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# QUOTIENT SINGULARITIES OF PRODUCTS OF TWO CURVES

**by Kentaro MITSUI (*)**

Abstract. — We give a method to resolve a quotient surface singularity which arises as the quotient of a product action of a finite group on two curves. In the characteristic zero case, the singularity is resolved by means of a continued fraction, which is known as the Hirzebruch–Jung desingularization. We develop the method in the positive characteristic case where the square of the characteristic does not divide the order of the group.

Résumé. — Nous donnons une méthode pour résoudre une singularité quotient de surface qui se présente comme le quotient d'une action produit d'un groupe fini sur deux courbes. En caractéristique nulle, la singularité est résolue au moyen d'une fraction continue (désingularisation de Hirzebruch–Jung). Nous développons la méthode dans le cas de la caractéristique strictement positive où le carré de la caractéristique ne divise pas l'ordre du groupe.

## 1. Introduction

We give a method to resolve a quotient surface singularity which arises as the quotient of a product action of a finite group on two curves. In the characteristic zero case, the singularity is resolved by means of a continued fraction, which is known as the Hirzebruch–Jung desingularization ([5], [7], [3, §10.2]). The intersection matrix of the exceptional locus is determined by this continued fraction, which gives formulas for invariants associated with the singularity. Nevertheless, few results are known in the positive characteristic case where the characteristic divides the order of the group (see [10, 1.4] for the history) while the existence of a desingularization of

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a two-dimensional excellent scheme is known ([2], [9]). In this paper, we give an explicit desingularization and calculate the intersection matrix and invariants by means of *plural* continued fractions in the case where the square of the characteristic does not divide the order of the group. Our result generalizes those in [6] and [10] as explained below. A special case of our result answers the question on the intersection matrix in [10, §1]. In the following, we explain our result by comparing it with the characteristic zero case or more generally the tame quotient case.

Let $k$ be an algebraically closed field of characteristic $p \geqslant 0$ and $G$ be a finite group of order $\#G$. Assume that $G$ faithfully acts on the complete discrete valuation ring $k[\![x_i]\!]$ over $k$ for any $i \in \{1, 2\}$. Put $R := k[\![x_1, x_2]\!]$ and $X := \operatorname{Spec} R$. Take the quotient

$$q\colon X \longrightarrow Y := X/G \tag{1.1}$$

of $X$ by the product action of $G$. The singularity of $Y$ may be resolved by a proper birational morphism $h\colon \widehat{Y} \to Y$. By $E_h$ we denote the exceptional locus of $h$ with reduced structure. The desingularization $h$ of $Y$ is called *good* (resp. *minimal good*) if any singularity of $E_h$ is a node, and any irreducible component of $E_h$ is regular (resp. $h$ is minimal among all good desingularizations, i.e., $h$ is good, and, if $h = h' \circ h''$, and $h'$ is a good desingularization, then $h''$ is an isomorphism). Let $\Omega_h$ be an intersection matrix of $E_h$. Put

$$\delta := |\det \Omega_h|, \tag{1.2}$$

which does not depend on the choice of $h$ whenever $h$ is good (Proposition 5.5). By $Z$ we denote the fundamental cycle of $h$ (Section 6). The *fundamental genus* $p_f$ (resp. the *geometric genus* $p_g$) *of the singularity of* $Y$ is defined as the arithmetic genus of $Z$ (resp. the dimension of $R^1 h_* \mathcal{O}_{\hat{Y}}$ over $k$). The singularity of $Y$ is said to be *rational* if $p_g = 0$, which is equivalent to the condition $p_f = 0$ [1, Theorem 3].

In the case $p \nmid \#G$, the Hirzebruch–Jung desingularization of $Y$ is minimal good whose exceptional locus is a chain of the projective lines. Moreover, the equalities $\delta = \#G$ and $p_f = p_g = 0$ hold. In particular, the singularity of $Y$ is rational.

Assume that $p > 0$ and $p \mid \#G$. Note that $G$ has the unique $p$-Sylow subgroup $H$. Although $H$ has a normal subgroup of order $p$, few results are known even in the simplest case $H \cong \mathbb{Z}/p\mathbb{Z}$. In the following, we assume that $H \cong \mathbb{Z}/p\mathbb{Z}$. Our main theorems give a minimal good desingularization of $Y$ whose exceptional locus is a star-shaped tree of the projective lines (Theorem 3.4; its proof is given in Section 4) and the intersection matrix of the exceptional locus by means of *three* continued fractions (Theorem 3.6; its proof is given in Section 5). As a corollary, we calculate $\delta$ (Theorem 1.2; its proof is given in Section 5) and give algorithms to obtain $Z$ (Section 6), $p_f$ (Section 6), and $p_g$ (Sections 7–8).

Take a generator $\sigma$ of $H$. For $i \in \{1,2\}$, we denote the maximal ideal of $k[\![x_i]\!]$ by $\mathfrak{m}_i$, take the valuation $v_i$ of $k[\![x_i]\!]$ with $v_i(k[\![x_i]\!] \setminus \{0\}) = \mathbb{Z}_{\geqslant 0}$, and put

$$\alpha_i \coloneqq v_i(\sigma x_i - x_i) - 1, \quad d \coloneqq \gcd(\alpha_1, \alpha_2), \quad \text{and} \quad a_i \coloneqq \frac{\alpha_i}{d}. \tag{1.3}$$

Note that $\alpha_i \in \mathbb{Z}_{\geqslant 1}$ since the action of $G$ on $k[\![x_i]\!]$ is faithful, and the multiplicative identity is the unique $p$-th root of unity in $k$. The definition of $\alpha_i$ does not depend on the choice of the generator $\sigma$ of $H$ or the uniformizer $x_i$ of $k[\![x_i]\!]$ since $\mathfrak{m}_i^{\alpha_i+1}$ is generated by $\{\tau x - x \,|\, \tau \in H, x \in \mathfrak{m}_i\}$. Since $\dim_k \mathfrak{m}_i/\mathfrak{m}_i^2 = 1$, we identify $\mathrm{Aut}_k(\mathfrak{m}_i/\mathfrak{m}_i^2)$ with $k^\times$. Then the action of $G$ on $\mathfrak{m}_i/\mathfrak{m}_i^2$ induces a character $\rho_i \colon G \to k^\times$. Put

$$m \coloneqq \#(G/H), \quad n \coloneqq \operatorname{ord} \rho_1^{a_1}\rho_2^{-a_2}, \quad \text{and} \quad d' \coloneqq \frac{d}{n}. \tag{1.4}$$

Note that $d' \in \mathbb{Z}$ (Lemma 3.1).

*Example 1.1.* — Assume that $p \geqslant 3$, $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $k[\![x_i]\!]$ is the extension of $k[\![y_i]\!]$ defined as $k[\![y_i]\!][x_i]/(P_i)$ for any $i \in \{1,2\}$, where

$$P_i \coloneqq x_i^p - y_i^{p-1}x_i - y_i.$$

Put $\sigma \coloneqq (1,0) \in G$ and $\tau \coloneqq (0,1) \in G$. Suppose that the equality

$$(\sigma x_i, \sigma y_i, \tau x_i, \tau y_i) = (x_i + y_i, y_i, -x_i, -y_i)$$

holds for any $i \in \{1,2\}$. Then $H \cong \mathbb{Z}/p\mathbb{Z}$, $G/H \cong \mathbb{Z}/2\mathbb{Z}$, $(\alpha_1,\alpha_2) = (p-1,p-1)$, $(a_1,a_2) = (1,1)$, and $(m,n,d,d') = (2,1,p-1,p-1)$. Put

$$\Phi \coloneqq \{(1,1),(1,2),(2,2)\}, \quad y_{i,j} \coloneqq y_i y_j \text{ for } (i,j) \in \Phi,$$

and

$$y \coloneqq x_1 y_2 - x_2 y_1.$$

Then $\{y_{i,j}\}_{(i,j)\in\Phi} \cup \{y\} \subset R^G$, and the equalities

$$\left(\frac{x_1}{y_1} - \frac{x_2}{y_2}\right)^p - \left(\frac{x_1}{y_1} - \frac{x_2}{y_2}\right) - (y_1^{1-p} - y_2^{1-p}) = \frac{P_1}{y_1^p} - \frac{P_2}{y_2^p} = 0$$

hold in $k(\!(x_1,x_2)\!)$. By multiplying both sides by $y_{1,2}^p$, we obtain the equality

$$y^p - y_{1,2}^{p-1} y + y_{1,2}\left(y_{1,1}^{\frac{p-1}{2}} - y_{2,2}^{\frac{p-1}{2}}\right) = 0$$

in $R$. Put

$$U := k[\![Y_{1,1}, Y_{2,2}]\!], \quad T := U[Y_{1,2}]/(Q), \quad \text{and} \quad S := T[Y]/(P),$$

where

$$Q := Y_{1,2}^2 - Y_{1,1}Y_{2,2} \quad \text{and} \quad P := Y^p - Y_{1,2}^{p-1}Y + Y_{1,2}\left(Y_{1,1}^{\frac{p-1}{2}} - Y_{2,2}^{\frac{p-1}{2}}\right).$$

Then $T$ (resp. $S$) is a Cohen–Macaulay local ring of dimension two whose singular locus is defined by the maximal ideal, which implies that $T$ (resp. $S$) is a normal integral domain by Serre's criterion for normality. We regard $S$ as a subring of $R$ by the injective $k$-algebra homomorphism $S \to R$, $Y \mapsto y$, $Y_{i,j} \mapsto y_{i,j}$ for $(i,j) \in \Phi$. Then $U \subset T \subset S \subset R^G \subset R$ are finite extensions of normal integral domains. For an extension $A'/A$ of integral domains, we denote the degree of the extension of their fields of fractions by $[A' : A]$. Then the equalities $[R : U] = 4p^2$, $[R : R^G] = 2p$, and $[T : U] = 2$ give the equality $[R^G : T] = p$. Thus, since $S \neq T$, the equality $[R^G : S] = 1$ holds, which implies that $R^G = S$. As a result, we obtain the isomorphism

$$R^G \cong k[\![Y, Y_{1,1}, Y_{1,2}, Y_{2,2}]\!]/(P, Q).$$

The explicit description of $R^G$ is complicated even in the case of the above simple action. We explain how to overcome this difficulty after stating our theorems on the invariants. We obtain a simple formula for $\delta$:

Theorem 1.2. — *The equality $\delta = p^{d'+1}m$ holds.*

Although the formula for $p_f$ is complicated in general (Corollary 6.1), the formula may be simplified in the case $G \cong \mathbb{Z}/p\mathbb{Z}$:

Theorem 1.3. — *Assume that $G \cong \mathbb{Z}/p\mathbb{Z}$. Then the equality*

$$p_f = \frac{(p-1)(\min\{\alpha_1, \alpha_2\} - 1)}{2}$$

*holds. In particular, the singularity of $Y$ is rational if and only if $\alpha_1 = 1$ or $\alpha_2 = 1$.*

In contrast to $\delta$ and $p_f$, the formula for $p_g$ is complicated even in the case $G \cong \mathbb{Z}/p\mathbb{Z}$ (Theorem 8.7). Nevertheless, the formula may be simplified in some special cases:

Theorem 1.4. — *Assume that $G \cong \mathbb{Z}/p\mathbb{Z}$, $\alpha_1 = \alpha_2$, and $\alpha_1 \mid (p-1)$. Put $\alpha := \alpha_1$. Then the equality*

$$p_g = \sum_{i=1}^{p-1} \left\lceil \frac{i\alpha}{p} \right\rceil^2 - \frac{(p-1)(\alpha+1)(\alpha+2)}{6}$$

*holds.*

When $G \cong \mathbb{Z}/p\mathbb{Z}$, our result in the local setting generalizes the previously known results in the global setting in the case $\alpha_1 = 1$ or $\alpha_2 = 1$ [10] and in the case $\alpha_1 = \alpha_2 = p - 1$ [6]. Our approach is different from those in [10] and [6]. The former [10] applies the Néron model of the Jacobian of a curve. The intersection matrix of the exceptional locus is calculated under certain conditions of the global geometry which are essentially used. The latter [6] uses an explicit defining equation of the invariant ring of the action on the product of two curves.

Let us briefly explain our method. We first take a proper birational morphism $\widetilde{X} \to X$ induced by a subdivision $\widetilde{\Delta}_0$ of a toric fan so that the action of $G$ on $X$ may be lifted to that on $\widetilde{X}$. Next, we take the quotient $\widetilde{Y} \coloneqq \widetilde{X}/G$. The point is that all singularities of $\widetilde{Y}$ are toric if we appropriately choose $\widetilde{\Delta}_0$. We may determine the fans of the toric singularities of $\widetilde{Y}$. The Hirzebruch–Jung desingularizations of these toric singularities give a minimal good desingularization of $Y$. We finally remark that, in other different situations, it has been observed that an appropriate blowing-up with a lifting of a group action may reduce a serious singularity to milder ones, e.g., Kirwan's partial desingularization of the quotient of a reductive group action on a complex projective variety. However, the blowing-up is non-singular in contrast to $\widetilde{X}$, which is *singular* whenever $\alpha_1 \neq \alpha_2$ (Remark 3.3). Our method may be regarded as a new variant.

## 2. Notation and Convention

Let $(B_i)_{i=1}^r$ be a sequence of integers greater than one. We denote the *Hirzebruch–Jung continued fraction*

$$B_1 - \cfrac{1}{B_2 - \cfrac{1}{\ddots - \cfrac{1}{B_r}}}$$

by $[B_i]_{i=1}^r = [B_1, \dots, B_r]$ [3, §10.2]. Put $B \coloneqq [B_i]_{i=1}^r$. Then $B \in \mathbb{Q}_{>1}$. Conversely, any rational number greater than one can be uniquely expressed as a Hirzebruch–Jung continued fraction. There exists a unique $(M_0, M_1) \in \mathbb{Z}_{>0}^2$ such that $B = M_0/M_1$ and $\gcd(M_0, M_1) = 1$. For $i \in \mathbb{Z}$ satisfying $1 \leqslant i \leqslant r$, we put

$$M_{i+1} \coloneqq M_i B_i - M_{i-1}. \tag{2.1}$$

Note that $M_i/M_{i+1} = [B_j]_{j=i+1}^{r}$ for any $i \in \mathbb{Z}$ satisfying $0 \leqslant i \leqslant r-1$, $M_r = 1$, and $M_{r+1} = 0$. The $(r+2)$-tuple

$$(M_i)_{i=0}^{r+1} = (M_0, M_1, \ldots, M_r, M_{r+1})$$

is called the *vector associated with* $B$, and the $r \times r$ matrix

$$\Omega := \begin{pmatrix} -B_1 & 1 & & & & \text{\Large 0} \\ 1 & -B_2 & 1 & & & \\ & 1 & -B_3 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -B_{r-1} & 1 \\ \text{\Large 0} & & & & 1 & -B_r \end{pmatrix}$$

is called the *matrix associated with* $B$. Note that $\det \Omega = (-1)^r M_0$.

Let $W$ be a regular $k$-scheme of dimension two. Take divisors $D$ and $D'$ on $W$. Assume that $D'$ is effective, and the support of $D'$ is proper over $k$. We define the *intersection number of* $D$ *and* $D'$ by $D \cdot D' := \chi(\mathcal{O}_W(D)|_{D'}) - \chi(\mathcal{O}_{D'})$. The definition may be $\mathbb{Q}$-linearly extended to the case where $D$ and $D'$ are $\mathbb{Q}$-divisors, and the support of $D'$ is proper over $k$ [8, 13.1.b]. Note that $D \cdot D' = D' \cdot D$ whenever the supports of both $D$ and $D'$ are proper over $k$ [8, 13.1.d]. We denote the self-intersection number $D \cdot D$ of $D$ by $D^2$.

We denote the number of the elements of a finite set $S$ by $\#S$.

## 3. Main Theorems

In this section, we state our main theorems. Their proofs are given in Sections 4–5. We use the notation introduced in Section 1. Assume that $p > 0$ and $H \cong \mathbb{Z}/p\mathbb{Z}$. Put

$$m' := \frac{m}{n}. \tag{3.1}$$

Lemma 3.1. — *The rational numbers $m'$ and $d'$ are integers.*

*Proof.* — The equalities $\operatorname{ord} \rho_1 = \operatorname{ord} \rho_2 = m$ show that $m' \in \mathbb{Z}$. Take $\tau \in G$ (resp. $u \in \mathbb{Z}$) so that the image of $\tau$ under the quotient homomorphism $G \to G/H$ is a generator of $G/H$ (resp. the equality $\tau^{-1}\sigma\tau = \sigma^u$ holds). Since $\sigma\tau = \tau\sigma^u$, the equality $\rho_i(\tau) = u\rho_i(\tau)^{\alpha_i+1}$ holds for any $i \in \{1, 2\}$. Thus, the equality $\rho_1^{\alpha_1} = \rho_2^{\alpha_2}$ holds, which concludes that $d' \in \mathbb{Z}$. □

For $i \in \{1,2\}$, we put

$$y_i \coloneqq \prod_{\tau \in H} \tau x_i. \tag{3.2}$$

Then $k[\![x_i]\!]^H = k[\![y_i]\!]$.

