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LINEARIZATION OF TRANSITION FUNCTIONS OF A SEMI-POSITIVE LINE BUNDLE ALONG A CERTAIN SUBMANIFOLD

by Takayuki KOIKE (*)

Abstract. — Let X be a complex manifold and L be a holomorphic line bundle on X. Assume that L is semi-positive, namely L admits a smooth Hermitian metric with semi-positive Chern curvature. Let Y be a compact Kähler submanifold of X such that the restriction of L to Y is topologically trivial. We investigate the obstruction for L to be unitary flat on a neighborhood of Y in X. As an application, for example, we show the existence of nef, big, and non semi-positive line bundle on a non-singular projective surface.

Résumé. — Soit X une variété complexe et L un fibré en droites holomorphe sur X. Supposons que L est semi-positive, à savoir L admet une métrique hermitienne lisse avec une courbure de Chern semi-positif. Soit Y une sous-variété kählérienne compacte de X telle que la restriction de L à Y est topologiquement triviale. Nous examinons l'obstruction pour que L soit plat unitaire sur un voisinage de Y. Comme application, par exemple, nous prouvons l'existence d'un fibré en droites nef, grand et non semi-positif sur une surface projective non singulière.

1. Introduction

Let X be a complex manifold and L be a holomorphic line bundle on X. Assume that L is *semi-positive*, namely there exists a C^∞ Hermitian metric h such that $\sqrt{-1}\Theta_h$ is semi-positive at any point of X, where Θ_h is the Chern curvature tensor of h. Let Y be a compact Kähler submanifold of X such that the restriction $L|_Y$ of L to Y is topologically trivial. In this case,

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it is known that $L|_Y$ is unitary flat (see § 2.1). Our interest is in the relation between the restriction of $L|_V$ of L to a small tubular neighborhood V of Y and the unitary flat line bundle $\widetilde{L}$ on V with $\widetilde{L}|_Y = L|_Y$ (*flat extension*, the existence of such a line bundle $\widetilde{L}$ follows by considering the isomorphism $H^1(V, \mathrm{U}(1)) \to H^1(Y, \mathrm{U}(1))$, where we denote by $\mathrm{U}(1)$ the unitary group of degree 1; i.e. $\mathrm{U}(1) := \{t \in \mathbb{C} \mid |t| = 1\}$). As a tentative answer, let us pose the following:

Conjecture 1.1. — *Let X, L, and Y be as above. Then $L|_V$ is unitary flat for a sufficiently small neighborhood V of Y in X.*

See also [11, Conjecture 2.1] and [9, Theorem 1.1]. Towards solving this conjecture, we investigate the difference between L and $\widetilde{L}$ in each finite order jet along Y in the present paper and give some partial affirmative results on this conjecture (Theorems 1.4, 1.5, and 1.6 below). As applications of these results and arguments used in the proofs of these, we show the following two results.

Theorem 1.2. — *There exists a nef and big line bundle on a non-singular projective surface which is not semi-positive.*

Theorem 1.3. — *Let X be a connected weakly 1-complete Kähler manifold of dimension 2 and $Y \subset X$ be a holomorphically embedded compact non-singular curve. Assume that the normal bundle $N_{Y/X}$ of Y is topologically trivial, the canonical bundle $K_{X\setminus Y}$ of $X \setminus Y$ is holomorphically trivial, and that there exists a C^∞ Hermitian metric h on $[Y]$ with semi-positive curvature, where $[Y]$ is the holomorphic line bundle on X which corresponds to the divisor Y. Then either the conditions* (i), (ii) *or* (iii) *holds:*

(i) *The surface $X \setminus Y$ is holomorphically convex,*
(ii) *There exists a transversally continuous foliation $\mathcal{F}$ on a neighborhood V of Y whose leaves are holomorphic such that Y is a leaf of $\mathcal{F}$, $\mathcal{F}|_{V\setminus Y}$ is of class C^∞, and that $i_L^*\Theta_h \equiv 0$ holds on any leaf L of $\mathcal{F}$, where $i_L\colon L \to X$ is the inclusion, or*
(iii) *It holds that $\Theta_h \wedge \Theta_h \equiv 0$ and that the function $\rho\colon X \to \mathbb{R}_{>0}$ which maps a point $p \in X$ to the trace of $\Theta_h|_p$ with respect to some Hermitian metric on X is flat at any point of Y: i.e. $\rho(p) = o((\mathrm{dist}\,(p, Y))^n)$ for any positive integer n as p approaches to Y, where “dist” is a local Euclidean distance.*

Note that, in the proof of Theorem 1.2, we prove that the line bundle L is not semi-positive for a variant of Grauert’s example (X, L) of a nef and big line bundle L on a non-singular projective surface X, see [6, Problem 2.2] and Example 4.1. Note also that the existence of such line bundles for higher dimensional projective manifolds has been known ([7, Example 2.14] and [3, Example 5.4], see also Remark 4.2). The motivation of Theorem 1.3 comes from the study of a neighborhood of an elliptic curve embedded in a Kähler surface S such that it is a suitable irreducible component of the divisor corresponding to the canonical bundle K_S of S when K_S or the anti-canonical bundle K_S^{-1} is semi-positive (see Question 5.1). In the proof of Theorem 1.3, [4, Proposition 2] plays an important role.

Our idea to compare the line bundle L with the flat extension $\widetilde{L}$ of $L|_Y$ comes from Ueda theory on the classification of the analytic structures of a neighborhood of a submanifold ([16], see also [13] or § 2.2 here). Let Y be a compact non-singular curve holomorphically embedded in a non-singular surface X such that the normal bundle $N_{Y/X}$ is topologically trivial. We denote by $[Y]$ the line bundle on X which corresponds to the divisor Y (or the invertible sheaf $\mathcal{O}_X(Y)$). Let $\widetilde{N}$ be the flat extension of $N_{Y/X}$, namely $\widetilde{N}$ is the unitary flat line bundle on a tubular neighborhood of Y such that $\widetilde{N}|_Y = N_{Y/X}$. As $[Y]|_Y$ also coincides with $N_{Y/X}$, one can consider the difference between $[Y]$ and $\widetilde{N}$ in the first order jet along Y, by which the first Ueda's obstruction class

$$u_1(Y,X) \in H^1\left(Y, N_{Y/X}^{-1}\right)$$

is defined. When $u_1(Y,X) = 0$, or equivalently when $[Y]$ and $\widetilde{N}$ coincide in the first order jet, one can define the second Ueda's obstruction class

$$u_2(Y,X) \in H^1\left(Y, N_{Y/X}^{-2}\right)$$

by comparing $[Y]$ and $\widetilde{N}$ in the second order jet along Y. In the case where all (similarly and inductively defined) Ueda's obstruction classes $u_n(Y,X)$'s vanish, Ueda gave a sufficient condition for the coincidence of $[Y]$ and $\widetilde{N}$ on a neighborhood of Y in X [16, Theorem 3]. As the coincidence of $[Y]$ and $\widetilde{N}$ can be interpreted as the vertical linearizability of a neighborhood of Y (or, more precisely, the linearizability of the transition functions of the system of local defining functions of Y), this Ueda's theorem can be regarded as a generalization of Arnold's linearization theorem [1] of a neighborhood of an elliptic curve. Note that the definition of Ueda's obstruction classes and this type of vertical linearization theorems can naturally be generalized into the cases of general dimensions if the normal bundle is unitary flat, see [10] or § 2.2 here.

Again, let Y be a compact non-singular curve holomorphically embedded in a non-singular surface X such that the normal bundle $N_{Y/X}$ is topologically trivial. When $u_n(Y,X)$ is a non-zero element of $H^1(Y, N_{Y/X}^{-n})$ for some positive integer n, the pair (Y,X) is said to be *of finite type*. Ueda investigated the details of the complex analytical properties of a neighborhood of Y also in this case [16, Theorem 1, 2]. By applying one of these results of Ueda, the author showed the non semi-positivity of the line bundle $[Y]$ when the pair (Y,X) is of finite type [9, Theorem 1.1], which is one of the biggest motivation of the present paper since it can be regarded as a partial answer to Conjecture 1.1.

Let X be a complex manifold, $Y \subset X$ be a compact Kähler submanifold, and L be a line bundle on X such that the restriction $L|_Y$ is topologically trivial. In § 3, according to the spirit of Ueda's classification, we pose an obstruction class

$$u_1(Y,X,L) \in H^1\left(Y, N_{Y/X}^*\right)$$

by comparing L and the flat extension $\widetilde{L}$ of $L|_Y$ in the first order jet along Y (so that it vanishes if these two line bundles coincide in the first order jet), where $N_{Y/X}^*$ is the dual vector bundle of the normal bundle. Note that, when Y is a hypersurface with topologically trivial normal bundle, the definition of two obstruction classes $u_1(Y,X)$ and $u_1(Y,X,L)$ coincide if $L=[Y]$. By using this first obstruction class, we show the following:

Theorem 1.4. — *Let X be a complex manifold, $Y \subset X$ be a compact Kähler submanifold, and L be a line bundle on X such that the restriction $L|_Y$ is topologically trivial. Assume that $u_1(Y,X,L) \neq 0$. Then L is not semi-positive.*

Theorem 1.5. — *Let X be a non-singular surface, L a holomorphic line bundle on X, and Y be a non-singular compact curve holomorphically embedded into X such that* $\deg N_{Y/X} \leqslant \min\{-1,\ 2-2g\}$, *where g is the genus of Y. Assume that* $\deg L|_Y = 0$ *and that $L \otimes [Y]^{-m}$ is semi-positive for some positive integer m. Then L is semi-positive if and only if $u_1(Y,X,L)=0$.*

Note that Theorem 1.2 follows from Theorem 1.4 and a concrete calculation of the obstruction class $u_1(Y,X,L)$ for a variant of Grauert's example (Example 4.1, see § 4.1). Note also that Theorem 1.5 can be regarded as a generalization of the semi-positivity result for the case of $\deg N_{Y/X} \leqslant \min\{0,\ 4-4g\}$ mentioned just after [8, Theorem 1.2], since $u_1(Y,X,L)=0$ automatically follows from Kodaira vanishing theorem when $\deg N_{Y/X} < 2-2g$.

Though it seems to be natural to consider the second obstruction class when $u_1(Y, X, L)$ vanishes, there is a difficulty in general on the well-definedness. More precisely, though one can define a class of the first cohomology group $H^1(Y, S^2N^*_{Y/X})$ of the second symmetric tensor product bundle

$$S^2N^*_{Y/X} \text{ of } N^*_{Y/X}$$

by comparing the difference between L and $\widetilde{L}$ in second order jet, this class may depend on the choice of a system of local defining functions of Y and frames of L. The same type of well-definedness problem also occurs in higher order jets. In § 3.2, we give a sufficient condition for the obstruction class $u_n(Y, X, L) \in H^1(Y, S^nN^*_{Y/X})$ we will inductively define to be well-defined (Proposition 3.2). Especially we observe the well-definedness of the obstruction classes and give a necessary condition for L to be semi-positive by using higher obstruction classes when Y is a hypersurface and $N^*_{Y/X}$ is either topologically trivial or not pseudo-effective in § 3.4 and 3.5. For example, we have the following for the case where $N_{Y/X}$ is topologically trivial.

Theorem 1.6. — *Let X be a complex manifold and Y be a non-singular compact hypersurface of X which is Kähler. Assume that the normal bundle $N_{Y/X}$ is topologically trivial, and that $[Y]$ is semi-positive. Then it holds that $u_1(Y, X) = u_2(Y, X) = 0$.*

The organization of the paper is as follows. In § 2, we will explain some previous or known results and fundamental facts on Hermitian metrics on line bundles and Ueda theory. In § 3, we define the obstruction classes $u_n(Y, X, L)$'s and investigate some properties of them especially when L is semi-positive. In § 4, we apply some results from § 3 to show Theorems 1.2, 1.3, and 1.5. In § 5, we make some discussion and pose some problems on our obstruction classes, neighborhoods of a submanifold, and the semi-positivity of the line bundles.

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Wuppertal at the beginning of 2020 and the author believe that it was a great inspiration for this work. Thanks are also due to Doctor Nicholas M^c^Cleerey for his helpful comments and warm encouragements, and to Professor Dano Kim for his kindly giving him information on his work on the existence of nef, big and not semi-positive line bundles for higher dimensional projective manifolds.

2. Preliminaries

2.1. Curvature semi-positivity of Hermitian metrics on line bundles

Let X be a complex manifold and L be a holomorphic line bundle on X. For a positive integer m, we denote by L^m the m^{th} tensor power $L^{\otimes m}$ and by L^{-m} the m^{th} tensor power $(L^*)^{\otimes m}$ of the dual bundle L^* of L.

