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THE SPLIT CASE OF THE PRASAD–TAKLOO-BIGHASH CONJECTURE FOR CUSPIDAL REPRESENTATIONS OF LEVEL ZERO

by Marion CHOMMAUX & Nadir MATRINGE

Abstract. — Let E/F be a quadratic extension of non archimedean local fields of odd residual characteristic. We prove a conjecture of Prasad and Takloo-Bighash, in the case of cuspidal representations of depth zero of $\mathrm{GL}_{2m}(F)$. This conjecture characterizes distinction for the pair $(\mathrm{GL}_{2m}(F), \mathrm{GL}_m(E))$ with respect to a character $\mu \circ \det$ of $\mathrm{GL}_m(E)$, in terms of certain conditions on Langlands paremeters, including an epsilon value. We also compute the multiplicity of the involved equivariant linear forms when E/F is unramified, and also when μ is tame. In both cases this multiplicity is at most one.

Résumé. — Soit E/F une extension quadratique de corps locaux nonarchimédiens de caractéristique résiduelle impaire. On prouve une conjecture de Prasad et Takloo-Bighash dans le cas des représentations cuspidales de niveau zéro de $\mathrm{GL}_{2m}(F)$. Cette conjecture caractérise la distinction pour la paire $(\mathrm{GL}_{2m}(F), \mathrm{GL}_m(E))$ selon un caractère $\mu \circ \det$ de $\mathrm{GL}_m(E)$, en termes de certaines conditions sur le paramètre de Langlands incluant une valeur spéciale de facteur epsilon. On montre aussi que l'espace des formes linéaires équivariantes vaut un lorsque E/F est non ramifiée, et aussi lorsque μ est modéré.

Introduction

Let E/F be a quadratic extension of non archimedean local fields. Let D be an F-division algebra of dimension d^2 and n be a positive integer such that nd is even. Set $\mathcal{M} = \mathcal{M}(n, D)$, so that E embeds into M uniquely up to inner automorphism. Set $C_E(\mathcal{M})$ to be the centralizer of E in $\mathcal{M}$, it is an E-central simple algebra. Let $G = \mathcal{M}^\times$ and $H = C_E(\mathcal{M})^\times$, for $\mu : E^* \to \mathbb{C}^*$ a smooth character, we denote by μ of the character H obtained by composing μ with the reduced norm on H. This paper is concerned with

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the following conjecture of Prasad and Takloo-Bighash [21, Conjecture 1] (the generic transfer assumption in [ibid.] has been shown to be unnecessary in [24]):

Conjecture 0.1. — *Let π be an irreducible admissible representation of $G = \mathrm{GL}(n, D)$ with central character ω_π. Let μ be a character of $E^\times$ such that $\mu^{\frac{nd}{2}}|_{F^\times} = \omega_\pi$. If the representation π is μ-distinguished by H, i.e. if $\mathrm{Hom}_H(\pi, \mu) \neq 0$, then:*

(1) *the Langlands parameter $\phi(\pi)$ of π takes values in $GSP_{nd}(\mathbb{C})$, with similitude factor $\mu_{|F^\times}$;*
(2) *the epsilon factor satisfies the relation*

$$\epsilon\left(\frac{1}{2}, \phi(\pi) \otimes \mathrm{Ind}_{W_E}^{W_F}\left(\mu^{-1}\right)\right) = (-1)^n \omega_{E/F}(-1)^{\frac{nd}{2}} \mu(-1)^{\frac{nd}{2}}$$

where $\omega_{E/F}$ is the quadratic character of $F^\times$ with kernel the norms of $E^\times$, and W stands for the Weil group.

If π is a disrete series representation of G, then the implication becomes an equivalence.

This conjecture is inspired by earlier results of J. Tunnel and H. Saito for $n = 2$ and $D = F$. In fact Tunnel was the first to consider the problem for $\mathrm{GL}(2, F)$, and to solve it when the residual characteristic of F is not 2 ([27, Theorem p. 1277]), then Saito found a simpler proof valid in characteristic different from 2 ([22, Theorem p. 99]).

The current status of the conjecture is the following: when $\mu = 1$ and F has characteristic zero and odd residual characteristic, the conjecture should be proved by the combination of [23, 24, 25, 29]. The paper [29] holds in characteristic zero, the paper [24] holds in characteristic not 2, and the parts of [23] which do not depend on [29] hold in residual characteristic not 2. Finally the reduction from discrete series to cuspidal representations done in [25] holds in characteristic zero (and assumes odd residual characteristic because the main theorem depends on [23]).

For general μ and F of characteristic not 2, the conjecture is proved by the first named author in [9] for Steinberg representations. In this paper, when the residual characteristic of F is not 2, we prove it for general μ and depth-zero cuspidal representations of F-split G.

Let us describe the how the paper is organized: we assume the residual characteristic of F to be different from 2, and suppose that $n \geqslant 4$ as in any case the conjecture we intend to prove is known for $n = 2$ from Tunnel and Saito's results.

In Section 2 we treat the case where μ is tame. By standard Mackey theory arguments, and an also standard argument of Hakim and Murnaghan, we characterize μ-distinction of depth-zero cuspidal representations in terms of their Langlands parameters (Theorem 2.4).

In Section 3, in order to characterize distinction when μ is not tame, we prove in Proposition 3.2 that a μ-distinguished cuspidal representation of any inner form of $\mathrm{GL}_n(F)$ is μ-selfdual, by a standard globalization argument.

In Section 4 we extend in Theorem 4.7 our characterization of μ-distinction depth-zero cuspidal representations of $\mathrm{GL}_n(F)$ in terms of their Langlands parameter to any character μ. Along the way we isolate the contribution of residual twisted Shalika models in Proposition 4.3, and show in Proposition 4.5 that when E/F is unramified, the only double coset contributing to distinction is the one isolated in Proposition 4.3. In particular this gives a multiplicity at most one statement when E/F is unramified.

In Section 5 we give an explicit characterization of μ-simplecticity of depth-zero cuspidal representations of $\mathrm{GL}_n(F)$ (see Corollary 5.6), which resembles (and in fact is implied by) our μ-distinction criterion.

Finally in Section 6 we prove the Prasad and Takloo-Bighash conjecture for depth-zero cuspidal representations of $\mathrm{GL}_n(F)$ (Corollary 6.2). With all the analysis done before, it reduces to a pleasant computation of the epsilon value of the conjecture for μ-symplectic depth-zero cuspidal representations of $\mathrm{GL}_n(F)$ (with an extra condition on the central character) which is done in particular thanks to a result of Fröhlich and Queyrut ([10]). The computation in question is performed in Theorem 6.1.

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1. Preliminary results

1.1. Notation / definitions

Let F be a non-archimedean local field of residual characteristic not 2 and D an F-central division algebra of dimension d^2 over F. We fix an algebraic closure which will contain all finite extensions of F under consideration, and similarly for the residual field k_F of F. For a finite extension $\bullet$ of F, we denote by the $\mathcal{O}_\bullet$, $\mathcal{P}_\bullet$, $\varpi_\bullet$, $k_\bullet$ and $q_\bullet$ the ring of integers, its maximal ideal, a fixed uniformizer, the residual field of $\bullet$. Whenever $\chi : \bullet^* \to \mathbb{C}^*$ is a (smooth) character, we say that it is tame if $\mu(1+\mathcal{P}_\bullet) = \{1\}$. Let E be a quadratic extension of F (we write $E = F[\delta]$ for a fixed δ in $E \setminus F$ such that δ^2 is in F and we set $\Delta = \delta^2$). We let $e(E/F)$ denote the ramification index of E/F. When E/F is ramified, we choose ϖ_E and ϖ_F such that $\varpi_F = \varpi_E^2$; when E/F is unramified, we choose $\varpi_F = \varpi_E$.

Throughout the paper we will have

$$nd = 2m$$

for m a natural number. In fact except in Proposition 3.2, we will have

$$D = F \Leftrightarrow d = 1 \Leftrightarrow n = 2m.$$

We will consider the group

$$G = \mathrm{GL}_n(F)$$

and its subgroup

$$H \simeq \mathrm{GL}_m(E)$$

embedded in G as we now explain. Let $(e_1, \ldots, e_m)$ be the canonical basis of E^m. Then E^m identifies to F^n as F-vector space via the basis $\mathcal{B} = (\delta e_1, \ldots, \delta e_m, e_1, \ldots, e_m)$. Now H embeds in G as the fixed points of G under the involution

$$\begin{array}{cccc} \theta & : & G & \longrightarrow & G \\ & & g & \mapsto & AgA^{-1} \end{array} \quad \text{where } A = \begin{pmatrix} & I_m \\ \Delta I_m & \end{pmatrix}.$$

We denote by $\det_E$ the determinant map on H identified with $\mathrm{GL}_m(E)$, with values in E^*. Hence any character μ of E^* defines a character which we still write μ of H, and in fact all characters of H are such.

1.2. Parametrization of depth-zero cuspidal representations

We call a depth-zero cuspidal representation of $\mathrm{GL}_n(F)$ an irreducible cuspidal representation of this group with a vector fixed by $I_n + \varpi_F \mathcal{M}_n(\mathcal{O}_F)$. One can parametrize depth-zero cuspidal representations by admissible tame pairs as we now recall (see [8, Part 5]).

- Let L/F be the unramified field extension of degree n, of ring of integers $\mathcal{O}_L$. Let χ be a character of L^* that satisfies:
  - χ is tame,
  - $\chi \circ \gamma = \chi \Rightarrow \gamma = id_L$ for all γ in $\mathrm{Gal}_F(L)$; we say that χ is regular.

  Such a pair (L, χ) is said to be *tame admissible*.
- As χ is trivial on $1 + \mathcal{P}_L$, (L, χ) induces a pair $(k_L, \overline{\chi})$ where $\overline{\chi}$ is a character of k_L^* which satisfies $\overline{\chi} \circ \overline{\gamma} = \overline{\chi} \Rightarrow \overline{\gamma} = id_{k_L}$ for all $\overline{\gamma}$ in $\mathrm{Gal}_{k_F}(k_L)$; $\overline{\chi}$ is said to be *regular*.

  By Green parametrization, one can associate to $(k_L, \overline{\chi})$ an irreducible cuspidal representation $(\overline{\pi}_{\overline{\chi}}, \mathcal{V})$ of $\mathrm{GL}_n(k_F)$ i.e. an irreducible representation of $\mathrm{GL}_n(k_F)$ such that for all proper parabolic subgroup P with Levi decomposition $P = MN$, the vector subspace of fixed points of $\mathcal{V}$ by N is trivial.

