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JOHN C. TAYLOR

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THE MARTIN BOUNDARIES OF EQUIVALENT SHEAVES

by J. C. TAYLOR

Introduction.

Let X be a locally compact space on which a sheaf $\underline{\underline{H}}$ of vector spaces of continuous real-valued functions is defined satisfying the basic axioms of Brelot [3](1). In addition, assume that the following conditions hold : X has a countable base ; there is a positive potential defined by $\underline{\underline{H}}$; and $\underline{\underline{H}}$ satisfies the hypothesis of proportionality, that is for $y \in X$ any two potentials with support $\{y\}$ are proportional.

Then, following the original construction of R.S. Martin [13], it is possible to define a Martin compactification of X. A priori this compactification depends on $\underline{\underline{H}}$. The purpose of this article is to initiate a study of the dependence.

The question is not an empty one as the following examples show. Let B be the closed unit ball in $\mathbf{R}^3$ and set $X = B \backslash (S \cup L)$, where S is the unit sphere and L is the closed line segment joining $(0, 0, 0)$ to $(0, 0, 1)$. Then, if $\underline{\underline{H}}$ is the sheaf defined by Laplace's equation $\Delta h = 0$, the Martin compactification of X is B. This follows from the fact that the Martin compactification of $B \backslash S$ is B and that L is a closed set of capacity zero.

Now let $Y = B \backslash (S \cup C)$, where C is the closed convex cone defined by $x^2 + y^2 \leqslant z^2$ and $z \geqslant 0$. Then from results of de la Vallée Poussin [17] it follows that the Martin compactification for Y associated with Laplace's equation is the closure $\overline{Y}$ of Y in $\mathbf{R}^3$. Following a suggestion of Choquet, the differential equation $\Delta h = 0$

(1) Throughout this article it will be assumed that X satisfies the customary connectivity conditions.

on Y can be transported to X by means of a diffeomorphism. This defines an elliptic operator L on X whose associated Martin compactification is clearly homeomorphic to $\overline{Y}$. The Martin boundary in this case is homeomorphic to S, which is not homeomorphic to $S \cup L$. Hence, these two Martin compactifications of X are distinct.

The principal result of this article is the result (theorem 2) that if two sheaves $\underline{\underline{H}}_1$ and $\underline{\underline{H}}_2$ are equivalent, that is they agree on the complement of a compact subset of X [11], then the Martin compactifications of X coincide. In this coincidence, the corresponding sets of minimal points coïncide (theorem 2). This has as a consequence the result that the cones $\underline{\underline{S}}_2^+$ and $\underline{\underline{S}}_2^+$, equipped with the T-topology, are isomorphic.

The last part of the article discusses the relation between the Martin compactification of X\A, A compact, and $\overline{X}$\A where $\overline{X}$ is the Martin compactification of X (corollary 3 to theorem 5). Further, it is shown that the Martin compactification is of type S (corollary 4 to theorem 5) and that the ends of X are related to direct decomposition of the cone of positive harmonic functions.

I wish to thank M. Sieveking for a very useful discussion in the course of which we obtained the proof of theorem 1.

I would also like to thank Professor M. Brelot for his continued interest in this work, and for his persistent belief that a proof for theorem 2 could be found without the use of adjoint harmonic functions [16].

2. Elementary properties of Q-compactification.

Let X be a locally compact space and let $(K_a)_{a \in A}$ be a family of continuous functions $K_a : X \longrightarrow \overline{R}$. Then, as is well known (c.f. [6]), there is a unique compactification $\overline{X}$ of X such that (1) each function K_a extends continuously to $\overline{X}$ and (2) the extended functions separate the points of $\Delta = \overline{X} \setminus X$ (2). The space $\overline{X}$ can be realized as the closure of the image of X under the embedding of X in $\Pi\{R_f \mid f \in \underline{\underline{C}}_K(X) + A\}$ by the mapping e, which is defined as

(2) "Unique" means that if $\overline{X}_1$ and $\overline{X}_2$ are any two compactifications satisfying (1) and (2), there is a homeomorphism $\varphi : \overline{X}_1 \longrightarrow \overline{X}_2$ with $\varphi(x) = x$ for each $x \in X$.

follows : $(pr_f \circ e)(x)$ equals $f(x)$ if $f \in \underline{\underline{C}}_K(X)$ and equals $K_\alpha(x)$ if $f = \alpha \in A$ (3).

The following formal properties of Q-compactifications are easily verified.

1) Let $\varphi : X \longrightarrow Y$, Y locally compact, be proper and let $(K_\alpha)_{\alpha \in A}$ be a family of continuous functions on X, $(L_\beta)_{\beta \in B}$ a family of continuous functions on Y. Denote by $\overline{X}$ and $\overline{Y}$ the corresponding compactifications. If for each $\beta \in B$ there is an $\alpha \in A$ with $L_\beta \circ \varphi = K_\alpha$, then there is a unique continuous map $\overline{\varphi} : \overline{X} \longrightarrow \overline{Y}$ with $\overline{\varphi}(x) = \varphi(x)$, for all $x \in X$.

2) Let $(K_\alpha)_{\alpha \in A}$ be a given family of continuous functions on X and denote by $(K'_\beta)_{\beta \in B}$ a second family such that each K'_β extends continuously to $\overline{X}$. Define $(K''_\gamma)_{\gamma \in A+B}$ by setting $K''_\gamma = K_\alpha$ if $\gamma = \alpha \in A$ and $K''_\gamma = K'_\beta$ if $\gamma = \beta \in B$. Let $\overline{X}''$ be the compactification determined by (K''_γ). Then $\overline{X}'' = \overline{X}'$.

3) Let φ, X, Y, $(K_\alpha)_{\alpha \in A}$ and $(L_\beta)_{\beta \in B}$ be as in 1. Assume that for each $\beta \in B$, $L_\beta \circ \varphi = K'_\beta$ extends continuously to $\overline{X}$. Then there is a unique continuous map $\overline{\varphi} : \overline{X} \longrightarrow \overline{Y}$ which extends φ.

4) Let $(K'_\alpha)_{\alpha \in A}$ and $(K''_\alpha)_{\alpha \in A}$ be two families of continuous functions such that, for each α, there is a compact set D_α with $K'_\alpha(x) = K''_\alpha(x)$ if $x \in X \backslash D_\alpha$. Let $\overline{X}'$ and $\overline{X}''$ be the corresponding compactifications of X. Then $\overline{X}' = \overline{X}''$.

These elementary properties established, it is easy to prove the following propositions.

PROPOSITION 1. — *Let* X *be a locally compact space and* $(K_\alpha)_{\alpha \in A}$ *a family of continuous functions* $K_\alpha : X \backslash D_\alpha \longrightarrow \overline{\mathbf{R}}$, *where* D_α *is a compact subset of* X. *Then there is a unique compactification* $\overline{X}$ *of* X *such that* :

1) *each* K_α *extends continuously to* $\overline{X} \backslash D_\alpha$; *and*

2) *the extended functions separate the points of* $\overline{X} \backslash X$.

Proof. — It is an immediate consequence of 4), since for each K_α there is a continuous function K'_α which agrees with K_α on the complement of a compact neighbourhood of D_α.