Let $\{V_j\}$ be an open covering of X. When $\{V_j\}$ is sufficiently fine, one can take a system of local frames $\{(V_j, e_j)\}$ of L; i.e. each e_j is a nowhere vanishing holomorphic section of L on V_j. The ratio $s_{jk} := e_j/e_k$, which is a nowhere vanishing holomorphic function on $V_{jk} := V_j \cap V_k$, can be regarded as the transition function of L. Let h be a C^∞ Hermitian metric on L. The function $\varphi_j := -\log |e_j|_h^2$ on each V_j is called the *local weight function* of h. It follows from a simple calculation that $\{(V_j, \partial\overline{\partial}\varphi_j)\}$ glue to each other to define a global $(1,1)$-form Θ_h, which coincides with the Chern curvature tensor of h. Therefore it follows that h is semi-positively curved if and only if any local weight function φ_j is plurisubharmonic.

We say that L is *unitary flat* if one can choose a system of local frames $\{(V_j, e_j)\}$ of L such that all the transition functions s_{jk}'s are elements of $\mathrm{U}(1)$ (by taking a refinement of $\{V_j\}$ if necessary). It is known that L is unitary flat if X is a compact Kähler manifold and L is topologically trivial (Kashiwara's theorem, see [16, § 1]). Assume that X is compact, L is unitary flat, and that a system $\{(V_j, e_j)\}$ satisfies that all the transition functions s_{jk}'s are elements of $\mathrm{U}(1)$. Then, for any Hermitian metric h on L, it clearly holds that $\{(V_j, -\log |e_j|_h^2)\}$ glue to each other to define a global function on X. When h is semi-positively curved, this function must be a constant function by the maximal principle.

Let D be a compact hypersurface of X. Then one can define a corresponding holomorphic line bundle $[D]$ on X so that the sheaf $\mathcal{O}_X([D])$ of holomorphic sections of $[D]$ is isomorphic to the invertible sheaf $\mathcal{O}_X(D)$ of rational functions on X which may have a pole only along D with degree at most one. For an open (Stein) covering $\{V_j\}$ of X, one can take a system of local frames $\{(V_j, e_j)\}$ of $[D]$ such that e_j corresponds, via the isomorphism $\mathcal{O}_X([D]) \cong \mathcal{O}_X(D)$, to the constant function 1 if $V_j \cap D = \emptyset$ and to the meromorphic function $1/w_j$ if $V_j \cap D \neq \emptyset$ for some local holomorphic defining function w_j of Y in V_j. In this case, $\{(V_j, w_j \cdot e_j)\}$ patch together to define a global section $f_D \colon X \to [D]$, which is called the *canonical section* of $[D]$. Assume that $[D]$ admits a C^∞ Hermitian metric h. Let ϕ_D be the locally L^1 function on X defined by $\phi_D := -\log|f_D|_h^2$. On each V_j, one clearly has that $\phi_D = \varphi_j$ if $V_j \cap D = \emptyset$ and $\phi_D = -\log|w_j|^2 + \varphi_j$ if $V_j \cap D \neq \emptyset$. Thus it follows that ϕ_D is a plurisubharmonic function on the complement $X \setminus D$ in this case. By considering this function ϕ_D, problems on the existence of a metric with semi-positive curvature on $[D]$ can be reworded to problems on the complex analytical convexity of the complement $X \setminus D$ (see the arguments in the proof of [9, Theorem 1.1] or [11, § 2.1]). Note that, even when D is smooth and $N_{D/X}$ is unitary flat, it may possible that $[D]$ is not semi-positive. Indeed, [5, Example 1.7] gives such an example. Note also that $[D]$ is semi-positive if there exists a neighborhood V of D in X such that $[D]|_V$ is unitary flat. This can be shown by using "regularized minimum construction", see [11, § 2.1] for the detail.

2.2. Ueda Theory

Let X be a complex manifold and $Y \subset X$ be a holomorphically embedded compact complex submanifold with unitary flat normal bundle.

In [16], Ueda investigated the complex analytic structure on a neighborhood of Y when X is a surface and Y a curve by defining obstruction classes as we shortly explained in § 1 (see also [13]). In [10], we investigated a higher codimensional analogue of Ueda theory. In this subsection, we will summarize our notions on this generalized version of Ueda theory for reader's convenience, see also [11, § 2.2].

Let X be a complex manifold and $Y \subset X$ be a compact complex submanifold of codimension $r \geqslant 1$ such that $N_{Y/X}$ is unitary flat. Take a finite open covering $\{U_j\}$ of Y and a neighborhood V_j of U_j in X and a defining function $w_j \colon V_j \to \mathbb{C}^r$ of U_j for each j: i.e. w_j is a holomorphic function on V_j such that $\mathrm{div}(w_j^\lambda)$ intersect transversally along U_j, where $w_j^\lambda \colon V_j \to \mathbb{C}$ is the composition of w_j and $\lambda^{-\mathrm{th}}$ projection map $\mathbb{C}^r \to \mathbb{C}$. By a simple argument, one may assume that $dw_j = S_{jk} dw_k$ holds on each $U_{jk} := U_j \cap U_k$ for some unitary matrix $S_{jk} \in \mathrm{U}(r)$ by changing w_j's if necessary, where

$$dw_j := \begin{pmatrix} dw_j^1 \\ dw_j^2 \\ \vdots \\ dw_j^r \end{pmatrix}.$$

We call such a system $\{(V_j, w_j)\}$ of local defining functions of Y a system of *type* 1. By shrinking V_j's if necessary again, we assume that, for each j, there exists a holomorphic surjection $\mathrm{Pr}_{U_j} : V_j \to U_j$ such that $(w_j, z_j \circ \mathrm{Pr}_{U_j})$ are coordinates of V_j, where z_j is a coordinate of U_j. In what follows, for any holomorphic function f on U_j, we denote by the same letter f the pull-back $\mathrm{Pr}_{U_j}^* f := f \circ Pr_{U_j}$. On U_j and U_k such that $U_{jk} \neq \emptyset$, one has the series expansion

$$S_{jk} \cdot \begin{pmatrix} w_k^1 \\ w_k^2 \\ \vdots \\ w_k^r \end{pmatrix} = \begin{pmatrix} w_j^1 \\ w_j^2 \\ \vdots \\ w_j^r \end{pmatrix} + \sum_{|a| \geqslant 2} \begin{pmatrix} f_{kj,\,a}^{(1)}(z_j) \\ f_{kj,\,a}^{(2)}(z_j) \\ \vdots \\ f_{kj,\,a}^{(r)}(z_j) \end{pmatrix} \cdot w_j^a,$$

where $a = (a_1, a_2, \ldots, a_r)$ is the multiple index running over all the elements of $(\mathbb{Z}_{\geqslant 0})^r$ with $|a| := \sum_{\lambda=1}^r a_\lambda$ is larger than or equal to 2, $f_{kj,\,a}^{(\lambda)}$'s are holomorphic functions on U_{jk} (we regard this also as a function defined by

$$\left(\Pr_{U_j}\Big|_{\mathrm{Pr}_{U_j}^{-1}(U_{jk})}\right)^* \left(f_{kj,\,a}^{(\lambda)}\right), \quad \text{and} \quad w_j^a := \prod_{\lambda=1}^r \left(w_j^\lambda\right)^{a_\lambda}.$$

For a positive integer m, we say that the system $\{(V_j, w_j)\}$ of local defining functions is of *type* m if $f_{kj,\,a} \equiv 0$ holds for any a with $|a| \leqslant m$ and any j, k with $U_{jk} \neq \emptyset$. If $\{(V_j, w_j)\}$ is of type m, it follows that

$$\left\{\left(U_{jk}, \sum_{\lambda=1}^r \sum_{|a|=m+1} f_{kj,\,a}^{(\lambda)} \frac{\partial}{\partial w_j^{(\lambda)}} \otimes dw_j^a\right)\right\}$$

satisfies the 1-cocycle condition, and thus it defines an element of $H^1(Y, N_{Y/X} \otimes S^{m+1} N_{Y/X}^*)$. We denote this cohomology class by $u_m(Y, X)$, which is the definition of the m^{th} Ueda class. Ueda classes $u_m(Y, X)$ are well-defined up to the action of $\mathrm{U}(r)$ on

$$H^1\left(Y, N_{Y/X} \otimes S^{m+1} N_{Y/X}^*\right),$$

namely

$$[u_m(Y, X)] \in H^1\left(Y, N_{Y/X} \otimes S^{m+1} N^*_{Y/X}\right) / \mathrm{U}(r)$$

does not depend on the choice of the system of type m.

From a simple observation, one has that there exists a system of type $m+1$ if and only if $u_m(Y, X) = 0$, in which case one can also define $u_{m+1}(Y, X)$. The pair (Y, X) is said to be of *finite type* if, for some positive integer n, there exists a system of type n such that $u_n(Y, X) \neq 0$. Otherwise, the pair is said to be of *infinite type*.

2.3. A fundamental lemma

Though the following lemma is fundamental, it plays an important role in the linearizing procedure we will see in next section.

LEMMA 2.1. — *Let Ω be a neighborhood of the origin in the complex plane $\mathbb{C}$ with the standard coordinate w, and $\Phi\colon \Omega \to \mathbb{R}$ be a function. Assume that $\Phi(w) \geqslant 0$ for any $w \in \Omega$ and that, for a positive integer n, Φ satisfies*

$$\Phi(w) = \sum_{p=0}^{n} c_p \cdot w^p \overline{w}^{n-p} + O\left(|w|^{n+1}\right)$$

as $|w| \to 0$, where c_p's are complex constants. Then the following holds:

(i) *When n is odd, $c_p = 0$ for any $p \in \{0, 1, 2, \ldots, n\}$.*

(ii) *When n is even, the constant $c_{n/2}$ is a non-negative real number. Any of the other constants are zero if $c_{n/2} = 0$.*

Proof. — We let $\Psi(w) := \sum_{p=0}^{n} c_p \cdot w^p \overline{w}^{n-p}$. As Φ is real valued, one has that $\Phi = \overline{\Phi}$, which implies that Ψ is also a real-valued function. By considering a function $f_w\colon \mathbb{R}_{>0} \to \mathbb{R}_{\geqslant 0}$ defined by $f_w(r) := \Phi(rw) = r^n \cdot \Psi(w) + O(r^{n+1})$ for each $w \in \mathbb{C}$, one has that $\Psi(w)$ is non-negative at any point in $\mathbb{C}$.

Denote by Δ the unit disc $\{w \in \mathbb{C} \mid |w| < 1\}$ of $\mathbb{C}$. Then one has that

$$\begin{aligned}
\int_\Delta \Psi(w)\, d\lambda &= \sum_{p=0}^{n} c_p \int_\Delta w^p \overline{w}^{n-p}\, d\lambda \\
&= \sum_{p=0}^{n} c_p \cdot \left(\int_0^1 \ell^n \cdot \ell\, d\ell\right) \cdot \int_0^{2\pi} e^{\sqrt{-1}(2p-n)\theta}\, d\theta \\
&= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{2\pi}{n+2} \cdot c_{n/2} & \text{if } n \text{ is even,} \end{cases}
\end{aligned}$$

where we denote by $d\lambda$ the Lebesgue measure, from which the assertions follow. □

3. Linearization of a metric with semi-positive curvature

3.1. Set-up and notation

Let X be a complex manifold with $\dim X = d + r$, $Y \subset X$ a compact Kähler submanifold of $\dim Y = d$ $(d, r \geqslant 1)$, and $L \to X$ be a holomorphic line bundle such that the restriction $L|_Y$ is topologically trivial. Note that, in what follows, the assumption that Y is compact Kähler is only needed to assure that $L|_Y$ is unitary flat, and that any unitary flat line bundle on Y is trivial as a unitary flat line bundle if it is analytically trivial (see § 2.1).

