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EXOTIC GROUP C^*-ALGEBRAS OF SIMPLE LIE GROUPS WITH REAL RANK ONE

by Tim DE LAAT & Timo SIEBENAND (*)

Abstract. — Exotic group C^*-algebras are C^*-algebras that lie between the universal and the reduced group C^*-algebra of a locally compact group. We consider simple Lie groups G with real rank one and investigate their exotic group C^*-algebras $C^*_{L^{p+}}(G)$, which are defined through L^p-integrability properties of matrix coefficients of unitary representations. First, we show that the subset of equivalence classes of irreducible unitary L^{p+}-representations forms a closed ideal of the unitary dual of these groups. This result holds more generally for groups with the Kunze–Stein property. Second, for every classical simple Lie group G with real rank one and every $2 \leqslant q < p \leqslant \infty$, we determine whether the canonical quotient map $C^*_{L^{p+}}(G) \twoheadrightarrow C^*_{L^{q+}}(G)$ has non-trivial kernel. Our results generalize, with different methods, recent results of Samei and Wiersma on exotic group C^*-algebras of $\mathrm{SO}_0(n,1)$ and $\mathrm{SU}(n,1)$. In particular, our approach also works for groups with property (T).

Résumé. — Les C^*-algèbres exotiques des groupes sont des C^*-algèbres, qui se situent entre la C^*-algèbre universelle et la C^*-algèbre réduite d'un groupe localement compact. Nous considérons des groupes de Lie simples G de rang réel un et nous étudions leurs C^*-algèbres exotiques $C^*_{L^{p+}}(G)$, qui sont définies par des propriétés d'intégrabilité L^p des coefficients des représentations unitaires. Nous montrons que les classes d'équivalence de représentations L^{p+} unitaires irréductibles forment un idéal fermé du dual unitaire de ces groupes. Ce résultat vaut plus généralement pour les groupes avec la propriété de Kunze–Stein. Pour chaque groupe de Lie simple classique G de rang un et chaque $2 \leqslant q < p \leqslant \infty$, nous déterminons si l'application canonique $C^*_{L^{p+}}(G) \twoheadrightarrow C^*_{L^{q+}}(G)$ a un noyau non trivial. Nos résultats généralisent, avec des méthodes différentes, des résultats récents de Samei et Wiersma sur les C^*-algèbres exotiques des groupes $\mathrm{SO}_0(n,1)$ et $\mathrm{SU}(n,1)$. En particulier, notre approche s'applique également à des groupes avec la propriété (T).

Keywords: group C^*-algebras, L^{p+}-representations, simple Lie groups.
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1. Introduction and main results

With every locally compact group G, we can associate two natural group C^*-algebras, namely the universal group C^*-algebra $C^*(G)$ and the reduced group C^*-algebra $C^*_r(G)$. The left-regular representation $\lambda\colon G \to \mathcal{B}(L^2(G))$ of G extends to a quotient map $C^*(G) \twoheadrightarrow C^*_r(G)$ and this quotient map is a *-isomorphism if and only if G is amenable.

If G is not amenable, there may be many group C^*-algebras lying "between" the universal and the reduced group C^*-algebra. An exotic group C^*-algebra of a locally compact group G is a C^*-completion A of $C_c(G)$ such that the identity map from $C_c(G)$ to itself extends to non-injective quotient maps from $C^*(G)$ to A and from A to $C^*_r(G)$:

$$C^*(G) \twoheadrightarrow A \twoheadrightarrow C^*_r(G).$$

In recent years, exotic group C^*-algebras and related constructions, such as exotic crossed products, have received an increased amount of attention, partly because of their relation with the Baum–Connes conjecture (see e.g. [1, 4, 5]).

The systematic study of exotic group C^*-algebras (of discrete groups) goes back to Brown and Guentner [3], who introduced and studied the notion of ideal completion. If Γ is a countable discrete group and D is an appropriate algebraic two-sided ideal of $\ell^\infty(\Gamma)$ (e.g. $D = \ell^p(\Gamma)$, with $1 \leqslant p \leqslant \infty$), the ideal completion $C^*_D(\Gamma)$, which is defined as the completion of the group ring of Γ with respect to the natural norm defined through all unitary representations with sufficiently many matrix coefficients lying in the ideal D, is a potentially exotic group C^*-algebra. Okayasu proved that for $2 \leqslant q < p \leqslant \infty$, the canonical quotient map $C^*_{\ell^p}(\mathbb{F}_d) \twoheadrightarrow C^*_{\ell^q}(\mathbb{F}_d)$ between ideal completions of the non-abelian free group $\mathbb{F}_d$ has non-trivial kernel [23]. This result was independently obtained by Higson and by Ozawa, and it was extended to discrete groups containing a non-abelian free group as a subgroup by Wiersma [29]. It is an open question whether every non-amenable (discrete) group admits exotic group C^*-algebras.

In the setting of (non-discrete) locally compact groups, the L^p-integrability properties (for different values of p) of matrix coefficients of unitary representations of the group also form an important source of potentially exotic group C^*-algebras. For $p \in [1,\infty]$, a unitary representation $\pi\colon G \to \mathcal{B}(\mathcal{H})$ is called an L^p-representation if suitable many of its matrix coefficients are elements of $L^p(G)$ (see Definition 3.1 for the precise definition). The representation π is called an L^{p+}-representation if π is an $L^{p+\varepsilon}$-representation for every $\varepsilon > 0$.

The aim of this article is to investigate the exotic group C^*-algebras $C^*_{L^{p+}}(G)$ (see Section 3 for the precise construction), which are constructed from the L^{p+}-representations of G, for connected simple Lie groups with real rank one and finite center. To this end, we first prove a structural result on the subset of the unitary dual $\widehat{G}$ consisting of (irreducible unitary) L^{p+}-representations of such groups, which more generally holds for groups with the Kunze–Stein property (see Section 4). This result captures a key idea of Samei and Wiersma (cf. [24, Theorem 5.3]) in the setting of the Fell topology.

First, recall that a subset $S \subset \widehat{G}$ is called an ideal if for every representation $\pi \in S$ and every unitary representation ρ of G, the unitary representation $\pi \otimes \rho$ is weakly contained in S.

Theorem 1.1. — *Let G be a Kunze–Stein group, and let $\widehat{G}_{L^{p+}}$ denote the subset of $\widehat{G}$ consisting of (equivalence classes of) L^{p+}-representations. Then $\widehat{G}_{L^{p+}}$ is a closed ideal of $\widehat{G}$.*

This result was already known for the groups $\mathrm{SO}_0(n,1)$ and $\mathrm{SU}(n,1)$, with $n \geqslant 2$, from the work of Shalom [26, Theorem 2.1]. Related results were shown in [7, 12, 18, 19, 21, 22].

