, at least theoretically, to classify finite subgroups of Cremona groups “up to abelian subgroups”. Indeed, we know that for each extension

$$1 \rightarrow A \rightarrow G \rightarrow G/A \rightarrow 1,$$

where $A \subset G$ is a normal abelian subgroup, the sizes of G/A are uniformly bounded by a universal constant depending only on n and \mathbb{k} . How large can be the list of possible G/A , i.e. what are precise values of $J(\text{Cr}_n(\mathbb{k}))$? There are some results in this direction.

THEOREM 1.2 ([40, Theorems 1.9, 1.10]). — *One has*

$$J(\text{Cr}_2(\mathbb{C})) = 7200, \quad J(\text{Cr}_2(\mathbb{R})) = 120.$$

THEOREM 1.3 ([28, Theorem 1.2.4]). — *Suppose that the field \mathbb{k} has characteristic 0. Then one has*

$$J(\text{Cr}_3(\mathbb{k})) \leq 107\,495\,424.$$

⁽¹⁾It was initially proved modulo so-called Borisov–Alexeev–Borisov conjecture, which was settled in any dimension in [3].

1.2. G -surfaces

Let us briefly recall a general strategy of classification of finite subgroups in $\text{Cr}_2(\mathbb{k})$. Throughout this paper G denotes a finite group. Let \mathbb{k} be a perfect field. We use the standard language of G -varieties (see e.g. [13] or [39]). The modern approach to classification is based on the following observations:

- For any finite subgroup $G \subset \text{Cr}_2(\mathbb{k})$ there exists a \mathbb{k} -rational smooth projective surface X , an injective homomorphism $\iota : G \rightarrow \text{Aut}_{\mathbb{k}}(X)$ and a birational G -equivariant \mathbb{k} -map $\psi : X \dashrightarrow \mathbb{P}_{\mathbb{k}}^2$, such that

$$G = \psi \circ \iota(G) \circ \psi^{-1}$$

This process of passing from a birational action of G on $\mathbb{P}_{\mathbb{k}}^2$ to a regular action on X is usually called the *regularization* of the G -action. On the other hand, for a \mathbb{k} -rational G -surface X a birational map $\psi : X \dashrightarrow \mathbb{P}_{\mathbb{k}}^2$ yields an injective homomorphism

$$i_{\psi} : G \rightarrow \text{Cr}_2(\mathbb{k}), \quad g \mapsto \psi \circ g \circ \psi^{-1}.$$

Moreover, two subgroups of $\text{Cr}_2(\mathbb{k})$ are conjugate if and only if the corresponding G -surfaces are birationally equivalent. So, there is a natural bijection between the conjugacy classes of finite subgroups $G \subset \text{Cr}_2(\mathbb{k})$ and birational isomorphism classes of smooth \mathbb{k} -rational G -surfaces (X, G) .

- For any projective geometrically smooth G -surface X over \mathbb{k} there exists a birational G -equivariant \mathbb{k} -morphism $X \rightarrow X_{\min}$ where the G -surface X_{\min} is G -minimal. The latter means that any birational G -equivariant \mathbb{k} -morphism $X_{\min} \rightarrow Z$ is an isomorphism. If the surface X is additionally $\bar{\mathbb{k}}$ -rational, then one of the following holds [14, Theorem 5]:
 - (1) X_{\min} admits a conic bundle structure with $\text{Pic}(X)^G \cong \mathbb{Z}^2$;
 - (2) X_{\min} is a del Pezzo surface with $\text{Pic}(X)^G \cong \mathbb{Z}$.

So, the classification of finite subgroups of $\text{Cr}_2(\mathbb{k})$ is equivalent to birational classification of minimal pairs (X, G) described above. The goal of this paper is to describe all the minimal pairs (X, G) with X a real del Pezzo surface, i.e. to complete the study of the first case in the previous dichotomy.

1.3. Some comments on the conic bundle case

The reader may wonder why do we focus only on the case of del Pezzo surfaces in this paper. The following example can serve as a partial explanation (or rather an excuse). Namely, it shows that there exist infinitely many pairwise non-conjugate involutions in $\text{Cr}_2(\mathbb{R})$, which are all conjugate over \mathbb{C} . So, the classification of finite subgroups up to conjugacy in $\text{Cr}_2(\mathbb{R})$ is a much more subtle question. For the philosophy of \mathbb{k} -birational unboundedness of conic bundles quotients standing behind this example see [32].

Example 1.4. — Consider the surface

$$Z_n : x^2 \prod_{k=1}^{2n} (t_0^2 + k^2 t_1^2) + y^2 t_0^{4n} + z^2 t_1^{4n} = 0$$

in $\text{Proj } \mathbb{R}[x, y, z] \times \text{Proj } \mathbb{R}[t_0, t_1] \cong \mathbb{P}_{\mathbb{R}}^2 \times \mathbb{P}_{\mathbb{R}}^1$. The projection to $\mathbb{P}_{\mathbb{R}}^1$ -factor defines a structure of a conic bundle on $\pi : Z_n \rightarrow \mathbb{P}^1$. Its geometrically singular fibers lie over the points $p_k = [ik : 1]$, $\bar{p}_k = [-ik : 1]$ (here $i = \sqrt{-1}$) and are given by $y^2 + z^2 = 0$.

Let $g_n \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$ be the involution $[t_0 : t_1] \mapsto [-t_0 : t_1]$. The complex involution σ and the automorphism g_n act on Z_n as shown on Figure 1.1. Note that

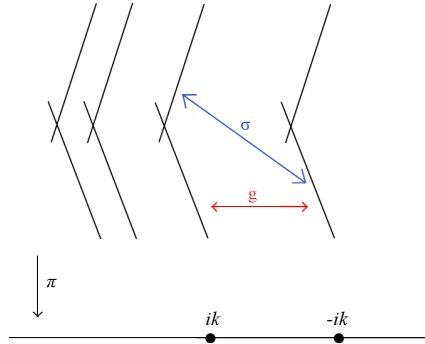


Figure 1.1. Involutions on Z_n .

- (1) Irreducible components of all singular fibers of Z_n can be Γ -equivariantly contracted on a conic bundle without singular fibers, hence Z_n is rational over \mathbb{R} . In particular, $g_n \in \text{Cr}_2(\mathbb{R})$.

- (2) Z_n is $\langle g_n \rangle$ -minimal. On the other hand, $Z_n \otimes \mathbb{C}$ is not $\langle g_n \rangle$ -minimal over \mathbb{C} , as we can contract disjoint irreducible components of all singular fibers onto some Hirzebruch surface equivariantly. Using elementary transformations between Hirzebruch surfaces (or just [4, Theorem 1]), we observe that all g_n are conjugate in $\mathrm{Cr}_2(\mathbb{C})$.
- (3) The surface $X_n = Z_n / \langle g_n \rangle$ has a structure of a conic bundle with $2n$ singular fibers, and irreducible components in each fiber are complex conjugate. In particular X_n is \mathbb{R} -minimal. Thus X_n is not rational over \mathbb{R} when $n > 3$ (e.g. by Iskovskikh's rationality criterion, see [17, § 4]).

Consider two finite subgroups $G_1, G_2 \subset \mathrm{Cr}_2(\mathbb{R})$ with regularizations (Y_1, G_1) and (Y_2, G_2) respectively. Assuming that G_1 is conjugate to G_2 , there exists a common equivariant resolution $Y \rightarrow Y_1, Y \rightarrow Y_2$ such that the actions of G_1 and G_2 coincide on Y . Therefore, Y_1/G_1 is birational to Y_2/G_2 . However, for $n, m > 3$ the conic bundles X_n and X_m are not pairwise birational to each other (see e.g. [16, Theorem 1.6] or [19, Theorem 4.3]). Therefore, involutions g_n and g_m are not conjugate in $\mathrm{Cr}_2(\mathbb{R})$.

This paper is organised as follows. Section 2 recalls some basic facts about del Pezzo surfaces, their topology and relation to Weyl groups; it also gathers some auxiliary results about Sarkisov program and classical linear groups that will be used later. The reader may skip this section and return to it later, if needed. In Sections 3–9 we study groups acting on real del Pezzo surfaces X with $K_X^2 \geq 3$, $K_X^2 \neq 7, 9$. The cases $K_X^2 = 9$ and $K_X^2 = 7$ are trivial. Indeed, a del Pezzo surface of degree 7 is never G -minimal, and a real del Pezzo surface X of degree 9 with $X(\mathbb{R}) \neq \emptyset$ is isomorphic to $\mathbb{P}_{\mathbb{R}}^2$, so finite groups acting on it are well known, see Lemma 2.5.

In comparison with the case $\mathbb{k} = \mathbb{C}$, we have to deal with *real forms* of del Pezzo surfaces (i.e. non-isomorphic real surfaces that become isomorphic over \mathbb{C}). Here we face an additional difficulty, since the complete classification of possible automorphism groups of del Pezzo surfaces is available only over the field of complex numbers; in fact, this classification was heavily used in the work of Dolgachev and Iskovskikh. So, in Sections 3–6 (i.e. $K_X^2 \geq 4$) we generally adapt the following strategy to classification: for each (\mathbb{R} -rational) real form of a del Pezzo surface X , we study the group $\mathrm{Aut}(X)$ (giving its precise description in many cases), and then determine possible finite groups $G \subset \mathrm{Aut}(X)$ that can act minimally on X . To find such groups G , we usually investigate the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \times G$ on $X \otimes \mathbb{C}$; for high degree del Pezzo surfaces, we look directly at the intersection graph of (-1) -curves, which is easy to analyze in these cases. For low degree surfaces

($K_X^2 \leq 3$), our approach becomes more combinatorial. Both real structure σ on X and automorphisms $G \subset \text{Aut}(X)$ can be considered as elements of the Weyl group \mathscr{W} associated to X (see Section 2.1). Using the classification of conjugacy classes in \mathscr{W} , we determine possible pairs (σ, G) such that the action of $\langle \sigma \rangle \times G$ on $X \otimes \mathbb{C}$ is minimal. In many cases we work with explicit equations of X and G (for example, in Section 7 we adapt for our purposes Sylvester's classical approach to cubic surfaces).

In Appendixes A and B we focus on some special classes of finite subgroups in $\text{Cr}_2(\mathbb{R})$ (being motivated by the study of those in [24, 26, 35]) and in particular classify non-solvable finite groups acting on real geometrically rational surfaces. Our goal is to demonstrate that:

- (1) this classification can be obtained independently of the “complete” classification of all finite subgroups and
- (2) the corresponding list is considerably shorter than in the case $\mathbb{k} = \mathbb{C}$. Finally, for the reader's convenience, some technical information about real invariants of some finite groups is included in Appendix C.

1.4. Notation and conventions

We use the following notation and conventions.

- In this paper, we say that a real del Pezzo surface X is *G-minimal*, or simply *G is minimal* (when it acts on X), if and only if any birational G -morphism $X \rightarrow Y$ of G -surfaces is an isomorphism. Further, we say that X is *strongly⁽²⁾ G-minimal*, or *G is strongly minimal*, if and only if $\text{rk Pic}(X)^G = 1$. Clearly, strong G -minimality implies G -minimality, but not vice versa (consider e.g. $X = \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$ with G acting preserving the factors).
- Moreover, all del Pezzo surfaces are assumed to be \mathbb{R} -rational (if not stated otherwise), and in particular their real loci $X(\mathbb{R})$ are not empty. The latter condition implies that

$$\text{Pic}(X_{\mathbb{C}})^{\Gamma} = \text{Pic}(X), \quad \text{hence } \text{Pic}(X_{\mathbb{C}})^{\Gamma \times G} = \text{Pic}(X)^G,$$

where $X_{\mathbb{C}} = X \otimes \mathbb{C}$, and Γ is the Galois group $\text{Gal}(\mathbb{C}/\mathbb{R})$ generated by the involution σ . Therefore, a real del Pezzo surface X is strongly G -minimal if and only if $X_{\mathbb{C}}$ is strongly $\Gamma \times G$ -minimal.

(2) This term is not conventional. We use it only for brevity.

- We denote by $Q_{r,s}$ the smooth quadric hypersurface

$$\{[x_1 : \dots : x_{r+s}] : x_1^2 + \dots + x_r^2 - x_{r+1}^2 - \dots - x_{r+s}^2 = 0\} \subset \mathbb{P}_{\mathbb{R}}^{r+s-1}.$$
- For a real del Pezzo surface X , we denote by $X(a, b)$ the blow-up of X at a real points and b pairs of complex conjugate points. We shall mostly use $\mathbb{P}_{\mathbb{R}}^2$, $Q_{3,1}$ or $Q_{2,2}$ as X .
- \mathbb{Z}/n or simply n is a cyclic group of order n ;
- D_n is a dihedral group of order $2n$;
- $\text{BD}_n = \langle a, x \mid a^{2n} = 1, x^2 = a^n, axa^{-1} = a^{-1} \rangle$ is the binary dihedral group of order $2n$;
- \mathfrak{S}_n is a symmetric group on n -letters.
- $A\Delta_D B$ is the diagonal product of A and B over their common homomorphic image D , i.e. the subgroup of $A \times B$ of pairs (a, b) such that $\alpha(a) = \beta(b)$ for some epimorphisms $\alpha : A \rightarrow D$, $\beta : B \rightarrow D$.
- $A_{\bullet} B$ is an extension of B by A ;
- When running the Sarkisov program (e.g. as in Proposition 3.4) we denote by \mathcal{D}_d (resp. \mathcal{C}_d) a del Pezzo surface (resp. a conic bundle) of degree d (resp. with $d = 8 - K_X^2$ singular fibers).
- I or I_n denotes the identity matrix of size $n \times n$.

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2. Some auxiliary results

2.1. A quick look at (real) del Pezzo surfaces

Let us briefly overview some important tools that shall be used in this paper. For a more comprehensive account see e.g. [11] or [20]. For the Minimal Model Program over \mathbb{R} and its relation to the topology of real rational surfaces see [19].

In this paper we are interested in the embedding of finite groups into $\text{Cr}_2(\mathbb{R})$, hence we focus on \mathbb{R} -rational surfaces in the first place. When X is

a non-singular real projective algebraic surface its set of real points $X(\mathbb{R})$ will be always regarded as a compact two-dimensional C^∞ -manifold with the usual Euclidean topology. The following characterization of \mathbb{R} -rational del Pezzo surfaces will be useful for us.

PROPOSITION 2.1. — *Let X be a smooth real del Pezzo surface. Then X is \mathbb{R} -rational if and only if $X(\mathbb{R})$ is nonempty and connected.*

Proof. — The result is classical and follows from [19, Theorem 1.9, Theorem 2.2, Lemma 3.2]. □

Remark 2.2. — In fact, for an \mathbb{R} -rational del Pezzo surface X , its real locus $X(\mathbb{R})$ is diffeomorphic to one of the following manifolds:

- (1) \mathbb{S}^2 if $X \cong Q_{3,1}(0, b)$;
- (2) \mathbb{T}^2 if $X \cong Q_{2,2}(0, b)$;
- (3) $N_g = \#_g \mathbb{R}P^2$ if $X \cong \mathbb{P}_{\mathbb{R}}^2(a, b)$ where $g = a + 1$ and $1 \leq g \leq 9$.

See [19] for details.

Another powerful tool for studying del Pezzo surfaces is the *Weyl groups*. Let $X_{\mathbb{C}}$ be a complex del Pezzo surface of degree $d \leq 6$, obtained by blowing up $\mathbb{P}_{\mathbb{C}}^2$ in $r = 9 - d$ points. The group $\text{Pic } X_{\mathbb{C}} \cong \mathbb{Z}^{r+1}$ has a basis e_0, e_1, \dots, e_r , where e_0 is the pull-back of the class of a line on $\mathbb{P}_{\mathbb{C}}^2$, and e_i are the classes of exceptional curves. Put

$$\Delta_r = \{s \in \text{Pic}(X_{\mathbb{C}}) : s^2 = -2, s \cdot K_{X_{\mathbb{C}}} = 0\}.$$

Then Δ_r is a root system in the orthogonal complement to $K_{X_{\mathbb{C}}}^{\perp} \subset \text{Pic}(X_{\mathbb{C}}) \otimes \mathbb{R}$. As usual, one can associate with Δ_r the Weyl group $\mathscr{W}(\Delta_r)$. Depending on degree d , the type of Δ_r and the size of $\mathscr{W}(\Delta_r)$ are the following:

Table 2.1. The Weyl groups

d	1	2	3	4	5	6
Δ_r	E_8	E_7	E_6	D_5	A_4	$A_1 \times A_2$
$ \mathscr{W}(\Delta_r) $	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^7 \cdot 3^4 \cdot 5$	$2^7 \cdot 3 \cdot 5$	$2^3 \cdot 3 \cdot 5$	12

Moreover, there are natural homomorphisms

$$\rho : \text{Aut}(X_{\mathbb{C}}) \rightarrow \mathscr{W}(\Delta_r), \quad \eta : \Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) \rightarrow \mathscr{W}(\Delta_r),$$

where ρ is an injection for $d \leq 5$. We denote by g^* the image of $g \in \Gamma \times G$ in the corresponding Weyl group.

Denote by \mathbb{E}_r the sublattice of $\text{Pic}(X_{\mathbb{C}})$ generated by the root system Δ_r . For an element $g^* \in \mathscr{W}(\Delta_r)$ denote by $\text{tr}(g^*)$ its trace on \mathbb{E}_r . To determine

whether a finite group $\Gamma \times G$ acts strongly minimally on $X_{\mathbb{C}}$, we use the well-known formula from the character theory of finite groups

$$(2.1) \quad \text{rk Pic}(X_{\mathbb{C}})^{\Gamma \times G} = 1 + \frac{1}{|\Gamma \times G|} \sum_{g \in \Gamma \times G} \text{tr}(g^*).$$

Thus the group $\Gamma \times G$ acts strongly minimally on $X_{\mathbb{C}}$ if and only if

$$\sum_{g \in \Gamma \times G} \text{tr}(g^*) = 0.$$

On the other hand, by the Lefschetz fixed point formula for any $h \in G$ we have,

$$(2.2) \quad \text{Eu}(X_{\mathbb{C}}^h) = \text{tr}(h^*) + 3.$$

Remark 2.3. — Note that a cyclic group always has a fixed point on a complex rational variety. This follows from the holomorphic Lefschetz fixed-point formula.

In this paper we shall use the known classification of conjugacy classes in the Weyl groups. These classes are indexed by *Carter graphs*, named e.g. A_1, A_1^2 , etc. Here we follow the terminology of [8] (used in [13]). Among other things, a Carter graph determines the characteristic polynomial of an element from a given class and its trace on $K_{X_{\mathbb{C}}}^{\perp}$, see [13, Table 2]. Another useful source of information about involutions in Weyl groups and real structures on del Pezzo surfaces is [38]. Note that Wall labels the conjugacy classes by Dynkin diagrams; in the situation where it can be confusing for the reader, we give the precise correspondence between these two different notations (e.g. in Table 9.1).

2.2. Sarkisov links

The main tool for exploring conjugacy in Cremona groups is the *Sarkisov program*. Here we very briefly recall how this tool looks like. For details see [13], [17] or [22] for the theory developed over \mathbb{R} .

We work in the category of G -surfaces over a perfect field \mathbb{k} . Similarly to the classical case of trivial G , any birational G -map between two G -surfaces can be decomposed into a sequence of birational G -morphisms and their inverses. A birational G -morphism $X \rightarrow Y$ can be thought of as a blow-up of a closed G -invariant 0-dimensional subscheme \mathfrak{p} of Y . Recall that $\text{deg}(\mathfrak{p}) = h^0(\mathcal{O}_{\mathfrak{p}})$. When \mathfrak{p} is reduced and consists of closed points y_1, \dots, y_n with residue fields $\kappa(y_i)$, one has $\text{deg } \mathfrak{p} = \sum \text{deg } y_i$ with $\text{deg } y_i = [\kappa(y_i) : \mathbb{k}]$.

If \mathfrak{p} is G -invariant, then it is a union of G -orbits. So, over the field of reals one can blow up orbits of real points and pairs of complex conjugate points.

In this paper we shall work with G -minimal del Pezzo surfaces and conic bundles (in the sense defined above). From the Mori theory’s point of view, these are rational Fano–Mori G -fibrations of dimension two (extremal contractions $\pi : X \rightarrow C$, where C is a point in the del Pezzo case, and C is a curve in the conic bundle case). A birational G -map f between Mori fibrations is a diagram of G -equivariant maps

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \pi \downarrow & & \downarrow \pi' \\ C & & C' \end{array}$$

Now, according to Sarkisov program, every birational map $f : X \dashrightarrow X'$ of rational minimal G -surfaces is factorized into a composition of *elementary Sarkisov links* of four types. For complete description of all such possible links we refer to [17].

2.3. Topological bounds

For a finite group $G \subset \text{Aut}(X_{\mathbb{C}})$, the representation

$$\rho : G \rightarrow \mathscr{W}(\Delta_r)$$

obviously restricts the order of G when $K_X^2 < 6$, which makes the classification of finite subgroups of $\text{Cr}_2(\mathbb{k})$ possible. It seems curious to us, that for real del Pezzo surfaces one can get some bounds on $|G|$ independently of the Weyl groups. We shall not use the following result, but in our opinion it is worth mentioning.

PROPOSITION 2.4. — *Let X be an \mathbb{R} -rational del Pezzo surface of degree d and $G \subset \text{Aut}(X)$ be a finite group. Then one of the following holds.*

- If $X \cong \mathbb{P}_{\mathbb{R}}^2(a, b)$, then

$$|G| \leq 84(8 - d)$$

for $a \geq 2$. For $a = 0$ the group G is isomorphic to

$$(2.3) \quad \mathbb{Z}/n, \quad D_n, \quad \mathfrak{A}_4, \quad \mathfrak{S}_4 \quad \text{or} \quad \mathfrak{A}_5.$$

For $a = 1$ one has $G \cong (n \times m)_{\bullet} k$, where $k \in \{1, 2, 3, 4, 6\}$.

- If $X \cong Q_{3,1}(0, b)$, then $G \cong H_{\bullet} 2^r$, where $r \in \{0, 1\}$ and H belongs to the list (2.3).

- If $X \cong Q_{2,2}(0, b)$, then $G \cong ((n \times m) \bullet k) \bullet 2^r$, where $r \in \{0, 1\}$ and $k \in \{1, 2, 3, 4, 6\}$.

Proof. — We may assume that G faithfully acts on $X(\mathbb{R})$ by diffeomorphisms. Let $X \cong \mathbb{P}_{\mathbb{R}}^2(a, b)$. Then $X(\mathbb{R}) \approx \#_{a+1} \mathbb{R}\mathbb{P}^2$. Denote its orientable double cover by Σ_a . By [6, Corollary 9.4] we may assume that G acts faithfully on Σ_a by orientation-preserving diffeomorphisms. Take any Riemannian metric on Σ_a and average it with respect to G action. The resulting G -invariant metric gives a complex G -invariant structure on Σ_a , and G can be regarded as a group of automorphisms of a Riemann surface of genus a .

Therefore, for $a = 0$ the group G embeds into $\text{Aut}(\Sigma_0) \cong \text{PSL}_2(\mathbb{C})$. We recall its subgroups in Lemma 2.5 below. For $a = 1$ the claim follows from a well-known classification of automorphisms of elliptic curves. Finally, for $a > 1$ the Hurwitz theorem implies

$$|G| \leq 84(a - 1),$$

so $a + 2b = 9 - d$ gives the result. Let $X \cong Q_{3,1}(0, b)$ or $X \cong Q_{2,2}(0, b)$. Again, G faithfully acts by diffeomorphisms of $X(\mathbb{R})$. Passing to an index 2 subgroup, we may assume that the action is orientation-preserving. Applying the same arguments as above, we finish the proof⁽³⁾. \square

2.4. Classical linear groups

The next result is classical and will be used throughout all the paper (see e.g. [2] or [5] for a modern treatment).

LEMMA 2.5. — *The following assertions hold.*

- (1) Any finite subgroup of $\text{PGL}_2(\mathbb{C})$ is one of the following:

(2.4) $\mathbb{Z}/n, D_n, n \geq 1, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5.$

Any finite subgroup of $\text{GL}_2(\mathbb{R})$ and $\text{PGL}_2(\mathbb{R})$ is either cyclic or dihedral.

- (2) One has $\text{PGL}_3(\mathbb{R}) \cong \text{SL}_3(\mathbb{R})$. Any finite subgroup of $\text{PGL}_3(\mathbb{R})$ is among the ones listed in (2.4).

Despite its simplicity, Lemma 2.5 has important consequences for classification of finite subgroups of $\text{Cr}_2(\mathbb{R})$ and, more generally, groups acting on real geometrically rational surfaces. For example, it “kills” almost all simple finite subgroups of $\text{Cr}_2(\mathbb{R})$, see Appendix A.

⁽³⁾ See also Remark 3.1.

3. Del Pezzo surfaces of degree 8

In this section X denotes a real del Pezzo surface of degree 8. We shall assume that $X_{\mathbb{C}} \cong \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ (the other surface of degree 8, the blow up of $\mathbb{P}_{\mathbb{R}}^2$ at one point, is never G -minimal), so either $X \cong Q_{3,1}$ or $X \cong Q_{2,2}$ [19, Lemma 1.16]. We treat these two cases separately.