Lemma 3.2. — *The integer $\alpha_i$ is coprime to $p$ for any $i \in \{1,2\}$.*

*Proof.* — Choose $i \in \{1,2\}$. Put

$$F(T) = T^p + \sum_{j=0}^{p-1} F_j T^j \coloneqq \prod_{j=0}^{p-1} (T - \sigma^j x_i) \in k[\![y_i]\!][T]$$

and

$$J \coloneqq \{j \in \mathbb{Z} \mid 0 \leqslant j \leqslant p-2, F_{j+1} \neq 0\}.$$

Since $v_i(k[\![y_i]\!]\backslash\{0\}) = p\mathbb{Z}_{\geqslant 0}$, the integers $(v_i(F_{j+1})+j)_{j\in J}$ are different from each other. Thus, by taking the valuations of both sides of the equalities

$$\sum_{j\in J} (j+1) F_{j+1} x_i^j = \frac{\mathrm{d}F}{\mathrm{d}T}(x_i) = \prod_{j=1}^{p-1} (x_i - \sigma^j x_i),$$

we conclude that there exists $j \in J$ such that $v_i(F_{j+1}) + j = (p-1)(\alpha_i+1)$, which implies that $p-1 \not\equiv (p-1)(\alpha_i+1) \bmod p$. Therefore, the integer $\alpha_i$ is coprime to $p$. □

Since $p \nmid d a_1 a_2$ (Lemma 3.2), there exists a unique $e \in \mathbb{Z}$ such that

$$p \mid e d a_1 a_2 + 1 \quad \text{and} \quad 0 < e < p. \tag{3.3}$$

We simply call an $N$-dimensional cone in $\mathbb{R}^2$ an *$N$-cone*. We define vectors $(v_i)_{i=0}^3$ on $\mathbb{R}^2$, lattices $(\Gamma_i)_{i=0}^3$ of $\mathbb{R}^2$, and 2-cones $(\Sigma_i)_{i=0}^3$ in the following way (Figure 3.1):

$$\begin{aligned}
&v_0 \coloneqq (a_2, a_1); \quad v_1 \coloneqq (1,0); \quad v_2 \coloneqq (0,1); \quad v_3 \coloneqq (1,e);\\
&\Gamma_0 \coloneqq \mathbb{Z}v_1 + \mathbb{Z}v_2; \quad \Gamma_3 \coloneqq \mathbb{Z}m' v_1 + \mathbb{Z}p v_2;\\
&\Gamma_1 \coloneqq \{(l_1,l_2) \in \Gamma_0 \mid l_1 \in p\mathbb{Z}, \rho_1^{l_2} = \rho_2^{l_1}\};\\
&\Gamma_2 \coloneqq \{(l_1,l_2) \in \Gamma_0 \mid l_2 \in p\mathbb{Z}, \rho_1^{l_2} = \rho_2^{l_1}\};\\
&\Sigma_0 \coloneqq \mathbb{R}_{\geqslant 0} v_1 + \mathbb{R}_{\geqslant 0} v_2; \quad \Sigma_1 \coloneqq \mathbb{R}_{\geqslant 0} v_0 + \mathbb{R}_{\geqslant 0} v_1;\\
&\Sigma_2 \coloneqq \mathbb{R}_{\geqslant 0} v_0 + \mathbb{R}_{\geqslant 0} v_2; \quad \Sigma_3 \coloneqq \mathbb{R}_{\geqslant 0} v_2 + \mathbb{R}_{\geqslant 0} v_3.
\end{aligned} \tag{3.4}$$

By $\Delta_0$ we denote the fan induced by the 2-cone $\Sigma_0$. We define a fan $\widetilde{\Delta}_0$ as the subdivision of $\Delta_0$ by the 1-cone $\mathbb{R}_{\geqslant 0} v_0$. The subdivision $\widetilde{\Delta}_0$ of $\Delta_0$ induces a proper birational morphism

$$\phi\colon \widetilde{\mathscr{X}} \longrightarrow \mathbb{A}_k^2 = \operatorname{Spec} k[\chi_1, \chi_2] \tag{3.5}$$

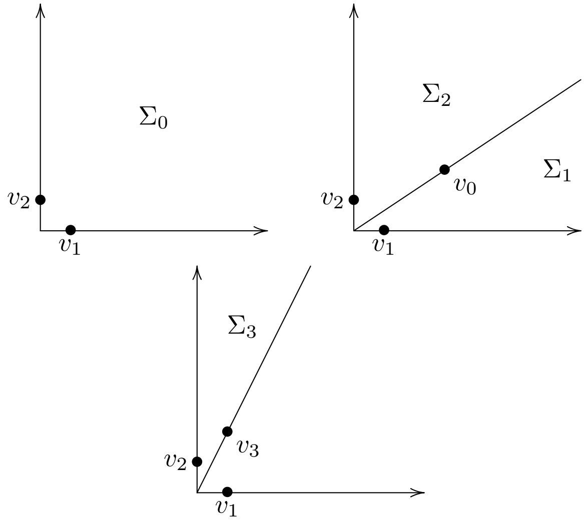


*Figure 3.1. The vectors* $(v_i)_{i=0}^3$ *and 2-cones* $(\Sigma_i)_{i=0}^3$

[3, 3.3.4.a and 3.4.11], where $(\chi_1, \chi_2)$ corresponds to the dual of the basis $(v_1, v_2)$ [3, 1.2.18]. Take the morphism

(3.6) $$\iota \colon X \longrightarrow \mathbb{A}_k^2$$

induced by the $k$-algebra homomorphism $k[\chi_1, \chi_2] \to R$, $\chi_i \mapsto x_i$ for $i \in \{1, 2\}$. By

(3.7) $$f \colon \widetilde{X} \longrightarrow X$$

we denote the base change of $\phi$ via $\iota$.

*Remark 3.3.* — The following statements are equivalent [3, 3.1.19.a]:

(1) $\widetilde{X}$ is regular;
(2) $\widetilde{\mathcal{X}}$ is smooth over $k$;
(3) $\widetilde{\Delta}_0$ is smooth [3, 3.1.18.a];
(4) $a_1 = a_2$;
(5) $\alpha_1 = \alpha_2$.

If the above equivalent statements hold, then any of $\phi$ and $f$ is a blowing-up at the origin [3, 3.3.12]. In the general case, the morphism $f$ is a blowing-up along the ideal generated by the monomials $\{x_1^{l_1} x_2^{l_2} \mid a_1 l_2 + a_2 l_1 \geqslant a_1 a_2\}$ in $x_1$ and $x_2$ [3, 7.1.13, 7.1.9.b, and 11.3.1]. In particular, the definition of $f$ does not depend on the choice of the uniformizer $x_i$ of $k[\![x_i]\!]$ for any $i \in \{1,2\}$.

The action of $G$ on $X$ uniquely lifts to that on $\widetilde{X}$ since $\tau x/x \in R^\times$ for any $\tau \in G$ and any monomial $x$ in $x_1$ and $x_2$. Take the quotient $\widetilde{q}\colon \widetilde{X} \to \widetilde{Y}$ of $\widetilde{X}$ by $G$ and the unique morphism $g\colon \widetilde{Y} \to Y$ satisfying $g \circ \widetilde{q} = q \circ f$ (1.1). By $E_f$ we denote the exceptional locus of $f$ with reduced structure. Put $\widetilde{E} \coloneqq \widetilde{q}(E_f)$. By $D_1$ (resp. $D_2$) we denote the divisor on $X$ defined by $x_1$ (resp. $x_2$). For $i \in \{1,2\}$, we take the strict transform $\widetilde{D}_i$ of $D_i$ via $f$ and put $Q_i \coloneqq \widetilde{E} \cap \widetilde{q}(\widetilde{D}_i)$. Put

$$I \coloneqq \{1,2,3\}, \quad I_3 \coloneqq \{i \in \mathbb{Z} \mid 3 \leqslant i \leqslant d'+2\}, \quad \text{and} \quad I_{\mathrm{all}} \coloneqq I \cup I_3.$$

For $i \in I$, by $Z_i$ we denote the completion of the toric singularity associated with $\Sigma_i$ in $\Gamma_i \otimes_{\mathbb{Z}} \mathbb{R}$ [3, 1.2.18].

Theorem 3.4. — *The following statements hold.*

(1) *All singular points of $\widetilde{Y}$ are contained in $\widetilde{E}$, and the number of these singular points is equal to $d'+2$. Both $Q_1$ and $Q_2$ are singular points of $\widetilde{Y}$. By $(Q_i)_{i \in I_{\mathrm{all}}}$ we denote the singular points of $\widetilde{Y}$. The completion of the singularity at $Q_1$ (resp. $Q_2$, resp. $Q_i$ for any $i \in I_3$) is isomorphic to $Z_1$ (resp. $Z_2$, resp. $Z_3$).*

(2) *Take the Hirzebruch–Jung desingularizations $\widetilde{h}\colon \widehat{Y} \to \widetilde{Y}$ of the singularities of $\widetilde{Y}$. Put $h \coloneqq g \circ \widetilde{h}$. We denote the exceptional locus of $h$ (resp. the preimage of $Q_i$ under $\widetilde{h}$ for $i \in I_{\mathrm{all}}$) with reduced structure by $E_h$ (resp. $E_i$) and the strict transform of $\widetilde{E}$ via $\widetilde{h}$ by $E_0$. Then $E_h$ is a union of the projective lines any of whose singularities is a node and whose dual graph is a star-shaped tree with central node (resp. $d'+2$ branches) corresponding to $E_0$ (resp. $(E_i)_{i \in I_{\mathrm{all}}}$).*

(3) *The desingularization $h\colon \widehat{Y} \to Y$ of $Y$ is minimal good.*

(4) *For $i \in \{1,2\}$, by $\widehat{D}_i$ we denote the strict transform of $q(D_i)$ via $h$. Then the equality $\widehat{D}_i \cdot E_h = 1$ holds, and the irreducible component of $E_h$ intersecting with $\widehat{D}_i$ corresponds to the end of the branch corresponding to $E_i$.*

We obtain the following diagram with commutative triangle and square:

$$\begin{array}{ccccc} \widehat{Y} & \xrightarrow{\tilde{h}} & \widetilde{Y} & \xleftarrow{\tilde{q}} & \widetilde{X} \\ & \searrow^{h} & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ & & Y & \xleftarrow{q} & X. \end{array} \tag{3.8}$$

For $i \in I_{\mathrm{all}}$, we denote the irreducible components of $E_i$ starting from the irreducible component intersecting with $E_0$ by

$$(E_{i,j})_{j=1}^{s_i} \tag{3.9}$$

(Figure 3.2). By $\Omega_h$ we denote the intersection matrix of $E_h$ with respect to the ordered basis $E_0$ followed by $(E_{i,j})_{i,j}$ with dictionary order.

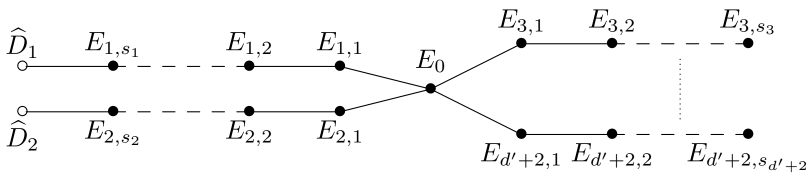


*Figure 3.2. The dual graph of $E_h \cup \widehat{D}_1 \cup \widehat{D}_2$*

Put

$$\nu := \frac{eda_1a_2+1}{p} \in \mathbb{Z} \tag{3.10}$$

(3.3). Since $\gcd(a_1,a_2)=1$ (1.3), the sequence of $\mathbb{Z}$-modules and homomorphisms

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\nu_1} \mathbb{Z}^2 \xrightarrow{\nu_2} \mathbb{Z} \longrightarrow 0$$

is exact, where $\nu_1(l_0)=l_0(a_2,-a_1)$ and $\nu_2(l_1,l_2)=a_1l_1+a_2l_2$. Thus, there exists a unique $(b_1,b_2,c_1,c_2)\in\mathbb{Z}^4$ such that

$$\nu_2(b_2,b_1)=\nu_2(c_2,c_1)=\nu, \quad 0<b_2\leqslant a_2, \quad \text{and} \quad 0<c_1\leqslant a_1. \tag{3.11}$$

Since $(b_2-c_2,b_1-c_1)\in \operatorname{Ker}\nu_2$, there exists a unique $n_0\in\mathbb{Z}$ such that

$$\nu_1(n_0)=(b_2-c_2,b_1-c_1). \tag{3.12}$$

Lemma 3.5. — *Take $i\in\{1,2\}$ and $(l_1,l_2)\in\Gamma_0$. Assume that $l_i\in p\mathbb{Z}$. Then there exists a minimum $n'\in\mathbb{Z}$ such that*

$$pn'v_0+m'(l_1,l_2)\in\Gamma_i\cap\Sigma_0.$$

*Moreover, the following holds:*

$$p(n'-n)v_0+m'(l_1,l_2)\in\Gamma_i\setminus\Sigma_0.$$

*Proof.* — The equalities $\operatorname{ord}\rho_1=\operatorname{ord}\rho_2=m=m'n$ (3.1) show that $\operatorname{ord}\rho_1^{m'l_2}\rho_2^{-m'l_1}\mid n$. Thus, since $v_0=(a_2,a_1)$ (3.4) and $\operatorname{ord}\rho_1^{pa_1}\rho_2^{-pa_2}=n$ (1.4), there exists $n'\in\mathbb{Z}$ such that

$$pn'v_0+m'(l_1,l_2)\in\Gamma_i. \tag{3.13}$$

Since $pnv_0 \in \Gamma_1 \cap \Gamma_2$, any element of $n' + n\mathbb{Z}$ satisfies (3.13). Thus, the lemma follows from the fact that $v_0$ is contained in the interior of $\Sigma_0$. $\square$

Put

$$v_1' := (pb_2, eda_1 - pb_1) \quad \text{and} \quad v_2' := (eda_2 - pc_2, pc_1). \tag{3.14}$$

These vectors appear in the definition of defining functions of $E_f$ (Lemma 4.3 (1)). For each $i \in \{1, 2\}$, Lemma 3.5 implies that there exists a minimum $n_i \in \mathbb{Z}$ such that

$$pn_i v_0 + m' v_i' \in \Gamma_i \cap \Sigma_0. \tag{3.15}$$

These vectors appear as a part of the data of the subdivisions of the fans corresponding to the Hirzebruch–Jung desingularizations $\widetilde{h}$ (Section 5). We define $(\widehat{b}_1, \widehat{b}_2, \widehat{c}_1, \widehat{c}_2) \in \mathbb{Z}^4$ by

$$(p\widehat{b}_2, \widehat{b}_1) = pn_1 v_0 + m' v_1' \quad \text{and} \quad (\widehat{c}_2, p\widehat{c}_1) = pn_2 v_0 + m' v_2'. \tag{3.16}$$

Put

$$e' := \left\lceil \frac{m'e}{p} \right\rceil, \quad b_0 := \frac{m'n_0 + n_1 + n_2}{n} + e'd', \tag{3.17}$$

and

$$(m_1, m_2, m_3, k_1, k_2, k_3) := (pna_1, pna_2, p, \widehat{b}_1, \widehat{c}_2, pe' - m'e). \tag{3.18}$$

For $i \in I$, by $\Omega_i$ we denote the $r_i \times r_i$ matrix associated with $m_i/k_i$ (Section 2), where $\gcd(m_i, k_i) = 1$ and $0 < k_i < m_i$ (Lemma 5.1 (4)). By $\Theta_i$ we denote the $1 \times r_i$ matrix whose first entry is the unique non-zero entry and equal to one.

Theorem 3.6. — *The equalities $s_1 = r_1$, $s_2 = r_2$, $s_i = r_3$ for any $i \in I_3$ (see (3.9) for $(s_i)_{i \in I_{\mathrm{all}}}$), and*

$$\Omega_h = \begin{pmatrix} -b_0 & \Theta_1 & \Theta_2 & \Theta_3 & \cdots & \Theta_3 \\ {}^t\Theta_1 & \Omega_1 & & & & \\ {}^t\Theta_2 & & \Omega_2 & & & 0 \\ {}^t\Theta_3 & & & \Omega_3 & & \\ \vdots & & & & \ddots & \\ {}^t\Theta_3 & 0 & & & & \Omega_3 \end{pmatrix}$$

*hold, where the number of each $\Theta_3$, ${}^t\Theta_3$, and $\Omega_3$ is equal to $d'$. Put*

$$r_{\mathrm{tot}} := 1 + r_1 + r_2 + d' r_3.$$

*Then the number of the irreducible components of $E_h$ is equal to $r_{\mathrm{tot}}$, and the equality*

$$\det \Omega_h = (-1)^{r_{\mathrm{tot}}} p^{d'+1} m$$

*holds.*

Corollary 3.7. — *Assume that* $\gcd(\alpha_1,\alpha_2)=1$*. Then* $d=d'=1$ *((1.3)–(1.4)) and* $\delta=p^2m$ *(1.2).*

Note that the following equalities and inequalities hold (3.11):

$$0<\frac{eda_1-pb_1}{pb_2}=\frac{a_1}{a_2}-\frac{1}{pa_2b_2}<\frac{a_1}{a_2}\leqslant\frac{a_1}{b_2}; \tag{3.19}$$

$$0<\frac{eda_2-pc_2}{pc_1}=\frac{a_2}{a_1}-\frac{1}{pa_1c_1}<\frac{a_2}{a_1}\leqslant\frac{a_2}{c_1}. \tag{3.20}$$

Corollary 3.8. — *Assume that* $G\cong\mathbb{Z}/p\mathbb{Z}$*. Then* $m=m'=n=\operatorname{ord}\rho_1=\operatorname{ord}\rho_2=1$ *((1.4) and (3.1)),* $d=d'$ *(1.4),* $e'=1$ *((3.17) and (3.3)),* $\Gamma_1=\mathbb{Z}pv_1+\mathbb{Z}v_2$*, and* $\Gamma_2=\Gamma_3=\mathbb{Z}v_1+\mathbb{Z}pv_2$*. In particular, we conclude that* $v_1'\in\Gamma_1$ *and* $v_2'\in\Gamma_2$ *(3.14), which implies that* $(n_0,n_1,n_2)=((c_1-b_1)/a_1,0,0)$ *((3.12) and (3.15)) since* $0<eda_1-pb_1<pa_1$ *(3.19) and* $0<eda_2-pc_2<pa_2$ *(3.20). Thus, we obtain the equalities* $(\widehat{b}_1,\widehat{b}_2,\widehat{c}_1,\widehat{c}_2)=(eda_1-pb_1,b_2,c_1,eda_2-pc_2)$ *(3.16),* $(m_1,m_2,m_3,k_1,k_2,k_3)=(pa_1,pa_2,p,eda_1-pb_1,eda_2-pc_2,p-e)$ *(3.18),* $E_0^2=((b_1-c_1)/a_1)-d$ *(3.17), and* $\delta=p^{d+1}$ *(1.2).*

*Example 3.9.* — Although $E_{i,j}^2\leqslant-2$ for any $(i,j)$, the equality $E_0^2=-1$ can hold. Assume that $G\cong\mathbb{Z}/p\mathbb{Z}$ and $(\alpha_1,\alpha_2)=(p+1,2p+1)$. Then $(d,e,a_1,a_2,b_1,b_2,c_1,c_2)=(1,p-1,p+1,2p+1,1,2p-3,1,2p-3)$ ((1.3), (3.3), and (3.11)), which implies that $E_0^2=-1$ (Corollary 3.8).