Take a sufficiently fine finite open covering $\{U_j\}$ of Y and a sufficiently small Stein open subset V_j of X such that $V_j \cap Y = U_j$. Denote by V the neighborhood $\bigcup_j V_j$ of Y. Let $w_j \colon V_j \to \mathbb{C}^r$ $(w_j = (w_j^1, w_j^2, \ldots, w_j^r))$ be local defining functions of Y, and $z_j = (z_j^1, z_j^2, \ldots, z_j^d)$ be coordinates of U_j. By shrinking V_j's if necessary, we assume that there exists a holomorphic surjection $\mathrm{Pr}_{U_j} \colon V_j \to U_j$ such that $(z_j^1 \circ \mathrm{Pr}_{U_j}, z_j^2 \circ \mathrm{Pr}_{U_j}, \ldots, z_j^d \circ \mathrm{Pr}_{U_j}, w_j)$ are coordinates of V_j. In what follows, for any holomorphic function f on U_j, we denote by the same letter f the pull-back $\mathrm{Pr}_{U_j}^* f := f \circ \mathrm{Pr}_{U_j}$ for each j. We also simply denote by (z_j, w_j) our local coordinates on V_j. Take a local frame e_j of L on each V_j. As $L|_Y$ admits a structure of unitary flat line bundle, one can take e_j's such that $t_{jk}^{-1} \cdot e_k|_{U_{jk}} = e_j|_{U_{jk}}$ holds for some $t_{jk} \in \mathrm{U}(1)$ on each $U_{jk} := U_j \cap U_k$. Then the expansion of the ratio function $t_{jk}^{-1} e_k / e_j$ is in the form

$$\frac{t_{jk}^{-1} e_k}{e_j} = 1 + \sum_{|\alpha| \geqslant 1} f_{kj,\alpha}(z_j) \cdot w_j^\alpha.$$

We say that the system $\{(V_j, e_j, w_j)\}$ is of *type n* if $f_{kj,\alpha} \equiv 0$ for any j, k and any α with $|\alpha| < n$.

Assume that L is semi-positive. Take a C^∞ Hermitian metric h on L with semi-positive curvature. Denote by $\varphi_j(z_j, w_j)$ the local weight function of h on each V_j. Note that φ_j's are plurisubharmonic. As $|t_{jk}| = 1$, one has that

$$\begin{aligned}\varphi_k(z_k, w_k) &= -\log |e_k|^2_{h(z_k, w_k)} \\ &= -\log |e_j|^2_{h(z_j, w_j)} - \log \left| 1 + \sum_{|\alpha| \geqslant 1} f_{kj,\alpha}(z_j) \cdot w_j^\alpha \right|^2\end{aligned}$$

if $(z_j, w_j) = \Phi_{kj}(z_k, w_k)$, where the diffeomorphisms Φ_{kj} are the transition functions for X. By considering Taylor expansion of the function $x \mapsto \log(1+x)$, one has that

$$\varphi_k - \varphi_j = -\sum_{|\alpha|=n} f_{kj,\alpha}(z_j)\cdot w_j^\alpha - \sum_{|\alpha|=n} \overline{f_{kj,\alpha}(z_j)}\cdot \overline{w_j^\alpha} + O\left(|w_j|^{n+1}\right) \tag{3.1}$$

holds if $\{(V_j, e_j, w_j)\}$ is a system of type n, where

$$|w_j| := \sqrt{\left|w_j^1\right|^2 + \left|w_j^2\right|^2 + \cdots \left|w_j^r\right|^2}.$$

3.2. Definition and well-definedness of the obstruction classes

Let X, Y, L, $\{(U_j, z_j)\}$ and $\{(V_j, e_j, w_j)\}$ be as in the previous subsection. Here we drop the assumption that L is semi-positive, and consider (inductive) linearization of the transition functions of e_j's in each finite order jet.

Assume that $\{(V_j, e_j, w_j)\}$ is a system of type n. Then the expansion of the function $t_{jk}^{-1}e_k/e_j$ around U_{jk} is in the form

$$\frac{t_{jk}^{-1}e_k}{e_j} = 1 + \sum_{|\alpha|=n} f_{kj,\alpha}(z_j)\cdot w_j^\alpha + O\left(|w_j|^{n+1}\right).$$

As

$$\frac{t_{jk}^{-1}e_k}{e_j}\cdot\frac{t_{k\ell}^{-1}e_\ell}{e_k}\cdot\frac{t_{\ell j}^{-1}e_j}{e_\ell} = 1,$$

one has that $\{(U_{jk}, \sum_{|\alpha|=n} f_{kj,\alpha}(z_j)\cdot(dw_j)^\alpha)\}$ satisfies the 1-cocycle condition, where $(dw_j)^\alpha := \bigotimes_{\lambda=1}^r (dw_j^\lambda)^{\otimes\alpha_\lambda}$. We denote by

$$u_n\big(Y, X, L; \{\,(V_j, e_j, w_j)\,\}\big)$$

the class

$$\left[\left\{\left(U_{jk},\ \sum_{|\alpha|=n} f_{kj,\alpha}(z_j)\cdot(dw_j)^\alpha\right)\right\}\right] \in \check{H}^1\left(\{U_j\}, \mathcal{O}_Y\left(S^n N_{Y/X}^*\right)\right),$$

and call it n^{th} *obstruction class* for the linearization of the transition functions of L.

In the rest of this subsection, we discuss the well-definedness of this class of $H^1(Y, S^n N_{Y/X}^*)$; i.e. the dependence of this class on the choice of a system $\{(V_j, e_j, w_j)\}$ of type n. First we show the following:

Lemma 3.1. — *Let $\{(V_j, e_j, w_j)\}$ be a system of type n. Take another local frame $\widehat{e}_j$ of L on V_j such that $\{(V_j, \widehat{e}_j, w_j)\}$ is also a system of type n. Assume $t_{jk}^{-1}e_k = e_j$ and $t_{jk}^{-1}\widehat{e}_k = \widehat{e}_j$ hold on each U_{jk}. Assume also that one of the following three conditions holds:*

(i) $n = 1$,

(ii) $N_{Y/X}$ *is unitary flat and the system $\{(V_j, w_j)\}$ of local defining functions of Y is of type n, or*

(iii) $H^0(Y, S^m N^*_{Y/X}) = 0$ *holds for any integer m with $1 \leqslant m \leqslant n-1$.*

Then it holds that $u_n(Y, X, L; \{(V_j, e_j, w_j)\}) = u_n(Y, X, L; \{(V_j, \widehat{e}_j, w_j)\})$.

Proof. — We let

$$\frac{t_{jk}^{-1}\widehat{e}_k}{\widehat{e}_j} = 1 + \sum_{|\alpha| \geqslant n} \widehat{f}_{kj,\alpha}(z_j) \cdot w_j^\alpha$$

and

$$\frac{\widehat{e}_j}{e_j} = \sum_{|\alpha| \geqslant 0} A_{j,\alpha}(z_j) \cdot w_j^\alpha$$

be the expansions. As it holds that

$$\frac{t_{jk}^{-1}\widehat{e}_k}{\widehat{e}_j} \cdot \frac{\widehat{e}_j}{e_j} = \frac{t_{jk}^{-1}e_k}{e_j} \cdot \frac{\widehat{e}_k}{e_k},$$

we will compare the left hand side

$$\frac{t_{jk}^{-1}\widehat{e}_k}{\widehat{e}_j} \cdot \frac{\widehat{e}_j}{e_j} = \left(1 + \sum_{|\alpha| \geqslant n} \widehat{f}_{kj,\alpha}(z_j) \cdot w_j^\alpha\right) \cdot \left(\sum_{|\alpha| \geqslant 0} A_{j,\alpha}(z_j) \cdot w_j^\alpha\right) \tag{3.2}$$

and right hand side

$$\frac{t_{jk}^{-1}e_k}{e_j} \cdot \frac{\widehat{e}_k}{e_k} = \left(1 + \sum_{|\alpha| \geqslant n} f_{kj,\alpha}(z_j) \cdot w_j^\alpha\right) \cdot \left(\sum_{|\alpha| \geqslant 0} A_{k,\alpha}(z_k) \cdot w_k^\alpha\right) \tag{3.3}$$

in each order.

First, by comparing 0^{th} order of the equations (3.2) and (3.3), one has that $A_{j,0} = A_{k,0}$ for each j and k, where we are simply denoting by 0 the multi-index $(0, 0, \ldots, 0)$. Thus we have that there exists a constant A_0 such that $A_{j,0} = A_0$ holds for each j. As both $\widehat{e}_j$ and e_j are local frames, one has that $A_0 \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Next, let us observe the case where $n = 1$. By comparing the equations (3.2) and (3.3), one have that

$$\sum_{|\alpha|=1} A_0 \cdot \widehat{f}_{kj,\alpha}(z_j) \cdot w_j^\alpha + \sum_{|\alpha|=1} A_{j,\alpha}(z_j) \cdot w_j^\alpha \\ = \sum_{|\alpha|=1} A_0 f_{kj,\alpha}(z_j) \cdot w_j^\alpha + \sum_{|\alpha|=1} A_{k,\alpha}(z_k) \cdot w_k^\alpha + O\left(|w_j|^2\right).$$

Thus one has that

$$\sum_{|\alpha|=1} \left(\widehat{f}_{kj,\alpha}(z_j) - f_{kj,\alpha}(z_j)\right) \cdot (dw_j)^\alpha \\ = \frac{1}{A_0} \cdot \left(- \sum_{|\alpha|=1} A_{j,\alpha}(z_j) \cdot (dw_j)^\alpha + \sum_{|\alpha|=1} A_{k,\alpha}(z_k) \cdot (dw_k)^\alpha \right),$$

which proves the assertion (i).

Finally, let us assume either the assumption (ii) or (iii). Take an integer m such that $1 \leqslant m \leqslant n-1$. As an inductive assumption, we assume the following condition $\mathbf{(H)}_\mu$ holds for $\mu = 1, 2, \ldots, m-1$.

$$((\mathbf{H})_\mu) \qquad \sum_{|\alpha| \leqslant \mu} A_{j,\alpha}(z_j) \cdot w_j^\alpha = \sum_{|\alpha| \leqslant \mu} A_{k,\alpha}(z_k) \cdot w_k^\alpha + O\left(|w_k|^{n+1}\right).$$

Note that $\{(U_j, \sum_{|\alpha|=\mu} A_{j,\alpha}(dw_j)^\alpha)\}$ glue to define an element of $H^0(Y, S^\mu N^*_{Y/X})$ for $\mu = 1, 2, \ldots, m-1$. By comparing the equations (3.2) and (3.3), one has

$$\sum_{|\alpha|=m} A_{j,\alpha}(z_j) \cdot w_j^\alpha = \sum_{|\alpha|=m} A_{k,\alpha}(z_k) \cdot w_k^\alpha + O\left(|w_k|^{m+1}\right).$$

Therefore we have that

$$\left\{ \left(U_j, \sum_{|\alpha|=m} A_{j,\alpha}(z_j) \cdot (dw_j)^\alpha \right) \right\}$$

also glue up to define a global section of $S^m N^*_{Y/X}$. Note that, for each α with $|\alpha| = m$, it especially follows that $A_{j,\alpha}$ is constant under the assumption (ii), and is zero under the assumption (iii) (Here we used the constantness result for a global section of a unitary flat vector bundle, see [10, Lemma 2.1, Remark 2.7]). Thus one has that the condition $\mathbf{(H)}_m$ also holds.

From the argument above, one has that

$$\sum_{|\alpha|=n}\left(\widehat{f}_{kj,\,\alpha}(z_j)-f_{kj,\,\alpha}(z_j)\right)\cdot(dw_j)^\alpha$$
$$=\frac{1}{A_0}\cdot\left(-\sum_{|\alpha|=n}A_{j,\,\alpha}(z_j)\cdot(dw_j)^\alpha+\sum_{|\alpha|=n}A_{k,\,\alpha}(z_k)\cdot(dw_k)^\alpha\right),$$

from which the assertions follow. □

Proposition 3.2. — *Let $\{(V_j,e_j,w_j)\}$ be a system of type n. Take another system $\{(\widehat{V}_j,\widehat{e}_j,\widehat{w}_j)\}$ of type n. Assume also that one of the following three conditions holds:*

(i) $n=1$,

(ii) $N_{Y/X}$ *is unitary flat and the system $\{(V_j,w_j)\}$ of local defining functions of Y is of type n, or*

(iii) $H^0(Y,S^mN^*_{Y/X})=0$ *holds for any integer m with $1\leqslant m\leqslant n-1$.*

Then it holds that $u_n(Y,X,L;\{(V_j,e_j,w_j)\})=u_n(Y,X,L;\{(\widehat{V}_j,\widehat{e}_j,\widehat{w}_j)\})$.

Based on this Proposition 3.2, we simply denote by $u_n(Y,X,L)$ the n^{th} obstruction class when either of three assumptions in the proposition holds.

Proof. — By taking refinements, we may assume that two open coverings $\{V_j\}$ and $\{\widehat{V}_j\}$ coincide.