Let $X = Q_{3,1}$. Since $Q_{3,1}$ is \mathbb{R} -minimal, any $G \subset \text{Aut}(X)$ acts strongly minimally on X . By definition, $\text{Aut}(X) = \text{PO}(3, 1)$, where

$$\text{PO}(3, 1) = \text{O}(3, 1)/\{\pm I\}.$$

On the other hand,

$$\text{O}(3, 1) = \text{O}(3, 1)^\uparrow \times \{\pm I\},$$

where $\text{O}(3, 1)^\uparrow$ is the subgroup preserving the future light cone. In particular, $\text{O}(3, 1)^\uparrow \cong \text{PO}(3, 1)$ and we may identify subgroups of $\text{PO}(3, 1)$ with subgroups of the Lorentz group $\text{O}(3, 1)$. Finite subgroups of $\text{O}(3, 1)$ were classified in [21]. The authors also indicated the smallest of the five locally isomorphic Lorentz groups which contains each finite subgroup. The group $\text{O}(3, 1)^\uparrow$ was denoted $\text{O}_1(3, 1)$. To list the finite subgroups of $\text{O}(3, 1)^\uparrow$ we then have to look at finite subgroups belonging to $\text{O}_1(3, 1)$ and $\text{DO}(3, 1)$ in the notation of [21]. It turns out that all our subgroups belong to class (i) in the cited paper, i.e. we may assume that they consist of elements of the form $g \oplus 1$, where $g \in \text{O}_3(\mathbb{R})$ and 1 is the identity acting on the time coordinate. The classification of finite subgroups of $\text{O}_3(\mathbb{R})$ (or *point groups* in three dimensions) is a very classical topic and we do not give the whole list here (one can consult [10, II] or apply Goursat’s lemma to $\text{O}_3(\mathbb{R}) = \text{SO}_3(\mathbb{R}) \times \{\pm I\}$). For an explicit description of these groups by matrices we refer the reader to [21].

Remark 3.1. — One can give a topological explanation of the embedding $G \hookrightarrow \text{O}_3(\mathbb{R})$. Indeed, the group G faithfully acts by diffeomorphisms of $Q_{3,1}(\mathbb{R}) \approx \mathbb{S}^2$. By the classical theorem of Brouwer–Kerekjarto–Eilenberg, every such action is equivalent (i.e. conjugate) to a linear one, see e.g. [41, § 2].

Now let $X = Q_{2,2}$. Then $X \cong \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$ and

$$\text{Aut}(X) \cong (\text{PGL}_2(\mathbb{R}) \times \text{PGL}_2(\mathbb{R})) \rtimes (\mathbb{Z}/2).$$

PROPOSITION 3.2. — *Let $G \subset \text{Aut}(Q_{2,2})$ be a finite subgroup such that $\text{Pic}(Q_{2,2})^G \cong \mathbb{Z}$. Then G is isomorphic to one of the following groups (which are all strongly minimal):*

$$(\mathbb{Z}/n \triangleleft_D \mathbb{Z}/n) \bullet 2 \cong (\mathbb{Z}/m \times \mathbb{Z}/k) \bullet \mathbb{Z}/2, \quad (\text{D}_n \triangleleft_D \text{D}_n) \bullet 2.$$

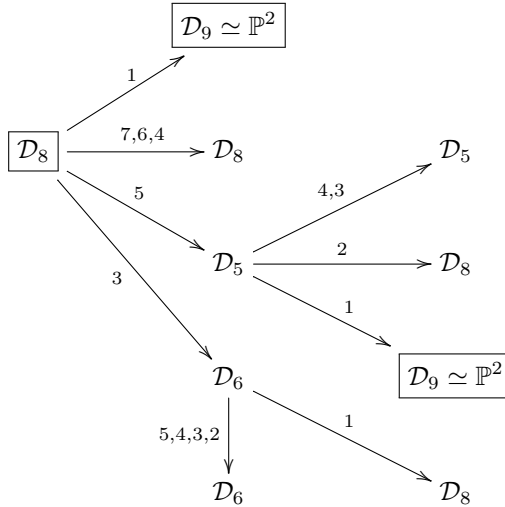
Proof. — The group $\widehat{G} = G \cap (\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{R}))$ naturally acts on the factors of $X = \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$ preserving them. Let \widehat{G}_1 and \widehat{G}_2 be the images of \widehat{G} under the projections of $\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{R})$ onto its factors. By Goursat’s lemma, $\widehat{G} = \widehat{G}_1 \triangle_D \widehat{G}_2$ for some D . As $\mathbb{Z}/2$ -component of $\mathrm{Aut}(Q_{2,2})$ acts on $\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$ by switching the factors, the groups \widehat{G}_1 and \widehat{G}_2 must be isomorphic: otherwise $G = \widehat{G}$ and $\mathrm{Pic}(X)^G \cong \mathbb{Z}^2$, a contradiction. Thus $\widehat{G} \cong H \triangle_D H$, where H is either cyclic, or dihedral. Note that a subgroup of a direct product of two cyclic groups is itself a direct product of at most two cyclic groups. Thus for H cyclic one can also write $G \cong (\mathbb{Z}/m \times \mathbb{Z}/k) \bullet \mathbb{Z}/2$, $m, k \geq 1$. For some isomorphic presentations of $D_n \triangle_D D_n$ see [13, Theorem 4.9]. \square

Remark 3.3. — Let X be a real del Pezzo surface of degree 8 with $X_{\mathbb{C}} \cong \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$, and $G \subset \mathrm{Aut}(X)$. If G has a real fixed point p on X , then G is linearizable. Indeed, blowing up p and contracting the strict transforms of the lines passing through p , we conjugate G to a subgroup of $\mathrm{Aut}(\mathbb{P}_{\mathbb{R}}^2)$.

PROPOSITION 3.4. — *Let X be a real del Pezzo surface of degree 8 and $G \subset \mathrm{Aut}(X)$ be a finite group with $\mathrm{Pic}(X)^G \simeq \mathbb{Z}$. Then G is linearizable if and only if either G has a real fixed point on X , or $G \simeq D_5$ acts on $X = Q_{2,2}$. In particular, a linearizable group is either cyclic, or dihedral.*

Proof. — If G has a real fixed point on X , then G is linearizable by Remark 3.3. Assume there is a birational map $f : (X, G) \dashrightarrow (\mathbb{P}_{\mathbb{R}}^2, G)$ and run the Sarkisov program on X to decompose f into a product of Sarkisov links; in what follows we refer to [17, Theorem 2.6] for description of these links (including group action in the picture is straightforward). The first link can connect \mathcal{D}_8 either with some \mathcal{D}_* (link of type II) or with \mathcal{C}_2 (link of type I; recall that here 2 stands for the number of singular fibers). In the latter case we can continue making the links in the class \mathcal{C} (e.g. of type II or IV), without creating new singular fibers, but at some point we have to link a conic bundle with a del Pezzo surface S . Same theorem shows that $S \in \mathcal{D}_8$. Since we do not want to return back to \mathcal{D}_8 , we may assume that the first link was actually of type II. In the diagram below we list all possibilities (regardless the base field or group action). We stop drawing arrows if we have to link our surface with some \mathcal{D}_* which already occurred in the diagram. The labels denote the degrees of points which we

blow up.



So, we see that we have only two possibilities to connect (X, G) with some $(\mathbb{P}^2_{\mathbb{R}}, G)$. The first one assumes that the link starts at a G -invariant point, which have to be real in our case. The second possibility is a combination of links of type II, namely $\mathcal{D}_8 \xrightarrow{5} \mathcal{D}_5 \xrightarrow{1} \mathcal{D}_9$. In particular, G must have a real fixed point on \mathcal{D}_5 , and hence either $G \simeq \mathbb{Z}/5$ or $G \simeq D_5$ (see Proposition 5.2). In the first case G must have a fixed point on X and $X \simeq Q_{3,1}$ (see e.g. [39, 4.4]). Let $G \simeq D_5$, and the link $\mathcal{D}_8 \rightarrow \mathcal{D}_5$ is as follows: $X \xleftarrow{f} Z \xrightarrow{g} Y$, where $X \in \mathcal{D}_8, Y \in \mathcal{D}_5, f$ is a blow up of the point η , $\deg \eta = 5$, and g is a contraction to a point ξ ; note that $\deg \xi = 2$ by [17, Theorem 2.6]. We now use the linearization argument given in Section 5 below (or [39, § 4.6]). If $X \simeq Q_{3,1}$, then g contracts two conjugate G -orbits, so ξ is a pair of conjugate G -fixed points, and we cannot proceed to $\mathbb{P}^2_{\mathbb{R}}$. If $X \simeq Q_{2,2}$, then g contracts two real G -orbits, so ξ is a pair of real G -fixed points. Such a group indeed can be further linearized.

Finally, if G has a fixed point $p \in X(\mathbb{R})$ then there is a faithful linear representation $G \hookrightarrow \text{GL}_2(T_p X)$, so G is either cyclic or dihedral by Lemma 2.5. □

4. Del Pezzo surfaces of degree 6

Let X be a real del Pezzo surface of degree 6. Then $X_{\mathbb{C}}$ can be obtained by blowing up $\mathbb{P}^2_{\mathbb{C}}$ in three noncollinear points p_1, p_2, p_3 . The set of (-1) -curves on $X_{\mathbb{C}}$ consists of six curves: the exceptional divisors of blow-up

$e_i = \pi^{-1}(p_i)$ and the strict transforms of the lines d_{ij} passing through p_i, p_j . In the anticanonical embedding $X_{\mathbb{C}} \hookrightarrow \mathbb{P}_{\mathbb{C}}^6$ these exceptional curves form a “hexagon” Σ . This yields a homomorphism to the symmetry group of this hexagon

$$\rho : \text{Aut}(X_{\mathbb{C}}) \rightarrow \text{Aut}(\Sigma) \cong \mathscr{W}(A_1 \times A_2) \cong D_6,$$

Since the set of all (-1) -curves on $X_{\mathbb{C}}$ is defined over \mathbb{R} , its complement T is isomorphic to a torus over \mathbb{C} . But $X(\mathbb{R}) \neq \emptyset$, so T is in fact an algebraic \mathbb{R} -torus. One can view it as the connected component of the identity of $\text{Aut}(X)$. There exist only 4 real forms of \mathbb{R} -rational del Pezzo surfaces of degree 6: $\mathbb{P}_{\mathbb{R}}^2(3, 0)$, $\mathbb{P}_{\mathbb{R}}^2(1, 1)$, $Q_{3,1}(0, 1)$, and $Q_{2,2}(0, 1)$. They correspond to real forms of T described by V. E .Voskresenkii [36, 10.1].

Table 4.1. Real forms of \mathbb{R} -rational del Pezzo surfaces of degree 6

$\Gamma : \Sigma$	id	Fig. 4.1a	Fig. 4.1b	Fig. 4.1c
X	$\mathbb{P}_{\mathbb{R}}^2(3, 0)$	$Q_{2,2}(0, 1)$	$\mathbb{P}_{\mathbb{R}}^2(1, 1)$	$Q_{3,1}(0, 1)$
$X(\mathbb{R})$	$\#4\mathbb{R}P^2$	\mathbb{T}^2	$\#2\mathbb{R}P^2$	\mathbb{S}^2

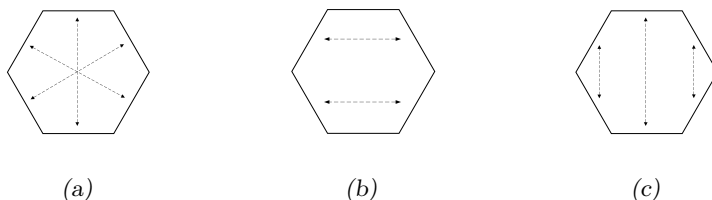


Figure 4.1. Action of Γ on Σ

PROPOSITION 4.1. — *Let X be a real del Pezzo surface of degree 6 and $G \subset \text{Aut}(X)$ be a finite group acting minimally on X . Then one of the following holds:*

- (1) *The surface X is isomorphic to $Q_{2,2}(2, 0) \cong \mathbb{P}_{\mathbb{R}}^2(3, 0)$ and can be given as*

$$\{([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}_{\mathbb{R}}^2 \times \mathbb{P}_{\mathbb{R}}^2 : x_0y_0 = x_1y_1 = x_2y_2\}$$

Its automorphism group fits into the short exact sequence

$$1 \rightarrow \text{Ker } \rho \rightarrow \text{Aut}(X) \xrightarrow{\rho} D_6 \rightarrow 1.$$

Here $\text{Ker } \rho \cong (\mathbb{R}^*)^2$ is the diagonal subgroup of $\text{PGL}_3(\mathbb{R})$, and $\rho(\text{Aut}(X)) \cong D_6$ is generated by the rotation $r = \rho(\alpha_1)$ and the reflection $s = \rho(\alpha_2)$, where

$$\begin{aligned} \alpha_1 &: ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([y_2 : y_0 : y_1], [x_2 : x_0 : x_1]), \\ \alpha_2 &: ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([x_1 : x_0 : x_2], [y_1 : y_0 : y_2]). \end{aligned}$$

The group G is of the form

$$(1a) \ H_\bullet \langle r \rangle \cong H_\bullet 6, \quad (1b) \ H_\bullet \langle r^2, s \rangle \cong H_\bullet \mathfrak{S}_3, \quad \text{or} \quad (1c) \ H_\bullet \langle r, s \rangle \cong H_\bullet D_6,$$

where $H \subset \text{Ker } \rho$ is isomorphic to a subgroup of $\mathbb{Z}/2 \times \mathbb{Z}/2$.

(2) The surface X is isomorphic to $Q_{2,2}(0, 1)$ and can be given as

$$\begin{aligned} \{([x_0 : x_1], [y_0 : y_1], [z_0, z_1]) \in \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 : \\ x_0 y_0 z_1 + x_0 y_1 z_0 + x_1 y_0 z_0 - x_1 y_1 z_1 = 0\} \end{aligned}$$

Its automorphism group fits into the short exact sequence

$$1 \rightarrow \text{Ker } \rho \rightarrow \text{Aut}(X) \xrightarrow{\rho} D_6 \rightarrow 1.$$

Here $\text{Ker } \rho \cong \text{SO}_2(\mathbb{R}) \times \text{SO}_2(\mathbb{R})$, and $\rho(\text{Aut}(X)) \cong D_6$ is generated by the rotation $r = \rho(\alpha_1)$ and the reflection $s = \rho(\alpha_2)$, where

$$\begin{aligned} \alpha_1 &: ([x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) \mapsto ([z_1 : z_0], [x_0 : -x_1], [y_1 : y_0]), \\ \alpha_2 &: ([x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) \mapsto ([y_0 : y_1], [x_0 : x_1], [z_0 : z_1]). \end{aligned}$$

The group G is one of the following:

$$\begin{aligned} (2a) \ H_\bullet \langle r \rangle \cong H_\bullet 6, \quad (2b) \ H_\bullet \langle r^2 \rangle \cong H_\bullet 3, \quad (2c) \ H_\bullet \langle r^2, s \rangle \cong H_\bullet \mathfrak{S}_3, \\ (2d) \ H_\bullet \langle r^2, rs \rangle \cong H_\bullet \mathfrak{S}_3, \quad (2e) \ H_\bullet \langle r, s \rangle \cong H_\bullet D_6, \end{aligned}$$

where $H \subset \text{Ker } \rho$ is a direct product of at most 2 cyclic groups of an arbitrary large order.

All listed groups do act minimally on the corresponding real surfaces.

Proof. — All statements about automorphism groups of real del Pezzo surfaces of degree 6 and their equations can be found in [30, Section 3]. Moreover, for $X = Q_{3,1}(0, 1)$ or $\mathbb{P}_{\mathbb{R}}^2(1, 1)$ the pair $(X, \text{Aut}(X))$ is not minimal, so we may assume that $X = Q_{2,2}(0, 1)$ or $X = \mathbb{P}_{\mathbb{R}}^2(3, 0)$. Up to conjugacy, the group $\rho(G) \subset D_6 = \langle r, s : r^6 = s^2 = 1, srs^{-1} = r^{-1} \rangle$ is one of the following:

- cyclic: $\langle r^k \rangle, \langle s \rangle, \langle rs \rangle, k = 0, 1, 2, 3;$
- dihedral: $\langle r, s \rangle, \langle r^2, s \rangle, \langle r^2, rs \rangle, \langle r^3, s \rangle.$

Case $X = \mathbb{P}_{\mathbb{R}}^2(3, 0)$. — All (-1) -curves on X are real. Thus a cyclic group $\rho(G) \cong \langle r^k \rangle$ acts minimally on X if and only if $k = 1$ (otherwise one can G -equivariantly contract an orbit which consists of disjoint (-1) -curves and is defined over \mathbb{R}). Following the same argument, it is easy to check that in the dihedral case only $\langle r^2, s \rangle$, and hence $\langle r, s \rangle$, act minimally on X . As any nontrivial finite subgroup of \mathbb{R}^* is isomorphic to $\mathbb{Z}/2$, we get the result.

Case $X = Q_{2,2}(0, 1)$. — The action of Γ on the hexagon is shown on Figure 4.1. Examining the action of G on Σ , one easily gets that only the groups $\langle r \rangle$, $\langle r^2 \rangle$, $\langle r^2, s \rangle$, $\langle r^2, rs \rangle$, $\langle r, s \rangle$ act minimally on X . \square

PROPOSITION 4.2. — *Let X be a real del Pezzo surface of degree 6, and $G \subset \text{Aut}(X)$ be a finite group acting minimally on X . Assume that G is linearizable. Then G is one of the following groups (in the notation of Proposition 4.1):*

- isomorphic to \mathfrak{S}_4 : (1b) and (2c), where H is a Klein 4-group;
- isomorphic to \mathfrak{A}_4 : (2b), where H is a Klein 4-group;
- dihedral:
 - $D_3 \cong \mathfrak{S}_3$: (1b), (2c), (2d);
 - D_6 : (1c), (1b), (2c), (2d), (2e);
 - D_{12} : (1c), (2c), (2d), (2e);
 - $D_{3k}, k \geq 2$: (2c), (2d);
 - $D_{6k}, k \geq 2$: (2e).
- cyclic:
 - (1a): $\mathbb{Z}/6$ and $\mathbb{Z}/12$;
 - (2a): $\mathbb{Z}/6k$;
 - (2b): $\mathbb{Z}/3k$.

Proof. — This is an elementary group theory. As \mathfrak{A}_5 is simple, none of the groups from Proposition 4.1 is isomorphic to \mathfrak{A}_5 . Let $G \cong \mathfrak{S}_4$. Note that \mathfrak{S}_4 has no normal subgroups H with G/H isomorphic to $\mathbb{Z}/3$, $\mathbb{Z}/6$ or D_6 . If $\mathfrak{S}_4/H \cong \mathfrak{S}_3$, then $H = \{e, (12)(34), (13)(24), (14)(23)\}$ is a Klein group. Let $G \cong \mathfrak{A}_4$. Note that \mathfrak{A}_4 has no normal subgroups H with quotient isomorphic to $\mathbb{Z}/6$, \mathfrak{S}_3 or D_6 . If $\mathfrak{A}_4/H \cong \mathbb{Z}/3$, then H is a Klein group.

Let $G \cong D_n$. We know that G has a normal subgroup H with G/H isomorphic to $\mathbb{Z}/3$, $\mathbb{Z}/6$, \mathfrak{S}_3 or D_6 . In particular H is cyclic (otherwise $[G : H] \leq 2$). On the other hand, a quotient of a dihedral group is again dihedral. In the case (1) of Proposition 4.1 we get that for $H = \text{id}$ the group G is D_3 (1b) or D_6 (1c), while for $H \cong \mathbb{Z}/2$ the group G is D_6 (1b) or D_{12} (1c). In the case (2) the cyclic group H can be of any order k , so either $G \cong D_{3k}$ and is of type (2c), (2d), or $G \cong D_{6k}$ and is of type (2e).

Finally, let $G \cong \mathbb{Z}/n$. Then H is cyclic. In the case (1) of Proposition 4.1 one has $|H| \leq 2$, and $G/H \cong \mathbb{Z}/6$. Thus $G \cong \mathbb{Z}/6$ or $\mathbb{Z}/12$. In the case (2) the order of H can be arbitrary large, hence G is isomorphic to $\mathbb{Z}/3k$ or $\mathbb{Z}/6k$. \square

Remark 4.3. — As was shown in [39, § 4.5] there exist infinitely many non-linearizable subgroups of type (2b) acting minimally on $Q_{2,2}(0,1)$. Moreover, we exhibited two non-conjugate embeddings of $G = (\mathbb{Z}/3)^2$ into $\text{Cr}_2(\mathbb{R})$: the one is a trivial extension of type (2b), and the other comes from the fiberwise G -action on the conic bundle $X = Q_{2,2} \cong \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$ with $\text{rk Pic}(X)^G = 2$.

5. Del Pezzo surfaces of degree 5

Each real del Pezzo surface X of degree 5 is isomorphic to $\mathbb{P}_{\mathbb{R}}^2(a,b)$, where $(a,b) \in \{(4,0), (2,1), (0,2)\}$ [19, Corollary 5.4]. There are 10, 4 or 2 real lines on X respectively. It is clear from the blow-up model of X that the configuration of Γ -orbits of exceptional curves is uniquely determined by the pair (a,b) . The incidence graph of such a configuration is the colored Petersen graph, where the lines in one Γ -orbit have the same color (and we additionally label by $*$ the real ones). We assume that X is the blow-up of $\mathbb{P}_{\mathbb{R}}^2$ at four points p_1, p_2, p_3, p_4 in general position, e_i is the exceptional divisor over the point p_i and d_{ij} is the proper transform of the line passing through the points p_i and p_j , see Figure 5.1.

Let us do some extra work and find all possibilities for $\text{Aut}(X)$.

PROPOSITION 5.1. — *Let X be a real del Pezzo surface of degree 5. Then*

- $\text{Aut}(X) \cong \mathfrak{S}_5$ if $X \cong \mathbb{P}_{\mathbb{R}}^2(4,0)$,
- $\text{Aut}(X) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ if $X \cong \mathbb{P}_{\mathbb{R}}^2(2,1)$,
- $\text{Aut}(X) \cong D_4$ if $X \cong \mathbb{P}_{\mathbb{R}}^2(0,2)$.

Proof. — The “split” case $X \cong \mathbb{P}_{\mathbb{R}}^2(4,0)$ is classical and can be found e.g. in [11, Theorem 8.5.8]. Denote by $\Pi_{a,b}$ the colored incidence graph of (-1) -curves on $X_{\mathbb{C}} = \mathbb{P}_{\mathbb{R}}^2(a,b) \otimes \mathbb{C}$. As $\text{Aut}(X)$ naturally acts on the exceptional lines preserving incidence relations, we have a homomorphism $\psi : \text{Aut}(X) \rightarrow \text{Aut}(\Pi_{a,b})$. It is injective, as any automorphism of $X_{\mathbb{C}}$ which fixes all (-1) -curves comes from an automorphism of $\mathbb{P}_{\mathbb{R}}^2$ that fixes 4 closed points p_i 's, so it must be trivial.

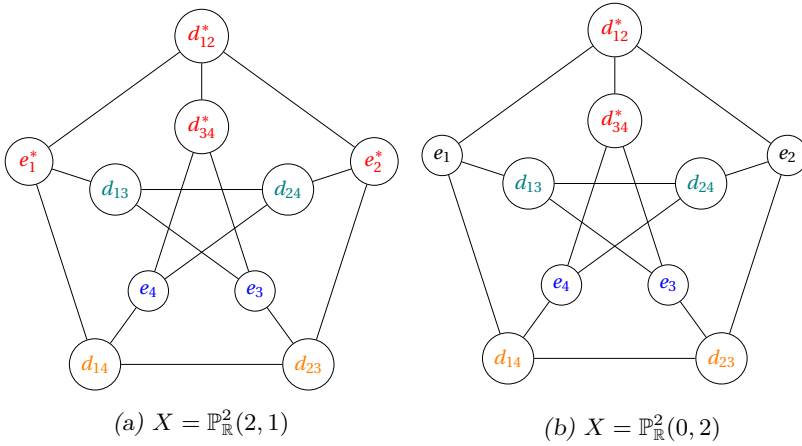


Figure 5.1. Graph of (-1) -curves on del Pezzo surface X of degree 5

Note that for each $\varphi \in \text{Aut}(\Pi_{a,b})$ and any two vertices v_1 and v_2 we must have:

- (1) if $\{v_1, v_2\}$ is Γ -invariant then $\{\varphi(v_1), \varphi(v_2)\}$ is Γ -invariant;
- (2) if v_1 and v_2 are incident then $\varphi(v_1)$ and $\varphi(v_2)$ are incident.