Corollary 3.10. — *Assume that* $\alpha_1=\alpha_2$ *and* $\alpha_1\mid(p-1)$*. Put* $\alpha:=\alpha_1$*. Then* $(d,e,a_1,a_2,b_1,b_2,c_1,c_2)=(\alpha,(p-1)/\alpha,1,1,0,1,1,0)$ *((1.3), (3.3), and (3.11)).*

(1) *Assume that* $G\cong\mathbb{Z}/p\mathbb{Z}$*. Then* $E_0^2=-\alpha-1$ *and* $\delta=p^{\alpha+1}$ *(Corollary 3.8). In particular, if* $\alpha=1$ *(resp.* $p-1$*), then* $E_0^2=-2$ *(resp.* $-p$*) and* $\delta=p^2$ *(resp.* $p^p$*).*

(2) *Assume that* $m=p-1$ *and* $n=1$*. Then* $m'=p-1$ *(3.1),* $d'=\alpha$ *(1.4),* $e'=e$ *((3.17) and (3.3)),* $(n_0,n_1,n_2)=(1,2-p,2-p)$ *((3.12) and (3.15)),* $(\widehat{b}_1,\widehat{b}_2,\widehat{c}_1,\widehat{c}_2)=(1,1,1,1)$ *(3.16),* $(m_1,m_2,m_3,k_1,k_2,k_3)=(p,p,p,1,1,e)$ *(3.18),* $E_0^2=-2$ *(3.17), and* $\delta=p^{\alpha+1}(p-1)$ *(1.2).*

## 4. Singularities

We use the notation introduced in Section 3. For each $i\in\{1,2\}$, the valuation $v_i$ and the action of $G$ on $k[\![x_i]\!]$ uniquely extend to those on

$k(\!(x_i)\!)$. Then $k(\!(x_i)\!)^H = k(\!(y_i)\!)$ (3.2). Put

$$\mathcal{K}_i := \bigoplus_{j \in \mathbb{Z}_{<0} \setminus p\mathbb{Z}_{<0}} k \cdot y_i^j \subset k[y_i^{-1}] \subset k(\!(y_i)\!).$$

Lemma 4.1. — *For any $i \in \{1,2\}$, there exists a unique $z_i \in \mathcal{K}_i$ satisfying the following condition: there exists $t_i \in k(\!(x_i)\!)$ such that $t_i^p - t_i = z_i$ and $\sigma t_i = t_i + 1$. In particular, the $k(\!(y_i)\!)$-algebra homomorphism*

$$k(\!(y_i)\!)[T]/(T^p - T - z_i) \longrightarrow k(\!(x_i)\!), \qquad T \longmapsto t_i$$

*is bijective.*

*Proof.* — Choose $i \in \{1,2\}$. Put $K := k(\!(y_i)\!)$. Choose a separable closure $K^{\mathrm{sep}}$ of $k(\!(x_i)\!)$. Put $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$. We define an endomorphism $\wp$ of the additive group $K^{\mathrm{sep}}$ by $x \mapsto x^p - x$. The exact sequence of $G_K$-modules and $G_K$-equivariant homomorphisms

$$0 \longrightarrow \mathbb{F}_p \longrightarrow K^{\mathrm{sep}} \xrightarrow{\wp} K^{\mathrm{sep}} \longrightarrow 0$$

induces $K/\wp(K) \cong H^1(K, \mathbb{F}_p) = \mathrm{Hom}(G_K, \mathbb{F}_p)$. We denote the composite of the quotient homomorphism $K \to K/\wp(K)$ and this isomorphism by $\psi\colon K \to \mathrm{Hom}(G_K, \mathbb{F}_p)$. Take $z \in K$. Choose $t \in K^{\mathrm{sep}}$ so that $t^p - t = z$. Then $\psi(z)(\tau) = \tau(t) - t \in \mathbb{F}_p$ for any $\tau \in G_K$, and the Galois extension corresponding to $\mathrm{Ker}\,\psi(z)$ is equal to $K(t) \subset K^{\mathrm{sep}}$. Note that $\wp(ay_i^n) = a^p y_i^{pn} - ay_i^n$ for any $(a,n) \in k \times \mathbb{Z}$, which implies that $k[\![y_i]\!] \subset \wp(K)$. Thus, the restriction of $\psi$ to $\mathcal{K}_i$ is bijective. Therefore, there exists a unique $z_i \in \mathcal{K}_i$ such that $\psi(z_i)$ is equal to the composite of the quotient homomorphism $G_K \to \mathrm{Gal}(k(\!(x_i)\!)/K)$ and the isomorphism $\mathrm{Gal}(k(\!(x_i)\!)/K) \to \mathbb{F}_p$, $\sigma \mapsto 1$, which concludes the proof. □

Lemma 4.2. — *For any $i \in \{1,2\}$, there exists a uniformizer $\widehat{x}_i$ of $k[\![x_i]\!]$ such that $\sigma(\widehat{x}_i)^{-\alpha_i} = (\widehat{x}_i)^{-\alpha_i} + 1$ in $k(\!(x_i)\!)$.*

*Proof.* — Choose $i \in \{1,2\}$. Take $z_i$ and $t_i$ given by Lemma 4.1. Then $p \nmid v_i(t_i)$ since $v_i(z_i) \in p\mathbb{Z}_{<0} \setminus p^2\mathbb{Z}_{<0}$. Thus, we may take $(a,b) \in \mathbb{Z}^2$ satisfying $av_i(t_i) + bp = 1$. Put $\widetilde{x}_i := t_i^a y_i^b$. Then $p \nmid a$ and $v_i(\widetilde{x}_i) = 1$, which gives the equalities

$$-v_i(t_i) = v_i(\widetilde{x}_i) - v_i(t_i) - 1 = v_i(\sigma\widetilde{x}_i - \widetilde{x}_i) - 1 = \alpha_i.$$

Thus, since $p \nmid \alpha_i$ (Lemma 3.2), there exists a uniformizer $\widehat{x}_i$ of $k[\![x_i]\!]$ such that $(\widehat{x}_i)^{\alpha_i} = t_i^{-1}$, which concludes the proof. □

By Lemma 4.2 and Remark 3.3, we may assume that the equality

$$\sigma x_i^{-\alpha_i} = x_i^{-\alpha_i} + 1 \tag{4.1}$$

holds in $k(\!(x_i)\!)$ for any $i \in \{1,2\}$ after replacing the uniformizer $x_i$ of $k[\![x_i]\!]$. For $i \in \{0,1,2,3\}$, we denote the dual lattice of $\Gamma_i$ by $\Gamma_i^\vee$ and the dual cone of $\Sigma_i$ by $\Sigma_i^\vee$. The toric variety $\widetilde{\mathcal{X}}$ (3.5) has an affine covering $(\widetilde{\mathcal{U}}_1, \widetilde{\mathcal{U}}_2)$, where $\widetilde{\mathcal{U}}_i \coloneqq \operatorname{Spec} k[\Sigma_i^\vee \cap \Gamma_0^\vee]$ for $i \in \{1,2\}$. The base changes of $\widetilde{\mathcal{U}}_1$ and $\widetilde{\mathcal{U}}_2$ via $\iota$ (3.6) give an affine covering $(\widetilde{U}_1, \widetilde{U}_2)$ of $\widetilde{X}$ (3.7). Put

$$\widetilde{U}_0 \coloneqq \widetilde{U}_1 \cap \widetilde{U}_2 \quad \text{and} \quad x_i' \coloneqq \begin{cases} x_1^{l_2} x_2^{-l_1} & \text{for } i \in \{0,2\}, \\ x_1^{-l_2} x_2^{l_1} & \text{for } i = 1, \end{cases}$$

where

$$l_1 v_1 + l_2 v_2 = \begin{cases} v_0 & \text{if } i = 0 \quad (3.4), \\ v_i' & \text{if } i \in \{1,2\} \quad (3.14). \end{cases}$$

Lemma 4.3. — *The following statements hold.*

(1) *Both $x_1'|_{\tilde{U}_0}$ and $x_2'|_{\tilde{U}_0}$ are defining functions of $E_f \cap \widetilde{U}_0$.*

(2) *The restriction $x_0'|_{E_f \cap \tilde{U}_1}$ (resp. $(x_0')^{-1}|_{E_f \cap \tilde{U}_2}$) is a parameter of $E_f \cap \widetilde{U}_1$ ($\cong \mathbb{A}_k^1$) (resp. $E_f \cap \widetilde{U}_2$ ($\cong \mathbb{A}_k^1$)) and a defining function of $E_f \cap \widetilde{D}_1$ (resp. $E_f \cap \widetilde{D}_2$).*

(3) *Take $(l_1, l_2) \in \mathbb{Z}^2$ and $\tau \in G$. Put $x \coloneqq x_1^{l_1} x_2^{l_2}$. Then the rational function $\tau x / x$ on $\widetilde{X}$ is a nowhere-zero regular function on $\widetilde{X}$ whose restriction to $E_f$ is equal to the constant function with value $\rho_1(\tau)^{l_1} \rho_2(\tau)^{l_2}$.*

*Proof.* — Let us show Statements (1) and (2). The following equalities hold ((3.11) and (3.14)):

$$\det \begin{pmatrix} v_0 \\ v_1' \end{pmatrix} = \det \begin{pmatrix} a_2 & a_1 \\ pb_2 & eda_1 - pb_1 \end{pmatrix} = -1; \tag{4.2}$$

$$\det \begin{pmatrix} v_0 \\ v_2' \end{pmatrix} = \det \begin{pmatrix} a_2 & a_1 \\ eda_2 - pc_2 & pc_1 \end{pmatrix} = 1. \tag{4.3}$$

Thus, the following holds:

$$\mathbb{Z} v_0 + \mathbb{Z} v_1' = \mathbb{Z} v_0 + \mathbb{Z} v_2' = \Gamma_0; \quad v_1' \in \Sigma_1; \quad v_2' \in \Sigma_2. \tag{4.4}$$

We define 2-cones in the following way (Figure 4.1):

$$\Sigma_1' \coloneqq \mathbb{R}_{\geqslant 0} v_0 + \mathbb{R}_{\geqslant 0} v_1' \subset \Sigma_1; \quad \Sigma_2' \coloneqq \mathbb{R}_{\geqslant 0} v_0 + \mathbb{R}_{\geqslant 0} v_2' \subset \Sigma_2.$$

For $i \in \{1,2\}$, we denote the dual cone of $\Sigma_i'$ by $(\Sigma_i')^\vee$. By $E_\phi$ we denote the exceptional locus of $\phi \colon \widetilde{\mathcal{X}} \to \mathbb{A}_k^2$ (3.5) with reduced structure. Take the Hirzebruch–Jung desingularizations $\widetilde{\phi} \colon \mathcal{X}' \to \widetilde{\mathcal{X}}$ of the toric blowing-up of $\widetilde{\mathcal{X}}$ by the 1-cones $\mathbb{R}_{\geqslant 0} v_1'$ and $\mathbb{R}_{\geqslant 0} v_2'$ [3, §10.2]. Then the strict transform of $E_\phi$ via $\widetilde{\phi}$ corresponds to the 1-cone $\mathbb{R}_{\geqslant 0} v_0$ [3, 3.2.6 and 3.3.21], which

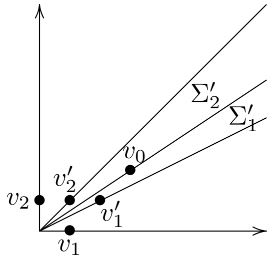


*Figure 4.1. The vectors $v_1'$ and $v_2'$ and the 2-cones $\Sigma_1'$ and $\Sigma_2'$*

is contained in the union of the two affine open subsets $\operatorname{Spec} k[(\Sigma_1')^\vee \cap \Gamma_0^\vee]$ ($\cong \mathbb{A}_k^2$) and $\operatorname{Spec} k[(\Sigma_2')^\vee \cap \Gamma_0^\vee]$ ($\cong \mathbb{A}_k^2$) of $\mathcal{X}'$ since $\widetilde{\phi}$ is induced by a subdivision of $\widetilde{\Delta}_0$ [3, 10.2.3] containing $\Sigma_1'$ and $\Sigma_2'$ (4.4). Taking the base change via $\iota$ (3.6), we obtain a desingularization $\widetilde{f}\colon X' \to \widetilde{X}$ of $\widetilde{X}$ and two affine open subsets $(U_1', U_2')$ of $X'$ satisfying the following conditions:

(a) $\widetilde{f}$ induces isomorphisms $U_1' \cap U_2' \cong \widetilde{U}_0$ and $E_f' \cap U_i' \cong E_f \cap \widetilde{U}_i$ for each $i \in \{1,2\}$, where $E_f'$ is the strict transform of $E_f$ via $\widetilde{f}$;

(b) $x_1'|_{U_1'}$ (resp. $x_2'|_{U_2'}$) is a defining function of $E_f' \cap U_1'$ (resp. $E_f' \cap U_2'$);

(c) $x_0'|_{E_f' \cap U_1'}$ (resp. $(x_0')^{-1}|_{E_f' \cap U_2'}$) is a parameter of $E_f' \cap U_1'$ ($\cong \mathbb{A}_k^1$) (resp. $E_f' \cap U_2'$ ($\cong \mathbb{A}_k^1$));

(d) $x_0'|_{U_1'}$ (resp. $(x_0')^{-1}|_{U_2'}$) is a defining function of $D_1' \cap U_1'$ (resp. $D_2' \cap U_2'$), where $D_1'$ (resp. $D_2'$) is the preimage of $\widetilde{D}_1$ (resp. $\widetilde{D}_2$) under $\widetilde{f}$ with reduced structure.

Thus, Statements (1) and (2) hold. Let us show Statement (3). We have only to show the case $x = x_i$ for $i \in \{1,2\}$. The equality $\tau\overline{x_i} = \rho_i(\tau)\overline{x_i}$ holds in $\mathfrak{m}_i/\mathfrak{m}_i^2$, where $\overline{x_i}$ is the image of $x_i$ under the quotient homomorphism $\mathfrak{m}_i \to \mathfrak{m}_i/\mathfrak{m}_i^2$. Thus, the rational function $\tau x_i/x_i$ on $X$ is a nowhere-zero regular function on $X$ whose value at $D_1 \cap D_2$ is equal to $\rho_i(\tau)$. Since $f(E_f) = D_1 \cap D_2$, Statement (3) holds. □

Although Lemma 4.3 (1) gives two defining functions of $E_f \cap \widetilde{U}_0$, we use only $x_1'$ in the following. Choose $c \in k \cup \{\infty\}$. Put

$$x_{(c)} := \begin{cases} x_0' - c & \text{if } c \in k, \\ (x_0')^{-1} & \text{otherwise.} \end{cases}$$

We introduce the following notation:

$\widetilde{P}_{(c)}$: the closed point on $E_f$ defined by $x_{(c)} = 0$;

$\widetilde{Q}_{(c)}$: the closed point $\widetilde{q}(\widetilde{P}_{(c)})$ on $\widetilde{E}$;

$\widetilde{R}_{(c)}$: the completion of $\mathcal{O}_{\tilde{X},\tilde{P}_{(c)}}$ with respect to the maximal ideal;
$E_{f,(c)}$: the base change of $E_f$ via the morphism $\operatorname{Spec} \widetilde{R}_{(c)} \to \widetilde{X}$;
$G_{(c)}$: the stabilizer subgroup of the action of $G$ at $\widetilde{P}_{(c)}$;
$N_{(c)}$: the order of $G_{(c)}$;
$G'$: the preimage of $n\mathbb{Z}/m\mathbb{Z}$ under the quotient homomorphism $G \to G/H \cong \mathbb{Z}/m\mathbb{Z}$ (1.4).

Then the equality

$$(G_{(c)}, N_{(c)}) = \begin{cases} (G, pm) & \text{if } c \in \{0,\infty\}, \\ (G', pm') & \text{otherwise} \quad (3.1) \end{cases} \tag{4.5}$$

holds. If $c \notin \{0,\infty\}$, then $E_{f,(c)}$ is the spectrum of the complete discrete valuation ring $\widetilde{R}_{(c)}/(x'_1)$ with uniformizer $x_{(c)}|_{E_{f,(c)}}$, and the action of $G_{(c)}$ on $E_{f,(c)}$ is trivial (Lemma 4.3). Put

$$C := \{\zeta \in k \mid \zeta^d = 1\}, \quad \widetilde{S}_{(c)} := (\widetilde{R}_{(c)})^H, \quad \text{and} \quad \widetilde{T}_{(c)} := (\widetilde{R}_{(c)})^{G_{(c)}}.$$

Then $\mathcal{O}_{\tilde{Y},\tilde{Q}_{(c)}} \subset \widetilde{T}_{(c)} \subset \widetilde{S}_{(c)} \subset \widetilde{R}_{(c)}$ are extensions of normal integral domains. In the following, we study $\widetilde{T}_{(c)}$, which is a completion of $\mathcal{O}_{\tilde{Y},\tilde{Q}_{(c)}}$ with respect to the maximal ideal. For an extension $A'/A$ of integral domains, we denote the degree of the extension of their fields of fractions by $[A' : A]$. Then the equalities

$$[\widetilde{R}_{(c)} : \widetilde{S}_{(c)}] = p \quad \text{and} \quad [\widetilde{R}_{(c)} : \widetilde{T}_{(c)}] = N_{(c)} \tag{4.6}$$

hold. We define elements of $k(\!(x_1, x_2)\!)^H$ by

$$y_{(c)} := \prod_{\tau \in H} \tau x_{(c)}, \quad y'_1 := \prod_{\tau \in H} \tau x'_1, \quad z := x_1^{-\alpha_1} - x_2^{-\alpha_2}$$

(4.1), and

$$z' := y_1^{b_1} y_2^{b_2} z^e$$

((3.2), (3.3), and (3.11)). Then $y_{(c)} \in \widetilde{S}_{(c)}$, $y'_1/(x'_1)^p \in (\widetilde{R}_{(c)})^\times$, and

$$x_i^{\alpha_i} z = \begin{cases} 1 - (x'_0)^d \in (\widetilde{R}_{(c)})^\times & \text{if } i = 1 \text{ and } c \notin C \cup \{\infty\}, \\ (x'_0)^{-d} - 1 \in (\widetilde{R}_{(c)})^\times & \text{if } i = 2 \text{ and } c \notin C \cup \{0\}. \end{cases} \tag{4.7}$$

If $c \in C$, then $x_1^{\alpha_1} z / x_{(c)} \in (\widetilde{R}_{(c)})^\times$, which implies that $y_{(c)}/x_{(c)}^p \in (\widetilde{R}_{(c)})^\times$. If $c \in C$ (resp. $c \notin C \cup \{\infty\}$, resp. $c \notin C \cup \{0\}$), then $z'/(x_{(c)}^e x'_1) \in (\widetilde{R}_{(c)})^\times$ (resp. $z'/x'_1 \in (\widetilde{R}_{(c)})^\times$, resp. $z'/((x'_0)^{ed} x'_1) \in (\widetilde{R}_{(c)})^\times$). In each case of (4.7), we may take $\widetilde{x}_{i(c)} \in \widetilde{R}_{(c)}$ so that $(\widetilde{x}_{i(c)})^{\alpha_i} = z^{-1}$ since $p \nmid \alpha_i$ (Lemma 3.2). Then $\widetilde{x}_{i(c)}/x_i \in (\widetilde{R}_{(c)})^\times$ and $\widetilde{x}_{i(c)} \in \widetilde{S}_{(c)}$ since $(\widetilde{x}_{i(c)})^{\alpha_i} \in \widetilde{S}_{(c)}$ and $\#H = p$.