Theorem 1.1 leads to a natural strategy to find and distinguish exotic group C^*-algebras. The idea is to determine for which values of p, the ideals $\widehat{G}_{L^{p+}}$ are pairwise different. To this end, it suffices to show that there are representations with sufficiently many matrix coefficients which are L^p-integrable for certain p, but not L^q-integrable when $p > q$.

Our second result is an application of this approach in the setting of simple Lie groups with real rank one. Let G be a connected simple Lie group with real rank one and finite center. Then G is locally isomorphic to one of the following groups: $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ (with $n \geqslant 2$) or $F_{4(-20)}$. The first three are called the (connected) classical simple Lie groups with real rank one, whereas $F_{4(-20)}$ is an exceptional Lie group.

Given a locally compact group G, we define $\Phi(G)$ as follows:

$$\Phi(G) := \inf \left\{ p \in [1,\infty] \mid \forall\ \pi \in \widehat{G} \setminus \{\tau_0\},\ \pi \text{ is an } L^{p+}\text{-representation} \right\},$$

where τ_0 denotes the trivial representation of G. The constant $\Phi(G)$ is known for the three aforementioned classical Lie groups and is given by

$$\Phi(G) = \begin{cases} \infty & \text{if } G = \mathrm{SO}_0(n,1), \\ \infty & \text{if } G = \mathrm{SU}(n,1), \\ 2n+1 & \text{if } G = \mathrm{Sp}(n,1). \end{cases} \tag{1.1}$$

The cases $\mathrm{SO}_0(n,1)$ and $\mathrm{SU}(n,1)$ essentially follow from Harish–Chandra's rich work. It also follows directly from Proposition 5.4.
The number $\Phi(\mathrm{Sp}(n,1))$ was computed in [18].

Our second result characterizes the exotic group C^*-algebras of the type $C^*_{L^{p+}}(G)$ for the classical simple Lie groups with real rank one.

Theorem 1.2. — *Let G be a (connected) classical simple Lie group with real rank one. Then for $2 \leqslant q < p \leqslant \Phi(G)$ (where $\Phi(G)$ is as in (1.1)), the canonical quotient map*

$$C^*_{L^{p+}}(G) \twoheadrightarrow C^*_{L^{q+}}(G)$$

has non-trivial kernel. Furthermore, for every $p, q \in [\Phi(G), \infty)$, we have

$$C^*_{L^{p+}}(G) = C^*_{L^{q+}}(G).$$

Furthermore, if H is a connected simple Lie group with finite center that is locally isomorphic to a classical simple Lie group G with real rank one, then the same result holds for H with $\Phi(H) = \Phi(G)$.

Additional to Theorem 1.2, we obtain partial results for the exceptional Lie group $F_{4(-20)}$ (see Theorem 5.7).

Exotic group C^*-algebras of Lie groups were considered before in [28], in which Wiersma proved that for $2 \leqslant q < p \leqslant \infty$, the quotient map $C^*_{L^{p+}}(\mathrm{SL}(2,\mathbb{R})) \twoheadrightarrow C^*_{L^{q+}}(\mathrm{SL}(2,\mathbb{R}))$ has non-trivial kernel, by studying the representation theory of $\mathrm{SL}(2,\mathbb{R})$. Note that $\mathrm{SL}(2,\mathbb{R})$ is locally isomorphic to $\mathrm{SO}_0(2,1)$, and hence included in Theorem 1.2. More recently, Samei and Wiersma deduced the existence of continua of exotic group C^*-algebras for certain groups having the "integrable Haagerup property" and the rapid decay property or the Kunze–Stein property. They obtained the cases $G = \mathrm{SO}_0(n,1)$ and $\mathrm{SU}(n,1)$ of Theorem 1.2 above. Their method, which is of geometric nature, is inherently unable to deal with groups with Kazhdan's property (T) (see [2] for background on property (T)), examples of which are $\mathrm{Sp}(n,1)$ and $F_{4(-20)}$. In the methods used in this article, the (integrable) Haagerup property does not play a role, and our method works equally well for the groups $\mathrm{Sp}(n,1)$ and $F_{4(-20)}$.

The article is organized as follows. After recalling some preliminaries in Section 2, we recall the algebras $C^*_{L^{p+}}(G)$ and prove some new results about them in Section 3. Section 4 is concerned with the unitary dual and the algebras $C^*_{L^{p+}}(G)$ of Kunze–Stein groups G. In particular, we prove Theorem 1.1 in that section. In Section 5, we prove Theorem 1.2.

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2. Preliminaries

2.1. Matrix coefficients and weak containment

Let G be a locally compact group, and let $\pi\colon G \to \mathcal{B}(\mathcal{H})$ be a unitary representation. Recall that a matrix coefficient of π is a function from G to $\mathbb{C}$ of the form $\pi_{\xi,\eta}\colon s \mapsto \langle \pi(s)\xi, \eta\rangle$, where $\xi, \eta \in \mathcal{H}$. Such functions are continuous bounded functions on G. Matrix coefficients of the form $\pi_{\xi,\xi}$ (i.e. with $\xi = \eta$) are called diagonal matrix coefficients.

A unitary representation π_1 of G is said to be weakly contained in a unitary representation π_2 of G if every diagonal matrix coefficient of π_1 can be approximated by finite sums of diagonal matrix coefficients of π_2 uniformly on compact subsets of G. For details, we refer to [9].

2.2. Unitary dual and Fell topology

Let G be a locally compact group, and let $\widehat{G}$ denote its unitary dual, i.e. the set of equivalence classes of irreducible unitary representations equipped with the Fell topology. If S is a subset of $\widehat{G}$, then the closure $\overline{S}$ of S in the Fell topology consists of all elements of $\widehat{G}$ which are weakly contained in S. Let $\widehat{G}_r$ denote the subspace of $\widehat{G}$ consisting of all elements of $\widehat{G}$ which are weakly contained in the left regular representation $\lambda\colon G \to \mathcal{B}(L^2(G))$. For details, we refer to [9].

2.3. Constructions of exotic group C^*-algebras

Let us recall two well-known constructions of exotic group C^*-algebras, which go back to [1, 13].