Put

$$\alpha : e_1 \leftrightarrow e_2, \quad d_{13} \leftrightarrow d_{24}, \quad d_{14} \leftrightarrow d_{23}, \quad e_3 \leftrightarrow e_4,$$

$$\beta : e_1 \leftrightarrow e_2, \quad d_{13} \leftrightarrow d_{23}, \quad d_{14} \leftrightarrow d_{24},$$

$$\varsigma : e_3 \leftrightarrow e_4, \quad d_{14} \leftrightarrow d_{13}, \quad d_{24} \leftrightarrow d_{23},$$

$$\varrho : d_{12} \leftrightarrow d_{34}, \quad e_1 \mapsto e_3, \quad e_2 \mapsto e_4, \quad e_3 \mapsto e_2, \quad e_4 \mapsto e_1,$$

$$d_{14} \mapsto d_{13}, \quad d_{23} \mapsto d_{24}, \quad d_{13} \mapsto d_{23}, \quad d_{24} \mapsto d_{14}.$$

(if a line is not indicated then it is stabilized). Note that $\alpha, \beta \in \text{Aut}(\Pi_{2,1})$ and $\varsigma, \varrho \in \text{Aut}(\Pi_{0,2})$. Then

$$\text{Aut}(\Pi_{2,1}) = \langle \alpha \rangle_2 \times \langle \beta \rangle_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2,$$

$$\text{Aut}(\Pi_{0,2}) = \langle \varsigma, \varrho \mid \varrho^4 = \varsigma^2 = \text{id}, \quad \varsigma^{-1} \varrho \varsigma = \varrho^{-1} \rangle \cong D_4.$$

Indeed, in the case of $\Pi_{0,2}$ one can use that $\text{Aut}(\Pi_{0,2})$ acts on the set $\{e_1, e_2, e_3, e_4\}$ and the kernel of this action is obviously trivial. On the other hand, $\text{Aut}(\Pi_{0,2})$ cannot be isomorphic to \mathfrak{S}_4 , as any automorphism of order 3 would fix d_{12} (hence e_1 and e_2), d_{34} (hence e_3 and e_4). Since $\text{Aut}(\Pi_{0,2})$ contains D_4 , we get $\text{Aut}(\Pi_{0,2}) \cong D_4$. The case of $\Pi_{2,1}$ is easy as well.

To show that ψ is surjective we explicitly construct the corresponding geometric actions. For this set

$$\alpha' : [x : y : z] \mapsto [x : y : -z], \quad \beta' : [x : y : z] \mapsto [-x : y : z], \quad \varsigma' = \alpha',$$

$$\varrho' : [x : y : z] \mapsto [z : y : -x].$$

We may also assume (after applying a suitable transformation from $\mathrm{PGL}_3(\mathbb{R})$) that the blown up points are

$$p_1 = [1 : 0 : 1], \quad p_2 = [1 : 0 : -1], \quad p_3 = [0 : 1 : i], \quad p_4 = [0 : 1 : -i],$$

when $X = \mathbb{P}_{\mathbb{R}}^2(2, 1)$,

$$p_1 = [1 : i : 0], \quad p_2 = [1 : -i : 0], \quad p_3 = [0 : 1 : i], \quad p_4 = [0 : 1 : -i],$$

when $X = \mathbb{P}_{\mathbb{R}}^2(0, 2)$.

Then the lifts of α' , β' , ς' and ϱ' act as α , β , ς and ϱ respectively on the corresponding $\Pi_{a,b}$. □

PROPOSITION 5.2. — *Let X be a real del Pezzo surface of degree 5 and $G \subset \mathrm{Aut}(X)$ be a finite group acting minimally on X . Then X is isomorphic to $\mathbb{P}_{\mathbb{R}}^2(4, 0)$, and the group G is one of the following:*

$$\mathfrak{S}_5, \quad \mathfrak{A}_5, \quad \mathbb{Z}/5 \rtimes \mathbb{Z}/4 = \langle a, b \mid a^5 = b^4 = 1, bab^{-1} = a^2 \rangle, \quad D_5, \quad \mathbb{Z}/5.$$

All listed groups do act minimally on X .

Proof. — In the case $X = \mathbb{P}_{\mathbb{R}}^2(4, 0)$ we argue exactly as if $\mathbb{k} = \mathbb{C}$, see [13, Theorem 6.4]. Assume that $X \cong \mathbb{P}_{\mathbb{R}}^2(2, 1)$. Note that the curve d_{12} is the only line on X intersecting 3 real lines. Thus it is stabilized by $\mathrm{Aut}(X)$ and can be equivariantly contracted, implying that the pair $(X, \mathrm{Aut}(X))$ is not minimal. Now let $X \cong \mathbb{P}_{\mathbb{R}}^2(0, 2)$. Then every automorphism in $\mathrm{Aut}(X)$ preserves the set $\{e_1, e_2, e_3, e_4\}$ consisting of 2 pairs of complex conjugate lines which are pairwise disjoint, and hence can be equivariantly contracted. So, $(X, \mathrm{Aut}(X))$ is not minimal. □

Let $S = \mathbb{P}_{\mathbb{R}}^2(4, 0)$. It follows from the classification of Sarkisov links that for $G = \mathfrak{S}_5$ or \mathfrak{A}_5 the pair (S, G) is superrigid, see [13, Proposition 7.12, 7.13]. Let $G = \langle a \rangle \cong \mathbb{Z}/5$. Then $S(\mathbb{C})^a$ consists of 2 points, whose blow-up is a del Pezzo surface Y of degree 3 with two skew lines ℓ_1 and ℓ_2 , either real or complex conjugate [39, § 4.6]. One can use the G -birational map

$$\ell_1 \times \ell_2 \dashrightarrow Y, \quad (p_1, p_2) \mapsto q, \quad \text{where } Y \cap \overline{p_1 p_2} = \{p_1, p_2, q\}$$

to conjugate G to a group acting on a quadric surface Q . If $\sigma(\ell_1) = \ell_2$, then $Q \cong Q_{3,1}$ and G can be further linearized [39, Proposition 4.18]. Now let G be $\mathbb{Z}/5 \rtimes \mathbb{Z}/4 = \langle a, b \mid a^5 = b^4 = 1, bab^{-1} = a^2 \rangle$ or $\mathbb{Z}/5 \times \mathbb{Z}/2 =$

$\langle a, b \mid a^5 = b^2 = 1, bab^{-1} = a^{-1} \rangle \cong D_5$. Then $S(\mathbb{C})^a$ consists of 2 points and this set is b -invariant. Therefore, we can again use the same birational map as above to conjugate G to a group acting on a quadric surface (this is a Sarkisov link of type II).

6. Del Pezzo surfaces of degree 4

6.1. Topology and equations

Throughout this section X denotes a real del Pezzo surface of degree 4. It is well-known that the linear system $|-K_X|$ embeds X into $\mathbb{P}_{\mathbb{R}}^4$ as a complete intersection of two quadrics, which we denote Q_0 and Q_∞ . If no confusion arises, we denote by the same letter a quadric, the corresponding quadratic form and its matrix. Let \mathcal{Q} be the pencil

$$\lambda Q_0(x_0, \dots, x_4) + \mu Q_\infty(x_0, \dots, x_4).$$

Its discriminant $\Delta(\mu, \lambda) \equiv \det(\lambda Q_0 + \mu Q_\infty)$ is a binary form of degree 5. Since we assume X smooth, the equation $\Delta = 0$ has five distinct roots $[\lambda_i : \mu_i]$, $i = 1, \dots, 5$. Equivalently, the matrix $Q_0^{-1}Q_\infty$ (we may suppose Q_0 nonsingular) has five distinct eigenvalues $-\lambda_i/\mu_i \in \mathbb{C}$. They correspond to the singular members of \mathcal{Q} , which we denote by Q_i , $i = 1, \dots, 5$.

Note that eigenspaces corresponding to different eigenvalues are orthogonal with respect to both Q_0 and Q_∞ . Over \mathbb{C} , we can find a basis of eigenvectors, making both Q_0 and Q_∞ diagonal, so the pencil takes the form

$$\sum_{i=0}^4 (\lambda a_i + \mu b_i) x_i^2$$

with $b_i/a_i = -\lambda_i/\mu_i$.

The complex conjugation permutes the eigenspaces. In a Γ -invariant one, we can pick a real vector for our basis, so the corresponding part of the pencil's equation has real coefficients a_i and b_i . For two complex conjugate eigenspaces, we get a two-dimensional real subspace W orthogonal to the other eigenspaces. If we pick an orthogonal basis $\{w, \bar{w}\}$ in $W \otimes \mathbb{C}$, where w is an eigenvector with eigenvalue $-b/a$, then

$$Q(z_1 w + z_2 \bar{w}) = (\lambda a + \mu b) z_1^2 + (\lambda \bar{a} + \mu \bar{b}) z_2^2.$$

Clearly, $z_1 w + z_2 \bar{w} \in W$ if and only if $z_1 = \bar{z}_2$. Put

$$z_1 = u + iv, \quad a = a_1 + ia_2, \quad b = b_1 + ib_2.$$

to get

$$2(\lambda a_1 + \mu b_1)(u^2 - v^2) - 4(\lambda a_2 + \mu b_2)uv.$$

Set

$$-b/a = \alpha + i\beta, \quad a = i/2$$

to obtain the normal form

$$\mu\beta(u^2 - v^2) + 2(\lambda - \alpha\mu)uv.$$

Let us summarize this discussion by stating the following classification result.

PROPOSITION 6.1. — *A real del Pezzo surface of degree 4 can be reduced to one of the following normal forms*

- (1) $\begin{cases} a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = 0 \\ b_0x_0^2 + b_1x_1^2 + b_2x_2^2 + b_3x_3^2 + b_4x_4^2 = 0 \end{cases}$
in $\text{Proj } \mathbb{R}[x_0, x_1, x_2, x_3, x_4]$;
- (2) $\begin{cases} a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + 2u_1v_1 = 0 \\ b_0x_0^2 + b_1x_1^2 + b_2x_2^2 + \beta_1(u_1^2 - v_1^2) - 2\alpha_1u_1v_1 = 0 \end{cases}$
in $\text{Proj } \mathbb{R}[x_0, x_1, x_2, u_1, v_1]$;
- (3) $\begin{cases} a_0x_0^2 + 2u_1v_1 + 2u_2v_2 = 0 \\ b_0x_0^2 + \beta_1(u_1^2 - v_1^2) - 2\alpha_1u_1v_1 + \beta_2(u_2^2 - v_2^2) - 2\alpha_2u_2v_2 = 0 \end{cases}$
in $\text{Proj } \mathbb{R}[x_0, u_1, v_1, u_2, v_2]$,

where $-b_i/a_i$ and $-(\alpha_i \pm i\beta_i)$ are eigenvalues of \mathcal{Q} .

Now let us describe how the topology of $X(\mathbb{R})$ depends on the equation of X . Nonsingular real pencils of quadrics were classified by C. T. C. Wall in [37] by an invariant called characteristic. In the notation of Proposition 6.1 set

$$a_k = r_k \cos \theta_k, \quad b_k = r_k \sin \theta_k, \quad r_k > 0.$$

and define points on the circle

$$P_k = (\cos \theta_k, \sin \theta_k), \quad Q_k = -P_k.$$

These points can be grouped in blocks: as we proceed anticlockwise around the circle we meet a block of m_1 points P_t , then a block of n_1 points Q_t , then a block of m_2 points P_t and so on. When we are half way round, we meet an opposite block of m_1 points Q_t , so one has $m_1 = n_{g+1}$ for some $g \geq 0$. This g is called the *genus* and the sequence (m_1, \dots, m_{2g+1}) in cyclic order the *characteristic* $\Xi(\mathcal{Q})$ of our pencil. Below we list some information about the topology and real lines on X , following [19, 37, 38] (we list only those surfaces which are rational over \mathbb{R}). Using Proposition 6.1, for each real form we also indicate the type of equation of X .

Table 6.1. Real forms of \mathbb{R} -rational del Pezzo surfaces of degree 4

Class of $\sigma^* \in \mathscr{W}(D_5)$	Eigenvalues of σ^*	$\Xi(Q)$	Equation type	X	$X(\mathbb{R})$	# real lines
id	1^5	$(1,1,1,1,1)$	I	$\mathbb{P}_{\mathbb{R}}^2(5, 0)$	$\#6\mathbb{R}\mathbb{P}^2$	16
A_1	$-1, 1^4$	$(1,1,1)$	II	$\mathbb{P}_{\mathbb{R}}^2(3, 1)$	$\#4\mathbb{R}\mathbb{P}^2$	8
A_1^2	$-1^2, 1^3$	(1)	III	$\mathbb{P}_{\mathbb{R}}^2(1, 2)$	$\#2\mathbb{R}\mathbb{P}^2$	4
$A_1^{2'}$	$-1^2, 1^3$	$(2,2,1)$	I	$Q_{2,2}(0, 2)$	\mathbb{T}^2	0
A_1^3	$-1^3, 1^2$	(3)	II	$Q_{3,1}(0, 2)$	\mathbb{S}^2	0

Remark 6.2. — Note that the sum of entries in $\Xi(Q)$ equals to the number of real eigenvalues of Q . In particular, there is no one-to-one correspondence between the numbers of real eigenvalues of Q and the real structures on X .

6.2. Automorphisms

Let v_i and Q_i^b denote the vertex and the base of the singular quadric Q_i respectively. Since Γ acts on the set $\{v_1, v_2, v_3, v_4, v_5\}$, there can be 1, 3 or 5 real v_i 's. As $Q_i^b \otimes \mathbb{C}$ has two pencils of lines, each Q_i has two pencils of planes, whose intersections with $X_{\mathbb{C}}$ give two complementary pencils of conics \mathcal{C}_i and \mathcal{C}'_i on $X_{\mathbb{C}}$. These pencils satisfy the conditions $\mathcal{C}_i \cdot \mathcal{C}'_i = 2$, $\mathcal{C}_i \cdot \mathcal{C}_j = \mathcal{C}_i \cdot \mathcal{C}'_j = 1$ for $i \neq j$, and $\mathcal{C}_i + \mathcal{C}'_i \sim -K_X$. Two complementary pencils define a double cover $\pi_i : X_{\mathbb{C}} \rightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$, which coincides with the projection of X from v_i . Depending on the type of real locus $Q_i^b(\mathbb{R})$ (i.e. on the realness of two pencils of lines on Q_i^b) one has either $\sigma(\mathcal{C}_i) = \mathcal{C}_i$, $\sigma(\mathcal{C}'_i) = \mathcal{C}'_i$ (if $Q_i^b \cong Q_{2,2}$), or $\sigma(\mathcal{C}_i) = \mathcal{C}'_i$ (if $Q_i^b \cong Q_{3,1}$).

The Galois involution of the double cover π_i induces an automorphism $\tau_i \in \text{Aut}(X_{\mathbb{C}})$. For real v_i both π_i and τ_i are defined over \mathbb{R} . As was explained in the beginning of this section, in a suitable system of complex coordinates both Q_0 and Q_{∞} can be brought to diagonal form, so the equations of X can be written in the form

$$\sum_{i=0}^4 x_i^2 = \sum_{i=0}^4 \theta_i x_i^2 = 0,$$

and then τ_i are given by $x_i \mapsto -x_i$. These five commuting involutions generate a normal abelian subgroup $A \subset \text{Aut}(X_{\mathbb{C}})$ with a unique relation

$\tau_1\tau_2\tau_3\tau_4\tau_5 = \text{id}$, hence

$$A = \{1, \tau_k, \tau_i\tau_j : 1 \leq k \leq 5, 1 \leq i < j \leq 5\}, \quad A \cong (\mathbb{Z}/2)^4.$$

In what follows it will be convenient for us to use the following description of this group, see [4, Lemma 9.11]:

$$A = \left\{ a = (a_1, a_2, a_3, a_4, a_5) \in (\mathbb{Z}/2)^5 : \sum_{i=1}^5 a_i = 0 \right\},$$

where an element (a_1, \dots, a_5) exchanges the two conic bundles \mathcal{C}_i and \mathcal{C}'_i if $a_i = 1$ and preserves each one if $a_i = 0$. In this terminology, the automorphism $a = (a_1, a_2, a_3, a_4, a_5)$ corresponds to the projective transformation

$$(6.1) \quad [x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [(-1)^{a_1}x_1 : (-1)^{a_2}x_2 : (-1)^{a_3}x_3 : (-1)^{a_4}x_4 : (-1)^{a_5}x_5],$$

so τ_1 corresponds to $(0, 1, 1, 1, 1)$, τ_2 corresponds to $(1, 0, 1, 1, 1)$ etc.

Further, the groups $\text{Aut}(X)$ and $\text{Aut}(X_{\mathbb{C}})$ act on the pencil \mathcal{Q} preserving the set of five degenerate quadrics or, equivalently, the set of pairs $\mathcal{R}_i = \{\mathcal{C}_i, \mathcal{C}'_i\}$. Thus we have two homomorphisms

$$(6.2) \quad \rho_1 : \text{Aut}(X_{\mathbb{C}}) \rightarrow \text{PGL}_2(\mathbb{C}), \quad \rho_2 : \text{Aut}(X_{\mathbb{C}}) \rightarrow \mathfrak{S}_5$$

with $\ker \rho_1 = \ker \rho_2 = A$. In fact, the exact sequence

$$\text{id} \rightarrow A \rightarrow \text{Aut}(X_{\mathbb{C}}) \rightarrow \text{Im } \rho_2 \rightarrow \text{id}$$

splits, and $\text{Aut}(X_{\mathbb{C}}) \cong A \rtimes \text{Im } \rho_2$. One can easily see [13, Section 6] that $\text{Aut}(X_{\mathbb{C}})/A$ is one of the following groups:

$$(6.3) \quad \text{id}, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/5, \mathfrak{S}_3, D_5.$$

Denote by ρ the restriction of ρ_2 on the real automorphism group $\text{Aut}(X)$. Set

$$A_o = \text{Ker } \rho = A \cap \text{Aut}(X), \quad A' = \text{Im}(\rho).$$

CONVENTION. — *In this paragraph every permutation $\tau \in \mathfrak{S}_5$ should be understood as a permutation of the set $\{\mathcal{R}_i : i = 1, \dots, 5\}$. For an automorphism $(a, \tau) \in \text{Aut}(X_{\mathbb{C}})$ we denote it simply by a if $\tau = \text{id}$, and by τ if $a = 0$.*

6.3. Groups acting minimally on real del Pezzo quartics

We now start to enumerate the groups acting minimally on real del Pezzo surfaces of degree 4. For each real form listed in Table 6.1, we first get some restrictions on the groups A_o and A' , and then list possible minimal groups $G \subset \text{Aut}(X)$. In all the cases we describe the structure of G , giving an explicit way of listing all its elements (a, τ) (or even list them ourselves, if it is not too long). It is straightforward to write down these automorphisms in coordinates using (6.1). To write down the equation of X , one may use Proposition 6.1 choosing coefficients in accordance with characteristic $\Xi(Q)$, see Table 6.1.

The split case $X \cong \mathbb{P}_{\mathbb{R}}^2(5, 0)$ immediately follows from the work of Dolgachev and Iskovskikh [13], as $\sigma^* = \text{id}$ and the whole groups $(\mathbb{Z}/2)^4$ and \mathfrak{S}_5 act by real transformations of $\mathbb{P}_{\mathbb{R}}^4$.

PROPOSITION 6.3 ([13, Theorem 6.9]). — *Let $X = \mathbb{P}_{\mathbb{R}}^2(5, 0)$ be a real del Pezzo surface of degree 4, and $G \subset \text{Aut}(X)$ be a group acting strongly minimally on X . Then G is isomorphic to one of the following:*

$$\begin{aligned}
 &(\mathbb{Z}/2)^k, \quad k = 2, 3, 4; \quad \mathbb{Z}/2 \times \mathbb{Z}/4, \quad D_4, \quad L_{16}, \quad (\mathbb{Z}/2)^4 \rtimes \mathbb{Z}/2, \quad \mathbb{Z}/8, \quad M_{16}, \\
 &(\mathbb{Z}/2)^4 \rtimes \mathbb{Z}/4, \quad (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/3, \quad \mathbb{Z}/2 \times \mathfrak{A}_4, \quad (\mathbb{Z}/2)^4 \rtimes \mathbb{Z}/3, \\
 &\mathbb{Z}/3 \rtimes \mathbb{Z}/4, \quad (\mathbb{Z}/2)^k \rtimes \mathfrak{S}_3, \quad k = 2, 3, 4; \\
 &L_{16} \rtimes \mathbb{Z}/3, \quad (\mathbb{Z}/2)^4 \rtimes D_5, \quad (\mathbb{Z}/2)^4 \rtimes \mathbb{Z}/5,
 \end{aligned}$$

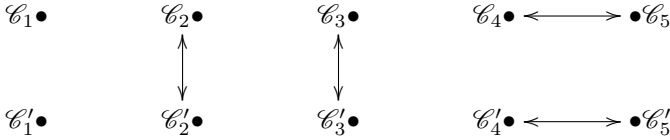
where $L_{16} = \langle a, b, c \mid a^4 = b^2 = c^2 = [c, a]b = [a, b] = [c, b] = 1 \rangle$ and $M_{16} = \langle a, b, c \mid a^8 = b^2 = [a, b]a^4 = 1 \rangle$ are non-abelian groups of order 16. Moreover, all listed groups act strongly minimally on X .

We now proceed with non-trivial real forms of del Pezzo quartics. Let $X = Q_{3,1}(0, 2)$ be the blow up of $Q_{3,1}$ at four points p, \bar{p}, q, \bar{q} . Denote by E_x the exceptional divisor over a point $x \in Q_{3,1}$, and by F the strict transform of a fiber. Then, in the notation as above, one has

$$\begin{aligned}
 \{\mathcal{C}_1, \mathcal{C}'_1\} &= \{F + \bar{F} - E_p - E_{\bar{p}}, F + \bar{F} - E_q - E_{\bar{q}}\}, \\
 \{\mathcal{C}_2, \mathcal{C}'_2\} &= \{F + \bar{F} - E_p - E_q, F + \bar{F} - E_{\bar{p}} - E_{\bar{q}}\}, \\
 \{\mathcal{C}_3, \mathcal{C}'_3\} &= \{F + \bar{F} - E_p - E_{\bar{q}}, F + \bar{F} - E_q - E_{\bar{p}}\}, \\
 \{\mathcal{C}_4, \mathcal{C}'_4\} &= \{F, F + 2\bar{F} - E_p - E_{\bar{p}} - E_q - E_{\bar{q}}\}, \\
 \{\mathcal{C}_5, \mathcal{C}'_5\} &= \{\bar{F}, 2F + \bar{F} - E_p - E_{\bar{p}} - E_q - E_{\bar{q}}\}.
 \end{aligned}$$

Note that in each pair one has $\mathcal{C}_i + \mathcal{C}'_i = -K_X = 2F + 2\bar{F} - E_p - E_{\bar{p}} - E_q - E_{\bar{q}}$. In what follows we shall depict the action of σ on five pairs of conic bundles

like this:



This example corresponds to description of pairs $\mathcal{R}_i = \{\mathcal{C}_i, \mathcal{C}'_i\}$ for $X = Q_{3,1}(0, 2)$ given above. No arrow means that the corresponding conic bundle is σ -invariant. We shall omit the bullets' labels in the future. Now it is easy to see that $A' \subset \{\text{id}, (23), (45), (23)(45)\}$ (however this inclusion is strict, as one can see from the list (6.3)), and any element $(a_1, a_2, a_3, a_4, a_5) \in \text{Ker } \rho$ has $a_4 = a_5$, so A_o embeds into $(\mathbb{Z}/2)^3$ (here and below we often use the fact that Γ commutes with automorphism).

PROPOSITION 6.4. — *Let $X \cong Q_{3,1}(0, 2)$ be a real del Pezzo surface of degree 4, and $G \subset \text{Aut}(X)$ be a group acting strongly minimally on X . Then the kernel A_o of $\rho : \text{Aut}(X) \rightarrow \mathfrak{S}_5$ is isomorphic to $(\mathbb{Z}/2)^3$, and generated by elements $\gamma_1 = (0, 1, 1, 0, 0)$, $\gamma_2 = (1, 0, 1, 0, 0)$ and $\gamma_3 = (0, 0, 0, 1, 1)$. Further, the image A' of ρ is either $\langle (23)(45) \rangle$ or trivial. Finally, the group G is one of the following:*

$$\mathbb{Z}/2, \quad (\mathbb{Z}/2)^2, \quad (\mathbb{Z}/2)^3, \quad \mathbb{Z}/4, \quad (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/2, \quad (\mathbb{Z}/2)^3 \rtimes \mathbb{Z}/2.$$

More precisely, the first group $\mathbb{Z}/2$ is generated by either $(\gamma_2 + \gamma_3, \text{id})$, or $(\gamma_1 + \gamma_2 + \gamma_3, \text{id})$. All other groups, except $\mathbb{Z}/4$, contain at least one of these elements. The group $\mathbb{Z}/4$ is generated by $((1, 0, 1, 1, 1), (23)(45))$. The first three groups lie in $\text{Ker } \rho$. Finally, all listed groups indeed act strongly minimally on X .

More information about the structure of the last three groups is given in the proof.