Definition 4.4. — *For each $i \in \{1,2\}$, the character $\rho_i\colon G \to k^\times$ factors through the quotient homomorphism $G \to G/H$ and induces a character $\bar{\rho}_i\colon G/H \to k^\times$ since* $\operatorname{Ker}\rho_i = H$*. Put $\bar{\rho} \coloneqq (\bar{\rho}_1)^{pb_1 - eda_1}(\bar{\rho}_2)^{pb_2}$.*

*Take $x \in \widetilde{S}_{(c)} \setminus \{0\}$. We consider one of the following cases:*

(1) *$c \in \{0,\infty\}$ and $\tau x/x \in (\widetilde{S}_{(c)})^\times$ for any $\tau \in G/H$;*
(2) *$c \notin \{0,\infty\}$.*

*Then we define the* linearization of *$x$ by*

$$Lx \coloneqq \sum_{\tau \in G_{(c)}/H} \zeta_\tau^{-1}\tau x,$$

*where $\zeta_\tau$ in Case* (1) *(resp. Case* (2)*) is the image of $\tau x/x$ in the residue field $k$ of $\widetilde{S}_{(c)}$ (resp. $\bar{\rho}(\tau)^l$ for the unique $l \in \mathbb{Z}_{\geqslant 0}$ satisfying $x/(x_1')^l \in \widetilde{R}_{(c)}$ and $(x/(x_1')^l)|_{E_{f,(c)}} \neq 0$).*

*Remark 4.5.* — The equality $\operatorname{ord}\bar{\rho}|_{G'/H} = m'$ holds ((1.4), (3.1), and (4.2)). The map $G_{(c)}/H \to k^\times$, $\tau \mapsto \zeta_\tau$ is a character, the equality $\tau(Lx) = \zeta_\tau Lx$ holds for any $\tau \in G_{(c)}/H$, and the following statements hold: in Case (1), $(Lx)/x \in (\widetilde{S}_{(c)})^\times$; in Case (2), $(Lx)/(x_1')^l \in \widetilde{R}_{(c)}$ and $((Lx)/(x_1')^l)|_{E_{f,(c)}} = m'(x/(x_1')^l)|_{E_{f,(c)}}$ (Lemma 4.3).

Proposition 4.6. — *Assume that $c \notin C \cup \{0,\infty\}$. Then the equality $\widetilde{T}_{(c)} = k[\![Ly_{(c)}, (Lz')^{m'}]\!]$ holds. In particular, the ring $\widetilde{T}_{(c)}$ is regular, and $Ly_{(c)}$ (resp. $(Lz')^{m'}$) is a parameter (resp. a defining function) of $\widetilde{E}$ at $\widetilde{Q}_{(c)}$.*

*Proof.* — Put $y_{1(c)} \coloneqq Lz' \in \widetilde{S}_{(c)}$. Since $y_{1(c)}/x_1' \in (\widetilde{R}_{(c)})^\times$, the equalities

$$\widetilde{R}_{(c)} = k[\![x_{(c)}, x_1']\!] = k[\![x_{(c)}, y_{1(c)}]\!]$$

hold (Lemma 4.3). Put

$$z_{0(c)} \coloneqq Ly_{(c)} \in \widetilde{T}_{(c)} \quad \text{and} \quad S_{(c)} \coloneqq k[\![z_{0(c)}, y_{1(c)}]\!] \subset \widetilde{S}_{(c)}.$$

Then $S_{(c)} \subset \widetilde{S}_{(c)} \subset \widetilde{R}_{(c)}$ are finite extensions of normal integral domains since the $S_{(c)}/(y_{1(c)})$-module $\widetilde{R}_{(c)}/(y_{1(c)})$ is generated by $\{x_{(c)}^i\}_{i=0}^{p-1}$. Thus, since the equalities

$$[\widetilde{R}_{(c)} : S_{(c)}] = p = [\widetilde{R}_{(c)} : \widetilde{S}_{(c)}]$$

hold (4.6), the equality $S_{(c)} = \widetilde{S}_{(c)}$ holds. Put

$$z_{1(c)} \coloneqq y_{1(c)}^{m'} \in \widetilde{T}_{(c)} \quad \text{and} \quad T_{(c)} \coloneqq k[\![z_{0(c)}, z_{1(c)}]\!] \subset \widetilde{T}_{(c)}.$$

Then $T_{(c)} \subset \widetilde{T}_{(c)} \subset \widetilde{S}_{(c)}$ are finite extensions of normal integral domains since the $T_{(c)}$-module $\widetilde{S}_{(c)}$ is generated by $\{y_{1(c)}^i\}_{i=0}^{m'-1}$. Thus, since the equalities

$$[\widetilde{S}_{(c)} : T_{(c)}] = m' = [\widetilde{S}_{(c)} : \widetilde{T}_{(c)}]$$

hold (4.6), the equality $T_{(c)} = \widetilde{T}_{(c)}$ holds. Since $x_{(c)}$ (resp. $x'_1$) is a parameter (resp. a defining function) of $E_f$ at $\widetilde{P}_{(c)}$ (Lemma 4.3), the regular function $z_{0(c)}$ (resp. $z_{1(c)}$) is a parameter (resp. a defining function) of $\widetilde{E}$ at $\widetilde{Q}_{(c)}$. □

Assume that $c \in C \cup \{0, \infty\}$. Put

$$j := \begin{cases} 1 & \text{if } c = 0, \\ 2 & \text{if } c = \infty, \\ 3 & \text{otherwise,} \end{cases} \qquad (w_1, w_2) := \begin{cases} (v_1, \frac{1}{p} v_2) & \text{if } c = 0, \\ (\frac{1}{p} v_1, v_2) & \text{otherwise,} \end{cases}$$

and

$$\Lambda_j := \mathbb{Z} w_1 + \mathbb{Z} w_2.$$

We denote the dual lattice of $\Lambda_j$ by $\Lambda_j^\vee$ and the dual of the basis $(v_1, v_2)$ (resp. $(w_1, w_2)$) by $(v_1^\vee, v_2^\vee)$ (resp. $(w_1^\vee, w_2^\vee)$). We define $v_0^\vee \in \Gamma_0^\vee$ and $w_0^\vee \in \Lambda_j^\vee$ in the following way:

(4.8)
$$v_0^\vee := \begin{cases} a_1 v_1^\vee - a_2 v_2^\vee & \text{if } c = 0, \\ -a_1 v_1^\vee + a_2 v_2^\vee & \text{if } c = \infty, \\ -e v_1^\vee + v_2^\vee & \text{otherwise;} \end{cases} \qquad w_0^\vee := \begin{cases} p a_1 w_1^\vee - a_2 w_2^\vee & \text{if } c = 0, \\ -a_1 w_1^\vee + p a_2 w_2^\vee & \text{if } c = \infty, \\ -e w_1^\vee + p w_2^\vee & \text{otherwise.} \end{cases}$$

Then the equality

$$(w_0^\vee, w_1^\vee, w_2^\vee) = \begin{cases} (p v_0^\vee, v_1^\vee, p v_2^\vee) & \text{if } c = 0, \\ (p v_0^\vee, p v_1^\vee, v_2^\vee) & \text{otherwise} \end{cases} \tag{4.9}$$

holds. We define submonoids of $\Gamma_0^\vee$ by

$$B_R := \sum_{i=0}^{2} \mathbb{Z}_{\geqslant 0} v_i^\vee \quad \text{and} \quad B_S := \sum_{i=0}^{2} \mathbb{Z}_{\geqslant 0} w_i^\vee.$$

Put

$$(x_{1(c)}, x_{2(c)}, y_{1(c)}, y_{2(c)}) := \begin{cases} (x_1, x_2, L\widetilde{x}_{1(0)}, L y_2) & \text{if } c = 0, \\ (x_1, x_2, L y_1, L\widetilde{x}_{2(\infty)}) & \text{if } c = \infty. \end{cases}$$

If $c \in C$, then we put

$$z_{(c)} := (y_{(c)} y'_1)^\nu (z')^{-d a_1 a_2}, \quad \widetilde{x}_{(c)} := (L z_{(c)})(x'_1)^{d a_1 a_2 - p\nu},$$

and

$$\begin{aligned}(x_{1(c)}, x_{2(c)}, y_{1(c)}, y_{2(c)}) \\ \coloneqq (\widetilde{x}_{(c)}, (\widetilde{x}_{(c)})^e x'_1, (Lz_{(c)})^p (Ly'_1)^{da_1a_2-p\nu}, (Lz_{(c)})^e (Ly'_1)^{(1-e)\nu})\end{aligned}$$

(3.10), where $z_{(c)}/(x_{(c)}(x'_1)^{p\nu-da_1a_2}) \in (\widetilde{R}_{(c)})^\times$, $\widetilde{x}_{(c)} \in \widetilde{R}_{(c)}$, and $\widetilde{x}_{(c)}|_{E_{f,(c)}}$ is a uniformizer of $\widetilde{R}_{(c)}/(x'_1)$. Put

$$x_{0(c)} \coloneqq x_{1(c)}^{l_1} x_{2(c)}^{l_2} \quad (\text{resp. } y_{0(c)} \coloneqq y_{1(c)}^{l_1} y_{2(c)}^{l_2}), \tag{4.10}$$

where $v_0^\vee = l_1 v_1^\vee + l_2 v_2^\vee$ (resp. $w_0^\vee = l_1 w_1^\vee + l_2 w_2^\vee$) (4.8). Note that the equality

$$(x_{0(c)}, x_{1(c)}, x_{2(c)}) = \begin{cases} (x_{(c)}, x_1, x_2) & \text{if } c \in \{0, \infty\}, \\ (x'_1, \widetilde{x}_{(c)}, (\widetilde{x}_{(c)})^e x'_1) & \text{otherwise} \end{cases} \tag{4.11}$$

holds, and $y_{i(c)}/x_{i(c)}^{l_i} \in (\widetilde{R}_{(c)})^\times$ for any $i \in \{0,1,2\}$, where $w_i^\vee = l_i v_i^\vee$ (4.9). Thus, we conclude that $x_{i(c)} \in \widetilde{R}_{(c)}$ and $y_{i(c)} \in \widetilde{S}_{(c)}$ for any $i \in \{0,1,2\}$. Put

$$R_{(c)} \coloneqq k[\![x_{0(c)}, x_{1(c)}, x_{2(c)}]\!] \subset \widetilde{R}_{(c)} \ \text{ and } \ S_{(c)} \coloneqq k[\![y_{0(c)}, y_{1(c)}, y_{2(c)}]\!] \subset \widetilde{S}_{(c)}.$$

For a commutative monoid $B$, by $k[\![B]\!]$ we denote the completion of the monoid ring $k[B]$ with respect to the ideal generated by $B \setminus \{0\}$.

Lemma 4.7. — *Take the following $k$-algebra homomorphisms:*

$$\begin{aligned} p_R &\colon k[\![X_0, X_1, X_2]\!] \longrightarrow k[\![B_R]\!], & X_i &\longmapsto v_i^\vee \text{ for } i \in \{0,1,2\}; \\ p_S &\colon k[\![Y_0, Y_1, Y_2]\!] \longrightarrow k[\![B_S]\!], & Y_i &\longmapsto w_i^\vee \text{ for } i \in \{0,1,2\}; \\ \phi_R &\colon k[\![X_0, X_1, X_2]\!] \longrightarrow R_{(c)}, & X_i &\longmapsto x_{i(c)} \text{ for } i \in \{0,1,2\}; \\ \phi_S &\colon k[\![Y_0, Y_1, Y_2]\!] \longrightarrow S_{(c)}, & Y_i &\longmapsto y_{i(c)} \text{ for } i \in \{0,1,2\}. \end{aligned}$$

*Then $\phi_R$ (resp. $\phi_S$) factors through $p_R$ (resp. $p_S$) and induces a $k$-algebra homomorphism $\bar{\phi}_R \colon k[\![B_R]\!] \to R_{(c)}$ (resp. $\bar{\phi}_S \colon k[\![B_S]\!] \to S_{(c)}$). Moreover, the homomorphisms $\bar{\phi}_R$ and $\bar{\phi}_S$ are bijective.*

*Proof.* — The equalities $\phi_R(b_R) = 0$ and $\phi_S(b_S) = 0$ hold (4.10), where

$$(b_R, b_S) \coloneqq \begin{cases} (X_1^{a_1} - X_0 X_2^{a_2}, Y_1^{pa_1} - Y_0 Y_2^{a_2}) & \text{if } c = 0, \\ (X_0 X_1^{a_1} - X_2^{a_2}, Y_0 Y_1^{a_1} - Y_2^{pa_2}) & \text{if } c = \infty, \\ (X_0 X_1^e - X_2, Y_0 Y_1^e - Y_2^p) & \text{otherwise.} \end{cases}$$

Thus, the equalities $\operatorname{Ker} p_R = (b_R)$ and $\operatorname{Ker} p_S = (b_S)$ (4.8) prove the first statement. Since $\bar{\phi}_R$ and $\bar{\phi}_S$ are surjective homomorphisms between Noetherian local integral domains of same dimension, they are bijective. $\square$

Take the normalization $\overline{R}_{(c)}$ (resp. $\overline{S}_{(c)}$) of $R_{(c)}$ (resp. $S_{(c)}$).

Lemma 4.8. — *The following statements hold.*

(1) *The $k$-algebra $\overline{R}_{(c)}$ (resp. $\overline{S}_{(c)}$) is isomorphic to $k[\![\Sigma_j^\vee \cap \Gamma_0^\vee]\!]$ (resp. $k[\![\Sigma_j^\vee \cap \Lambda_j^\vee]\!]$), where $x_{i(c)}$ (resp. $y_{i(c)}$) maps to $v_i^\vee$ (resp. $w_i^\vee$) for $i \in \{0,1,2\}$.*

(2) *The equalities $\overline{R}_{(c)} = \widetilde{R}_{(c)}$ and $\overline{S}_{(c)} = \widetilde{S}_{(c)}$ hold.*

*Proof.* — By Lemma 4.7, we identify $R_{(c)}$ (resp. $S_{(c)}$) with $k[\![B_R]\!]$ (resp. $k[\![B_S]\!]$), where $x_{i(c)}$ (resp. $y_{i(c)}$) is identified with $v_i^\vee$ (resp. $w_i^\vee$) for $i \in \{0,1,2\}$. Put

$$\overline{B}_R \coloneqq \Sigma_j^\vee \cap \Gamma_0^\vee \quad \text{and} \quad \overline{B}_S \coloneqq \Sigma_j^\vee \cap \Lambda_j^\vee.$$

Since $\overline{B}_R$ (resp. $\overline{B}_S$) is the saturation of the commutative monoid $B_R$ (resp. $B_S$), the $k$-algebra $k[\overline{B}_R]$ (resp. $k[\overline{B}_S]$) is the normalization of $k[B_R]$ (resp. $k[B_S]$) [3, 1.3.8], which implies that $k[\![\overline{B}_R]\!]$ (resp. $k[\![\overline{B}_S]\!]$) is the normalization of $k[\![B_R]\!]$ (resp. $k[\![B_S]\!]$). Thus, the equalities $\overline{R}_{(c)} = k[\![\overline{B}_R]\!]$ and $\overline{S}_{(c)} = k[\![\overline{B}_S]\!]$ hold, which proves Statement (1). Therefore, since $k[\![\overline{B}_R]\!] = \widetilde{R}_{(c)}$ (4.11), the equality $\overline{R}_{(c)} = \widetilde{R}_{(c)}$ holds. Note that $\overline{S}_{(c)} \subset \widetilde{S}_{(c)} \subset \widetilde{R}_{(c)}$ are finite extensions of normal integral domains since the equality $\overline{B}_R = \bigcup (v^\vee + \overline{B}_S)$ holds, where $v^\vee$ runs through the finite set

$$\left\{ l_1 v_1^\vee + l_2 v_2^\vee \in \overline{B}_R \;\middle|\; \max\{|l_1|, |l_2|\} < p \max\{a_1, a_2, e\} \right\}.$$

Thus, since the equalities

$$[\widetilde{R}_{(c)} : \overline{S}_{(c)}] = [\Gamma_0^\vee : \Lambda_j^\vee] = [\Lambda_j : \Gamma_0] = p = [\widetilde{R}_{(c)} : \widetilde{S}_{(c)}]$$

hold (4.6), the equality $\overline{S}_{(c)} = \widetilde{S}_{(c)}$ holds, which proves Statement (2). □

Lemma 4.9. — *The equality*

$$\Gamma_j^\vee = \begin{cases} N_{(c)}^{-1}\{l_1 w_1^\vee + l_2 w_2^\vee \in \Lambda_j^\vee \mid \rho_1^{l_1}\rho_2^{pl_2} = 1\} & \text{if } c = 0, \\ N_{(c)}^{-1}\{l_1 w_1^\vee + l_2 w_2^\vee \in \Lambda_j^\vee \mid \rho_1^{pl_1}\rho_2^{l_2} = 1\} & \text{if } c = \infty, \\ N_{(c)}^{-1}\{l_1 w_1^\vee + l_2 w_2^\vee \in \Lambda_j^\vee \mid l_2 \in m'\mathbb{Z}\} & \text{otherwise} \end{cases}$$