(3) If A, B are sets, then A + B denotes their disjoint sum.

PROPOSITION 2. – *Let* X, $(K_\alpha)_{\alpha\in A}$ *be as in proposition 1 and let* Y *be a locally compact space with a family* $(L_\beta)_{\beta\in B}$ *of functions* $L_\beta : Y\backslash E_\beta \longrightarrow \overline{R}$, E_β *compact in* Y.

If φ *is proper and such that each* $L_\beta \circ \varphi$ *extends continuously to* $\overline{X}\backslash \varphi^{-1}(E_\beta)$, *then there is a unique continuous extension* $\overline{\varphi}$: $\overline{X} \longrightarrow \overline{Y}$ of φ.

Proof. – It is a consequence of 3) and 4).

If $\overline{X}$ is the compactification of X determined by $(K_\alpha)_{\alpha\in A}$ let $\overline{K}_\alpha$ denote the extended function. These functions separate the points of $\overline{X}\backslash X$ *strictly* if for $x_1 \neq x_2$, two points of $\overline{X}\backslash X$, there exist α_1, α_2 with $\overline{K}_{\alpha_1}(x_1)\,\overline{K}_{\alpha_2}(x_2) \neq \overline{K}_{\alpha_1}(x_2)\,\overline{K}_{\alpha_2}(x_1)$.

3. A general theorem.

Let X_1, X_2, be locally compact spaces with countable bases and denote by $\underline{\underline{H}}_1$ and $\underline{\underline{H}}_2$ sheaves on the corresponding spaces which satisfy the axioms of Bauer [2] and are such that both harmonic spaces are strict.

Let $K^i : X_i \times X_i \longrightarrow \overline{R}^+$ be a function with the following properties :

1) $y \longrightarrow K^i(x, y) = {}^*K^i_x(y)$ is continuous outside a compact set A_x ;

2) $x \longrightarrow K^i(x, y) = K^i_y(x)$ is superharmonic.

Denote by $\varphi : X_1 \longrightarrow X_2$ a surjective proper map and let $A_2 \subset X_2$, and $A_1 = \varphi^{-1}(A_2)$, be compact sets such that for $y \in X_1 \backslash A_1$:

3) $K^1_y - P_A K^1_y = f(y)\,[K^2_{\varphi(y)} - P_A K^2_{\varphi(y)}] \circ \varphi + d_y(x)$ where $d_y(x)$ tends to zero in x as y tends to the point at infinity and, for any superharmonic function v, $P_E v = R^E_v$.

THEOREM 1. – *Let* $\overline{X}_i$ *be the compactification of* X_i *determined by the family* $({}^*K^i_x)_{x\in X}$. *Assume that the extensions of the functions* ${}^*K^2_x$ *separate the points of* $\overline{X}_2\backslash X_2$ *strictly. There is a unique continuous extension* $\overline{\varphi} : \overline{X}_1 \longrightarrow \overline{X}_2$ *of* φ *if the following holds for* $i = 1,2$:

4) if a net (y_γ) converges to a point $y \in \overline{X}_i \backslash X_i$, then the functions $K^i_{y_\gamma}$ converge pointwise to a harmonic function, the convergence being uniform on A_i, where $K^i_y(x) = K^i(x, y)$.

Proof. – In view of proposition 2, it suffices to show that if the net (y_γ) on X_1 converges to a point $y_1 \in \overline{X}_1 \backslash X_1$, then $\lim_\gamma {}^*K^2_x(\varphi(y_\gamma))$ exists for each $x \in X_2$.

Since y_γ converges to y_1, the superharmonic functions $K^1_{y_\gamma}$ converge pointwise to a harmonic function h_1. The convergence being uniform on A_1, the functions $P_{A_1} K^1_{y_\gamma}$ converge to $P_A h_1$. It follows from 3) that $f(y_\gamma)[K^2_{\varphi(y_\gamma)} - P_{A_2} K^2_{\varphi(y_\gamma)}]$ converges on $X_2 \backslash A_2$. For convenience denote $\varphi(y)$ by z and let $(z_{\gamma'})$ and $(z_{\gamma''})$ be two subnets of the net (z_γ) which converge to points z' and z'' of $\overline{X}_2$, necessarily in $\overline{X}_2 \backslash X_2$ as φ is proper. Denote by h'_2 and h''_2 the corresponding harmonic functions.

Condition 4) implies that $P_{A_2} K^2_{z_{\gamma'}}$ converges to $P_{A_2} h'_2$ and that $P_{A_2} K^2_{z_{\gamma''}}$ converges to $P_{A_2} h''_2$. However, since

$$f(y_\gamma)[K^2_{z_\gamma} - P_{A_2} K^2_{z_\gamma}]$$

converges, it follows that f converges along both subnets and that $\alpha'[h'_2 - P_{A_2} h'_2] = \alpha''[h''_2 - P_{A_2} h''_2]$ where

$$\alpha' = \lim_{\gamma'} f(y_{\gamma'}) \quad \text{and} \quad \alpha'' = \lim_{\gamma''} f(y_{\gamma''}) .$$

Rewriting this $\alpha' h'_2 + \alpha'' P_{A_2} h''_2 = \alpha'' h''_2 + \alpha' P_{A_2} h'_2$, it follows from the Riesz Decomposition theorem that $\alpha' h'_2 = \alpha'' h''_2$ (regularize both sides). Since $h'_2(x) = \lim_{\gamma'} {}^*K^2_x(y_{\gamma'})$ and similarly for $h''_2(x)$, it follows from the strict separation assumption that $h'_2 = h''_2$.

In other words, (z_γ) converges in $\overline{X}_2$, that is, for each

$$x \in X_2, \ \lim_\gamma {}^*K^2_x(\varphi(y_\gamma)) \text{ exists.}$$

Remark. – The assumption of a countable base for X does not enter into the proof. It is made so as to fulfil the hypotheses of the theory of Bauer. The result holds for the theory without this assumption.

4. Application to the Martin Boundary.

Let X be a non compact locally compact space with a countable base, and let $\underline{\underline{H}}$ be a sheaf on X that satisfies the basic axioms of Brelot [3](4). Assume that a positive potential is defined by $\underline{\underline{H}}$ and that $\underline{\underline{H}}$ satisfies the hypothesis of proportionality.

Madame Hervé [10] (Proposition 18.1) proved the existence of a lower semi-continuous function $G : X \times X \longrightarrow \overline{R}^+$, continuous off the diagonal and such that for each

$$y \in X, \quad x \longrightarrow G(x, y) = p_y(x)$$

is a potential with support $\{y\}$. Such a function will be called a Green's function for $\underline{\underline{H}}$. If f is continuous and strictly positive on X define $G f(x, y) = G(x, y) f(y)$. Then Gf is a Green's function and every Green's function has this form.

Let $x_0 \in X$ and define $K(x, y)$ to be 1 if $x = x_0 = y$ and to be $G(x, y)/G(x_0, y)$ otherwise.

