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Akihiko MIYACHI *&* NAOHITO TOMITA

Bilinear pseudo-differential operators with exotic symbols

CENTRE
MERSENNE

BILINEAR PSEUDO-DIFFERENTIAL OPERATORS WITH EXOTIC SYMBOLS

by Akihiko MIYACHI & Naohito TOMITA

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ABSTRACT. — The boundedness from $L^p \times L^q$ to L^r, $1 < p, q \leqslant \infty$, $0 < 1/p + 1/q = 1/r \leqslant 1$, of bilinear pseudo-differential operators with symbols in the bilinear Hörmander class $BS_{\rho,\rho}^m$, $0 \leqslant \rho < 1$, is proved for the critical order m. Related results for the cases $p = 1$, $q = 1$ or $r = \infty$ are also obtained.

RÉSUMÉ. — On considère des opérateurs pseudo-différentiels avec des symboles dans la classe exotique de Hörmander. On prouve des estimations dans des espaces de Lebesgue pour ces opérateurs, sous l'hypothèse que leurs symboles soient dans la classe exotique de Hörmander d'ordre critique. On donne aussi des résultats reliés pour les espaces de Hardy et BMO.

1. Introduction

Let $m \in \mathbb{R}$ and $0 \leqslant \delta \leqslant \rho \leqslant 1$. We say that a function $\sigma(x, \xi, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ belongs to the bilinear Hörmander symbol class $BS_{\rho,\delta}^m = BS_{\rho,\delta}^m(\mathbb{R}^n)$ if for every triple of multi-indices $\alpha, \beta, \gamma \in \mathbb{N}_0^n = \{0, 1, 2, \ldots\}^n$ there exists a constant $C_{\alpha,\beta,\gamma} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \leqslant C_{\alpha,\beta,\gamma}(1 + |\xi| + |\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}.$$

For a symbol $\sigma \in BS_{\rho,\delta}^m$, the bilinear pseudo-differential operator T_σ is defined by

$$T_\sigma(f, g)(x) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{ix \cdot (\xi+\eta)} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) \, \mathrm{d}\xi \mathrm{d}\eta, \quad f, g \in \mathcal{S}(\mathbb{R}^n).$$

The study of bilinear operators T_σ with σ in the bilinear Hörmander class $BS_{\rho,\delta}^m$ was initiated by Bényi, Maldonado, Naibo, and Torres in [2], where in particular the symbolic calculus of the operators T_σ, $\sigma \in BS_{\rho,\delta}^m$,

was established. The boundedness properties of those operators have been considered in many works, some of which will be mentioned below. In the present paper, we shall also consider the boundedness property of the operators T_σ, $\sigma \in BS_{\rho,\delta}^m$. For the boundedness of the operators T_σ, we shall use the following terminology. If X, Y, Z are function spaces on $\mathbb{R}^n$ equipped with quasi-norms $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_Z$ and if there exists a constant A_σ such that the estimate

$$(1.1) \qquad \|T_\sigma(f,g)\|_Z \leqslant A_\sigma \|f\|_X \|g\|_Y, \quad f \in \mathcal{S} \cap X, \quad g \in \mathcal{S} \cap Y,$$

holds, then we shall simply say that T_σ is bounded from $X \times Y$ to Z and write

$$T_\sigma : X \times Y \to Z.$$

The smallest constant A_σ of (1.1) is denoted by $\|T_\sigma\|_{X \times Y \to Z}$.

In the case $\rho = 1$, bilinear pseudo-differential operators with symbols in $BS_{1,\delta}^0$, $\delta < 1$, fall into the bilinear Calderón–Zygmund theory in the sense of Grafakos–Torres [8] and their boundedness properties are well-understood; see, e.g., Coifman–Meyer [6], Bényi–Torres [3], and Bényi–Maldonado–Naibo–Torres [2]. In the case $\rho < 1$, however, we cannot reduce the corresponding operators to bilinear Calderón–Zygmund operators and there are some interesting features peculiar to the bilinear case. For example, in contrast to the well-known Calderón–Vaillancourt theorem ([5]) for linear pseudo-differential operators, the condition $\sigma \in BS_{\rho,\rho}^0$, $0 \leqslant \rho < 1$, does not assure any boundedness of the corresponding bilinear operator. This gap between the linear and bilinear cases was first pointed out by Bényi–Torres [4] for the case $\rho = 0$.

The subject of the present paper concerns with the estimate

$$(1.2) \qquad T_\sigma : H^p \times H^q \to L^r, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \quad \sigma \in BS_{\rho,\rho}^m, \quad 0 \leqslant \rho < 1,$$
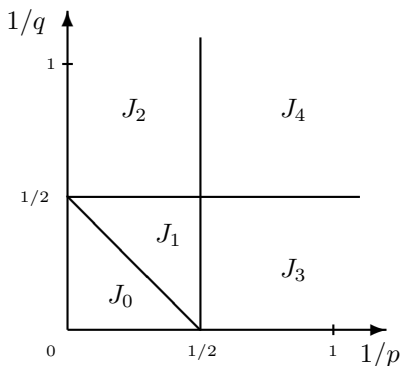
where H^p denotes Hardy space and L^r denotes Lebesgue space. In the case $p = q = r = \infty$, instead of $L^\infty \times L^\infty \to L^\infty$, we shall consider $L^\infty \times L^\infty \to \mathrm{BMO}$.

For $0 \leqslant \rho < 1$ and for $0 < p, q, r \leqslant \infty$ satisfying $1/p + 1/q = 1/r$, we define

$$m_\rho(p,q) = (1-\rho)m_0(p,q),$$

$$m_0(p,q) = -n \left(\max \left\{ \frac{1}{2}, \frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{r}, \frac{1}{r} - \frac{1}{2} \right\} \right).$$

Here is an expression of $m_0(p, q)$ that will be easy to see. We divide the region of $(1/p, 1/q)$ into 5 regions $J_0, \ldots, J_4$ as follows:

Then

$$
m_0(p, q) = \begin{cases}
\frac{n}{r} - n & \text{if } \left(\frac{1}{p}, \frac{1}{q}\right) \in J_0; \\
-\frac{n}{2} & \text{if } \left(\frac{1}{p}, \frac{1}{q}\right) \in J_1; \\
-\frac{n}{q} & \text{if } \left(\frac{1}{p}, \frac{1}{q}\right) \in J_2; \\
-\frac{n}{p} & \text{if } \left(\frac{1}{p}, \frac{1}{q}\right) \in J_3; \\
\frac{n}{2} - \frac{n}{r} & \text{if } \left(\frac{1}{p}, \frac{1}{q}\right) \in J_4,
\end{cases}
$$

where $1/p + 1/q = 1/r$.

The number $m_\rho(p, q)$ is the critical order as the following proposition shows. A proof of this proposition will be given in Appendix of this paper.

PROPOSITION 1.1. — Let $0 \leqslant \rho < 1$, $0 < p, q, r \leqslant \infty$, and suppose $1/p + 1/q = 1/r$. If $r < \infty$, then

$$
m_\rho(p, q) = \sup\{m \in \mathbb{R} : T_\sigma : H^p \times H^q \to L^r \text{ for all } \sigma \in BS^m_{\rho,\rho}\}.
$$

When $p = q = r = \infty$, the above equality holds if we replace $H^p \times H^q \to L^r$ by $L^\infty \times L^\infty \to$ BMO.

It should be an interesting problem to prove the boundedness of bilinear pseudo-differential operators in the critical class $BS^m_{\rho,\rho}$, $m = m_\rho(p, q)$. For the case $\rho = 0$, this problem was solved by the authors in [10]. For the case $0 < \rho < 1$, to the best of the authors' knowledge, the only known result for the problem is due to Naibo [11], which however is restricted to the case $0 < \rho < 1/2$ and $p = q = r = \infty$. The purpose of the present paper is to solve the problem in the range $0 \leqslant 1/p + 1/q = 1/r \leqslant 1$.

The following are the main results of this paper.

THEOREM 1.2. — Let $0 \leqslant \rho < 1$ and $m = -(1-\rho)n/2$. Then all bilinear pseudo-differential operators with symbols in $BS^m_{\rho,\rho}(\mathbb{R}^n)$ are bounded from $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

THEOREM 1.3. — Let $0 \leqslant \rho < 1$ and $m = -(1 - \rho)n$. Then all bilinear pseudo-differential operators with symbols in $BS^m_{\rho,\rho}(\mathbb{R}^n)$ are bounded from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$.

COROLLARY 1.4. — Let $0 \leqslant \rho < 1$, $1 \leqslant p, q, r \leqslant \infty$, $1/p + 1/q = 1/r$, and $m = m_\rho(p, q)$. Then all bilinear pseudo-differential operators with symbols in $BS^m_{\rho,\rho}(\mathbb{R}^n)$ are bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, where $L^p(\mathbb{R}^n)$ (respectively, $L^q(\mathbb{R}^n)$) should be replaced by $H^p(\mathbb{R}^n)$ (respectively, $H^q(\mathbb{R}^n)$) if $p = 1$ (respectively, $q = 1$) and $L^r(\mathbb{R}^n)$ should be replaced by $\mathrm{BMO}(\mathbb{R}^n)$ if $r = \infty$.

Here are some comments on the previous works related to the above results. For the subcritical case $m < m_\rho(p, q)$, the boundedness (1.2) were obtained by Michalowski–Rule–Staubach [9] (for $(1/p, 1/q)$ in the triangle with vertices $(1/2, 1/2)$, $(1/2, 0)$, $(0, 1/2)$) and by Bényi–Bernicot–Maldonado–Naibo–Torres [1] (in the range $1/p + 1/q \leqslant 1$). As we mentioned above, the case $m = m_\rho(p, q)$ with $\rho = 0$ was obtained by the authors [10]. In fact, [10, Theorem 1.1] gives a sharper version of the above Corollary 1.4 for $\rho = 0$ and covers the full range $0 < p, q, r \leqslant \infty$. Naibo [11] has proved the claim of Theorem 1.3 in the case $0 < \rho < 1/2$.

Theorem 1.2 should be one of the key estimates to consider the critical case $m = m_\rho(p, q)$ in the whole range $0 < p, q \leqslant \infty$. Here is a comment concerning the method of proof of Theorem 1.2. As we mentioned above this theorem for the case $\rho = 0$ was already proved in [10]. However, the method of the present paper is totally different from that of [10]. The method of [10] seems to work only in the case $\rho = 0$, but the method of the present paper covers all $0 \leqslant \rho < 1$.

The contents of this paper are as follows. In Section 2, we recall some preliminary facts. In Sections 3, 4 and 5, we prove Theorems 1.2, 1.3 and Corollary 1.4, respectively. In Appendix A, we prove Proposition 1.1.

2. Preliminaries

For two nonnegative quantities A and B, the notation $A \lesssim B$ means that $A \leqslant CB$ for some unspecified constant $C > 0$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. We denote by $\mathbb{1}_S$ the characteristic function of a set S, and by $|S|$ the Lebesgue measure of a measurable set S in $\mathbb{R}^n$.

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, \mathrm{d}x \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, \mathrm{d}\xi.$$

For $m \in L^\infty(\mathbb{R}^n)$, the linear Fourier multiplier operator $m(D)$ is defined by

$$m(D)f(x) = \mathcal{F}^{-1}[m\widehat{f}](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} m(\xi)\widehat{f}(\xi)\, \mathrm{d}\xi, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

We recall the definition of Hardy spaces and the space BMO on $\mathbb{R}^n$ (see [12, Chapters 3 and 4]). Let $0 < p \leqslant \infty$, and let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \phi(x)\, \mathrm{d}x \neq 0$. Then the Hardy space $H^p(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^p} = \left\| \sup_{0<t<\infty} |\phi_t * f| \right\|_{L^p} < \infty,$$

where $\phi_t(x) = t^{-n}\phi(x/t)$. It is known that $H^p(\mathbb{R}^n)$ does not depend on the choice of the function ϕ and $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p \leqslant \infty$. The space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, \mathrm{d}x < \infty,$$

where f_Q is the average of f on Q and the supremum is taken over all cubes Q in $\mathbb{R}^n$. It is known that the dual spaces of $H^1(\mathbb{R}^n)$ is $\mathrm{BMO}(\mathbb{R}^n)$.

We end this section by quoting the following, which we shall call *Schur's lemma*. For a proof, see, e.g., [7, Appendix I].