*holds* (4.5). *In particular, the equality $\mathbb{R}_{\geqslant 0} w_i^\vee \cap \Gamma_j^\vee = \mathbb{Z}_{\geqslant 0} N_{(c)}^{-1} l w_i^\vee$ holds, where*

$$l \coloneqq \begin{cases} n & \text{if } c \in \{0,\infty\} \text{ and } i = 0, \\ m & \text{if } c \in \{0,\infty\} \text{ and } i \in \{1,2\}, \\ m' & \text{if } c \in C \text{ and } i \in \{0,2\}, \\ 1 & \text{if } c \in C \text{ and } i = 1. \end{cases}$$

*Proof.* — Take $e_1 \in \mathbb{Z}$ (resp. $e_2 \in \mathbb{Z}$) satisfying $\rho_1^{e_1} = \rho_2^p$ (resp. $\rho_1^p = \rho_2^{e_2}$). Then the equality

$$\Gamma_j = \begin{cases} \mathbb{Z}pmv_1 + \mathbb{Z}mv_2 + \mathbb{Z}(pv_1 + e_1 v_2) & \text{if } c = 0, \\ \mathbb{Z}mv_1 + \mathbb{Z}pmv_2 + \mathbb{Z}(e_2 v_1 + pv_2) & \text{if } c = \infty, \\ \mathbb{Z}m'v_1 + \mathbb{Z}pv_2 & \text{otherwise} \end{cases}$$

holds, which gives the equality

$$N_{(c)}^{-1}\Gamma_j = \begin{cases} \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}\frac{1}{m}(w_1 + e_1 w_2) & \text{if } c = 0, \\ \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}\frac{1}{m}(e_2 w_1 + w_2) & \text{if } c = \infty, \\ \mathbb{Z}w_1 + \mathbb{Z}\frac{1}{m'}w_2 & \text{otherwise.} \end{cases}$$

Thus, the equalities

$$N_{(c)}\Gamma_j^\vee = (N_{(c)}^{-1}\Gamma_j)^\vee = \begin{cases} \mathbb{Z}mw_1^\vee + \mathbb{Z}mw_2^\vee + \mathbb{Z}(e_1 w_1^\vee - w_2^\vee) & \text{if } c = 0, \\ \mathbb{Z}mw_1^\vee + \mathbb{Z}mw_2^\vee + \mathbb{Z}(w_1^\vee - e_2 w_2^\vee) & \text{if } c = \infty, \\ \mathbb{Z}w_1^\vee + \mathbb{Z}m'w_2^\vee & \text{otherwise} \end{cases}$$

hold, which concludes the proof. □

Proposition 4.10. — *The $k$-algebra $\widetilde{T}_{(c)}$ is isomorphic to $k[\![\Sigma_j^\vee \cap \Gamma_j^\vee]\!]$, where $y_{1(c)}^{l_1} y_{2(c)}^{l_2}$ maps to $w^\vee$ for any $w^\vee = N_{(c)}^{-1}(l_1 w_1^\vee + l_2 w_2^\vee) \in \Sigma_j^\vee \cap \Gamma_j^\vee$ (4.5).*

*Proof.* — By multiplying $\Gamma_j^\vee$ by $N_{(c)}$, we have only to show that the $k$-algebra $\widetilde{T}_{(c)}$ is isomorphic to $k[\![\Sigma_j^\vee \cap N_{(c)}\Gamma_j^\vee]\!]$, where $y_{1(c)}^{l_1} y_{2(c)}^{l_2}$ maps to $w^\vee$ for any $w^\vee = l_1 w_1^\vee + l_2 w_2^\vee \in \Sigma_j^\vee \cap N_{(c)}\Gamma_j^\vee$. Put

$$\overline{B}_S := \Sigma_j^\vee \cap \Lambda_j^\vee, \quad \overline{B}_T := \Sigma_j^\vee \cap N_{(c)}\Gamma_j^\vee, \quad \text{and} \quad \overline{T}_{(c)} := k[\![\overline{B}_T]\!].$$

By Lemma 4.8, we identify $\widetilde{S}_{(c)}$ with $k[\![\overline{B}_S]\!]$, where $y_{i(c)}$ is identified with $w_i^\vee$ for $i \in \{0,1,2\}$. Since $N_{(c)}\Lambda_j \subset \Gamma_j$, we conclude that $\overline{B}_T \subset \overline{B}_S$. Thus, we may regard $\overline{T}_{(c)}$ as a subring of $\widetilde{S}_{(c)}$. Note that $\overline{T}_{(c)} \subset \widetilde{T}_{(c)} \subset \widetilde{S}_{(c)}$ are finite extensions of normal integral domains (Lemma 4.9) since the equality $\overline{B}_S = \bigcup(w^\vee + \overline{B}_T)$ holds, where $w^\vee$ runs through the finite set

$$\left\{ l_1 w_1^\vee + l_2 w_2^\vee \in \overline{B}_S \;\middle|\; \max\{|l_1|, |l_2|\} < N_{(c)} \max\{a_1, a_2\} \right\}.$$

Thus, since the equalities

$$\begin{aligned} [\widetilde{S}_{(c)} : \overline{T}_{(c)}] &= [\Lambda_j^\vee : N_{(c)}\Gamma_j^\vee] = \frac{N_{(c)}^2}{[\Gamma_j^\vee : \Lambda_j^\vee]} = \frac{N_{(c)}^2}{[\Lambda_j : \Gamma_0][\Gamma_0 : \Gamma_j]} \\ &= \frac{N_{(c)}}{p} = [\widetilde{S}_{(c)} : \widetilde{T}_{(c)}] \end{aligned}$$

hold (4.6), the equality $\overline{T}_{(c)} = \widetilde{T}_{(c)}$ holds, which concludes the proof. □

*Proof of Theorem 3.4.* — Lemma 4.3 shows that the action of $G$ on $E_f$ induces a faithful action of $G/G'$ ($\cong \mathbb{Z}/n\mathbb{Z}$) on $E_f$ ($\cong \mathbb{P}^1_k$) with fixed locus $\{\widetilde{P}_{(0)}, \widetilde{P}_{(\infty)}\}$. Moreover, the equalities $Q_1 = \widetilde{Q}_{(0)}$ and $Q_2 = \widetilde{Q}_{(\infty)}$ hold. Thus, Propositions 4.6 and 4.10 show the following. For any $i \in \{1,2\}$, the scheme $\widetilde{Y}$ has a singularity at $Q_i$ whose completion is isomorphic to $Z_i$. The other singularities are contained in $\widetilde{E}$, their number is equal to $d'$, and any of their completions is isomorphic to $Z_3$. Therefore, Statement (1) holds. For $c \in C \cup \{0, \infty\}$, we denote the preimage of $\widetilde{Q}_{(c)}$ under $\widetilde{h}$ with reduced structure by $E_{(c)}$. Since $y_{0(c)}/x^p_{(c)} \in (\widetilde{R}_{(c)})^\times$ for any $c \in \{0,\infty\}$ (resp. $y_{1(c)}/(\widetilde{x}_{(c)})^p \in (\widetilde{R}_{(c)})^\times$ for any $c \in C$), Proposition 4.10 and Lemma 4.9 show that $y^n_{0(c)}$ (resp. $y_{1(c)}$) is a parameter of $E_0$ and a defining function of $E_{(c)}$ at the closed point $E_0 \cap E_{(c)}$ for any $c \in \{0,\infty\}$ (resp. $c \in C$), which proves Statement (2). In particular, the desingularization $h$ of $Y$ is good. Moreover, since the desingularization $\widetilde{h}$ of $\widetilde{Y}$ is minimal good, and $E_0$ intersects with more than two irreducible components $\{E_{i,1}\}_{i=1}^{d'+2}$ of $E_h$, the desingularization $h$ of $Y$ is minimal good, which proves Statement (3). Since $y_{1(\infty)}/x^p_1 \in (\widetilde{R}_{(\infty)})^\times$ (resp. $y_{2(0)}/x^p_2 \in (\widetilde{R}_{(0)})^\times$), Proposition 4.10 and Lemma 4.9 show that $y^m_{1(\infty)}$ (resp. $y^m_{2(0)}$) is a parameter of $\widehat{D}_2$ (resp. $\widehat{D}_1$) and a defining function of $E_h$ at the closed point $\widehat{D}_2 \cap E_h$ (resp. $\widehat{D}_1 \cap E_h$), which proves Statement (4). □

## 5. Intersection Matrix

We use the notation introduced in Section 4. Recall that $(v_i, \Gamma_i, \Sigma_i)_{i=0}^3$ (resp. $(m_i, k_i)_{i \in I}$) is introduced in (3.4) (resp. (3.18)). Put

$$\widehat{v}_i := \begin{cases} (p\widehat{b}_2, \widehat{b}_1) = pn_1 v_0 + m' v'_1 & \text{for } i = 1 \quad (3.16), \\ (\widehat{c}_2, p\widehat{c}_1) = pn_2 v_0 + m' v'_2 & \text{for } i = 2 \quad (3.16), \\ (m', pe') & \text{for } i = 3 \quad ((3.1) \text{ and } (3.17)). \end{cases}$$

Lemma 5.1. — *The following holds:*

1. $\mathbb{Z}\widehat{v}_1 + \mathbb{Z}pnv_0 = \Gamma_1$, $\mathbb{Z}\widehat{v}_2 + \mathbb{Z}pnv_0 = \Gamma_2$, *and* $\mathbb{Z}\widehat{v}_3 + \mathbb{Z}pv_2 = \Gamma_3$*;*
2. $\widehat{v}_i \in \Sigma_i$ *for any* $i \in I$*;*
3. $m_i\widehat{v}_i - k_i pnv_0 = pmv_i$ *for any* $i \in \{1,2\}$ *and* $m_3\widehat{v}_3 - k_3 pv_2 = pm'v_3$*;*
4. $\gcd(m_i, k_i) = 1$ *and* $0 < k_i < m_i$ *for any* $i \in I$*.*

*Proof.* — We denote the left hand side in (1) by $\Gamma'_i$ in each case $i \in I$. Note that the following equalities hold ((4.2)–(4.3)):

$$\det\begin{pmatrix} v_0 \\ \widehat{v}_1 \end{pmatrix} = m' \det\begin{pmatrix} v_0 \\ v'_1 \end{pmatrix} = -m'; \quad \det\begin{pmatrix} v_0 \\ \widehat{v}_2 \end{pmatrix} = m' \det\begin{pmatrix} v_0 \\ v'_2 \end{pmatrix} = m'; \tag{5.1}$$

$$\det\begin{pmatrix} v_2 \\ \widehat{v}_3 \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ m' & pe' \end{pmatrix} = -m'; \tag{5.2}$$

$$\det\begin{pmatrix} v_3 \\ \widehat{v}_3 \end{pmatrix} = \det\begin{pmatrix} 1 & e \\ m' & pe' \end{pmatrix} = pe' - m'e. \tag{5.3}$$

Equalities (5.1)–(5.2) give the following equalities:

$$\begin{aligned} [\Gamma_0 : \Gamma'_1] &= pnm' = pm = [\Gamma_0 : \Gamma_1]; \\ [\Gamma_0 : \Gamma'_2] &= pnm' = pm = [\Gamma_0 : \Gamma_2]; \\ [\Gamma_0 : \Gamma'_3] &= pm' = [\Gamma_0 : \Gamma_3]. \end{aligned}$$

Thus, since $\Gamma'_i \subset \Gamma_i$, the equality $\Gamma'_i = \Gamma_i$ holds for any $i \in I$, which proves (1). Since $m' > 0$, Equalities (5.1) show that $\widehat{v}_i \in \Sigma_i$ for any $i \in \{1, 2\}$. Since $p \not| m'e$ (3.3) and $e' = \lceil m'e/p \rceil$ (3.17), the inequalities

$$0 < pe' - m'e < p \tag{5.4}$$

hold. In particular, Equalities (5.3) show that $\widehat{v}_3 \in \Sigma_3$, which concludes the proof of (2). Equalities (5.1) show (3). Thus, Lemma 3.5, Inequalities (5.4), and (1)–(3) show (4). □

For $c \in k \cup \{\infty\}$, by $Q_{(c)}$ we denote the closed point $E_0 \cap (\widetilde{h})^{-1}(\widetilde{Q}_{(c)})$ on $\widehat{Y}$. By $E_{\tilde{h}}$ we denote the exceptional locus of $\widetilde{h}$ with reduced structure. Put

$$(\widehat{y}_{1(c)}, \widehat{y}_{2(c)}) := \begin{cases} (y^n_{0(0)}, y^{-\hat{b}_1}_{1(0)} y^{\hat{b}_2}_{2(0)}) & \text{for } c = 0, \\ (y^n_{0(\infty)}, y^{\hat{c}_1}_{1(\infty)} y^{-\hat{c}_2}_{2(\infty)}) & \text{for } c = \infty, \\ (y_{1(c)}, y^{-e'}_{1(c)} y^{m'}_{2(c)}) & \text{for } c \in C. \end{cases}$$

Proposition 4.10 and Lemmas 4.9 and 5.1 (1)–(2) show the following:

Lemma 5.2. — *For any $c \in C \cup \{0, \infty\}$, the rational function $\widehat{y}_{1(c)}$ (resp. $\widehat{y}_{2(c)}$) is regular and defines $E_{\tilde{h}}$ (resp. $E_0$) at $Q_{(c)}$.*

Lemma 5.3. — *The equality $E_0^2 = -b_0$ holds* (3.17).

*Proof.* — We define an element of $k((x_1, x_2))^G$ by

$$\widehat{z} := \prod_{\tau \in G/G'} \tau \left( \sum_{\tau' \in G'/H} \bar{\rho}(\tau')^{-1} \tau' z' \right)^{m'}$$

(Definition 4.4). For any $c \in k \setminus (C \cup \{0\})$, Proposition 4.6 implies that $\widehat{z}$ is a defining function of $n\widetilde{E}$ at $\widetilde{Q}_{(c)}$, which implies that $\widehat{z}$ is a defining function of $nE_0$ at $Q_{(c)}$ since the restriction $\widehat{Y} \setminus E_{\widetilde{h}} \to \widetilde{Y} \setminus \{Q_i\}_{i \in I_{\mathrm{all}}}$ of $\widetilde{h}$ is an isomorphism (Theorem 3.4 (2)). Note that the following holds:

$$\left\{ \frac{\widehat{y}_{1(0)}}{(x_0')^{pn}}, \frac{\widehat{y}_{2(0)}}{x_1^{-\hat{b}_1} x_2^{p\hat{b}_2}}, \frac{\widehat{z}}{(x_1')^m} \right\} \subset (\widetilde{R}_{(0)})^\times;$$

$$\left\{ \frac{\widehat{y}_{1(\infty)}}{(x_0')^{-pn}}, \frac{\widehat{y}_{2(\infty)}}{x_1^{p\hat{c}_1} x_2^{-\hat{c}_2}}, \frac{\widehat{z}}{((x_0')^{ed} x_1')^m} \right\} \subset (\widetilde{R}_{(\infty)})^\times;$$

$$\left\{ \frac{\widehat{y}_{1(c)}}{(\widetilde{x}_{(c)})^p}, \frac{\widehat{y}_{2(c)}}{(\widetilde{x}_{(c)})^{m'e-pe'}(x_1')^{m'}}, \frac{(\widehat{y}_{1(c)})^{ne'}(\widehat{y}_{2(c)})^n}{(\widetilde{x}_{(c)})^{me}(x_1')^m} \right\} \subset (\widetilde{R}_{(c)})^\times,$$

$$\left\{ \frac{(\widehat{y}_{1(c)})^{ne'}(\widehat{y}_{2(c)})^n}{(x_1')^m}, \frac{\widehat{z}}{(x_1')^m} \right\} \subset \widetilde{R}_{(c)},$$

and

$$\left\{ \frac{((\widehat{y}_{1(c)})^{ne'}(\widehat{y}_{2(c)})^n (x_1')^{-m})|_{E_{f,(c)}}}{(\widetilde{x}_{(c)})^{me}|_{E_{f,(c)}}}, \frac{(\widehat{z}(x_1')^{-m})|_{E_{f,(c)}}}{(\widetilde{x}_{(c)})^{me}|_{E_{f,(c)}}} \right\} \subset (\widetilde{R}_{(c)}/(x_1'))^\times$$

for any $c \in C$. The equality $n\widehat{v}_i = pnn_i v_0 + mv_i'$ holds for any $i \in \{1,2\}$. Thus, since $v_1' + v_2' = (ed + pn_0)v_0$ ((3.12) and (3.14)), the equality $n\widehat{v}_2 = (pnn_2' + med)v_0 - mv_1'$ holds, where $n_2' := m'n_0 + n_2$. Therefore, we conclude that

$$\frac{\widehat{z}}{(\widehat{y}_{1(0)})^{n_1}(\widehat{y}_{2(0)})^n} \in (\widetilde{R}_{(0)})^\times, \quad \frac{\widehat{z}}{(\widehat{y}_{1(\infty)})^{n_2'}(\widehat{y}_{2(\infty)})^n} \in (\widetilde{R}_{(\infty)})^\times,$$

and

$$\frac{(\widehat{z}(x_1')^{-m})|_{E_{f,(c)}}}{((\widehat{y}_{1(c)})^{ne'}(\widehat{y}_{2(c)})^n(x_1')^{-m})|_{E_{f,(c)}}} \in (\widetilde{R}_{(c)}/(x_1'))^\times$$

for any $c \in C$. Thus, Lemma 5.2 implies that there exist an open neighborhood $U$ of $E_0$ in $\widehat{Y}$ and a divisor $D_U$ on $U$ such that the intersection of the supports of $D_U$ and $E_0$ is contained in $\{Q_{(c)}\}_{c \in C}$, $D_U|_{E_0} = 0$, and $\widehat{z}_U \in H^0(U, \mathcal{L})$, where $\widehat{z}_U := \widehat{z}|_U$ and $\mathcal{L} := \mathcal{O}_U(D_U - nE_0)$. Put $\mathcal{T} := (\mathcal{L}/\widehat{z}_U \mathcal{O}_U)|_{E_0}$. Since $\#\{Q_{(c)}\}_{c \in C} = d'$ (1.4), the equalities

$$h^0(\mathcal{T}) = n_1 + n_2' + ne'd' = m'n_0 + n_1 + n_2 + ne'd'$$

hold. Therefore, the equalities

$$-nE_0^2 = (D_U - nE_0) \cdot E_0 = \deg \mathcal{L}|_{E_0} = h^0(\mathcal{T})$$

conclude the proof. □

Put

$$(d_1, d_2, d_3) \coloneqq (1, 1, d') \tag{5.5}$$

(1.4), where $d_i$ is equal to the number of the singular points of $\widetilde{Y}$ corresponding to $Z_i$ in Theorem 3.4 (1) for any $i \in I$.