First, we show the proposition by assuming $w_j=\widehat{w}_j$ on each $V_j(=\widehat{V}_j)$. Let $\widehat{t}_{jk}$ be an element of $\mathrm{U}(1)$ such that $\widehat{t}_{jk}^{-1}\widehat{e}_k=\widehat{e}_j$ holds on each U_{jk}. As both $\{t_{jk}\}$ and $\{\widehat{t}_{jk}\}$ are transition functions of $L|_Y$, there exists $c_j\in\mathrm{U}(1)$ such that

$$\frac{\widehat{t}_{jk}}{t_{jk}}=\frac{c_k}{c_j}.$$

Note that here we used the fact that any unitary flat line bundle on Y is trivial as unitary flat line bundle if it is analytically trivial: i.e. $H^1(Y,\mathrm{U}(1))\to H^1(Y,\mathcal{O}_Y^*)$ is injective. Consider a new local frame defined by $\varepsilon_j:=c_je_j$. Then, as it holds that

$$\frac{\varepsilon_k}{\varepsilon_j}=\frac{e_k}{e_j}\cdot\frac{c_k}{c_j}=t_{jk}\cdot\frac{\widehat{t}_{jk}}{t_{jk}}=\widehat{t}_{jk}$$

on each U_{jk}, it follows from Lemma 3.1 that the proof of the assertion is reduced to show that $u_n(Y,X,L;\{(V_j,e_j,w_j)\})=u_n(Y,X,L;\{(V_j,\varepsilon_j,w_j)\})$, which follows from

$$\frac{\widehat{t}_{jk}^{-1}\varepsilon_k}{\varepsilon_j}=\frac{c_k^{-1}}{c_j^{-1}}\cdot t_{jk}^{-1}\cdot\frac{\varepsilon_k}{\varepsilon_j}=t_{jk}^{-1}\cdot\frac{e_k}{e_j}$$

on each V_{jk}.

Again, let $\{(V_j, e_j, w_j)\}$ and $\{(V_j, \widehat{e}_j, \widehat{w}_j)\}$ be systems of type n. From the argument above, one has that

$$u_n\big(Y, X, L; \{\, (V_j, \widehat{e}_j, \widehat{w}_j)\, \}\big) = u_n\big(Y, X, L; \{\, (V_j, e_j, \widehat{w}_j)\, \}\big).$$

As it clearly holds that

$$u_n\big(Y, X, L; \{\, (V_j, e_j, \widehat{w}_j)\, \}\big) = u_n\big(Y, X, L; \{\, (V_j, e_j, w_j)\, \}\big)$$

by definition, the Proposition 3.2 holds. □

Remark 3.3. — Let Y be a hypersurface of X such that the normal bundle $N_{Y/X}$ is unitary flat and that $L = [Y]$. Assume that there exists a system $\{(V_j, w_j)\}$ of local defining functions of Y of type n: i.e. it holds that

$$t_{jk} w_k = w_j + \sum_{\nu=n+1}^{\infty} g_{kj,\,\nu}(z_j) \cdot w_j^{\nu}$$

on each V_{jk} for some holomorphic functions $g_{kj,\nu}$'s. In this case, n^{th} Ueda class $u_n(Y, X)$ is the class defined by the 1-cocycle

$$\big\{U_{jk}, g_{kj,\,n+1}(z_j) \cdot (dw_j)^n\big\}.$$

Let $\{(V_j, e_j)\}$ be a system of local frames of $[Y]$ such that e_j corresponds to the meromorphic function $1/w_j$ via the isomorphism $\mathcal{O}_X([Y]) \cong \mathcal{O}_X(Y)$. Then, as

$$\begin{aligned}\frac{t_{jk}^{-1} e_k}{e_j} = \frac{w_j}{t_{jk} w_k} &= \left(1 + \sum_{\nu=n}^{\infty} g_{kj,\,\nu+1}(z_j) \cdot w_j^{\nu}\right)^{-1} \\ &= 1 - g_{kj,\,n+1}(z_j) \cdot w_j^n + O\left(w_j^{n+1}\right),\end{aligned}$$

one has that our n^{th} obstruction class $u_n(Y, X, L)$ coincides with n^{th} Ueda class $u_n(Y, X)$ up to multiplication by a $\mathbb{C}^*$-constant.

3.3. Linearization in the first order jet

In what follows, we assume that L is semi-positive. Let h be a C^∞ Hermitian metric with semi-positive curvature. Denote by φ_j the local weight function of h on each V_j. By Taylor's theorem, there exist a C^∞'ly smooth functions $R_j^{(2)}(z_j, w_j)$ on each V_j such that

$$\begin{cases} \varphi_j\left(z_j, w_j\right) = \\ \qquad \varphi_j^{(0)}(z_j) + \displaystyle\sum_{\lambda=1}^{r} \left(\varphi_j^{(\lambda)}(z_j) \cdot w_j^{\lambda} + \overline{\varphi_j^{(\lambda)}(z_j)} \cdot \overline{w_j^{\lambda}}\right) + R_j^{(2)}\left(z_j, w_j\right) \\ R_j^{(2)}\left(z_j, w_j\right) = O\left(|w_j|^2\right) \text{ as } |w_j| \to 0. \end{cases}$$

Then we have that

$$\varphi_k - \varphi_j = \left(\varphi_k^{(0)} - \varphi_j^{(0)}\right) + \sum_{\lambda=1}^{r} \left(\varphi_k^{(\lambda)} \cdot w_k^\lambda - \varphi_j^{(\lambda)} \cdot w_j^\lambda + \overline{\varphi_k^{(\lambda)}} \cdot \overline{w_k^\lambda} - \overline{\varphi_j^{(\lambda)}} \cdot \overline{w_j^\lambda}\right) + O\left(|w_j|^2\right).$$

By the equation (3.1), we have that

$$\begin{aligned} -\sum_{|\alpha|=1} f_{kj,\alpha}(z_j) \cdot w_j^\alpha - \sum_{|\alpha|=1} \overline{f_{kj,\alpha}(z_j)} \cdot \overline{w_j^\alpha} &= \left(\varphi_k^{(0)} - \varphi_j^{(0)}\right) + \sum_{\lambda=1}^{r} \left(\varphi_k^{(\lambda)} \cdot w_k^\lambda - \varphi_j^{(\lambda)} \cdot w_j^\lambda\right) \\ &\quad + \sum_{\lambda=1}^{r} \left(\overline{\varphi_k^{(\lambda)}} \cdot \overline{w_k^\lambda} - \overline{\varphi_j^{(\lambda)}} \cdot \overline{w_j^\lambda}\right) + O\left(|w_j|^2\right), \end{aligned}$$

from which it follows that

$$\varphi_k^{(0)} = \varphi_j^{(0)} \tag{3.4}$$

and

$$-\sum_{|\alpha|=1} f_{kj,\alpha}(dw_j)^\alpha = \sum_{\lambda=1}^{r} \varphi_k^{(\lambda)} \cdot (dw_k)^\lambda - \sum_{\lambda=1}^{r} \varphi_j^{(\lambda)} \cdot (dw_j)^\lambda. \tag{3.5}$$

The equation (3.4) implies that $\{(U_j, \varphi_j^{(0)})\}$ defines a global function $\varphi^{(0)} \colon Y \to \mathbb{R}$. As the Chern curvature of $h|_Y$ is semi-positive, it clearly holds that $\varphi^{(0)}$ is plurisubharmonic. The compactness of Y implies that $\varphi^{(0)}$ is constant. By changing h by multiplying a constant, we always assume that $\varphi^{(0)} \equiv 0$ in what follows.

LEMMA 3.4. — *For any $\lambda = 1, 2, \ldots, r$ and any j, $\varphi_j^{(\lambda)}$ is a holomorphic function on U_j.*

Proof. — For any $\lambda = 1, 2, \ldots, r$, $\nu = 1, 2, \ldots, d$, and any j, we show that

$$\left(\varphi_j^{(\lambda)}\right)_{\overline{z_j^\nu}}(z_j) := \frac{\partial}{\partial \overline{z_j^\nu}} \varphi_j^{(\lambda)}(z_j) \equiv 0$$

holds for all $z_j \in U_j$. Take a point $z_j \in U_j$, and consider the Hermitian form $\langle -, -\rangle_{\lambda,\nu} \colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ defined by

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle_{\lambda,\nu} := (a,b) \begin{pmatrix} (\varphi_j)_{w_j^\lambda \overline{w_j^\lambda}}(z_j,0) & (\varphi_j)_{w_j^\lambda \overline{z_j^\nu}}(z_j,0) \\ (\varphi_j)_{z_j^\nu \overline{w_j^\lambda}}(z_j,0) & (\varphi_j)_{z_j^\nu \overline{z_j^\nu}}(z_j,0) \end{pmatrix} \begin{pmatrix} \overline{c} \\ \overline{d} \end{pmatrix}.$$

As is clearly follows from the semi-positivity assumption for the Chern curvature of h, the Hermitian form $\langle -, -\rangle_{\lambda,\nu}$ is positive semi-definite. Thus one has that

$$\det \begin{pmatrix} (\varphi_j)_{w_j^\lambda \overline{w_j^\lambda}}(z_j,0) & (\varphi_j)_{w_j^\lambda \overline{z_j^\nu}}(z_j,0) \\ (\varphi_j)_{z_j^\nu \overline{w_j^\lambda}}(z_j,0) & (\varphi_j)_{z_j^\nu \overline{z_j^\nu}}(z_j,0) \end{pmatrix} \geqslant 0.$$

As $\varphi_j^{(0)} \equiv 0$, it follows that $(\varphi_j)_{z_j^\nu \overline{z_j^\nu}}(z_j,0) = 0$. Therefore, we have that $(\varphi_j)_{w_j^\lambda \overline{z_j^\nu}}(-,0) \equiv 0$. The assertion follows from the calculation

$$(\varphi_j)_{w_j^\lambda \overline{z_j^\nu}} = \left(\varphi_j^{(\lambda)}\right)_{\overline{z_j^\nu}} + \left(R_j^{(2)}\right)_{w_j^\lambda \overline{z_j^\nu}} = \left(\varphi_j^{(\lambda)}\right)_{\overline{z_j^\nu}} + O(|w_j|). \qquad \square$$

Proof of Theorem 1.4. — Assume that L is semi-positive. We use the notation as above and show that $u_1(Y,X,L) = 0$. Let $\widehat{e}_j$ be the section of L on V_j defined by

$$\widehat{e}_j = e_j \cdot \left(1 + \sum_{\lambda=1}^{r} \varphi_j^{(\lambda)}(z_j) \cdot w_j^\lambda\right)$$

for each j after shrinking V_j if necessary (so that $\widehat{e}_j \neq 0$ at any point of V_j). By Lemma 3.4, $\widehat{e}_j$'s are also (holomorphic) local frames. Theorem 1.4 follows by calculating $u_1(Y,X,L)$ by using them. $\square$

Let $\widehat{e}_j$ be the one in the proof of Theorem 1.4. Denote by $\widehat{\varphi}_j$ the corresponding local weight function of h. Then one has that

$$\begin{aligned}\widehat{\varphi}_j(z_j, w_j) &= -\log|\widehat{e}_j|^2_{h(z_j,\,w_j)}\\ &= -\log|e_j|^2_{h(z_j,\,w_j)} - \log\left|1+\sum_{\lambda=1}^{r}\varphi_j^{(\lambda)}(z_j)\cdot w_j^\lambda\right|^2\\ &= \varphi_j(z_j,w_j) - \log\left|1+\sum_{\lambda=1}^{r}\varphi_j^{(\lambda)}(z_j)\cdot w_j^\lambda\right|^2\\ &= \varphi_j(z_j,w_j) - \sum_{\lambda=1}^{r}\varphi_j^{(\lambda)}(z_j)\cdot w_j^\lambda + \sum_{\lambda=1}^{r}\overline{\varphi_j^{(\lambda)}(z_j)}\cdot\overline{w_j^\lambda} + O\left(|w_j|^2\right)\\ &= O\left(|w_j|^2\right).\end{aligned}$$

Thus we have that, by leaving w_j's as they were and changing only e_j's, one may assume that $\varphi_j(z_j, w_j) = O(|w_j|^2)$ holds as $|w_j| \to 0$ for each j.

3.4. Linearization in the second order jet when Y is a hypersurface

In what follows, we assume that Y is a hypersurface of X: i.e. $r = 1$. We simply denote by w_j the function w_j^1.

By the argument in the previous subsection, we may assume that the expansion of φ_j is in the form

$$\begin{aligned}&\varphi_j(z_j,w_j) =\\ &\quad \varphi_j^{(2,\,0)}(z_j)\cdot w_j^2 + \varphi_j^{(1,\,1)}(z_j)\cdot|w_j|^2 + \varphi_j^{(0,\,2)}(z_j)\cdot\overline{w_j}^2 + R_j^{(3)}(z_j,w_j),\end{aligned}$$

where $R_j^{(3)}(z_j,w_j)$ is a C^∞'ly smooth function on V_j such that $R_j^{(3)}(z_j,w_j) = O(|w_j|^3)$ as $|w_j| \to 0$. Note that, as φ_j is real valued, one has that $\varphi_j^{(0,\,2)} = \overline{\varphi_j^{(2,\,0)}}$.