  More precisely, if one defines an equivalence relation $\sim$ on regular characters of k_L^* by

$$\bar{\chi}_1 \sim \bar{\chi}_2 \quad \text{if and only if} \quad \exists\, \bar{\gamma} \in \mathrm{Gal}_{k_F}(k_L) \quad \text{such that} \quad \bar{\chi}_2 = \bar{\chi}_1 \circ \bar{\gamma},$$

  one has a bijection:

$$\begin{aligned} &\left\{\begin{matrix} \text{equivalence classes for } \sim \\ \text{of regular characters of } k_L^* \end{matrix}\right\} \\ &\qquad\qquad\longrightarrow \qquad \left\{\begin{matrix} \text{equivalence classes of irreducible} \\ \text{cuspidal representations of } \mathrm{GL}_n(k_F) \end{matrix}\right\} \\ &\qquad\quad \bar{\chi} \mapsto \quad \bar{\pi}_{\bar{\chi}} \end{aligned}$$

- As $\mathrm{GL}_n(k_F) \simeq \mathrm{GL}_n(\mathcal{O}_F)/1 + \varpi_F \mathcal{M}_n(\mathcal{O}_F)$, $\bar{\pi}_{\bar{\chi}}$ can be seen as a representation of $\mathrm{GL}_n(\mathcal{O}_F)$ that is trivial on $1+\varpi_F\mathcal{M}_n(\mathcal{O}_F)$. Then, one can define a representation of $F^*\mathrm{GL}_n(\mathcal{O}_F)$, denoted by λ_χ, in the following way:

$$\lambda_\chi(xk) = \chi_{|F^*}(x)\bar{\pi}_{\bar{\chi}}(k) \text{ for all } x \in F^*, k \in \mathrm{GL}_n(\mathcal{O}_F).$$

- Finally, we set $\pi(\chi) := c - \mathrm{Ind}_{F^*\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)}(\lambda_\chi)$ (c-Ind refers to compact induction), it is a depth zero cuspidal representation of G.

If we denote again by $\sim$ the equivalence relation between admissible tame pairs of degree n by $\chi_1 \sim \chi_2$ if and only if $\exists\, \gamma \in \mathrm{Gal}_F(L)$ such that $\chi_2 = \chi_1 \circ \gamma$, one gets a bijection:

$$\begin{aligned} &\left\{\begin{matrix}\text{equivalence classes for } \sim \text{ of} \\ \text{admissible tame pairs of degree } n\end{matrix}\right\} \\ &\qquad\qquad\qquad \longrightarrow \left\{\begin{matrix}\text{equivalence classes of} \\ \text{depth 0 cuspidal} \\ \text{representations of } \mathrm{GL}_n(F)\end{matrix}\right\} \\ &\qquad\qquad (L,\chi) \mapsto \quad \pi(\chi) \end{aligned}$$

Let us recall that the central character of $\pi(\chi)$ is $\chi_{|F^*}$ and its contragredient is $\pi(\chi)^\vee \simeq \pi(\chi^{-1})$.

1.3. Reminder about the building of $\mathrm{GL}_n(F)$

Let us recall how to describe the Bruhat–Tits building of $\mathrm{GL}_n(F)$ with lattice chains.

Definition 1.1. — *An $\boldsymbol{\mathcal{O}_F}$-lattice chain in F^n is a strictly decreasing sequence (for inclusion) $\mathcal{L} = (L_k)_{k\in\mathbb{Z}}$ of lattices such that there exists a unique positive integer T that satisfies: for any uniformizer ϖ_F, $\varpi_F L_k = L_{k+T}$ for all $k \in \mathbb{Z}$. The integer T (or $T(\mathcal{L})$) is called the period of $\mathcal{L}$.*

It is known that T is at most n, and that there are lattice chains with period n. The group $\mathrm{GL}_n(F)$ naturally acts on the set of lattice chains $(L_k)_{k\in\mathbb{Z}}$ by $g \cdot (L_k)_k = (g \cdot L_k)_k$ for $g \in \mathrm{GL}_n(F)$, and we say that two lattice chains are *equivalent* if they are in the same Z-orbit, for Z the center of $\mathrm{GL}_n(F)$.

Definition 1.2. — *As a simplicial complex, the Bruhat–Tits building of $\mathrm{GL}_n(F)$, X_G, is defined as the the set of equivalence classes of lattice chains. The $(T-1)$-dimensional simplex being the equivalence classes of lattice chains of period T.*

We identify lattice chains of period one with Z-orbits of lattices in F^n, and denote by $[L]$ the Z-orbit of the lattice L: by definition they from the set X_G° of *vertices of* X_G. Clearly the group $\mathrm{GL}_n(F)/Z$, hence $\mathrm{GL}_n(F)$ acts on X_G by respecting its simplicial structure. Let $\mathcal{K}$ denote the maximal compact modulo center subgroup $F^*\mathrm{GL}_n(\mathcal{O}_F)$ and let s_0 be the vertex of X_G that is stabilized by $\mathcal{K}$ i.e. the standard lattice chain of period 1; the vertex s_0 is called the *standard vertex* of X_G. We recall the following G-set isomorphism:

$$\begin{array}{ccc} X_G^\circ & \xrightarrow{\sim} & G/\mathcal{K} \\ g\cdot s_0 & \longmapsto & g\mathcal{K} \end{array} \qquad \text{for } g\in G. \tag{1.1}$$

We will need the geometric realization of X_G, denoted by $|X_G|$. Each $T-1$-dimensional simplex of X_G is embedded in $\mathbb{R}^{T-1}$ with the following property: if we consider a $T-1$-dimensional simplex, the points of its geometric realization in $|X_G|$ are given by the set of all barycenters of its vertices. We will use the geometric realization of the building X_G given by lattice-functions. The definition comes from [4, Section I.2].

Definition 1.3. — *A* lattice-function *of F^n is a map*

$$\Lambda : \mathbb{R} \longrightarrow \{\textit{lattices of } F^n\}$$

satisfying:

- $\varpi_F\Lambda(r) = \Lambda(r+1)$;
- Λ *is decreasing: for all* $r \geqslant s$, $\Lambda(r) \subseteq \Lambda(s)$;
- Λ *is left-continuous for the discrete topology on lattices.*

Let us explain with more details how the set of lattice-functions allows to realize geometrically the building of $\mathrm{GL}_n(F)$. Let Λ be a lattice-function of F^n, then its image is a lattice chain $\mathcal{L} = (L_k)_{k\in\mathbb{Z}}$ with period T. If we denote by λ_k the length of the interval defined by $\{r\in\mathbb{R}, \Lambda(r) = L_k\}$, then the point x_Λ of $|X_G|$ associated to Λ is the barycenter of the weighted points $([L_0],\lambda_0), ([L_1],\lambda_1), \ldots, ([L_{T-1}],\lambda_{T-1})$. Two lattice-functions Λ_1 and Λ_2 are said to be *equivalent* if there exists a real number r_0 such that $\Lambda_1(r) = \Lambda_2(r+r_0)$ for all $r\in\mathbb{R}$, in which case they realize the same point of the building. We denote by $\overline{\Lambda}$ the class of a lattice-function Λ. Moreover, the group $\mathrm{GL}_n(F)$ naturally acts on the set of lattice-functions by: $(g\cdot\Lambda)(r) = g\cdot(\Lambda(r))$ for every lattice-function Λ, every $g\in\mathrm{GL}_n(F)$ and every real number r. Thus, one has the following G-set isomorphism:

$$\begin{array}{ccc} \{\text{equivalence classes of lattice-functions of } F^n\} & \xrightarrow{\sim} & |X_G| \\ \overline{\Lambda} & \longmapsto & x_\Lambda \end{array}$$

Of course, all these reminders are valid for the construction of the building of $\mathrm{GL}_m(E)$, X_H.

1.4. Vertices of the building fixed by the involution

First we recall the relation between $|X_G|$ and $|X_H|$, we will use the following terminology from type theory.

Definition 1.4. — *Let $u \in G$ such that $F_1 := F[u]$ is a field; let us denote by v_{F_1} the normalized valuation of F_1 and by $e(F_1/F)$ the ramification index of F_1/F. One says that u is minimal on F if:*

(1) $\gcd(v_{F_1}(u), e(F_1/F)) = 1$,
(2) $\varpi_F^{-v_{F_1}(u)} u^{e(F_1/F)} + \mathcal{P}_{F_1}$ *generates the residual field extension* k_{F_1}/k_F.

Recall that $E = F[\delta]$ and let us show that δ can be chosen minimal.

- If E/F is ramified, we recall that $\varpi_F := \varpi_E^2$. If we choose $\delta = \varpi_E$, then we do have $E = F[\varpi_E]$ and δ is minimal. Indeed, $v_E(\varpi_E) = 1$ so $\gcd(v_E(\varpi_E), e(E/F)) = 1$ and moreover $\varpi_F^{-1}\varpi_E^2 + \mathcal{P}_E = 1 + \mathcal{P}_E$ which generates k_E/k_F (because $k_E = k_F$ in the ramified case).
- If E/F is unramified (i.e. $e(E/F) = 1$), then k_E is an extension of k_F with cardinality q_F^2 and there exists $\xi \in E^*$ a primitive $(q_F^2-1)^{\text{th}}$ root of unity which generates E over F. Set $\delta := \xi^{\frac{q_F+1}{2}}$. As the order of δ is $2(q_F - 1)$, then $\delta \notin F$ but $\delta^2 \in F$, so that we do have $E = F[\delta]$ with $\delta^2 \in F$. Moreover, δ is a minimal element because $v_E(\delta) = 0$ so $\gcd(v_E(\delta), e(E/F)) = 1$ and moreover, $\varpi_F^0\delta^1 + \mathcal{P}_E = \delta + \mathcal{P}_E$ generates k_E/k_F (see Weil [28, Theorem 7 and Corollary 3 of Chapter 1, § 4]).

From now on, we choose $\delta = \varpi_E$ if E/F is ramified and $\delta = \xi^{\frac{q_F+1}{2}}$ (for ξ a primitive $(q_F^2-1)^{\text{th}}$ root of unity) if E/F is unramified, thus δ is minimal. Then by [6, Lemma XII.4.2] we have:

Lemma 1.5. — *We have $|X_G|^\theta = |X_G|^{E^*}$.*

Note that an $\mathcal{O}_E$-lattice of E^m can always be seen as an $\mathcal{O}_F$-lattice of F^{2m} because $\mathcal{O}_E$ is an $\mathcal{O}_F$-lattice in F^2. [4, Theorem 1.1] then asserts:

Theorem 1.6.

(1) *There exists a unique map $j : |X_H| \longrightarrow |X_G|$ that is H-equivariant and affine.*
(2) *It is injective and $j(|X_H|) = |X_G|^{E^*}$, the set of points that are fixed by E^*.*
(3) *If $x \in |X_H|$ is associated to the lattice-function $r \mapsto \Lambda(r)$, then $j(x)$ is associated to the lattice-function $r \mapsto \Lambda(e(E/F)r)$.*

The theorem above enables us to determine the H-orbits of θ-fixed vertices in ${X_G}^\circ$ depending on the ramification of E/F.