Remark 6.5. — The case when $X \cong Q_{3,1}$ and G is a group of prime order was investigated in [29, § 4.3]. It was shown that

- (1) X can be given by the equations

$$\begin{cases}
 (\mu - \mu\bar{\mu} + \bar{\mu})x_1^2 - 2x_1x_2 + x_2^2 + (1 - \bar{\mu} + \mu\bar{\mu} - \mu)x_3^2 + x_4^2 = 0, \\
 \mu\bar{\mu}x_1^2 - 2\mu\bar{\mu}x_1x_2 + (\mu - 1 + \bar{\mu})x_2^2 + \mu\bar{\mu}x_4^2 + (1 - \bar{\mu} + \mu\bar{\mu} - \mu)x_5^2.
 \end{cases}$$

- (2) A_o is isomorphic to $(\mathbb{Z}/2)^3$, and generated by elements γ_1, γ_2 and γ_3 .
- (3) A' is either $\langle (23)(45) \rangle$ or trivial. Moreover, the former happens if and only if $|\mu| = 1$.

To save some space, we shall use these results below referring to [29] for their proofs .

Proof of Proposition 6.4. — In the light of the previous Remark, we may proceed with determining minimal groups. In what follows we denote the elements of $\text{Ker } \rho$ as $a \equiv (a, \text{id})$, where $a = (a_1, \dots, a_5) \in (\mathbb{Z}/2)^5$, and the elements of $\text{Im } \rho$ are denoted as $\tau \equiv (0, \tau)$, $\tau \in \mathfrak{S}_5$. Below we shall use the following trivial observation several times. Assume that elements of G

- either all have $a_1 = 0$;
- or all have $a_4 = a_5 = 0$.

Then G is not strongly minimal. Indeed, in the first case G fixes σ -invariant \mathcal{C}_1 and \mathcal{C}'_1 , hence we have $\text{rk Pic}(X)^G > 1$. In the second case G fixes $F + \overline{F}$, which is not a multiple of K_X ; hence $\text{rk Pic}(X)^G > 1$. For brevity, we will call any of the two conditions above a \star -condition (it will be always clear from the context which one we actually mean).

Case $G \subset \text{Ker } \rho$. — Assume $G = \langle g = (a, \text{id}) \rangle$ acts strongly minimally. By the previous remark $a_1 = 1$ and $a_4 = a_5 = 1$. We see that g is one of the following elements: $\alpha_1 = (1, 0, 1, 1, 1)$ or $\alpha_2 = (1, 1, 0, 1, 1)$. One can easily write down these automorphisms in homogeneous coordinates of $\mathbb{P}^4_{\mathbb{R}}$, see [29, Proposition 4.11]. In particular, the whole group $G = \text{Ker } \rho$ acts strongly minimally on X .

Now assume $G \cong (\mathbb{Z}/2)^2$. We need to consider only those G which do not contain α_1 or α_2 . Denote by $\beta_1 = (0, 1, 1, 1, 1)$ and $\beta_2 = (1, 1, 0, 0, 0)$ remaining non-trivial elements of $\text{Ker } \rho$. Then G is one of the following:

$$G_1 = \langle \beta_1, \gamma_1 \rangle = \{\text{id}, \beta_1, \gamma_1, \gamma_3\}, \quad G_2 = \langle \beta_2, \gamma_1 \rangle = \{\text{id}, \gamma_1, \gamma_2, \beta_2\}.$$

Each element of G_1 has $a_1 = 0$, so G_1 is not strongly minimal. On the other hand, each element of G_2 has $a_4 = a_5 = 0$, G_2 is not strongly minimal either.

Case $G \not\subset \text{Ker } \rho$. — We start with the case of a cyclic group. $G = \langle g \rangle$. Set $\tau = (23)(45)$. Clearly, $g = (0, \tau)$ does not act strongly minimally, so we may suppose $g = (a, \tau)$, $a \neq 0$. Again we must have $a_1 = a_4 = a_5 = 1$, so g has the form

$$((1, 0, 1, 1, 1), \tau), \quad \text{or} \quad ((1, 1, 0, 1, 1), \tau) = ((1, 0, 1, 1, 1), \tau)^3.$$

In particular, g is of order 4, and we may assume that we are in the first case. A simple calculation shows that the action of g and σ on $\text{Pic}(X) =$

$\langle F, \overline{F}, E_p, \overline{E}_p, E_q, \overline{E}_q \rangle \cong \mathbb{Z}^6$ is

$$g^* = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 & -1 \end{pmatrix} \quad \text{and} \quad \sigma^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

From this one can easily get that $\text{rk Pic}(X_{\mathbb{C}})^{\Gamma \times G} = (\sum_{h \in \Gamma \times G} \text{tr}(h^*)) / |\Gamma \times G| = 1$, so G is strongly minimal.

Now suppose that G is not cyclic and set $G_0 = G \cap \text{Ker } \rho$. We may assume that $|G_0|$ equals 2 or 4, as otherwise G contains $\text{Ker } \rho$ and hence is strongly minimal. First consider the case $|G_0| = 2$. Since we already considered the cyclic case, we may assume that $G = \langle g \rangle \times \langle h \rangle$, where $h = (0, \tau)$ and $g = (a, \text{id})$. The condition $gh = hg$ gives $(a, \tau) = (\tau \cdot a, \tau)$, which means $a_2 = a_3, a_4 = a_5$. From $a_1 + a_2 + a_3 = 0$ we obtain $a_1 = 0$ which implies the same for all elements of G , showing that we have a \star -condition.

Finally, suppose $G_0 = \langle (a, \text{id}) \rangle \times \langle (b, \text{id}) \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and put $H = \langle (0, \tau) \rangle$. The condition $G_0 H = H G_0$ implies that the set $\{(a, \tau), (b, \tau), (a + b, \tau)\}$ coincides with $\{(\tau \cdot a, \tau), (\tau \cdot b, \tau), (\tau \cdot (a + b), \tau)\}$. Let us examine possible cases.

Case $\tau \cdot a = a$. — Then $a_4 = a_5$, and $a_2 = a_3$ implies $a_1 = 0$. We have the following possibilities for a :

- (i) $(0, 0, 0, 1, 1)$,
- (ii) $(0, 1, 1, 0, 0)$,
- (iii) $(0, 1, 1, 1, 1)$.

Subcase $\tau \cdot b = b$. — As above, this implies $b_1 = 0$, and we are in the \star -condition, contradicting minimality.

Subcase $\tau \cdot b = a + b$. — According to each possibility for a , we have

- (i) $(b_1, b_3, b_2, b_5, b_4) = (b_1, b_2, b_3, b_4 + 1, b_5 + 1)$, so $b_4 + b_5 = 2b_4 + 1 = 1$, a contradiction.
- (ii) $(b_1, b_3, b_2, b_5, b_4) = (b_1, b_2 + 1, b_3 + 1, b_4, b_5)$, so $0 = b_1 + b_2 + b_3 = b_1 + 2b_2 + 1$ implies $b_1 = 1$. As we may assume $b_4 = b_5 = 1$ (otherwise the \star -condition is satisfied), there are only two possibilities for b :

$$\alpha_1 = (1, 0, 1, 1, 1) \quad \text{or} \quad \alpha_2 = (1, 1, 0, 1, 1).$$

Both these elements indeed give minimal automorphisms, as was noticed in the very beginning of the proof.

- (iii) $(b_1, b_3, b_2, b_5, b_4) = (b_1, b_2 + 1, b_3 + 1, b_4 + 1, b_5 + 1)$. Then $b_4 + b_5 = 2b_4 + 1 = 1$, a contradiction.

Case $\tau \cdot a = b$. — We have $a_1 = b_1, a_2 = b_3, a_3 = b_2, a_4 = b_5, a_5 = b_4$. The \star -condition implies that one may assume $a_1 = b_1 = 1$ and $a_4 = b_4 = a_5 = b_5 = 1$. We conclude that $\{a, b\} = \{\alpha_1, \alpha_2\}$, so G is minimal.

Case $\tau \cdot a = a + b$. — is obtained from the first one by switching the roles of a and b .

Finally, let us stress once again that all groups of Proposition 6.4 are defined over \mathbb{R} and strongly minimal. □

We next pass to the case when $X \cong \mathbb{P}_{\mathbb{R}}^2(a, b)$.

PROPOSITION 6.6. — *Let $X \cong \mathbb{P}_{\mathbb{R}}^2(1, 2)$ be a real del Pezzo surface of degree 4. Then $A_o \cong (\mathbb{Z}/2)^2$ and A' lies in the group*

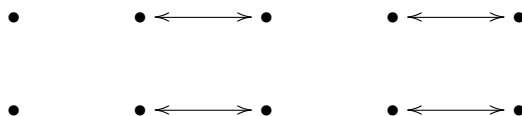
$$\langle s = (23), r = (2435) \mid s^2 = r^4 = 1, srs = r^{-1} \rangle \cong D_4.$$

There are no strongly minimal groups G acting on X .

Proof. — One has

$$\begin{aligned} \{\mathcal{C}_1, \mathcal{C}'_1\} &= \{L - E_1, -K_X - L + E_1\}; \\ \{\mathcal{C}_2, \mathcal{C}'_2\} &= \{L - E_2, -K_X - L + E_2\}; \\ \{\mathcal{C}_3, \mathcal{C}'_3\} &= \{L - \bar{E}_2, -K_X - L + \bar{E}_2\}; \\ \{\mathcal{C}_4, \mathcal{C}'_4\} &= \{L - E_3, -K_X - L + E_3\}; \\ \{\mathcal{C}_5, \mathcal{C}'_5\} &= \{L - \bar{E}_3, -K_X - L + \bar{E}_3\}. \end{aligned}$$

The complex involution acts as



This immediately gives the statement about A' . Moreover, for any element $(a_1, a_2, a_3, a_4, a_5) \in A_o$ one has $a_2 = a_3$ and $a_4 = a_5$. Thus A_o is a subgroup of

$$\{(0, 0, 0, 0, 0), (0, 0, 0, 1, 1), (0, 1, 1, 0, 0), (0, 1, 1, 1, 1)\} \cong (\mathbb{Z}/2)^2,$$

and a_1 always equals to 0. This implies that both \mathcal{C}_1 and \mathcal{C}'_1 are (real and) G -invariant for any group G , hence $\text{rk Pic}(X)^G > 1$. □

PROPOSITION 6.7. — *Let $X \cong \mathbb{P}_{\mathbb{R}}^2(3, 1)$ be a real del Pezzo surface of degree 4. Then A_o lies inside the group*

$$\{(a_1, a_2, a_3, a_4, a_5) \in (\mathbb{Z}/2)^5 : a_4 + a_5 = a_1 + a_2 + a_3 = 0\} \cong (\mathbb{Z}/2)^3,$$

and A' is a subgroup of

$$\text{Sym}\{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3\} \times \text{Sym}\{\mathcal{R}_4, \mathcal{R}_5\} \cong \mathfrak{S}_3 \times \mathbb{Z}/2 \cong D_6$$

isomorphic to id , $\mathbb{Z}/2$, $\mathbb{Z}/3$ or \mathfrak{S}_3 . Moreover, each group acting strongly minimally on X is either contained in A_o and isomorphic to

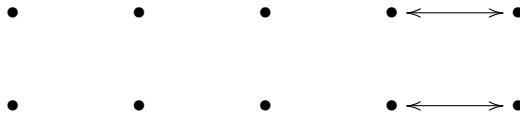
$$(\mathbb{Z}/2)^2, \quad (\mathbb{Z}/2)^3,$$

or is an extension of such a group by a subgroup of A' isomorphic to $\mathbb{Z}/2$, $\mathbb{Z}/3$ or \mathfrak{S}_3 (and every such group actually occurs as a strongly minimal group), or is a group of mixed type of order > 2 .

Proof. — We have

$$\begin{aligned}
\{\mathcal{C}_1, \mathcal{C}'_1\} &= \{L - E_1, -K_X - L + E_1\}; \\
\{\mathcal{C}_2, \mathcal{C}'_2\} &= \{L - E_2, -K_X - L + E_2\}; \\
\{\mathcal{C}_3, \mathcal{C}'_3\} &= \{L - E_3, -K_X - L + E_3\}; \\
\{\mathcal{C}_4, \mathcal{C}'_4\} &= \{L - E_4, -K_X - L + E_4\}; \\
\{\mathcal{C}_5, \mathcal{C}'_5\} &= \{L - \overline{E}_4, -K_X - L + \overline{E}_4\}.
\end{aligned}$$

The complex involution acts as



Thus, for any element $(a_1, a_2, a_3, a_4, a_5) \in A_o$ one has $a_4 = a_5$ and $A_o \subset (\mathbb{Z}/2)^3$ is the group described in the statement. Embedding $A' \hookrightarrow D_6$ indicated therein is clear as well; to exclude some possibilities for A' one consults (6.3).

Now let $G \subset \text{Aut}(X)$ be a strongly minimal group. Note that $G \not\subset A'$. Indeed, otherwise $\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 = 3L - E_1 - E_2 - E_3$ is defined over \mathbb{R} , G -invariant and not a multiple of $-K_X$.

Assume that $G \subset A_o$. Note that G cannot be of order 2, as every element $g = ((a_1, a_2, a_3, a_4, a_5), \text{id}) \in A_o$ has $a_i = 0$ for some $i \in \{1, 2, 3\}$, so that the corresponding \mathcal{C}_i is $\langle g \rangle$ -invariant. By the same principle, every strongly minimal group $G \subset A_o$ of order 4 must consist of elements which do not share 0 on the same i -th place with $i \in \{1, 2, 3\}$. This leaves the following

possibilities for G :

$$G_o^1 = \{(0, 0, 0, 0, 0), (0, 1, 1, 0, 0), (1, 0, 1, 1, 1), (1, 1, 0, 1, 1)\},$$

$$G_o^2 = \{(0, 0, 0, 0, 0), (1, 1, 0, 0, 0), (0, 1, 1, 1, 1), (1, 0, 1, 1, 1)\},$$

$$G_o^3 = \{(0, 0, 0, 0, 0), (1, 0, 1, 0, 0), (0, 1, 1, 1, 1), (1, 1, 0, 1, 1)\}.$$

All these groups are in fact strongly minimal. To check this, it is convenient to assume that $\text{Pic}(X_{\mathbb{C}}) \otimes \mathbb{R}$ is spanned by $e_0 = -K_X$, $e_i = \mathcal{C}_i$, $i = 1, \dots, 5$. In this basis, the actions of $a = (a_1, a_2, a_3, a_4, a_5)$ and $\sigma \circ a$ are given by

$$a^* = \begin{pmatrix} 1 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & (-1)^{a_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & (-1)^{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & (-1)^{a_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & (-1)^{a_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & (-1)^{a_5} \end{pmatrix}$$

$$\sigma^* a^* = \begin{pmatrix} 1 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & (-1)^{a_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & (-1)^{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & (-1)^{a_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (-1)^{a_5} \\ 0 & 0 & 0 & 0 & (-1)^{a_4} & 0 \end{pmatrix}$$

Thus for a group $G \subseteq A_o$ one has

$$\begin{aligned} \text{rk Pic}(X_{\mathbb{C}})^{\Gamma \times G} &= 1 + \frac{1}{2|G|} \sum_{h \in \Gamma \times G} \text{tr } h^* \\ &= 1 + \frac{1}{2|G|} \sum_{a \in G} (2(-1)^{a_1} + 2(-1)^{a_2} + 2(-1)^{a_3} \\ &\hspace{15em} + (-1)^{a_4} + (-1)^{a_5}) \\ &= 1 + \frac{1}{2|G|} (2(\delta_0 - \delta_1) + (\varepsilon_0 - \varepsilon_1)), \end{aligned}$$

where δ_i is the total number of i 's occurring at the first three positions of all $a \in G$, and ε_i is the total number of i 's occurring at the last two positions of a . Hence G is strongly minimal if and only if $2(\delta_0 - \delta_1) + (\varepsilon_0 - \varepsilon_1) = 0$. It is now straightforward to check the latter condition for G_o^1 , G_o^2 and G_o^3 , so all listed groups do act strongly minimally on X . □

PROPOSITION 6.8. — *Let $X \cong Q_{2,2}(0, 2)$ be a real del Pezzo surface of degree 4, and $G \subset \text{Aut}(X)$ be a strongly minimal group. Then A' is a subgroup of*

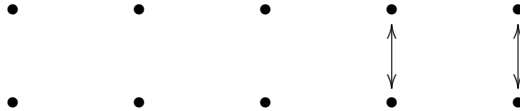
$$\text{Sym}\{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3\} \times \text{Sym}\{\mathcal{R}_4, \mathcal{R}_5\} \cong \mathfrak{S}_3 \times \mathbb{Z}/2 \cong D_6$$

isomorphic to id , $\mathbb{Z}/2$, $\mathbb{Z}/3$ or \mathfrak{S}_3 . Moreover, each strongly minimal group is either contained in A_o and isomorphic to $(\mathbb{Z}/2)^k$, $k = 1, 2, 3, 4$, or is an extension of such a group by a subgroup of A' isomorphic to $\mathbb{Z}/2$, $\mathbb{Z}/3$ or \mathfrak{S}_3 (and all listed groups indeed occur as strongly minimal groups), or is a group of mixed type. More information about the structure of G is given in the proof.

Proof. — As above, denote by E_x the exceptional divisor over a point $x \in Q_{2,2}$, and by F_1 and F_2 the strict transforms of fibers. Then one has

$$\begin{aligned} \{\mathcal{C}_1, \mathcal{C}'_1\} &= \{F_1 + F_2 - E_p - E_{\bar{p}}, F_1 + F_2 - E_q - E_{\bar{q}}\}, \\ \{\mathcal{C}_2, \mathcal{C}'_2\} &= \{F_1, F_1 + 2F_2 - E_p - E_{\bar{p}} - E_q - E_{\bar{q}}\}, \\ \{\mathcal{C}_3, \mathcal{C}'_3\} &= \{F_2, 2F_1 + F_2 - E_p - E_{\bar{p}} - E_q - E_{\bar{q}}\}, \\ \{\mathcal{C}_4, \mathcal{C}'_4\} &= \{F_1 + F_2 - E_p - E_q, F_1 + F_2 - E_{\bar{p}} - E_{\bar{q}}\}, \\ \{\mathcal{C}_5, \mathcal{C}'_5\} &= \{F_1 + F_2 - E_p - E_{\bar{q}}, F_1 + F_2 - E_q - E_{\bar{p}}\}. \end{aligned}$$

The complex involution acts as



and the same reasoning as in the proof of Proposition 6.7 applies to A' to get restrictions on this group. We proceed with enumerating minimal subgroups of A_o . One can use the same basis for $\text{Pic}(X_{\mathbb{C}}) \otimes \mathbb{R}$ as in the proof of Proposition 6.7 and see that a^* remains unchanged, while $\sigma^* a^*$ sends e_i to $(1 - a_i)e_0 + (-1)^{a_i+1}e_i$ for $i = 4, 5$. Thus for $G \subset A_o$ we have

$$\begin{aligned} \text{rk Pic}(X_{\mathbb{C}})^{\Gamma \times G} &= 1 + \frac{1}{2|G|} \sum_{a \in G} (2(-1)^{a_1} + 2(-1)^{a_2} + 2(-1)^{a_3} \\ &\quad + (-1)^{a_4} + (-1)^{a_5} + (-1)^{a_4+1} + (-1)^{a_5+1}) \\ &= 1 + \frac{1}{|G|} \sum_{a \in G} ((-1)^{a_1} + (-1)^{a_2} + (-1)^{a_3}) \\ &= 1 + \frac{1}{|G|} (\delta_0 - \delta_1), \end{aligned}$$

implying that G is strongly minimal if and only if $\delta_0 = \delta_1$. We leave to the interested reader to write down all the possibilities for such G . \square

6.4. Conjugacy problem

It follows from the classification of Sarkisov links [17, Theorem 2.6] that if G has no real fixed points on X , then every G -link starting on X is of type II and leads to the same (isomorphic) surface. Therefore, the conjugacy class of G in $\text{Cr}_2(\mathbb{R})$ is determined by the conjugacy class of G in $\text{Aut}(X)$; we do not carry out the detailed classification of the latter ones here. For each real form, some information about A_o and A' can be found in the previous section.

If G has a real fixed point on X , then its blow up induces a link of type I and gives a G -minimal conic bundle of degree 3. Note that according to Lemma 2.5 and the description of possible groups from the previous section, G can be one of the following:

$$\mathbb{Z}/2, (\mathbb{Z}/2)^2, \mathbb{Z}/4, \mathbb{Z}/8, D_4.$$

7. Del Pezzo surfaces of degree 3: cubic surfaces

Before stating our main result, let us briefly recall some facts about real cubic surfaces. It is classically known that any smooth complex cubic surface contains exactly 27 lines. It is probably less known that in [31] Segre divided real lines on smooth real cubic into two species called *hyperbolic* and *elliptic*. Consider a real line on the cubic surface. Any plane passing through this line intersects the surface in the line itself and a further residual conic. This conic intersects the line in two points. Define an involution on the line by exchanging these two points of intersection. The fixed points of this involution are called *parabolic points*. The real line is called *hyperbolic* if the involution has two real parabolic points. The real line is called *elliptic* if it has a pair of complex conjugate parabolic points. In the following table we collect the information about possible types of $\sigma^* \in \mathscr{W}(E_6)$ (including Segre's notation F_i), real lines, tritangent planes⁽⁴⁾ and topology of $X(\mathbb{R})$.

Table 7.1. Real lines and real structures on cubic surfaces

σ^*	[31]	# of real lines/tritangent planes	# of elliptic/hyperbolic lines	Topology of $X(\mathbb{R})$
id	F_1	(27,45)	12/15	$\#7\mathbb{RP}^2$
A_1	F_2	(15,15)	6/9	$\#5\mathbb{RP}^2$
A_1^2	F_3	(7,5)	2/5	$\#3\mathbb{RP}^2$
A_1^3	F_4	(3,7)	0/3	\mathbb{RP}^2
A_1^4	F_5	(3,13)	0/3	$\mathbb{RP}^2 \sqcup \mathbb{S}^2$

This information can be used to get some restrictions on group actions on X .

LEMMA 7.1. — *Let G be a p -group, where p is any prime number, and X be a real \mathbb{R} -rational cubic surface. Then X is not G -minimal.*

Proof. — It was shown in [39] that a group of odd order cannot act minimally on a \mathbb{R} -rational cubic surface. On the other hand, a 2-group cannot act minimally on a real cubic surface, as there exists an invariant real line. \square

LEMMA 7.2. — *There are no finite groups acting minimally on a real cubic surface with σ^* of type A_1^2 .*

Proof. — According to Table 7.1 such a surface contains 2 elliptic lines. Assume that G acts minimally on X . Then it permutes elliptic lines which must intersect at a point. The plane passing through these lines intersects X in the third real line, which must be G -invariant, a contradiction. \square

LEMMA 7.3. — *Let X be a real cubic surface with σ^* of type A_1^3 or A_1^4 , and $G \subset \text{Aut}(X)$ be a finite group acting minimally on X . Then G is one of the following groups:*

$$k = 0 : \mathfrak{S}_3,$$

$$k = 1 : \mathbb{Z}/6, D_6, \text{BD}_6 \cong \mathbb{Z}/3 \times \mathbb{Z}/4,$$

$$k = 2 : \mathfrak{A}_4, \mathbb{Z}/6 \times \mathbb{Z}/2, D_6 \times \mathbb{Z}/2, \mathfrak{S}_4, \mathfrak{A}_4 \times \mathbb{Z}/2, \text{BD}_6 \times \mathbb{Z}/2,$$

(see the meaning of k in the proof).

(4) Recall that a tritangent plane is a plane intersecting a smooth cubic surface in three lines.

Proof. — If σ^* is of type A_1^3 or A_1^4 , then X has exactly 3 real lines. In both cases these lines form a triangle, possibly a degenerate one (i.e. all lines meet at one Eckardt point). Indeed, in the first case this is obvious, as X dominates $\mathbb{P}_{\mathbb{R}}^2$. In the second case X is non-rational over \mathbb{R} , so it cannot contain two disjoint real lines.

So, the group G acts on the set of three real lines, say ℓ_1, ℓ_2, ℓ_3 , and one has a homomorphism $\delta : G \rightarrow \mathfrak{S}_3$. The minimality condition implies that $\text{Im } \delta$ contains an element of order 3. The kernel $\text{Ker } \delta$ consists of automorphisms that preserve each ℓ_i , and in particular either fix three real points $p = \ell_1 \cap \ell_2, \ell_1 \cap \ell_3, \ell_2 \cap \ell_3$, or preserve the unique point $p = \ell_1 \cap \ell_2 \cap \ell_3$. In both cases $\text{Ker } \delta$ embeds into $\text{GL}(T_p X) = \text{GL}_2(\mathbb{R})$ and acts on $T_p X$ with two real eigenvectors, hence must be isomorphic to $(\mathbb{Z}/2)^k, k = 0, 1, 2$. Now a simple exercise in group theory and Lemma 7.1 give the list of groups in the statement. \square

We are ready to state the main result of this section. Note that in the following theorem we do not classify all possible automorphism groups of real cubic surfaces. It is more convenient for us to go through classification of possible $\text{Aut}(X_{\mathbb{C}})$ instead. The latter can be found in [11, 9.5] and [13]; see also⁽⁵⁾ [15, 31]. For reader’s convenience, we collect this description in the table below.