LEMMA 2.1 (Schur's lemma). — *Let $\{A_{j,k}\}_{j,k\geqslant 0}$ be a sequence of non-negative numbers satisfying*

$$\sup_{j\geqslant 0} \sum_{k\geqslant 0} A_{j,k} \leqslant 1 \quad \text{and} \quad \sup_{k\geqslant 0} \sum_{j\geqslant 0} A_{j,k} \leqslant 1.$$

Then

$$\sum_{j,k\geqslant 0} A_{j,k} b_j c_k \leqslant \left(\sum_{j\geqslant 0} b_j^2 \right)^{1/2} \left(\sum_{k\geqslant 0} c_k^2 \right)^{1/2}$$

for all nonnegative sequences $\{b_j\}$ and $\{c_k\}$.

3. Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. The argument is divided into three subsections. Although the proof for general $\sigma \in BS_{\rho,\rho}^m$ is somewhat complicated, the main idea already consists in the special case that $\sigma(x, \xi, \eta)$ is independent of x, namely the bilinear Fourier multiplier case. In this case, $\sigma_{j,k,\nu}$ to be introduced in Subsection 3.1 reduces to

$$\sigma_{j,k,\nu} = \begin{cases} \sigma_{j,\nu} & \text{if } k = 0 \\ 0 & \text{if } k \geqslant 1, \end{cases}$$

and the argument will be simple.

We use the following notation and terminology. For a finite set Λ, we write $|\Lambda|$ to denote the number of elements of Λ. The following are cubes in $\mathbb{R}^n$:

$$Q = [-1, 1]^n, \quad aQ = [-a, a]^n, \quad a > 0,$$
$$x + aQ = \{x + y : y \in aQ\}, \quad x \in \mathbb{R}^n.$$

If σ is a function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, then

$$\text{supp}^* \sigma = \text{ closure of } \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : \sigma(x, \xi, \eta) \neq 0 \text{ for some } x \in \mathbb{R}^n\}.$$

The usual inner product of $f, h \in L^2 = L^2(\mathbb{R}^n)$ is denoted by $\langle f, g \rangle$. If $\{E_\alpha\}$ is a finite family of subsets of $\mathbb{R}^n$, L is a positive integer, and if

$$|\{\beta : E_\beta \cap E_\alpha \neq \emptyset\}| \leqslant L \quad \text{for all} \quad \alpha,$$

then we say that the *interaction* of the family $\{E_\alpha\}$ is bounded by L.

3.1. Decomposition of the symbol and some preliminaries

We use the following two types of partitions of unity. One is the dyadic decomposition:

(3.1)
$$\text{supp} \, \psi_0 \subset \{\zeta \in \mathbb{R}^d : |\zeta| \leqslant 2\},$$
$$\text{supp} \, \psi_j \subset \{\zeta \in \mathbb{R}^d : 2^{j-1} \leqslant |\zeta| \leqslant 2^{j+1}\}, \quad j \geqslant 1,$$
$$\|\partial^\alpha \psi_j\|_{L^\infty} \lesssim 2^{-j|\alpha|}, \quad \alpha \in \mathbb{N}_0^d, \, j \geqslant 0,$$
$$\sum_{j \geqslant 0} \psi_j(\zeta) = 1, \quad \zeta \in \mathbb{R}^d.$$

The other is the uniform decomposition:

$$\operatorname{supp}\varphi \subset Q,$$

(3.2)
$$\sum_{\nu \in \mathbb{Z}^n} \varphi(\xi - \nu) = 1, \quad \xi \in \mathbb{R}^n.$$

Here ψ_j, $j \geqslant 0$, and φ are smooth real-valued functions. We shall use (3.1) with $d = 2n$ and $d = n$. We write Ψ_j to denote the function ψ_j of (3.1) with $d = 2n$ and write ψ_j to denote the function of (3.1) with $d = n$. We shall use (3.2) only on $\mathbb{R}^n$.

In this subsection, we assume $\sigma \in BS^m_{\rho,\rho}$ with $m \in \mathbb{R}$ and $0 \leqslant \rho \leqslant 1$. (The conditions on m and ρ as in Theorem 1.2 are not necessary in this subsection.) We decompose σ as

$$\sigma(x, \xi, \eta) = \sum_{j \geqslant 0} \sum_{\nu = (\nu_1, \nu_2) \in \mathbb{Z}^n \times \mathbb{Z}^n} \sigma_{j,\nu}(x, \xi, \eta)$$

(3.3)
$$= \sum_{j \geqslant 0} \sum_{k \geqslant 0} \sum_{\nu = (\nu_1, \nu_2) \in \mathbb{Z}^n \times \mathbb{Z}^n} \sigma_{j,k,\nu}(x, \xi, \eta),$$

where

(3.4) $$\sigma_{j,\nu}(x, \xi, \eta) = \sigma(x, \xi, \eta)\varphi(2^{-j\rho}\xi - \nu_1)\varphi(2^{-j\rho}\eta - \nu_2)\Psi_j(\xi, \eta)$$

and

$$\sigma_{j,k,\nu}(x, \xi, \eta) = [\psi_k(2^{-j\rho}D_x)\sigma_{j,\nu}](x, \xi, \eta)$$

(3.5)
$$= 2^{j\rho n} \int_{\mathbb{R}^n} [\mathcal{F}^{-1}\psi_k](2^{j\rho}y)\sigma_{j,\nu}(x - y, \xi, \eta)\, dy.$$

Notice the following facts. First, if we write the projections as

$$\pi_1(\xi, \eta) = \xi, \quad \pi_2(\xi, \eta) = \eta,$$

then it is obvious that

(3.6) $$T_{\sigma_{j,k,\nu}}(f, g) = T_{\sigma_{j,k,\nu}}(f^{(1)}, g) = T_{\sigma_{j,k,\nu}}(f, g^{(1)})$$

whenever $f^{(1)}$ and $g^{(1)}$ satisfy $(f^{(1)})^\wedge = \widehat{f}$ on $\pi_1(\operatorname{supp}^*(\sigma_{j,\nu}))$ and $(g^{(1)})^\wedge = \widehat{g}$ on $\pi_2(\operatorname{supp}^*(\sigma_{j,\nu}))$. Secondly, the Fourier transform of $T_{\sigma_{j,k,\nu}}(f, g)$ is given by

(3.7) $$\mathcal{F}[T_{\sigma_{j,k,\nu}}(f, g)](\zeta) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n \times \mathbb{R}^n} \psi_k(2^{-j\rho}(\zeta - \xi - \eta))$$

$$\times [\mathcal{F}_x \sigma_{j,\nu}](\zeta - \xi - \eta, \xi, \eta)\widehat{f}(\xi)\widehat{g}(\eta)\, d\xi d\eta,$$

where $\zeta \in \mathbb{R}^n$ and $\mathcal{F}_x \sigma_{j,\nu}$ denotes the partial Fourier transform of $\sigma_{j,\nu}(x, \xi, \eta)$ with respect to the x-variable. From this we see that

$$(3.8) \quad \operatorname{supp} \mathcal{F}[T_{\sigma_{j,k,\nu}}(f,g)] \subset \bigcup_{(\xi,\eta) \in \operatorname{supp}^*(\sigma_{j,\nu})} \{\zeta : |\zeta - \xi - \eta| \leqslant 2^{j\rho + k + 1}\}.$$

Hence, we have

$$(3.9) \qquad \langle T_{\sigma_{j,k,\nu}}(f,g), h \rangle = \langle T_{\sigma_{j,k,\nu}}(f,g), h^{(1)} \rangle$$

whenever $h^{(1)}$ satisfy $(h^{(1)})^\wedge = \widehat{h}$ on the set on the right-hand side of (3.8). In the argument to follow, we shall use (3.6) and (3.9) by choosing the functions $f^{(1)}$, $g^{(1)}$, $h^{(1)}$ according to several different situations.

We also use the following general lemma for nearly orthogonal functions and operators.

LEMMA 3.1.

(1) If $\{f_\alpha\}$ is a finite family of functions in L^2, L is a positive integer, and if $|\{\beta : \langle f_\beta, f_\alpha \rangle \neq 0\}| \leqslant L$ for all α, then $\|\sum_\alpha f_\alpha\|_{L^2}^2 \leqslant L \sum_\alpha \|f_\alpha\|_{L^2}^2$.

(2) If $\{T_\alpha\}$ is a finite family of bounded linear operators in L^2, L is a positive integer, and if $|\{\beta : T_\beta^* T_\alpha \neq 0\}| \leqslant L$ for all α, then $\|\sum_\alpha T_\alpha\|_{L^2 \to L^2}^2 \leqslant L \sum_\alpha \|T_\alpha\|_{L^2 \to L^2}^2$.

(3) If $\{T_\alpha\}$ is a finite family of bounded linear operators in L^2, L is a positive integer, and if $|\{\beta : T_\beta T_\alpha^* \neq 0\}| \leqslant L$ for all α, then $\|\sum_\alpha T_\alpha\|_{L^2 \to L^2}^2 \leqslant L \sum_\alpha \|T_\alpha\|_{L^2 \to L^2}^2$.

Proof. — To prove (1), we write

$$\left\| \sum_\alpha f_\alpha \right\|_{L^2}^2 = \sum_\alpha \sum_\beta \langle f_\alpha, f_\beta \rangle \leqslant \sum_\alpha \sum_\beta \mathbb{1}\{\langle f_\alpha, f_\beta \rangle \neq 0\} \|f_\alpha\|_{L^2} \|f_\beta\|_{L^2}.$$

Applying Schur's lemma, we obtain the desired inequality. We can prove (2) by applying (1) to $f_\alpha = T_\alpha f$. The assertion (3) follows from (2) since the norms of an operator and its adjoint are the same. □

3.2. Basic estimates

In this subsection, except in the last lemma, Lemma 3.7, we only assume $\sigma \in BS^m_{\rho,\rho}$ with $m \in \mathbb{R}$ and $0 \leqslant \rho \leqslant 1$.

We shall give some basic estimates which will be used later. We use the following notation

$$S_a(f)(x) = a^n \int_{\mathbb{R}^n} \frac{|f(y)|}{(1 + a|x - y|)^{n+1}} \, \mathrm{d}y, \quad a > 0, \ x \in \mathbb{R}^n.$$

Let us start with the estimate for the square function of $\widetilde{\varphi}(2^{-j\rho}D - \ell)f$ with respect to $\ell \in \mathbb{Z}^n$. Although this is known to many people, we shall give the proof for the reader's convenience.

LEMMA 3.2. — Let $\widetilde{\varphi} \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\left(\sum_{\ell \in \mathbb{Z}^n} |\widetilde{\varphi}(2^{-j\rho}D - \ell)f(x)|^2 \right)^{1/2} \lesssim S_{2^{j\rho}}(f^2)(x)^{1/2}$$

holds for $j \geqslant 0$ and $x \in \mathbb{R}^n$.