Proposition 1.7. — *When E/F is unramified, the set $(X_G^\circ)^\theta$ consists of a unique H-orbit, namely that of the standard vertex s_0 fixed by $\mathcal{K}$, whereas when E/F is ramified $(X_G^\circ)^\theta$ is empty.*

Proof. — When E/F is unramified, the map j is simply the identity on lattice-functions and is simplicial. Thus by, $(X_G^\circ)^\theta = j(X_H^\circ)$ whence $H\backslash(X_G^\circ)^\theta = H\backslash j(X_H^\circ) = j(H\backslash X_H^\circ)$ by Theorem 1.6. As H acts transitively on X_H°, we deduce that $(X_G^\circ)^\theta$ consists of a unique H-orbit. Moreover it is that of s_0 because s_0 is the image of the standard vertex in X_H° under j. When E/F is ramified, then by Theorem 1.6 the map j sends an equivalence class of lattice functions with image a lattice chain of of period 1 to an equivalence class of lattice functions with image a lattice chain of of period $e(E/F) = 2$, i.e. it sends a vertex to an interior point of a simplex of dimension $\geqslant 1$, so $j(X_H) \cap X_G^\circ$ is empty and the resutl follows again from Theorem 1.6. □

1.5. Properties of local constants

Let K'/K be a finite separable extension of non-archimedean local fields, if ψ is a non-trivial character of K, we denote by $\psi_{K'}$ the character $\psi \circ \mathrm{Tr}_{K'/K}$. We call *the conductor of* ψ the smallest integer $d(\psi)$ such that ψ is trivial on $\mathcal{P}_K^{d(\psi)}$. Similarly if χ is a character of K^*, we call *the conductor of* χ the integer $c(\chi)$ equal to zero if χ is unramified, or equal to the smallest integer such that χ is trivial on $1+\mathcal{P}_{K'}^{c(\psi)}$ if χ is ramified. We say that χ is tame when $c(\chi) \leqslant 1$. When K'/K *is unramified*, it follows from [28, Chapter 8, Corollary 3] that

$$d(\psi_{K'}) = d(\psi). \tag{1.2}$$

If ϕ is a representation of W_K of finite dimension, and ψ is a non-trivial character of K, we refer to [26, 3.6.4] for the definition of the root number $\epsilon(1/2, \phi, \psi)$ (denoted ϵ_L there). One then defines the Langlands λ-constant:

$$\lambda\left(K'/K, \psi\right) = \frac{\epsilon\left(1/2, \mathrm{Ind}_{W_{K'}}^{W_K}\left(\mathbf{1}_{W_K}\right), \psi\right)}{\epsilon\left(1/2, \mathbf{1}_{W_L}, \psi_{K'}\right)}.$$

We set

$$\omega_{K'/K} = \det \circ \mathrm{Ind}_{W_{K'}}^{W_K}\left(\mathbf{1}_{W_K}\right),$$

it identifies with the quadratic character of K^* with kernel the norms of K'^* when K'/K is quadratic. For $a \in K^\times$, we set $\psi_a = \psi(a\,.)$. These constants enjoy the following list of properties, which we will freely use later in the paper.

(1) $\epsilon(1/2,\phi\oplus\phi',\psi)=\epsilon(1/2,\phi,\psi)\epsilon(1/2,\phi',\psi)$ where ϕ' is another finite dimensional representation of W_K [26, (3.4.2)].
(2) $\epsilon(1/2,\phi,\psi_a)=\det(\phi(a))\epsilon(1/2,\phi,\psi)$ ([26, (3.6.6)]).
(3) $\epsilon(1/2,\phi^\sigma,\psi^\sigma)=\epsilon(1/2,\phi,\psi)$ whenever σ is a finite order field automorphism of K, as can be checked by the definition of the epsilon factor.
(4) $\epsilon(1/2,\phi,\psi)\epsilon(1/2,\phi^\vee,\psi^{-1})=1$ ([26, (3.6.7)]).
(5) If χ is a character of K^*, and μ is an unramified character of K^*, by [26, (3.6.5)]:

$$\epsilon\left(1/2,\mu\chi,\psi\right)=\mu\left(\varpi_K^{d(\psi)+c(\chi)}\right)\epsilon\left(1/2,\chi,\psi\right).$$

(6) If K'/K is a quadratic, $\delta\in\ker(\mathrm{Tr}_{K'/K})-\{0\}$, χ is a character of K'^* with $\chi_{|K^*}=1$, then by [10, Theorem 3]:

$$\epsilon\left(1/2,\chi,\psi_{K'}\right)=\chi(\delta).$$

(7) If $\phi_{K'}$ is an r-dimensional representation of $W_{K'}$, then

$$\epsilon\left(1/2,\mathrm{Ind}_{W_{K'}}^{W_K}\left(\phi_{K'}\right),\psi\right)=\lambda\left(K'/K,\psi\right)^r\epsilon\left(1/2,\phi_{K'},\psi_{K'}\right)$$

([7, (30.4.2)]).
(8) If K'/K is unramified with $[K'/K]=n$:

$$\lambda\left(K'/K,\psi\right)=(-1)^{d(\psi)(n-1)}$$

(for example [16] and 2., together with Equation (1.2).)
(9) If K'' is a field with $K\subset K''\subset K'$, then

$$\lambda\left(K'/K,\psi\right)=\lambda\left(K'/K'',\psi_{K''}\right)\lambda\left(K''/K,\psi\right)^{[K':K'']}$$

([14]).
(10) $\lambda(K'/K,\psi)^2=\omega_{K'/K}(-1)$ ([7, (30.4.3)]).

2. Distinction of depth-zero cuspidal representations when μ is tame

This case is the easiest case, and we use the proof of [13, Proposition 5.20] to determine multiplicities. We fix $\pi(\chi)$ a cuspidal representation of $\mathrm{GL}_n(F)$ of depth-zero, and μ is a character of E^*.

Lemma 2.1 ([13]). — *Let $x\in X_G^\circ$ a vertex such that $\theta(x)\neq x$. Let $\mathcal{K}_x$ be the stabilizer of x in G, K_x the maximal compact sugbroup of $\mathcal{K}_x$ and $K_x^1\subseteq K_x$ its pro-unipotent radical. Let $\overline{\sigma}$ be a cuspidal representation of K_x/K_x^1, let σ be the inflation of $\overline{\sigma}$ to K_x. Suppose that μ is tame and set $\rho:=\mu$, then $\mathrm{Hom}_{K_x\cap H}(\sigma,\rho)=\{0\}$.*

Proof. — By the proof of [13, Proposition 5.20], if $\theta(x) \neq x$ there is a group $K_x^1 \subset U \subset K_x$, such that $\overline{U} := U/K_x^1 \subset K_x/K_x^1$ is the unipotent radical of a proper parabolic subgroup of $K_x/K_x^1 \simeq \mathrm{GL}_n(k_F)$ and which satisfies $U = U^\theta K_x^1$ (where the exponent denotes fixed points). Suppose for the sake of contradiction that $\mathrm{Hom}_{K_x \cap H}(\sigma, \rho) \neq \{0\}$, this first implies that $\rho_{|K_x^1 \cap H} = 1$ because σ is trivial on K_x^1. Now for $h \in U \cap H$, there exists $\alpha \geqslant 0$ such that $h^{p^\alpha} \in K_x^1 \cap H$, which implies that $\rho(h^{p^\alpha}) = 1$. Thus, $\mu(\det_E(h))^{p^\alpha} = 1$ where $\det_E(h) \in \mathcal{O}_E^\times$. Yet μ is tame so $\mu_{|\mathcal{O}_E^\times}$ factors through $\mathcal{O}_E^\times/(1+\mathcal{P}_E)$ which is a finite group of order prime to p, hence $\mu(det(h)) = 1$. So $\rho_{|U \cap H} = 1$ and

$$\{0\} \neq \mathrm{Hom}_{K_x \cap H}(\sigma, \rho) \subset \mathrm{Hom}_{U^\theta}(\sigma, 1) \simeq \mathrm{Hom}_{\overline{U}}(\overline{\sigma}, 1)$$

as $U = U^\theta K_x^1$, contradicting the cuspidality of σ. □

In other words, as each vertex x in X_G° is of the form $g \cdot s_0$ for a certain g in G and its stabilizer is $g\mathcal{K}g^{-1}$, this amounts to the following lemma.

Lemma 2.2 ([13]). — *If $g \in H\backslash G/\mathcal{K}$ satisfies $\mathrm{Hom}_{H \cap g\mathcal{K}g^{-1}}({}^g\lambda_\chi, \mu) \neq \{0\}$ (where ${}^g\lambda_\chi(x) = \lambda_\chi(g^{-1}xg)$ for all x in $g\mathcal{K}g^{-1}$), then $g\mathcal{K}g^{-1}$ is stable by θ.*

The next step is:

Lemma 2.3. — *There is an isomorphism of $\mathbb{C}$-vector spaces:*

$$\mathrm{Hom}_H\left(\pi(\chi), \mu\right) \simeq \prod_{g \cdot s_0 \in H\backslash \left(X_G^\circ\right)^\theta} \mathrm{Hom}_{H \cap g\mathcal{K}g^{-1}}\left({}^g\lambda_\chi, \mu\right).$$

Proof. — Write successively:

$$\begin{aligned}\mathrm{Hom}_H\left(\pi(\chi), \mu\right) &= \mathrm{Hom}_H\left(c - \mathrm{Ind}_{\mathcal{K}}^G(\lambda_\chi), \mu\right) \\ &\simeq \mathrm{Hom}_H\left(\bigoplus_{g \in H/G\backslash\mathcal{K}} c - \mathrm{Ind}_{H \cap g\mathcal{K}g^{-1}}^{\mathcal{K}}\left({}^g\lambda_\chi\right), \mu\right)\end{aligned}$$

by Mackey's restriction formula

$$\simeq \prod_{g \in H\backslash G/\mathcal{K}} \mathrm{Hom}_{H \cap g\mathcal{K}g^{-1}}\left({}^g\lambda_\chi, \mu\right)$$

by Frobenius reciprocity on the left, for compact induction from a compact modulo center open subgroup

$$\simeq \prod_{g\cdot s_0\,\in\, H\backslash X_G^\circ} \mathrm{Hom}_{H\,\cap\, g\,\mathcal{K}\,g^{-1}}\left({}^g\lambda_\chi,\mu\right) \quad \text{thanks to Isomorphism (1.1)}$$

$$\simeq \prod_{g\cdot s_0\,\in\, H\backslash \left(X_G^\circ\right)^\theta} \mathrm{Hom}_{H\,\cap\, g\mathcal{K}g^{-1}}\left({}^g\lambda_\chi,\mu\right) \text{ thanks to Lemma 2.2}. \qquad \square$$

We denote by L_0 the unramified extension of F of degree m. Thanks to Theorem 1.6 and the recent paper [19] we obtain:

Theorem 2.4. — *When μ is tame and $n \geqslant 4$, we have* $\mathrm{Hom}_H(\pi(\chi),\mu) \neq \{0\}$ *if and only if E/F is unramified and $\chi_{|L_0^*} = \mu_{|F^*} \circ N_{L_0/F}$, in which case* $\mathrm{Hom}_H(\pi(\chi),\mu) \simeq \mathbb{C}$.