Let H_j be the $(d+1)\times(d+1)$ matrix with entries $(H_j)^a_b$ with $0 \leqslant a, b \leqslant d$ defined by

$$(H_j)^a_b = \begin{cases}\frac{\partial^2\varphi_j}{\partial w_j\partial\overline{w_j}} & \text{if } a=b=0\\ \frac{\partial^2\varphi_j}{\partial w_j\partial\overline{z_j^b}} & \text{if } a=0, b>0\\ \frac{\partial^2\varphi_j}{\partial z_j^a\partial\overline{w_j}} & \text{if } a>0, b=0\\ \frac{\partial^2\varphi_j}{\partial z_j^a\partial\overline{z_j^b}} & \text{if } a,b>0\end{cases}$$

i.e. H_j is the complex Hessian of φ_j. By the assumption that the curvature of h is semi-positive, we have that H_j is positive semi-definite definite. By a simple computation, one has that

$$(H_j)^a_b = \begin{cases} \varphi_j^{(1,\,1)} + O(|w_j|) \\ \qquad \text{if } a = b = 0 \\ 2\left(\varphi_j^{(2,\,0)}\right)_{\overline{z_j^b}} \cdot w_j + \left(\varphi_j^{(1,\,1)}\right)_{\overline{z_j^b}} \cdot \overline{w_j} + O\left(|w_j|^2\right) \\ \qquad \text{if } a = 0, b > 0 \\ \left(\varphi_j^{(1,\,1)}\right)_{z_j^a} \cdot w_j + 2\left(\varphi_j^{(0,\,2)}\right)_{z_j^a} \cdot \overline{w_j} + O\left(|w_j|^2\right) \\ \qquad \text{if } a > 0, b = 0 \\ \left(\varphi_j^{(2,\,0)}\right)_{z_j^a\overline{z_j^b}} \cdot w_j^2 + \left(\varphi_j^{(1,\,1)}\right)_{z_j^a\overline{z_j^b}} \cdot |w_j|^2 + \left(\varphi_j^{(0,\,2)}\right)_{z_j^a\overline{z_j^b}} \cdot \overline{w_j}^2 + O\left(|w_j|^3\right) \\ \qquad \text{if } a, b > 0 \end{cases}.$$

As H_j is positive semi-definite, one has that $(H_j)^0_0$ is non-negative. Thus one has that $\varphi_j^{(1,\,1)}$ is also non-negative.

Lemma 3.5. — *The function $\varphi_j^{(1,1)}$ is plurisubharmonic on U_j.*

Proof. — Take d complex constants $\xi_1, \ldots, \xi_d \in \mathbb{C}$. Denote by v_ξ the vector $(0, \xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{C}^{d+1}$. Then, as H_j is positive semi-definite, one has that

$$v_\xi H_j {}^t\overline{v_\xi} = \sum_{a=1}^{d} \sum_{b=1}^{d} \xi_a (H_j)^a_b \overline{\xi_b},$$

is non-negative. By applying Lemma 2.1 to this function, one has that

$$\sum_{\nu=1}^{d} \sum_{\mu=1}^{d} \xi_\nu \left(\varphi_j^{(1,\,1)}\right)_{z_j^\nu \overline{z_j^\mu}} \overline{\xi_\mu} \geqslant 0,$$

from which the assertion follows. □

Proposition 3.6. — *Assume that $\varphi_j^{(1,\,1)} \not\equiv 0$. Then the function $z_j \mapsto \log \varphi_j^{(1,\,1)}(z_j)$ is plurisubharmonic.*

Proof. — For a positive real number ε, we consider the function $\psi_\varepsilon(z_j) := \log(\varphi_j^{(1,\,1)}(z_j) + \varepsilon)$. As ψ_ε's monotonically approximate the function log $\varphi_j^{(1,\,1)}(z_j)$ from above, it is sufficient to show that each ψ_ε is plurisubharmonic.

Let $\xi=(\xi_1,\xi_2,\ldots,\xi_d)\in\mathbb{C}^d$ be a vector. As

$$\begin{aligned}\partial\overline{\partial}\psi_\varepsilon=&\frac{1}{\left(\varphi_j^{(1,1)}+\varepsilon\right)^2}\\&\sum_{\nu=1}^{d}\sum_{\mu=1}^{d}\left(\varphi_j^{(1,1)}\cdot\left(\varphi_j^{(1,1)}\right)_{z_j^\nu\overline{z_j^\mu}}-\left(\varphi_j^{(1,1)}\right)_{z_j^\nu}\cdot\left(\varphi_j^{(1,1)}\right)_{\overline{z_j^\mu}}\right)\\&dz_j^\nu\wedge d\overline{z_j^\mu}+\frac{\varepsilon}{\left(\varphi_j^{(1,1)}+\varepsilon\right)^2}\sum_{\nu=1}^{d}\sum_{\mu=1}^{d}\left(\varphi_j^{(1,1)}\right)_{z_j^\nu\overline{z_j^\mu}}dz_j^\nu\wedge d\overline{z_j^\mu},\end{aligned}$$

it follows from Lemma 3.5 that it is sufficient to show the inequality

$$\sum_{\nu=1}^{d}\sum_{\mu=1}^{d}\xi_\nu\overline{\xi_\mu}\cdot\left(\varphi_j^{(1,1)}\cdot\left(\varphi_j^{(1,1)}\right)_{z_j^\nu\overline{z_j^\mu}}-\left(\varphi_j^{(1,1)}\right)_{z_j^\nu}\cdot\left(\varphi_j^{(1,1)}\right)_{\overline{z_j^\mu}}\right)\geqslant 0. \tag{3.6}$$

We set $u:=(1,0,0,\ldots,0)\in\mathbb{C}^{d+1}$ and $v_\xi:=(0,\xi_1,\xi_2,\ldots,\xi_d)\in\mathbb{C}^{d+1}$. As the quadratic form

$$\left\langle\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}c\\d\end{pmatrix}\right\rangle:=(au+bv_\xi)\,H_j\,{}^t\overline{(cu+dv_\xi)}$$

is semi-positive definite, one has that

$$\det\begin{pmatrix}uH_j{}^t\overline{u}&uH_j{}^t\overline{v_\xi}\\v_\xi H_j{}^t\overline{u}&v_\xi H_j{}^t\overline{v_\xi}\end{pmatrix}\geqslant 0.$$

As it holds that $uH_j{}^t\overline{u}=(H_j)_0^0=\varphi_j^{(1,1)}+O(|w_j|)$ and

$$\begin{aligned}uH_j{}^t\overline{v_\xi}&=\sum_{\nu=1}^{d}(H_j)_\nu^0\overline{\xi_\nu}\\&=\sum_{\nu=1}^{d}\overline{\xi_\nu}\cdot\left(2\left(\varphi_j^{(2,0)}\right)_{\overline{z_j^\nu}}\cdot w_j+\left(\varphi_j^{(1,1)}\right)_{\overline{z_j^\nu}}\cdot\overline{w_j}\right)+O\left(|w_j|^2\right)\\&=2\left(\sum_{\nu=1}^{d}\overline{\xi_\nu}\cdot\left(\varphi_j^{(2,0)}\right)_{\overline{z_j^\nu}}\right)\cdot w_j\\&\qquad+\left(\sum_{\nu=1}^{d}\overline{\xi_\nu}\cdot\left(\varphi_j^{(1,1)}\right)_{\overline{z_j^\nu}}\right)\cdot\overline{w_j}+O\left(|w_j|^2\right),\end{aligned}$$

it follows from Lemma 2.1 that

$$\begin{aligned}
&\varphi_j^{(1,1)} \cdot \sum_{\nu=1}^{d}\sum_{\mu=1}^{d} \xi_\nu \overline{\xi_\mu} \cdot \left(\varphi_j^{(1,1)}\right)_{z_j^\nu \overline{z_j^\mu}} \\
&\qquad - \left| \sum_{\nu=1}^{d} \overline{\xi_\nu} \cdot \left(\varphi_j^{(1,1)}\right)_{\overline{z_j^\nu}} \right|^2 - 4 \cdot \left| \sum_{\nu=1}^{d} \overline{\xi_\nu} \cdot \left(\varphi_j^{(2,0)}\right)_{\overline{z_j^\nu}} \right|^2 \\
&\quad = \sum_{\nu=1}^{d}\sum_{\mu=1}^{d} \xi_\nu \overline{\xi_\mu} \cdot \left(\varphi_j^{(1,1)} \cdot \left(\varphi_j^{(1,1)}\right)_{z_j^\nu \overline{z_j^\mu}} - \left(\varphi_j^{(1,1)}\right)_{z_j^\nu} \cdot \left(\varphi_j^{(1,1)}\right)_{\overline{z_j^\mu}} \right) \\
&\qquad - 4 \cdot \left| \sum_{\nu=1}^{d} \overline{\xi_\nu} \cdot \left(\varphi_j^{(2,0)}\right)_{\overline{z_j^\nu}} \right|^2
\end{aligned}$$

is non-negative, which proves the inequality (3.6). □

From Proposition 3.6 and the last inequality of the proof of it, one can easily deduce the following:

Theorem 3.7. — *Let X be a complex manifold and Y be a non-singular compact hypersurface of X which is Kähler. Let L be a semi-positive line bundle on X such that $L|_Y$ is topologically trivial. Assume that the conormal bundle $N_{Y/X}^{-1}$ is not pseudo-effective. Then there exists a system of local trivializations $\{(\widehat{V}_j, \widehat{e}_j, w_j)\}$ of type 3. Especially, it holds that $u_1(Y, X, L) = u_2(Y, X, L) = 0$.*

Note that the second obstruction class $u_2(Y, X, L)$ is well-defined in this case, since $H^0(Y, N_{Y/X}^*) = 0$ (Here we used Proposition 3.2 (iii)).

Proof of Theorem 3.7. — Assume that $\varphi_j^{(1,1)} \not\equiv 0$. It follows from the equation (3.1) with $n = 2$ that $\{(U_j, \varphi_j^{(1,1)} dw_j \wedge d\overline{w}_j)\}$ glue up to define a global form on Y. By Proposition 3.6, one has that this form is non-zero outside of a pluripolar subset of Y. Thus, again by using Proposition 3.6, it turns out that one may regard the system $\{(U_j, \varphi_j^{(1,1)})\}$ as a singular Hermitian metric on $N_{Y/X}$, say h_N, and that the curvature current $\sqrt{-1}\Theta_{h_N^{-1}}$ of the dual metric h_N^{-1} on $N_{Y/X}^{-1}$ is semi-positive. Thus, as $N_{Y/X}^{-1}$ is not pseudo-effective, one has that $\varphi_j^{(1,1)} \equiv 0$.

If follows from the same argument as in the proof of Proposition 3.6 that $(\varphi_j^{(2,0)})_{\overline{z_j^\nu}} \equiv 0$ for any $\nu = 1, 2, \ldots, d$. Therefore we have that $\varphi^{(2,0)}$ is a holomorphic function on U_j.

Let $\widehat{e}_j$ be the section of L on V_j defined by

$$\widehat{e}_j = e_j \cdot \left(1 + \varphi_j^{(2,0)}(z_j) \cdot w_j^2\right)$$

for each j after shrinking V_j if necessary (so that $\widehat{e}_j \neq 0$ at any point of V_j). Then, by a simple computation, one has that the system $\{(V_j, \widehat{e}_j, w_j)\}$ is of type 3. $\square$

Let $\widehat{e}_j$ be as in the proof of Theorem 3.7. Denote by $\widehat{\varphi}_j$ the corresponding local weight function of h. Then one has that

$$\begin{aligned}\widehat{\varphi}_j\,(z_j, w_j) &= -\log|\widehat{e}_j|^2_{h(z_j,\,w_j)}\\ &= -\log|e_j|^2_{h(z_j,\,w_j)} - \log\left|1+\varphi_j^{(2,\,0)}(z_j)\cdot w_j^2\right|^2\\ &= \varphi_j\,(z_j, w_j) - \log\left|1+\varphi_j^{(2,\,0)}(z_j)\cdot w_j^2\right|^2\\ &= O\left(|w_j|^3\right)\end{aligned}$$

as $|w_j| \to 0$. Thus we have that, by leaving w_j's as it was and changing only e_j's, one may assume that $\varphi_j(z_j, w_j) = O(|w_j|^3)$ holds as $|w_j| \to 0$ for each j when $r = 1$ and $N_{Y/X}^{-1}$ is not pseudo-effective. The similar argument also runs when $r = 1$, $N_{Y/X}$ is topologically trivial, and $L = [Y]$, which will be generalized in the following subsection.

3.5. Linearization in the higher order jets when Y is a hypersurface with topologically trivial normal bundle

Fix a positive integer n. In this section, we assume that Y is a hypersurface of X (i.e. $r = 1$) and that $N_{Y/X}$ is a unitary flat line bundle. In what follows, we let L be the line bundle $[Y]$.