Proof. — Since $\widetilde{\varphi}(2^{-j\rho}D - \ell)f(x) = \widetilde{\varphi}(D - \ell)[f(2^{-j\rho} \cdot)](2^{j\rho}x)$, by a scaling argument, it is sufficient to prove the case $j = 0$. By a periodization technique, we can write

$$\widetilde{\varphi}(D - \ell)f(x) = \int_{\mathbb{R}^n} e^{i\ell \cdot (x-y)} \widetilde{\Phi}(x - y)f(y) \, \mathrm{d}y$$

$$= \sum_{\widetilde{\ell} \in \mathbb{Z}^n} \int_{2\pi\widetilde{\ell}+[-\pi,\pi]^n} e^{i\ell \cdot (x-y)} \widetilde{\Phi}(x - y)f(y) \, \mathrm{d}y$$

$$= e^{i\ell \cdot x} \int_{[-\pi,\pi]^n} e^{-i\ell \cdot y} \left(\sum_{\widetilde{\ell} \in \mathbb{Z}^n} \widetilde{\Phi}(x - y - 2\pi\widetilde{\ell})f(y + 2\pi\widetilde{\ell}) \right) \mathrm{d}y,$$

where $\widetilde{\Phi} = \mathcal{F}^{-1}\widetilde{\varphi}$. This means that $|\widetilde{\varphi}(D - \ell)f(x)|$ is equal to $(2\pi)^n$ times the absolute value of the ℓ-th Fourier coefficient of the $(2\pi\mathbb{Z})^n$-periodic function $\sum_{\widetilde{\ell} \in \mathbb{Z}^n} \widetilde{\Phi}(x - y - 2\pi\widetilde{\ell})f(y + 2\pi\widetilde{\ell})$ of the y-variable. Hence, it follows from Parseval's identity that

$$\sum_{\ell \in \mathbb{Z}^n} |\widetilde{\varphi}(D - \ell)f(x)|^2 = (2\pi)^n \int_{[-\pi,\pi]^n} \left| \sum_{\widetilde{\ell} \in \mathbb{Z}^n} \widetilde{\Phi}(x - y - 2\pi\widetilde{\ell})f(y + 2\pi\widetilde{\ell}) \right|^2 \mathrm{d}y.$$

Since $\sup_{z \in \mathbb{R}^n} \left(\sum_{\widetilde{\ell} \in \mathbb{Z}^n} |\widetilde{\Phi}(z - 2\pi\widetilde{\ell})| \right) < \infty$, by Schwarz's inequality, the right-hand side of this identity is estimated by

$$\int_{[-\pi,\pi]^n} \sum_{\widetilde{\ell} \in \mathbb{Z}^n} |\widetilde{\Phi}(x - y - 2\pi\widetilde{\ell})||f(y + 2\pi\widetilde{\ell})|^2 \, \mathrm{d}y = \int_{\mathbb{R}^n} |\widetilde{\Phi}(x - y)||f(y)|^2 \, \mathrm{d}y.$$

Therefore, the rapidly decreasing property of $\widetilde{\Phi}$ gives the desired estimate. $\qquad\square$

LEMMA 3.3. — For each $N \in \mathbb{N}_0$ and $\beta, \gamma \in \mathbb{N}_0^n$, the estimate

$$|\partial_\xi^\beta \partial_\eta^\gamma \sigma_{j,k,\nu}(x, \xi, \eta)| \lesssim 2^{jm-kN} 2^{-j\rho(|\beta|+|\gamma|)}$$

holds for $j, k \geqslant 0$ and $\nu \in \mathbb{Z}^n \times \mathbb{Z}^n$.

Proof. — First, suppose $k \geqslant 1$. Then by the moment condition of $\mathcal{F}^{-1}\psi_k$ and Taylor's formula, we can write (3.5) as

$$
\sigma_{j,k,\nu}(x,\xi,\eta)
$$
$$
= 2^{j\rho n} \int_{\mathbb{R}^n} [\mathcal{F}^{-1}\psi_k](2^{j\rho}y)
$$
$$
\times \left(\sigma_{j,\nu}(x-y,\xi,\eta) - \sum_{|\alpha|<N} \frac{\partial_x^\alpha \sigma_{j,\nu}(x,\xi,\eta)}{\alpha!}(-y)^\alpha \right) dy
$$
$$
= 2^{j\rho n} \int_{\mathbb{R}^n} [\mathcal{F}^{-1}\psi_k](2^{j\rho}y)
$$
$$
\times \left(N \sum_{|\alpha|=N} \frac{(-y)^\alpha}{\alpha!} \int_0^1 (1-t)^{N-1}[\partial_x^\alpha \sigma_{j,\nu}](x-ty,\xi,\eta)\, dt \right) dy.
$$

Using the fact that $1+|\xi|+|\eta| \approx 2^j$ for $(\xi,\eta) \in \mathrm{supp}^*(\sigma_{j,\nu})$, we have

$$
|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma_{j,\nu}(x,\xi,\eta)| \lesssim 2^{jm+j\rho(|\alpha|-|\beta|-|\gamma|)}.
$$

On the other hand, it follows from (3.1) that

$$
|\mathcal{F}^{-1}\psi_k(y)| \lesssim 2^{kn}(1+2^k|y|)^{-(N+n+1)}.
$$

Hence

$$
|\partial_\xi^\beta \partial_\eta^\gamma \sigma_{j,k,\nu}(x,\xi,\eta)|
$$
$$
\lesssim \sum_{|\alpha|=N} 2^{j\rho n} \int_{\mathbb{R}^n} \int_0^1 \left| [\mathcal{F}^{-1}\psi_k](2^{j\rho}y)y^\alpha[\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma_{j,\nu}](x-ty,\xi,\eta) \right| dt dy
$$
$$
\lesssim 2^{(j\rho+k)n} \int_{\mathbb{R}^n} (1+2^{j\rho+k}|y|)^{-(N+n+1)}|y|^N 2^{jm+j\rho(N-|\beta|-|\gamma|)}\, dy
$$
$$
\approx 2^{jm-kN} 2^{-j\rho(|\beta|+|\gamma|)}.
$$

If $k = 0$, then using (3.5) and slightly modifying the above argument, we obtain the desired estimate. □

LEMMA 3.4. — For each $N \in \mathbb{N}_0$, the estimate

$$
|T_{\sigma_{j,k,\nu}}(f,g)(x)| \lesssim 2^{jm-kN} S_{2^{j\rho}}(f)(x)S_{2^{j\rho}}(g)(x)
$$

holds for $j,k \geqslant 0$ and $\nu \in \mathbb{Z}^n \times \mathbb{Z}^n$.

Proof. — We write

$$
T_{\sigma_{j,k,\nu}}(f,g)(x) = \int_{\mathbb{R}^n \times \mathbb{R}^n} K_{j,k,\nu}(x,x-y,x-z)f(y)g(z)\, dydz,
$$

where

$$K_{j,k,\nu}(x,y,z) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(y\cdot\xi + z\cdot\eta)} \sigma_{j,k,\nu}(x,\xi,\eta) \, d\xi d\eta.$$

Since $|\xi - 2^{j\rho}\nu_1| \lesssim 2^{j\rho}$ and $|\eta - 2^{j\rho}\nu_2| \lesssim 2^{j\rho}$ for $(\xi, \eta) \in \text{supp}^*(\sigma_{j,k,\nu})$, it follows from Lemma 3.3 and integration by parts that

$$|K_{j,k,\nu}(x,y,z)| \lesssim 2^{jm-kN} \frac{2^{j\rho n}}{(1 + 2^{j\rho}|y|)^{n+1}} \frac{2^{j\rho n}}{(1 + 2^{j\rho}|z|)^{n+1}}.$$

From this the desired estimate follows. $\qquad\square$

The estimate

$$(3.10) \qquad \|T_{\sigma_{j,k,\nu}}(f,g)\|_{L^2} \lesssim 2^{jm-kN} \|f\|_{L^2} \|g\|_{L^\infty}$$

immediately follows from Lemma 3.4. In the lemmas below, we shall derive finer estimates by utilizing orthogonality.

Lemma 3.5.

(1) For each $N \in \mathbb{N}_0$, the estimate

$$\left\| \sum_{\nu_2 \in \mathbb{Z}^n} T_{\sigma_{j,k,\nu}}(f,g) \right\|_{L^2} \lesssim 2^{jm-kN} \|f\|_{L^2} \|g\|_{L^\infty}$$

holds for all $j, k \geqslant 0$ and all $\nu_1 \in \mathbb{Z}^n$.

(2) For each $N \in \mathbb{N}_0$, the estimate

$$\left\| \sum_{\nu_1 + \nu_2 = \mu} T_{\sigma_{j,k,\nu}}(f,g) \right\|_{L^2} \lesssim 2^{jm-kN} \|f\|_{L^2} \|g\|_{L^\infty}$$

holds for all $j, k \geqslant 0$ and all $\mu \in \mathbb{Z}^n$.

Proof.

(1). — Take a function $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^n)$ such that $\widetilde{\varphi}(\xi) = 1$ on $\text{supp}\,\varphi$. Then, by (3.6),

$$(3.11) \qquad T_{\sigma_{j,k,\nu}}(f,g) = T_{\sigma_{j,k,\nu}}(f_{j,\nu_1}, g) = T_{\sigma_{j,k,\nu}}(f, g_{j,\nu_2})$$

with

$$(3.12) \qquad f_{j,\nu_1} = \widetilde{\varphi}(2^{-j\rho}D - \nu_1)f, \quad g_{j,\nu_2} = \widetilde{\varphi}(2^{-j\rho}D - \nu_2)g.$$

From (3.8), we see that

$$(3.13) \qquad \text{supp}\,\mathcal{F}[T_{\sigma_{j,k,\nu}}(f,g)] \subset 2^{j\rho}(\nu_1 + \nu_2) + 2^{j\rho+k+2}Q.$$

Notice that for fixed $\nu_1 \in \mathbb{Z}^n$ the interaction of the family $\{2^{j\rho}(\nu_1 + \nu_2) + 2^{j\rho+k+2}Q\}_{\nu_2 \in \mathbb{Z}^n}$ is $\lesssim 2^{kn}$. Hence, by Lemma 3.1 (1) and Lemma 3.4, we have

$$
\left\| \sum_{\nu_2 \in \mathbb{Z}^n} T_{\sigma_{j,k,\nu}}(f,g) \right\|_{L^2}^2 = \left\| \sum_{\nu_2 \in \mathbb{Z}^n} T_{\sigma_{j,k,\nu}}(f,g_{j,\nu_2}) \right\|_{L^2}^2
$$

$$
\lesssim 2^{kn} \sum_{\nu_2 \in \mathbb{Z}^n} \| T_{\sigma_{j,k,\nu}}(f,g_{j,\nu_2}) \|_{L^2}^2
$$

$$
\text{(3.14)} \qquad \lesssim 2^{kn+2(jm-kN)} \sum_{\nu_2 \in \mathbb{Z}^n} \int_{\mathbb{R}^n} S_{2^{j\rho}}(f)(x)^2 S_{2^{j\rho}}(g_{j,\nu_2})(x)^2 \, \mathrm{d}x.
$$

By Schwarz's inequality and Lemma 3.2,

$$
\sum_{\nu_2 \in \mathbb{Z}^n} \int_{\mathbb{R}^n} S_{2^{j\rho}}(f)(x)^2 S_{2^{j\rho}}(g_{j,\nu_2})(x)^2 \, \mathrm{d}x
$$

$$
\lesssim \sum_{\nu_2 \in \mathbb{Z}^n} \int_{\mathbb{R}^n} S_{2^{j\rho}}(f^2)(x) S_{2^{j\rho}}(g_{j,\nu_2}^2)(x) \, \mathrm{d}x
$$

$$
\lesssim \int_{\mathbb{R}^n} S_{2^{j\rho}}(f^2)(x) S_{2^{j\rho}}\big[S_{2^{j\rho}}(g^2)\big](x) \, \mathrm{d}x
$$

$$
\lesssim \|f\|_{L^2}^2 \|g\|_{L^\infty}^2.
$$

Since N can be taken arbitrarily large, we obtain the desired estimate.

(2). — By (3.11)-(3.12), Lemma 3.4, and Schwarz's inequality, we have

$$
\left| \sum_{\nu_1+\nu_2=\mu} T_{\sigma_{j,k,\nu}}(f,g)(x) \right| \leqslant \sum_{\nu_1+\nu_2=\mu} |T_{\sigma_{j,k,\nu}}(f_{j,\nu_1}, g_{j,\nu_2})(x)|
$$

$$
\lesssim 2^{jm-kN} \sum_{\nu_1+\nu_2=\mu} S_{2^{j\rho}}(f_{j,\nu_1})(x) S_{2^{j\rho}}(g_{j,\nu_2})(x)
$$

$$
\leqslant 2^{jm-kN} \left(\sum_{\nu_1} S_{2^{j\rho}}(f_{j,\nu_1})(x)^2 \right)^{1/2} \left(\sum_{\nu_1} S_{2^{j\rho}}(g_{j,\mu-\nu_1})(x)^2 \right)^{1/2}
$$

$$
\lesssim 2^{jm-kN} \left(\sum_{\nu_1} S_{2^{j\rho}}(f_{j,\nu_1}^2)(x) \right)^{1/2} \left(\sum_{\nu_1} S_{2^{j\rho}}(g_{j,\mu-\nu_1}^2)(x) \right)^{1/2}.
$$

By Lemma 3.2, we have

$$
\left(\sum_{\nu_1} S_{2^{j\rho}}(f_{j,\nu_1}^2)(x) \right)^{1/2} \lesssim S_{2^{j\rho}}\big[S_{2^{j\rho}}(f^2)\big](x)^{1/2} \approx S_{2^{j\rho}}(f^2)(x)^{1/2}.
$$

Similarly,

$$\left(\sum_{\nu_1} S_{2^{j\rho}}(g^2_{j,\mu-\nu_1})(x)\right)^{1/2} \lesssim S_{2^{j\rho}}(g^2)(x)^{1/2} \lesssim \|g\|_{L^\infty}.$$

Thus we have the pointwise estimate

$$\left|\sum_{\nu_1+\nu_2=\mu} T_{\sigma_{j,k,\nu}}(f,g)(x)\right| \lesssim 2^{jm-kN} S_{2^{j\rho}}(f^2)(x)^{1/2}\|g\|_{L^\infty},$$

from which the desired L^2 inequality follows. $\qquad\square$

LEMMA 3.6.