The compactification of X defined by $({}^*K_x)_{x \in X}$, where

$${}^*K_x(y) = K(x, y),$$

will be called the *Martin compactification of* X and will be denoted by $M(X, \underline{\underline{H}})$ or $\overline{X}$. It is clearly independent of the choice of Green's function G.

Let Λ be a compact base for the cone $\underline{\underline{S}}^+$ equipped with the T-topology [10]. For $y \in X$ denote by p_y the unique potential in Λ with support $\{y\}$. Gowrisankaran [9] (theorem IV.I) proved that the mapping $y \longrightarrow p_y$ embeds X in Λ. Identifying X with its image let $\widetilde{X}$ denote the closure of X in Λ. It is not hard to see from Scolie 21.1 of [10] that $\widetilde{X}$ is the compactification of X determined by $(p_x^*)_{x \in X}$, where $p_x^*(y) = p_y(x)$.

PROPOSITION 3. – *The compactifications* $\overline{X}$ *and* $\widetilde{X}$ *coincide. Hence,* $\overline{X}$ *is independent of the choice of* x_0.

(4) As was pointed out by C. Constantinescu, the assumption of a countable base is not necessary. However, in order to avoid it it is necessary to establish some lemmas corresponding to results of Madame Hervé [10]. These lemmas are established in the appendix.

Proof. – According to proposition 22.1 of [10], the function $G(x, y) = p_y(x)$ is a Green's kernel for $\underline{\underline{H}}$. Let $x_0 \in X$ and define $K(x, y)$ as above. Then as long as $y \notin \{x, x_0\}$, $p_y(x_0)\, K_y(x) = p_y(x)$ or $p^*_{x_0}(y)\, K^*_x(y) = p^*_x(y)$. Since $y \longrightarrow p^*_{x_0}(y)$ never vanishes on $\overline{X}\backslash X$ (note that $x \longrightarrow p^*_x(y)$ is harmonic on X for each $y \in \overline{X}\backslash X$), it follows that all the functions $^*K_x$ extend continuously to $\widetilde{X}\backslash\{x, x_0\}$ as functions $^*\widetilde{K}_x$.

Suppose that y_1, y_2 are two points in $\widetilde{X}\backslash X$ and that for all $x \in X$, $^*\widetilde{K}_x(y_1) = {}^*\widetilde{K}_x(y_2)$. Denote by p_{y_1} and p_{y_2} the corresponding harmonic functions in Λ. Then $[p_{y_1}(x)/p_{y_1}(x_0)] = [p_{y_2}(x)/p_{y_2}(x_0)]$ for all $x \in X$. Since the functions p_{y_i} are in Λ, it follows that $p_{y_1} = p_{y_2}$ and so $y_1 = y_2$. Consequently, $\widetilde{X}$ and $\overline{X}$ are the same compactification of X in the sense defined in 2.

The general theorem is now applied to prove the following result.

THEOREM 2. – *Let $\underline{\underline{H}}_1$ and $\underline{\underline{H}}_2$ be two sheaves on* X *that satisfy the above hypotheses. Assume there is a compact set* $A \subset X$ *such that the sheaves coincide on* $X\backslash A$. *Then* $M(X, \underline{\underline{H}}_1) = M(X, \underline{\underline{H}}_2)$.

Proof. – Let G^i be a Green's function for $\underline{\underline{H}}_i$ and let

$$p^i_y(x) = G^i(x, y) .$$

Define $q^i_y = p^i_y - P_A\, p^i_y$, $y \in X\backslash A$. Then by theorem 16.4 of [6] q^i_y is a potential on $X\backslash A$ of support $\{y\}$ which is positive only on the connected component of A that contains y. Furthermore, by the same theorem, the hypothesis of proportionality is satisfied on $\complement A$.

Pick $x_0 \in A$ and consider the two functions K^1 and K^2 defined by G^1 and G^2. Since $K^i_y - P_A\, K^i_y = [1/p^i_y(x_0)]\, q^i_y$ there is a continuous function $f(y)$ with $K^1_y - P_A\, K^1_y = f(y)\, [K^2_y - P_A\, K^2_y]$.

Since, in the case of Martin compactifications, the harmonic functions corresponding to the boundary points all take 1 at x_0, it follows that the extensions to $M(X, \underline{\underline{H}}_i)$ of the functions $^*K^i_x$ separate strictly the points of the ideal boundary.

It is well known that condition 4) is satisfied and so the conditions of theorem 1 are satisfied with $\varphi(x) = x$ for all $x \in X$.

Hence, there is a unique continuous map $\overline{\varphi}$:

$$M(X, \underline{\underline{H}}_1) \longrightarrow M(X, \underline{\underline{H}}_2)$$

which extends the identity map on X. The argument being symmetrical, it follows formally from the denseness of X in a compactification and the uniqueness condition that φ is a homeomorphism. In other words, the compactifications coincide.

Remarks. – Since theorem 1 holds in Bauer's theory, it is reasonable to ask for a similar theorem there. A slight modification of Sieveking's definition of a Martin space [15] leads to similar results.

However, the non compactness of $X \cup \Delta$ requires a hypothesis [4′] that ensures if a net (y_γ) converges to a point in Δ^1, then it converges to a point in Δ^2, Δ^i being boundaries for $\underline{\underline{H}}_i$, $i = 1, 2$.

5. The extension of harmonic functions.

Let $\underline{\underline{H}}$ be a sheaf satisfying the axioms of Bauer [2] which is strict. It can be assumed that 1 is superharmonic.

Denote by A, B compact subsets of X and by O a relatively compact open set with $A \subset \overset{\circ}{B} \subset B \subset O$.

If U is open in X denote by H_U the kernel defined by the Dirichlet problem for U, that is, if φ is a continuous function with compact support on X then $H_U(x, \varphi)$ equals $\varphi(x)$ if $x \notin U$ and equals $H_f^U(x)$ where $f = \varphi \mid \partial U$ if $x \in U$ (see [11]).

The open set O can be chosen so that $H_O H_{CB} 1 \leqslant \lambda < 1$ on B since $H_{CB} 1$ coincides with a potential except possibly on ∂B.

Define $T : \underline{\underline{C}}(\partial O) \longrightarrow \underline{\underline{C}}(\partial O)$ by setting $Tf = (H_{CB} H_O f) \mid \partial O$ and define $S : \underline{\underline{C}}(\partial B) \longrightarrow \underline{\underline{C}}(\partial B)$ by setting

$$Sg = (H_O H_{CB} g) \mid \partial B .$$

Then S and T are positive linear operators such that $\|S\| \leqslant \lambda$, $\|T\| \leqslant 1$ and $T^n = H_{CB} S^{n-1} H_O$. Hence $\|T^n\| \leqslant \lambda^{n-1}$. As a result the series

$\sum_{n \geq 0} T^n$ converges to an operator which is the inverse of $(I - T)$. Therefore $(I - T)^{-1}$ exists and is a positive operator.

Using these results, it is easy to prove the following proposition due to Nakai [14] in Brelot's theory.