Table 7.2. Automorphism groups of complex cubic surfaces

Type [13]	$\text{Aut}(X_{\mathbb{C}})$	Equation
I	$(\mathbb{Z}/3)^3 \rtimes \mathfrak{S}_4$	$x_0^3 + x_1^3 + x_2^3 + x_3^3$
II	\mathfrak{S}_5	$x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_0$
III	$\mathcal{H}_3(3) \rtimes \mathbb{Z}/4$	$x_0^3 + x_1^3 + x_2^3 + x_3^3 + 6ax_1 x_2 x_3$
IV	$\mathcal{H}_3(3) \rtimes \mathbb{Z}/2$	$x_0^3 + x_1^3 + x_2^3 + x_3^3 + 6ax_1 x_2 x_3$
V	\mathfrak{S}_4	$x_0^3 + x_0(x_1^2 + x_2^2 + x_3^2) + ax_1 x_2 x_3$
VI	$\mathfrak{S}_3 \times \mathbb{Z}/2$	$x_2^3 + x_3^3 + ax_2 x_3(x_0 + x_1) + x_0^3 + x_1^3$
VII	$\mathbb{Z}/8$	$x_3^2 x_2 + x_2^2 x_1 + x_0^3 + x_0 x_1^2$
VIII	\mathfrak{S}_3	$x_2^3 + x_3^3 + x_2 x_3(ax_0 + bx_1) + x_0^3 + x_1^3$
IX	$\mathbb{Z}/4$	$x_3^2 x_2 + x_2^2 x_1 + x_0^3 + x_0 x_1^2 + ax_1^3$
X	$(\mathbb{Z}/2)^2$	$x_0^2(x_1 + x_2 + ax_3) + x_1^3 + x_2^3 + x_3^3 + 6bx_1 x_2 x_3$
XI	$\mathbb{Z}/2$	$x_1^3 + x_2^3 + x_3^3 + 6ax_1 x_2 x_3 + x_0^2(x_1 + bx_2 + cx_3)$

⁽⁵⁾Note that Segre’s classification is known to be incorrect in some places. For example, the class VII is missing in his classification.

THEOREM 7.4. — *Let X be a smooth real \mathbb{R} -rational cubic surface, and $G \subset \text{Aut}(X)$ be a group acting minimally on X . Then, according to the type of $X_{\mathbb{C}}$, one of the following cases holds:*

Type I: X is a real form of the Fermat cubic surface

$$(7.1) \quad x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$

There are 3 real forms of the Fermat cubic, denoted F_{id} , $F_{(12)}$ and $F_{(12)(34)}$ (see Section 7.3). In the first two cases, σ^* is of type A_1^3 , and for $F_{(12)(34)}$ it is of type A_1 . For the real form F_{id} , the group G is \mathfrak{S}_3 , \mathfrak{A}_4 or \mathfrak{S}_4 (acting by permutation of coordinates in (7.1)); the groups \mathfrak{S}_4 and \mathfrak{S}_3 do occur as minimal⁽⁶⁾. For the real form $F_{(12)}$ the group G is \mathfrak{S}_3 , $\mathbb{Z}/6$ or D_6 , while for the real form $F_{(12)(34)}$ it embeds into $(\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \mathbb{Z}/2$ (in the last two cases we do not claim that all such subgroups are minimal).

Type II: X is the real Clebsch cubic surface

$$x_0 + x_1 + x_2 + x_3 + x_4 = x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

Moreover, $\sigma^* = \text{id}$, $\text{Aut}(X) \cong \mathfrak{S}_5$ and G is either \mathfrak{S}_4 , or \mathfrak{S}_5 (both cases occur).

Type III and Type IV: Then X is a real form of the cyclic cubic surface

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + ax_1x_2x_3 = 0,$$

and $G = \text{Aut}(X) \cong \mathfrak{S}_3$ acts minimally by permuting the coordinates x_1, x_2 and x_3 . The real structure σ^* is of type A_1^3 .

Type V: X is the real cubic surface

$$\alpha x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0, \quad x_0 + x_1 + x_2 + x_3 + x_4 = 0.$$

and depending on the parameter α , we have one of the following cases (all groups, except possibly \mathfrak{A}_4 , are indeed minimal):

id	($\alpha > 1/4$)	$\mathfrak{S}_4, \mathfrak{S}_3$
A_1^3	($\alpha < 1/16$)	$\mathfrak{S}_3, \mathfrak{A}_4, \mathfrak{S}_4$.

Type VI: Then $\text{Aut}(X) \cong \mathfrak{S}_3 \times \mathbb{Z}/2$, and X is a real form of the surface given by

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + ax_2x_3(x_0 + x_1) = 0$$

The group G is one of the following:

$$\mathfrak{S}_3, \quad \mathbb{Z}/6 \simeq \mathbb{Z}/3 \times \mathbb{Z}/2, \quad \mathfrak{S}_3 \times \mathbb{Z}/2$$

⁽⁶⁾ Meaning that we do not know about \mathfrak{A}_4 .

(all groups act minimally). Possible types of σ^* are: id , A_1 , and A_1^3 (more information is given in the proof).

Type VIII: Then $\text{Aut}(X) \cong \mathfrak{S}_3$, $G = \text{Aut}(X)$, and X is a real form of the surface given by

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_2x_3(ax_0 + bx_1) = 0.$$

($\text{Aut}(X)$ acts minimally). Possible types of σ^* are id , A_1 and A_1^3 (more information is given in the proof).

Proof. — Here we give an overview of the proof, referring the reader to subsequent paragraphs for details. First we notice that Lemma 7.1 implies that Types VII, IX, X and XI are not relevant for us, as G would be a p -group.

Next we look at surfaces with comparatively “large” automorphism groups. Cubics of Types II and V are studied in Sections 7.1 and 7.2 respectively. Type I is discussed in Section 7.3. Types III and IV are discussed in Section 7.4.

Now let us consider the case when X is a surface of Type VIII, i.e. $\text{Aut}(X_{\mathbb{C}}) \cong \mathfrak{S}_3$. Then we must have $G = \text{Aut}(X) \cong \mathfrak{S}_3$ and G is minimal by [13, Theorem 6.14 (7)]. Let us find possible real structures. Note that X has 3 real Eckardt points p_1, p_2 and p_3 (recall that there is a bijective correspondence between the set of Eckardt points on a smooth cubic surface, and the set of its involutions whose fixed loci consist of a hyperplane section and an isolated point [11, Proposition 9.1.23]). Note that p_i are collinear and do not lie on a line contained in X [11, Proposition 9.1.26]. Let us say that a real Eckardt point is of the *first type* if all three lines passing through this point are real, and of the *second type* if there are one real R_i and two complex conjugate. We may assume that one of the following cases hold:

- (1) p_1, p_2 are of 1st type and p_3 is of 2nd type. Then clearly G preserves R_3 , hence is not minimal.
- (2) p_1 and p_2 are of 2nd type and p_3 is of 1st type. We may assume that G permutes R_1 and R_2 . If $R_1 \cap R_2 = \emptyset$, then G is not minimal. Otherwise the plane $\langle R_1, R_2 \rangle$ intersects X at some real line which is G -invariant.
- (3) All points are of 2nd type. Then G acts on R_i . Note that these lines cannot intersect in one point, as it would be another Eckardt point. Therefore, R_i form a triangle. Now a surface containing a point of 2nd type can be of Segre types F_3, F_4 or F_5 [31, p. 153]. In our

situation, \mathbb{R} -rationality assumption and Lemma 7.2 imply that σ is of type A_1^3 .

- (4) Finally, if all points are of 1st type, then we have at least 9 real lines on X and hence σ^* is of type id or A_1 .

Finally, assume that X is a real cubic of type VI, i.e. $\text{Aut}(X_{\mathbb{C}}) \cong \mathfrak{S}_3 \times \mathbb{Z}/2$. Then $X_{\mathbb{C}}$ has 4 Eckardt points: 3 collinear points p_1, p_2, p_3 , and the fourth point q . The lines $\overline{qp_i}$, $i = 1, 2, 3$, lie on $X_{\mathbb{C}}$ (otherwise, $X_{\mathbb{C}} \cap \overline{qp_i} = \{q, p_i, r_i\}$ with r_i being an Eckardt point by [11, Proposition 9.1.26]). Since both Γ and G preserve collinearity, q is real and G -fixed. Assume that not all p_i 's are real, and let p_1 be the only real point among them. Then the real line $\overline{qp_1}$ is G -invariant for any $G \subset \text{Aut}(X)$. Thus we may assume that all Eckardt points on X are real. In particular, $\text{Aut}(X) \cong \mathfrak{S}_3 \times \mathbb{Z}/2$ (otherwise we do not have enough real involutions), and then [13, Theorem 6.14 (6)] shows that minimal subgroups are $\mathfrak{S}_3, \mathbb{Z}/6$ and the whole $\text{Aut}(X)$. Same considerations as in the previous case show that p_i have the same type. One can show that σ can be of types id, A_1 and A_1^3 , see [31, § 106]. \square

Remark 7.5. — The classification given in Theorem 7.4 can be also formulated in terms of elements of the Weyl group $\mathscr{W}(E_6)$. Let X be a real \mathbb{R} -rational cubic surface, and $G \subset \text{Aut}(X)$ be a group acting minimally on X . Recall that $|\mathscr{W}(E_6)| = 2^7 3^4 5$. By Lemma 7.1 we may assume that G contains an element of order 3 or 5. We have the following cases:

- G has an element of order 5. Then X is isomorphic to the Clebsch diagonal cubic over \mathbb{R} , see Proposition 7.6.
- G has an element of order 3. Let $g \in G$ be an element of order 3. Then g^* is of type⁽⁷⁾ A_2, A_2^2 or A_2^3 in $\mathscr{W}(E_6)$. On the other hand, if g^* is of type A_2^3 , then $\text{tr } g^* = -3$, hence $\text{Eu}(X(\mathbb{C})^g) = 0$. This is possible if and only if $X(\mathbb{C})^g$ consists of an elliptic curve, a section by a fixed hyperplane of g in $\mathbb{P}_{\mathbb{C}}^3$. But $g \in \text{PGL}_4(\mathbb{R})$, so it cannot have such a hyperplane (as it would correspond to an eigenvalue of multiplicity 3). So, we may assume that g^* is either of type A_2 , or of type A_2^2 in $\mathscr{W}(E_6)$.
 - G has an element of type A_2 . As was shown in [11, 9.5.1], in this case $X_{\mathbb{C}}$ is isomorphic to the Fermat cubic surface

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0,$$

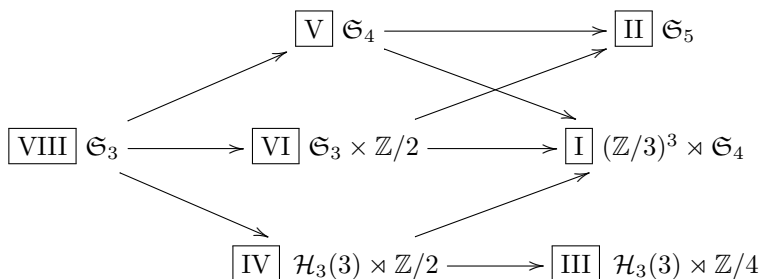
whose real forms are studied in Section 7.3.

⁽⁷⁾ 3C, 3D and 3A respectively in ATLAS notation.

– G has an element of type A_2^2 . Then $X_{\mathbb{C}}$ is isomorphic to the surface

$$(7.2) \quad x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_0x_1(ax_2 + bx_3) = 0.$$

whose complex automorphism groups is \mathfrak{S}_3 for general values of parameters a and b . For special values we get more automorphisms, which can be illustrated as follows (the arrows denote specialization, and the numbers denote the type of surface according to [13]):



Note that the Types III, IV, and I correspond to the situation when the surface (7.2) specializes to a cyclic cubic surface (defined later). Such surfaces are logically divided in three distinct types: harmonic, equianharmonic and the rest, see below. The equianharmonic case, namely the Fermat cubic (I), is discussed in Section 7.3. The types III and IV correspond to harmonic and neither harmonic nor equianharmonic cubics respectively and are discussed in Section 7.4.

In the next few paragraphs we discuss cubic surfaces of Types I-V. For this we first need to recall Sylvester’s classical approach to cubic forms.

Sylvester non-degenerate cubic surfaces

Recall that by the classical result of J. Sylvester (see [11, Corollary 9.4.2] for modern exposition) a general homogeneous cubic form $F(x_0, x_1, x_2, x_3)$ can be written as a sum

$$(7.3) \quad F = L_0^3 + L_1^3 + L_2^3 + L_3^3 + L_4^3,$$

over \mathbb{C} , where $L_i(x_0, x_1, x_2, x_3)$ are linear forms, no four are linearly dependent. Moreover, these forms are defined uniquely, up to scaling by a cubic root of unity. The corresponding planes $L_i = 0$ cut out so-called *Sylvester*

pentahedron. Let $\alpha_0 L_0 + \dots + \alpha_4 L_4 = 0$ be a unique, up to proportionality, linear relation. Consider the embedding

$$\begin{aligned} \mathbb{P}^3 &\hookrightarrow \mathbb{P}^4, \quad \iota: [x_0 : x_1 : x_2 : x_3] \\ &\mapsto [y_0 : y_1 : y_2 : y_3 : y_4] \stackrel{\text{def}}{=} [L_0 : L_1 : L_2 : L_3 : L_4]. \end{aligned}$$

If $S = \{F = 0\} \subset \mathbb{P}^3$ is the cubic surface given by F , one has

$$\iota(S) = \left\{ \sum_{i=0}^4 y_i^3 = \sum_{i=0}^4 \alpha_i y_i = 0 \right\}.$$

Let us further make the change of coordinates $z_i = \alpha_i y_i$ and assume that our surface is given by

$$(7.4) \quad \sum_{i=0}^4 \lambda_i z_i^3 = \sum_{i=0}^4 z_i = 0,$$

where $\lambda_i = 1/\alpha_i^3$. These parameters are uniquely determined up to permutation and common scaling by the isomorphism class of the surface.

Representation (7.4) is called the *Sylvester form* of a cubic surface. So, a *general* cubic surface admits a unique⁽⁸⁾ Sylvester form. We call such surfaces *Sylvester nondegenerate* (and *Sylvester degenerate* otherwise).

One can show that the automorphism group of any surface given by (7.4) is a subgroup of the group \mathfrak{S}_5 which acts by permuting coordinates (or, equivalently, the sides of the Sylvester pentahedron) [12, Theorem 6.1]. Moreover, the equation

$$\sum_{i=0}^4 \lambda_i z_i^3 = 0$$

must be transformed into itself under any such permutation τ , i.e. the constant $\zeta \in \mathbb{C}$ by which this equation is multiplied equals to 1. Indeed, otherwise it is easy to see that ζ must be a 5th primitive root of unity, and τ is a cycle of length 5. The equation then necessarily reduces to

$$z_0^3 + \zeta z_1^3 + \zeta^2 z_2^3 + \zeta^3 z_3^3 + \zeta^4 z_4^3 = 0,$$

which defines a Sylvester degenerate cubic surface. So, in order to have some nontrivial permutation among the z_i 's transforming (7.4) into itself, the parameters λ_i 's must be not all distinct. As was noticed already in [31], the corresponding automorphism groups are generated by permutations of z_i 's with the same values of λ_i 's (e.g. if $\lambda_0 = \lambda_1 = \lambda_2$, then $\text{Aut}(X_{\mathbb{C}}) \cong \mathfrak{S}_3$ is the group of permutations of z_0, z_1 and z_2):

⁽⁸⁾In the sense mentioned above.

Table 7.3. Automorphism groups of Sylvester non-degenerate cubic surfaces

Name in [13]	$\text{Aut}(X_{\mathbb{C}})$	Relations between λ_i 's
II	\mathfrak{S}_5	$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$
V	\mathfrak{S}_4	$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$
VI	$\mathfrak{S}_3 \times \mathbb{Z}/2$	$\lambda_0 = \lambda_1 = \lambda_2, \lambda_3 = \lambda_4$
VIII	\mathfrak{S}_3	$\lambda_0 = \lambda_1 = \lambda_2$
X	$(\mathbb{Z}/2)^2$	$\lambda_1 = \lambda_2, \lambda_3 = \lambda_4$
XI	$\mathbb{Z}/2$	$\lambda_3 = \lambda_4$

7.1. Clebsch diagonal cubic

(see also Segre’s account [31, § 102])

PROPOSITION 7.6. — *Let X be a real \mathbb{R} -rational cubic surface with $\text{Aut}(X_{\mathbb{C}}) \cong \mathfrak{S}_5$, and $G \subset \text{Aut}(X)$ be a group acting minimally on X . Then X is isomorphic to the Clebsch diagonal cubic*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0, \quad x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0,$$

in $\mathbb{P}_{\mathbb{R}}^4$, $\text{Aut}(X) \cong \mathfrak{S}_5$ and G is either \mathfrak{S}_5 or \mathfrak{S}_4 (both groups occur).

Proof. — It is well known that $X_{\mathbb{C}}$ is \mathbb{C} -isomorphic to the Clebsch cubic surface [11, Theorem 9.5.8]. Note that \mathfrak{S}_5 acts on $X_{\mathbb{C}}$ acts by permuting coordinates x_1, \dots, x_5 , and $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R})$ acts on \mathfrak{S}_5 trivially. Thus $H^1(\Gamma, \mathfrak{S}_5) = \text{Hom}(\Gamma, \mathfrak{S}_5) / \sim$, where \sim denotes conjugation by elements of \mathfrak{S}_5 . Since \mathfrak{S}_5 has exactly 2 conjugacy classes of involutions with representatives (12) and (12)(34), we see that the Clebsch cubic has 3 real forms, which we denote by X_{id} , $X_{(12)}$ and $X_{(12)(34)}$. We are now going to calculate the number of real lines on each nontrivial real form of the Clebsch cubic. First start with their description on $X_{\mathbb{C}}$. Put

$$L_{ijk} = \{x_i = x_j + x_k = 0\},$$

where either $i = 1$ and $(jk) \in \{(23), (24), (25)\}$, or $i \in \{2, 3, 4, 5\}$, $j = 1$ and $k \neq i, k \neq 1$ (clearly, some permutation of indexes give same lines). Further, define

$$L_{ijkl} = \{x_i + \zeta x_j + x_k = x_j + \zeta x_i + x_l = \zeta x_i + \zeta x_j - x_5 = 0\}$$

where $i, j, k, l \in \{1, 2, 3, 4\}$, $i < j$ and $\zeta = (1 + \sqrt{5})/2$. An easy calculation shows that we have 15 different lines L_{ijk} and 12 different lines L_{ijkl} .

Now we are interested in those lines $L \subset X_{\mathbb{C}}$ for which $\sigma(L) = L$. It means that in the case of the real form corresponding to the cycle (12) we are looking for those lines among L_{ijk} and L_{ijkl} which are (12)-invariant. So, we get just L_{312} , L_{412} and L_{512} . An easy calculation shows that $X_{(12)}$ has 13 real tritangent planes (these correspond to (12)-invariant pairs of lines). In particular, $X_{(12)}$ is not rational over \mathbb{R} (see Table 7.1), so X is not isomorphic to $X_{(12)}$. Similarly, for the cycle (12)(34) the real lines on the corresponding real form are L_{512} , L_{513} , L_{514} , L_{1234} , L_{1243} , L_{3412} and L_{3421} . Our calculations are summed up in Table 7.4.

Table 7.4. Real lines on Clebsch cubic surface

Form	X_{id}	$X_{(12)}$	$X_{(12)(34)}$
Number of real lines	27	3 (not \mathbb{R} -rational)	7

Finally, if X has exactly 7 real lines, then it cannot be G -minimal for any group G by Lemma 7.2. If all 27 lines are real, then G is either \mathfrak{S}_4 or \mathfrak{S}_5 by [13, Theorem 6.14]. \square

7.2. Cubic surfaces with automorphism group \mathfrak{S}_4

(see also [31, § 107]) Let X be a real \mathbb{R} -rational cubic surface with $\text{Aut}(X_{\mathbb{C}}) \cong \mathfrak{S}_4$. The Sylvester form of $X_{\mathbb{C}}$ is

$$(7.5) \quad \alpha z_0^3 + z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0, \quad z_0 + z_1 + z_2 + z_3 + z_4 = 0,$$

where we put $\alpha = \lambda_0/\lambda_1$. Both $\text{Aut}(X)$ and Γ act on the faces of the Sylvester pentahedron, which we denote by π_0, \dots, π_4 . In particular, we have three distinct cases:

- (1) all π_i 's are real;
- (2) π_0, π_1, π_2 are real, and $\pi_3 = \sigma(\pi_4)$;
- (3) π_0 is real, and $\pi_1 = \sigma(\pi_2)$, $\pi_3 = \sigma(\pi_4)$.

Note that in the last two cases $\text{Aut}(X)$ faithfully acts on the set of pairs $\{\pi_1, \pi_2\}$ and $\{\pi_3, \pi_4\}$, so $\text{Aut}(X)$ will be a 2-group, and X is then never G -minimal, see [31, Table on p. 161].

Thus we may assume that the Sylvester presentation is real, i.e. all π_i are defined over \mathbb{R} and X varies in the real pencil of cubic surfaces X_{α} , $\alpha \in \mathbb{R}\mathbb{P}^1$. A simple calculation shows that X_{α} is a smooth cubic surface for all $\alpha \in \mathbb{C}$, except $\alpha = 1/4$ and $\alpha = 1/16$. The surface $X_{1/16}$ has the unique singular

point $[-4 : 1 : 1 : 1 : 1]$, and the surface $X_{1/4}$ has exactly four singular points $[2 : 1 : -1 : -1 : -1]$, $[2 : -1 : -1 : 1 : -1]$, $[2 : -1 : 1 : -1 : -1]$, and $[2 : -1 : -1 : -1 : 1]$. By Ehresmann’s fibration theorem, the surfaces X_α arising from α ’s lying between a consecutive pair of the values $-\infty$, $1/16$, $1/4$, $+\infty$ are homeomorphic to each other. The special cases $\alpha = 1$ and $\alpha = 0$ yield Clebsch and Fermat cubic surfaces respectively. Their real forms are studied in Sections 7.1 and 7.3. Finally, it can be shown that for $1/16 < \alpha < 1/4$ the real loci $X_\alpha(\mathbb{R})$ are disconnected, so σ^* is of type A_1^4 for such surfaces. We can illustrate the situation as follows:

$$\alpha : \quad \infty \xrightarrow{A_1^3} 1/16 \xrightarrow{A_1^4} 1/4 \xrightarrow{\text{id}} +\infty$$

By [13, Theorem 6.14 (5)] the groups $\text{Aut}(X) \simeq \mathfrak{S}_4$ and \mathfrak{S}_3 act minimally on (7.5) by permuting coordinates. For the real structure of type A_1^3 we might have a new group \mathfrak{A}_4 (however, we do not address this question here).

Sylvester degenerate cubic surfaces

We are now going to study those real cubic surfaces which either do not admit the Sylvester form at all, or this form is not unique. The latter ones are called *cyclic* surfaces. These are the surfaces for which four of the five L_i ’s are linearly dependent, and after a suitable change of variables the equation becomes

$$(7.6) \quad F = x_0^3 + G_3(x_1, x_2, x_3),$$

where G_3 is a ternary cubic form (so, our surface is a Galois triple cover of \mathbb{P}^2). Consider the cubic curve E given by $G_3 = 0$. Following [11, Definition 3.1.2, Theorem 3.1.3], we call E and the corresponding cyclic surface *harmonic* if the absolute invariant $j(E) = 1728$, and *equianharmonic* if $j(E) = 0$.

7.3. Equianharmonic case: Fermat cubic

(compare [31, § 103]) Any equianharmonic cubic is projectively isomorphic to the Fermat cubic over \mathbb{C} . In this subsection X denotes the Fermat cubic surface

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

Recall that $\text{Aut}(X_{\mathbb{C}}) \cong (\mathbb{Z}/3)^3 \rtimes \mathfrak{S}_4$, where one can view $(\mathbb{Z}/3)^3$ as the group

$$\{ \omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in \mathbb{C}^4 : \omega_i^3 = 1 \text{ for all } i, \text{ and } \omega_1\omega_2\omega_3\omega_4 = 1 \}$$

with an obvious action ψ of \mathfrak{S}_4 on $(\mathbb{Z}/3)^3$. The group Γ acts on $\text{Aut}(X_{\mathbb{C}})$ as

$$\sigma \cdot ((\omega_1, \omega_2, \omega_3, \omega_4), \tau) = ((\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4), \tau).$$

Any 1-cocycle $c : \Gamma \rightarrow \text{Aut}(X_{\mathbb{C}})$ is given by $c(\sigma) = (\omega, \tau)$ such that $c(\sigma) \cdot \sigma(c(\sigma)) = 1$, i.e.