Lemma 5.4. — *The equality*

$$\left(b_0 - \sum_{i\in I} \frac{d_i k_i}{m_i}\right) pna_1a_2 = m'$$

*holds (*(1.3)–(1.4), (3.1), (3.17)–(3.18), *and* (5.5)*).*

*Proof.* — The following equalities hold:

$$b_0 pna_1a_2 = pm'a_2(c_1 - b_1) + p(n_1 + n_2)a_1a_2 + pe'da_1a_2;$$

$$\begin{aligned}\left(\sum_{i\in I} \frac{d_i k_i}{m_i}\right) pna_1a_2 &= \widehat{b}_1 a_2 + \widehat{c}_2 a_1 + (pe' - m'e)da_1a_2 \\ &= -pm'(a_2b_1 + a_1c_2) + p(n_1 + n_2)a_1a_2 \\ &\quad + (pe' + m'e)da_1a_2.\end{aligned}$$

By subtracting the second from the first, we obtain the equalities

$$\left(b_0 - \sum_{i\in I} \frac{d_i k_i}{m_i}\right) pna_1a_2 = pm'(a_1c_2 + a_2c_1) - m'eda_1a_2 = m',$$

which concludes the proof. □

*Proof of Theorem 3.6.* — Let us show the first statement. Theorem 3.4 (2) shows that $h$ is a good desingularization, and the dual graph of $E_h$ is a star-shaped tree with central node (resp. $d' + 2$ branches) corresponding to $E_0$ (resp. the exceptional loci $(E_l)_{l\in I_{\text{all}}}$ of the Hirzebruch–Jung desingularizations). Take $l \in I_{\text{all}}$. Put $i \coloneqq \min\{l, 3\}$. Proposition 4.10 and Lemma 5.1 show that the intersection matrix of $E_l$ with respect to the ordered basis $(E_{l,j})_{j=1}^{s_l}$ is equal to $\Omega_i$ [3, 10.2.3 and 10.4.4], which is the $r_i \times r_i$ matrix associated with $m_i/k_i$ (Section 2). In particular, the equality $s_l = r_i$ holds. Moreover, Lemma 5.3 gives the equality $E_0^2 = -b_0$, which concludes the proof of the first statement.

Let us show the last equality. By $\Omega_i'$ we denote the submatrix of $\Omega_i$ formed by deleting the first row and the first column. By $I'$ we denote the multiset consisting of $d_i$ copies of $i$ for $i \in I$. Then the equality

$$\det \Omega_h = -b_0 \prod_{i\in I'} \det \Omega_i - \sum_{i'\in I'} \det \Omega_{i'}' \prod_{i\in I'\setminus\{i'\}} \det \Omega_i$$

holds. Since the equalities

$$\det \Omega_i = (-1)^{r_i} m_i \quad \text{and} \quad \det \Omega'_i = (-1)^{r_i - 1} k_i$$

hold for any $i \in I$ (Section 2), the equality

$$\det \Omega_h = (-1)^{r_{\mathrm{tot}}} \left( b_0 - \sum_{i \in I} \frac{d_i k_i}{m_i} \right) m_1 m_2 m_3^{d'}$$

holds. Thus, since $(m_1, m_2, m_3) = (pna_1, pna_2, p)$ (3.18), Lemma 5.4 concludes the proof. □

*Proof of Theorem 1.2.* — By taking the absolute values of both sides of the equality

$$\det \Omega_h = (-1)^{r_{\mathrm{tot}}} p^{d'+1} m$$

(Theorem 3.6), we obtain the desired equality $\delta = p^{d'+1} m$. □

Finally, we show that $\delta$ does not depend on the choice of a good desingularization.

Proposition 5.5. — *Let $X_0$ be the spectrum of an excellent normal local ring of dimension two with algebraically closed residue field. For $i \in \{1, 2\}$, let $f_i \colon X_i \to X_0$ be a good desingularization of $X_0$. By $E_{f_i}$ we denote the exceptional locus of $f_i$ with reduced structure. Let $\Omega_{f_i}$ be an intersection matrix of $E_{f_i}$ (with respect to an ordered basis of the irreducible components of $E_{f_i}$). Then the equality $|\det \Omega_{f_1}| = |\det \Omega_{f_2}|$ holds.*

*Proof.* — The good desingularizations $f_1$ and $f_2$ of $X_0$ are dominated by a good desingularization of $X_0$ via proper birational morphisms, each of which is a finite succession of blowing-ups at closed points. Thus, we may assume that $f_2 = f_1 \circ f_3$, and $f_3$ is a blowing-up at a closed point $P$ on $X_1$. Let $A$ be an $r \times r$ intersection matrix of the irreducible components of $E_{f_1}$ that do not contain $P$. By $s$ we denote the number of the irreducible components of $E_{f_1}$ that contain $P$. Since the absolute value of the determinant of an intersection matrix does not depend on the choice of an ordered basis of the irreducible components, we may assume that the equalities

$$\Omega_{f_1} = \begin{pmatrix} A & B \\ {}^t B & C \end{pmatrix} \quad \text{and} \quad \Omega_{f_2} = \begin{pmatrix} A & B & 0 \\ {}^t B & C' & D \\ 0 & {}^t D & -1 \end{pmatrix}$$

hold, where $B$ is an $r \times s$ matrix, $C$ and $C'$ are $s \times s$ matrices, $D$ is an $s \times 1$ matrix, and all entries of $C - C'$ and $D$ are equal to one. Thus, the equality $\det \Omega_{f_2} = -\det \Omega_{f_1}$ holds, which concludes the proof. □

## 6. Fundamental Cycle

We use the notation introduced in Section 3. By $\mathrm{Div}_h^+$ we denote the set of positive divisors on $\widehat{Y}$ whose supports are contained in $E_h$. The *fundamental cycle $Z$ of $h$* is the minimum divisor in

$$\{D \in \mathrm{Div}_h^+ \mid \forall\, D' \in \mathrm{Div}_h^+,\ D \cdot D' \leqslant 0\},$$

which always exists [1, p. 132]. We may write

$$Z = \lambda_0 E_0 + \sum_{i \in I_{\mathrm{all}}} \sum_{j=1}^{s_i} \lambda_{i,j} E_{i,j},$$

where $\lambda_0 \in \mathbb{Z}_{\geqslant 0}$ and $\lambda_{i,j} \in \mathbb{Z}_{\geqslant 0}$. Since $Z$ is minimum, the equality

$$(\lambda_{i_1,j})_{j=1}^{r_3} = (\lambda_{i_2,j})_{j=1}^{r_3}$$

holds for any $(i_1, i_2) \in I_3^2$ (Theorem 3.6). Thus, in the following, we study $\lambda_0$ and $(\lambda_{i,j})_{j=1}^{r_i}$ for $i \in I$. Recall that the actions of $G$ on $k[\![x_1]\!]$ and $k[\![x_2]\!]$ determine the integers $a_1$, $a_2$, $m$, $n$, $b_0$, and $(m_i, k_i, d_i)_{i \in I}$ ((1.3)–(1.4), (3.17)–(3.18), and (5.5)). For $u \in \mathbb{Z}$, we put

$$\kappa_u \coloneqq u b_0 - \sum_{i \in I} d_i \left\lceil \frac{u k_i}{m_i} \right\rceil.$$

Then Lemma 5.4 gives the equality

$$\kappa_u = \frac{um}{pn^2 a_1 a_2} - \sum_{i \in I} d_i \left\langle \frac{u k_i}{m_i} \right\rangle, \tag{6.1}$$

where $\langle x \rangle \coloneqq \lceil x \rceil - x$ for $x \in \mathbb{R}$. Since $E_0 \cong \mathbb{P}_k^1$ (Theorem 3.4(2)), we may calculate $Z$ and $p_f$ by means of the formulas for a desingularization with star-shaped dual graph whose branches are induced by the Hirzebruch–Jung desingularizations [12, §3]:

Corollary 6.1. — *For $i \in I$, we denote the vector associated with $m_i/k_i$ by $(v_{i,j})_{j=0}^{r_i+1}$ (Section 2) and put $\lambda_{i,0} \coloneqq \lambda_0$. Then the equalities*

$$\lambda_0 = \min\{u \in \mathbb{Z} \mid u \geqslant 1 \text{ and } \kappa_u \geqslant 0\} \quad \text{and} \quad \lambda_{i,j} = \left\lceil \frac{\lambda_{i,j-1} v_{i,j}}{v_{i,j-1}} \right\rceil$$

*hold for any $(i, j) \in I \times \mathbb{Z}$ satisfying $1 \leqslant j \leqslant r_i$. Moreover, the equalities*

$$\begin{aligned} p_f &= -(\lambda_0 - 1)\left(\frac{\lambda_0 b_0}{2} + 1\right) + \sum_{i \in I} d_i \sum_{u=1}^{\lambda_0 - 1} \left\lceil \frac{u k_i}{m_i} \right\rceil \\ &= -(\lambda_0 - 1)\left(\frac{\lambda_0 m}{2pn^2 a_1 a_2} + 1\right) + \sum_{i \in I} d_i \sum_{u=1}^{\lambda_0 - 1} \left\langle \frac{u k_i}{m_i} \right\rangle \end{aligned}$$

*hold.*

*Proof.* — The first two equalities follow from Theorem 3.6 and [12, §3, pp. 282–283]. Take a closed point $P$ on $E_0$. Since $E_0 \cong \mathbb{P}^1_k$ (Theorem 3.4 (2)), Theorem 3.6 gives the equality $p_f = \sum_{u=1}^{\lambda_0-1} h^1(\mathcal{O}_{E_0}(\kappa_u P))$ [12, 3.1]. Since $h^1(\mathcal{O}_{E_0}(\kappa P)) = -\kappa - 1$ for any $\kappa \in \mathbb{Z}_{<0}$, the equality $p_f = -\sum_{u=1}^{\lambda_0-1}(\kappa_u + 1)$ holds, which concludes the proof. □

*Example 6.2.* — Assume that $\alpha_1 = \alpha_2 = m = p-1$ and $n = 1$. Then $d' = p-1$, $(m_i, k_i) = (p, 1)$ for any $i \in I$, and $b_0 = 2$ (Corollary 3.10), which implies that $\kappa_u = 2u - (p+1)\lceil u/p \rceil$. If $p = 2$ (resp. $p > 2$), then $\kappa_1 = -1$ and $\kappa_2 = 1$ (resp. $\kappa_u = 2u - (p+1) \leqslant -2$ for any $u \in \mathbb{Z}$ satisfying $1 \leqslant u \leqslant (p-1)/2$ and $\kappa_{(p+1)/2} = 0$). Thus, the equalities

$$\lambda_0 = \left\lceil \frac{p+1}{2} \right\rceil \quad \text{and} \quad p_f = (\lambda_0 - 1)(p - \lambda_0) = \left( \left\lceil \frac{p+1}{2} \right\rceil - 1 \right) \left( p - \left\lceil \frac{p+1}{2} \right\rceil \right)$$

hold. In particular, the singularity of $Y$ is rational if and only if $p = 2$.

Put $\lambda_i := \lambda_{i,1}$ for $i \in I$. Let us estimate the quotient $\lambda_i/\lambda_0$.

Lemma 6.3. — *The following inequalities hold:*

$$\frac{\lambda_i}{\lambda_0} \geqslant \frac{k_i}{m_i} \text{ for any } i \in I; \tag{6.2}$$

$$\sum_{i \in I} d_i \left( \frac{\lambda_i}{\lambda_0} - \frac{k_i}{m_i} \right) \leqslant \frac{m}{pn^2 a_1 a_2}; \tag{6.3}$$

$$-\frac{m}{pn^2 a_1 a_2} \leqslant \frac{\lambda_1}{\lambda_0} - \frac{\widehat{b}_2}{na_2} \leqslant 0; \tag{6.4}$$

$$-\frac{m}{pn^2 a_1 a_2} \leqslant \frac{\lambda_2}{\lambda_0} - \frac{\widehat{c}_1}{na_1} \leqslant 0. \tag{6.5}$$

*Moreover, the following statements hold:*

1. *the inequality in* (6.2) *is an equality for $i \in I$ if and only if $m_i$ divides $\lambda_0$;*
2. *the last inequality in* (6.4) *(resp.* (6.5)*) is an equality if and only if the inequality in* (6.2) *is an equality for any $i \in \{2, 3\}$ (resp. $\{1, 3\}$);*
3. *if the equivalent statements in* (2) *hold, then the first inequality in* (6.5) *(resp.* (6.4)*) is an equality.*

*Proof.* — Since $\lambda_0 \geqslant 1$ and $\lambda_i = \lceil \lambda_0 k_i / m_i \rceil$ for any $i \in I$ (Corollary 6.1), the equality

$$\frac{\lambda_i}{\lambda_0} - \frac{k_i}{m_i} = \frac{1}{\lambda_0} \left\langle \frac{\lambda_0 k_i}{m_i} \right\rangle$$

holds, which proves (6.2) and (1). Since $\kappa_{\lambda_0} \geqslant 0$ (Corollary 6.1), the inequality

$$\sum_{i\in I} \frac{d_i}{\lambda_0}\left\langle \frac{\lambda_0 k_i}{m_i}\right\rangle \leqslant \frac{m}{pn^2 a_1 a_2}$$

holds (6.1), which proves (6.3). In particular, the inequalities

$$0 \leqslant \frac{\lambda_i}{\lambda_0} - \frac{k_i}{m_i} \leqslant \frac{m}{pn^2 d_i a_1 a_2}$$

hold for any $i \in I$. Thus, the equalities

$$\frac{k_1}{m_1} + \frac{m}{pn^2 a_1 a_2} = \frac{\widehat{b}_2}{na_2} \quad \text{and} \quad \frac{k_2}{m_2} + \frac{m}{pn^2 a_1 a_2} = \frac{\widehat{c}_1}{na_1}$$

((3.11), (3.16), and (3.18)) show the other statements. □

Theorem 6.4. — *Assume that $G \cong \mathbb{Z}/p\mathbb{Z}$. Then the following equality holds:*

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \begin{cases} (pa_1, eda_1 - pb_1, pc_1, (p-e)a_1) & \text{if } a_1 < a_2, \\ (pa_2, pb_2, eda_2 - pc_2, (p-e)a_2) & \text{if } a_1 > a_2, \\ (p, p-1, p-1, p-e) & \text{otherwise.} \end{cases}$$

*Remark that $a_1 = a_2 = 1$ if $a_1 = a_2$ (1.3).*

*Proof.* — By $(\lambda_0', \lambda_1', \lambda_2', \lambda_3')$ we denote the right hand side. We use Corollary 3.8. The equality

$$\kappa_u = \frac{u}{pa_1a_2} - \left\langle \frac{u(eda_1 - pb_1)}{pa_1}\right\rangle - \left\langle \frac{u(eda_2 - pc_2)}{pa_2}\right\rangle - d\left\langle \frac{u(p-e)}{p}\right\rangle$$

holds (6.1). If $a_1 = a_2$, then $\kappa_{\lambda_0'} = 1$. Since the equalities

$$\frac{u(eda_1 - pb_1)}{pa_1} = \frac{u(pa_1b_2 - 1)}{pa_1a_2} \quad \text{and} \quad \frac{u(eda_2 - pc_2)}{pa_2} = \frac{u(pa_2c_1 - 1)}{pa_1a_2}$$

hold (3.11), the equality $\kappa_{\lambda_0'} = 0$ holds if $a_1 \neq a_2$. Thus, since $\kappa_{\lambda_0'} \geqslant 0$, the inequality $\lambda_0 \leqslant \lambda_0'$ holds. Lemma 6.3 (3) implies that either $\lambda_1/\lambda_0 \neq b_2/a_2$ or $\lambda_2/\lambda_0 \neq c_1/a_1$ holds. In the former case (resp. the latter case), the inequality $\lambda_0 \geqslant pa_1$ (resp. $\lambda_0 \geqslant pa_2$) holds (6.4) (resp. (6.5)). Since $\lambda_0' = p\min\{a_1, a_2\}$, we obtain the inequality $\lambda_0 \geqslant \lambda_0'$. Thus, the equality $\lambda_0 = \lambda_0'$ holds. Therefore, the equality $\lambda_i = \lambda_i'$ for any $i \in I$ follows from Lemma 6.3 (1)–(2). □

Lemma 6.5. — *Take* $(a,b,c)\in\mathbb{Z}_{\geqslant 1}\times\mathbb{Z}_{\geqslant 1}\times\mathbb{Z}$. *Assume that* $\gcd(b,c)=1$. *Then the equalities*

$$\sum_{u=0}^{ab-1}\left\langle\frac{uc}{b}\right\rangle=\frac{a(b-1)}{2}\quad\text{and}\quad\sum_{u=0}^{ab-1}\left\lceil\frac{uc}{b}\right\rceil=\frac{a(abc+b-c-1)}{2}$$

*hold.*

*Proof.* — Since $(uc)_{u=u_0}^{u_0+b-1}$ is a complete system of representatives of $\mathbb{Z}/b\mathbb{Z}$ in $\mathbb{Z}$ for any $u_0\in\mathbb{Z}$, the equalities

$$\sum_{u=0}^{ab-1}\left\langle\frac{uc}{b}\right\rangle=a\sum_{u=0}^{b-1}\frac{u}{b}=\frac{a(b-1)}{2}$$

hold. Thus, the equalities

$$\sum_{u=0}^{ab-1}\left\lceil\frac{uc}{b}\right\rceil=\sum_{u=0}^{ab-1}\left(\frac{uc}{b}+\left\langle\frac{uc}{b}\right\rangle\right)=\frac{a(abc+b-c-1)}{2}$$

hold. □

*Proof of Theorem 1.3.* — We may assume that $a_1\leqslant a_2$. Theorem 6.4 gives the equality $\lambda_0=pa_1$. Since the equalities

$$\frac{k_2}{m_2}=\frac{eda_2-pc_2}{pa_2}=\frac{c_1}{a_1}-\frac{1}{pa_1a_2}$$

hold (3.20), the equality

$$\left\langle\frac{uk_2}{m_2}\right\rangle=\left\langle\frac{uc_1}{a_1}\right\rangle+\frac{u}{pa_1a_2}$$

holds for any $u\in\mathbb{Z}$ satisfying $0<u<pa_1$. Thus, Corollary 6.1 and Lemma 6.5 give the equalities

$$\begin{aligned}p_f&=-(pa_1-1)\left(\frac{1}{2a_2}+1\right)\\&\quad+\sum_{u=1}^{pa_1-1}\left(\left\langle\frac{uk_1}{pa_1}\right\rangle+\left\langle\frac{uc_1}{a_1}\right\rangle+\frac{u}{pa_1a_2}+d\left\langle\frac{uk_3}{p}\right\rangle\right)\\&=-\frac{pa_1-1}{2a_2}-(pa_1-1)\\&\quad+\frac{pa_1-1}{2}+\frac{p(a_1-1)}{2}+\frac{pa_1-1}{2a_2}+\frac{(p-1)\alpha_1}{2}\\&=\frac{(p-1)(\alpha_1-1)}{2},\end{aligned}$$

which concludes the proof. □

## 7. Canonical Divisor

We use the notation introduced in Section 3. A *canonical divisor $K_h$ of $h$* is a $\mathbb{Q}$-divisor on $\widehat{Y}$ satisfying the following conditions:

(1) the support of $K_h$ is contained in $E_h$;
(2) for any integral exceptional divisor $E$ of $h$, the adjunction formula $K_h \cdot E + E^2 = -2$ holds.