PROPOSITION 4. – *Let h be a continuous function on $X\backslash A$, harmonic on $X\backslash B$. Then there is a unique harmonic function $\bar{h}$ on X such that $\bar{h} - H_{CB}\bar{h} = h - H_{CB} h$. The function $\bar{h}$ is positive if h is positive.*

Proof. – Let f be the unique continuous function on ∂O with $(I - T) f = (h - H_{CB} h)|\partial O$. Define $\bar{h}$ by setting $\bar{h}(x)$ equal to $H_O(x, f)$ if $x \in O$ and equal to

$$(H_{CB} H_O)(x, f) + h(x) - H_{CB}(x, h) \quad \text{if} \quad x \notin B .$$

Consider the harmonic function

$$h_1 = H_{CB} H_O f + h - H_{CB} h = h + H_{CB}(H_O f - h)$$

and let $O_1 = O\backslash B$. Now the continuity of h implies that $H_O h_1 = h$ and from the comparison theorem it follows that

$$H_{O_1} H_{CB}(H_O f - h) = H_{CB}(H_O f - h) .$$

Hence, $h_1 = H_{O_1} h_1$. On ∂O_1, $h_1 = H_O f$ and so on O_1,

$$h_1 = H_{O_1} H_O f = H_O f .$$

As a result, $\bar{h}$ is a well defined harmonic function on X. Furthermore, $H_{CB}\bar{h} = H_{CB} H_O f$, and so $\bar{h} - H_{CB}\bar{h} = h - H_{CB} h$.

Assume that $\bar{h}$ is a harmonic function on X for which

$$\bar{h} - H_{CB}\bar{h} = h - H_{CB} h .$$

Then $h - H_{CB} h = \bar{h} - H_{CB} H_O \bar{h}$, which coincides on ∂O with $(I - T)\bar{h}$. Consequently, $\bar{h}$ is uniquely defined.

If $h \geq 0$ then $h - H_{CB} h \geq 0$ and as a result $(I - T)\bar{h}$, and consequently $\bar{h}$ are also positive.

It follows from the proposition that $\bar{h}$ is independent of the set O containing B. Let B_1 be compact $B \subset B_1$. The fact that

$$H_{CB_1} H_{CB} = H_{CB}$$

implies that $\bar{h}$ is independent of the compact set $B \supset A$ provided h is harmonic on $X \backslash B$. Define the linear operator E :

$$\underline{\underline{H}}(X \backslash A) \longrightarrow \underline{\underline{H}}(X)$$

by setting $E h = \bar{h}$.

In what follows, B and O will be sets such that $A \subset \mathring{B} \subset B \subset O$, B compact, O open, relatively compact with $H_O H_{CB} 1 \leqslant \lambda < 1$ on B.

Proposition 5. – *The operator* E *has the following properties* :

1) *it is linear and positive ;*

2) *it is continuous and open in the topology of uniform convergence on compact sets ;*

3) if $h = h' | X \backslash A$, $h' \in \underline{\underline{H}}(X)$, then $E(h) = h'$;

4) $E(h) = 0$, *if and only if for some compact* $B \supset A$, $h = H_{CB} h$.

Proof. – Statements 1) and 3) have been proved. To prove 2), let (h_n) be a sequence in $\underline{\underline{H}}(X \backslash A)$ converging to h. Then $h_n - H_{CB} h_n$ converges uniformly to $h - H_{CB} h$ on the compact subsets of $X \backslash B$. The continuity of $(I - T)^{-1}$ implies that $\bar{h}_n$ converges to $\bar{h}$.

Let P be open in $\underline{\underline{H}}(X \backslash A)$ and let $\bar{h}_0 = E(h_0)$, $h_0 \in P$. There exist $\varepsilon > 0$ and K compact in $X \backslash A$ with $h \in P$ if $|h(x) - h_0(x)| < \varepsilon$ for all $x \in K$. Clearly K can be assumed to contain ∂O. Let $D = K \cup \bar{O}$.

If $\bar{h} \in \underline{\underline{H}}(X)$ and $|\bar{h}(x) - \bar{h}_0(x)| < \varepsilon$ for all $x \in D$ then

$$(\bar{h} - \bar{h}_0) | (X \backslash A) + h_0 = h \in \underline{\underline{H}}(X \backslash A)$$

is in P. From 3) it follows that $E(h) = \bar{h}$ and so E is open.

Assume that $h = H_{CB} h$ for some compact set $B \supset A$. Then $O = h - H_{CB} h$ which implies $(I - T) \bar{h} = 0$. As a result, $\bar{h} = 0$. Assume now that $\bar{h} = 0$. Then since $h - H_{CB} h = \bar{h} - H_{CB} \bar{h}$ it follows that $h = H_{CB} h$.

Proposition 6. – *Let* $\underline{\underline{H}}_1$ *and* $\underline{\underline{H}}_2$ *be two strict harmonic sheaves on* X*. Assume that there is a compact set* $A \subset X$ *such that the sheaves agnee on* $X \backslash A$.

Then with respect to the topology of uniform convergence on compact sets, the topological vector spaces $\underline{\underline{H}}_1(X)$ *and* $\underline{\underline{H}}_2(X)$ *are isomorphic.*

Proof. – Let v_i be a continuous strictly positive superharmoni-function on X (relative to $\underline{\underline{H}}_i$), $i = 1, 2$. Define E_i :

$$\underline{\underline{H}}_i(X \setminus A) \longrightarrow H_i(X)$$

by setting $E_i(h) = v_i\, E(h/v_i)$. These operators are continuous and open.

Assume $\varphi(x) > 0$ for all $x \in X$ is a continuous real-valued function. If U is open in X and if H_U^{φ} is the kernel defined by the Dirichlet problem for U relative to the sheaf $\varphi^{-1}\, \underline{\underline{H}}$, then

$$\varphi[H_U^{\varphi}(f/\varphi)] = H_U\, f\,.$$

Consequently, E_1 and E_2 have the same kernel.

This shows that the mapping $J : \underline{\underline{H}}_1(X) \longrightarrow \underline{\underline{H}}_2(X)$ defined by setting $J(E_1(h)) = E_2(h)$, for all $h \in \underline{\underline{H}}_1(X \setminus A)$ is an isomorphism.

Remark. – Again the assumption of a countable base is not necessary. For example, the arguments hold in the theory of Brelot without this assumption.

6. Applications.

All sheaves considered here will be assumed to satisfy the initial hypotheses of section 3.

Let $\underline{\underline{H}}_1$ and $\underline{\underline{H}}_2$ be two sheaves on X that satisfy the hypothesis of theorem 2. Denote by $\overline{X}$ the common Martin compactification of X and by Δ the Martin boundary $\overline{X} \setminus X$.

THEOREM 3. – *A point* $y_0 \in \Delta$ *is minimal with respect to* $\underline{\underline{H}}_1$ *if and only if it is minimal with respect to* $\underline{\underline{H}}_2$.

Proof. – In the proof of theorem 2, it was shown that there is a continuous function f with $K_y^1 - P_A\, K_y^1 = f(y)\, [K_y^2 - P_A\, K_y^2]$ for all $y \notin A$.