Let G be a locally compact group, and let μ_G be a Haar measure on G. A group C^*-algebra associated with G is a C^*-completion A of $C_c(G)$ with respect to a C^*-norm $\|.\|_\mu$ which satisfies $\|f\|_u \geqslant \|f\|_\mu \geqslant \|f\|_r$ for all $f \in C_c(G)$, where $\|f\|_u$ and $\|f\|_r$ denote the universal and the reduced

C^*-norm, respectively. In this case, the identity map $C_c(G) \to C_c(G)$ induces canonical surjective *-homomorphisms $C^*(G) \twoheadrightarrow A$ and $A \twoheadrightarrow C^*_r(G)$. If both these quotient maps have non-trivial kernel, then A is called an exotic group C^*-algebra. This is equivalent to the definition in Section 1.

Let G be a locally compact group, and let $\widehat{G}$ and $\widehat{G}_r$ be as before. A subset $S \subset \widehat{G}$ is called admissible if $\widehat{G}_r \subset \overline{S}$. If S is admissible, then

$$\|f\|_S := \sup\{\|\pi(f)\| | \pi \in S\}$$

defines a C^*-norm on $C_c(G)$. The corresponding completion $C^*_S(G)$ is a potentially exotic group C^*-algebra. Furthermore, if S is admissible, we have $\widehat{C^*_S(G)} = \overline{S}$, where $\widehat{C^*_S(G)}$ is the spectrum of the group C^*-algebra $C^*_S(G)$. More precisely, there is a bijective map from S to $\widehat{C^*_S(G)}$ mapping a representation $\pi\colon G \to \mathcal{B}(\mathcal{H})$ to the corresponding irreducible *-representation $\pi_*\colon C^*_S(G) \to \mathcal{B}(\mathcal{H})$ given by $\pi_*(f) = \int f(s)\pi(s)\,\mathrm{d}\mu_G(s)$ for $f \in C_c(G)$.

A subset $S \subset \widehat{G}$ is said to be an ideal if for every representation $\pi \in S$ and every unitary representation ρ of G, the unitary representation $\pi \otimes \rho$ is weakly contained in S. Note that the Fell absorption principle implies that every non-empty ideal $S \subset \widehat{G}$ is admissible. For more details on the above construction, we refer to [1].

Let us now discuss another approach, from [13]. Recall that the Fourier–Stieltjes algebra $B(G)$, consisting of all matrix coefficients of unitary representations of G, is a subalgebra of the algebra of continuous bounded functions on G. It can be canonically identified with the dual space $C^*(G)^*$ of the universal group C^*-algebra $C^*(G)$ through the pairing induced by

$$\langle \varphi, f\rangle = \int \varphi f \mathrm{d}\mu_G$$

for $\varphi \in B(G)$ and $f \in C_c(G) \subset C^*(G)$. Furthermore, $B(G)$ admits, in a canonical way, a left and a right G-action (see [13, Section 3]). Let $B_r(G) \subset B(G)$ denote the dual space of the reduced group C^*-algebra $C^*_r(G)$. It was shown in [13, Lemma 3.1] that if $E \subset B(G)$ is a weak*-closed, G-invariant subspace of $B(G)$ containing $B_r(G)$, then

$$C^*_E(G) = C^*(G)/{}^\perp E$$

is a group C^*-algebra, where ${}^\perp E = \{x \in C^*(G) \mid \langle \varphi, x\rangle = 0 \ \ \forall\, \varphi \in E\}$ is the pre-annihilator of E.

Let us explain the connection between the two approaches mentioned above. This connection is essentially contained in [13], but we give a proof which is in line with our framework and conventions.

Proposition 2.1. — *A non-empty closed set $S \subset \widehat{G}$ is an ideal if and only if the canonical comultiplication $\Delta\colon C^*(G) \to \mathcal{M}(C^*(G) \otimes C^*(G))$ factors through a coaction $\Delta_S\colon C^*_S(G) \to \mathcal{M}(C^*_S(G) \otimes C^*(G))$. Here $C^*_S(G) \otimes C^*(G)$ denotes the minimal tensor product of the C^*-algebra $C^*_S(G)$ and $C^*(G)$.*

Proof. — First, suppose that S is an ideal, and let $q\colon C^*(G) \to C^*_S(G)$ be the canonical quotient map and $\sigma\colon G \to \mathcal{B}(\mathcal{H})$ a unitary representation of G such that the integrated form $\sigma_*\colon C^*(G) \to \mathcal{B}(\mathcal{H})$ is a faithful (non-degenerate) *-representation of $C^*(G)$. The *-representation $\pi_* = (\bigoplus_{\rho \in S} \rho)_* \otimes \sigma_*$ is a faithful (non-degenerate) *-representation of $C^*_S(G) \otimes C^*(G)$. Therefore, it extends to a faithful *-representation of the multiplier algebra of the algebra $C^*_S(G) \otimes C^*(G)$, which we denote with π_* again. Note that the *-representation $\pi_* \circ (q \otimes \mathrm{id}) \circ \Delta$ of $C^*(G)$ is the integrated form of $\bigoplus_{\rho \in S} \rho \otimes \sigma$. Since S is assumed to be an ideal, this implies that $\pi_* \circ (q \otimes \mathrm{id}) \circ \Delta$ factors through the canonical quotient map $q\colon C^*(G) \to C^*_S(G)$. Finally, since π_* is faithful, we obtain that $(q \otimes \mathrm{id}) \circ \Delta$ factors through q, which proves the first direction.

Now, suppose that $C^*_S(G)$ is a group C^*-algebra with $\widehat{C^*_S(G)} = S$ such that the comultiplication $\Delta\colon C^*(G) \to \mathcal{M}(C^*(G) \otimes C^*(G))$ factors through a coaction $\Delta_S\colon C^*_S(G) \to \mathcal{M}(C^*_S(G) \otimes C^*(G))$. Let $\rho \in S$, and let π be any unitary representation of G. Again, we denote the integrated form of ρ and π by ρ_* and π_*, respectively. Then $\pi_* \otimes \rho_*$ is a non-degenerate *-representation of $C^*(G) \otimes C^*_S(G)$. Hence, it extends to a non-degenerate *-representation of the multiplier algebra $\mathcal{M}(C^*(G) \otimes C^*_S(G))$, which we denote with $\pi_* \otimes \rho_*$ again. The *-representation $(\pi_* \otimes \rho_*) \circ \Delta_S$ is the integrated form of the unitary representation $\pi \otimes \rho$ of G. This shows that $\pi \otimes \rho$ is weakly contained in S. □

Combining this proposition with [13, Corollary 3.13], we obtain the following result, which is probably well known to experts, but to our knowledge not explicitly contained in the literature.