(1) For each $N \in \mathbb{N}_0$, the estimate

$$\left\|\sum_{\nu_1\in\Lambda}\sum_{\nu_2\in\mathbb{Z}^n} T_{\sigma_{j,k,\nu}}(f,g)\right\|_{L^2} \lesssim |\Lambda|^{1/2} 2^{jm-kN}\|f\|_{L^2}\|g\|_{L^\infty}$$

holds for all $j,k \geqslant 0$ and all finite sets $\Lambda \subset \mathbb{Z}^n$.

(2) For each $N \in \mathbb{N}_0$, the estimate

$$\left\|\sum_{\mu\in\Lambda}\sum_{\nu_1+\nu_2=\mu} T_{\sigma_{j,k,\nu}}(f,g)\right\|_{L^2} \lesssim |\Lambda|^{1/2} 2^{jm-kN}\|f\|_{L^2}\|g\|_{L^\infty}$$

holds for all $j,k \geqslant 0$ and all finite sets $\Lambda \subset \mathbb{Z}^n$.

Proof. — For the proof of (1) and (2), we freeze $g \in L^\infty$ and consider the linear operator $T_{\sigma_{j,k,\nu}}(\,\cdot\,,g)$ defined by $[T_{\sigma_{j,k,\nu}}(\,\cdot\,,g)](f) = T_{\sigma_{j,k,\nu}}(f,g)$ for $j,k \geqslant 0$ and $\nu \in \mathbb{Z}^n \times \mathbb{Z}^n$. By (3.10), $T_{\sigma_{j,k,\nu}}(\,\cdot\,,g)$ is a bounded linear operator in L^2.

(1). — Since

$$\operatorname{supp}^*\left(\sum_{\nu_2\in Z^n} \sigma_{j,k,\nu}\right) \subset \operatorname{supp}\varphi(2^{-j\rho}\cdot-\nu_1) \times \mathbb{R}^n,$$

we have

$$\sum_{\nu_2\in\mathbb{Z}^n} T_{\sigma_{j,k,\nu}}(f,g) = \sum_{\nu_2\in\mathbb{Z}^n} T_{\sigma_{j,k,\nu}}(\mathbb{1}_{\operatorname{supp}\varphi(2^{-j\rho}\cdot-\nu_1)}(D)f,g).$$

In terms of the linear operator, this can be written as

$$\sum_{\nu_2\in\mathbb{Z}^n} T_{\sigma_{j,k,\nu}}(\,\cdot\,,g) = \left(\sum_{\nu_2\in\mathbb{Z}^n} T_{\sigma_{j,k,\nu}}(\,\cdot\,,g)\right)\mathbb{1}_{\operatorname{supp}\varphi(2^{-j\rho}\cdot-\nu_1)}(D).$$

Since the interaction of the family $\{\mathrm{supp}\,\varphi(2^{-j\rho}\cdot-\nu_1)\}_{\nu_1}$ is $\lesssim 1$, we see that

$$\left|\left\{\tilde{\nu}_1\in\mathbb{Z}^n:\left(\sum_{\nu_2\in\mathbb{Z}^n}T_{\sigma_{j,k,(\tilde{\nu}_1,\nu_2)}}(\,\cdot\,,g)\right)\left(\sum_{\nu_2\in\mathbb{Z}^n}T_{\sigma_{j,k,(\nu_1,\nu_2)}}(\,\cdot\,,g)\right)^*\neq 0\right\}\right|\lesssim 1$$

for all $\nu_1\in\mathbb{Z}^n$. Thus Lemma 3.1 (3) yields

$$\left\|\sum_{\nu_1\in\Lambda}\sum_{\nu_2\in\mathbb{Z}^n}T_{\sigma_{j,k,\nu}}(\,\cdot\,,g)\right\|_{L^2\to L^2}^2\lesssim\sum_{\nu_1\in\Lambda}\left\|\sum_{\nu_2\in\mathbb{Z}^n}T_{\sigma_{j,k,\nu}}(\,\cdot\,,g)\right\|_{L^2\to L^2}^2.$$

By Lemma 3.5 (1), the right-hand side of the above is $\lesssim 2^{2(jm-kN)}|\Lambda|\|g\|_{L^\infty}^2$, which implies the desired estimate.

(2). — As in (3.13), the formula (3.8) implies

$$\mathrm{supp}\,\mathcal{F}\left[\sum_{\nu_1+\nu_2=\mu}T_{\sigma_{j,k,\nu}}(f,g)\right]\subset 2^{j\rho}\mu+2^{j\rho+k+2}Q,$$

which gives

$$\sum_{\nu_1+\nu_2=\mu}T_{\sigma_{j,k,\nu}}(f,g)=\mathbb{1}_{2^{j\rho}\mu+2^{j\rho+k+2}Q}(D)\left(\sum_{\nu_1+\nu_2=\mu}T_{\sigma_{j,k,\nu}}(f,g)\right).$$

Thus, in terms of the linear operator,

$$\sum_{\nu_1+\nu_2=\mu}T_{\sigma_{j,k,\nu}}(\,\cdot\,,g)=\mathbb{1}_{2^{j\rho}\mu+2^{j\rho+k+2}Q}(D)\left(\sum_{\nu_1+\nu_2=\mu}T_{\sigma_{j,k,\nu}}(\,\cdot\,,g)\right).$$

Since the interaction of the family $\{2^{j\rho}\mu+2^{j\rho+k+2}Q\}_{\mu\in\mathbb{Z}^n}$ is $\lesssim 2^{kn}$, we see that

$$\left|\left\{\tilde{\mu}\in\mathbb{Z}^n:\left(\sum_{\nu_1+\nu_2=\tilde{\mu}}T_{\sigma_{j,k,\nu}}(\,\cdot\,,g)\right)^*\left(\sum_{\nu_1+\nu_2=\mu}T_{\sigma_{j,k,\nu}}(\,\cdot\,,g)\right)\neq 0\right\}\right|\lesssim 2^{kn}$$

for all $\mu\in\mathbb{Z}^n$. Hence Lemma 3.1 (2) yields

$$\left\|\sum_{\mu\in\Lambda}\sum_{\nu_1+\nu_2=\mu}T_{\sigma_{j,k,\nu}}(\,\cdot\,,g)\right\|_{L^2\to L^2}^2\lesssim 2^{kn}\sum_{\mu\in\Lambda}\left\|\sum_{\nu_1+\nu_2=\mu}T_{\sigma_{j,k,\nu}}(\,\cdot\,,g)\right\|_{L^2\to L^2}^2.$$

By Lemma 3.5 (2), the right-hand side of the above is $\lesssim 2^{kn+2(jm-kN)}\times|\Lambda|\|g\|_{L^\infty}^2$. Since N can be taken arbitrarily large, we obtain the desired estimate. $\square$

Notice that $\sigma_{j,k,\nu}\neq 0$ only for $|\nu_1|\lesssim 2^{j(1-\rho)}$ and $|\nu_2|\lesssim 2^{j(1-\rho)}$ and hence $\sum_{\nu\in\mathbb{Z}^n\times\mathbb{Z}^n}T_{\sigma_{j,k,\nu}}$ can be written as the sum of Lemma 3.6 (1) or (2) with $|\Lambda|\approx 2^{j(1-\rho)n}$. Hence the following lemma directly follows from Lemma 3.6.

LEMMA 3.7. — *If $m = -(1-\rho)n/2$ and $0 \leqslant \rho \leqslant 1$, then for each $N \in \mathbb{N}_0$ the estimate*

$$\left\| \sum_{\nu \in \mathbb{Z}^n \times \mathbb{Z}^n} T_{\sigma_{j,k,\nu}}(f,g) \right\|_{L^2} \lesssim 2^{-kN} \|f\|_{L^2} \|g\|_{L^\infty}$$

holds for $j, k \geqslant 0$.

3.3. Proof of Theorem 1.2

Throughout this subsection, we assume m, ρ, and σ satisfy the conditions of Theorem 1.2, namely, $0 \leqslant \rho < 1$, $m = -(1-\rho)n/2$, and $\sigma \in BS^m_{\rho,\rho}(\mathbb{R}^n)$.

Before proceeding to the main argument, we shall see that it is sufficient to consider the case where $\operatorname{supp}^* \sigma$ is included in a cone minus a ball centered at the origin.

To see this, take a function $\Theta \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\Theta(\xi, \eta) = 1$ on $\{(|\xi|^2 + |\eta|^2)^{1/2} \leqslant 2\}$ and $\operatorname{supp} \Theta \subset \{(|\xi|^2 + |\eta|^2)^{1/2} \leqslant 4\}$, and write σ as

$$\sigma(x, \xi, \eta) = \sigma(x, \xi, \eta)\Theta(\xi, \eta) + \sigma(x, \xi, \eta)(1 - \Theta(\xi, \eta)).$$

By simply summing the estimate of Lemma 3.7 over $k \geqslant 0$ and $0 \leqslant j \leqslant 2$, we obtain

$$\|T_{\sigma\Theta}(f,g)\|_{L^2} \lesssim \|f\|_{L^2}\|g\|_{L^\infty}.$$

Hence it is sufficient to treat only $T_{\sigma(1-\Theta)}$. Next, if (ξ, η) belongs to the unit sphere Σ of $\mathbb{R}^n \times \mathbb{R}^n$, then either $\xi + \eta \neq 0$ or $\xi \neq 0$. By the compactness of Σ, this implies that there exists a constant $c > 0$ such that Σ is covered by the two open sets

$$V_1 = \{(\xi, \eta) \in \Sigma : |\xi + \eta| > c\}, \quad V_2 = \{(\xi, \eta) \in \Sigma : |\xi| > c\}.$$

Taking a smooth partition of unity Φ_i, $i = 1, 2$, on Σ such that $\operatorname{supp} \Phi_i \subset V_i$, we decompose $\sigma(1 - \Theta)$ as

$$\sigma(x, \zeta)(1 - \Theta(\zeta)) = \sum_{i=1}^{2} \sigma(x, \zeta)(1 - \Theta(\zeta))\Phi_i(\zeta/|\zeta|) = \sum_{i=1}^{2} \sigma^{(i)}(x, \zeta),$$

$$\zeta = (\xi, \eta).$$

It is sufficient to prove the estimate for each $T_{\sigma^{(i)}}$, $i = 1, 2$. Obviously $\sigma^{(i)} \in BS^m_{\rho,\rho}(\mathbb{R}^n)$.

To sum up, by writing $\sigma^{(i)}$ simply as σ, we may assume that σ satisfies the additional condition

$$\operatorname{supp}^* \sigma \subset \Gamma(V_i) = \{\zeta \in \mathbb{R}^{2n} : \zeta/|\zeta| \in V_i, \ |\zeta| \geqslant 2\} \quad \text{for } i = 1 \text{ or } 2.$$

For such σ, we have $\sigma_{j,k,\nu} = 0$ for $j = 0$ and thus the decomposition (3.3) takes the form

$$(3.15) \qquad \sigma = \sum_{j \geqslant 1} \sum_{\nu = (\nu_1, \nu_2) \in \mathbb{Z}^n \times \mathbb{Z}^n} \sigma_{j,\nu} = \sum_{j \geqslant 1} \sum_{k \geqslant 0} \sum_{\nu = (\nu_1, \nu_2) \in \mathbb{Z}^n \times \mathbb{Z}^n} \sigma_{j,k,\nu}.$$

In the rest of the proof, we shall consider the two cases

$$\mathrm{supp}^* \, \sigma \subset \Gamma(V_1), \quad \mathrm{supp}^* \, \sigma \subset \Gamma(V_2)$$

separately.