Proof. — Multiplicity zero in the ramified case is immediate from Lemma 2.3 and Theorem 1.6. When E/F is unramified in Lemma 2.3 and Theorem 1.6 imply that

$$\mathrm{Hom}_H(\pi,\mu) = \mathrm{Hom}_{H\,\cap\,\mathcal{K}}\left(\lambda_\chi,\mu\right),$$

which is zero if $\chi_{|F^*} \neq \mu^m_{|F^*}$. If $\chi_{|F^*} = \mu^m_{|F^*}$ (which is in particular true when $\chi_{|L_0^*} = \mu_{|F^*} \circ N_{L_0/F}$) we obtain

$$\mathrm{Hom}_{H\,\cap\,\mathcal{K}}\left(\lambda_\chi,\mu\right) = \mathrm{Hom}_{H\,\cap\,K}\left(\lambda_\chi,\mu\right) = \mathrm{Hom}_{\overline{H}}\left(\overline{\pi}_{\overline{\chi}},\overline{\mu}\right).$$

The result then follows from [19, Proposition 4.3] (which has the assumption $n \geqslant 4$). $\square$

3. On μ-selfduality of μ-distinguished representations

Now we take μ any character of E^* with no restriction on its conductor. We intend to prove that μ-distinguished representations of cuspidal (of any level) representations of any inner form of $\mathrm{GL}_n(F)$ is μ-selfdual automatically. Our result will follow from a classical globalization argument, and the case of principal series for split inner forms.

Proposition 3.1. — *Let π be a generic principal series of* $\mathrm{GL}_n(F)$ *(induced from a character of a Borel subgroup), and μ_1 be a character of $F^* \times F^*$, and μ_2 be a character of E^*. Let H_1 be the block diagonal subgroup* $\mathrm{GL}_m(F) \times \mathrm{GL}_m(F)$ *and H_2 be the subgroup* $H \simeq \mathrm{GL}_m(E)$ *of* $\mathrm{GL}_n(F)$. *Then if π is μ_i-distinguished by H_i, then*

$$\pi \simeq \mu_{i_{|F^*}} \otimes \pi^\vee$$

(where F^* is diagonally embedded in $F^* \times F^*$ in the first case)

Proof. — We only do the case (H_1, μ_1), as the argument for (H_2, μ_2) is completely similar but simpler due to simplification of quotients of modulus characters (see [9] for the parametrization of double cossets involved there, and [5, (5.3) and Remark 5.4] for the modulus characters involved). Here we rather consider distinction by the conjugate H of H_1 by the matrix w_n of [15, p. 121], and set $h(g_1, g_1) = w_n^{-1}\mathrm{diag}(g_1, g_2)w_n$ for $g_i \in \mathrm{GL}_m(F)$. The character μ_1 is of the form $\mu_{\alpha,\beta}(h(g_1, g_1)) = \alpha(\det(g_1))\beta(\det(g_2))$ for α and β characters of F^*. Let B be the upper triangular Borel subgroup of $G = \mathrm{GL}_n(F)$ and χ be a character of the diagonal torus A of G such that $\pi = Ind_B^G(\chi)$ is generic. We want to show that if π is $\mu_{\alpha,\,\beta}$-distinguished, then

$$\pi \simeq \alpha\beta \otimes \pi^\vee.$$

This amounts to prove that there is a permutation $\sigma \in S_n$ such that

$$\alpha\beta\chi^{-\sigma} = \chi, \tag{3.1}$$

where by abuse of notation

$$(\alpha\beta)\big(\mathrm{diag}(a_1, \ldots, a_n)\big) = \prod_{i=1}^{n} (\alpha\beta)(a_i).$$

We will do this by using Mackey theory, i.e. the natural filtration of $Ind_B^G(\chi)_{|H}$ with sub-quotients

$$ind_{u^{-1}Bu\,\cap\,H}^H \left(\left(\delta_B^{1/2}\chi\right)^{u^{-1}}\right)$$

when u varies through a set of representatives of $B\backslash G/H$

$$(\text{and} \quad \left(\delta_B^{1/2}\chi\right)^{u^{-1}} := \delta_B^{1/2}\chi\left(u\,.\,u^{-1}\right)),$$

so that if

$$\mathrm{Hom}_H\left(Ind_B^G(\chi), \mu_{\alpha,\,\beta}\right) \neq \{0\}$$

then some space

$$\begin{aligned}\mathrm{Hom}_H\left(ind_{u^{-1}Bu\,\cap\,H}^H\left(\left(\delta_B^{1/2}\chi\right)^{u^{-1}}\right), \mu_{\alpha,\,\beta}\right) \\ \simeq \mathrm{Hom}_{B\cap uHu^{-1}}\left(\delta_B^{1/2}\chi, \delta_{B\,\cap\,uHu^{-1}}\mu_{\alpha,\,\beta}^u\right) \\ \subseteq \mathrm{Hom}_{A\,\cap\,uHu^{-1}}\left(\chi, \frac{\delta_{B\,\cap\,uHu^{-1}}}{\delta_B^{1/2}}\mu_{\alpha,\,\beta}^u\right)\end{aligned}$$

must be nonzero (in fact with the representatives u given in [15, Section 3.2] the last inclusion is an equality). This will tell us that χ has to be of a particular form which will give the existence of σ such that Equation (3.1) is satisfied.

We set

$$\epsilon = \operatorname{diag}(1, -1, \ldots, 1, -1) \in G$$

so that H is the subgroup of G fixed under the conjugation θ_ϵ by ϵ. By a re-interpretation of the discussion in [15, Section 3.2], the double cosets $B\backslash G/H$ are parametrized by couples $s = (w_s, x_s)$ where $w_s \in S_n \subset G$ is an involution, and x_s is a map from the set of fixed points $\mathrm{Fix}(w_s)$ of w_s in $\{1, \ldots, n\}$ to $\{\pm 1\}$, such that $|x_s^{-1}(\{-1\})| = |x_s^{-1}(\{1\})| = \frac{|\mathrm{Fix}(w_s)|}{2}$. The corresponding representative u_s in $B\backslash G/H$ in particular satisfies $u_s \epsilon u_s^{-1} \epsilon^{-1} = w_s$, and we set

$$\theta_s(x) = w_s \theta_\epsilon(x) w_s^{-1} = u_s \theta_\epsilon\left(u_s^{-1} x u_s\right) u_s^{-1}$$

for $x \in G$. Conjugation by u_s stabilizes A, and θ_s as well. Suppose that π is $\mu_{\alpha,\beta}$-distinguished, by the Mackey strategy discussed above (see also before Theorem [15, Theorem 3.14]), there is s such that

$$\chi_{|A^{\theta_s}} = \left(\delta_{B^{\theta_s}} \delta_B^{-1/2} \mu_{\alpha,\beta}^{u_s}\right)_{|A^{\theta_s}},$$

where the exponent θ_s denotes the fixed points of θ_s in the corresponding set (which is not necessarily θ_s-stable, for example B), and $\mu_{\alpha,\beta}^{u_s}(a_s) = \mu_{\alpha,\beta}(u_s^{-1} a_s u_s)$ for $a_s \in A^{\theta_s}$. The character $\delta_{B^{\theta_s}} \delta_B^{-1/2}$ restricted to A^{θ_s} is computed in [15, Proposition 3.6]. We extend x_s from $\mathrm{Fix}(w_s)$ to $\{1, \ldots, n\}$ by 0 outside $\mathrm{Fix}(w_s)$. Then for $a = \operatorname{diag}(a_1, \ldots, a_n) \in A^{\theta_s}$ one has:

$$\delta_{B^{\theta_s}} \delta_B^{-1/2}(a) = \prod_{1 \leqslant i < j \leqslant n} |a_i|^{\frac{x_s(i)x_s(j)}{2}} |a_j|^{-\frac{x_s(i)x_s(j)}{2}}.$$

On the other hand by a computation similar to that done in the proof of [15, proposition 3.6], we have for $a = \operatorname{diag}(a_1, \ldots, a_n) \in A^{\theta_s}$ (note that for any i one has $a_{w_s(i)} = a_i$):

$$\mu_{\alpha,\beta}^{u_s}(a) = \prod_{i \in x_s^{-1}(\{1\})} \alpha(a_i) \prod_{i \in x_s^{-1}(\{-1\})} \beta(a_i) \prod_{i \in x_s^{-1}(\{0\}),\, i < w_s(i)} \alpha\beta(a_i).$$

For $a \in A$ we set $w_s(a) = w_s a w_s^{-1}$, so that $a w_s(a) \in A^{\theta_s}$, then from the relations above it follows that for $a \in A$ (note that $x_s \circ w_s = -x_s$ and is order reversing on $\{1, \ldots, n\} - \mathrm{Fix}(w_s)$):

$$\chi(a w_s(a)) = \alpha(a)\beta(a), \quad \text{i.e.} \quad \chi\chi^{w_s} = \alpha\beta$$

so we can choose the sought $\sigma \in S_n$ to be w_s. □

As in [5, Proposition 5.2], we deduce from Proposition 3.1, using the globalization results of [20] and [11] together with the strong multiplicity one theorems from [1] and [2], the following result.

Proposition 3.2. — *Let D be an F-division algebra of index d and n a positive integer such that nd is even, let H be the centralizer of E in $G = \mathrm{GL}_n(D)$. Let μ be a character of E^* identified via the reduced norm to a character of H, then a cuspidal representation π of G which is μ-distinguished satisfies*

$$\pi \simeq \mu_{|F^*} \otimes \pi^\vee.$$

Here are two important corollaries for depth-zero cuspidal representations of $\mathrm{GL}_n(F)$.

Corollary 3.3. — *Let π be a cuspidal representation of $\mathrm{GL}_n(F)$ which is of depth zero, and μ-distinguished, then automatically $\mu_{|F^*}$ is tame (i.e. $\mu(1+\mathcal{P}_F)=1$).*

Proof. — Write $\pi = \pi(\chi)$. By Proposition 3.2 we have $\chi^\gamma = \mu_{|F^*} \circ N_{L/F}.\chi^{-1}$ for some $\gamma \in \mathrm{Gal}_F(L)$. But because χ^γ and χ^{-1} are both tame, the result follows from the fact that $N_{L/F}(1+\mathcal{P}_L) = 1+\mathcal{P}_F$. □

We denote by L_0 the unramified extension of F of degree m.

Corollary 3.4. — *Suppose that $n \geqslant 4$. Let $\pi(\chi)$ be a cuspidal μ-distinguished representation of $\mathrm{GL}_n(F)$ of depth zero. Then*

$$\chi_{|L_0^*} = \mu_{|F^*} \circ N_{L_0/F}.$$

Proof. — Thanks to Proposition 3.2, there is $\gamma \in \mathrm{Gal}_F(L)$ such that $\chi^\gamma = \mu_{|F^*} \circ N_{L/F}\chi^{-1}$. Because χ and and $\mu_{|F^*}$ are tame this this reduces to $\overline{\chi}^{\overline{\gamma}} = \overline{\mu_{|F^*}} \circ N_{L/F}\overline{\chi}^{-1}$. This implies that $\chi^{\gamma^2} = \chi$, hence that γ has order dividing two because χ is regular. If γ was trivial one would have $\chi^2 = \mu_{|F^*} \circ N_{L/F}$. Because χ and $\mu_{|F^*}$ are tame this would imply

$$\overline{\chi}^2 = \overline{\mu} \circ N_{k_L/k_F}.$$

But the group of characters of the form $\alpha \circ N_{k_L/k_F}$ for α a character of k_F^* form a group of order $q_F - 1$ so one should have $\overline{\chi}^{2(q_F-1)} = 1$. But because χ is regular and $n \geqslant 4$, the character $\overline{\chi}^{q_F^2-1}$ must be nontrivial, hence $\overline{\chi}^{2(q_F-1)} \neq 1$. Thus γ is the conjugation of L/L_0 so $\chi \circ N_{L/L_0} = \mu_{|F^*} \circ N_{L/F}$, and χ and $\mu_{|F^*} \circ N_{L_0/F}$ agree on the units of L_0^* because L/L_0 is unramified. Finally they also agree on ϖ_F by central character considerations. □

4. Distinction of depth-zero cuspidal representations

We want to show that the necessary condition obtained in the above section is also sufficient when μ is not tame. By Proposition 1.7 and Lemma 2.2, the contribution to distinction in Mackey formula will in this case arise from double cosets in $H\backslash G/\mathcal{K}$ corresponding to H-orbits of non θ-fixed vertices of X_G. For such double cosets, the distinction problem reduces residually to the existence of a twisted Shalika model, which have been studied by Prasad in [18]. We recall his result.