Proposition 2.2. — *Let G be a locally compact group and $C^*_\mu(G)$ a group C^*-algebra of G. The following are equivalent:*

(i) *The set $\widehat{C^*_\mu(G)} \subset \widehat{G}$ is a closed ideal in $\widehat{G}$.*
(ii) *The dual space $C^*_\mu(G)^*$ of $C^*_\mu(G)$ is a G-invariant ideal in $B(G)$.*

2.4. Covering groups of Lie groups

A covering group of a connected Lie group G is a Lie group $\widetilde{G}$ with a surjective Lie group homomorphism $\sigma\colon \widetilde{G} \to G$ in such a way that $(\widetilde{G}, \sigma)$ is a (topological) covering space of G.

A universal covering space is a covering space which is simply connected. Every connected Lie group G admits a universal covering space $\widetilde{G}$, which admits a canonical Lie group structure. Such a universal covering group $\widetilde{G}$ of G is uniquely determined up to isomorphism. Universal covering groups satisfy the exact sequence $1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$, where $\pi_1(G)$ denotes the fundamental group of G. For details, we refer to [15, Section I.11].

2.5. Class one representations and spherical functions

We briefly recall Gelfand pairs, spherical functions and class one representations, which we will need for the proof of Theorem 1.2.

Let G be a locally compact group, let μ_G be a Haar measure, and let $K < G$ be a compact subgroup of G. A function $\varphi\colon G \to \mathbb{C}$ is called K-bi-invariant if $\varphi(k_1 s k_2) = \varphi(s)$ for all $s \in G$ and $k_1, k_2 \in K$. Let $C_c(K\backslash G/K)$ denote the $*$-subalgebra of the convolution algebra $C_c(G)$ consisting of all K-bi-invariant functions on G with compact support. The pair (G, K) is called a Gelfand pair if the algebra $C_c(K\backslash G/K)$ is commutative.

Let (G, K) be a Gelfand pair. A continuous K-bi-invariant function $\varphi\colon G \to \mathbb{C}$ is called a spherical function if

$$\chi\colon f \mapsto \int f(s)\varphi\left(s^{-1}\right) d\mu_G(s)$$

defines a non-trivial character of $C_c(K\backslash G/K)$. Here, μ_G denotes a Haar measure on G.

A pair (G, K) consisting of a locally compact group G with a compact subgroup K is a Gelfand pair if and only if for every irreducible unitary representation $\pi\colon G \to \mathcal{B}(\mathcal{H})$, the subspace $\mathcal{H}^K$ of $\mathcal{H}$ consisting of $\pi(K)$-invariant vectors is at most one-dimensional. An irreducible unitary representation $\pi\colon G \to \mathcal{B}(\mathcal{H})$ for which $\dim(\mathcal{H}^K) = 1$ is called a class one representation. For a Gelfand pair (G, K), we write $(\widehat{G}_K)_1$ for the (equivalence classes of) class one representations in $\widehat{G}$. The space $(\widehat{G}_K)_1$ is also called the spherical unitary dual.

Let $\pi\colon G \to \mathcal{B}(\mathcal{H})$ be a class one representation of the Gelfand pair (G, K), and let $\xi \in \mathcal{H}^K$ be a $\pi(K)$-invariant vector of norm one. Then the diagonal matrix coefficient $\pi_{\xi,\xi}$ is a positive definite spherical function. This assignment defines a bijection between $(\widehat{G}_K)_1$ and the set of all positive definite spherical functions of the Gelfand pair (G, K).

For an introduction to the theory of Gelfand pairs, spherical functions and class one representations, we refer to [8].

3. The algebras $C^*_{L^{p+}}(G)$

We now recall and discuss the algebras $C^*_{L^{p+}}(G)$, which are the (potentially) exotic group C^*-algebras of our interest. To this end, we first recall L^p-representations and L^{p+}-representations.

Definition 3.1. — *Let G be a locally compact group, let $\pi\colon G \to \mathcal{B}(\mathcal{H})$ be a unitary representation, and let $p \in [1, \infty]$.*

(i) *We say that π is an L^p-representation if there is a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $\pi_{\xi,\eta} \in L^p(G)$ for all $\xi, \eta \in \mathcal{H}_0$.*
(ii) *We say that π is an L^{p+}-representation if π is an $L^{p+\varepsilon}$-representation for all $\varepsilon > 0$.*

Recall that whenever a function $f \in C_b(G)$ is contained in $L^p(G)$ for some $p \in [1, \infty]$, then f is contained in $L^q(G)$ for all $q \geqslant p$. In particular, this is true for matrix coefficients of unitary representations.

Remark 3.2. — Suppose that $\pi\colon G \to \mathcal{B}(\mathcal{H})$ is a unitary representation and that $p \in [1, \infty]$. It follows by the polarization identity that π is an L^p-representation if and only if there is a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $\pi_{\xi,\xi} \in L^p(G)$ for all $\xi \in \mathcal{H}_0$. This definition of L^p-representation is used in [24].

Also, note that in the literature different terminology, e.g. strongly L^p- and strongly L^{p+}-representation, is used for what we call L^p-representation and L^{p+}-representation.

The following result gives a sufficient condition for a cyclic representation to be an L^p-representation.

Proposition 3.3. — *Let $\pi\colon G \to \mathcal{B}(\mathcal{H})$ be a cyclic unitary representation with cyclic vector $\xi \in \mathcal{H}$ such that $\pi_{\xi,\xi} \in L^p(G)$ for some $p \in [1, \infty]$. Then π is an L^p-representation.*

Proof. — The case $p = \infty$ is trivial, so suppose that $p \in [1, \infty)$.