(7.7)

$$(\omega, \tau) \cdot (\bar{\omega}, \tau) = ((\omega_1 \bar{\omega}_{\tau^{-1}(1)}, \omega_2 \bar{\omega}_{\tau^{-1}(2)}, \omega_3 \bar{\omega}_{\tau^{-1}(3)}, \omega_4 \bar{\omega}_{\tau^{-1}(4)}), \tau^2) = 1.$$

In particular, τ is either trivial, or of order 2. If $c \sim c'$, then τ and τ' (corresponding to $c(\sigma)$ and $c'(\sigma)$) are conjugate in \mathfrak{S}_4 , thus we may assume that τ is one of the following: id, (12) or (12)(34). A slightly tedious computation shows that this indeed corresponds to partition of the set of 1-cocylces into 3 conjugacy classes with representatives (1, id), (1, (12)) and (1, (12)(34)), so the Fermat cubic surface has 3 real forms. We refer to these cases as F_{id} , $F_{(12)}$ and $F_{(12)(34)}$ respectively. The 27 lines on $X_{\mathbb{C}}$ are given by

$$\begin{aligned} \alpha_{kj} : x_1 + \omega^k x_4 &= x_2 + \omega^j x_3 = 0, \\ \beta_{kj} : x_1 + \omega^k x_3 &= x_4 + \omega^j x_2 = 0, \\ \gamma_{kj} : x_1 + \omega^k x_2 &= x_4 + \omega^j x_3 = 0, \end{aligned}$$

where $0 \leq j, k \leq 2$, and ω is a primitive 3rd root of unity. One can easily check that (σ, g) -invariant lines (i.e. real ones) are

$$\begin{aligned} \alpha_{00}, \beta_{00}, \gamma_{00} &\text{ for } g = (1, \text{id}), \\ \gamma_{00}, \gamma_{10}, \gamma_{20} &\text{ for } g = (1, (12)), \\ \text{all } \gamma_{kj}, \alpha_{00}, \beta_{00}, \alpha_{12}, \beta_{11}, \alpha_{21}, \beta_{22} &\text{ for } g = (1, (12)(34)), \end{aligned}$$

We see that there are 3 real lines on F_{id} and $F_{(12)}$, and 15 real lines on $F_{(12)(34)}$. Note that three real lines on F_{id} form a triangle, while on $F_{(12)}$ they intersect at an Eckardt point. A real cubic surface with 15 real lines is always rational over \mathbb{R} . To determine which of the forms F_{id} and $F_{(12)}$ are \mathbb{R} -rational, one can compute the number of real tritangent planes. These are given by

$$\begin{aligned} x_1 + \omega^i x_2 + \omega^j x_3 + \omega^k x_4 &= 0, \\ x_s + \omega^l x_p &= 0, \end{aligned}$$

where $s < p$, and $i, j, k, l \in \mathbb{Z}/3$. So, in each of two cases the number of real planes is 7, which means that all real forms of the Fermat cubic are rational over \mathbb{R} (see Table 7.1).

Finally, let us determine which groups can act minimally on a real Fermat cubic. The surface (7.1) corresponds to the real form F_{id} . Thus, a minimal group G embeds into \mathfrak{S}_4 (acting by permutation of coordinates).

The groups \mathfrak{S}_4 and \mathfrak{S}_3 do act minimally by [13, Theorem 6.14 (1)]. Since we have a non-trivial real structure of type A_1^3 , it could be possible that \mathfrak{A}_4 (i.e. the only remaining non- p -group) acts minimally on such X ; we do not investigate this question here. For the real form $F_{(12)}$, our group G must embed into the tangent space of a real Eckardt point; it is easy to see, using Lemma 7.3 or directly [31, § 103, case II], that G is \mathfrak{S}_3 , $\mathbb{Z}/6$ or D_6 . We leave it to the interested reader to write down an explicit equation of $F_{(12)}$ and to find which of these groups are actually minimal. Consider now the form $F_{(12)(34)}$. By [31, § 43, § 103] the group $\text{Aut}(X)$ is a group of order 72, having $\mathfrak{S}_3 \times \mathfrak{S}_3$ as an index 2 subgroup. In fact, it is straightforward to give a real cubic surfaces acted by $(\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \mathbb{Z}/2$, just by considering the surface

$$S : g_3(x, y) + g_3(z, t) = 0,$$

where $g_3(x, y) = x^3 - 3xy^2$ is the absolute invariant of $D_3 \cong \mathfrak{S}_3$. Note that S is automatically the real form of the Fermat cubic, since only the automorphism group of the latter one can contain a copy of $\mathfrak{S}_3 \times \mathfrak{S}_3$ (see Table 7.2; the case of $\mathcal{H}_3(3) \rtimes \mathbb{Z}/4$ is easily excluded).

7.4. Non-equianharmonic case

A cyclic non-singular and non-equianharmonic cubic surface has the canonical equation [31, § 88] over \mathbb{C}

$$(7.8) \quad x_0^3 + (x_1^3 + x_2^3 + x_3^3 - 3\lambda x_1 x_2 x_3) = 0$$

with $\lambda(\lambda^3 + 8)(\lambda^3 - 1) \neq 0$. It corresponds to Segre types (viii) and (ix) [31, § 100] and Types III-IV of [13]. So, the equation (7.8) describes a cyclic cubic surface varying in a pencil whose real members correspond to $\lambda \in \mathbb{R}$. There are only two real singular surfaces in this pencil, arising from $\lambda = \infty$ and $\lambda = 1$. It can be checked⁽⁹⁾ that σ^* is always of type A_1^3 (see also [31, § 104])

Let f be a homogeneous polynomial defining a hypersurface Z in \mathbb{P}^n . Recall that the hypersurface $\text{Hess}(Z) = \{ \det \text{Hess}(f) = 0 \}$ is called the Hessian hypersurface of Z . The Hessian of a cyclic cubic surface is the union of a *fundamental plane* $\Pi = \{x_0 = 0\}$ and the cone over a cubic curve. Thus each automorphism of X is a linear map operating separately on x_0 and x_1, x_2, x_3 . One can show that $\text{Aut}(X)$ is isomorphic to a subgroup of

⁽⁹⁾One can pick a specific value of $\lambda > 1$ and $\lambda < 1$ and calculate the number of real lines and tritangent planes, and then use Table 7.1.

\mathfrak{S}_3 [31, § 104]. So, a minimal group G must be isomorphic to \mathfrak{S}_3 ; note that such a group indeed acts minimally on X (since it is already minimal over \mathbb{C} , [13, Theorem 6.14]).

Remark 7.7. — Recall that the intersection $C = \Pi \cap X_{\mathbb{C}}$ is a cubic curve, whose 9 inflection points correspond to 9 Eckardt points of $X_{\mathbb{C}}$. Obviously, in our case C is defined over \mathbb{R} (as Π is Γ -invariant, being the only plane component of the Hessian). It is well known that a real cubic curve has exactly 3 real inflection points, and these points are collinear. In terminology of the proof of Theorem 7.4, the corresponding Eckardt points on X are of type 2 (these automatically follows from the type of σ^* , or can be easily seen from the explicit description of lines on X , see [11, Example 9.1.24]).

7.5. Non-cyclic Sylvester degenerate surfaces

A detailed description of the automorphism groups of such surfaces can be found in [31, § 100] (cases x–xvii). After excluding 2-groups, we are left just with two types (xi) and (xiv), having (complex) automorphism groups \mathfrak{S}_3 and $\mathfrak{S}_3 \times \mathbb{Z}/2$ respectively. Such groups were already discussed in the proof of Theorem 7.4.

7.6. Conjugacy classes

Classification of links in [17] shows that del Pezzo cubic surfaces are rigid, and hence the conjugacy class of G in $\mathrm{Cr}_2(\mathbb{R})$ is determined by the conjugacy class of G in $\mathrm{Aut}(X)$.

8. Del Pezzo surfaces of degree 2

Throughout this section X (or X_B^{sgn} , see below) denotes a real del Pezzo surface of degree 2. The anticanonical map $\varphi_{|-K_X|} : X \rightarrow \mathbb{P}_{\mathbb{R}}^2$ is a double cover branched over a smooth quartic $B \subset \mathbb{P}_{\mathbb{R}}^2$. The Galois involution γ of the double cover is called the *Geiser involution*. Note that $B(\mathbb{R})$ divides $\mathbb{R}\mathbb{P}^2$ into connected open sets and only one of these is non-orientable. Choose an equation $F(x, y, z) = 0$ of B such that F is negative on that non-orientable set. One can associate two different degree 2 del Pezzo surfaces to B , namely

$$X_B^{\mathrm{sgn}} = \{[x : y : z : w] \in \mathbb{P}_{\mathbb{R}}(1, 1, 1, 2) : w^2 = \mathrm{sgn} \cdot F(x, y, z)\},$$

where $\text{sgn} \in \{1, -1\}$. It is classically known that there are 6 topological types of degree 4 smooth real plane curves. Correspondingly there are 12 topological types of degree 2 real del Pezzo surfaces. The following table lists only those $X = X_B^{\text{sgn}}$ which are rational over \mathbb{R} (see [19] or [38] for details):

Table 8.1. Involutions in $\mathcal{W}(E_7)$ and real forms of \mathbb{R} -rational del Pezzo surfaces of degree 2

Conjugacy class of $\sigma^* \in \mathcal{W}(E_7)$	Eigenvalues of σ^*	$\text{tr } \sigma^*$	sgn	$X_B^{\text{sgn}}(\mathbb{R})$	$B(\mathbb{R})$	# real lines
id	1^7	7	-	$\#8\mathbb{R}P^2$	$\bigcirc \bigcirc \bigcirc \bigcirc$	56
A_1	$-1, 1^6$	5	-	$\#6\mathbb{R}P^2$	$\bigcirc \bigcirc \bigcirc$	32
A_1^2	$-1^2, 1^5$	3	-	$\#4\mathbb{R}P^2$	$\bigcirc \bigcirc$	16
$A_1^{3''}$	$-1^3, 1^4$	1	-	$\#2\mathbb{R}P^2$	\bigcirc	8
$A_1^{3'}$	$-1^3, 1^4$	1	+	\mathbb{T}^2	\odot	0
$A_1^{4'}$	$-1^4, 1^3$	-1	+	\mathbb{S}^2	\bigcirc	0

The Geiser involution is contained in the center of $\text{Aut}(X)$ and fits into the short exact sequence

$$1 \longrightarrow \langle \gamma \rangle \longrightarrow \text{Aut}(X) \longrightarrow \text{Aut}(B) \longrightarrow 1,$$

It is well known that this exact sequence splits, i.e. $\text{Aut}(X) \cong \text{Aut}(B) \times \langle \gamma \rangle$. In particular, we have the following possibilities for the group G :

- $\gamma \notin G$. Then G is isomorphic to a subgroup $G_B \subset \text{Aut}(B) \subset \text{PGL}_3(\mathbb{R})$. Possible automorphism groups of real algebraic curves of genus 3 (considered as Klein surfaces) were described⁽¹⁰⁾ in [7]. Excluding those which do not embed into $\text{PGL}_3(\mathbb{R})$ we get the following list:

$$\mathbb{Z}/2, \mathbb{Z}/2 \times \mathbb{Z}/2, D_3, D_4, D_6, \mathfrak{S}_4.$$

Since our quartic lies in $\mathbb{P}_{\mathbb{R}}^2$ it is not difficult to obtain this classification using invariant theory, see Appendix C for description of some invariants. This also shows that our curve cannot admit an

⁽¹⁰⁾ In fact, for each automorphism group the authors even provide some restrictions on the number of real ovals.

automorphism of order 6: otherwise the equation of B reduces to the form

$$z^4 + Az^2(x^2 + y^2) + B(x^2 + y^2)^2 = 0,$$

which is singular. Further, by [39, Theorem 1.2] there is no $H = \mathbb{Z}/3$ -action on an \mathbb{R} -rational del Pezzo surface of degree 2 with $\text{Pic}(X)^H \simeq \mathbb{Z}$. Therefore, if G does not contain γ and acts strongly minimally on an \mathbb{R} -rational del Pezzo surface of degree 2, then it is isomorphic to one of the following groups:

(8.1) $\mathbb{Z}/2, \mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/4, D_3, D_4, \mathfrak{A}_4, \mathfrak{S}_4.$

- $\gamma \in G$. Then G is of the form $\langle \gamma \rangle \times G_B$, where G_B is one of those listed in (8.1) (if not trivial), and the group $\mathbb{Z}/6$ containing γ . Recall that for every real del Pezzo surface of degree 2 we have $\text{Pic}(X)^\gamma \simeq \mathbb{Z}$. Therefore, any group $G \subset \text{Aut}(X)$ containing γ is automatically strongly minimal.

The main result of this section is the following.

PROPOSITION 8.1. — *Let (X, σ) be a real del Pezzo surface of degree 2 and $G \subset \text{Aut}(X)$ be a group acting strongly minimally on X and not containing the Geiser involution. Then one of the following possibilities holds.*

(1) G is a cyclic group $\langle g \rangle_n$

- $n = 2$:

(2⁺) $g : [x : y : z : w] \mapsto [x : y : -z : w]$, g^* has type $A_1^{4'}$, and σ^* is of type $A_1^{4'}$, $A_1^{3'}$ or $A_1^{3''}$. The equation of X has the form

$$\pm w^2 = z^4 + f_2(x, y)z^2 + f_4(x, y),$$

where f_2 and f_4 are some binary forms of degrees 2 and 4 respectively which are chosen⁽¹¹⁾ in accordance with Table 8.1 (as well as the sign of w).

- $n = 4$:

(4⁺) $g : [x : y : z : w] \mapsto [-y : x : z : w]$, g^* has type A_3^2 , and X is of the form

(8.2) $\pm w^2 = z^4 + Az^2(x^2 + y^2) + B(x^4 + y^4) + Cx^2y^2 + D(x^3y - xy^3)$

for some $A, B, C, D \in \mathbb{R}$.

⁽¹¹⁾ Unfortunately, we do not know if there is any characterization of $B(\mathbb{R})$ in terms of the coefficients of F . However, it should not be difficult to determine the topology of $B(\mathbb{R})$ for a given equation.

(4⁻) $g : [x : y : z : w] \mapsto [-y : x : z : -w]$, g^* has type $D_4(a_1) \times A_1$, and X is of the form

$$(8.3) \quad \pm w^2 = z^4 + Az^2(x^2 + y^2) + B(x^4 + y^4) + Cx^2y^2$$

for some $A, B, C \in \mathbb{R}$.

In each case, except possibly 2^+ and σ of type $A_1^{3''}$, it is indeed possible to choose the coefficients in the equation of X , such that G is strongly minimal.

- (2) G is isomorphic to one of the groups $(\mathbb{Z}/2)^2, \mathfrak{S}_3, D_4, \mathfrak{A}_4$ or \mathfrak{S}_4 and contains at least one of the elements described in (1). In particular, all groups occur (but we do not find all possibilities for compatible real structures).

In what follows we assume that $\gamma \notin G$.

8.1. Case $G \cong \mathbb{Z}/2$

Let g be an involution generating G . Without loss of generality we may assume that g acts on $\mathbb{P}_{\mathbb{R}}^2$ as $[x : y : z] \mapsto [x : y : -z]$ and then the equation of $X_{\mathbb{C}}$ has the form

$$\pm w^2 = z^4 + 2f_2(x, y)z^2 + f_4(x, y),$$

where f_4 has no multiple factors (since B is smooth).

Assume that $g \neq \gamma$ and X is strongly $\langle g \rangle$ -minimal. Then $\sigma^* \neq \text{id}$. Otherwise, $\text{rk Pic}(X_{\mathbb{C}})^{\langle g \rangle} = 1$ implies $\text{tr id}^* + \text{tr } g^* = 7 + \text{tr } g^* = 0$, so g acts as $-id$ on E_7 , i.e. coincides with γ . Thus we may assume that $\text{tr } \sigma^* \in \{-1, 1, 3, 5\}$.

For the action on X we have two possibilities

$$(2^+) \quad [x : y : z : w] \mapsto [x : y : -z : w],$$

or

$$(2^-) \quad [x : y : z : w] \mapsto [x : y : -z : -w].$$

We consider these two cases separately.

2⁻ The fixed locus $X_{\mathbb{C}}^g$ consists of 4 points $[x : y : 0 : 0]$ where $f_4(x, y) = 0$. Thus $\text{tr } g^* = 1$, g^* is of type A_1^3 in $\mathscr{W}(E_7)$ and $\text{tr id}^* + \text{tr } g^* + \text{tr } \sigma^* = 8 + \text{tr } \sigma^* \in \{7, 9, 11, 13\}$. Since $X_{\mathbb{C}}$ is assumed to be $\Gamma \times G$ -minimal, we must have $\text{tr}(\sigma^* g^*) \in \{-7, -9, -11, -13\}$. The last three values are impossible in $\mathscr{W}(E_7)$, so we may assume that σ^* is of type $A_1^{4'}$ and $\sigma^* = \gamma^* \circ g^*$. We are going to show that this case does not occur.

Indeed, run two H -equivariant minimal model programs on $X_{\mathbb{C}}$, one with $H = \langle \sigma \rangle$ and the other with $H = \langle g \circ \gamma \rangle$. Their common result will be some del Pezzo surface Z . Since $g \circ \gamma$ is of type (2^+) it fixes an elliptic curve on X (see below), so we have $K_{\mathbb{Z}}^2 \leq 4$ (it is easy to check that a del Pezzo surface Z with $K_{\mathbb{Z}}^2 > 4$ cannot contain a fixed-point elliptic curve). On the other hand, Z is minimal over \mathbb{R} , hence is not \mathbb{R} -rational. But then X is non-rational over \mathbb{R} too, a contradiction.

2⁺ Then $X_{\mathbb{C}}^g$ consists of 2 real points $[0 : 0 : 1 : \pm 1]$ and a smooth genus 1 curve $\mathcal{E} = \{w^2 = f_4(x, y)\}$. Thus $\text{tr } g^* = -1$, g^* is of type $A_1^{4'}$ and $\text{tr id}^* + \text{tr } g^* + \text{tr } \sigma^* \in \{5, 7, 9, 11\}$. Again, the last two options are not possible for $\mathscr{W}(E_7)$. Thus either σ^* is of type $A_1^{4'}$ (with $X(\mathbb{R}) \approx \mathbb{S}^2$), or $A_1^{3'}$ ($X(\mathbb{R}) \approx \mathbb{T}^2$), or $A_1^{3''}$ ($X(\mathbb{R})$ is a Klein bottle).

The first two possibilities do occur. The first one is considered in Example 8.2. The second one is obtained by applying the same construction to $Q_{2,2}$.

Example 8.2. — Consider a quadric surface $Q = \{t_0^2 + t_1^2 + t_2^2 = t_3^2\} \subset \text{Proj } \mathbb{R}[t_0, t_1, t_2, t_3]$ with $X(\mathbb{R}) \approx \mathbb{S}^2$ and three pairs of complex conjugate points

$$p_{\pm} = [\pm i : \pm i\sqrt{2} : 2 : 1], \quad s_{\pm} = [\pm i : 0 : 1 : 0], \quad r_{\pm} = [0 : \pm i : \sqrt{2} : 1],$$

lying on Q . Let $\widehat{g} \in \text{Aut}(Q)$ be the automorphism given by

$$[t_0 : t_1 : t_2 : t_3] \mapsto [-t_0 : -t_1 : t_2 : t_3]$$

(a “rotation” of \mathbb{S}^2 by 180°). Then $\widehat{g}(p_+) = p_-$, $\widehat{g}(s_+) = s_-$, $\widehat{g}(r_+) = r_-$. Denote by $\pi : X \rightarrow Q$ the blow up of Q at our six points and by \widetilde{g} the lift of \widehat{g} on X . We claim that

- (1) X is a smooth real del Pezzo surface of degree 2,
- (2) The involution σ^* on X is of type A_1^4 (in particular, $X(\mathbb{R}) \approx \mathbb{S}^2$),
- (3) X is strongly minimal with respect to $g = \gamma \circ \widetilde{g}$.

Let us assume that (1) holds for a moment. Note that $\text{Pic}(X_{\mathbb{C}})$ is generated by three pairs of complex conjugate exceptional divisors $E_{p_{\pm}}, E_{s_{\pm}}, E_{r_{\pm}}$ and a pair of complex conjugate divisors F, \overline{F} , where $F = \pi^*(\ell)$, $\overline{F} = \pi^*(\overline{\ell})$, $\text{Pic}(Q_{\mathbb{C}}) = \mathbb{Z}[\ell] \oplus \mathbb{Z}[\overline{\ell}]$. Note that σ permutes the members in each pair (which implies (2)), while \widetilde{g} permutes the members in each pair of exceptional divisors and preserves F and \overline{F} . So, $\widetilde{g}^* \circ \sigma^*$ acts with trace equal to 6 in $\text{Pic}(X_{\mathbb{C}}) \otimes \mathbb{R}$, hence with trace equal to 5 in $K_X^{\perp} \otimes \mathbb{R}$. Put $g = \widetilde{g} \circ \gamma$. Since γ^* acts as $-\text{id}$ in $K_X^{\perp} \otimes \mathbb{R}$ one has $\text{tr}((\gamma \circ g)^* \circ \sigma^*) = -5$, so $X_{\mathbb{C}}$ is strongly $\langle g \rangle \times \Gamma$ -minimal.

Finally, let us prove (1). Note that X is a smooth del Pezzo if our six points do not lie on divisors of bidegree $(1, 0)$, $(0, 1)$, $(1, 1)$, $(1, 2)$, $(2, 1)$ or $(2, 2)$. For convenience of calculation, let us make the linear change of coordinates

$$T_0 = t_3 - t_2, \quad T_1 = t_0 - it_1, \quad T_2 = t_0 + it_1, \quad T_3 = t_3 + t_2.$$

Then $Q = \{t_3^2 - t_2^2 = t_0^2 + t_1^2\}$ is given by $T_0T_3 = T_1T_2$ in $\text{Proj } \mathbb{R}[T_0, T_1, T_2, T_3]$ and the blown up points are

$$\begin{aligned} p &= [-1 : i + \sqrt{2} : i - \sqrt{2} : 3], & \tilde{p} &= [-1 : -i - \sqrt{2} : -i + \sqrt{2} : 3], \\ s &= [-1 : i : i : 1], & \tilde{s} &= [-1 : -i : -i : 1], \\ r &= [1 - \sqrt{2} : 1 : -1 : 1 + \sqrt{2}], & \tilde{r} &= [1 - \sqrt{2} : -1 : 1 : 1 + \sqrt{2}]. \end{aligned}$$

Divisors of bidegree $(1, 0)$ and $(0, 1)$ on Q are the lines $T_1 = tT_0$, $T_3 = tT_2$ and $T_2 = tT_0$, $T_3 = tT_1$. It can be easily checked that no two points from above lie on such lines. Further, divisors of bidegree $(1, 1)$ are hyperplane sections of Q , but our points do not simultaneously satisfy the equation $\alpha T_0 + \beta T_1 + \gamma T_2 + \delta T_3$. Next, assume that our six points lie on the curve C of bidegree $(1, 2)$ (note that C is smooth). Then $g(C)$ is a curve of bidegree $(2, 1)$ still containing all six points. But $C \cdot g(C) = 5$, a contradiction. Finally, assume that the six points lie on a curve E of bidegree $(2, 2)$. Note that E has at most one ordinary double point and $E^2 = 8$. So, the self-intersection of a strict transform of E after the blow-up is at least -1 .

8.2. Case $G \cong \mathbb{Z}/4$

Let g be a generator of $G_B \subset \text{PGL}_3(\mathbb{R}) \cong \text{SL}_3(\mathbb{R})$. We may assume that g acts as $[x : y : z] \mapsto [-y : x : z]$. The equation of B then has the form

$$(8.4) \quad z^4 + Az^2(x^2 + y^2) + B(x^4 + y^4) + Cx^2y^2 + D(x^3y - xy^3) = 0,$$

There are two possible lifts of g to an automorphism of X , namely

$$(4^+) \quad [x : y : z : w] \mapsto [-y : x : z : w]$$

or

$$(4^-) \quad [x : y : z : w] \mapsto [-y : x : z : -w].$$

We treat these two cases separately.

4^+ Let $p = [x : y : z : w] \in \mathbb{P}(1, 1, 1, 2)$ be a point fixed by g . Then either $p = [0 : 0 : z : w]$ or $p = [1 : \pm i : 0 : 0]$. The condition $p \in X$ implies

$w/z^2 = \pm 1$ in the former case, and $2B - C \pm 2iD = 0$ in the latter. It follows that in the second case we have $C = 2B, D = 0$, so F reduces to the form

$$(8.5) \quad z^4 + Az^2(x^2 + y^2) + B(x^2 + y^2)^2 = 0,$$

which is singular. Therefore $\text{Fix}(g, X_{\mathbb{C}}) = \{[0 : 0 : 1 : \pm 1]\}$ and $\text{tr } g^* = -1$. It follows that the conjugacy class of g^* in $\mathscr{W}(E_7)$ is $A_3 \times A_1^2$ or A_3^2 .

In the first case $\text{Sp}(g^*) = \{\pm i, -1^3, 1^2\}$, so $\text{Sp}(g^2)^* = \{-1^2, 1^5\}$ and $(g^2)^*$ cannot be of type A_1^4 , a contradiction. Thus only the case of A_3^2 remains. This case does occur, see Example 8.3

4⁻ As above, let $p = [x : y : z : w] \in \mathbb{P}(1, 1, 1, 2)$ be a point fixed by g . Then $p = [1 : \pm i : 0 : \beta]$, where $\beta = w/x^2$. By Remark 2.3 the set $X(\mathbb{C})^g$ is not empty. It is easy to check then that $X(\mathbb{C})^g$ consists of 4 points $[1 : \pm i : 0 : \beta]$, where $\beta^2 = 2B - C + 2iD$ and $\beta^2 = 2B - C - 2iD$, so $D = 0$ ($2B - C \neq 0$: otherwise $2B = C, D = 0$, which gives a singular quartic).