4.1. Twisted Shalika models over finite fields

Let π be an irreducible representation of $\mathrm{GL}_n(k_F)$, and α be a character of k_F^*, and ψ be a nontrivial character of k_F. We recall that we call the Shalika subgroup of $\mathrm{GL}_n(k_F)$ the group:

$$S_n(k_F) = \left\{ \begin{pmatrix} g & \\ & g \end{pmatrix} \begin{pmatrix} I_m & x \\ & I_m \end{pmatrix}, \quad g \in \mathrm{GL}_m(k_F),\ x \in \mathcal{M}_m(k_F) \right\}.$$

On then defines the character Ψ_α of $S_n(k_F)$ by the formula:

$$\Psi_\alpha \left(\begin{pmatrix} g & \\ & g \end{pmatrix} \begin{pmatrix} I_m & x \\ & I_m \end{pmatrix} \right) = \alpha(\det(g))\psi(\mathrm{Tr}(x)).$$

We say that π has an *α-twisted Shalika model* if

$$\mathrm{Hom}_{S_n(k_F)}(\pi, \Psi_\alpha) \neq 0,$$

and this does not depend on the choice of ψ. The following proposition is due to Prasad ([18, Theorem 1]).

Proposition 4.1. — *Let $\overline{\pi}_{\overline{\chi}}$ be a cuspidal representation of $\mathrm{GL}_n(k_F)$, then $\overline{\pi}_{\overline{\chi}}$ has an α-twisted Shalika model if and only if $\overline{\chi}_{|k_{L_0}^*} = \alpha \circ N_{L_0/F}$ in which case $\mathrm{Hom}_{S_n(k_F)}(\overline{\pi}_{\overline{\chi}}, \Psi_\alpha) \simeq \mathbb{C}$.*

Proof. — We denote by N the subgroup of matrices

$$n(x) = \begin{pmatrix} I_m & x \\ & I_m \end{pmatrix}$$

in $\mathrm{GL}_n(k_F)$ and by $(\overline{\pi}_{\overline{\chi}})_{N,\Psi}$ the quotient of $\overline{\pi}_{\overline{\chi}}$ by $\{v - \psi(\mathrm{Tr}(x))v,\ n(x) \in N,\ v \in \overline{\pi}_{\overline{\chi}}\}$. The space $(\overline{\pi}_{\overline{\chi}})_{N,\Psi}$ is a $\mathrm{GL}_n(k_F)$-module (for diagonal action). Then by [18, Theorem 1], we have

$$(\overline{\pi}_{\overline{\chi}})_{N,\Psi} = \mathrm{Ind}_{k_{L_0}^*}^{\mathrm{GL}_n(k_F)} \left(\overline{\chi}_{|k_{L_0}^*} \right).$$

Now by definition we have

$$\mathrm{Hom}_{S_n(k_F)}\left(\overline{\pi}_{\overline{\chi}}, \Psi_\alpha\right) \simeq \mathrm{Hom}_{\mathrm{GL}(m,\, k_F)}\left(\mathrm{Ind}_{k_{L_0}^*}^{\mathrm{GL}_n(k_F)}\left(\overline{\chi}_{|k_{L_0}^*}\right), \alpha\circ\det\right)$$

and this latter space is isomorphic to

$$\mathrm{Hom}_{k_{L_0}^*}\left(\overline{\chi}_{|k_{L_0}^*}, \alpha\circ\det\right) = \mathrm{Hom}_{k_{L_0}^*}\left(\overline{\chi}_{|k_{L_0}^*}, \alpha\circ N_{L_0/F}\right),$$

and the statement follows. □

Remark 4.2. — The condition in Proposition 4.1 is also equivalent to $\overline{\pi}_{\overline{\chi}} \simeq \alpha\otimes\overline{\pi}_{\overline{\chi}}^\vee$.

4.2. Double cosets contributing to distinction

Take $\Delta \in F^\times$ with square root δ generating E/F, which we take of valuation 0 when E/F is unramified and of valuation 1 when E/F is ramified. The subgroup H of $\mathrm{GL}_n(F)$ consists of invertible matrices of the form $\left(\begin{smallmatrix} a & b \\ \Delta b & a\end{smallmatrix}\right)$. The character μ of H satisfies

$$\mu\begin{pmatrix} a & b \\ \Delta b & a\end{pmatrix} = \mu(\det(a+\delta b)).$$

First we identify a non trivial double coset contributing to distinction when μ has conductor $\geqslant 2$. Note that when E/F is ramified, if μ has conductor $l\geqslant 2$ and is trivial on $1+\mathcal{P}_F$, then it has an even conductor, because of the isomorphism $\overline{x}\mapsto 1+\varpi_F^d\overline{x}$ between $k_F=k_E$ and $\frac{1+\mathcal{P}_F^{2d}}{1+\mathcal{P}_F^{2d+1}}$ for any $d\geqslant 1$.

Proposition 4.3. — *Suppose μ has conductor $r+1\geqslant 2$ but satisfies $\mu(1+\mathcal{P}_F)=1$. We set $l=r$ if E/F is unramified, whereas we set $l=(r-1)/2$ when E/F is ramified. Set $d_l=\mathrm{diag}(\varpi_F^l I_m, I_m)$ and suppose that $\chi_{|L_0^*}=\mu_{|F^*}\circ N_{L_0/F}$, then*

$$\mathrm{Hom}_{\mathcal{K}\cap d_l^{-1}Hd_l}\left(\lambda_\chi, \mu^{d_l}\right)\neq 0,$$

where

$$\mu^{d_l}(x) = \mu\left(d_l x d_l^{-1}\right).$$

Proof. — First the condition $\chi_{|L_0^*}=\mu_{|F^*}\circ N_{L_0/F}$ implies that $\chi_{|F^*}=\mu_{|F^*}^m$, hence

$$\mathrm{Hom}_{\mathcal{K}\cap d_l^{-1}Hd_l}\left(\lambda_\chi, \mu^{d_l}\right)\simeq \mathrm{Hom}_{K\cap d_l^{-1}Hd_l}\left(\lambda_\chi, \mu^{d_l}\right).$$

The group $K \cap d_l^{-1} H d_l$ is the set of matrices

$$\begin{pmatrix} a & \varpi_F^{-l} b \\ \varpi_F^l \Delta b & a \end{pmatrix}$$

with $a \in \mathrm{GL}_m(\mathcal{O}_F)$ and $b \in \mathcal{M}_m(\mathcal{P}_F^l)$, and

$$\mu^{d_l} \begin{pmatrix} a & \varpi_F^{-l} b \\ \varpi_F^l \Delta b & a \end{pmatrix} = \mu(\det(a + \delta b)).$$

But

$$\begin{aligned} \det(a+\delta b) &= \det(a)\det\left(I_m + \delta a^{-1} b\right) \\ &= \det(a)\left(1 + \mathrm{Tr}\left(\delta a^{-1} b\right)\right)\left[\mathcal{M}_m\left(\mathcal{P}_F^{l+1}\right)\right] \end{aligned}$$

so

$$\mu^{d_l} \begin{pmatrix} a & \varpi_F^{-l} b \\ \varpi_F^l \Delta b & a \end{pmatrix} = \mu(\det(a))\mu\left(1 + \mathrm{Tr}\left(\delta a^{-1} b\right)\right),$$

where the dependences are in fact in $\overline{a} \in \mathrm{GL}_m(k_F)$ and $\overline{b} \in \mathcal{M}_m(\mathcal{P}_F^l/\mathcal{P}_F^{l+1})$. So in fact for $a \in \mathrm{GL}_m(\mathcal{O}_F)$ and $b \in \mathcal{M}_m(\mathcal{O}_F)$ we have

$$\mu^{d_l} \begin{pmatrix} a & b \\ \varpi_F^{2l} \Delta b & a \end{pmatrix} = \overline{\mu^{d_l}} \begin{pmatrix} \overline{a} & \overline{b} \\ & \overline{a} \end{pmatrix} = \overline{\mu_{|F^*}}(\det(a))\mu\left(1 + \varpi_F^l \delta \mathrm{Tr}\left(\overline{a}^{-1}\overline{b}\right)\right).$$

The character $\psi(x) = \mu(1 + \varpi_F^l \delta \overline{x})$ is a nontrivial character of k_F because $\mu(1 + \mathcal{P}_F) = 1$ whereas μ has conductor $r+1$. On the other hand

$$\lambda_\chi \begin{pmatrix} a & b \\ \varpi_F^{2l} \Delta b & a \end{pmatrix} = \overline{\pi}_{\overline{\chi}} \begin{pmatrix} \overline{a} & \overline{b} \\ & \overline{a} \end{pmatrix}.$$

Hence $\overline{\pi}_{\overline{\chi}}$ has a α-twisted Shalika model and the result follows from Proposition 4.1. □

4.3. Multiplicity one when E/F is unramified

In this section E/F is ramified. We denote by Λ_m^+ the sequences of integers $(\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 \geqslant \ldots \geqslant \lambda_m \geqslant 0$ in $\mathbb{Z}^m$, and set for $\lambda \in \Lambda_m^+$:

$$d_\lambda = \mathrm{diag}\left(\varpi_F^{\lambda_1}, \ldots, \varpi_F^{\lambda_m}, 1, \ldots, 1\right) \in G.$$

We recall from [17] the following result:

Proposition 4.4. —

$$G = \coprod_{\lambda \in \Lambda_m^+} K d_\lambda H.$$

Proof. — For $\lambda \in \Lambda_m^+$ we set $\varpi_F^\lambda = \mathrm{diag}(\varpi_F^{\lambda_1}, \ldots, \varpi_F^{\lambda_m})$, we also set

$$w_m = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in \mathrm{GL}_m(F)$$

and $w = \mathrm{diag}(I_m, w_m)$. It follows from [12] that the map $p : x \mapsto xAx^{-1}$ identifies G/H with the conjugacy class of A. The matrix d_λ is sent by p to

$$\begin{pmatrix} & \varpi_F^\lambda \\ \Delta \varpi_F^{-\lambda} & \end{pmatrix},$$

the result now follows from [17, Proposition 4], noting that the group H in [17] is equal the centralizer of wAw^{-1} whereas here it is the centralizer of A. □

One has the following multiplicity one result:

PROPOSITION 4.5. — *Let $\pi(\chi)$ be a cuspidal representation of $\mathrm{GL}_n(F)$ of depth zero for $n \geqslant 4$, and suppose that E/F is unramified. If it is μ-distinguished, then $\mathrm{Hom}_{\mathrm{GL}_m(E)}(\pi(\chi), \mu) \simeq \mathbb{C}$.*

Proof. — Suppose that $\pi(\chi)$ is μ-distinguished so that $\mu_{|F^*} = \alpha \circ N_{L_0/F}$ thanks to Corollary 3.4. The result follows from Theorem 2.4 when μ is tame so we suppose that μ has conductor $l+1 \geqslant 2$. By Mackey theory, the result will follow from Propositions 4.1, 4.3 and 4.4 if we show that if

$$\mathrm{Hom}_{K \cap d_\lambda^{-1} H d_\lambda}\left(\lambda_\chi, \mu^{d_\lambda}\right) = \mathrm{Hom}_{K \cap d_\lambda^{-1} H d_\lambda}\left(\lambda_\chi, \mu^{d_\lambda}\right) \neq 0$$

for $\lambda \in \Lambda_m^+$, then $\lambda = (l, \ldots, l)$. Note that $K \cap d_l^{-1} H d_l$ is the set of matrices

$$\begin{pmatrix} a & \varpi_F^{-\lambda} b \\ \varpi_F^\lambda \Delta b & a \end{pmatrix}$$

with $a \in \mathrm{GL}_m(\mathcal{O}_F)$ and $l_i(b) \in (\mathcal{P}_F^{\lambda_i})^m$ for $i = 1, \ldots, m$, where $l_i(b)$ is i^{th} row of b. So we assume that $\mathrm{Hom}_{K \cap d_\lambda^{-1} H d_\lambda}(\lambda_\chi, \mu^{d_\lambda}) \neq 0$.