Note that the right hand side $-2$ is equal to the degree of a canonical divisor of $E$ ($\cong \mathbb{P}^1_k$). By Condition (1), we may write

$$K_h = \mu_0 E_0 + \sum_{i \in I_{\mathrm{all}}} \sum_{j=1}^{s_i} \mu_{i,j} E_{i,j},$$

where $\mu_0 \in \mathbb{Q}$ and $\mu_{i,j} \in \mathbb{Q}$. In this section, we show the unique existence of $K_h$ and calculate $K_h$ and $K_h^2$.

Lemma 7.1. — *Let $M$ and $K$ be integers satisfying* $\gcd(M, K) = 1$ *and $0 < K < M$. Take the Hirzebruch–Jung continued fraction $[B_j]_{j=1}^r$ of $M/K$ (Section 2) and the unique $K' \in \mathbb{Z}$ satisfying $M \mid KK' - 1$ and $0 < K' < M$. We denote the vector associated with $M/K$ (resp. $M/K'$) by $V$ (resp. $V'$) (Section 2) and the vector whose entries are the reverse of the entries of $V'$ by $W$:*

$$V = (M, K, \dots, 1, 0) \quad \text{and} \quad W = (0, 1, \dots, K', M).$$

*Then $\{V, W\}$ is a basis of the kernel of the $\mathbb{Q}$-homomorphism*

$$L\colon \mathbb{Q}^{r+2} \longrightarrow \mathbb{Q}^r, \qquad (A_j)_{j=0}^{r+1} \longmapsto (A_{j-1} - A_j B_j + A_{j+1})_{j=1}^r.$$

*Take $\mu \in \mathbb{Q}$. Then there exists a unique $U = (U_j)_{j=0}^{r+1} \in \mathbb{Q}^{r+2}$ such that $L(U) = (B_j - 2)_{j=1}^r$ and $(U_0, U_{r+1}) = (\mu, 0)$. Moreover, the equalities*

$$\begin{aligned} U &= \frac{\mu+1}{M} V + \frac{1}{M} W + U' \\ &= \left(\mu, \frac{(\mu+1)K + 1}{M} - 1, \dots, \frac{\mu + 1 + K'}{M} - 1, 0\right) \end{aligned}$$

*hold, where $U' \coloneqq (-1)_{j=0}^{r+1} \in \mathbb{Q}^{r+2}$.*

*Proof.* — By the definition of the entries of $V$ and $V'$ (2.1) and the equality $M/K' = [B_{r+1-j}]_{j=1}^r$ [3, 10.2.6], we conclude that $\{V, W\} \subset \operatorname{Ker} L$. Thus, since $V$ and $W$ are linearly independent over $\mathbb{Q}$, and the equality $\dim_{\mathbb{Q}} \operatorname{Ker} L = 2$ holds, the set $\{V, W\}$ is a basis of $\operatorname{Ker} L$. Take $U = (U_j)_{j=0}^{r+1} \in \mathbb{Q}^{r+2}$. Since $L(U') = (B_j - 2)_{j=1}^r$, the following statements are equivalent:

(1) $L(U) = (B_j - 2)_{j=1}^r$;
(2) $U - U' \in \operatorname{Ker} L$;
(3) there exists a unique $(a, b) \in \mathbb{Q}^2$ such that $U = aV + bW + U'$.

Assume that the above equivalent statements hold. Then $(U_0, U_{r+1}) = (aM - 1, bM - 1)$. Thus, the equality $(U_0, U_{r+1}) = (\mu, 0)$ holds if and only if $(a, b) = ((\mu + 1)/M, 1/M)$, which concludes the proof. □

Definition 7.2. — *We use the notation introduced in Lemma 7.1. By $U(\mu, M, K)$ we denote the vector $(U_j)_{j=1}^r \in \mathbb{Q}^r$ formed by deleting the first and last entries of the unique $U = (U_j)_{j=0}^{r+1} \in \mathbb{Q}^{r+2}$ satisfying $L(U) = (B_j - 2)_{j=1}^r$ and $(U_0, U_{r+1}) = (\mu, 0)$.*

Theorem 7.3. — *There exists a unique canonical divisor of $h$. Moreover, the equalities*

$$\mu_0 = \frac{a_1 + a_2 - (p-1)da_1a_2}{m'} - 1$$

*((1.3) and (3.1)) and*

$$(7.1) \qquad (\mu_{l,j})_{j=1}^{s_l} = U(\mu_0, m_i, k_i)$$

*(3.18) hold for any $l \in I_{\mathrm{all}}$, where $i \coloneqq \min\{l, 3\}$.*

*Proof.* — Take the Hirzebruch–Jung continued fraction $[b_{i,j}]_{j=1}^{r_i}$ of $m_i/k_i$ for $i \in I$. Put $\mu_{l,0} \coloneqq \mu_0$ and $\mu_{l,s_l+1} \coloneqq 0$ for $l \in I_{\mathrm{all}}$. Condition (2) on $K_h$ for $E_{l,j}$ is satisfied if and only if the equality

$$\mu_{l,j-1} - \mu_{l,j}b_{i,j} + \mu_{l,j+1} = b_{i,j} - 2$$

holds, where $i \coloneqq \min\{l, 3\}$ (Theorem 3.6). Thus, Condition (2) on $K_h$ for all $E_{l,j}$ is satisfied if and only if Equality (7.1) holds for any $l \in I_{\mathrm{all}}$ (Lemma 7.1). If these equivalent statements hold, then the equality

$$\mu_{l,1} = \frac{(\mu_0 + 1)k_i + 1}{m_i} - 1$$

holds for any $l \in I_{\mathrm{all}}$ (Lemma 7.1), which gives the equalities

$$\sum_{l \in I_{\mathrm{all}}} \mu_{l,1} = \sum_{i \in I} d_i\mu_{i,1} = -2 - d' + \sum_{i \in I} \left( (\mu_0 + 1)\frac{d_ik_i}{m_i} + \frac{d_i}{m_i} \right)$$

(5.5). Condition (2) on $K_h$ for $E_0$ is satisfied if and only if the equality

$$-(\mu_0 + 1)b_0 + \sum_{l \in I_{\mathrm{all}}} \mu_{l,1} = -2$$

holds (Theorem 3.6). Thus, Condition (2) on $K_h$ is satisfied if and only if Equality (7.1) holds for any $l \in I_{\mathrm{all}}$, and the rational number $\mu_0$ is a solution of the equation

$$(\mu_0+1)\left(b_0-\sum_{i\in I}\frac{d_ik_i}{m_i}\right)=-d'+\sum_{i\in I}\frac{d_i}{m_i}.$$

Lemma 5.4 gives a unique solution of this equation

$$\begin{aligned}\mu_0&=\frac{pna_1a_2}{m'}\left(-d'+\frac{1}{pna_1}+\frac{1}{pna_2}+\frac{d'}{p}\right)-1\\&=\frac{a_1+a_2-(p-1)da_1a_2}{m'}-1,\end{aligned}$$

which concludes the proof. □

Lemma 7.4. — *We use the notation introduced in Lemma 7.1. Then the equality*

$$\sum_{j=1}^{r}U_j(B_j-2)=U_0+2-\frac{(U_0+1)(K+1)+K'+1}{M}+\sum_{j=1}^{r}(2-B_j)$$

*holds.*

*Proof.* — Lemma 7.1 gives the following equalities for any $j\in\mathbb{Z}$ satisfying $1\leqslant j\leqslant r$:

$$\begin{gathered}-U_{j-1}+U_jB_j-U_{j+1}=2-B_j;\quad U_{r+1}=0;\\ \frac{(U_0+1)(K+1)+K'+1}{M}-2=U_1+U_r.\end{gathered}$$

By adding both sides, we obtain the desired equality. □

Theorem 7.5. — *We use the notation introduced in Theorem 7.3. For $i\in I$, we take the Hirzebruch–Jung continued fraction $[b_{i,j}]_{j=1}^{r_i}$ of $m_i/k_i$ and the unique $k_i'\in\mathbb{Z}$ satisfying $m_i\mid k_ik_i'-1$ and $0<k_i'<m_i$. Then the equalities*

$$\begin{aligned}K_h^2&=\mu_0(b_0-2)+\sum_{i\in I}d_i\sum_{j=1}^{r_i}\mu_{i,j}(b_{i,j}-2)\\&=\mu_0(b_0-2)+(d'+2)(\mu_0+2)\\&\quad+\sum_{i\in I}d_i\left(-\frac{(\mu_0+1)(k_i+1)+k_i'+1}{m_i}+\sum_{j=1}^{r_i}(2-b_{i,j})\right)\end{aligned}$$

*hold ((3.17) and (5.5)).*

*Proof.* — Since $K_h \cdot E = -E^2 - 2$ for any integral exceptional divisor $E$ of $h$, the first equality follows from Theorem 3.6. Since $(\mu_{i,j})_{j=1}^{r_i} = U(\mu_0, m_i, k_i)$ for any $i \in I$, the last equality follows from Lemma 7.4. □

## 8. Geometric Genus

We use the notation introduced in Section 3. The geometric genus $p_g$ of the singularity of $Y$ is most difficult to compute. The difficulty derives from that of the calculation of the dimension of the differential forms on the product of two curves invariant under the product action of $G$ (Lemma 8.3). In this section, we calculate $p_g$ in the case $G \cong \mathbb{Z}/p\mathbb{Z}$ by generalizing the method in [6].

Lemma 8.1. — *Let $\alpha$ be a positive integer coprime to $p$ and $z$ be a rational function on $\mathbb{P}^1_k$. Assume that $z$ is regular on $\mathbb{P}^1_k \setminus \{0\}$, and the order of the pole of $z$ at $0$ is equal to $\alpha$. We denote the normal model of the equation $T^p - T = z$ by $\pi\colon C \to \mathbb{P}^1_k$ and the pull-back of the coordinate function of $\mathbb{A}^1_k = \mathbb{P}^1_k \setminus \{\infty\}$ via $\pi$ by $y$. Choose a rational function $x$ on $C$ satisfying $x^p - x = z$. For $(i, j) \in \mathbb{Z}^2$, we put*

$$\omega_{i,j} := x^{i-1}y^{-j-1}\mathrm{d}y.$$

*Put $O := \pi^{-1}(0)$, where $\pi$ is totally ramified. For a non-zero rational differential form $\omega$ on $C$, by $v_O(\omega)$ we denote the order of the zero of $\omega$ at $O$, which is negative if $\omega$ has a pole at $O$. Then the equality*

$$v_O(\omega_{i,j}) = (p - i)\alpha - jp - 1$$

*holds. In particular, if $v_O(\omega_{i,j}) = v_O(\omega_{i',j'})$ and $|i - i'| \leqslant p - 1$, then $(i, j) = (i', j')$.*

*Proof.* — Since $x^p - x = z$, the equality $v_O(\mathrm{d}x) = v_O(y^{-\alpha-1}\mathrm{d}y)$ holds, and the order of the zero of $x^{-1}$ (resp. $y$) at $O$ is equal to $\alpha$ (resp. $p$). Thus, the equalities

$$v_O(\mathrm{d}x) = -\alpha - 1 \quad \text{and} \quad v_O(\mathrm{d}y) = p(\alpha + 1) + v_O(\mathrm{d}x) = (p - 1)(\alpha + 1)$$

hold. Since $\omega_{i,j} = (x^{-1})^{1-i}y^{-j-1}\mathrm{d}y$, the equality $v_O(\omega_{i,j}) = (p-i)\alpha-jp-1$ holds. Therefore, since $p \not| \alpha$, the last statement holds. □

We denote the genus of a proper smooth $k$-curve $W$ by $g(W)$.

Lemma 8.2. — *We use the notation introduced in Lemma 8.1. By $G_\pi$ we denote the Galois group of $\pi$, which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. By $V_C$ we denote the $k$-vector space $H^0(C,\Omega^1_C)$ with $G_\pi$-module structure. Put*

$$\Phi_\alpha := \left\{(i,j)\in\mathbb{Z}^2 \;\middle|\; 1\leqslant i\leqslant p-1 \text{ and } 1\leqslant j\leqslant \left\lceil\frac{(p-i)\alpha}{p}\right\rceil -1\right\}.$$

*Then $\omega_{i,j}\in V_C$ for any $(i,j)\in\Phi_\alpha$. Put $J:=\{j\mid (1,j)\in\Phi_\alpha\}$. For $j\in J$, by $V_{C,j}$ we denote the $k$-subspace of $V_C$ generated by $\{\omega_{i,j}\mid (i,j)\in\Phi_\alpha\}$. Then the following statements hold.*

(1) *For any $i\in\mathbb{Z}$ satisfying $1\leqslant i\leqslant p-1$, the equality*

$$\#\{j\mid \dim_k V_{C,j}\geqslant i\} = \left\lceil\frac{(p-i)\alpha}{p}\right\rceil -1$$

*holds.*

(2) *The rational differential form $\omega_{1,1}$ $(= y^{-2}\mathrm{d}y)$ on $C$ is regular and nowhere-zero on $C\setminus\{O\}$. In particular, the equality $2g(C)-2 = v_O(\omega_{1,1})$ holds.*

(3) *The equalities*

$$g(C)=\frac{(p-1)(\alpha-1)}{2}=\#\Phi_\alpha$$

*hold.*

(4) *The family $(\omega_{i,j})_{(i,j)\in\Phi_\alpha}$ is a basis of $V_C$, and the inclusions $V_{C,j}\to V_C$ for all $j\in J$ induce an isomorphism*

$$V_C\cong\bigoplus_{j\in J}V_{C,j}.$$

(5) *For any $j\in J$, the $G_\pi$-module $V_{C,j}$ is indecomposable, and the $G_\pi$-invariant $k$-subspace $V_{C,j}^{G_\pi}$ is generated by $\omega_{1,j}$.*

*Proof.* — The inequalities

$$\frac{(p-i)\alpha-1}{p}\geqslant\left\lceil\frac{(p-i)\alpha}{p}\right\rceil-1\geqslant j$$

hold for any $(i,j)\in\Phi_\alpha$ since $p\nmid\alpha$ and $1\leqslant i\leqslant p-1$. Thus, since the equalities

$$v_O(\omega_{i,j})=(p-i)\alpha-jp-1=p\left(\frac{(p-i)\alpha-1}{p}-j\right)$$

hold for any $(i,j)\in\Phi_\alpha$ (Lemma 8.1), the integers $(v_O(\omega_{i,j}))_{(i,j)\in\Phi_\alpha}$ are non-negative and different from each other (Lemma 8.1). Thus, since $x$, $y^{-1}$, and $y^{-2}\mathrm{d}y$ are regular on $C\setminus\{O\}$, we conclude that $\omega_{i,j}\in V_C$ for any

$(i,j) \in \Phi_\alpha$, and the elements of $(\omega_{i,j})_{(i,j)\in\Phi_\alpha}$ are linearly independent. In particular, for any $i \in \mathbb{Z}$ satisfying $1 \leqslant i \leqslant p-1$, the equalities

$$\#\{j \mid \dim_k V_{C,j} \geqslant i\} = \#\{j \mid (i,j) \in \Phi_\alpha\} = \left\lceil \frac{(p-i)\alpha}{p} \right\rceil - 1$$

hold, which proves Statement (1). Since the restriction $C\setminus\{O\} \to \mathbb{P}^1_k\setminus\{0\}$ of $\pi$ is étale, and $\omega_{1,1}$ is equal to the pull-back via $\pi$ of a rational differential form on $\mathbb{P}^1_k$ that is regular and nowhere-zero on $\mathbb{P}^1_k \setminus \{0\}$, Statement (2) holds. Thus, Lemma 8.1 gives the first equality of Statement (3). Lemma 6.5 gives the equalities

$$\begin{aligned}\#\Phi_\alpha &= \sum_{i=1}^{p-1}\left(\left\lceil \frac{(p-i)\alpha}{p} \right\rceil - 1\right) \\ &= \frac{(p-1)(\alpha+1)}{2} - (p-1) = \frac{(p-1)(\alpha-1)}{2},\end{aligned}$$

which concludes the proof of Statement (3). Since $\dim_k V_C = g(C)$, Statement (4) follows from Statement (3). Take a generator $\sigma_\pi$ of $G_\pi$ so that $\sigma_\pi(x) = x+1$. For any $(i,j) \in \Phi_\alpha$, the equalities

$$\sigma_\pi(\omega_{i,j}) = (x+1)^{i-1}y^{-j-1}\mathrm{d}y = \sum_{i'=1}^{i}\binom{i-1}{i'-1}\omega_{i',j}$$

hold, which proves Statement (5). □

Assume that $G \cong \mathbb{Z}/p\mathbb{Z}$. Lemma 4.1 shows that, for each $i \in \{1,2\}$, we may take $z_i \in k[y_i^{-1}]$ so that the completion of the normal model $\pi_i \colon C_i \to \mathbb{P}^1_k$ of the equation $T^p - T = z_i$ at 0 induces the extension $k[\![x_i]\!]/k[\![y_i]\!]$. Then the action of $G$ on $\operatorname{Spec} k[\![x_i]\!]$ extends to that on $C_i$ with fixed locus $\{O_i\}$, where $O_i \coloneqq \pi_i^{-1}(0)$. Moreover, the action of $G$ on $\operatorname{Spec} k[\![x_1,x_2]\!]$ extends to the product action of $G$ on $C_1 \times C_2$ with fixed locus $\{O_1 \times O_2\}$.