Since π has cyclic vector $\xi \in \mathcal{H}$, the subspace $\mathcal{H}_0 = \operatorname{span}\{\pi(s)\xi \mid s \in G\}$ is dense in $\mathcal{H}$. Let s_1 and s_2 be arbitrary elements of G, and let $\zeta_1 := \pi(s_1)\xi$ and $\zeta_2 := \pi(s_2)\xi$. Then

$$\int |\pi_{\zeta_1,\zeta_2}|^p \, d\mu_G = \int \left|\left\langle \pi\left(s_2^{-1} t s_1\right)\xi, \xi\right\rangle\right|^p d\mu_G(t) = \Delta_G\left(s_1^{-1}\right) \int |\pi_{\xi,\xi}|^p \, d\mu_G,$$

where μ_G denotes a Haar measure on G and Δ_G denotes the associated modular function. Hence, $\pi_{\zeta_1,\zeta_2} \in L^p(G)$ for all $\zeta_1, \zeta_2 \in \{\pi(s)\xi \mid s \in G\}$. This implies that $\pi_{\zeta_1,\zeta_2} \in L^p(G)$ for all $\zeta_1, \zeta_2 \in \mathcal{H}_0$, which completes the proof of Proposition 3.3. □

Recall the construction of the potentially exotic group C^*-algebra $C^*_S(G)$ from Section 2.3. Our main interest is to consider the subset S of $\widehat{G}$ consisting of (equivalence classes of) L^{p+}-representations (or L^p-representations) of a locally compact group G. Note that in general, however, these sets may be empty, which is for example the case for non-compact locally compact abelian groups.

For a locally compact group G and $p \in [2,\infty]$, let $C^*_{L^p}(G)$ and $C^*_{L^{p+}}(G)$ be the group C^*-algebras obtained as the completions of $C_c(G)$ with respect to the C^*-norms

$$\|\cdot\|_{L^p} \colon C_c(G) \to [0,\infty),\ f \mapsto \sup\{\|\pi(f)\| | \pi \text{ is a } L^p\text{-representation}\}$$

and

$$\|\cdot\|_{L^{p+}} \colon C_c(G) \to [0,\infty),\ f \mapsto \sup\left\{\|\pi(f)\| \middle| \pi \text{ is a } L^{p+}\text{-representation}\right\},$$

respectively.

It is well known that $C^*(G) = C^*_{L^\infty}(G)$ and $C^*_r(G) = C^*_{L^2}(G)$. The following result follows directly from Proposition 2.2.

Proposition 3.4. — *Let $p \in [2,\infty]$. Then the dual spaces $\widehat{C^*_{L^p}(G)}$ and $\widehat{C^*_{L^{p+}}(G)}$ of $C^*_{L^p}(G)$ and $C^*_{L^{p+}}(G)$, respectively, are ideals in $\widehat{G}$.*

4. On the unitary dual of Kunze–Stein groups

Recall that a locally compact group G is called a Kunze-Stein group if the convolution product $m\colon C_c(G)\times C_c(G) \to C_c(G)$, $(f,g) \mapsto f*g$ extends to a bounded bilinear map $L^q(G) \times L^2(G) \to L^2(G)$ for all $q \in [1,2)$.

This property originated from the work of Kunze and Stein [17], who showed the above property for the group $\mathrm{SL}(2,\mathbb{R})$. In 1978, Cowling proved that connected semisimple Lie groups with finite center are Kunze–Stein groups [6]. Other classes of Kunze–Stein groups were provided in [6, 20] and [27].

The aim of this section is to prove Theorem 1.1, starting from the following result by Samei and Wiersma (see [24, Theorem 5.3]).

Theorem 4.1 (Samei–Wiersma). — *Let G be a Kunze–Stein group, and let $2 \leqslant p < \infty$. For a unitary representation $\pi\colon G \to \mathcal{B}(\mathcal{H})$, the following are equivalent:*

(i) *The representation π extends to a *-representation of $C^*_{L^{p+}}(G)$.*
(ii) *We have $B_\pi \subset L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$,*

where B_π denotes the closure in the weak-topology (on $B(G)$) of the linear span of all matrix coefficients of π, i.e. the functions $\pi_{\xi,\eta}$, with $\xi, \eta \in \mathcal{H}$.*

We first prove the following result.

Proposition 4.2. — *Let G be a Kunze–Stein group, let $p \in [2, \infty]$, let π be an irreducible unitary representation of G, and let $\xi \in \mathcal{H}$ be a nonzero vector. Then π is an L^{p+}-representation if and only if $\pi_{\xi,\xi} \in L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$.*

Proof. — The case $p = \infty$ is trivial, so suppose $p \in [2, \infty)$. If π is an L^{p+}-representation then $\pi_{\xi,\xi} \in L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$ by Theorem 4.1. On the other hand, suppose that $\pi_{\xi,\xi} \in L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$. Since ξ is a cyclic vector, the representation π is an L^{p+}-representation by Proposition 3.3. □

Proposition 4.3. — *Let G be a Kunze–Stein group, let $p \in [2, \infty)$, let π be an L^{p+}-representation of G and ρ another unitary representation of G which is weakly contained in π. Then ρ is an L^{p+}-representation.*

Proof. — Let π_* and ρ_* denote the integrated forms of π and ρ, respectively. Since π is an L^{p+}-representation, π_* factors through the canonical quotient map $C^*(G) \to C^*_{L^{p+}}(G)$. Since ρ is weakly contained in π, we have that $\ker \pi_* \subset \ker \rho_*$. Hence ρ_* factors through the quotient map $C^*(G) \to C^*_{L^{p+}}(G)$ as well. Equivalently, ρ extends to a *-representation of $C^*_{L^{p+}}(G)$. By Theorem 4.1, it follows that ρ is an L^{p+}-representation. □

For $p \in [2, \infty]$, let

$$\widehat{G}_{L^p} := \left\{ [\pi] \in \widehat{G} \,\middle|\, \pi \text{ is an } L^p\text{-representation} \right\}$$

and

$$\widehat{G}_{L^{p+}} := \left\{ [\pi] \in \widehat{G} \,\middle|\, \pi \text{ is an } L^{p+}\text{-representation} \right\}$$

If $2 \leqslant q < p \leqslant \infty$, then we have

$$\widehat{G}_{L^q} \subset \widehat{G}_{L^{q+}} \subset \widehat{G}_{L^p}.$$

As mentioned before, the sets $\widehat{G}_{L^p}$ and $\widehat{G}_{L^{p+}}$ could, in general, be empty. However, for Kunze–Stein groups, we can prove Theorem 1.1. First, we identify the dual space $\widehat{C^*_{L_{p+}}(G)}$ with $\widehat{G}_{L^{p+}}$.