So, $\text{tr } g^* = 1$ and g^* belongs to the class $(A_3 \times A_1)'$, $(A_3 \times A_1)''$ or $D_4(a_1) \times A_1$ in $\mathscr{W}(E_7)$. The same arguments as in the **(4⁺)**-case exclude the first two possibilities. The remaining case does occur, see Example 8.3.

Example 8.3. — Consider a smooth real del Pezzo surface

$$X = \{[x : y : z : w] : x^4 + 6x^2y^2 + y^4 - 2z^4 = w^2\} \subset \text{Proj } \mathbb{P}_{\mathbb{R}}(1, 1, 1, 2).$$

The curve $B = \{[x : y : z] : x^4 + 6x^2y^2 + y^4 - 2z^4 = 0\}$ is a smooth plane quartic with one oval. One can easily find all 28 bitangents of B and 56 lines on $X_{\mathbb{C}}$:

$$\begin{aligned} & \{w = \pm\sqrt{2}iz^2, \quad x = \alpha_1y\}, \quad \alpha_1 = i(\pm 1 \pm \sqrt{2}), \\ & \{w = \pm(x^2 + 3y^2), \quad z = \alpha_2y\} \\ & \quad \text{and } \{w = \pm(3x^2 + y^2), \quad z = \alpha_2x\}, \quad \alpha_2^2 = \pm 2i, \\ & \left\{w = \pm \frac{1}{\sqrt{2}}(x - y)^2, \quad z = \alpha_3(x + y)\right\} \\ & \quad \text{and } \left\{w = \pm \frac{1}{\sqrt{2}}(x + y)^2, \quad z = \alpha_3(x - y)\right\}, \quad \alpha_3^2 = \pm \frac{1}{2}, \\ & \{w = \pm i(x^2 + 4ixy - y^2), \quad z = \alpha_4(x + iy)\} \\ & \quad \text{and } \{w = \pm i(x^2 - 4ixy - y^2), \quad z = \alpha_4(x - iy)\}, \quad \alpha_4^2 = \pm 1. \end{aligned}$$

(in [33] these sets of lines are called θ -, η -, σ -, and τ -lines respectively). Consider the automorphism $g : [x : y : z : w] \mapsto [-y : x : z : \pm w]$. It is

easy to check that there are no disjoint $\langle g \rangle$ -orbits defined over \mathbb{R} , so X is $\langle g \rangle$ -minimal.

8.3. Other groups

We may assume that the groups listed in (8.1) contain an element of order 2 or 4, described above. This shows that all these groups can act strongly minimally on a real del Pezzo surface of degree 2. We however do not find all compatible real structures here.

8.4. G -links

To classify isomorphism classes of (X, G) , we use classification of Sarkisov links [13, Corollary 7.11] or [17, Theorem 2.6] which says that if a del Pezzo surface of degree d has no orbits of length $< d$, then X is superrigid. In particular a Del Pezzo surface of degree 2 is superrigid unless G has a fixed point, which must be real in our case. The only possible link is a birational Bertini involution (see the next paragraph). From Lemma 2.5 we conclude that X is G -superrigid for groups $\mathfrak{A}_4, \mathfrak{S}_4$ (not containing γ), or the groups $H \times (\mathbb{Z}/2)^2, H \times \mathbb{Z}/4, H \times \mathfrak{S}_3, H \times D_4, H \times \mathfrak{A}_4, H \times \mathfrak{S}_4$, where H is generated by the Geiser involution.

9. Del Pezzo surfaces of degree 1

Let X be a del Pezzo surface of degree 1 over a field \mathbb{k} . Its anticanonical model

$$\text{Proj} \bigoplus_{n \geq 0} H^0(X, -nK_X)$$

is a smooth sextic hypersurface $f(w, x, y, z) = 0$ in $\mathbb{P}_{\mathbb{k}}(3, 1, 1, 2)$. Write $f(w, x, y, z) = w^2 - g_3(x, y, z)w - g_6(x, y, z)$, where $g_i \in \mathbb{k}[x, y, z]$ is a polynomial of (graded) degree i . If $\text{char } \mathbb{k} \neq 2$ one can make the change of variables $w \mapsto w + g_3/2$ and reduce the equation to the form

$$w^2 = Az^3 + z^2h_2(x, y) + zh_4(x, y) + h_6(x, y).$$

When $\mathbb{k} = \mathbb{R}$ one can make the change of variables $z \mapsto z/\sqrt[3]{A} - B/3\sqrt[3]{A^2}$ and reduce the equation of X to the form

$$(9.1) \quad w^2 = z^3 + f_4(x, y)z + f_6(x, y).$$

The linear system $|-2K_X|$ has no base points and exhibits X as a double cover of a quadratic cone $Q \subset \text{Proj } \mathbb{R}[x, y, z]$. The corresponding Galois involution β is called the *Bertini involution* and acts as

$$[w : x : y : z] \mapsto [-w : x : y : z] = [w : -x : -y : z].$$

Its fixed point locus X^β is the union of a curve $R \subset Q$ of genus 4 and a single point q . This point is the unique base point of the elliptic pencil $|-K_X|$, so in particular $q \in X(\mathbb{R})$.

Remark 9.1. — In Table 9.1 below we collect some information about real structures on del Pezzo surfaces of degree 1. This time we do not restrict ourselves to \mathbb{R} -rational surfaces only, because — as will become clear in Section 9.1 — we should have a closer look at involutions' conjugacy classes in $\mathscr{W}(\mathbb{E}_8)$, and deal with the fact that sometimes the Carter graph does not determine an involution up to conjugacy.

For an irreducible reflection group \mathscr{W} acting on a vector space V , and involution $\sigma^* \in \mathscr{W}$, define $i(\sigma^*) = \dim V^-$, where $V = V^+ \oplus V^-$ is the decomposition into eigenspaces. In the notation of Table 9.1, $i(\sigma^*)$ is the sum of lower indices. Note that there is a central involution $-\text{id}$ in $\mathscr{W}(\mathbb{E}_8)$, which induces a correspondence of each σ^* with σ_t^* (called the *Bertini twist* of σ^* in Section 9.1), where $i(\sigma_t^*) = 8 - i(\sigma^*)$. It will be important for us that two classes with $i(\sigma^*) = 4$ are both self-corresponding under this, see [38, § 2] for details.

Since $\text{Aut}(X)$ fixes q , we have the natural faithful representation

$$\text{Aut}(X) \hookrightarrow \text{GL}(T_q X) \cong \text{GL}_2(\mathbb{R}),$$

so either $\text{Aut}(X) \cong \mathbb{Z}/n$ or $\text{Aut}(X) \cong D_n$.

Let $G \subset \text{Aut}(X)$. The action of G on the pencil $|-K_X|$ induces the action on

$$C = \text{Proj } \mathbb{R}[x, y] \cong \mathbb{P}_{\mathbb{R}}^1$$

(recall that by construction $\{x, y\}$ is a basis in $H^0(X, -K_X)$). This gives us the natural homomorphism

$$v : G \rightarrow \text{Aut}(C) = \text{PGL}_2(\mathbb{R}).$$

Put $G_0 = \text{Ker } v$. Every element of G_0 acts on $\mathbb{P}_{\mathbb{R}}(3, 1, 1, 2)$ as $\text{diag}\{\alpha, \varepsilon, \varepsilon, \beta\}$, where $\varepsilon = \pm 1$, and $\varepsilon^6 = \alpha^2 = \beta^3$. Thus $\beta = 1$, $\alpha = \pm 1$, and $|G_0| \leq 2$. Moreover, $G_0 \cong \mathbb{Z}/2$ means that $\beta \in G$.

Since each $g \in G \subset \text{Aut}(X)$ leaves the equation (9.1) invariant, g must have the form

$$[w : x : y : z] \mapsto [w : ax + by : cx + dy : z], \quad a, b, c, d \in \mathbb{R},$$

Table 9.1. Involution in $\mathscr{W}(E_8)$ and real forms of del Pezzo surfaces of degree 1

[38]	[8]	Eigenvalues of σ^*	$\text{tr } \sigma^*$	$X(\mathbb{R})$	number of real lines on X
1	\emptyset	1^8	8	$\#9\mathbb{R}P^2$	240
A_1	A_1	$-1, 1^7$	6	$\#7\mathbb{R}P^2$	126
A_1^2	A_1^2	$-1^2, 1^6$	4	$\#5\mathbb{R}P^2$	60
A_1^3	A_1^3	$-1^3, 1^5$	2	$\#3\mathbb{R}P^2$	26
A_1^4	$A_1^{4''}$	$-1^4, 1^4$	0	$\mathbb{R}P^2$	8
D_4	$A_1^{4'}$	$-1^4, 1^4$	0	$\mathbb{S}^2 \sqcup \#3\mathbb{R}P^2$	24
$A_1 \times D_4$	A_1^5	$-1^5, 1^3$	-2	$\mathbb{S}^2 \sqcup \mathbb{R}P^2$	6
D_6	A_1^6	$-1^6, 1^2$	-4	$2\mathbb{S}^2 \sqcup \mathbb{R}P^2$	4
E_7	A_1^7	$-1^7, 1$	-6	$3\mathbb{S}^2 \sqcup \mathbb{R}P^2$	2
E_8	A_1^8	-1^8	-8	$4\mathbb{S}^2 \sqcup \mathbb{R}P^2$	0

(unlike the case $\mathbb{k} = \mathbb{C}$). In particular, $f_4(x, y)$ and $f_6(x, y)$ are absolute invariants of G . From the list of basis invariants in Appendix C we get that for $n > 4, n \neq 6$, one has $f_4 = a(x^2 + y^2)^2, f_6 = b(x^2 + y^2)^3$, so

$$27f_6^2 + 4f_4^3 = (27b^2 + 4a^3)(x^2 + y^2)^6,$$

and hence X is singular. Moreover, X does not admit $H = \mathbb{Z}/3$ -action with $\text{Pic}(X)^H \simeq \mathbb{Z}$ by [39, Theorem 1.2]. Therefore, a strongly minimal G can be isomorphic to one of the following groups:

$$(9.2) \quad \mathbb{Z}/2, \mathbb{Z}/4, \mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/6, D_3, D_4, D_6.$$

Note that $\text{rk}(X_{\mathbb{C}})^\beta = 1$, hence to classify groups acting strongly minimally on X , we can focus only on those that do not contain the Bertini involution.

PROPOSITION 9.2. — *Let X be a real \mathbb{R} -rational del Pezzo surface of degree 1 and G be a finite group acting strongly minimally on X . Then G contains the Bertini involution and we are in one of the following cases (all groups indeed act on X strongly minimally):*

Table 9.2. Minimal automorphism groups of del Pezzo surfaces of degree 1.

G	Generators	$f_4(x, y)$	$f_6(x, y)$
Cyclic groups $G = \langle r \rangle$			
$\mathbb{Z}/2$	β	$f_4(x, y)$	$f_6(x, y)$
$\mathbb{Z}/4$	R_4	$ax^4 + bx^2y^2 + ay^4 + cxy(x^2 - y^2)$	$(x^2 + y^2)(a'x^4 + d'x^3y + b'x^2y^2 - d'xy^3 + a'y^4)$
$\mathbb{Z}/6$	R_6	$a(x^2 + y^2)^2$	$b(x^2 + y^2)^3 + c(x^6 - 15x^4y^2 + 15x^2y^4 - y^6) + d(6x^5y - 20x^3y^3 + 6xy^5)$.
Dihedral groups $G = \langle R_n, S \mid R_n^n = S^2 = 1, SR_nS^{-1} = R_n^{-1} \rangle$			
$(\mathbb{Z}/2)^2$	R_2, S	$ax^4 + bx^2y^2 + cy^4$	$a'x^6 + b'x^4y^2 + c'x^2y^4 + d'y^6$
D_4	R_4, S	$ax^4 + bx^2y^2 + cy^4$	$(x^2 + y^2)(a'x^4 + b'x^2y^2 + a'y^4)$
D_6	R_6, S	$a(x^2 + y^2)^2$	$b(x^2 + y^2)^3 + c(x^6 - 15x^4y^2 + 15x^2y^4 - y^6)$

Proof. — All the groups listed in Table 9.2 do contain the Bertini involution, so they act strongly minimally on X . To write down the corresponding equation one should consult Appendix C. It remains to exclude the groups which do not contain the Bertini involution. Below we assume that $\beta \notin G$ and $G \not\cong \mathfrak{S}_3$. The case $G \cong \mathfrak{S}_3$ requires more thorough analysis and is excluded in Section 9.1.

Case $G = \mathbb{Z}/2$. — Denote by g an involution which generates G . We may assume that g acts on $T_qX \cong \mathbb{R}^2$ as $\text{diag}\{-1, 1\}$. The set $X_{\mathbb{C}}^g$ is the disjoint union of the elliptic curve $x = 0$ and 2 or 3 points $w = y = 0$, so $\text{tr } g^* \in \{-1, 0\}$. However, there are no involutions in $\mathscr{W}(E_8)$ whose trace equals -1 , so we assume $\text{tr } g^* = 0$. If $X_{\mathbb{C}}$ is strongly $\Gamma \times G$ -minimal, then $\text{tr id}^* + \text{tr } g^* + \text{tr } \sigma^* + \text{tr}(\sigma \circ g)^* = 8 + \text{tr } \sigma^* + \text{tr}(\sigma \circ g)^* = 0$. Hence $\text{tr } \sigma^* = 0$, $\text{tr}(\sigma \circ g)^* = -8$ (see Table 9.1). The latter equality implies that $\sigma^* = g^* \circ \beta^*$.

Now run two H -equivariant minimal model programs on $X_{\mathbb{C}}$, one with $H = \langle \sigma \rangle$ and the other with $H = \langle g \circ \beta \rangle$. Their common result will be some del Pezzo surface Z . Since $g \circ \beta$ fixes an elliptic curve on X (as well as g), we have $K_Z^2 \leq 4$ (it is easy to check that a del Pezzo surface Z with $K_Z^2 > 4$ cannot contain a fixed-point elliptic curve). On the other hand, Z

is minimal over \mathbb{R} , hence is not \mathbb{R} -rational. But then X is non-rational over \mathbb{R} too, a contradiction.

Case $G = \mathbb{Z}/2n, n \geq 2$. — Let g generate G . As G does not contain the Bertini involution, we may assume that g^n acts as $\text{diag}\{-1, 1\}$ on $T_q X$, so $\det g^n = -1$. But each $h \in \text{GL}_2(\mathbb{R})$ with $2 < \text{ord } h < \infty$ has determinant equal to 1, a contradiction.

Case $G = D_n, n \geq 2$. — It is easy to see that $G = D_2 \cong (\mathbb{Z}/2)^2$ always contains the Bertini involution. So, we assume that $n > 2$ and n is even. Then G contains an element of order n whose $(n/2)^{\text{th}}$ -power is not the Bertini involution. The same argument as before shows that this is impossible. □

9.1. Geometry of \star -configurations and \mathfrak{S}_3 -actions

We now apply the techniques of [34] to analyze \mathfrak{S}_3 -actions on real del Pezzo surfaces of degree 1. More precisely, we now show that if $\beta \notin G \cong \mathfrak{S}_3$, then G cannot act on any \mathbb{R} -rational del Pezzo surface of degree 1 with invariant Picard number equal to one (the \mathbb{R} -rationality assumption is crucial).

So, assume $G = \langle g, h \mid g^3 = h^2 = 1, gh = hg^{-1} \rangle$ and $\text{rk Pic}(X_{\mathbb{C}})^{\Gamma \times G} = 1$. Since we suppose $\beta \notin G$, all involutions in G have zero traces on K_X^{\perp} (i.e. of types $A_1^{4'}$ or $A_1^{4''}$). All elements of order 3 in G are of type A_2^2 , with trace equal 2, as was shown in [39, § 5.4]. The formula (2.1) implies

$$(9.3) \quad \text{rk Pic}(X_{\mathbb{C}})^G = 3.$$

Following the terminology of [34], we say that six (-1) -curves H_1, \dots, H_6 on a del Pezzo surface of degree 1 form a \star -configuration, if

$$H_i \cdot H_{i+1} = 0, \quad H_i \cdot H_{i+2} = 2, \quad H_i \cdot H_{i+3} = 3$$

(all subscripts are modulo 6). In fact,

$$(9.4) \quad \begin{aligned} H_i + H_{i+3} &\sim -2K_X, & H_i + H_{i+2} + H_{i+4} &\sim -3K_X, \\ \sum_{i=1}^6 H_i &\sim -6K_X, \end{aligned}$$

and $H_{i+3} = \beta(H_i)$. The graph of a \star -configuration looks like \star , where the vertices denote (-1) -curves, the edges denote intersections with multiplicity 2, and the curves corresponding to opposite vertices intersect with multiplicity 3. Two \star -configurations $\star = \{H_1, \dots, H_6\}$ and $\star' = \{H'_1, \dots, H'_6\}$ are called *asynchronized* if $H_i \cdot H'_j = 1$ for any i, j .

By [33, Lemma 4.12], for every element g^* of type A_2^2 there are twelve g -invariant (-1) -curves on $X_{\mathbb{C}}$ forming two \heartsuit -configurations, and two g -invariant \heartsuit -configurations on which g acts faithfully. These four configurations are pairwise asynchronized. Denote the first two configurations by $\heartsuit_A = \{A_1, \dots, A_6\}$ and $\heartsuit_B = \{B_1, \dots, B_6\}$, and the last two (where g acts faithfully) by $\heartsuit_C = \{C_1, \dots, C_6\}$ and $\heartsuit_D = \{D_1, \dots, D_6\}$. Let us choose the numbering in every 6-tuple so that the first two entries are disjoint (i.e. neighbors in \heartsuit graph). By the proof of [33, Lemma 4.15], the classes $a_i = A_i + K_X$, $b_i = B_i + K_X$, $c_i = C_i + K_X$ and $d_i = D_i + K_X$, $i = 1, 2$, form a basis of the vector space $V = \text{Pic}(X_{\mathbb{C}}) \otimes \mathbb{R} \cap K_X^{\perp}$. We may assume that g acts on \heartsuit_C and \heartsuit_D by rotating them (more precisely, the “triangles” \triangle and ∇) counterclockwise. Using relations (9.4), one easily finds the matrix of g^* in our basis:

$$g^* = I_4 \oplus \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad (g^{-1})^* = I_4 \oplus \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

Now the involution $h \in G$ acts on the \heartsuit -configurations, and it is easy to see that condition (9.3) implies that there is a G -invariant \heartsuit -configuration among our four (the incidence relation in \heartsuit together with $gh = hg^{-1}$ show that h acts either trivially, or as a central symmetry). We call this invariant configuration \heartsuit_0 .

Similarly, the $\Gamma \times G$ -minimality of $X_{\mathbb{C}}$ implies that Γ acts by central symmetry on \heartsuit_0 . Denote by σ^* the image of the complex involution on X in the Weyl group $\mathscr{W}(E_8)$, and assume that X is given by equation (9.1). Changing the sign of w^2 in that equation gives another del Pezzo surface of degree 1, which we denote $X[\beta]$ and call the *Bertini twist* of X . If σ_t is the complex involution on $X[\beta]$, then its image in $\mathscr{W}(E_8)$ equals $\sigma_t^* = \beta^* \circ \sigma^*$. Note that β^* acts as $-\text{id}$ on K_X^{\perp} , and therefore $\text{tr}(\sigma_t^*) = -\text{tr} \sigma^*$. In particular,

$$\text{rk Pic } X[\beta]_{\mathbb{C}}^{\Gamma_t \times G} = 3,$$

where $\Gamma_t = \langle \sigma_t \rangle$. The output of G -Minimal Model Program on $X[\beta]$ is a real G -minimal del Pezzo cubic surface Y . Note that now Γ_t stabilizes the vertices of \heartsuit_0 , so σ_t^* has the same type in $\mathscr{W}(E_8)$ as the type of the complex involution on Y — i.e. id , A_1 , A_1^2 , A_1^3 , or some lift of A_1^4 (see Section 7). Therefore the original involution σ^* is of type A_1^8 , A_1^7 , A_1^6 , A_1^5 , $A_1^{4'}$ or $A_1^{4''}$. The first four correspond to non- \mathbb{R} -rational del Pezzo surfaces. So, we may assume that the complex involution on Y is of type A_1^4 , and hence both Y and $X[\beta]$ are not \mathbb{R} -rational. As was noticed in Remark 9.1, both classes $A_1^{4'}$ and $A_1^{4''}$ are self-dual under the Bertini twist, hence X is not rational over \mathbb{R} either.

9.2. Embedding into $\text{Cr}_2(\mathbb{R})$ and conjugacy classes

A priori it is not clear that one can choose the coefficients of f_4 and f_6 in Table 9.2 in such a way that the corresponding surfaces are \mathbb{R} -rational. Here is one of the possible approaches to this problem.

Let \tilde{X} denote the blow-up of X at q . By Proposition 2.1, the surface X is \mathbb{R} -rational if and only if $\tilde{X}(\mathbb{R})$ is connected. The surface \tilde{X} is an elliptic fibration over $\mathbb{P}^1_{\mathbb{R}}$ with a real section (coming from the exceptional divisor of the blow-up). As shown in [38, § 5] the set $\tilde{X}(\mathbb{R})$ is connected if⁽¹²⁾ $\text{Eu}(\tilde{X}(\mathbb{R})) < 0$. Now $\text{Eu}(\tilde{X}(\mathbb{R}))$ is the sum of Euler characteristics of singular fibers.

Recall that every geometrically singular member of $|-K_X|$ is an irreducible curve of arithmetic genus 1. Therefore, each singular fiber of the fibration $\tilde{X} \rightarrow \mathbb{P}^1_{\mathbb{R}}$ is a rational curve with a unique singularity which is either a node or a simple cusp. From the point of view of Euler characteristic only the nodes do matter: we have contributions $+1$ from each acnode (a singularity which is equivalent to the singularity $y^2 = x^3 - x^2$ over \mathbb{R}) and -1 for each crunode (those which are equivalent to $y^2 = x^3 + x^2$), see Figure 9.1.

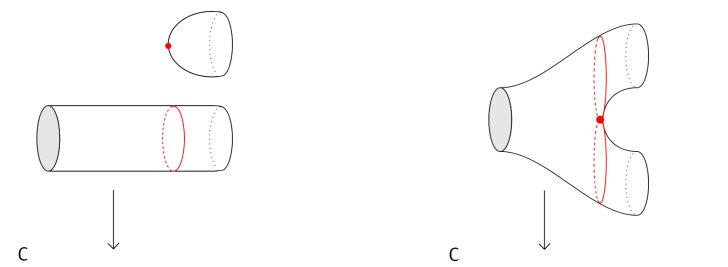


Figure 9.1. Transition between fibers with 1 component and those with 2 components through an acnode (left) and a crunode (right).

Now an easy calculation shows that the coefficients of the binary forms $f_4(x, y)$ and $f_6(x, y)$ from Table 9.2 can be chosen in such a way that the number of crunodal curves in (9.1) is greater than acnodal ones, so $\tilde{X}(\mathbb{R}) < 0$.

⁽¹²⁾This condition is sufficient but not necessary. In fact $\tilde{X}(\mathbb{R})$ is non-connected if $\text{Eu}(\tilde{X}(\mathbb{R})) > 0$, but the case $\text{Eu}(\tilde{X}(\mathbb{R})) = 0$ is more subtle and we will not discuss it here.

So, all the groups G from Table 9.2 do embed into $\mathrm{Cr}_2(\mathbb{R})$. Also note that any del Pezzo surfaces of degree 1 is G -superrigid (see [13, Corollary 7.11] and [17, Theorem 2.6]). In particular, none of the groups listed in Proposition 9.2 is linearizable.

Appendix A. Simple groups and p -groups acting on real rational surfaces

In this Appendix we show that classification of subgroups of some *particular types* in $\mathrm{Cr}_2(\mathbb{R})$ can be much simpler than the analogous question in complex settings. Also, these results can be obtained directly, i.e. avoiding the complete classification.

In this section X denotes a real smooth *geometrically rational* (not necessarily \mathbb{R} -rational) surface. Let $G \subset \mathrm{Bir}(X)$ be a finite group. Then, applying G -equivariant minimal model program to X , we can assume that X is either a real del Pezzo surface with $\mathrm{Pic}(X)^G \cong \mathbb{Z}$, or a real surface with G -equivariant conic bundle structure and $\mathrm{Pic}(X)^G \cong \mathbb{Z}^2$ [14, Theorem 5].

Our goal is to classify simple groups acting on real geometrically rational surfaces. Let us first recall the situation in the case $\mathbb{k} = \mathbb{C}$. As a by-product result of [13] one has the next theorem.

THEOREM A.1. — *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a finite non-abelian simple group. Then G is isomorphic to one of the following groups:*

$$\mathfrak{A}_5, \quad \mathfrak{A}_6, \quad \mathrm{PSL}_2(\mathbb{F}_7).$$

More precisely, we have the following characterization of these groups.