We shall prove the following estimate for the trilinear form:

$$|\langle T_\sigma(f,g), h \rangle| \lesssim \|f\|_{L^2} \|g\|_{L^\infty} \|h\|_{L^2},$$

which is equivalent to the desired estimate for the operator T_σ.

$$\text{The case } \mathrm{supp}^* \, \sigma \subset \Gamma(V_1)$$

In this case, all $(\xi, \eta) \in \mathrm{supp}^* \, \sigma$ satisfy $|\xi + \eta| \approx (|\xi|^2 + |\eta|^2)^{1/2}$ (but $|\xi|$ may be small compared with $(|\xi|^2 + |\eta|^2)^{1/2}$). We take a positive integer a such that

$$(3.16) \qquad (\xi, \eta) \in \mathrm{supp}^*(\sigma_{j,k,\nu}) \Rightarrow 2^{j-a} \leqslant |\xi + \eta| \leqslant 2^{j+a}.$$

Using this a, we write (3.15) as

$$\sigma = \sum_{j \geqslant 1} \sum_{k \geqslant 0} \sum_\nu \sigma_{j,k,\nu} = \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k \leqslant j(1-\rho) - a - 2}} \sum_\nu \sigma_{j,k,\nu} + \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k > j(1-\rho) - a - 2}} \sum_\nu \sigma_{j,k,\nu}.$$

According to this decomposition of σ, we write the trilinear form as

$$\langle T_\sigma(f,g), h \rangle$$
$$= \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k \leqslant j(1-\rho) - a - 2}} \sum_\nu \langle T_{\sigma_{j,k,\nu}}(f,g), h \rangle + \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k > j(1-\rho) - a - 2}} \sum_\nu \langle T_{\sigma_{j,k,\nu}}(f,g), h \rangle$$
$$= X_1 + X_2, \quad \text{say}.$$

The estimate for the second term X_2 is easy. In fact, Lemma 3.7 gives

$$|X_2| \leqslant \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k > j(1-\rho) - a - 2}} \left\| \sum_\nu T_{\sigma_{j,k,\nu}}(f,g) \right\|_{L^2} \|h\|_{L^2}$$

$$\lesssim \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k > j(1-\rho) - a - 2}} 2^{-kN} \|f\|_{L^2} \|g\|_{L^\infty} \|h\|_{L^2} \approx \|f\|_{L^2} \|g\|_{L^\infty} \|h\|_{L^2},$$

where we used the assumption $\rho < 1$ in the last $\approx$.

In order to estimate X_1, we use the decomposition

$$f = \sum_\ell f_\ell, \quad f_\ell = \psi_\ell(D)f,$$

and write

$$X_1 = \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k \leqslant j(1-\rho)-a-2}} \sum_{\ell \geqslant 0} \sum_\nu \langle T_{\sigma_{j,k,\nu}}(f_\ell, g), h \rangle.$$

Here we make simple observations. First, if $k \leqslant j(1-\rho) - a - 2$, then from (3.8) and (3.16) we see that

$$\operatorname{supp} \mathcal{F}[T_{\sigma_{j,k,\nu}}(f,g)] \subset \bigcup_{2^{j-a} \leqslant |\xi+\eta| \leqslant 2^{j+a}} \{\zeta \in \mathbb{R}^n : |\zeta - \xi - \eta| \leqslant 2^{j\rho+k+1}\}$$

$$\subset \{\zeta \in \mathbb{R}^n : 2^{j-a-1} \leqslant |\zeta| \leqslant 2^{j+a+1}\}.$$

Hence, by (3.9), $\langle T_{\sigma_{j,k,\nu}}(f_\ell, g), h \rangle$ in X_1 can be written as

$$\langle T_{\sigma_{j,k,\nu}}(f_\ell, g), h \rangle = \langle T_{\sigma_{j,k,\nu}}(f_\ell, g), h_j \rangle, \quad h_j = \theta(2^{-j}D)h,$$

where θ is an appropriate function supported in an annulus. Secondly, since $\operatorname{supp} \widehat{f}_\ell \subset \{2^{\ell-1} \leqslant |\xi| \leqslant 2^{\ell+1}\}$ for $\ell > 0$ and since $\operatorname{supp}^*(\sigma_{j,k,\nu}) \subset \{|\xi| \leqslant 2^{j+1}\}$, it follows that $T_{\sigma_{j,k,\nu}}(f_\ell, g) = 0$ for $\ell > j+1$. Thirdly, since $\operatorname{supp}^*(\sigma_{j,k,\nu}) \subset \operatorname{supp} \varphi(2^{-j\rho} \cdot -\nu_1) \times \mathbb{R}^n$ and $\operatorname{supp} \widehat{f}_\ell \subset \operatorname{supp} \psi_\ell$, we have

$$T_{\sigma_{j,k,\nu}}(f_\ell, g) \neq 0 \; \Rightarrow \; \operatorname{supp} \varphi(2^{-j\rho} \cdot -\nu_1) \cap \operatorname{supp} \psi_\ell \neq \emptyset.$$

Combining these observations, we see that X_1 can be written as

$$(3.17) \qquad X_1 = \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k \leqslant j(1-\rho)-a-2}} \sum_{\ell=0}^{j+1} \sum_{\nu_1 \in \Lambda_{j,\ell}} \sum_{\nu_2 \in \mathbb{Z}^n} \langle T_{\sigma_{j,k,\nu}}(f_\ell, g), h_j \rangle,$$

where

$$\Lambda_{j,\ell} = \{\nu_1 \in \mathbb{Z}^n : \operatorname{supp} \varphi(2^{-j\rho} \cdot -\nu_1) \cap \operatorname{supp} \psi_\ell \neq \emptyset\}.$$

The number of elements of $\Lambda_{j,\ell}$ satisfies

$$|\Lambda_{j,\ell}| \lesssim (\max\{1, 2^{\ell-j\rho}\})^n.$$

Thus Lemma 3.6 (1) gives

$$\left| \sum_{\nu_1 \in \Lambda_{j,\ell}} \sum_{\nu_2 \in \mathbb{Z}^n} \langle T_{\sigma_{j,k,\nu}}(f_\ell, g), h_j \rangle \right|$$

$$\leqslant \left\| \sum_{\nu_1 \in \Lambda_{j,\ell}} \sum_{\nu_2 \in \mathbb{Z}^n} T_{\sigma_{j,k,\nu}}(f_\ell, g) \right\|_{L^2} \|h_j\|_{L^2}$$

$$\lesssim \max\{1, 2^{(\ell-j\rho)n/2}\} 2^{jm-kN} \|f_\ell\|_{L^2} \|g\|_{L^\infty} \|h_j\|_{L^2}.$$

Hence

$$|X_1| \lesssim \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k \leqslant j(1-\rho)-a-2}} \sum_{\ell=0}^{j+1} \max\{1, 2^{(\ell-j\rho)n/2}\} 2^{jm-kN} \|f_\ell\|_{L^2} \|g\|_{L^\infty} \|h_j\|_{L^2}$$

$$(3.18) \quad \leqslant \sum_{k \geqslant 0} \sum_{\substack{j \geqslant 1, \, \ell \geqslant 0 \\ \ell \leqslant j+1}} \max\{1, 2^{(\ell-j\rho)n/2}\} 2^{jm-kN} \|f_\ell\|_{L^2} \|g\|_{L^\infty} \|h_j\|_{L^2}.$$

Under our assumption $m = -(1-\rho)n/2 < 0$, it holds that

$$(3.19) \quad \sum_{j \geqslant 1} \mathbb{1}\{\ell \leqslant j+1\} \max\{1, 2^{(\ell-j\rho)n/2}\} 2^{jm} \approx 1 \quad \text{for all} \quad \ell \geqslant 0$$

and

$$(3.20) \quad \sum_{\ell \geqslant 0} \mathbb{1}\{\ell \leqslant j+1\} \max\{1, 2^{(\ell-j\rho)n/2}\} 2^{jm} \approx 1 \quad \text{for all} \quad j \geqslant 1.$$

Hence, by Schur's lemma, (3.18) is bounded by

$$\sum_{k \geqslant 0} 2^{-kN} \left(\sum_{\ell \geqslant 0} \|f_\ell\|_{L^2}^2 \right)^{1/2} \left(\sum_{j \geqslant 1} \|h_j\|_{L^2}^2 \right)^{1/2} \|g\|_{L^\infty} \lesssim \|f\|_{L^2} \|h\|_{L^2} \|g\|_{L^\infty}.$$

This completes the proof for the first case.

$$\text{The case } \operatorname{supp}^* \sigma \subset \Gamma(V_2)$$

In this case, all $(\xi, \eta) \in \operatorname{supp}^* \sigma$ satisfy $|\xi| \approx (|\xi|^2 + |\eta|^2)^{1/2}$ (but $|\xi + \eta|$ may be small compared with $(|\xi|^2 + |\eta|^2)^{1/2}$). We divide the sum over (j, k) in (3.15) into two parts $k \leqslant j(1-\rho)$ and $k > j(1-\rho)$ and write the trilinear form $\langle T_\sigma(f, g), h \rangle$ as

$$\langle T_\sigma(f, g), h \rangle = \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k \leqslant j(1-\rho)}} \sum_\nu \langle T_{\sigma_{j,k,\nu}}(f, g), h \rangle + \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k > j(1-\rho)}} \sum_\nu \langle T_{\sigma_{j,k,\nu}}(f, g), h \rangle$$

$$= Y_1 + Y_2, \quad \text{say}.$$

As in the first case, the estimate for the second term Y_2 is easy. In fact, Lemma 3.7 gives

$$|Y_2| \leqslant \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k > j(1-\rho)}} \left\| \sum_\nu T_{\sigma_{j,k,\nu}}(f, g) \right\|_{L^2} \|h\|_{L^2}$$

$$\lesssim \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k > j(1-\rho)}} 2^{-kN} \|f\|_{L^2} \|g\|_{L^\infty} \|h\|_{L^2} \approx \|f\|_{L^2} \|g\|_{L^\infty} \|h\|_{L^2},$$

where the last $\approx$ holds because $1 - \rho > 0$.

In order to estimate Y_1, we use the decomposition

$$h = \sum_\ell h_\ell, \quad h_\ell = \psi_\ell(D)h,$$

and write

$$Y_1 = \sum_{\substack{j \geqslant 1,\, k \geqslant 0 \\ k \leqslant j(1-\rho)}} \sum_{\ell \geqslant 0} \sum_{\nu = (\nu_1, \nu_2) \in \mathbb{Z}^n \times \mathbb{Z}^n} \langle T_{\sigma_{j,k,\nu}}(f, g), h_\ell \rangle.$$

Here observe the following. Firstly, since $|\xi| \approx (|\xi|^2 + |\eta|^2)^{1/2}$ for $(\xi, \eta) \in \operatorname{supp}^* \sigma$, there exists a positive integer b such that $\operatorname{supp}^*(\sigma_{j,k,\nu}) \subset \{(\xi, \eta) : 2^{j-b} \leqslant |\xi| \leqslant 2^{j+b}\}$. Hence, by (3.6),

$$T_{\sigma_{j,k,\nu}}(f, g) = T_{\sigma_{j,k,\nu}}(f_j, g), \quad f_j = \theta(2^{-j}D)f,$$

where θ is an appropriate function supported in an annulus. Secondly, if $k \leqslant j(1 - \rho)$, then (3.8) yields

$$\operatorname{supp} \mathcal{F}[T_{\sigma_{j,k,\nu}}(f_j, g)] \subset \bigcup_{(|\xi|^2 + |\eta|^2)^{1/2} \leqslant 2^{j+1}} \{\zeta \in \mathbb{R}^n : |\zeta - \xi - \eta| \leqslant 2^{j\rho + k + 1}\}$$

$$\subset \{\zeta \in \mathbb{R}^n : |\zeta| \leqslant 2^{j+3}\},$$

which together with the fact $\operatorname{supp} \widehat{h}_\ell \subset \operatorname{supp} \psi_\ell$ implies $\langle T_{\sigma_{j,k,\nu}}(f_j, g), h_\ell \rangle = 0$ for $\ell > j + 3$. Thirdly, as we have already seen, (3.13) holds, and hence, by (3.9),

$$\langle T_{\sigma_{j,k,\nu}}(f_j, g), h_\ell \rangle \neq 0 \implies (2^{j\rho}(\nu_1 + \nu_2) + 2^{j\rho + k + 2}Q) \cap \operatorname{supp} \psi_\ell \neq \emptyset.$$