From the proof of theorem 1, it follows that if a net (y_γ) on X converges to $\bar{y} \in \Delta$ then $\lim_\gamma f(y_\gamma) = f(\bar{y})$ exists. Hence, if

$$K^i_{\bar{y}}(x) = \lim_\gamma {}^*K^i_x(y_\gamma),$$

it follows that $K^1_{\bar{y}} - P_A K^1_{\bar{y}} = f(\bar{y})\,[K^2_{\bar{y}} - P_A K^2_{\bar{y}}]$.

Since $P_A = H_{\complement A}$, this shows that $J(K^1_{\bar{y}}) = f(\bar{y})\,K^2_{\bar{y}}$ where J is the isomorphism of proposition 4. As J is a positive operator $K^1_{\bar{y}}$ is minimal if and only if $K^2_{\bar{y}}$ is minimal.

Denote by $\Lambda_i = (l_i = 1)$ compact bases of $\underline{\underline{S}}^+_i$ for $i = 1, 2$ where l_1 and l_2 are positive continuous linear functionals. Then theorem 2 and proposition 3 imply that there is a unique homeomorphism φ : $\overline{\mathcal{E}(\Lambda_1)} \longrightarrow \overline{\mathcal{E}(\Lambda_2)}$ with $\varphi(p^1_y) = p^2_y$ for all $y \in X$ (p^i_y being the potential with support $\{y\}$ in Λ_i). The fact, proved in theorem 3, that $J(K^1_{\bar{y}}) = f(\bar{y})\,K^2_{\bar{y}}$ implies the following result.

LEMMA. – *If* $h_1 \in \overline{\mathcal{E}(\Lambda_1)} \cap \underline{\underline{H}}^+_1$, *then*

$$\varphi(h_1) = [1/(l_2 \circ J)(h_1)]\,J(h_1).$$

Proof. – Let $h_2 = \varphi(h_1)$ and for any positive harmonic function h set $h^0 = [1/h(x_0)]\,h$. Let K^i denote the kernel defined by $(p^i_y)_{y\in X}$ and x_0. Then $\varphi(h_1) = h_2$ if $\lim_\gamma K^1_{y_\gamma} = h^0_1$ implies $\lim_\gamma K^2_{y_\gamma} = h^0_2$.

Hence, if $\alpha = \lim_\gamma f(y_\gamma)$ then $J(h^0_1) = \alpha\, h^0_2$. Since $h_i = [1/l_i(h^0_i)]\; h^0_i$ if $h_i \in \Lambda_i$, this implies that $h_2 = [1/(l_2 \circ J)(h_1)]\,J(h_1)$.

THEOREM 4. – *Let* $\underline{\underline{H}}_1$ *and* $\underline{\underline{H}}_2$ *be equivalent sheaves on* X. *Then the topological cones* $\underline{\underline{S}}^+_1$ *and* $\underline{\underline{S}}^+_2$ *are isomorphic.*

Proof. – According to a remark of Alfsen [1] (p. 120), there is a continuous affine map $\Phi : \Lambda_1 \longrightarrow \Lambda_2$ extending φ providing φ has the following property : if $\mu, \mu' \in \mathcal{M}^+_1(\overline{\mathcal{E}(\Lambda_1)})$ have the same barycentre then the image measures $\hat{\varphi}_\mu = \nu$ and $\varphi_{\mu'} = \nu'$ have the same barycentre.

Let $\nu_1 = p_1 + h_1$ be the barycentre of μ and of μ'. Then both measures coincide on $\{p^1_y \mid y \in X\}$. Since φ maps this set on $\{p^2_y \mid y \in X\}$, it follows that ν and ν' coincide on this set. By the lemma φ coincides on $\overline{\mathcal{E}(\Lambda_1)} \cap \underline{\underline{H}}^+_1$ with the composition of J with the affine map

$$h_2 \longrightarrow [1/l_2(h_2)]\, h_2 \ .$$

Hence, it follows that if $\nu_2 = p_2 + h_2$ and $\nu'_2 = p'_2 + h'_2$ are the barycentres of ν and ν', then $p_2 = p'_2$ and $h_2 = h'_2$.

Extending Φ to $\underline{\underline{S}}_1^+$ by setting $\Phi(\nu) = l_1(\nu) \cdot \Phi([1/l_1(\nu)]\, \nu)$, gives the required isomorphism.

Remark. – This theorem implies theorems 2 and 3. It would therefore be desirable to have a direct proof of this result.

7. The Martin Compactification of X\A.

Let $\underline{\underline{H}}$ be a sheaf on X satisfying the hypotheses of section 4 for which 1 is superharmonic. Let $A \subset X$ be a compact set and let O denote a connected component of X\A which is not relatively compact.

Pick $x_0 \in O$ and let $K(x, y)$ be the kernel obtained by normalizing the potentials p_y. Then if $q_y = p_y - P_A\, p_y$, the kernel $K^O(x, y)$ on O x O defined by normalizing the potentials q_y equals

$f(y)\, [K_y(x) - P_A\, K_y(x)]$, with $f(y) = p_y(x_0)/q_y(x_0)$ if $q_y(x_0)$ is finite and 1 if $q_y(x_0)$ equals $+\infty$.

Let $\overline{X} = X \cup \Delta$ denote the Martin compactification of X and let $\overline{O}$ denote the closure of O in $\overline{X}$.

LEMMA 1. – *The functions* $^*K^O_x$, $x \in O$, *extend continuously to* $\overline{O}\backslash A$ *and separate the points of* $\overline{O} \cap \Delta$.

Proof. – Since X is locally connected $[\partial\overline{O}]\backslash A \subset \Delta$. Let (y_γ) be a net on O which converges in $\overline{X}$ to $y \in \Delta$. The functions

$$[1/f(y_\gamma)]\, K^O_{y_\gamma}$$

converge on O to the harmonic function $h - P_A\, h$, where h corresponds to y.

Let $(y_{\gamma'})$ and $(y_{\gamma''})$ be two subnets for which $h' = \lim_{\gamma'} K^O_{y_{\gamma'}}$ and $h'' = \lim_{\gamma''} K^O_{y_{\gamma''}}$ exist. Then $\alpha' = \lim_{\gamma'} f(y_{\gamma'})$ and $\alpha'' = \lim_{\gamma''} f(y_{\gamma''})$

both exist and are non zero. Now $\alpha''h' = \alpha'h''$ and as

$$h'(x_0) = h''(x_0) = 1 ,$$

it follows that $h' = h''$. Hence, $\lim_{\gamma} {}^*\mathrm{K}_x^{\mathrm{O}}(y_\gamma)$ exists for each $x \in \mathrm{O}$.

Let y, y' be two points of $\overline{\mathrm{O}} \cap \Delta$ and let h, h' be the corresponding harmonic functions on X. The function f extends continuously to $\overline{\mathrm{O}}\backslash\mathrm{A}$ as do the functions ${}^*\mathrm{K}_x^{\mathrm{O}}$. If the extensions of these functions do not distinguish y from y', then

$$f(y)\,[h - \mathrm{P_A}\,h] = f(y')\,[h' - \mathrm{P_A}\,h'] .$$

Consequently $f(y)\,h = f(y')\,h'$ and so $f(y) = f(y')$. As a result, $h = h'$ and so $y = y'$.