Suppose first that $\lambda_m \leqslant l-1$ and denote by $\mathcal{M}_m(\mathcal{O}_F)^-$ the space of matrices in $\mathcal{M}_m(\mathcal{O}_F)$ with $l_i(b) = 0$ for $i = 1, \ldots, m-1$ and $l_m(b) \in (\mathcal{P}_F^{\lambda_m+1})^m$. Because $\pi(\chi)$ is tame, if $\mathrm{Hom}_{K \cap d_\lambda^{-1} H d_\lambda}(\lambda_\chi, \mu^{d_\lambda})$ was nonzero this would imply that

$$1 = \mu^{d_l} \begin{pmatrix} I_m & \varpi_F^{-\lambda} b \\ \varpi_F^\lambda \Delta b & I_m \end{pmatrix} = \mu\big(\det(I_m + \delta l_m(b))\big)$$

for all $b \in \mathcal{M}_m(\mathcal{O}_F)^-$, hence that $\mu(1 + \delta \mathcal{P}_F^l) = \{1\}$. Because $\mu(1 + \mathcal{P}_F) = \{1\}$ as well, this would in turn imply that $\mu(1 + \mathcal{P}_E^l) = \mu(1 + \mathcal{P}_F^l + \delta \mathcal{P}_F^l) = \{1\}$, contradicting the definition of l, hence $\lambda_m \geqslant l$. Now let s be the smallest integer between 1 and m such that $\lambda_s = \lambda_m$, by the arguments of Proposition 4.3 we obtain that

$$\mu^{d_\lambda}\begin{pmatrix} a & b \\ \varpi_F^{2\lambda}\Delta b & a \end{pmatrix} = \mu(\det(a))\mu\left(1 + \mathrm{Tr}\left(\delta a^{-1}\varpi_F^{\lambda} b\right)\right)$$

for $a \in \mathrm{GL}_m(\mathcal{O}_F)$ and $b \in \mathcal{M}_m(\mathcal{O}_F)$. By reduction we deduce that

$$\begin{aligned}\mu^{d_\lambda}\begin{pmatrix} \overline{a} & \overline{b} \\ & \overline{a} \end{pmatrix} &= \mu(\det(\overline{a}))\,\mu\,\left(1 + \mathrm{Tr}\left(\delta \overline{a}^{-1}\varpi_F^{\lambda}\overline{b}\right)\right) \\ &= \mu(\det(\overline{a}))\,\mu\,\left(1 + \varpi_F^{\lambda_m}\mathrm{Tr}\left(\delta\overline{a}^{-1}\mathrm{diag}\left(0_{s-1}, I_{m-s+1}\right)\overline{b}\right)\right)\end{aligned}$$

for $a \in \mathrm{GL}_m(k_F)$ and $b \in \mathcal{M}_m(k_F)$. However the identity

$$\overline{\pi}_{\overline{\chi}}\begin{pmatrix} \overline{a} & \overline{b} \\ & \overline{a} \end{pmatrix} = \mu^{d_\lambda}\begin{pmatrix} \overline{a} & \overline{b} \\ & \overline{a} \end{pmatrix}\mathrm{Id}$$

first implies that if $\lambda_m > l$ then the unipotent radical of type (m, m) acts trivially on the space $\overline{\pi}_{\overline{\chi}}$ contradicting its cuspidality, hence $\lambda_m = l$. It also implies that

$$\overline{b} \mapsto \mu\left(1 + \delta\varpi_F^{\lambda_m}\mathrm{Tr}\left(\mathrm{diag}\left(0_{s-1}, I_{m-s+1}\right)\overline{b}\right)\right)$$

must be invariant under conjugation by $\mathrm{GL}_m(k_F)$, which in turn implies that $s = 1$ hence $\lambda_1 = \cdots = \lambda_m = l$ □

Remark 4.6. — A similar analysis could certainly be done when E/F is ramified but we don't have at our disposal the description of the double coset representatives given by [17] in the unramified case. As we can still prove the Prasad and Takloo-Bighash conjecture in this case, without computing the exact multiplicity, we do not pursue this direction.

4.4. Characterization of distinction of level zero cuspidal representations

The spaces

$$\mathrm{Hom}_{\mathcal{K}\cap d_{l'}^{-1}Hd_{l'}}\left(\lambda_\chi, \mu^{d_{l'}}\right)$$

are isomorphic to a subspace of $\mathrm{Hom}_H(\pi(\chi), \mu)$ thanks to Mackey theory for compact induction from open subgroups. Hence as a corollary of Theorem 2.4, Corollary 3.4, Propositions 4.3 and 4.5, we deduce the all assertions of the following theorem.

THEOREM 4.7. — *For $n \geqslant 4$, the depth-zero cuspidal representation $\pi(\chi)$ of $\mathrm{GL}_n(F)$ is μ-distinguished if and only if $\chi_{|L_0^*} = \mu_{|F^*} \circ N_{L_0/F}$, except when E/F is ramified and μ is tame, in which case $\pi(\chi)$ is never μ-distinguished. When μ is tame or E/F is unramified, the dimension of $\mathrm{Hom}_H(\pi(\chi), \mu)$ is one when nonzero.*

5. On $\mu_{|F^*}$-symplecticity for Langlands parameters

In this section, we will freely identify characters of the Weil group W_K and characters of K^* for any finite extension K of F via the Artin map. We fix α a character of F^*. For ϕ a finite dimensional irreducible representation of W_F, we say that ϕ is *α-selfdual* if

$$\phi \simeq \alpha \otimes \phi^\vee.$$

This is equivalent to say that there exists a non-zero bilinear form B (necessarily non-degenerate) which satisfies

$$B\big(\phi(w)v, \phi(w)v'\big) = \alpha(w)B\,(v, v')$$

for all $v,\ v'$ in V_ϕ and $w \in W_F$. By Schur's Lemma the space of such bilinear forms B is one dimensional hence B is either symmetric or alternate, but not both. In the first case we say that ϕ is *α-orthogonal* and in the second case we say that it is *α-symplectic*. As as we shall see later, one way to discriminate between α-orthogonal and α-symplectic tame irreducible representations is the determinant. The proof of the following lemma which we present here is that of the referee, strictly speaking we only need its $m = 1$ case to obtain Theorem 5.4 which is the main result of the section.

LEMMA 5.1. — *Let ϕ be an irreducible representation of W_F of dimension n which is α-symplectic, then $\det(\phi) = \alpha^m$. Moreover if $m = 1$ the converse is true.*

Proof. — Let B be the nonzero alternate form on the space V of ϕ with respect to which ϕ is α-symplectic. View B as an element of the exterior product $\bigwedge^2 V^*$ (where $V^*$is the dual of V), because B is non degenerate the vector $\bigwedge^m B$ is a nonzero element of the line $\bigwedge^n V^*$ on which W_F acts at the same time by $\det(\phi)$ and α^m. The converse when $m = 1$ is also clear now. □

We recall that irreducible representations of the form $\mathrm{Ind}_{W_L}^{W_F}(\chi')$ with $\chi'(1 + \mathcal{P}_L) = 1$ are exactly the *tame* n-dimensional irreducible representation of W_F, i.e. those with trivial restriction to the wild inertia subgroup of W_F. Characterizing α-selfdual tame irreducible representations of W_F is easy for $n \geqslant 4$.

Lemma 5.2. — *Let $\phi = Ind_{W_L}^{W_F}(\chi')$ be a tame irreducible representation of W_F of dimension $n \geqslant 4$. Then ϕ is α-selfdual if and only if $\chi' \circ N_{L/L_0} = \alpha \circ N_{L/F}$, i.e. if and only if $\chi'_{|L_0^*} = \alpha \circ N_{L_0/F}$ or $\chi'_{|L_0^*} = \alpha \circ N_{L_0/F}.\eta_{L/L_0}$ where η_{L/L_0} is the unramified quadratic character of L_0^*.*

Proof. — One direction is obvious. For the other if ϕ is α-selfdual, then $\chi'^{\gamma} \simeq \alpha \circ N_{L/F}\chi'^{-1}$ for some $\gamma \in \mathrm{Gal}_F(L)$. We conclude as in the proof of Corollary 3.4 that $\gamma = \sigma_{L/L_0}$ the Galois involution attached to L/L_0. □

A less obvious task is to distinguished between α-symplectic and α-orthogonal representations in the statement above. The following lemma, which was given to us by the referee, turns out to be useful for this.

Lemma 5.3. — *Let K be a subgroup of finite index of a group G, $\alpha : G \to \mathbb{C}^*$ a character. Let W be a finite dimensional representation of K and $V = Ind_K^G(W)$. If W is $\alpha_{|K}$-selfdual, resp. $\alpha_{|K}$-orthogonal, resp. $\alpha_{|K}$-symplectic, then V is α-selfdual, resp. α-orthogonal, resp. α-symplectic.*

Proof. — Take B a non-degenerate bilinear form on W such that $B(k.w, k.w') = \alpha(k)B(w,w')$ for $(k,w,w') \in K \times W \times W$. Then the bilinear form defined on $V \times V$ by

$$\widetilde{B}(f,f') = \sum_{g \in K\backslash G} \alpha^{-1}(g)B(f(g),f'(g))$$

is non-degenerate (because for each $g_0 \in G$ and $w \in W$ there is a function $f_{g_0,w} \in Ind_K^G(W)$ supported on Kg_0 such that $f_{g_0,w}(g_0) = w$), α-equivariant on $V \times V$, of the same type as B. □

Lemmas 5.1, 5.2 and 5.3 imply the following theorem (when α is trivial similar arguments are used in [3, Section 6.1] where Lemma 5.3 is tacitly used). We thank the referee for the simplifications provided in its proof.