On each V_j, we use the local frame e_j of L which corresponds to the meromorphic function $1/w_j$ via the isomorphism $\mathcal{O}_X([Y]) \cong \mathcal{O}_X(Y)$, where w_j is a local holomorphic defining function of Y in V_j. In this case, as is explained in § 2.1, $\{(V_j, w_j \cdot e_j)\}$ patches to each other to define the canonical section $f_Y \in H^0(V, [Y])$. In what follows, whenever we change a system of the local frames e_j's, we will also change a system of local defining functions w_j's so that the canonical section f_Y itself never changes. Let h be a C^∞ Hermitian metric on $[Y]$ with semi-positive curvature and φ_j be the local weight function with respect to the local frame e_j.

Consider the set $Z(Y, X, h; f_Y)$ of all positive integers n which satisfies the following condition: There exists a system $\{(V_j, e_j, w_j)\}$ of type n with $f_Y = w_j \cdot e_j$ such that the corresponding local weight function φ_j satisfies $\varphi_j(z_j, w_j) = O(|w_j|^n)$ as $|w_j| \to 0$ for each j. It clearly follows from the arguments in § 3.3 and Theorem 1.4 that $1, 2 \in Z(Y, X, h; f_Y)$.

Take an element $n \in Z(Y, X, h; f_Y)$ and a system $\{(V_j, e_j, w_j)\}$ of type n as above. Let

$$\varphi_j(z_j, w_j) = \sum_{p,\, q \geqslant 0,\, p+q=n} \varphi_j^{(p,\, q)}(z_j) \cdot w_j^p \overline{w_j}^q + R_j^{(n+1)}(z_j, w_j)$$

be the expression obtained by considering Taylor expansion of φ_j, where $R_j^{(n+1)}(z_j, w_j)$ is a smooth function with

$$R_j^{(n+1)}(z_j, w_j) = O(|w_j|^{n+1}) \text{ as } |w_j| \to 0.$$

Note that, by equation (3.1), one has that $\{(U_j, \varphi_j^{(p,\, q)}(z_j)\, dw_j^p \otimes d\overline{w_j}^q)\}$ patch to define a global section of $N_{Y/X}^{-p+q}$ for each pair (p, q) with $p, q > 0$ and $p + q = n$, and that

$$t_{jk}^{-n} \cdot \varphi_k^{(n,\, 0)}(z_k) - \varphi_j^{(n,\, 0)}(z_j) = -f_{kj,\, n}(z_j)$$

on each U_{jk}, where we are letting

$$\frac{t_{jk}^{-1} e_k}{e_j} = 1 + f_{kj,\, n}(z_j) \cdot w_j^n + O\left(w_j^{n+1}\right)$$

be the expansion.

Lemma 3.8. — *For $(p, q) = (n-1, 1), (n-2, 2), \ldots, (2, n-2), (1, n-1)$, there exists a constant $A_j^{(p,\, q)} \in \mathbb{C}$ such that $\varphi_j^{(p,\, q)} \equiv A_j^{(p,\, q)}$ holds on each U_j. Moreover, it holds that $A_j^{(p,\, q)}$ is non-negative if $p = q$, and that $A_j^{(p,\, q)} = 0$ if the line bundle $N_{Y/X}^{-p+q}$ is not analytically trivial.*

Proof. — Let $M_j := ((M_j)^a_b)_{a,\, b}$ be the $d \times d$-matrix with entries

$$(M_j)^\nu_\mu = \frac{\partial^2 \varphi_j}{\partial z_j^\nu \partial \overline{z_j^\mu}}(z_j) = \sum_{p,\, q \geqslant 0,\, p+q=n} \left(\varphi_j^{(p,\, q)}\right)_{z_j^\nu \overline{z_j^\mu}}(z_j) \cdot w_j^p \overline{w_j}^q + O\left(|w_j|^{n+1}\right)$$

at a point $z_j \in U_j$. As the curvature of h is semi-positive, one has that M_j is also positive semi-definite. Take an element $\xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{C}^d$. Then it easily follows from the semi-positivity that the sum

$$\sum_{p,\, q \geqslant 0,\, p+q=n} \left(\sum_{\nu=1}^{d} \sum_{\mu=1}^{d} \xi_\nu \overline{\xi_\mu} \cdot \left(\varphi_j^{(p,\, q)}\right)_{z_j^\nu \overline{z_j^\mu}}\right) \cdot w_j^p \overline{w_j}^q$$

is non-negative.

Assume that n is even. Set $n = 2m$. By Lemma 2.1, one has that

$$\sum_{\nu=1}^{d} \sum_{\mu=1}^{d} \xi_\nu \overline{\xi_\mu} \cdot \left(\varphi_j^{(m,\, m)}\right)_{z_j^\nu \overline{z_j^\mu}} \geqslant 0.$$

As ξ is arbitrary chosen, one has that the function $\varphi_j^{(m,\,m)}$ is plurisubharmonic. As $\varphi_j^{(m,\,m)}$ is a global function on a compact complex manifold Y, one has that it is constant. Therefore one obtains that

$$\left(\varphi_j^{(m,\,m)}\right)_{z_j^\nu \overline{z_j^\mu}} \equiv 0.$$

Thus, again by Lemma 2.1, one has that

$$\sum_{\nu=1}^{d}\sum_{\mu=1}^{d}\xi_\nu\overline{\xi_\mu}\cdot\left(\varphi_j^{(p,\,q)}\right)_{z_j^\nu \overline{z_j^\mu}} \equiv 0$$

holds for any element $\xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{C}^d$, and any pair (p,q) with $p, q \geqslant 0$ and $p+q=n$ in both the cases where n is even/odd. As one can regard $\{(U_j, \varphi_j^{(p,\,q)})\}$'s as pluriharmonic sections of unitary flat line bundles if $(p,q) \neq (n,0), (0,n)$, one can repeat the same argument as we have done for $(p,q)=(m,m)$ to obtain that each $\varphi_j^{(p,\,q)}$ is constant. When $n=2m$, the non-negativity of $\varphi_j^{(m,\,m)}$ follows by applying Lemma 2.1 to the function $(\varphi_j)_{w_j\overline{w_j}}$. When there exists j such that $\varphi_j^{(p,\,q)} \not\equiv 0$, we have that $\varphi_k^{(p,\,q)} \not\equiv 0$ for any k, which means that $N_{Y/X}^{-p+q}$ has a global nowhere vanishing section. $\square$

By using the constants as in Lemma 3.8, one can rewrite the expansion as

$$\begin{aligned}\varphi_j\left(z_j, w_j\right) = \sum_{p,\,q>0,\,p+q=n} & A_j^{(p,\,q)}\cdot w_j^p\overline{w_j}^q + \\ & \varphi_j^{(n,\,0)}(z_j)\cdot w_j^n + \varphi_j^{(0,\,n)}(z_j)\cdot\overline{w_j}^n + R_j^{(n+1)}\left(z_j, w_j\right).\end{aligned}$$

Note that $\varphi_j^{(0,\,n)} = \overline{\varphi_j^{(n,\,0)}}$ holds, since φ_j is real valued.

Lemma 3.9. — *The function $\varphi_j^{(n,\,0)}$ is holomorphic.*

Proof. — Let H_j be the $(d+1)\times(d+1)$ matrix with entries $(H_j)_b^a$ with $0 \leqslant a, b \leqslant d$ defined by

$$(H_j)_b^a = \begin{cases} \frac{\partial^2\varphi_j}{\partial w_j\partial\overline{w_j}} & \text{if } a=b=0 \\ \frac{\partial^2\varphi_j}{\partial w_j\partial\overline{z_j^b}} & \text{if } a=0, b>0 \\ \frac{\partial^2\varphi_j}{\partial z_j^a\partial\overline{w_j}} & \text{if } a>0, b=0 \\ \frac{\partial^2\varphi_j}{\partial z_j^a\partial\overline{z_j^b}} & \text{if } a,b>0 \end{cases}$$

i.e. H_j is the complex Hessian of φ_j. By the assumption that the curvature of h is semi-positive, we have that H_j is semi-positive definite. By a simple computation, one has that

$$(H_j)^a_b = \begin{cases} \sum_{p,\,q>0,\,p+q=} pqA_j^{(p,\,q)} \cdot w_j^{p-1}\overline{w_j}^{q-1} + O\left(|w_j|^{n-1}\right) & \text{if } a=b=0 \\ n\left(\varphi_j^{(n,\,0)}\right)_{\overline{z_j^b}} \cdot w_j^{n-1} + O\left(|w_j|^n\right) & \text{if } a=0, b>0 \\ n\left(\varphi_j^{(0,\,n)}\right)_{z_j^a} \cdot \overline{w_j}^{n-1} + O\left(|w_j|^n\right) & \text{if } a>0, b=0 \\ \left(\varphi_j^{(n,\,0)}\right)_{z_j^a\overline{z_j^b}} \cdot w_j^n + \left(\varphi_j^{(0,\,n)}\right)_{z_j^a\overline{z_j^b}} \cdot \overline{w_j}^n + O\left(|w_j|^{n+1}\right) & \text{if } a,b>0. \end{cases}$$

We set $u := (1, 0, 0, \dots, 0) \in \mathbb{C}^{d+1}$ and $v_\xi := (0, \xi_1, \xi_2, \dots, \xi_d) \in \mathbb{C}^{d+1}$. As the quadratic form

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle := (au + bv_\xi)\, H_j\, {}^t\overline{(cu + dv_\xi)}$$

is semi-positive definite, one has that

$$\det \begin{pmatrix} uH_j{}^t\overline{u} & uH_j{}^t\overline{v_\xi} \\ v_\xi H_j{}^t\overline{u} & v_\xi H_j{}^t\overline{v_\xi} \end{pmatrix} \geqslant 0.$$

As it hold that

$$\det \begin{pmatrix} uH_j{}^t\overline{u} & uH_j{}^t\overline{v_\xi} \\ v_\xi H_j{}^t\overline{u} & v_\xi H_j{}^t\overline{v_\xi} \end{pmatrix} = O\left(|w_j|^{2n-2}\right)$$

and the coefficient of $|w_j|^{2n-2}$ in Taylor expansion of the left hand side is

$$-n^2 \left| \sum_{\nu=1}^{d} \overline{\xi_\nu} \left(\varphi_j^{(n,\,0)}\right)_{\overline{z_j^\nu}} \right|^2,$$

the assertion follows from Lemma 2.1. □

By using these lemmata, one has the following:

Proposition 3.10. — *Let n be an element of $Z(Y, X, h; f_Y)$ and $\{(V_j, e_j, w_j)\}$ be a system of type n with $f_Y = w_j \cdot e_j$ such that the corresponding local weight function φ_j satisfies $\varphi_j(z_j, w_j) = O(|w_j|^n)$ as $|w_j| \to 0$ for each j. Then the following holds:*

(i) $u_n(Y, X) = u_n(Y, X, L) = 0$.
(ii) *When n is odd, it holds that $n+1 \in Z(Y, X, h; f_Y)$.*
(iii) *When n is even and $A_j^{(m,\,m)} \equiv 0$ for $m := n/2$, it holds that $n+1 \in Z(Y, X, h; f_Y)$.*

Proof. — Let $\widehat{e}_j$ be the section of L on V_j defined by

$$\widehat{e}_j = e_j \cdot \left(1 + \varphi_j^{(n,\,0)}(z_j) \cdot w_j^n\right)$$

for each j after shrinking V_j if necessary (so that $\widehat{e}_j \neq 0$ at any point of V_j). Denote by $\widehat{w}_j$ the corresponding defining function of U_j on V_j: i.e.

$$\widehat{w}_j = w_j \cdot \left(1 + \varphi_j^{(n,0)}(z_j) \cdot w_j^n\right)^{-1}.$$

Then, by a simple calculation and Lemma 3.9, one has that the system $\{(V_j, w_j)\}$ is of type $n+1$, which shows the assertion (i). In what follows we assume that $\{(V_j, w_j)\}$ is of type $n+1$ by replacing w_j with $\widehat{w}_j$.