Lemma 4.4. — *Let G be a Kunze–Stein group and $p \in [2,\infty]$. Then*

$$\widehat{C^*_{L_{p+}}(G)} = \widehat{G}_{L^{p+}}.$$

Proof. — Let $\pi \in \widehat{G}_{L^{p+}}$ be an irreducible unitary representation. By definition of $C^*_{L^{p+}}(G)$, the representation π extends to $C^*_{L^{p+}}(G)$. This implies that the integrated form π_* of π lies in $\widehat{C^*_{L^{p+}}(G)}$. On the other hand, if $\pi_* \in \widehat{C^*_{L^{p+}}(G)}$ then by Theorem 4.1, π is an L^{p+}-representation and hence $\pi \in \widehat{G}_{L^{p+}}$. □

Proof of Theorem 1.1. — The statement of the theorem now follows directly from Lemma 4.4, Proposition 3.4 and Proposition 4.3 (see Section 2.2 for a description of the closure of a subset of $\widehat{G}$). □

5. Simple Lie groups of real rank one

In this section, we investigate the group C^*-algebras $C^*_{L^{p+}}(G)$ for connected simple Lie groups G with real rank one and finite center. In particular, we prove Theorem 1.2.

From now on, we assume G to be a connected non-compact simple Lie group with finite center and K a maximal compact subgroup of G. It is well known that a pair (G,K) consisting of such groups is a Gelfand pair (see [10, Corollary 1.5.6] or [11, Chapter VI, Theorem 1.1]). Recall also that all maximal compact subgroups of G are conjugate under an inner automorphism of G.

Let $\mathfrak{g}$ be the Lie algebra of G, let $\mathfrak{k}$ be the Lie algebra of K, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. After fixing a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and choosing a positive root system $\Delta^+(\mathfrak{g},\mathfrak{a})$ from the root system $\Delta(\mathfrak{g},\mathfrak{a})$, we obtain the polar decomposition $G = K\overline{A^+}K$ of G, where $A^+ = \exp(\mathfrak{a}^+)$ and $\mathfrak{a}^+$ is the Weyl chamber corresponding to $\Delta^+(\mathfrak{g},\mathfrak{a})$.

The Haar measure μ_G on G can be normalized in such a way that for every $f \in C_c(G)$, we have

$$\int_G f d\mu_G = \int_K \int_{\overline{A^+}} \int_K f(k_1 a k_2) J(a)\, d\mu_K(k_1)\, d\mu_A(a)\, d\mu_K(k_2) \tag{5.1}$$

where μ_K is the normalized Haar measure on K, the measure μ_A is the Haar measure on A, and for $a \in A$,

$$J(a) = \prod_{\alpha \in \Delta^+(\mathfrak{g},\, \mathfrak{a})} \left(e^{\alpha(\log a)} - e^{-\alpha(\log a)}\right)^{\dim(\mathfrak{g}_\alpha)}, \tag{5.2}$$

where $\mathfrak{g}_\alpha$ is the root space corresponding to α (see [10, Proposition 2.4.6]). For details on the assertions made above, we refer to [10], [11] and [15].

There is an intimate relationship between the class one representations of a connected simple Lie group and the class one representations of a finite covering group of this Lie group, as is shown by the following result. This result is certainly known, but since we could not find an appropriate reference, we give a proof.

Lemma 5.1. — *Let G be a connected non-compact simple Lie group with finite center, $K < G$ a maximal compact subgroup and $q\colon \widetilde{G} \to G$ a connected finite covering group. Then $\widetilde{G}$ is a connected non-compact simple Lie group, $\widetilde{K} := q^{-1}(K)$ is a maximal compact subgroup of $\widetilde{G}$, and q induces a bijection*

$$q^*\colon \left(\widehat{G}_K\right)_1 \to \left(\widehat{\widetilde{G}}_{\tilde{K}}\right)_1, \, [\pi] \mapsto [\pi \circ q].$$

Furthermore, suppose that $p \in [2, \infty)$. Then $[\pi] \in (\widehat{G}_K)_1$ is represented by an L^{p+}-representation if and only if $q^*([\pi])$ is represented by an L^{p+}-representation.

Proof. — Since q is a finite covering, q is a proper map, and hence $\widetilde{K} = q^{-1}(K)$ is a compact subgroup of $\widetilde{G}$. Suppose C is a compact subgroup of $\widetilde{G}$ with $\widetilde{K} \subset C$. Then $q(C)$ is a compact subgroup of G and $K = q(\widetilde{K}) \subset q(C)$. The maximality of K implies $K = q(C)$. Hence, $C \subset q^{-1}(K) = \widetilde{K}$.

Note that q^* is well defined. Furthermore, q^* is obviously injective. It remains to show that q^* is surjective. To this end, let $\pi\colon \widetilde{G} \to \mathcal{B}(\mathcal{H}_\pi)$ be a class one representation. Then there is a $\widetilde{K}$-invariant vector $\xi \in \mathcal{H}_\pi$ of norm one, and $\widetilde{\omega} = \langle \pi(\cdot)\xi, \xi\rangle$ is a positive definite spherical function. Since G is a quotient group of $\widetilde{G}$ by a subgroup $\Gamma < Z(\widetilde{G})$, we know that $\widetilde{\omega}$ factors through $q\colon \widetilde{G} \to G$, by, say, $\omega\colon G \to \mathbb{C}$. It is immediate that ω is a positive definite normalized function as well. Let $\pi'\colon G \to \mathcal{B}(\mathcal{H})$ be the cyclic unitary representation with cyclic vector $\xi' \in \mathcal{H}$ of G and $\omega = \langle \pi'(\cdot)\xi', \xi'\rangle$. Then $\pi' \circ q$ is a cyclic unitary representation with $\langle \pi' \circ q(\cdot)\xi', \xi'\rangle = \omega \circ q = \widetilde{\omega} = \langle \pi(\cdot)\xi, \xi\rangle$. Hence $\pi' \circ q$ and π are unitary equivalent. This implies that π factors through $q\colon \widetilde{G} \to G$, which concludes the proof of the assertion that q^* is a bijection.