Theorem 5.4. — *Let $\phi = Ind_{W_L}^{W_F}(\chi')$ be a tame irreducible representation of W_F of dimension $n \geqslant 4$. Then ϕ is α-symplectic if and only if $\chi'_{|L_0^*} = \alpha \circ N_{L_0/F}.\eta_{L/L_0}$.*

Proof. — Let's first have a look at $\phi_0 := Ind_{W_L}^{W_{L_0}}(\chi'_{|L_0^*})$. According to Lemma 5.1 the representation ϕ_0 is $\alpha \circ N_{L_0/F}$-symplectic if and only if its determinant is equal to $\alpha \circ N_{L_0/F}$. This is well-known to be the same as $\chi'_{|L_0^*} = \alpha \circ N_{L_0/F}.\eta_{L/L_0}$ (see for example [7, Proposition 29.2]). In particular if $\chi'_{|L_0^*} = \alpha \circ N_{L_0/F}.\eta_{L/L_0}$ then ϕ is α-symplectic thanks to Lemma 5.3.

Conversely if ϕ is α-symplectic then by Lemma 5.2 we have $\chi'_{|L_0^*} = \alpha \circ N_{L_0/F}$ or $\chi'_{|L_0^*} = \alpha \circ N_{L_0/F}.\eta_{L/L_0}$. If we were in the first case then ϕ_0 would be $\alpha \circ N_{L_0/F}$-orthogonal hence ϕ in turn α-orthogonal by Lemma 5.3, which is not possible as ϕ is irreducible. Hence $\chi'_{|L_0^*} = \alpha \circ N_{L_0/F}.\eta_{L/L_0}$. □

Remark 5.5. — In fact Lemma 5.1 allows another criterion to discriminate between α-symplectic and α-orthogonal representations in Lemma 5.2, namely ϕ is α-symplectic if and only if $\det(\phi) = \alpha^m$. Indeed if we had $\chi'_{|L_0^*} = \alpha \circ N_{L_0/F}$, retricting to F^* we would get $\chi'_{|F^*} = \alpha^m$ and this would contradict Lemma 5.1 because according to [7, Proposition 29.2] one has $det(\phi) = \eta_{K/F}\chi'_{|F^*}$ for K/F the quadratic unramified extension of F and $\eta_{K/F}$ its corresponding quadratic character.

The Langlands parameter of the representation $\pi(\chi)$is given by [8, Theorem 2]: it is

$$\phi\left(\pi(\chi)\right) := \mathrm{Ind}_{W_L}^{W_F}(\eta\chi)$$

where η is the unramified quadratic character of L^*. Hence Theorem 5.4 has the following immediate corollary.

COROLLARY 5.6. — *For $n \geqslant 4$ (i.e. $m \geqslant 2$), the representation $\pi(\chi)$ is $\mu_{|F^*}$-symplectic if and only if $\chi_{|L_0^*} = \mu_{|F^*} \circ N_{L_0/F}$.*

6. The Prasad and Takloo-Bighash conjecture

We recall that the conjecture of Prasad and Takloo-Bighash has been proved by Tunnel and also Saito when $n = 2$ ([27, Theorem p. 1277] in residual characteristic not 2, [22, Theorem p. 99] in characteristic not 2), *hence in this Section we assume $n \geqslant 4$*. So comparing the statements of Theorem 4.7 and Corollary 5.6, it is enough to compute the Prasad and Takloo-Bighash ϵ value of a cuspidal depth-zero representation $\pi(\chi)$ with $\chi_{|L_0^*} = \mu_{|F^*} \circ N_{L_0/F}$, and to show that it is as expected by the conjecture when E/F is unramified or E/F is ramified and μ is not tame, and differs from the expected value when E/F is ramified and μ is tame. Again in the proof we will freely confuse characters of Weil groups and of multiplicative groups of local fields (hence restrictions will be often written as composition with the norm map).

Let's do some preliminary computations before computing the ϵ factor of the Prasad and Takloo-Bighash conjecture. When E/F is *unramified* we have:

$$\begin{aligned}\mathrm{Ind}_{W_L}^{W_F}(\eta\chi)\otimes\mathrm{Ind}_{W_E}^{W_F}\left(\mu^{-1}\right)\\ &=\mathrm{Ind}_{W_L}^{W_F}\left(\eta\chi\otimes\mathrm{Ind}_{W_E}^{W_F}\left(\mu^{-1}\right)_{|W_L}\right)\\ &=\mathrm{Ind}_{W_L}^{W_F}\left(\eta\chi\left(\mu^{-1}\circ N_{L/E}\right)\oplus\eta\chi\left(\mu^{-\sigma_{E/F}}\circ N_{L/E}\right)\right)\end{aligned}$$

by Mackey's restriction formula with

$$\begin{aligned}\left\langle\sigma_{E/F}\right\rangle=\mathrm{Gal}_F(E)\\ &=\mathrm{Ind}_{W_L}^{W_F}\left(\eta\chi\left(\mu^{-1}\circ N_{L/E}\right)\right)\oplus\mathrm{Ind}_{W_L}^{W_F}\left(\eta\chi\left(\mu^{-\sigma_{E/F}}\circ N_{L/E}\right)\right).\end{aligned}$$

When E/F is *ramified* we have:

$$\begin{aligned}\mathrm{Ind}_{W_L}^{W_F}(\eta\chi)\otimes\mathrm{Ind}_{W_E}^{W_F}\left(\mu^{-1}\right)&=\mathrm{Ind}_{W_L}^{W_F}\left(\eta\chi\otimes\mathrm{Ind}_{W_E}^{W_F}\left(\mu^{-1}\right)_{|W_L}\right)\\ &=\mathrm{Ind}_{W_E}^{W_F}\left(\eta\chi\otimes\mathrm{Ind}_{W_M}^{W_L}\left(\mu^{-1}\circ N_{M/E}\right)\right)\end{aligned}$$

by Mackey's restriction formula with

$$M=\langle L,E\rangle=\mathrm{Ind}_{W_M}^{W_F}\left((\eta\chi)\circ N_{M/L}.\mu^{-1}\circ N_{M/E}\right).$$

Theorem 6.1. — *Let $\pi(\chi)$ be a depth-zero cuspidal representation of $\mathrm{GL}_n(F)$, such that $\chi_{|L_0^*}=\mu_{|F^*}\circ N_{L_0/F}$. Let ψ be a non-trivial additive character of F.*

- *If E/F is unramified, then*

$$\epsilon\left(\frac{1}{2},\pi(\chi)\otimes\mathrm{Ind}_{W_E}^{W_F}(1),\psi\right)=\omega_{E/F}(-1)^m\mu(-1)^m.$$

- *If E/F is ramified:*
  - *If μ is tame then*

$$\epsilon\left(\frac{1}{2},\pi(\chi)\otimes\mathrm{Ind}_{W_E}^{W_F}(1),\psi\right)=-\omega_{E/F}(-1)^m\mu(-1)^m.$$

  - *If μ is not tame then*

$$\epsilon\left(\frac{1}{2},\pi(\chi)\otimes\mathrm{Ind}_{W_E}^{W_F}(1),\psi\right)=\omega_{E/F}(-1)^m\mu(-1)^m.$$

Proof. — If L/K is a separable quadratic extension of non Archimedean local fields, we denote by $\sigma_{L/K}$ the associated Galois involution. We distinguish the ramified and the unramified case in our computations.

Figure 6.1. Diagram of the extensions involved - E/F unramified case m odd (in the left) and m even (in the right)

When E/F is unramified. We recall the situation: E is included in L and possibly in L_0 according to the parity of m.

$$\epsilon\left(\frac{1}{2}, \mathrm{Ind}_{W_L}^{W_F}(\eta\chi) \otimes \mathrm{Ind}_{W_E}^{W_F}\left(\mu^{-1}\right), \psi\right)$$

$$= \epsilon\left(\frac{1}{2}, \mathrm{Ind}_{W_L}^{W_F}\left(\eta\chi\left(\mu^{-1} \circ N_{L/E}\right)\right), \psi\right)$$

$$\epsilon\left(\mathrm{Ind}_{W_L}^{W_F}\left(\eta\chi\left(\mu^{-\sigma_{E/F}} \circ N_{L/E}\right)\right), \psi\right) \text{ by Section 1.5 (1).}$$

$$= \lambda_{L/F}^2(\psi)\epsilon\left(\frac{1}{2}, \eta\chi\left(\mu^{-1} \circ N_{L/E}\right), \psi_L\right)$$

$$\epsilon\left(\frac{1}{2}, \eta\chi\left(\mu^{-\sigma_{E/F}} \circ N_{L/E}\right), \psi_L\right) \text{ by Section 1.5 (7).}$$

$$= \lambda_{L/E}^2(\psi_E)\lambda_{E/F}^n(\psi)\eta^2\left(\varpi_L^{d(\psi_L)}\right)$$

$$\epsilon\left(\frac{1}{2}, \chi\left(\mu^{-1} \circ N_{L/E}\right), \psi_L\right)\epsilon\left(\frac{1}{2}, \chi\left(\mu^{-\sigma_{E/F}} \circ N_{L/E}\right), \psi_L\right)$$

by Section 1.5 (9).

$$= \omega_{E/F}(-1)^m\epsilon\left(\frac{1}{2}, \chi\left(\mu^{-1} \circ N_{L/E}\right), \psi_L\right)$$

$$\epsilon\left(\frac{1}{2}, \chi\left(\mu^{-\sigma_{E/F}} \circ N_{L/E}\right), \psi_L\right)$$

by Section 1.5 (10) and (8) and because n is even.

Now we distinguish between two cases:

(1) m is even: then

$$\begin{aligned}&\epsilon\left(\frac{1}{2},\chi\left(\mu^{-1}\circ N_{L/E}\right),\psi_L\right)\epsilon\left(\frac{1}{2},\chi\left(\mu^{-\sigma_{E/F}}\circ N_{L/E}\right),\psi_L\right)\\&=\epsilon\left(\frac{1}{2},\chi^{\sigma_{L/L_0}}\left(\mu^{-1}\circ N_{L/E}\right),\psi_L\right)\epsilon\left(\frac{1}{2},\chi\left(\mu^{-\sigma_{E/F}}\circ N_{L/E}\right),\psi_L\right)\end{aligned}$$

according to Section 1.5 (3). because $\psi_L=\psi_L^{\sigma_{L/L_0}}$ and $\mu^{-1}\circ N_{L/E}$ is also σ_{L/L_0}-invariant as $E\subset L_0\subset L$.

$$=\epsilon\left(\frac{1}{2},\chi^{\sigma_{L/L_0}}\left(\mu^{-1}\circ N_{L/E}\right),\psi_L^{-1}\right)\epsilon\left(\frac{1}{2},\chi\left(\mu^{-\sigma_{E/F}}\circ N_{L/E}\right),\psi_L\right)$$

from Section 1.5 (2) because

$$\begin{aligned}&\left(\chi^{\sigma_{L/L_0}}\left(\mu^{-1}\circ N_{L/E}\right)\right)(-1)\\&\qquad=\left(\mu_{|F^*}\circ N_{L_0/F}\right)(-1)\left(\mu^{-1}\circ N_{L/E}\right)(-1)\\&\qquad=\mu(-1)^m\mu(-1)^{-m}=1\end{aligned}$$

But then because

$$\begin{aligned}&\chi^{\sigma_{L/L_0}}\left(\mu^{-1}\circ N_{L/E}\right)\chi\left(\mu^{-\sigma_{E/F}}\circ N_{L/E}\right)\\&\qquad=\chi\circ N_{L/L_0}.\mu_{|F^*}^{-1}\circ N_{L/F}\\&\qquad=\mu_{|F^*}\circ N_{L/F}.\mu_{|F^*}^{-1}\circ N_{L/F}=1,\end{aligned}$$

Section 1.5, 4. implies that

$$\epsilon\left(\frac{1}{2},\chi^{\sigma_{L/L_0}}\left(\mu^{-1}\circ N_{L/E}\right),\psi_L^{-1}\right)\epsilon\left(\frac{1}{2},\chi\left(\mu^{-\sigma_{E/F}}\circ N_{L/E}\right),\psi_L\right)=1,$$

and we recognize the expected value

$$\epsilon\left(\frac{1}{2},\mathrm{Ind}_{W_L}^{W_F}(\eta\chi)\otimes\mathrm{Ind}_{W_E}^{W_F}\left(\mu^{-1}\right),\psi\right)=\omega_{E/F}^m(-1)^m\mu(-1)^m$$

because m is even.