Denote by ${}^*\overline{\mathrm{K}}_x^{\mathrm{O}}$ the extension of ${}^*\mathrm{K}_x^{\mathrm{O}}$ to $\overline{\mathrm{O}}\backslash\mathrm{A}$ and let Y be the compactification of O obtained by compactifying $\overline{\mathrm{O}}\backslash\mathrm{A}$ with respect to $({}^*\overline{\mathrm{K}}_x^{\mathrm{O}})_{x\in\mathrm{O}}$.

LEMMA 2. — Y *is the Martin compactification of* O.

Proof. — The Martin compactification of O is the one defined by $({}^*\mathrm{K}_x^{\mathrm{O}})_{x\in\mathrm{O}}$. Since these functions clearly extend continuously to Y, it suffices to show that if $y \in \overline{\mathrm{O}} \cap \Delta$ and $y' \in \mathrm{Y}\backslash[\overline{\mathrm{O}}\backslash\mathrm{A}]$ their extensions distinguish these two points.

Assume this is false. Then there are nets (y_γ) and $(y_{\gamma'})$ on O such that $y = \lim_{\gamma} y_\gamma$ and $y' = \lim_{\gamma'} y_{\gamma'}$, with $h^{\mathrm{O}} = \lim_{\gamma} \mathrm{K}_{y_\gamma}^{\mathrm{O}} = \lim_{\gamma'} \mathrm{K}_{y_{\gamma'}}^{\mathrm{O}}$, a harmonic function on O.

Because $y \in \Delta$, $y = \lim_{\gamma} y_\gamma$ in $\overline{\mathrm{X}}$ and so, if h is the corresponding harmonic function on X, $f(y)\,[h - \mathrm{P_A}\,h] = h^{\mathrm{O}}$ on O. Further since $h = \lim_{\gamma} \mathrm{K}_{y_\gamma}$, $h - \mathrm{P_A}\,h = 0$ on $\complement\mathrm{O}$. Adopting the convention of extending all functions on O to X$\backslash$A by defining them to be zero off O, it follows that $f(y)\,[h - \mathrm{P_A}\,h] = h^{\mathrm{O}}$ on X$\backslash$A. Hence,

$$f(y)\,h = \mathrm{E}(h^{\mathrm{O}}) ,$$

E the operator of proposition 5.

Since Δ is a compact subset of Y it follows that the limit points in O of all the convergent subnets of $(y_{\gamma'})$ lie on $\partial\mathrm{A}$. Let $(y_{\gamma''})$ be such a net. Let B be a compact neighbourhood of A and let E :

$\underline{\underline{H}}(X\backslash B) \longrightarrow \underline{\underline{H}}(X)$ be the operator of proposition 5. Then, viewing all functions as defined on $X\backslash B$,

$$E(h^O) = \lim_{\gamma''} E(K^O_{y_{\gamma''}}) = \lim_{\gamma''} E(f(y_{\gamma''})\,[K_{y_{\gamma''}} - P_A K_{y_{\gamma''}}]) =$$
$$= \lim_{\gamma''} E(f(y_{\gamma''})\,K_{y_{\gamma''}}) \;.$$

However, if $y_{\gamma''}$ is close enough to A, $P_B K_{y_{\gamma''}} = K_{y_{\gamma''}}$ and so $E(K_{y_{\gamma''}}) = 0$. Therefore, $h = 0$, which is a contradiction.

This completes the proof of the first part of the following result.

THEOREM 5. – *$\overline{O}\backslash A$ is an open subspace of the Martin compactification of* O. *Further, a point $y \in \overline{O} \cap \Delta$ is minimal if and only if it is minimal as a point of the Martin boundary of* O.

Proof. – The proof of the second assertion uses the following lemma, E being the operator of proposition 5.

LEMMA 3. – *Let $h' \geqslant 0$ be harmonic on $X\backslash A$ and such that $P_B h'$ decreases to zero as* B, *a compact neighbourhood of* A, *decreases to* A. *Then if $h = E(h')$, $h - P_A h = h'$.*

Assuming the lemma 3 let $y \in \overline{O} \cap \Delta$. It corresponds to a positive harmonic function h on X. In the identification of $\overline{O} \cap \Delta$ with a subset of the Martin boundary of O, the function h is replaced by

$$f(y)\,[h - P_A h] \;.$$

The second assertion states that h is minimal on X if and only if $h - P_A h$ is minimal on O.

Assume h is minimal on X and let $h - P_A h \geqslant h' \geqslant 0$ where h' is harmonic on O. If $h_1 = E(h')$, viewing h' as extended by zero to $X\backslash A$, there exists λ, $0 \leqslant \lambda \leqslant 1$, with $h_1 = \lambda h$. Hence,

$$h_1 - P_A h_1 = \lambda (h - P_A h) \;.$$

If $B \supset A$ is compact, $P_B(h - P_A h) = P_B h - P_A h$ and so $P_B h'$ decreases to zero as $B \downarrow A$. Lemma 3 implies that $h_1 - P_A h_1 = h'$.

Assume now that $h - P_A h = h'$ is minimal on O and that $h \geqslant h_1 \geqslant 0$. Then if $h_1 - P_A h_1 = h'_1$, $0 \leqslant h'_1 \leqslant h'$ and so $h'_1 = \lambda h'$, $0 \leqslant \lambda \leqslant 1$. As a result, $h_1 = E(h'_1) = \lambda E(h') = \lambda h$.

To prove the lemma, note that for any compact neighbourhood B of A, $h' - h = P_B h' - P_B h$ (this follows from the definition of E). Since $P_B h$ decreases to $P_A h$, it follows that $h' = h - P_A h$.

COROLLARY 1. – *If* $X \backslash A$ *is connected and* $\overline{X}$ *is the Martin compactification of* X, *then* $\overline{X} \backslash A$ *is an open subspace of the Martin compactification of* $X \backslash A$. *Further, a point* $y \in \Delta = \overline{X} \backslash X$ *is minimal if and only if it is minimal as a point of the Martin boundary of* $X \backslash A$.

COROLLARY 2. – *Let* O, O′ *be two connected components of* $X \backslash A$ *that are not relatively compact. Then* $\overline{O} \cap \overline{O}' \cap \Delta = \emptyset$.

Proof. – If h corresponds to $y \in \overline{O} \cap \overline{O}' \cap \Delta$, then $h - P_A h = 0$. To see this note that $h - P_A h$ is a limit of functions of the form $K_y - P_A K_y$, $y \in O$ (respectively, $y \in O'$) which vanish on $\complement O$ (respectively, $\complement O'$).

The following lemma together with the above corollary imply that the Martin compactification is of type S (See [6] p. 99).

LEMMA 4. – *Let* $A \subset X$ *be compact. Then* $X \backslash A$ *has only a finite number of connected components which are not relatively compact. Further, if* $y \in \Delta$ *there is a unique component* O *of this type with* $y \in \overline{O}$.

Proof. – Let U be a relatively compact open set containing A. Cover ∂U with a finite number $U_1, \ldots, U_n$ of connected open sets with $\overline{U}_i \cap A = \emptyset$.