In order to simplify the notation, we use the same notation in the global case as in the local case. Put $X \coloneqq C_1 \times C_2$. Take the quotient $q \colon X \to Y \coloneqq X/G$ of $X$ by $G$. Then $Y$ is a normal surface, which implies that $Y$ is Cohen–Macaulay. By $\omega_X$ (resp. $\omega_Y$) we denote the dualizing sheaf of $X$ (resp. $Y$). Put $X' \coloneqq X \setminus \{O_1 \times O_2\}$ and $Y' \coloneqq q(X')$.

Lemma 8.3. — *The equality*

$$h^0(\omega_Y) = \sum_{i=1}^{p-1}\left\lceil \frac{i\alpha_1}{p} \right\rceil \left\lceil \frac{i\alpha_2}{p} \right\rceil - \frac{(p-1)(\alpha_1+\alpha_2)}{2}$$

*holds.*

*Proof.* — The diagram

$$\begin{array}{ccc} H^0(Y,\omega_Y) & \longrightarrow & H^0(Y',\omega_Y|_{Y'}) \\ \downarrow & & \downarrow \\ H^0(X,\omega_X)^G & \longrightarrow & H^0(X',\omega_X|_{X'})^G \end{array}$$

is commutative, where the horizontal arrows are induced by the restrictions, and the vertical arrows are induced by the pull-back via $q$. Since both $X$ and $Y$ are normal surfaces, the dualizing sheaves $\omega_X$ and $\omega_Y$ are reflexive. Thus, the horizontal arrows in the above diagram are bijective. Since the restriction $X' \to Y'$ of $q$ is étale, the right vertical arrow is bijective. Therefore, the equality $h^0(\omega_Y) = \dim_k H^0(X,\omega_X)^G$ holds. For $i \in \{1,2\}$, we take the decomposition $H^0(C_i,\Omega^1_{C_i}) = \bigoplus_{j\in J_i} V_{C_i,j}$ given by Lemma 8.2 (4). Then the equality

$$\dim_k H^0(X,\omega_X)^G = \sum_{(j_1,j_2)\in J_1\times J_2} \min\{\dim_k V_{C_1,j_1}, \dim_k V_{C_2,j_2}\}$$

holds (see the second paragraph of the proof of [6, 2.4]). Since the right hand side is equal to

$$\sum_{i=1}^{p-1} \#\{j_1 \mid \dim_k V_{C_1,j_1} \geqslant i\}\cdot\#\{j_2 \mid \dim_k V_{C_2,j_2} \geqslant i\},$$

Lemma 8.2 (1) gives the equality

$$h^0(\omega_Y) = \sum_{i=1}^{p-1}\left(\left\lceil\frac{i\alpha_1}{p}\right\rceil - 1\right)\left(\left\lceil\frac{i\alpha_2}{p}\right\rceil - 1\right).$$

Thus, Lemma 6.5 concludes the proof. □

Theorem 3.4 gives a desingularization of $Y$. We use the same notation in the global case as in the local case (3.8).

Lemma 8.4. — *The equality $h^1(\mathcal{O}_{\hat{Y}}) = 0$ holds.*

*Proof.* — Since $\widetilde{Y}$ has only rational singularities, the equality $h^1(\mathcal{O}_{\hat{Y}}) = h^1(\mathcal{O}_{\tilde{Y}})$ holds. Thus, we have only to show that $h^1(\mathcal{O}_{\tilde{Y}}) = 0$. Put $F'_X := (C_1 \times O_2) \cup (O_1 \times C_2) \subset X$. Take the normalization $F_X$ of $F'_X$ and the normalization $F_{\tilde{X}}$ (resp. $F_{\tilde{Y}}$) of the strict transform of $F'_X$ (resp. $q(F'_X)$)

via $f$ (resp. $g$). Then we obtain the diagram with commutative squares

$$\begin{array}{ccccc} \widetilde{Y} & \xleftarrow{\tilde{q}} & \widetilde{X} & \xrightarrow{f} & X \\ \uparrow & & \uparrow & & \uparrow \\ F_{\tilde{Y}} & \xleftarrow{\tilde{q}_F} & F_{\tilde{X}} & \xrightarrow{f_F} & F_X \end{array}$$

where $\widetilde{q}_F$ (resp. $f_F$) is induced by $\widetilde{q}$ (resp. $f$), and the vertical arrows are the projections. The above diagram induces a diagram with commutative squares

$$\begin{array}{ccccc} H^1(\widetilde{Y}, \mathcal{O}_{\tilde{Y}}) & \longrightarrow & H^1(\widetilde{X}, \mathcal{O}_{\tilde{X}}) & \longleftarrow & H^1(X, \mathcal{O}_X) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(F_{\tilde{Y}}, \mathcal{O}_{F_{\tilde{Y}}}) & \longrightarrow & H^1(F_{\tilde{X}}, \mathcal{O}_{F_{\tilde{X}}}) & \longleftarrow & H^1(F_X, \mathcal{O}_{F_X}) \end{array}$$

Since $f_F \colon F_{\tilde{X}} \cong F_X$ ($\cong C_1 \sqcup C_2$), the right lower arrow is bijective. Since $X$ is regular, the right upper arrow is bijective. Since $X = C_1 \times C_2$, the right vertical arrow is bijective. Thus, the middle arrow is bijective. Since $F_{\tilde{Y}} \cong \mathbb{P}^1_k \sqcup \mathbb{P}^1_k$. the equality $H^1(F_{\tilde{Y}}, \mathcal{O}_{F_{\tilde{Y}}}) = 0$ holds. Therefore, the lemma follows from the fact that the left upper arrow is injective [6, 4.2]. □

We denote the topological Euler characteristic of a proper curve or a proper smooth surface $W$ over a separably closed field by $e(W)$. Recall that the number of the irreducible components of $E_h$ is equal to $r_{\mathrm{tot}}$ (Theorem 3.6).

Lemma 8.5. — *The equality*

$$e(\widehat{Y}) = (p-1)(\alpha_1 - 1)(\alpha_2 - 1) + r_{\mathrm{tot}} + 4$$

*holds.*

*Proof.* — The first projection $X = C_1 \times C_2 \to C_1$ induces a morphism $\widehat{Y} \to \mathbb{P}^1_k$. Take the generic point $\eta$ (resp. $\theta$) of $\mathbb{P}^1_k$ (resp. $C_1$) and a geometric generic point $\overline{\eta}$ of $C_1$, where the composite $\overline{\eta} \to \theta \to \eta$ is induced by a separable closure of the function field of $\mathbb{P}^1_k$ with Galois group $G_\eta$. For $\xi \in \{0, \eta, \theta, \overline{\eta}\}$, we put $\widehat{Y}_\xi := \widehat{Y} \times_{\mathbb{P}^1_k} \xi$. Since $\operatorname{Pic} \widehat{Y}$ is finitely generated (Lemma 8.4), and the homomorphism $\operatorname{Pic} \widehat{Y} \to \operatorname{Pic} \widehat{Y}_\eta$ induced by the first projection $\widehat{Y}_\eta = \widehat{Y} \times_{\mathbb{P}^1_k} \eta \to \widehat{Y}$ is surjective, the group $\operatorname{Pic} \widehat{Y}_\eta$ is finitely generated. Note that $\operatorname{Pic} \widehat{Y}_\eta \cong \operatorname{Pic}_{\hat{Y}_\eta/\eta}(\eta)$ since the Leray spectral sequence for the second projection $\widehat{Y}_\eta = \widehat{Y} \times_{\mathbb{P}^1_k} \eta \to \eta$ and $\mathbb{G}_{m, \hat{Y}_\eta}$ induces an exact sequence of commutative groups and homomorphisms

$$0 \longrightarrow \operatorname{Pic}\eta \longrightarrow \operatorname{Pic}\widehat{Y}_\eta \longrightarrow \operatorname{Pic}_{\hat{Y}_\eta/\eta}(\eta) \longrightarrow \operatorname{Br}\eta,$$

$\operatorname{Pic}\eta = 0$ by Hilbert's theorem 90, and $\operatorname{Br}\eta = 0$ by Tsen's theorem. Thus, we may take a prime number $l$ different from $p$ so that $\operatorname{Pic}_{\hat{Y}_\eta/\eta}(\eta)[l] = 0$, where we denote the $l$-torsion subgroup of a commutative group $P$ by $P[l]$. We define a $G_\eta$-module by $M := H^1(\widehat{Y}_{\bar\eta}, \mu_{l,\hat{Y}_{\bar\eta}})$. Since $\widehat{Y}_\theta \cong C_2 \times_{\operatorname{Spec} k} \theta$ over $\theta$, the equality $\dim_{\mathbb{F}_l} M = 2g(C_2)$ holds, and the action of $G_\eta$ on $M$ induces that of $G$. The Kummer sequence

$$1 \longrightarrow \mu_{l,\hat{Y}_\eta} \longrightarrow \mathbb{G}_{m,\hat{Y}_\eta} \xrightarrow{\ l\ } \mathbb{G}_{m,\hat{Y}_\eta} \longrightarrow 1$$

induces a $G_\eta$-equivariant isomorphism $M \cong \operatorname{Pic}_{\hat{Y}_\eta/\eta}(\bar\eta)[l]$, which implies that $M^G \cong \operatorname{Pic}_{\hat{Y}_\eta/\eta}(\eta)[l] = 0$. Therefore, Serre's measure of wild ramification of $M$ at $0 \in \mathbb{P}^1_k$ [11, §I, p. 3] is given by

$$\delta_0 := \sum_{i\geqslant 1} \frac{1}{[G:G_i]} \dim_{\mathbb{F}_l} M/M^{G_i} = 2g(C_2)\alpha_1,$$

where the $i$-th ramification group of $\theta/\eta$ is given by

$$G_i := \{\tau \in G \mid v_1(\tau x_1 - x_1) \geqslant i+1\} = \begin{cases} G & \text{if } i \leqslant \alpha_1, \\ 1 & \text{otherwise.} \end{cases}$$

The reduction of $\widehat{Y}_0$ is a union of the $r_{\mathrm{tot}}+1$ projective lines any of whose singularities is a node and whose dual graph is a tree (Theorem 3.4 (2) and (4)), which implies that the equality $e(\widehat{Y}_0) = r_{\mathrm{tot}} + 2$ holds. Since $\widehat{Y}_{\bar\eta} \cong C_2 \times_{\operatorname{Spec} k} \bar\eta$ over $\bar\eta$, the equalities $e(\widehat{Y}_{\bar\eta}) = e(C_2) = 2 - 2g(C_2)$ hold. Thus, Dolgachev's formula [4, Theorem 1.1] gives the equalities

$$e(\widehat{Y}) = e(\widehat{Y}_{\bar\eta})e(\mathbb{P}^1_k) + e(\widehat{Y}_0) - e(\widehat{Y}_{\bar\eta}) + \delta_0 = 2(\alpha_1 - 1)g(C_2) + r_{\mathrm{tot}} + 4.$$

Therefore, Lemma 8.2 (3) concludes the proof. □

Take the canonical divisor

$$K_h = \mu_0 E_0 + \sum_{i=1}^{d+2} \sum_{j=1}^{s_i} \mu_{i,j} E_{i,j}$$

of $h$ (Theorem 7.3). For $i \in \{1,2\}$, by $y_i$ (resp. $F_i$) we denote the pull-back of the coordinate function of $\mathbb{A}^1_k = \mathbb{P}^1_k \setminus \{\infty\}$ (resp. the prime divisor on $\mathbb{P}^1_k$ with support $0 \in \mathbb{P}^1_k$) via the morphism $\widehat{Y} \to \mathbb{P}^1_k$ induced by the $i$-th projection $X = C_1 \times C_2 \to C_i$. By $K_{\hat{Y}}$ we denote the canonical divisor of $\widehat{Y}$ defined by the rational differential form $y_1^{-2}\mathrm{d}y_1 \wedge y_2^{-2}\mathrm{d}y_2$ on $\widehat{Y}$.

PROPOSITION 8.6. — *The following equalities hold:*

$$\begin{aligned} K_{\hat{Y}} &= K_h + \frac{2g(C_1)-2}{p}F_1 + \frac{2g(C_2)-2}{p}F_2 \\ &= K_h + \left(\frac{(p-1)(\alpha_1+1)}{p} - 2\right)F_1 + \left(\frac{(p-1)(\alpha_2+1)}{p} - 2\right)F_2; \\ K_{\hat{Y}}^2 &= K_h^2 + \frac{2(p-1)^2(\alpha_1+1)(\alpha_2+1)}{p} - 4(p-1)(\alpha_1+\alpha_2) + 8. \end{aligned}$$

*Proof.* — By $F$ we denote the right hand side of the first equality minus $K_h$. Since the restriction $X' \to Y'$ of $q$ (resp. $h^{-1}(Y') \to Y'$ of $h$) is étale (resp. an isomorphism), Lemma 8.2 (2) shows that the support of $K_{\hat{Y}} - F$ is contained in the exceptional locus of $h$. For any integral exceptional divisor $E$ of $h$, the adjunction formula gives the equalities

$$(K_{\hat{Y}} - F) \cdot E = K_{\hat{Y}} \cdot E = -E^2 - 2 = K_h \cdot E$$

since $F_i \cdot E = 0$ for any $i \in \{1, 2\}$. Thus, the first equality follows from the uniqueness of $K_h$ (Theorem 7.3). The second equality follows from Lemma 8.2 (3). Therefore, the last equality follows from the equalities $F_1 \cdot F_2 = p$ and $K_h \cdot F_i = F_i \cdot F_i = 0$ for any $i \in \{1, 2\}$. □

THEOREM 8.7. — *Assume that $G \cong \mathbb{Z}/p\mathbb{Z}$. Then the equality*

$$p_g = \sum_{i=1}^{p-1} \left\lceil \frac{i\alpha_1}{p} \right\rceil \left\lceil \frac{i\alpha_2}{p} \right\rceil - \frac{(p-1)(3p-2)(\alpha_1+1)(\alpha_2+1)}{12p} - \frac{K_h^2 + r_{\mathrm{tot}}}{12}$$

*holds.*

*Remark 8.8.* — The last term of the above equality is determined by $\alpha_1$ and $\alpha_2$ (Theorems 7.5 and 3.6 and Corollary 3.8).

*Proof.* — Since $h^1(\mathcal{O}_{\hat{Y}}) = 0$ (Lemma 8.4), the Leray spectral sequence for $h\colon \widehat{Y} \to Y$ and $\mathcal{O}_{\hat{Y}}$ and the Grothendieck duality give the equalities

$$p_g = h^2(\mathcal{O}_Y) - h^2(\mathcal{O}_{\hat{Y}}) = h^0(\omega_Y) - \chi(\mathcal{O}_{\hat{Y}}) + 1,$$

respectively. Thus, since $12\chi(\mathcal{O}_{\hat{Y}}) = K_{\hat{Y}}^2 + e(\widehat{Y})$ by Noether's formula, Lemmas 8.3 and 8.5 and Proposition 8.6 give the equalities

$$p_g = h^0(\omega_Y) - \frac{K_{\hat{Y}}^2 + e(\widehat{Y})}{12} + 1 = \sum_{i=1}^{p-1} \left\lceil \frac{i\alpha_1}{p} \right\rceil \left\lceil \frac{i\alpha_2}{p} \right\rceil - \frac{S}{12p} - \frac{K_h^2 + r_{\mathrm{tot}}}{12},$$

where

$$\begin{aligned}S &\coloneqq 6p(p-1)(\alpha_1+\alpha_2)\\ &\quad + 2(p-1)^2(\alpha_1+1)(\alpha_2+1) - 4p(p-1)(\alpha_1+\alpha_2) + 8p\\ &\quad + p(p-1)(\alpha_1-1)(\alpha_2-1) + 4p - 12p\\ &= (p-1)(3p-2)(\alpha_1+1)(\alpha_2+1),\end{aligned}$$

which concludes the proof. □

*Proof of Theorem 1.4.* — We use the notation introduced in Theorems 7.3 and 7.5. Corollaries 3.8 and 3.10 give the equalities $m = m' = 1$, $d = d'$, $(d, e, a_1, a_2) = (\alpha, (p-1)/\alpha, 1, 1)$, $(m_1, m_2, m_3, k_1, k_2, k_3) = (p, p, p, p-1, p-1, p-e)$, and $b_0 = \alpha+1$. Thus, the equalities $(d_1, d_2, d_3) = (1, 1, \alpha)$, $(k'_1, k'_2, k'_3) = (p-1, p-1, \alpha)$, and $\mu_0 = 1-(p-1)\alpha$ hold. Since $p/(p-1) = [2, \dots, 2]$ ($p-1$ copies of 2) and $p/(p-e) = [2, \dots, 2, e+1]$ ($\alpha - 1$ copies of 2 followed by $e+1$), the equalities

$$r_{\text{tot}} = 1 + 2(p-1) + \alpha^2 \quad \text{and} \quad \sum_{i\in I} d_i \sum_{j=1}^{r_i} (2 - b_{i,j}) = \alpha - p + 1$$

hold. Thus, Theorem 7.5 gives the equalities

$$\begin{aligned}K_h^2 &= \mu_0(b_0-2) + (\alpha+2)(\mu_0+2) + \alpha - p + 1\\ &\quad - 2\cdot\frac{(\mu_0+1)p+p}{p} - \alpha\cdot\frac{(\mu_0+1)(p-e+1)+\alpha+1}{p}\\ &= \frac{(p-1)(-p\alpha^2+\alpha^2+4\alpha+2)-\alpha^2-p^2}{p},\end{aligned}$$

which gives the equality

$$K_h^2 + r_{\text{tot}} = \frac{(p-1)(\alpha+1)(-p\alpha+2\alpha+p+2)}{p}.$$

Therefore, Theorem 8.7 concludes the proof. □

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