(2) m is odd: then we notice that both $\chi(\mu^{-1}\circ N_{L/E})$ and $\chi(\mu^{-\sigma_{E/F}}\circ N_{L/E})$ restrict to L_0^* as $\chi_{|L_0^*}(\mu_{|F^*}\circ N_{L_0/F})=1$. Hence by Section 1.5, (6), for $v\in L-L_0$ such that $v^2\in L_0$, we have

$$\epsilon\left(\frac{1}{2},\chi\left(\mu^{-1}\circ N_{L/E}\right),\psi_L\right)=\chi(v)\mu^{-1}(N_{L/E}(v))$$

and

$$\epsilon\left(\frac{1}{2},\chi\left(\mu^{-\sigma_{E/F}}\circ N_{L/E}\right),\psi_L\right)=\chi(v)\mu^{-\sigma_{E/F}}(N_{L/E}(v)),$$

so that

$$\begin{aligned}\epsilon\left(\frac{1}{2},\chi\left(\mu^{-1}\circ N_{L/E}\right),\psi_L\right)\epsilon\left(\frac{1}{2},\chi\left(\mu^{-\sigma_{E/F}}\circ N_{L/E}\right),\psi_L\right)\\ =\chi\left(v^2\right)\mu^{-1}(N_{L/F}(v))=\mu\left(N_{L_0/F}\left(v^2\right)N_{L/F}(v)^{-1}\right)\\ =\mu\left(N_{L_0/F}\left(v^2N_{L/L_0}(v)^{-1}\right)\right)=\mu\left(N_{L_0/F}(-1)\right)\end{aligned}$$

because $\sigma_{L/L_0}(v)=-v$, hence finally

$$\epsilon\left(\frac{1}{2},\mathrm{Ind}_{W_L}^{W_F}(\eta\chi)\otimes\mathrm{Ind}_{W_E}^{W_F}\left(\mu^{-1}\right),\psi\right)=\omega_{E/F}(-1)^m\mu(-1)^m$$

which is again the expected value.

When E/F is ramified. In this case, E is not included in L. Set M to be the extension of L generated by L and E, M is therefore unramified n-dimensional on E. We also set $L_1=\langle E,L_0\rangle$ so that M is an unramified quadratic extension of L_1. The situation is as follows.

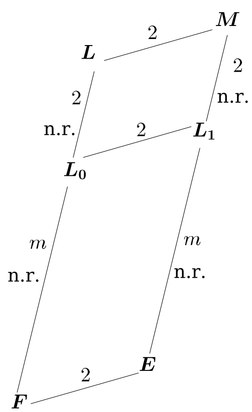

Figure 6.2. Diagram of the extensions involved – E/F ramified case

$$\begin{aligned}
&\epsilon\left(\frac{1}{2},\mathrm{Ind}_{W_L}^{W_F}(\eta\chi)\otimes\mathrm{Ind}_{W_E}^{W_F}\left(\mu^{-1}\right),\psi\right)\\
&=\epsilon\left(\frac{1}{2},\mathrm{Ind}_{W_M}^{W_F}\left((\eta\chi)\circ N_{M/L}.\mu^{-1}\circ N_{M/E}\right),\psi\right)\\
&=\lambda_{M/F}(\psi)\epsilon\left(\frac{1}{2},(\eta\chi)\circ N_{M/L}.\mu^{-1}\circ N_{M/E},\psi_M\right)\text{ by Section 1.5 (7).}\\
&=\lambda_{M/E}(\psi_E)\lambda_{E/F}^n(\psi)\epsilon\left(\frac{1}{2},(\eta\chi)\circ N_{M/L}.\mu^{-1}\circ N_{M/E},\psi_M\right)\\
&\qquad\qquad\text{by Section 1.5 (9).}\\
&=(-1)^{d(\psi_E)(n-1)}\omega_{E/F}(-1)^m\epsilon\left(\frac{1}{2},\omega_{M'/M}.\chi\circ N_{M/L}.\mu^{-1}\circ N_{M/E},\psi_M\right)
\end{aligned}$$

by Section 1.5 (8) and (10). where M'/M is quadratic unramified.

Before proceeding further with the computation let's discuss the conductor of the character $\chi\circ N_{M/L}.\mu^{-1}\circ N_{M/E}$.

- If μ is not tame then $\chi\circ N_{M/L}.\mu^{-1}\circ N_{M/E}$ clearly has the same conductor as $\mu^{-1}\circ N_{M/E}$ which is also not tame as it has the same conductor as μ, by surjectivity of $N_{M/E}$ from $1+\mathcal{P}_M^d$ onto $1+\mathcal{P}_E^d$ for any $d\geqslant 1$. In partcular $\chi\circ N_{M/L}.\mu^{-1}\circ N_{M/E}$ has conductor $c(\mu)$ which is even as we saw in Section 4.2.
- If μ is tame let us show that the character $\chi\circ N_{M/L}.\mu^{-1}\circ N_{M/E}$ has conductor 1. Clearly it is trivial on $1+\mathcal{P}_M$ because χ and μ are tame, but if it was unramified, going backwards one would deduce that $\mathrm{Ind}_{W_L}^{W_F}(\chi)\otimes\mathrm{Ind}_{W_E}^{W_F}(\mu^{-1})$ would be unramified, hence a direct sum of unramified characters. But $\mathrm{Ind}_{W_L}^{W_F}(\chi)\otimes\mathrm{Ind}_{W_E}^{W_F}(\mu^{-1})$ cannot contain any character, otherwise by irreducibility of $\mathrm{Ind}_{W_L}^{W_F}(\chi)$, it would appear as sub-representation of a character twist of $\mathrm{Ind}_{W_E}^{W_F}(\mu)$, which is impossible for dimension reasons (remember that we suppose $n\geqslant 3$). Hence $\chi\circ N_{M/L}.\mu^{-1}\circ N_{M/E}$ has conductor 1.

Hence setting $c'(\mu)=c(\mu)$ when $c(\mu)\geqslant 1$ and $c'(\mu)=1$ when μ is unramified, we obtain $c(\chi\circ N_{M/L}.\mu^{-1}\circ N_{M/E})=c'(\mu)$, which is even as soon as $c'(\mu)>1$. Finally we obtain:

$$\begin{aligned}\epsilon&\left(\frac{1}{2},\mathrm{Ind}_{W_L}^{W_F}(\eta\chi)\otimes\mathrm{Ind}_{W_E}^{W_F}\left(\mu^{-1}\right),\psi\right)\\&=(-1)^{d(\psi_M)(n-1)}\omega_{E/F}(-1)^m\omega_{M'/M}\left(\varpi_M^{d(\psi_M)+c'(\mu)}\right)\\&\qquad\epsilon\left(\frac{1}{2},\chi\circ N_{M/L}.\mu^{-1}\circ N_{M/E},\psi_M\right)\end{aligned}$$

thanks to Section 1.5 (5)

$$\begin{aligned}&=(-1)^{d(\psi_M)(n-1)}\omega_{E/F}(-1)^m(-1)^{d(\psi_M)+c'(\mu)}\\&\qquad\epsilon\left(\frac{1}{2},\chi\circ N_{M/L}.\mu^{-1}\circ N_{M/E},\psi_M\right)\\&=(-1)^{c'(\mu)}\omega_{E/F}(-1)^m\epsilon\left(\frac{1}{2},\chi\circ N_{M/L}.\mu^{-1}\circ N_{M/E},\psi_M\right)\end{aligned}$$

because n is even.

Note that M/L_0 is bi-quadratic, so there is one more quadratic extension L_2 of L_0 under M. Now the restriction of $\chi\circ N_{M/L}$ to L_2^* is equal to $\chi\circ N_{L_2/L_0}=\mu_{|F^*}\circ N_{L_2/F}$, whereas that of $\mu^{-1}\circ N_{M/E}$ is equal to $\mu^{-1}\circ N_{L_2/F}$, hence $\chi\circ N_{M/L}.\mu^{-1}\circ N_{M/E}$ restricts trivially to L_2^*.

Take $v\in L\setminus L_0$ with $v^2\in L_0$. Then $M=L_2[v]$ and we can apply Section 1.5 (6):

$$\begin{aligned}\epsilon\left(\frac{1}{2},\chi\circ N_{M/L}.\mu^{-1}\circ N_{M/E},\psi_M\right)&=\chi\circ N_{M/L}(v).\mu^{-1}\circ N_{M/E}(v)\\&=\chi\left(v^2\right)\mu^{-1}\circ N_{L_1/E}\left(-v^2\right)=\chi\left(v^2\right)\mu_{|F^*}^{-1}\circ N_{L_0/F}\left(-v^2\right)\\&=\mu_{|F^*}\circ N_{L_0/F}(v^2)\mu_{|F^*}^{-1}\circ N_{L_0/F}\left(-v^2\right)=\mu(-1)^m\end{aligned}$$

Thus

$$\epsilon\left(\frac{1}{2},\mathrm{Ind}_{W_L}^{W_F}(\eta\chi)\otimes\mathrm{Ind}_{W_E}^{W_F}(1),\psi\right)=(-1)^{c'(\mu)}\omega_{E/F}^m(-1)\mu(-1)^m,$$

as expected. □

As a corollary, we obtain:

Corollary 6.2. — *Let $\pi(\chi)$ be a depth 0 cuspidal representation of $\mathrm{GL}_n(F)$, let μ be a character of E^*, then $\pi(\chi)$ is $\mu\circ\det_{\mathrm{GL}_m(E)}$-distinguished by $H=\mathrm{GL}_m(E)$ if and only if*

(1) $\pi(\chi)$ is $\mu_{|F^*}$-symplectic ;

(2) $$\epsilon\left(\frac{1}{2}, \mathrm{Ind}_{W_L}^{W_F}(\eta\chi)\otimes \mathrm{Ind}_{W_E}^{W_F}\left(\mu^{-1}\right)\right) = \omega_{E/F}(-1)^m\mu(-1)^m.$$

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