Since X is connected, for any connected component O of $X \backslash A$, $\overline{O} \cap \partial A \neq \emptyset$. If in addition $O \cap [X \backslash U] \neq \emptyset$, then O meets some U_i. Consequently, at most n connected components of X A meet $X \backslash \overline{U}$. The uniqueness in the last statement follows from corollary 2 to theorem 5.

COROLLARY 3. – *Let* Σ *be the topological sum of the Martin compactification of the connected components of* $X \backslash A$. *Then* $\overline{X} \backslash A$ *is an open subspace of* Σ.

Proof. – It follows from corollary 2 and lemma 4 that the connected components of $\overline{X} \backslash A$ are the sets $\overline{O} \backslash A$, where O is a connected component of $X \backslash A$.

In view of theorem 5, each set $\overline{O} \backslash A$ can be identified with an open subspace of the Martin compactification of O (trivially if $\overline{O} \subset X$).

COROLLARY 4. – *The Martin compactification of* $\overline{X}$ *is of type* S.

Proof. – If f is a continuous function on X such that for some compact set $A \subset X$, f is constant on the connected components of $X \backslash A$, then clearly f extends continuously to $\overline{X}$. The result follows from Satz 9.1 of [6].

COROLLARY 5. – *For* $y \in \Delta$ *let* $O(A, y)$ *be the unique connected component* O *of* $X \backslash A$ *with* $y \in \overline{O}$. *Denote by* $\Gamma(y)$ *the intersection of the sets* $\overline{O(A, y)}$ *as* A *runs over the compact subsets of* X. *Then* $\Gamma(y)$ *is the connected component of* y *in* Δ.

Proof. – The sets $\Gamma(y)$, $y \in \Delta$, are connected and pairwise disjoint. Corollary 2 to theorem 5 implies that for each $y \in \Delta$ $\Gamma(y)$ contains the connected component of y in Δ.

COROLLARY 6. – *The cardinal number of the set of connected components of* Δ *is at most* $2^{\aleph_0}$

Remarks. – 1) The requirement that 1 be superharmonic is no restriction since for an arbitrary sheaf $\underline{\underline{H}}$ satisfying the hypotheses of section 4, there exists a positive continuous superharmonic function.

2) In [6] corollary 4 was proved for hyperbolic Riemann surfaces. The proof there depends on a description of the Martin boundary which does not apply in general. For it to hold, the sheaf $\underline{\underline{H}}$ has to have an adjoint (see [16].

3) Corollary 6 holds without the assumption of a countable base (as do the other results) in view of the result of Cornea [8] which ensures that X is σ-compact.

8. Direct Decomposition and the Ends of X.

The points of the compactification of X determined by the functions defined in corollary 4 of theorem 5 are often called the *ends* of the space X. As is pointed out in [6], since the Martin compactification is of type S, there is a one-one correspondence between the ends of X and the connected components of Δ.

Denote by C the cone of positive harmonic functions on X. Let $(h_i)_{i\in I}$ be a family of elements of C. Assume that

$$\left\{ \sum_{i\in F} h_i \mid F \subset I \text{ finite} \right\}$$

is bounded above in C. Then the supremum of this family of finite sums will be defined to be $\sum_{i\in I} h_i$. Using this concept of infinite sum, the cone C is said to be the *direct sum of the family* $(C_i)_{i\in I}$ of convex subcones C_i of C if for each $h \in C$ there is a unique family $(h_i)_{i\in I}$ with $h_i \in C_i$, $\forall i \in I$ and $h = \sum_{i\in I} h_i$.

A convex subcone C_1 of C will be said to be a *direct summand* of C if there is a convex subcone C_2 with C the direct sum $C_1 \oplus C_2$ of C_1 and C_2. If C is the direct sum of $(C_i)_{i\in I}$ then each C_i is a direct summand.

If $C = C_1 \oplus C_2$ and both C_1, C_2 are closed in the topology of uniform convergence on compact sets then C will be called a *topological direct sum.* In this case C_1, C_2 will be called *topological direct summands.*

The purpose of this section is to discuss the relationship between direct sums and the ends of X. The basic result is the following theorem.

THEOREM 6. – *Let* $A \subset X$ *be compact and let* $O_1, \ldots, O_n$ *be the connected components of* $X\setminus A$ *which are not relatively compact. Set* $D_i = \Delta \cap \overline{O}_1$ *and let* C_i *denote the set of all positive harmonic functions on* X *represented by measures whose support lies in* D_i. *Then* C_i *is a topological direct summand of* C *and* d $C = C_1 \oplus \ldots \oplus C_n$.

Proof. – Since $\Delta = \bigcup_{i=1}^{n} D_i$ it follows that if $h \in C$ then $h = \sum_{i=1}^{n} h_i$ with $h_i \in C_i$, which is clearly a closed convex subcone of C.

Let $h \in C$ be represented by two measures μ and μ' and let $\mu_i = \mu \mid D_i$, $\mu'_i = \mu' \mid D_i$. Then μ_i and μ'_i represent the same harmonic function. Consider $h = \sum_{i=1}^{n} h_i = \sum_{i=1}^{n} h'_i$ where h_i corresponds to μ_i and h'_i to μ'_i (i.e. $h_i(x) = \int_{D_i} K_y(x)\, d\mu(y)$ etc.). Then

$$h - P_A h = \sum_{i=1}^{n} (h_i - P_A h_i) = \sum_{i=1}^{n} (h'_i - P_A h'_i) .$$

Since $h_i - P_A h_i$ and $h'_i - P_A h'_i$ both equal $h - P_A h$ on O_i and zero on $\complement O_i$ (this follows because $K_y - P_A K_y$ vanishes on $\complement O_i$ if $y \in D_i$) as a result $h_i - P_A h_i = h'_i - P_A h'_i$. Hence, $h_i = h'_i$.

Consequently, $C = C_1 \oplus \cdots \oplus C_n$. Since $C_2 \oplus \cdots \oplus C_n$ is closed, it follows that C_1 (and similarly each C_i) is a topological direct summand.

COROLLARY 1. – *Let* $D \subset \Delta$ *be closed and a union of connected components. Let* μ, μ' *be two measures on* Δ *that represent the same positive harmonic function. Then* $\mu | D$ *and* $\mu' | D$ *represent the same harmonic function.*

Proof. – From the theorem it follows that this is true if D is a finite union of sets of the form D_i.

Denote by $D(A)$, $A \subset X$ compact, the union of the sets D_i (defined in the theorem) that meet D. Then if $(A_n)_n$ is an increasing sequence of compact sets with $X = \cup A_n$, $D = \cap D(A_n)$ and

$$D(A_n) \supset D(A_{n+1})$$

for all n. Let $\mu_n = \mu | D(A_n)$. Then μ_n converges weakly to $\mu | D$. Since the same is true for μ', the result follows.

COROLLARY 2. – *Let* Γ *be a connected component of* Δ *and let* C_Γ *be the set of all positive harmonic functions represented by measures supported by* Γ. *Then* C *is the direct sum of the cones* C_Γ, Γ *running through the collection of connected components of* Δ.

Proof. – Using the notation of the previous corollary,

$$\Gamma = \cap \Gamma(A_n) .$$

Hence, by corollary 1 if $h \in C$ is represented by μ and μ', then $\mu | \Gamma$ and $\mu' | \Gamma$ represent the same function h_Γ.