Let $\pi\colon G \to \mathcal{B}(\mathcal{H}_\pi)$ be a class one representation, let $\xi \in \mathcal{H}_\pi$ be a $\pi(K)$-invariant vector of norm one, and let $\omega = \pi_{\xi,\xi}$ the associated positive definite spherical function. If π is an L^{p+}-representation, then, due to Theorem 4.1, $\omega \in L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$. It follows that $\omega \circ q \in L^{p+\varepsilon}(\widetilde{G})$ for all $\varepsilon > 0$, by the quotient integral formula for Haar integrals. Hence, $\pi \circ q$ is an L^{p+}-representation (see Proposition 4.2). A similar argument shows the opposite direction. $\square$

We now specialize to the real rank one case. From now on, let G be a connected simple Lie group with real rank one and finite center. It is well known that such a G is locally isomorphic to one of the following Lie groups: $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$, with $n \geqslant 2$, or to the exceptional Lie group $\mathrm{F}_{4(-20)}$. The three of these groups arise as the isometry groups of the classical rank one symmetric spaces of the non-compact type. Explicitly, they are given by

$$\begin{aligned}
\mathrm{SO}(n,1) &= \{g \in \mathrm{SL}(n+1,\mathbb{R}) | g^* I_{n,1} g = I_{n,1}\}\,,\\
\mathrm{SU}(n,1) &= \{g \in \mathrm{SL}(n+1,\mathbb{C}) | g^* I_{n,1} g = I_{n,1}\}\,,\\
\mathrm{Sp}(n,1) &= \{g \in \mathrm{GL}(n+1,\mathbb{H}) | g^* I_{n,1} g = I_{n,1}\}\,,
\end{aligned}$$

where $I_{n,1} = \mathrm{diag}(1, \ldots, 1, -1)$, i.e. the diagonal $(n+1) \times (n+1)$-matrix with the first n diagonal entries equal to 1 and the $n+1^{\text{th}}$ entry equal to -1. These three groups are called the classical simple Lie groups with real rank one. The group $\mathrm{F}_{4(-20)}$ is an exceptional Lie group. We refer to [15] for more details.

Remark 5.2. — The universal covering group of $\mathrm{SO}_0(n,1)$ $(n \geqslant 3)$ has finite center, the group $\mathrm{Sp}(n,1)$ $(n \geqslant 2)$ is itself simply connected (so it is its own universal covering group) and has finite center, and the group $\mathrm{F}_{4(-20)}$ is simply connected and has trivial center (see e.g. [15] or [30]).

The universal covering group of $\mathrm{SU}(n,1)$ $(n \geqslant 2)$ and $\mathrm{SO}(2,1)$ have center isomorphic to $\mathbb{Z}$. Since every non-trivial quotient group of $\mathbb{Z}$ is a finite group, every group which is locally isomorphic to $\mathrm{SU}(n,1)$ $(n \geqslant 2)$ or $\mathrm{SO}(2,1)$ must either have finite center or be isomorphic to $\widetilde{\mathrm{SU}}(n,1)$ $(n \geqslant 2)$ or to $\widetilde{\mathrm{SO}}(2,1)$. Because of the accidental local isomorphism $\mathrm{SO}(2,1) \approx \mathrm{SU}(1,1)$, it follows that if G is a connected simple Lie group with real rank one and infinite center, then it is isomorphic to $\widetilde{\mathrm{SU}}(n,1)$ for some $n \geqslant 1$. For details, see e.g. [15].

Much of the theory recalled below goes back to Harish–Chandra's seminal work. For the purposes of this article, however, the exposition in the monograph by Gangolli and Varadarajan [10] seems more suitable, and we use this monograph as a reference.

Let G be a classical simple Lie group with real rank one, and let $\Phi(G)$ be as in (1.1). Let $G = KAN$ be an Iwasawa decomposition of G, and let $P = MAN$ be a minimal parabolic subgroup. Here M is the centralizer of A in K. We write $\mathfrak{a}$ for the Lie algebra of A and $\mathfrak{a}^*$ (resp. $\mathfrak{a}^*_{\mathbb{C}}$) for the dual space $\hom_{\mathbb{R}}(\mathfrak{a}, \mathbb{R})$ (resp. $\hom_{\mathbb{R}}(\mathfrak{a}, \mathbb{C})$) of $\mathfrak{a}$ (resp. $\mathfrak{a}_{\mathbb{C}}$). Finally, let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})} \dim(\mathfrak{g}_\alpha)\, \alpha.$$

The induced representation π_λ of the character

$$P = MAN \to \mathbb{C},\ man \mapsto e^{\lambda(\log a)}$$

to G, with $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, yields a not necessarily unitary group representation (which is unitary if $\lambda \in i\mathfrak{a}^*$) with a K-invariant vector ξ_λ of norm one. Hence, the matrix coefficient

$$\psi_\lambda = \langle \pi_\lambda(\cdot)\xi_\lambda, \xi_\lambda \rangle$$

defines a spherical function for (G, K).

In what follows, we identify $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{C}$ via the linear map from $\mathfrak{a}^*_{\mathbb{C}}$ to $\mathbb{C}$ mapping an element $\gamma \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ with $\frac{1}{2}\gamma \notin \Delta^+(\mathfrak{g}, \mathfrak{a})$ to 1.

As above let $(\widehat{G}_K)_1$ denote the set of all irreducible unitary class one representations. We recall the following result by Kostant [16] (see also [25, Lemma 5.2]).

Lemma 5.3.

(i) *The following holds:*

$$\rho = \begin{cases} \frac{n-1}{2} & \textit{if } G = \mathrm{SO}_0(n,1), \\ n & \textit{if } G = \mathrm{SU}(n,1), \\ 2n+1 & \textit{if } G = \mathrm{Sp}(n,1), \\ 11 & \textit{if } G = \mathrm{F}_{4(-20)}. \end{cases}$$

(ii) *The mapping*

$$i\mathfrak{a}^*_+ \cup [0,\, \rho_0(G)) \to \left(\widehat{G}_K\right)_1 \setminus \{\tau_0\},\ \lambda \mapsto \pi_\lambda$$

is bijective, where

$$\rho_0(G) = \begin{cases} \frac{n-1}{2} & \text{if } G = \mathrm{SO}_0(n,1), \\ n & \text{if } G = \mathrm{SU}(n,1), \\ 2n-1 & \text{if } G = \mathrm{Sp}(n,1), \\ 5 & \text{if } G = \mathrm{F}_{4(-20)}. \end{cases}$$

We now characterize which of the class one representations π_λ recalled above, with $\lambda \in [0, \rho_0(G))$, are L^{p+}-representations.

Proposition 5.4. — *Let G be $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ or $\mathrm{F}_{4(-20)}$, and let $\lambda \in [0, \rho_0(G))$. Then the class one representation π_λ is an L^{p+}-representation if and only if λ satisfies*

$$p(\rho - \lambda) \geqslant 2\rho.$$

The proof of this result follows from [14, Theorems 8.47 and 8.48]. Specifically, the result we need can be found in [25, p. 847–848]. It follows from the known asymptotic behaviour of $\psi_\lambda = \langle \pi_\lambda(\cdot)\xi_\lambda, \xi_\lambda \rangle$ on A^+. For the convenience of the reader, we sketch the proof.