- *There are 2 conjugacy classes of subgroups isomorphic to $\mathrm{PSL}_2(\mathbb{F}_7)$. First, $\mathrm{PSL}_2(\mathbb{F}_7)$ embeds into $\mathrm{PGL}_3(\mathbb{C})$ and preserves the Klein quartic $x^3y + y^3z + z^3x = 0$. Second, it embeds as a group of automorphisms of the double cover of $\mathbb{P}_{\mathbb{C}}^2$, ramified along that Klein quartic (i.e. a del Pezzo surface of degree 2).*
- *There are 3 embeddings of \mathfrak{A}_5 into $\mathrm{Cr}_2(\mathbb{C})$, up to conjugacy. The first is in $\mathrm{PGL}_2(\mathbb{C})$, the second is in $\mathrm{PGL}_3(\mathbb{C})$, and the third is in the group of automorphisms of a del Pezzo surface of degree 5.*
- *Up to conjugacy, there is a unique copy of the Valentiner group \mathfrak{A}_6 , acting linearly on $\mathbb{P}_{\mathbb{C}}^2$ and preserving the sextic curve*

$$10x^3y^3 + 9x^5z + 9y^5z + 27z^6 - 45x^2y^2z^2 - 135xyz^4 = 0.$$

Remark A.2. — Although a complete classification of finite subgroups in $\mathrm{Cr}_3(\mathbb{C})$ seems to be out of reach, Yu. Prokhorov managed to find all finite

simple non-abelian subgroups of $\text{Cr}_3(\mathbb{C})$. Besides \mathfrak{A}_5 , \mathfrak{A}_6 and $\text{PSL}_2(\mathbb{F}_7)$, we have three new simple groups

$$\text{SL}_2(\mathbb{F}_8), \quad \mathfrak{A}_7, \quad \text{PSp}_4(\mathbb{F}_3).$$

in $\text{Cr}_3(\mathbb{C})$, see [24] for details.

By contrast, over \mathbb{R} the following holds.

THEOREM A.3. — *Let X be a real geometrically rational surface with $X(\mathbb{R}) \neq \emptyset$ acted by a simple non-abelian group G . Then $G \cong \mathfrak{A}_5$. It has 3 embeddings into $\text{Cr}_2(\mathbb{R})$ up to conjugacy. One in $\text{PGL}_3(\mathbb{R})$, one in $\text{Aut}(Q_{3,1}) \cong \text{PO}(3,1)$, and one in the group of automorphisms of a del Pezzo surface of degree 5 obtained by blowing up $\mathbb{P}_{\mathbb{R}}^2$ at the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, $[1 : 1 : 1]$.*

Proof. — Assume first that G is minimally regularized on a real conic bundle $\pi : X \rightarrow \mathbb{P}_{\mathbb{R}}^1$. Since $X(\mathbb{R}) \neq \emptyset$ one has $C(\mathbb{R}) \neq \emptyset$, so $C \cong \mathbb{P}_{\mathbb{R}}^1$. The homomorphism $G \rightarrow \text{Aut}(\mathbb{P}_{\mathbb{R}}^1) \cong \text{PGL}_2(\mathbb{R})$ is either injective or trivial. In both cases G embeds into automorphism group of some $\mathbb{P}_{\mathbb{R}}^1$ (which is either a base or a fiber), hence must be cyclic or dihedral by Lemma 2.5 (1).

Now assume that G is minimally regularized on a real del Pezzo surface X of degree $d = K_X^2 \neq 7$. We consider each d separately.

$d = 9$: Then X is a Severi–Brauer variety. As $X(\mathbb{R}) \neq \emptyset$, we have $X \cong \mathbb{P}_{\mathbb{R}}^2$ and $\text{Aut}(X) \cong \text{PGL}_3(\mathbb{R})$, so $G \cong \mathfrak{A}_5$ by Proposition 2.5 (2). This is where the Valentiner group \mathfrak{A}_6 is excluded (it does not embed into $\text{PGL}_3(\mathbb{R})$).

$d = 8$: The surface $\mathbb{P}_{\mathbb{R}}^2(1,0)$ is never G -minimal. If $X \cong \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$, we argue as in the conic bundle case. If $X \cong Q_{3,1} = \{x^2 + y^2 + z^2 = w^2\}$, then $G \cong \mathfrak{A}_5$ realized as the automorphism group of an icosahedron inscribed in the sphere

$$\left(\frac{x}{w}\right)^2 + \left(\frac{y}{w}\right)^2 + \left(\frac{z}{w}\right)^2 = 1.$$

The action is minimal since $\text{Pic}(Q_{3,1}) \cong \mathbb{Z}$.

$d = 6$: Then G is a subgroup of $\text{Aut}(X_{\mathbb{C}}) \cong (\mathbb{C}^*)^2 \rtimes D_6$, so it maps isomorphically to a subgroup of D_6 . So, this case does not occur.

$d = 5$: Then G is a subgroup of $\text{Aut}(X_{\mathbb{C}}) \cong \mathfrak{S}_5$, so $G \cong \mathfrak{A}_5$. The action of this group can be defined over \mathbb{R} and is always minimal, since G contains a minimal element of order 5, see [39, 4.6] or Section 5

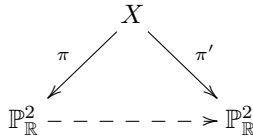
$d = 4$: Then $X = Q_1 \cap Q_2 \subset \mathbb{P}_{\mathbb{R}}^4$ is an intersection of two quadrics and G acts on the pencil $\mathcal{Q} = \langle Q_1, Q_2 \rangle \cong \mathbb{P}_{\mathbb{R}}^1$. Thus G acts trivially on \mathcal{Q}

and fixes the vertices of its singular members. But these vertices generate $\mathbb{P}_{\mathbb{C}}^4$, hence G is abelian, a contradiction.

$d = 3$: All possible groups of automorphisms of complex cubic surfaces are well-known, see [11, Table 9.6] or Section 7. The only group to consider is $G \cong \mathfrak{A}_5$. This group acts faithfully on $H^0(X, -K_X) \cong \mathbb{R}^4$. It is known that there exists only one real 4-dimensional irreducible representation⁽¹³⁾ of \mathfrak{A}_5 . Thus there exists a unique \mathfrak{A}_5 -invariant cubic surface in $\mathbb{P}H^0(X, -K_X)$; we may assume that it is given by

$$x_0 + x_1 + x_2 + x_3 + x_4 = x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$$

in $\text{Proj } \mathbb{R}[x_0, x_1, x_2, x_3, x_4]$ and the set S of (-1) -curves on X consists of 27 real lines (real forms of the Clebsch cubic were described in Section 7.1). Moreover $S = S_6 \sqcup S'_6 \sqcup S_{15}$, $|S_k| = k$, where the lines inside both S_6 and S'_6 are disjoint. Further, there exists a commutative diagram



such that π (resp. π') is a birational \mathfrak{A}_5 -morphism contracting S_6 (resp. S'_6) to the unique \mathfrak{A}_5 -orbit of length 6 in $\mathbb{P}_{\mathbb{R}}^2$. It follows that $\text{rk Pic}(X)^G = 2$, so X is not strongly G -minimal (in fact, it is not G -minimal either, as the conic bundle structures on a cubic surface are given by projecting away from a line).

$d = 2$: Then G embeds into $\text{Aut}(B) \subset \text{PGL}_3(\mathbb{R})$, where B is a smooth quartic curve. By Lemma 2.5 (2), we need to consider only $G \cong \mathfrak{A}_5$. But, as is well known, a genus 3 curve has no automorphisms of order 5, so this case does not occur.

$d = 1$: Note that any group $G \subset \text{Aut}(X)$ fixes a unique base point $p \in X(\mathbb{R})$ of an elliptic pencil $| -K_X |$. Thus we have a faithful representation $G \rightarrow \text{GL}(T_p X) \cong \text{GL}_2(\mathbb{R})$, and G cannot be simple by Lemma 2.5 (1).

⁽¹³⁾ Namely, the number of real irreducible representations of a finite group G equals to the number of equivalence classes under real conjugacy. By definition, two elements are equivalent if they are either in the same conjugacy class or if the inverse of one element is in the conjugacy class of the other. It is known that for \mathfrak{A}_5 (as a particular case of an ambivalent group) the number of such equivalence classes equals to the number of usual conjugacy classes, i.e. the number of irreducible complex representations (and all these representations can be defined over \mathbb{R}), see [18, Chapter 16].

Finally, let us remark that all three conjugacy classes of \mathfrak{A}_5 in $\mathrm{Cr}_2(\mathbb{R})$ (corresponding to $d = 5, 8, 9$) are different, since they are different in $\mathrm{Cr}_2(\mathbb{C})$, see e.g. [9, Theorem B2]. \square

We can generalize Theorem A.3 a little bit. Recall that a group G is called *quasisimple* if $[G, G] = G$ and $G/Z(G)$ is a simple group. It was proved in [26] that every finite quasisimple non-simple subgroup of $G \subset \mathrm{Cr}_2(\mathbb{C})$ is isomorphic to $2 \bullet \mathfrak{A}_5 \cong \mathrm{SL}_2(\mathbb{F}_5)$ and the embedding $G \subset \mathrm{Cr}_2(\mathbb{C})$ is induced by action either on \mathbb{P}^2 , or on a conic bundle. In contrast with this situation, we have

PROPOSITION A.4. — *Every quasisimple subgroup of $\mathrm{Cr}_2(\mathbb{R})$ is simple (and is described in Theorem A.3).*

Proof. — Let $G \subset \mathrm{Cr}_2(\mathbb{R})$ be a finite quasisimple non-simple group. As usual, we assume that G acts biregularly on some \mathbb{R} -rational surface X . The simple group $H = G/Z(G)$ acts on $Y = X/Z(G)$. The surface Y is clearly unirational over \mathbb{C} , hence is \mathbb{C} -rational by Castelnuovo's theorem. Thus $H \cong \mathfrak{A}_5$ by Theorem A.3. Same group-theoretic arguments as in [26, Proposition 2.1] imply that $Z(G) \cong \mathbb{Z}/2$ and G is the binary icosahedral group $2 \bullet \mathfrak{A}_5$.

Suppose that X is a G -equivariant conic bundle over $B \cong \mathbb{P}_{\mathbb{R}}^1$. The kernel of the homomorphism $G \rightarrow \mathrm{Aut}(B)$ coincides with $Z(G) = \mathbb{Z}/2$, as this is the only proper normal subgroup of $2 \bullet \mathfrak{A}_5$. Thus \mathfrak{A}_5 acts faithfully on the general fiber, which is impossible.

Now let X be a del Pezzo G -surface with $\mathrm{rk} \mathrm{Pic}(X)^G = 1$. We then argue as in the proof of Theorem A.3. Note that the image of every nontrivial homomorphism from $2 \bullet \mathfrak{A}_5$ either contains \mathfrak{A}_5 , or coincides with the whole group. This observation helps us to exclude all the cases⁽¹⁴⁾ except $d = 3$. It remains to notice that $2 \bullet \mathfrak{A}_5$ does not act on any cubic surface [11, Table 9.6]. \square

A.1. p -subgroups in $\mathrm{Cr}_2(\mathbb{R})$

Recall that a p -group is a finite group of order p^k , where p is a prime. From the group-theoretic point of view, these groups are somewhat opposite

⁽¹⁴⁾It follows from [21] that $\mathrm{PO}(3, 1)$ does not contain $2 \bullet \mathfrak{A}_5$. Alternative proof: assume that $G = 2 \bullet \mathfrak{A}_5 \subset \mathrm{Aut}(Q_{3,1})$. As G has no index 2 subgroups, it faithfully acts by orientation-preserving diffeomorphisms of \mathbb{S}^2 , and hence embeds into $\mathrm{SO}_3(\mathbb{R})$, see Remark 3.1. But this is impossible by Lemma 2.5.

to simple non-abelian groups. It follows from [39] that for $p \geq 3$, every p -subgroup $G \subset \text{Cr}_2(\mathbb{R})$ is conjugate either to a direct product of at most two cyclic groups, regularized on $X = \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$ with $\text{rk Pic}(X)^G = 2$, or to a cyclic subgroup of $\text{PGL}_3(\mathbb{R})$, or to $(\mathbb{Z}/3^k \times \mathbb{Z}/3^l) \rtimes (\mathbb{Z}/3)$ acting on a del Pezzo surface of degree 6, or to $\mathbb{Z}/5$ acting on a del Pezzo surface of degree 5 (with invariant Picard numbers equal to one). As the reader can see from present paper, the classification of 2-subgroups of $\text{Cr}_2(\mathbb{R})$ is much more extensive. We leave it to the interested reader to extract this classification for del Pezzo surfaces.

Instead we give a bound on the number of generators of an abelian p -subgroup $G \subset \text{Cr}_2(\mathbb{R})$ in the spirit of [1] (where it was done for $\mathbb{k} = \mathbb{C}$; note that a priori Beauville’s bound might fail to be sharp over \mathbb{R}).

PROPOSITION A.5. — *Let $G \subset \text{Cr}_2(\mathbb{R})$ be an abelian p -subgroup. Then it is generated by at most r elements, where*

$$r \leq \begin{cases} 4 & \text{if } p = 2, \\ 3 & \text{if } p = 3, \\ 2 & \text{if } p \geq 5. \end{cases}$$

If G is elementary, then $r \leq 2$ for $p = 3$. For any p , these bounds are attained by some abelian p -subgroups $G \subset \text{Cr}_2(\mathbb{R})$.

Proof. — If G is minimally regularized on a real conic bundle $X \rightarrow B$, then G fits into the short exact sequence

$$(A.1) \quad 1 \rightarrow G_F \rightarrow G \rightarrow G_B \rightarrow 1,$$

where $G_B \subset \text{Aut}(B) \cong \text{PGL}_2(\mathbb{R})$, and G_F acts by automorphisms of the generic fiber F . Since G is finite, G_F is a subgroup of $\text{PGL}_2(\mathbb{R})$. So, both G_F and G_B are cyclic or dihedral, and hence G is generated by at most 4 elements. Note that if $G \cong (\mathbb{Z}/p)^r$ and $p > 2$, then $r = 1$ or 2.

The remaining cases directly follow from the results of this paper. Note that for $p = 2$ the value $r = 4$ is achieved for a del Pezzo quartic surface isomorphic to $\mathbb{P}_{\mathbb{R}}^2(5, 0)$ or $Q_{2,2}(0, 2)$. The bound $r = 3$ for $p = 3$ is attained on a del Pezzo surface of degree 6 isomorphic to $Q_{2,2}(0, 1)$ (so G is a group of type 2b). □

Appendix B. Non-solvable groups

Another interesting class of subgroups of the Cremona group is non-solvable groups. As was shown in [13] and [35] the plane Cremona group

(over \mathbb{C}) already contains eight sporadic insoluble subgroups

$$\mathfrak{S}_5, \text{PSL}_2(\mathbb{F}_7), \text{PSL}_2(\mathbb{F}_7) \times \mathbb{Z}/2, \\ \mathfrak{A}_6, \mathfrak{A}_5 \times \mathfrak{A}_4, \mathfrak{A}_5 \times \mathfrak{S}_4, \mathfrak{A}_5 \times \mathfrak{A}_5, (\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2,$$

and four infinite series $\mathfrak{A}_5 \times \mathbb{Z}/n, \mathfrak{A}_5 \times D_n, \text{SL}_2(\mathbb{F}_5) \times \mathbb{Z}/n$ and $\text{SL}_2(\mathbb{F}_5) \times D_n$. By contrast, the following holds over \mathbb{R} .

THEOREM B.1. — *Let X be a real geometrically rational surface with $X(\mathbb{R}) \neq \emptyset$, and G be a finite non-solvable group acting on it. Then the pair (X, G) is isomorphic to one (and only one) of the following pairs*

- $(\mathbb{P}_{\mathbb{R}}^2, \mathfrak{A}_5)$;
- $(Q_{3,1}, \mathfrak{A}_5)$ or $(Q_{3,1}, \mathfrak{A}_5 \times \mathbb{Z}/2)$;
- $(\mathbb{P}_{\mathbb{R}}^2(4, 0), \mathfrak{A}_5)$ or $(\mathbb{P}_{\mathbb{R}}^2(4, 0), \mathfrak{S}_5)$;
- (Y, \mathfrak{S}_5) , where Y is the Clebsch diagonal cubic.

Proof. — If G is minimally regularized on a conic bundle, then we again have the short exact sequence (A.1) with both G_F and G_B cyclic or dihedral. Thus G is solvable.

So, we may assume that G acts on a real del Pezzo surface X of degree d with $\text{Pic}(X)^G \cong \mathbb{Z}$. If $d = 9$, then $G \cong \mathfrak{A}_5$. If $d = 8$ and $X \cong Q_{2,2}$, then $G \cong H_{\bullet}(\mathbb{Z}/2)^r$, where $r \in \{0, 1\}$ and H is a subgroup of $H_1 \times H_2$ with H_1 and H_2 being cyclic or dihedral. Clearly G is solvable in this case. If $d = 8$ and $X \cong Q_{3,1}$ we have $G \subset \text{PO}(3, 1)$, so $G \cong \mathfrak{A}_5$ or $\mathfrak{A}_5 \times \mathbb{Z}/2$, see [21]. When $d = 6$, Proposition 4.1 tells us that $G \cong H_{\bullet}N$, where H is an abelian group and N is a group of order at most 6, so G is solvable. For $d = 5$ we have either $G \cong \mathfrak{A}_5$, or $G \cong \mathfrak{S}_5$ by Proposition 5.2. Let $d = 4$. Then G is a subgroup of $\mathscr{W}(D_5) \cong (\mathbb{Z}/2)^4 \rtimes \mathfrak{S}_5$, so $G \cong A_{\bullet}H$, where A is an abelian group, and $H \subset \mathfrak{S}_5$. In fact it is known that $|H| < 10$ [11, Theorem 8.6.8], so G is solvable.

Let $d = 3$. Then [11, Table 9.6] shows that $G \cong \mathfrak{A}_5$ or \mathfrak{S}_5 . Moreover, we already know (see the proof of Theorem A.3) that X is the Clebsch cubic and it is not \mathfrak{A}_5 -minimal. Further, in the notation of that proof let $\ell_i, i = 1, \dots, 6$, be the elements of S_6 (note that all the lines on the Clebsch cubic are real). It is known that the divisor classes of ℓ_i and K_X span $\text{Pic}(X) \otimes \mathbb{R}$, so $\text{Pic}(X)^{\mathfrak{A}_5} \otimes \mathbb{R}$ is spanned by K_X and the sum $\sum \ell_i$. Since \mathfrak{S}_5 does not leave this sum invariant, the group \mathfrak{S}_5 acts minimally on X .

For $d = 2$ we have $G \cong (\mathbb{Z}/2)^r \times H$, where $r \in \{0, 1\}$, and H is either cyclic, or dihedral, or has order < 60 . So, G is solvable. When $d = 1$ the group G embeds into $\text{GL}(T_q X) \cong \text{GL}_2(\mathbb{R})$ (see Section 9), so it is solvable.

Finally as we noted in Theorem A.3 the pairs $(\mathbb{P}_{\mathbb{R}}^2, \mathfrak{A}_5)$, $(Q_{3,1}, \mathfrak{A}_5)$ and $(\mathbb{P}_{\mathbb{R}}^2(4, 0), \mathfrak{A}_5)$ are pairwise non-isomorphic. The pair $(\mathbb{P}_{\mathbb{R}}^2(4, 0), \mathfrak{S}_5)$ is known to be superrigid, see [13, 8.1]. \square

Appendix C. Real invariants of some finite groups

In this appendix we collect some results concerning invariant theory of finite groups over \mathbb{R} . They should be known to experts, but we decided to include them because we do not know proper references.

Let V be a real m -dimensional vector space and x_1, \dots, x_m be a standard dual basis of V^* . Let $\rho : G \rightarrow \text{GL}(V)$ be a faithful linear representation of a finite group G and $\eta : G \rightarrow \text{GL}(V \otimes \mathbb{C})$ be some faithful complex representation equivalent to ρ , i.e. $\rho(g) = T \circ \eta(g) \circ T^{-1}$ for each $g \in G$ and some $T \in \text{GL}_m(\mathbb{C})$.

Recall that every finite subgroups of $\text{GL}_2(\mathbb{R})$ is either cyclic $\mathbb{Z}/n \cong \langle R_n \rangle$ or dihedral

$$D_n \cong \langle R_n, S \mid R_n^n = S^2 = 1, SR_nS^{-1} = R_n^{-1} \rangle.$$

In the sequel by *standard representations* of \mathbb{Z}/n and D_n we mean

$$\rho : R_n \mapsto \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}, \quad S \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In order to construct G -invariant del Pezzo surfaces in Sections 8 and 9 we need to know G -invariant binary forms $f_k(x, y)$ of degrees $k = 2, 4$ and 6 . They are listed below for different groups (in their standard representation).

Cyclic groups

Let $G = \mathbb{Z}/n$. We claim that

$$(C.1) \quad \mathbb{R}[x, y]^{\rho(\mathbb{Z}/n)} = \mathbb{R}[x^2 + y^2, \text{Re}(x + iy)^n, \text{Im}(x + iy)^n]$$

Denote by ω a primitive n th root of unity. The representation ρ is equivalent to $\eta : R_n \mapsto \text{diag}\{\omega, \bar{\omega}\}$ via the map $T : x \mapsto z = x + iy, y \mapsto w = x - iy$. For each $g \in G$ we have $\eta(g)(Tf) = T\rho(g)T^{-1}Tf = Tf$, so $Tf \in \mathbb{C}[x, y]^{\eta(\mathbb{Z}/n)}$. It is well known that $\mathbb{C}[z, w]^{\eta(\mathbb{Z}/n)} = \mathbb{C}[z^n, zw, w^n]$, so

$$Tf = \sum c_{jkl} z^{nj} (zw)^k w^{nl}, \quad \text{and} \quad f = \sum c_{jkl} (x + iy)^{nj} (x^2 + y^2)^k (x - iy)^{nl}.$$

Separating the real part, we get the list of basic invariants. Below we use (C.1) mostly as a starting point for finding a nicer list of generators.

$$\boxed{\mathbb{Z}/2} \quad \mathbb{R}[x, y]^{\mathbb{Z}/2} = \mathbb{R}[x^2 + y^2, x^2 - y^2, 2xy] = \mathbb{R}[x^2, xy, y^2]$$

$f_{2k}(x, y)$ is invariant for all $k \geq 1$.

$$\boxed{\mathbb{Z}/4} \quad \mathbb{R}[x, y]^{\mathbb{Z}/4} = \mathbb{R}[x^2 + y^2, x^4 - 6x^2y^2 + y^4, 4x^3y - 4xy^3] = \mathbb{R}[x^2 + y^2, x^2y^2, x^3y - xy^3]$$

$$f_2(x, y) : a(x^2 + y^2);$$

$$f_4(x, y) : ax^4 + bx^2y^2 + ay^4 + cxy(x^2 - y^2);$$

$$f_6(x, y) : (x^2 + y^2)(ax^4 + dx^3y + bx^2y^2 - dxy^3 + ay^4)$$

$$\boxed{\mathbb{Z}/8} \quad \mathbb{R}[x, y]^{\mathbb{Z}/8} = \mathbb{R}[x^2 + y^2, x^8 - 28x^6y^2 + 70x^4y^4 - 28x^2y^6 + y^8, 8x^7y - 56x^5y^3 + 56x^3y^5 - 8xy^7] = \mathbb{R}[x^2 + y^2, xy(x^2 - y^2)(x^4 - 6x^2y^2 + y^4), x^2y^2(x^2 - y^2)^2]$$

$$f_2(x, y) : a(x^2 + y^2);$$

$$f_4(x, y) : a(x^2 + y^2)^2;$$

$$f_6(x, y) : a(x^2 + y^2)^3.$$

Dihedral groups

One has

$$\mathbb{R}[x, y]^{\rho(D_n)} = \mathbb{R}[x^2 + y^2, \operatorname{Re}(x + iy)^n].$$

$$\boxed{D_2} \quad \mathbb{R}[x, y]^{D_2} = \mathbb{R}[x^2 + y^2, x^2 - y^2] = \mathbb{R}[x^2, y^2]$$

$$f_2(x, y) : ax^2 + by^2;$$

$$f_4(x, y) : ax^4 + bx^2y^2 + cy^4;$$

$$f_6(x, y) : ax^6 + bx^4y^2 + cx^2y^4 + dy^6.$$

$$\boxed{D_4} \quad \mathbb{R}[x, y]^{D_4} = \mathbb{R}[x^2 + y^2, x^4 - 6x^2y^2 + y^4] = \mathbb{R}[x^2 + y^2, x^2y^2]$$

$$f_2(x, y) : a(x^2 + y^2);$$

$$f_4(x, y) : ax^4 + bx^2y^2 + ay^4;$$

$$f_6(x, y) : (x^2 + y^2)(ax^4 + bx^2y^2 + ay^4).$$

$$\boxed{D_8} \quad \mathbb{R}[x, y]^{D_8} = \mathbb{R}[x^2 + y^2, x^8 - 28x^6y^2 + 70x^4y^4 - 28x^2y^6 + y^8] = \mathbb{R}[x^2 + y^2, x^2y^2(x^2 - y^2)^2]$$

$$f_2(x, y) : a(x^2 + y^2);$$

$$f_4(x, y) : a(x^2 + y^2)^2;$$

$$f_6(x, y) : a(x^2 + y^2)^3.$$

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