Combining these observations, we see that Y_1 can be written as

$$(3.21) \qquad Y_1 = \sum_{\substack{j \geqslant 1,\, k \geqslant 0 \\ k \leqslant j(1-\rho)}} \sum_{\ell=0}^{j+3} \sum_{\mu \in \Lambda_{j,k,\ell}} \sum_{\nu_1 + \nu_2 = \mu} \langle T_{\sigma_{j,k,\nu}}(f_j, g), h_\ell \rangle,$$

where

$$\Lambda_{j,k,\ell} = \{\mu \in \mathbb{Z}^n : (2^{j\rho}\mu + 2^{j\rho + k + 2}Q) \cap \operatorname{supp} \psi_\ell \neq \emptyset\}.$$

The number of elements of $\Lambda_{j,k,\ell}$ is estimated by

$$|\Lambda_{j,k,\ell}| \lesssim (\max\{2^k, 2^{\ell - j\rho}\})^n \lesssim 2^{kn} \max\{1, 2^{(\ell - j\rho)n}\}.$$

Thus Lemma 3.6 (2) gives

$$\left| \sum_{\mu \in \Lambda_{j,k,\ell}} \sum_{\nu_1 + \nu_2 = \mu} \langle T_{\sigma_{j,k,\nu}}(f_\ell, g), h_j \rangle \right|$$

$$\leqslant \left\| \sum_{\mu \in \Lambda_{j,k,\ell}} \sum_{\nu_1 + \nu_2 = \mu} T_{\sigma_{j,k,\nu}}(f_\ell, g) \right\|_{L^2} \|h_j\|_{L^2}$$

$$\lesssim \max\{1, 2^{(\ell - j\rho)n/2}\} 2^{jm - k(N - n/2)} \|f_\ell\|_{L^2} \|g\|_{L^\infty} \|h_j\|_{L^2}.$$

Hence

$$|Y_1| \lesssim \sum_{\substack{j \geqslant 1, \, k \geqslant 0 \\ k \leqslant j(1-\rho)}} \sum_{\ell=0}^{j+3} \max\{1, 2^{(\ell - j\rho)n/2}\} 2^{jm - k(N - n/2)} \|f_\ell\|_{L^2} \|g\|_{L^\infty} \|h_j\|_{L^2}$$

$$(3.22) \quad \leqslant \sum_{k \geqslant 0} \sum_{\substack{j \geqslant 1, \, \ell \geqslant 0 \\ \ell \leqslant j+3}} \max\{1, 2^{(\ell - j\rho)n/2}\} 2^{jm - k(N - n/2)} \|f_\ell\|_{L^2} \|g\|_{L^\infty} \|h_j\|_{L^2}.$$

Since (3.19) and (3.20) hold if $\ell \leqslant j + 1$ is replaced by $\ell \leqslant j + 3$, by Schur's lemma, (3.22) is bounded by

$$\sum_{k \geqslant 0} 2^{-k(N - n/2)} \left(\sum_{\ell \geqslant 0} \|f_\ell\|_{L^2}^2 \right)^{1/2} \left(\sum_{j \geqslant 1} \|h_j\|_{L^2}^2 \right)^{1/2} \|g\|_{L^\infty}$$

$$\lesssim \|f\|_{L^2} \|h\|_{L^2} \|g\|_{L^\infty},$$

which gives the desired estimate for Y_1. This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3. The main scheme of the arguments is the same as that of Naibo [11]. In the last step, we introduce a new idea of using weak type estimates.

Since the theorem is already proved in the case $\rho = 0$ (see [10]), for the rest of this section, we assume $0 < \rho < 1$, $m = -(1 - \rho)n$, and $\sigma \in BS_{\rho,\rho}^m(\mathbb{R}^n)$.

Using the function Ψ_j of Subsection 3.1, we decompose σ as

$$(4.1) \qquad \sigma(x, \xi, \eta) = \sum_{j=0}^{\infty} \sigma_j(x, \xi, \eta),$$

$$(4.2) \qquad \sigma_j(x, \xi, \eta) = \sigma(x, \xi, \eta) \Psi_j(\xi, \eta).$$

We write the inverse Fourier transform of σ_j with respect to (ξ, η) as

$$K_j(x, y, z) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(y \cdot \xi + z \cdot \eta)} \sigma_j(x, \xi, \eta) \, \mathrm{d}\xi \mathrm{d}\eta.$$

First, we shall prove that K_j satisfy the following estimates:

$$(4.3) \qquad \|(1 + 2^{j\rho}|y|)^{N_1}(1 + 2^{j\rho}|z|)^{N_2} K_j(x, y, z)\|_{L^2_{y,z}} \lesssim 2^{j(m+n)},$$

$$(4.4) \qquad \|(1 + 2^{j\rho}|y|)^{N_1}(1 + 2^{j\rho}|z|)^{N_2} \nabla_x K_j(x, y, z)\|_{L^2_{y,z}} \lesssim 2^{j(\rho+m+n)},$$

$$(4.5) \qquad \|(1 + 2^{j\rho}|y|)^{N_1}(1 + 2^{j\rho}|z|)^{N_2} \nabla_y K_j(x, y, z)\|_{L^2_{y,z}} \lesssim 2^{j(1+m+n)},$$

$$(4.6) \qquad \|(1 + 2^{j\rho}|y|)^{N_1}(1 + 2^{j\rho}|z|)^{N_2} \nabla_z K_j(x, y, z)\|_{L^2_{y,z}} \lesssim 2^{j(1+m+n)},$$

where $\nabla_x, \nabla_y, \nabla_z$ denote the gradient operator with respect to x, y, z respectively, and N_1 and N_2 can be arbitrary nonnegative real numbers.

To prove (4.3), observe that $1 + |\xi| + |\eta| \approx 2^j$ for all $(\xi, \eta) \in \mathrm{supp}^*(\sigma_j)$ and σ_j satisfies the estimate

$$|\partial_\xi^\beta \partial_\eta^\gamma \sigma_j(x, \xi, \eta)| \lesssim (2^j)^{m - \rho|\beta| - \rho|\gamma|} \mathbb{1}\{1 + |\xi| + |\eta| \approx 2^j\}.$$

Taking inverse Fourier transform with respect to (ξ, η) and using Plancherel's theorem, we obtain

$$\|(2^{j\rho}y)^\beta (2^{j\rho}z)^\gamma K_j(x, y, z)\|_{L^2_{y,z}} \lesssim (2^j)^{m+n},$$

from which (4.3) follows. The estimates (4.4), (4.5), and (4.6) can be derived from the estimates

$$|\partial_\xi^\beta \partial_\eta^\gamma \nabla_x \sigma_j(x, \xi, \eta)| \lesssim (2^j)^{m+\rho-\rho|\beta|-\rho|\gamma|} \mathbb{1}\{1 + |\xi| + |\eta| \approx 2^j\},$$

$$|\partial_\xi^\beta \partial_\eta^\gamma \{\xi \sigma_j(x, \xi, \eta)\}| \lesssim (2^j)^{m+1-\rho|\beta|-\rho|\gamma|} \mathbb{1}\{1 + |\xi| + |\eta| \approx 2^j\},$$

$$|\partial_\xi^\beta \partial_\eta^\gamma \{\eta \sigma_j(x, \xi, \eta)\}| \lesssim (2^j)^{m+1-\rho|\beta|-\rho|\gamma|} \mathbb{1}\{1 + |\xi| + |\eta| \approx 2^j\}$$

in the same way.

Now we proceed to the proof of the $L^\infty \times L^\infty \to \mathrm{BMO}$ boundedness of T_σ. Let f, g be functions satisfying $\|f\|_{L^\infty} = \|g\|_{L^\infty} = 1$ and let Q be a cube in $\mathbb{R}^n$. We denote by $\ell(Q)$ the side length of Q, and by x_Q the center of Q. It is sufficient to prove that there exists a complex number C_Q such that

$$\frac{1}{|Q|} \int_Q |T_\sigma(f, g)(x) - C_Q| \, \mathrm{d}x \lesssim 1.$$

We write $h = \ell(Q)$ and take the cube $\widetilde{Q}$ with the same center as Q and with the sidelength

$$\ell(\widetilde{Q}) = \begin{cases} 2h^\rho & \text{if } h \leqslant 1, \\ 2h & \text{if } h > 1. \end{cases}$$

We divide f and g as

$$f = f1_{\tilde{Q}} + f1_{\tilde{Q}^c} = f^{(0)} + f^{(1)},$$
$$g = g1_{\tilde{Q}} + g1_{\tilde{Q}^c} = g^{(0)} + g^{(1)},$$

and divide $T_\sigma(f, g)$ into four parts

$$T_\sigma(f, g) = T_\sigma(f^{(0)}, g^{(0)}) + T_\sigma(f^{(0)}, g^{(1)}) + T_\sigma(f^{(1)}, g^{(0)}) + T_\sigma(f^{(1)}, g^{(1)})$$
$$= F^{(1)} + F^{(2)} + F^{(3)} + F^{(4)}, \quad \text{say.}$$

For each $i = 1, 2, 3, 4$, we shall show that there exists a complex number $C_Q^{(i)}$ such that

$$(4.7) \qquad \frac{1}{|Q|} \int_Q |F^{(i)}(x) - C_Q^{(i)}| \, dx \lesssim 1.$$

We divide the argument into two cases, $h > 1$ and $h \leqslant 1$.

The case $h = \ell(Q) > 1$

In this case, we shall prove (4.7) with $C_Q^{(i)} = 0$ for all i.

Estimate for $F^{(4)}$. — We have

$$F^{(4)}(x) = T_\sigma(f^{(1)}, g^{(1)})(x) = \sum_{j=0}^{\infty} T_{\sigma_j}(f^{(1)}, g^{(1)})(x).$$

Using the kernel K_j and using Schwarz's inequality, we have

$$|T_{\sigma_j}(f^{(1)}, g^{(1)})(x)| = \left| \int_{\substack{y \in \tilde{Q}^c \\ z \in \tilde{Q}^c}} K_j(x, x - y, x - z) f(y) g(z) \, dydz \right|$$

$$\leqslant \left\| h^n \left(\frac{|x-y|}{h} \right)^{N_1} \left(\frac{|x-z|}{h} \right)^{N_2} K_j(x, x - y, x - z) \right\|_{L^2(y \in \tilde{Q}^c, \, z \in \tilde{Q}^c)}$$

$$\times \left\| h^{-n} \left(\frac{|x-y|}{h} \right)^{-N_1} \left(\frac{|x-z|}{h} \right)^{-N_2} f(y) g(z) \right\|_{L^2(y \in \tilde{Q}^c, \, z \in \tilde{Q}^c)},$$

where $N_1, N_2 \geqslant 0$ can be taken arbitrarily. The first L^2-norm above is estimated by (4.3) as

$$
\left\| h^n \left(\frac{|x-y|}{h} \right)^{N_1} \left(\frac{|x-z|}{h} \right)^{N_2} K_j(x, x-y, x-z) \right\|_{L^2(y \in \tilde{Q}^c, z \in \tilde{Q}^c)}
$$
$$
\leqslant h^n (2^{j\rho}h)^{-N_1}(2^{j\rho}h)^{-N_2} \|(2^{j\rho}|y|)^{N_1}(2^{j\rho}|z|)^{N_2} K_j(x, y, z)\|_{L^2_{y,z}}
$$
$$
\lesssim h^n (2^{j\rho}h)^{-N_1}(2^{j\rho}h)^{-N_2} 2^{j(m+n)} = (2^{j\rho}h)^{-N_1-N_2+n},
$$

where the last equality holds because of our assumption $m = -(1-\rho)n$. If we take $N_1, N_2 > n/2$, then, for $x \in Q$, the second L^2-norm is estimated as

$$
\left\| h^{-n} \left(\frac{|x-z|}{h} \right)^{-N_1} \left(\frac{|x-z|}{h} \right)^{-N_2} f(y)g(z) \right\|_{L^2(y \in \tilde{Q}^c, z \in \tilde{Q}^c)}
$$
$$
\leqslant \left\| h^{-n} \left(\frac{|x-y|}{h} \right)^{-N_1} \left(\frac{|x-z|}{h} \right)^{-N_2} \right\|_{L^2(y \in \tilde{Q}^c, z \in \tilde{Q}^c)} \approx 1.
$$

Thus, by taking $N_1, N_2 > n/2$, we obtain the pointwise estimate

$$
|F^{(4)}(x)| \leqslant \sum_{j=0}^{\infty} |T_{\sigma_j}(f^{(1)}, g^{(1)})(x)| \lesssim \sum_{j=0}^{\infty} (2^{j\rho}h)^{-N_1-N_2+n} \approx h^{-N_1-N_2+n} \leqslant 1
$$

for all $x \in Q$. This certainly implies (4.7) for $i = 4$ with $C_Q^{(4)} = 0$.