When n is odd or when $n = 2m$ is even and $A_j^{(m,m)} \equiv 0$, it follows by applying Lemma 2.1 to the function $(\varphi_j)_{w_j\overline{w_j}}$ that $A^{(p,q)} \equiv 0$ for any (p, q) with $p+q = n$ and $p, q > 0$, from which the assertions (ii) and (iii) hold. □

Proof of Theorem 1.6. — Theorem follows from Theorem 1.4, Proposition 3.10 (i), and the fact that $2 \in Z(Y, X, h; f_Y)$. □

Note that the set $Z(Y, X, h; f_Y)$ need not to coincide with $\mathbb{Z}_{>0}$ even if the pair (Y, X) is of infinite type. For example, let X be a surface and $Y \subset X$ be a holomorphically embedded non-singular compact curve with topologically trivial normal bundle such that the pair (Y, X) is of type (β') or (β'') in the classification of [16, § 5]; i.e. there exists an open covering $\{U_j\}$ of Y and a local defining function w_j of Y on a neighborhood V_j of U_j such that $t_{jk}w_k = w_j$ holds on each V_{jk} ($t_{jk} \in \mathrm{U}(1)$). Consider a C^∞ Hermitian metric h of $[Y]$ whose local weight functions φ_j on each V_j with respect to the local frame e_j which corresponds to the meromorphic function $1/w_j$ satisfies $\varphi_j = |w_j|^2$. In this case, $3 \notin Z(Y, X, h; f_Y)$ whereas the pair (Y, X) is of infinite type.

4. Applications

4.1. Proof of Theorem 1.2

For proving Theorem 1.2, let us first explain our variant of Grauert's example (see also [6, Problem 2.2]).

Example 4.1. — Let C be a Riemann surface of genus 2. Let $\{U_j\}$ be a finite Stein cover of C and z_j be the coordinate of U_j which comes from the standard coordinate of a connected component of the inverse image of U_i by the universal covering $\mathbb{H} \to C$ ($\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$). Define a transition function $g_{jk}\colon U_{jk} \to \mathbb{C}^*$ of K_C by $dz_j = g_{jk}(z_k) \cdot dz_k$. Set

$$\rho_j(z_j) := \frac{\sqrt{-1}}{z_j - \overline{z_j}} = \frac{1}{2 \cdot \mathrm{Im}\, z_j}$$

on each U_i and

$$\xi_{jk} := \rho_j dz_j - \rho_k dz_k = (g_{jk} \cdot \rho_j - \rho_k) \cdot dz_k$$

on each U_{jk}. As

$$\overline{\partial}\left(\rho_j \cdot dz_j\right) = \frac{\sqrt{-1} dz_j \wedge d\overline{z_j}}{4\left(Im z_j\right)^2}$$

holds on each U_j and the $(1,1)$-forms as the right hand side glue up to define a non-trivial element of $H^1(C, K_C)$, one has that each ξ_{jk} is $\overline{\partial}$-closed and that

$$\xi := \left[\,\{(U_{jk}, \xi_{jk})\}\,\right] \in \check{H}^1\left(\{U_j\}, K_C\right)$$

is a non-trivial element (it follows by considering the Čech–Dolbeault correspondence).

We let X be the ruled surface which is the natural compactification of the affine bundle $\widetilde{X} := \bigcup_j K_{U_j}/\sim$ by considering the infinity section, where "$\sim$" is the relation generated by the following: $(x_j, v_j dz_j) \sim (x_k, v_k dz_k)$ holds if and only if $x_j = x_k \in U_{jk}$ and $v_j dz_j = v_k dz_k + \xi_{jk}$, or equivalently, $g_{jk} \cdot v_j = v_k + (g_{jk} \cdot \rho_j - \rho_k)$. Denote by $Y \subset X$ the infinity section. We have that $N_{Y/X} \cong K_C^{-1}$ and $(Y^2) = -2$. Denote by $\pi \colon X \to C$ the projection. We let L be a line bundle defined by $L := [Y] \otimes \pi^* K_C$. Note that $(L^2) = 2$ and $(L.Y) = 0$. Note also that it is easily observed that L is nef and big.

Regard the fiber coordinate function v_j as a local frame of the line bundle $[Y]$, and dz_j as of K_C. Then naturally one can regard $e_j := v_j \otimes \pi^* dz_j$ as a local frame of L. In what follows, we use $w_j := v_j^{-1}$ as a local defining function of Y and (z_j, w_j) as local coordinates on a neighborhood of $\pi^{-1}(U_j) \cap Y$. Note that

$$\frac{e_k}{e_j} = \frac{\pi^* dz_k}{\pi^* dz_j} \cdot \frac{v_k}{v_j} = g_{jk}^{-1} \cdot \frac{g_{jk} v_j - (g_{jk} \cdot \rho_j - \rho_k)}{v_j} = 1 - \left(\rho_j - g_{jk}^{-1} \cdot \rho_k\right) \cdot w_j$$

holds. Therefore one has that the first obstruction class $u_1(Y, X, L) \in H^1(Y, N_{Y/X}^{-1})$ coincides with the class $\xi \in H^1(Y, K_C)$ via the isomorphism between $N_{Y/X}^{-1}$ and K_C induced by $dw_j \mapsto dz_j$. Thus it follows that $u_1(Y, X, L) \neq 0$.

Proof of Theorem 1.2. — We show that the triple (Y, X, L) described in Example 4.1 satisfies that L is not semi-positive. Assume that L is semi-positive. Then, by Theorem 1.4, one has that $u_1(Y, X, L) = 0$, which contradicts to the fact mentioned in Example 4.1. □

Remark 4.2. — On higher dimensional manifolds, the existence of nef, big and non semi-positive line bundle have already been shown by [7] and [3]. Let us investigate [3]'s version of such an example ([3, Example 5.4]) here. In this example, the line bundle L is on the total space of the projective plane bundle $p\colon X \to C$ over an elliptic surface C which corresponds to a vector bundle $E \oplus A^{-1}$, where E is the non-trivial extension of the trivial line bundle by the trivial line bundle on C, and A is an ample line bundle. The line bundle L is the relative $\mathcal{O}(1)$-bundle (see also [8, Example 4.2]). Let Y be the image of the section of p which corresponds to the trivial subbundle of rank 1 included in E. Then one can easily calculate that the first obstruction class $u_1(Y, X, L)$ coincides with $(\xi, 0) \in H^1(C, \mathcal{O}_C) \oplus H^1(C, \mathcal{O}_C(A^{-1}))$ via the isomorphism $p|_Y\colon Y \cong C$, where ξ is the extension class of $0 \to \mathcal{O}_C \to \mathcal{O}_C(E) \to \mathcal{O}_C \to 0$.

4.2. Proof of Theorem 1.3

Proof. — Here we show Theorem 1.3. As $d = 1$ under the configuration of this theorem, we denote z_j^1 simply by z_j.

According to [9, Theorem 1.1], it follows from the semi-positivity of $[Y]$ that the pair (Y, X) is of infinite type. When $N_{Y/X}$ is a torsion element of the Picard variety, it follows from [16, Theorem 3] that there exists a neighborhood V of Y in X and a proper surjective holomorphic map $\pi\colon V \to \Delta$ with $Y = \pi^{-1}(0)$, where Δ is the unit disc of $\mathbb{C}$ (Y may be a multiple fiber). In this case, the assertion (ii) holds by regarding this fibration as a foliation and by considering the maximal principle on each fiber (=leaf).

Therefore, the problem is reduced to the case where $N_{Y/X}$ is non-torsion in the Picard variety. As is mentioned in § 3.5, it follows from the arguments in § 3.3 that $2 \in Z(Y, X, h; f_Y)$. Take a system $\{(V_j, e_j, w_j)\}$ of type 2 such that the corresponding local weight function φ_j satisfies $\varphi_j(z_j, w_j) = O(|w_j|^2)$ as $|w_j| \to 0$ for each j. Consider the set S of all positive integers m such that

$$\left.\frac{\partial^{2m}\varphi_j}{\partial w_j^m \partial \overline{w_j}^m}\right|_{U_j} \not\equiv 0.$$

First, let us consider the case where $S = \emptyset$. In this case, by Lemma 4.3 below, one has that there exists a pluriharmonic function $\eta_j^{(n)}$ for any positive integer n such that $\varphi_j = \eta_j^{(n)} + O(|w_j|^n)$ as $|w_j| \to 0$. Therefore, when $S = \emptyset$, the assertion (iii) holds if $\Theta_h \wedge \Theta_h \equiv 0$. When there exists a point of X at which $\Theta_h \wedge \Theta_h \neq 0$, it follows from [4, Proposition 2] that the assertion (i) holds (see Remark 4.4 below for details).

Next, consider the case where $S \neq \emptyset$. Denote by m the minimum of S. Take a holomorphic function $f_j^{(m-1)}$ as in Lemma 4.3. Then the expansion of φ_j is in the form

$$\begin{aligned}\varphi_j\left(z_j, w_j\right) = f_j^{(m-1)}\left(z_j, w_j\right) + \\ \overline{f_j^{(m-1)}\left(z_j, w_j\right)} + \sum_{p,\, q \geqslant 0,\, p+q=2m} \varphi_j^{(p,\, q)} \cdot w_j^p \overline{w_j}^q + O\left(|w_j|^{2m+1}\right).\end{aligned}$$

Define a new system

$$\{(V_j, \widehat{e}_j, \widehat{w}_j)\} \quad \text{by} \quad \widehat{e}_j := e_j \cdot \exp\left(f_j^{(m-1)}\right) \quad \text{and} \quad \widehat{w}_j := w_j \cdot \exp\left(-f_j^{(m-1)}\right).$$

Then one has that the corresponding (new) local weight function $\widehat{\varphi}_j$ satisfies

$$\begin{aligned}\widehat{\varphi}_j &= \varphi_j - \log\left|\exp\left(f_j^{(m-1)}\right)\right|^2 \\ &= \sum_{p,\, q \geqslant 0,\, p+q=2m} \varphi_j^{(p,\, q)} \cdot w_j^p \overline{w_j}^q + O\left(|w_j|^{2m+1}\right) \\ &= \sum_{p,\, q \geqslant 0,\, p+q=2m} \varphi_j^{(p,\, q)} \cdot \widehat{w}_j^p \overline{\widehat{w}_j}^q + O\left(|\widehat{w}_j|^{2m+1}\right).\end{aligned}$$

Therefore, by replacing our system $\{(V_j, e_j, w_j)\}$ with $\{(V_j, \widehat{e}_j, \widehat{w}_j)\}$, one may assume that $\{(V_j, w_j)\}$ is a system of type $2m-1$ and that the expansion of the local weight function φ_j with respect to the local frame e_j is in the form

$$\begin{aligned}\varphi_j\left(z_j, w_j\right) = \sum_{p,\, q > 0,\, p+q=2m} A_j^{(p,\, q)} \cdot w_j^p \overline{w_j}^q + \\ \varphi_j^{(2m,\, 0)}(z_j) \cdot w_j^{2m} + \varphi_j^{(2m,\, 0)}(z_j) \cdot \overline{w_j}^{2m} + O\left(|w_j|^{2m+1}\right).\end{aligned}$$

By Lemma 3.8 and the assumption that $N_{Y/X}$ is non-torsion, one has that $A_j^{(p,\, q)} \equiv 0$ if $p \neq q$. Again by Lemma 3.8 and our definition of m, one has that there exists a positive constant A such that $A_j^{(m,\, m)} \equiv A$. Therefore, by Lemma 3.9 and an argument in the proof of Proposition 3.10 (i), we can rewrite the above expansion into the form

$$\varphi_j\left(z_j, w_j\right) = A \cdot |w_j|^{2m} + O\left(|w_j|^{2m+1}\right)$$

by changing the system $\{(V_j, w_j)\}$ again if necessary. Thus one can calculate the complex Hessian as follows:

$$\begin{pmatrix} (\varphi_j)_{w_j\overline{w_j}} & (\varphi_j)_{w_j\overline{z_j}} \\ (\varphi_j)_{z_j\overline{w_j}} & (\varphi_j)_{z_j\overline{z_j}} \end{pmatrix} = \begin{pmatrix} m^2A|w_j|^{2m-2} + O\left(|w_j|^{2m-1}\right) & O\left(|w_j|^{2m}\right) \\ O\left(|w_j|^{2m}\right) & O\left(|w_j|^{2m+1}\right) \end{pmatrix}.$$

From this calculation, one has that there exists a neighborhood V of Y such that the curvature Θ_h has at least one positive eigenvalue at each point of $V \setminus Y$.

In the case where there exists a point in $V \setminus Y$ at which Θ_h has two positive eigenvalues, it follows from [4, Proposition 2] that the assertion (i) holds (see Remark 4.4 for details). Thus, in what follows, we may assume that the rank of the complex Hessian of each φ_j is one at each point of $V \setminus Y$. Then it follows from [15] and [2, Theorem 2.4] that there exists a (Monge–Ampère) foliation $\mathcal{G}$ on $V \setminus Y$ whose leaves are holomorphic. Let $G \subset T_{V\setminus Y}$ be the corresponding subbundle: $G := T_{\mathcal{G}}$. Denote by $s\colon V \setminus Y \to \mathbf{P}(T_{V\setminus Y})$ the smooth section whose image coincides with $\mathbf{P}(G)$, where we denote by $\mathbf{P}$ the relative projectivization.