Clearly, $h = \Sigma_\Gamma h_\Gamma$ and as this representation is unique C is the direct sum of the cones C_Γ.

COROLLARY 3. – *Each connected component Γ of Δ contains minimal points.*

Proof. – If $h \in C_\Gamma$ and μ represents h it follows from the uniqueness of its representation that μ is supported by Γ. Let $y \in \Gamma$ and let h be the corresponding harmonic function. Since it is represented by a canonical measure μ carried by Δ_1 the result follows.

The cones C_Γ, corresponding to the connected components of Δ are in some sense "canonical". The following result leads to a characterization of the C_Γ in terms of the cone C when $\overline{\Delta}_1 = \Delta$.

PROPOSITION 7. – *Let C be the topological direct sum of C_1 and C_2. Then there are disjoint closed sets D_i with $D_1 \cup D_2 = \Delta$ and C_i equal to the cone C_{D_i} of positive harmonic functions represented by measures carried by D_i providing Δ_1 is dense in Δ.*

Conversely, if $\Delta = D_1 \cup D_2$ with D_1 and D_2 disjoint closed sets then C is the topological direct sum of C_{D_1} and C_{D_2}.

Proof. – The converse follows from corollary 1 to theorem 6.

Viewing Δ as a subset of C let $D_i = C_i \cap \overline{\Delta}_1$. The sets D_1, D_2 are disjoint, closed and since their union contains $\Delta_1, D_1 \cup D_2 = \overline{\Delta}_1 = \Delta$. Hence, $C = C_{D_1} \oplus C_{D_2}$.

Let C'_i be the cone of functions whose canonical measure is carried by D_i. Then $C = C'_1 \oplus C'_2$ and as $C'_i \subset C_{D_i}$, $C'_i = C_{D_i}$ for $i = 1,2$. Further, since C_i is closed it follows that $C'_i \subset C_i$. As a result, $C_i = C_{D_i}$ for $i = 1,2$.

In view of this proposition, it follows that the cones C_Γ have the following property : if C is the topological direct sum of C_1

and C_2 then $C_\Gamma \subset C_1$ or $C_\Gamma \subset C_2$. Subcones of C with this property will be called *compatible.*

THEOREM 7. – *Assume* $\bar{\Delta}_1 = \Delta$. *Then the cones* C_Γ *are the maximal compatible subcones of* C. *If* C_Γ *is a topological direct summand it is a minimal one.*

Further, the following statements are equivalent :

1) X *has a finite number of ends ;*

2) Δ *has a finite number of connected components ;*

3) C *has only a finite number of topological direct summands ;*

4) *each* C_Γ *is a topological direct summand.*

In this case, C *is the topological direct sum of the cones* C_Γ.

Proof. – It follows from proposition 7 that if a cone C_Γ is a topological direct summand, then it is minimal.

Let C_0 be a compatible subcone of C. If $A \subset X$ is compact then there is a unique non relatively compact connected component O of $X \backslash A$ with $C_0 \subset C_D$, where $D = \bar{O} \cap \Delta$ and C_D is defined as in proposition 7. Hence if Γ is the intersection of these sets D, $C_0 \subset C_\Gamma$.

It is clear that 1), 2) and 3) are equivalent and that 2) implies 4). Assume Δ has an infinite number of connected components $(\Gamma_\alpha)_{\alpha \in I}$. For each α let $y_\alpha \in \Gamma_\alpha$ and let y be a limit point of $\{y_\alpha \mid \alpha \in I\}$. Assume that Γ_{α_0} is the connected component of y. Since $C_{\Gamma_{\alpha_0}}$ is a topological direct summand $\sum_{\alpha \neq \alpha_0} C_{\Gamma_\alpha} = C_1$ is a closed subcone. This contradicts the fact that y is a limit point since the harmonic functions h_α corresponding to y_α lie in C_1 if $\alpha \neq \alpha_0$.

When Δ has a finite number of connected components then, for some compact $A \subset X$, the connected components of Δ coincide with the sets $\bar{O} \cap \Delta$, O a non relatively compact connected component of $X \backslash A$. Consequently, the last statement follows from theorem 6.

Example. – Let B be an open ball in $\mathbf{R}^n$ $(n \geqslant 2)$ and let $D \subset B$ be a sequence of points all of whose limit points lie outside B.

Set $X = B \backslash D$. The Martin boundary of X equals $D \cup S$, S the sphere bounding B. Each $x \in D$ is a connected component of Δ whose corresponding subcone is a topological direct summand. The cone corresponding to the component S does not have this property.

Remarks. – In [4] Constantinescu and Cornea show that for any integer $n > 0$, there exists a hyperbolic Riemann surface X with one end and Δ_1 a set of n points. Since Δ is connected

$$\bar{\Delta}_1 = \Delta_1 \neq \Delta \ .$$

Furthermore, for these examples theorem 7 clearly breaks down.

It would be of interest to have sufficient conditions that ensure $\bar{\Delta}_1 = \Delta$.

9. Appendix.

Let X and $\underline{\underline{H}}$ be as in section 4 without the assumption of a countable base. Denote by $\underline{\underline{S}}^+$ the cone of positive superharmonic functions equipped with the T-topology. Then $\underline{\underline{S}}^+$ has a compact base Λ by corollary 3.2 of [7].

Let E^+ denote the set of superharmonic functions that are either harmonic or potentials with point support.

LEMMA 5. – $E^+ \cap \Lambda$ *is compact.*

Proof. – Let $\mathfrak{U}$ be an ultrafilter on $E^+ \cap \Lambda$ and denote by s_u the function defined in [7] (p. 1335). Define $\varphi : E^+ \longrightarrow X \cup \{a\}$, a the Alexandroff point at infinity by setting $\varphi(s) = a$ if s is harmonic and $\varphi(s) = y$ if the support of s is $\{y\}$.

The image ultrafilter $\varphi\mathfrak{U}$ converges to a or to a point $y \in X$. In the first case s_u is harmonic, and in the second it has its support contained in $\{y\}$ (see theorem 2.1 of [7].

Since in theorem 3.1 of [7] it is shown that $\mathfrak{U}$ converges to s_u in the T-topology, the result follows.

It was proved in [6] (theorem 5) that for each $y \in X$ there is a potential with support $\{y\}$, and so there is a unique potential p_y in Λ with this property.

PROPOSITION 8. – *The mapping* $y \longrightarrow p_y$ *embeds* X *in* Λ. *Further the function* $G(x, y) = p_y(x)$ *is a lower semicontinuous function on* $X \times X$, *continuous off the diagonal.*

Proof. – The proofs of propositions 18.1 and 19.1 of [10] do not use the second axiom of countability.

Lemma 5 therefore allows the argument of proposition 22.1 of [10] to apply without change.

It remains to note that proposition 19.1 of [10] implies that $p_y \longrightarrow y$ is a continuous function.

Remark. – From what has been said above, proposition 3 holds without the second axiom of countability.

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J.C. TAYLOR
Department of Mathematics
Mc Gill University
Montréal, Québec (Canada)