Estimate for $F^{(2)}$ and $F^{(3)}$. — By symmetry, we consider only $F^{(2)}$. We write

$$
F^{(2)}(x) = T_\sigma(f^{(0)}, g^{(1)})(x) = \sum_{j=0}^{\infty} T_{\sigma_j}(f^{(0)}, g^{(1)})(x).
$$

By Schwarz's inequality, we have

$$
|T_{\sigma_j}(f^{(0)}, g^{(1)})(x)| = \left| \iint_{\substack{y \in \tilde{Q} \\ z \in \tilde{Q}^c}} K_j(x, x-y, x-z) f(y)g(z) \, dydz \right|
$$
$$
\leqslant \left\| h^n \left(\frac{|x-z|}{h} \right)^{N_2} K_j(x, x-y, x-z) \right\|_{L^2(y \in \tilde{Q}, z \in \tilde{Q}^c)}
$$
$$
\times \left\| h^{-n} \left(\frac{|x-z|}{h} \right)^{-N_2} f(y)g(z) \right\|_{L^2(y \in \tilde{Q}, z \in \tilde{Q}^c)},
$$

where $N_2 \geqslant 0$ can be taken arbitrarily. The first L^2-norm above is estimated by (4.3) as

$$\left\| h^n \left(\frac{|x-z|}{h} \right)^{N_2} K_j(x, x-y, x-z) \right\|_{L^2(y \in \tilde{Q}, \, z \in \tilde{Q}^c)}$$
$$\leqslant h^n (2^{j\rho} h)^{-N_2} \|(2^{j\rho}|z|)^{N_2} K_j(x, y, z)\|_{L^2_{y,z}}$$
$$\lesssim h^n (2^{j\rho} h)^{-N_2} 2^{j(m+n)} = (2^{j\rho} h)^{-N_2+n}.$$

If we take $N_2 > n/2$, then, for $x \in Q$, the second L^2-norm is estimated as

$$\left\| h^{-n} \left(\frac{|x-z|}{h} \right)^{-N_2} f(y)g(z) \right\|_{L^2(y \in \tilde{Q}, \, z \in \tilde{Q}^c)}$$
$$\leqslant \left\| h^{-n} \left(\frac{|x-z|}{h} \right)^{-N_2} \right\|_{L^2(y \in \tilde{Q}, \, z \in \tilde{Q}^c)} \approx 1.$$

Thus, by taking $N_2 > n$, we obtain

$$|F^{(2)}(x)| \leqslant \sum_{j=0}^{\infty} |T_{\sigma_j}(f^{(0)}, g^{(1)})(x)| \lesssim \sum_{j=0}^{\infty} (2^{j\rho} h)^{-N_2+n} \approx h^{-N_2+n} \leqslant 1$$

for all $x \in Q$. This implies (4.7) for $i = 2$ with $C_Q^{(2)} = 0$.

Estimate for $F^{(1)}$. — Since $m = -(1-\rho)n < -(1-\rho)n/2 = m_\rho(2,2)$, the operator T_σ is bounded in $L^2 \times L^2 \to L^1$ (see Proposition 1.1). Hence

$$\frac{1}{|Q|} \int_Q |F^{(1)}(x)| \, \mathrm{d}x \leqslant |Q|^{-1} \|T_\sigma(f^{(0)}, g^{(0)})\|_{L^1} \lesssim |Q|^{-1} \|f^{(0)}\|_{L^2} \|g^{(0)}\|_{L^2} \lesssim 1.$$

The case $h = \ell(Q) \leqslant 1$

Estimate for $F^{(4)}$. — We shall prove the estimate (4.7) for $i = 4$ with $C_Q^{(4)} = F^{(4)}(x_Q)$. In the following, x always denotes arbitrary point in Q.

To estimate $F^{(4)}(x) - F^{(4)}(x_Q)$, we write

$$F^{(4)}(x) - F^{(4)}(x_Q) = \sum_{j=0}^{\infty} (T_{\sigma_j}(f^{(1)}, g^{(1)})(x) - T_{\sigma_j}(f^{(1)}, g^{(1)})(x_Q))$$
$$= \sum_{j=0}^{\infty} \int_{\substack{y \in \tilde{Q}^c \\ z \in \tilde{Q}^c}} H_{j,Q}(x, y, z) f(y) g(z) \, \mathrm{d}y \mathrm{d}z,$$

where

$$(4.8) \qquad H_{j,Q}(x,y,z) = K_j(x, x-y, x-z) - K_j(x_Q, x_Q - y, x_Q - z).$$

By Schwarz's inequality,

$$(4.9) \quad |T_{\sigma_j}(f^{(1)}, g^{(1)})(x) - T_{\sigma_j}(f^{(1)}, g^{(1)})(x_Q)|$$

$$\leqslant \left\| h^{\rho n} \left(\frac{|x-y|}{h^\rho} \right)^{N_1} \left(\frac{|x-z|}{h^\rho} \right)^{N_2} H_{j,Q}(x,y,z) \right\|_{L^2(y \in \tilde{Q}^c, z \in \tilde{Q}^c)}$$

$$\times \left\| h^{-\rho n} \left(\frac{|x-y|}{h^\rho} \right)^{-N_1} \left(\frac{|x-z|}{h^\rho} \right)^{-N_2} f(y)g(z) \right\|_{L^2(y \in \tilde{Q}^c, z \in \tilde{Q}^c)}.$$

Since $\|f\|_\infty = \|g\|_\infty = 1$, if we take $N_1, N_2 > n/2$, the latter L^2-norm of (4.9) is $\lesssim 1$.

In order to estimate the former L^2-norm of (4.9), we write

$$(4.10) \quad H_{j,Q}(x,y,z)$$

$$= \int_0^1 \nabla K_j(x(t), x(t) - y, x(t) - z) \cdot (x - x_Q, x - x_Q, x - x_Q) \, dt,$$

where we used the notation $x(t) = x_Q + t(x - x_Q)$ and

$$\nabla K_j(x,y,z) \cdot (u,v,w)$$
$$= \nabla_x K_j(x,y,z) \cdot u + \nabla_y K_j(x,y,z) \cdot v + \nabla_z K_j(x,y,z) \cdot w.$$

Notice that $|x-y| \approx |x(t) - y|$ and $|x-z| \approx |x(t) - z|$ for all $y, z \in \tilde{Q}^c$ and $0 < t < 1$. Hence, by (4.10) and by (4.4), (4.5), (4.6), we can estimate the former L^2-norm of (4.9) as follows: (here $\|\cdots\|_{L^2(*)}$ means $\|\cdots\|_{L^2(y \in \tilde{Q}^c, z \in \tilde{Q}^c)}$)

$$\left\| h^{\rho n} \left(\frac{|x-y|}{h^\rho} \right)^{N_1} \left(\frac{|x-z|}{h^\rho} \right)^{N_2} H_{j,Q}(x,y,z) \right\|_{L^2(*)}$$

$$\lesssim \left\| h^{1+\rho n} \left(\frac{|x-y|}{h^\rho} \right)^{N_1} \left(\frac{|x-z|}{h^\rho} \right)^{N_2} \int_0^1 |\nabla K_j(x(t), x(t) - y, x(t) - z)| \, dt \right\|_{L^2(*)}$$

$$\approx \left\| h^{1+\rho n} \int_0^1 \left(\frac{|x(t)-y|}{h^\rho} \right)^{N_1} \left(\frac{|x(t)-z|}{h^\rho} \right)^{N_2} |\nabla K_j(x(t), x(t)-y, x(t)-z)| \, dt \right\|_{L^2(*)}$$

$$\leqslant h^{1+\rho n} \int_0^1 \left\| \left(\frac{|x(t)-y|}{h^\rho} \right)^{N_1} \left(\frac{|x(t)-z|}{h^\rho} \right)^{N_2} \nabla K_j(x(t), x(t)-y, x(t)-z) \right\|_{L^2(*)} dt$$

$$\lesssim h^{1+\rho n} (2^{j\rho} h^\rho)^{-N_1} (2^{j\rho} h^\rho)^{-N_2} 2^{j(1+m+n)} = (2^{j\rho} h^\rho)^{-N_1 - N_2 + n + 1/\rho},$$

where we used the assumption $m = -(1-\rho)n$ to obtain the last equality. On the other hand, if we use (4.3), then we can estimate the former L^2-norm of (4.9) as follows: (the notation $\|\cdots\|_{L^2(*)}$ is the same as above)

$$\left\| h^{\rho n} \left(\frac{|x-y|}{h^\rho} \right)^{N_1} \left(\frac{|x-z|}{h^\rho} \right)^{N_2} H_{j,Q}(x,y,z) \right\|_{L^2(*)}$$

$$\leqslant \left\| h^{\rho n} \left(\frac{|x-y|}{h^\rho} \right)^{N_1} \left(\frac{|x-z|}{h^\rho} \right)^{N_2} K_j(x, x-y, x-z) \right\|_{L^2(*)}$$

$$+ \left\| h^{\rho n} \left(\frac{|x-y|}{h^\rho} \right)^{N_1} \left(\frac{|x-z|}{h^\rho} \right)^{N_2} K_j(x_Q, x_Q-y, x_Q-z) \right\|_{L^2(*)}$$

$$\approx \left\| h^{\rho n} \left(\frac{|x-y|}{h^\rho} \right)^{N_1} \left(\frac{|x-z|}{h^\rho} \right)^{N_2} K_j(x, x-y, x-z) \right\|_{L^2(*)}$$

$$+ \left\| h^{\rho n} \left(\frac{|x_Q-y|}{h^\rho} \right)^{N_1} \left(\frac{|x_Q-z|}{h^\rho} \right)^{N_2} K_j(x_Q, x_Q-y, x_Q-z) \right\|_{L^2(*)}$$

$$\lesssim h^{\rho n} (2^{j\rho} h^\rho)^{-N_1} (2^{j\rho} h^\rho)^{-N_2} 2^{j(m+n)} = (2^{j\rho} h^\rho)^{-N_1-N_2+n}.$$

Combining the above estimates, we have the following estimates for arbitrary $N_1, N_2 > n/2$:

$$|T_{\sigma_j}(f^{(1)}, g^{(1)})(x) - T_{\sigma_j}(f^{(1)}, g^{(1)})(x_Q)|$$
$$\lesssim \min\{ (2^{j\rho} h^\rho)^{-N_1-N_2+n+1/\rho}, (2^{j\rho} h^\rho)^{-N_1-N_2+n} \}.$$

Now we take $N_1 = N_2 = N$ such that $-2N + n + 1/\rho > 0 > -2N + n$. Then taking the sum of the above estimates over $j \geqslant 0$, we obtain

$$|F^{(4)}(x) - F^{(4)}(x_Q)|$$
$$\leqslant \sum_{j=0}^{\infty} |T_{\sigma_j}(f^{(1)}, g^{(1)})(x) - T_{\sigma_j}(f^{(1)}, g^{(1)})(x_Q)| \lesssim 1, \quad x \in Q,$$

which a fortiori implies (4.7) for $i = 4$ with $C_Q^{(4)} = F^{(4)}(x_Q)$.

Estimate for $F^{(2)}$ and $F^{(3)}$. — By symmetry, we consider only $F^{(2)}$. We shall prove the estimate (4.7) for $i = 2$ with $C_Q^{(2)} = F^{(2)}(x_Q)$. In the following, x always denotes arbitrary point in Q.

We write

$$F^{(2)}(x) - F^{(2)}(x_Q) = \sum_{j=0}^{\infty} (T_{\sigma_j}(f^{(0)}, g^{(1)})(x) - T_{\sigma_j}(f^{(0)}, g^{(1)})(x_Q))$$

$$= \sum_{j=0}^{\infty} \int_{\substack{y \in \tilde{Q} \\ z \in \tilde{Q}^c}} H_{j,Q}(x, y, z) f(y) g(z) \, \mathrm{d}y\mathrm{d}z$$

with $H_{j,Q}$ given by (4.8).