For each point $p \in V_j \cap (V \setminus Y)$, take complex numbers $a_j(p)$ and $b_j(p)$ such that

$$s(p) = \left[a_j(p) \cdot \frac{\partial}{\partial w_j} + b_j(p) \cdot \frac{\partial}{\partial z_j}\right].$$

By the definition of Monge–Ampère foliation, one has that

$$\begin{pmatrix} (\varphi_j)_{w_j\overline{w_j}}(p) & (\varphi_j)_{w_j\overline{z_j}}(p) \\ (\varphi_j)_{z_j\overline{w_j}}(p) & (\varphi_j)_{z_j\overline{z_j}}(p) \end{pmatrix} \begin{pmatrix} a_j(p) \\ b_j(p) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

holds at each point of $V_j \cap (V \setminus Y)$. Therefore, one has the estimate $a_j(p)/b_j(p) = O(|w_j(p)|)$ as p approaches to U_j. Thus it follows that the section $\widetilde{s}\colon V \to \mathbf{P}(T_V)$ defined by

$$\widetilde{s}(p) := \begin{cases} \left[\frac{\partial}{\partial z_j}\right] & \text{if } p \in Y \\ s(p) & \text{if } p \in V \setminus Y \end{cases}$$

is continuous.

Denote by $F \subset T_V$ the subbundle which corresponds to the image of $\widetilde{s}$. This subbundle F is integrable at a point p of V in both the cases of $p \in Y$ and $p \in V \setminus Y$, from which the Theorem 1.3 follows. □

Lemma 4.3. — *Let X, Y, L, and h be as in Theorem 1.3. Assume that $N_{Y/X}$ is non-torsion in the Picard variety of Y. Take a system $\{(V_j, e_j, w_j)\}$*

of type 2 such that the corresponding local weight function φ_j satisfies $\varphi_j(z_j, w_j) = O(|w_j|^2)$ as $|w_j| \to 0$ for each j, and an integer $m \geqslant 0$ which is less than any element of S, where S is the set as in the proof of Theorem 1.3. Then we have

$$\varphi_j\left(z_j, w_j\right) = f_j^{(m)}\left(z_j, w_j\right) + \overline{f_j^{(m)}\left(z_j, w_j\right)} + O\left(\left|w_j^{2m+2}\right|\right),$$

where $f_j^{(m)}$ is a holomorphic function on V_j in the form

$$f_j^{(m)}\left(z_j, w_j\right) = \sum_{\nu=1}^{2m+1} \varphi_j^{(\nu,0)}(z_j) \cdot w_j^{\nu}.$$

Here $\varphi_j^{(\nu,0)}$ is a holomorphic function on U_j for each ν with $1 \leqslant \nu \leqslant 2m+1$.

Proof. — The assertion clearly holds for $m = 0$. Assume the assertion for an integer μ with $0 \leqslant \mu < m$ as the inductive assumption. Assume also that m is less than any element of S. Then, it follows by applying Lemma 2.1 for $(H_j)_0^0$ that the expansion of φ_j is in the form

$$\begin{aligned}\varphi_j =& f_j^{(m-1)}\left(z_j, w_j\right) + \overline{f_j^{(m-1)}\left(z_j, w_j\right)} + \varphi_j^{(2m,0)}(z_j) \cdot w_j^{2m} \\ &+ \overline{\varphi_j^{(2m,0)}(z_j)} \cdot \overline{w_j}^{2m} + \varphi_j^{(2m+1,0)}(z_j) \cdot w_j^{2m+1} \\ &+ \overline{\varphi_j^{(2m+1,0)}(z_j)} \cdot \overline{w_j}^{2m+1} + O\left(|w_j|^{2m+2}\right),\end{aligned}$$

where H_j is the 2×2 matrix with entries $(H_j)_b^a$ with $0 \leqslant a, b \leqslant 1$ defined by

$$(H_j)_b^a = \begin{cases} \frac{\partial^2 \varphi_j}{\partial w_j \partial \overline{w_j}} & \text{if } a = b = 0 \\ \frac{\partial^2 \varphi_j}{\partial w_j \partial \overline{z_j}} & \text{if } a = 0, b = 1 \\ \frac{\partial^2 \varphi_j}{\partial z_j \partial \overline{w_j}} & \text{if } a = 1, b = 0 \\ \frac{\partial^2 \varphi_j}{\partial z_j \partial \overline{z_j}} & \text{if } a = b = 1 \end{cases}$$

i.e. H_j is the complex Hessian of φ_j. As it holds that det $H_j = O(|w_j|^{4m-2})$ and the coefficient of $|w_j|^{4m-2}$ of the expansion of this function is

$$-4m^2 \left|\left(\varphi_j^{(2m,0)}\right)_{\overline{z_j}}\right|^2,$$

It follows again from Lemma 2.1 that $\varphi_j^{(2m,0)}$ is holomorphic. By a similar argument, one also has that $\varphi_j^{(2m+1,0)}$ is holomorphic. □

Remark 4.4. — In order to conclude that $X \setminus Y$ is holomorphically convex by using [4, Proposition 2], as $X \setminus Y$ is a connected Kähler surface with trivial canonical bundle, one need to show the existence of a C^∞ plurisubharmonic exhaustion function $f\colon X \setminus Y \to \mathbb{R}$ which is strictly plurisubharmonic at some point. To show this, we use the existence of a point $p \in X \setminus Y$ at which the curvature Θ_h has two positive eigenvalues. Let ψ be a C^∞ plurisubharmonic exhaustion function of X (we need to assume that X is weakly 1-complete in order to assure the existence of this). Then the function $f := \psi - \log |f_Y|_h^2$ enjoys the condition above, where f_Y is the canonical section of $[Y]$.

4.3. Proof of Theorem 1.5

In this subsection, we prove the following:

Theorem 4.5. — *Let Y be a compact non-singular hypersurface of a complex manifold X and L be a holomorphic line bundle on X. Assume that the restriction $L|_Y$ of L to Y is unitary flat, the line bundle $K_Y^{-1} \otimes N_{Y/X}^{-1}$ is semi-positive, and that, for any neighborhood V of Y in X, there exists a 1-convex neighborhood of Y in V whose maximal compact analytic set is Y. Then the restriction of L to a neighborhood is unitary flat if $u_1(Y, X, L) = 0$.*

Proof. — Take an open cover $\{U_j\}$ of Y, small open subset V_j of X such that $U_j = V_j \cap Y$, a local defining function w_j of U_j in V_j, and a local frame e_j of L on V_j as in § 3. Assume that $u_1(Y, X, L) = 0$. Then, from the same argument as in the proof of Theorem 1.4, it follows that one may assume that the system $\{(V_j, e_j, w_j)\}$ is of type 2 by modifying them if necessary; namely there exists a constant $t_{jk} \in \mathrm{U}(1)$ such that $t_{jk}^{-1} e_k/e_j = 1 + O(w_j^2)$ on each V_{jk}. By shrinking V_j's, we may also assume that $|t_{jk}^{-1} e_k/e_j - 1| < 1$ on each V_{jk}. Define a function $a_{jk}\colon V_{jk} \to \mathbb{C}$ by

$$a_{jk} := \log \left(\frac{t_{jk}^{-1} e_k}{e_j} \right),$$

where we are using the branch such that $\log 1 = 0$. It can easily be observed that $a_{jk} = O(w_j^2)$ and that $a_{jk} + a_{k\ell} + a_{\ell j} = 0$ holds on each $V_j \cap V_k \cap V_\ell$. Therefore, $\{(V_{jk}, a_{jk})\}$ defines an element of Čech cohomology group $\check{H}^1(\{V_j\}, \mathcal{O}_V(-2Y))$.

By shrinking if necessary, we will assume that $V := \bigcup_j V_j$ is a 1-convex neighborhood of Y whose maximal compact analytic set is Y in what follows. Then, according to Ohsawa's vanishing theorem [14, Theorem 4.5], it follows that $H^1(V, [Y]^{-2}) = 0$, since

$$\left([Y]^{-2} \otimes K_V^{-1}\right)\big|_Y \cong N_{Y/X}^{-2} \otimes \left(N_{Y/X}^{-1} \otimes K_Y\right)^{-1} = N_{Y/X}^{-1} \otimes K_Y^{-1}$$

is semi-positive. Therefore one has that the Čech cohomology class defined by $\{(V_{jk}, a_{jk})\}$ is trivial, which means that there exists a holomorphic function $b_j \colon V_j \to \mathbb{C}$ on each V_j such that $b_j = O(w_j^2)$ and that $-b_j + b_k = a_{jk}$ holds on each V_{jk}.

Let $\widehat{e}_j$ be a new frame defined by $\widehat{e}_j := e_j \cdot \exp(-b_j)$. Then it follows from a simple calculation that $t_{jk}^{-1}\widehat{e}_k = \widehat{e}_j$, which prove the Theorem 4.5. □

Proof of Theorem 1.5. — By Theorem 1.4, it is sufficient to show that L is semi-positive by assuming that $u_1(Y, X, L) = 0$. As $\deg N_{Y/X} < 0$, Y admits a fundamental system of neighborhoods which consists of 1-convex neighborhoods of Y whose maximal compact analytic sets are Y. The degree of the line bundle $K_Y^{-1} \otimes N_{Y/X}^{-1}$ is non-negative, since $\deg N_{Y/X} \leqslant 2-2g$, from which it follows that $K_Y^{-1} \otimes N_{Y/X}^{-1}$ is semi-positive. Therefore it follows from Theorem 4.5 that there exists a neighborhood V of Y such that the restriction $L|_V$ is unitary flat.

Take a C^∞ Hermitian metric h_0 on $L \otimes [Y]^{-m}$ with semi-positive curvature. Let f_Y be the canonical section of $[Y]$. Denote by h_Y the singular Hermitian metric on $[Y]$ defined by $|f_Y|_{h_Y}^2 \equiv 1$. Then a C^∞ Hermitian metric on $L = (L \otimes [Y]^{-m}) \otimes [Y]^m$ can be constructed from the flat metric on $L|_V$ and the singular Hermitian metric $h_0 \cdot h_Y^m$ on L by using "regularized minimum construction", see [11, § 2.1] for the detail. □

5. Problems

Towards solving Conjecture 1.1 or [11, Conjecture 2.1], here we make some discussion.

First, consider one of the simplest cases: when Y is a non-singular compact curve holomorphically embedded into a non-singular surface X such that the normal bundle is topologically trivial. Assume that the line bundle $[Y]$ is semi-positive. Take a C^∞ Hermitian metric h with semi-positive curvature. As it follows from [9, Theorem 1.1] that the pair (Y, X) is of infinite type, we are interested in determining whether or not there exists such an example of (Y, X) of type (γ) in Ueda's classification [16, § 5]. From the viewpoint of Theorem 1.3 and the study on the relation between the holonomy of the foliation and Ueda type we investigated in [12], we are interested in the following question at least when h is real analytic:

Question 5.1. — Let Y be a smooth elliptic curve holomorphically embedded into a non-singular Kähler surface S. Assume that the canonical bundle K_S or the anti-canonical bundle K_S^{-1} is semi-positive. Assume also that there exists an integer m and a divisor D of S such that $K_S = [mY+D]$ and that the support of D does not intersect Y. Let X be a sufficiently small weakly 1-complete neighborhood of Y.

(i) Is there an example of (Y, X) of type (γ) such that $X \setminus Y$ is holomorphically convex?

(ii) How is the holonomy of the foliation $\mathcal{F}$ as in Theorem 1.3 (ii) (if exists)?

Note that many things are known for Question 5.1 when S is projective and K_S is semi-positive, since abundance conjecture is affirmative for projective surfaces.

Next, let us consider the case of general dimensions. According to [9, Theorem 1.1] and Theorem 1.6, it seems to be natural to pose the following:

Conjecture 5.2. — *Let X be a complex manifold and Y be a non-singular compact hypersurface of X which is Kähler. Assume that the normal bundle $N_{Y/X}$ is topologically trivial, and that $[Y]$ is semi-positive. Then the pair (Y, X) is of infinite type.*

We are also interested in the case where Y is a hypersurface and $N_{Y/X}^{-1}$ is not pseudo-effective, since in this case, by Proposition 3.2 (iii), the well-definedness of the n^{th} obstruction classes are assured for any n whenever there exists a system of type n. By Theorem 3.7, it seems to natural to ask the following:

Problem 5.3. — *Let X be a complex manifold and Y be a non-singular compact hypersurface of X which is Kähler. Let L be a semi-positive line bundle on X such that $L|_Y$ is topologically trivial. Assume that the conormal bundle $N_{Y/X}^{-1}$ is not pseudo-effective. Then, does it hold that $u_n(Y, X, L) = 0$ for any positive integer n?*

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