Proof. — The asymptotic behaviour of the spherical function ψ_λ on A^+ is given by

$$\psi_\lambda(a) \sim e^{(\lambda-\rho)\log a}$$

as $a \to \infty$ (see [25, p. 847]), and the asymptotic behaviour of the function J from (5.2) on A^+ is given by

$$J(a) \sim e^{2\rho \log a}$$

as $a \to \infty$ (see [25, p. 848]). Using (5.1), it follows that

$$\psi_\lambda \in L^{p+\varepsilon}(G)$$

for all $\varepsilon > 0$ and all $\lambda \in [0, \rho_0(G)]$ if and only if

$$p(\rho - \lambda) \geqslant 2\rho. \qquad \square$$

For any Gelfand pair (G, K), let $\Phi_K(G) \in [1, \infty]$ be defined as

$$\Phi_K(G) := \inf\left\{ p \in [1,\infty] \,\middle|\, \forall\, \pi \in \left(\widehat{G}_K\right)_1 \setminus \{\tau_0\},\ \pi \text{ is an } L^{p+}\text{-representation} \right\},$$

where τ_0 denotes the trivial unitary representation of G. It is clear that $\Phi_K(G) \leqslant \Phi(G)$.

PROPOSITION 5.5. — *Let G be a connected simple Lie group with real rank one and finite center that is not locally isomorphic to* $\mathrm{F}_{4(-20)}$. *Then, we have* $\Phi_K(G) = \Phi(G)$.

Proof. — Suppose that G is a connected Lie group with finite center locally isomorphic to $\mathrm{SO}_0(n,1)$ or $\mathrm{SU}(n,1)$. Then Lemma 5.1 and Proposition 5.4 immediately imply that $\Phi_K(G) = \infty \geqslant \Phi(G)$.

From Proposition 5.4 and the identity $\Phi(\mathrm{Sp}(n,1)) = 2n+1$, it follows that $\Phi(\mathrm{Sp}(n,1)) = \Phi_K(\mathrm{Sp}(n,1))$. Since $\mathrm{Sp}(n,1)$ is simply connected, every connected Lie group G that is locally isomorphic to $\mathrm{Sp}(n,1)$ is isomorphic to a quotient group of $\mathrm{Sp}(n,1)$, with $\mathrm{Sp}(n,1)$ as its universal covering group. Hence, we have $\Phi(G) \leqslant \Phi(\mathrm{Sp}(n,1)) = \Phi_K(\mathrm{Sp}(n,1)) = \Phi_K(G)$. Here the last equality follows by Lemma 5.1 again. □

Remark 5.6. — Note that Proposition 5.5 and Lemma 5.1 imply that $\Phi(G) = \Phi(G')$ for locally isomorphic connected Lie groups G and G' with real rank one and finite center whenever G is not locally isomorphic to $\mathrm{F}_{4(-20)}$.

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2. — For the proof of the first assertion we only need to show that

$$\widehat{G}_{L^{p+}} \subsetneq \widehat{G}_{L^{q+}}$$

for $2 \leqslant p < q \leqslant \Phi(G)$ (see Lemma 4.4). To this end, let $p, q \in [2, \Phi(G)]$ with $p < q$. By Proposition 4.2, an irreducible unitary representation is an L^{p+}-representation if and only if a non-trivial vector state of the representation lies in $L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$. Furthermore, the positive definite spherical functions are in one-to-one correspondence with class one irreducible unitary representations (see Section 2.5). By Lemma 4.4, together with the fact that the GNS-construction of every positive definite spherical function is an irreducible unitary representation, it suffices to show that there is a positive definite spherical function ψ for the Gelfand pair (G,K) that lies in $L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$ but not in $L^{q+\varepsilon}(G)$ for all $\varepsilon > 0$. Lemma 5.1 implies that we can restrict ourselves to the classical cases, i.e. the cases where G is equal to $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$ or $\mathrm{Sp}(n,1)$. Now Lemma 5.3 and Proposition 5.4 complete the proof of the first assertion. The second assertion follows from the definition of $\Phi(G)$. □

Note that for every connected simple Lie group G with finite center and property (T), it was already known from the work of Cowling [7] that there exists a $p_0 \in [2, \infty)$ (depending on G) such that every matrix coefficient of a unitary representation of G is in $L^p(G)$ for every $p \geqslant p_0$.

Let us now consider the exceptional Lie group $\mathrm{F}_{4(-20)}$. Unfortunately, we cannot obtain a description as complete as for the classical Lie groups as in Theorem 1.2. The reason for this is that we do not know what the value of $\Phi(\mathrm{F}_{4(-20)})$ is. However, from considering the class one representations of $\mathrm{F}_{4(-20)}$ in combination of a result of Cowling, we still obtain a partial result.

Theorem 5.7. — *For $2 \leqslant q < p \leqslant \frac{11}{3}$, the canonical quotient map*

$$C^*_{L^{p+}}\left(\mathrm{F}_{4(-20)}\right) \twoheadrightarrow C^*_{L^{q+}}\left(\mathrm{F}_{4(-20)}\right)$$

has non-trivial kernel. Furthermore, for every $p,q \in [\frac{22}{3},\infty)$, we have

$$C^*_{L^{p+}}\left(\mathrm{F}_{4(-20)}\right) = C^*_{L^{q+}}\left(\mathrm{F}_{4(-20)}\right).$$

For the proof of this theorem, we will use the following Lemma, which is a special case of a result due to Cowling (see [7, Lemme 2.2.6]).

Lemma 5.8. — *Let G be a connected simple Lie group with finite center, and let K be a maximal compact subgroup of G. Suppose that there exists a $p \in [2,\infty)$ such that all non-constant positive-definite spherical functions of the Gelfand pair (G,K) belong to $L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$. Then every non-trivial irreducible unitary representation of G is an L^{2p+}-representation.*

Proof of Theorem 5.7. — The first assertion follows in the same way as the first part of Theorem 1.2, relying on Lemma 5.3 and Proposition 5.4.

For the second assertion, we use Lemma 5.8. Indeed, note that by Proposition 5.4, every non-trivial class one representation of $G = F_{4(-20)}$ is an $L^{\frac{11}{3}+}$-representation, so every non-constant positive-definite spherical function belongs to $L^{\frac{11}{3}+\varepsilon}$ for all $\varepsilon > 0$. Hence, by the lemma, every non-trivial irreducible unitary representation of G is an $L^{\frac{22}{3}+}$-representation, which implies the second assertion. $\square$

In order to improve Theorem 5.7, one would need to study the asymptotic behaviour of the isolated series representations.

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