By Schwarz's inequality, we have

$$(4.11) \quad |T_{\sigma_j}(f^{(0)}, g^{(1)})(x) - T_{\sigma_j}(f^{(0)}, g^{(1)})(x_Q)|$$

$$\leqslant \left\| h^{\rho n} \left(\frac{|x - z|}{h^\rho} \right)^{N_2} H_{j,Q}(x, y, z) \right\|_{L^2(y \in \tilde{Q}, \, z \in \tilde{Q}^c)}$$

$$\times \left\| h^{-\rho n} \left(\frac{|x - z|}{h^\rho} \right)^{-N_2} f(y) g(z) \right\|_{L^2(y \in \tilde{Q}, \, z \in \tilde{Q}^c)}.$$

Since $\|f\|_\infty = \|g\|_\infty = 1$, if we take $N_2 > n/2$, then the latter L^2-norm of (4.11) is $\lesssim 1$.

By (4.10) and by (4.4), (4.5), and (4.6), we can estimate the former L^2-norm of (4.11) as

$$\left\| h^{\rho n} \left(\frac{|x - z|}{h^\rho} \right)^{N_2} H_{j,Q}(x, y, z) \right\|_{L^2(y \in \tilde{Q}, \, z \in \tilde{Q}^c)}$$

$$\lesssim \left\| h^{1+\rho n} \left(\frac{|x - z|}{h^\rho} \right)^{N_2} \int_0^1 |\nabla K_j(x(t), x(t) - y, x(t) - z)| \, \mathrm{d}t \right\|_{L^2(y \in \tilde{Q}, \, z \in \tilde{Q}^c)}$$

$$\approx \left\| h^{1+\rho n} \int_0^1 \left(\frac{|x(t) - z|}{h^\rho} \right)^{N_2} |\nabla K_j(x(t), x(t) - y, x(t) - z)| \, \mathrm{d}t \right\|_{L^2(y \in \tilde{Q}, \, z \in \tilde{Q}^c)}$$

$$\leqslant h^{1+\rho n} \int_0^1 \left\| \left(\frac{|x(t) - z|}{h^\rho} \right)^{N_2} \nabla K_j(x(t), x(t) - y, x(t) - z) \right\|_{L^2(y \in \tilde{Q}, \, z \in \tilde{Q}^c)} \, \mathrm{d}t$$

$$\lesssim h^{1+\rho n} (2^{j\rho} h^\rho)^{-N_2} 2^{j(1+m+n)} = (2^{j\rho} h^\rho)^{-N_2 + n + 1/\rho}.$$

On the other hand, using (4.3), we can estimate the former L^2-norm of (4.11) as

$$\left\| h^{\rho n}\left(\frac{|x-z|}{h^\rho}\right)^{N_2} H_{j,Q}(x,y,z)\right\|_{L^2(y\in\tilde{Q},\,z\in\tilde{Q}^c)}$$

$$\leqslant \left\| h^{\rho n}\left(\frac{|x-z|}{h^\rho}\right)^{N_2} K_j(x,x-y,x-z)\right\|_{L^2(y\in\tilde{Q},\,z\in\tilde{Q}^c)}$$

$$+ \left\| h^{\rho n}\left(\frac{|x-z|}{h^\rho}\right)^{N_2} K_j(x_Q,x_Q-y,x_Q-z)\right\|_{L^2(y\in\tilde{Q},\,z\in\tilde{Q}^c)}$$

$$\approx \left\| h^{\rho n}\left(\frac{|x-z|}{h^\rho}\right)^{N_2} K_j(x,x-y,x-z)\right\|_{L^2(y\in\tilde{Q},\,z\in\tilde{Q}^c)}$$

$$+ \left\| h^{\rho n}\left(\frac{|x_Q-z|}{h^\rho}\right)^{N_2} K_j(x_Q,x_Q-y,x_Q-z)\right\|_{L^2(y\in\tilde{Q},\,z\in\tilde{Q}^c)}$$

$$\lesssim h^{\rho n}(2^{j\rho}h^\rho)^{-N_2}2^{j(m+n)} = (2^{j\rho}h^\rho)^{-N_2+n}.$$

Combining the above estimates, we have the estimates

$$|T_{\sigma_j}(f^{(0)},g^{(1)})(x) - T_{\sigma_j}(f^{(0)},g^{(1)})(x_Q)|$$
$$\lesssim \min\{(2^{j\rho}h^\rho)^{-N_2+n+1/\rho}, (2^{j\rho}h^\rho)^{-N_2+n}\}$$

for arbitrary $N_2 > n/2$.

Now we take N_2 such that $-N_2 + n + 1/\rho > 0 > -N_2 + n$ and take the sum of the above estimates over $j \geqslant 0$ to obtain

$$|F^{(2)}(x) - F^{(2)}(x_Q)|$$
$$\leqslant \sum_{j=0}^\infty |T_{\sigma_j}(f^{(0)},g^{(1)})(x) - T_{\sigma_j}(f^{(0)},g^{(1)})(x_Q)| \lesssim 1, \quad x\in Q,$$

which a fortiori implies (4.7) for $i=2$ with $C_Q^{(2)} = F^{(2)}(x_Q)$.

Estimate for $F^{(1)}$. — We first prove an L^2 estimate of $T_{\sigma_j}(f^{(0)},g^{(0)})$. Let $\tilde{\sigma}_j$ be the symbol

$$(4.12) \qquad \tilde{\sigma}_j(x,\xi,\eta) = \sigma_j(2^{-j\rho}x, 2^{j\rho}\xi, 2^{j\rho}\eta).$$

Then a simple change of variables gives

$$(4.13) \qquad T_\sigma(a,b)(2^{-j\rho}x) = T_{\tilde{\sigma}}(a(2^{-j\rho}\cdot), b(2^{-j\rho}\cdot))(x),$$

which implies

$$\|T_{\sigma_j}\|_{L^2 \times L^\infty \to L^2} = \|T_{\widetilde{\sigma}_j}\|_{L^2 \times L^\infty \to L^2}.$$

Since $1 + |\xi| + |\eta| \approx 2^j$ for all $(\xi, \eta) \in \operatorname{supp}^*(\sigma_j)$, we see that $\widetilde{\sigma}_j$ satisfies

(4.14)
$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \widetilde{\sigma}_j(x, \xi, \eta)| \lesssim 2^{jm} \mathbb{1}\{1 + |\xi| + |\eta| \approx 2^{j(1-\rho)}\}$$
$$\lesssim 2^{-j(1-\rho)n/2}(1 + |\xi| + |\eta|)^{-n/2}.$$

Hence the theorem of [10] or the case $\rho = 0$ of Theorem 1.2 yields

$$\|T_{\widetilde{\sigma}_j}\|_{L^2 \times L^\infty \to L^2} \lesssim 2^{-j(1-\rho)n/2}.$$

Thus we obtain

(4.15)
$$\|T_{\sigma_j}(f^{(0)}, g^{(0)})\|_{L^2} \lesssim 2^{-j(1-\rho)n/2}\|f^{(0)}\|_{L^2}\|g^{(0)}\|_{L^\infty}$$
$$\leqslant 2^{-j(1-\rho)n/2}|\widetilde{Q}|^{1/2} \approx 2^{-j(1-\rho)n/2}|Q|^{\rho/2}.$$

Next we prove an L^∞ estimate of $T_{\sigma_j}(f^{(0)}, g^{(0)})$. From the formula

$$T_{\sigma_j}(a, b)(x) = \int_{\mathbb{R}^n} K_j(x, x - y, x - z)a(y)b(z)\,\mathrm{d}y\mathrm{d}z$$

and from (4.3), we have

$$|T_{\sigma_j}(a, b)(x)| \leqslant \|K_j(x, x - y, x - z)\|_{L^2_{y,z}}\|a(y)b(z)\|_{L^2_{y,z}}$$
$$\lesssim 2^{j(m+n)}\|a\|_{L^2}\|b\|_{L^2} = 2^{j\rho n}\|a\|_{L^2}\|b\|_{L^2}.$$

Hence

(4.16) $\quad \|T_{\sigma_j}(f^{(0)}, g^{(0)})\|_{L^\infty} \lesssim 2^{j\rho n}\|f^{(0)}\|_2\|g^{(0)}\|_2 \leqslant 2^{j\rho n}|\widetilde{Q}| \approx 2^{j\rho n}|Q|^\rho.$

Now by a characterization of weak L^p functions (see Lemma 4.1 to be given below), the estimates (4.15) and (4.16) imply the following weak type estimate for $F^{(1)} = \sum_{j=0}^\infty T_{\sigma_j}(f^{(0)}, g^{(0)})$:

$$|\{x \in \mathbb{R}^n : |F^{(1)}(x)| > \lambda\}| \lesssim |Q|\lambda^{-1-1/\rho}, \quad \lambda > 0.$$

From this we obtain

$$\frac{1}{|Q|}\int_Q |F^{(1)}(x)|\,\mathrm{d}x = \int_0^\infty |Q|^{-1}|\{x \in Q : |F^{(1)}(x)| > \lambda\}|\,\mathrm{d}\lambda$$
$$\leqslant \int_0^\infty \min\{1, \lambda^{-1-1/\rho}\}\,\mathrm{d}\lambda \approx 1,$$

which is the estimate (4.7) for $i = 1$ and $C_Q^{(1)} = 0$. This completes the proof of Theorem 1.3.

Finally we shall give a proof of the fact that was used at the last part of the above argument. Here we shall give a slightly general lemma. This lemma is equivalent to the fact that the space $L^{(p,\infty)}$ is equal to the real interpolation space $[L^\infty, L^r]_{\theta,\infty}$, $1/p = \theta/r$, combined with the characterization of the latter space by the J-method. Although this may be known to many people, we shall give a proof for reader's convenience.

LEMMA 4.1. — Let $0 < r < p < \infty$, $\alpha, \beta \in (0,\infty)$, and $0 < \theta < 1$ satisfy $1/p = \theta/r$ and $\alpha/(\alpha + \beta) = \theta$. Then for nonnegative measurable functions f on a measure space the following two conditions are equivalent:

(1) there exists constants $A, B \in (0,\infty)$ and a sequence of nonnegative measurable functions $\{f_j\}_{j \in \mathbb{Z}}$ such that $\|f_j\|_{L^\infty} \leqslant A2^{j\alpha}$, $\|f_j\|_{L^r} \leqslant B2^{-j\beta}$, and $f = \sum_{j \in \mathbb{Z}} f_j$.

(2) $f \in L^{(p,\infty)}$, i.e., there exists a constant $C \in (0,\infty)$ such that $|\{x : f(x) > \lambda\}| \leqslant (C\lambda^{-1})^p$ for all $\lambda > 0$.

To be precise, if (1) holds then (2) holds with $C = c(p,r,\alpha,\beta)A^{1-\theta}B^\theta$, and, conversely, if (2) holds then (1) holds with $A, B \in (0,\infty)$ such that $A^{1-\theta}B^\theta = c(p,r,\alpha,\beta)C$.

Proof.

$(1) \Rightarrow (2)$. — Suppose (1) holds and write $\gamma = \alpha + \beta$. Take an integer j_0 such that $A2^{j_0\alpha} \approx B2^{-j_0\beta}$ and set $C = A2^{j_0\alpha}$. Then $C \approx A^{1-\theta}B^\theta$, $\|f_{j+j_0}\|_{L^\infty} \lesssim C2^{j\gamma\theta}$, and $\|f_{j+j_0}\|_{L^r} \lesssim C2^{-j\gamma(1-\theta)}$. For $\lambda \in (0,\infty)$ given, take an integer j_1 such that $C2^{j_1\gamma\theta} \approx \lambda$ and decompose f as

$$f = \sum_{j \leqslant j_1} f_{j+j_0} + \sum_{j > j_1} f_{j+j_0} = f^{(0)} + f^{(1)}.$$

Then $\|f^{(0)}\|_{L^\infty} \lesssim C2^{j_1\gamma\theta} \approx \lambda$ and $\|f^{(1)}\|_{L^r} \lesssim C2^{-j_1\gamma(1-\theta)} \approx C^{1/\theta}\lambda^{1-1/\theta}$. Hence, if we take a sufficiently large constant c_0, which depends only on p, r, α, β, then we have

$$|\{x : f(x) > c_0\lambda\}| \leqslant |\{x : f^{(1)}(x) > \lambda\}|$$
$$\leqslant \|f^{(1)}\|_{L^r}^r \lambda^{-r} \lesssim (C^{1/\theta}\lambda^{1-1/\theta})^r \lambda^{-r} = (C\lambda^{-1})^p.$$

$(2) \Rightarrow (1)$. — Suppose (2) holds. Take an $A \in (0,\infty)$ and decompose f as

$$f(x) = \sum_{j \in \mathbb{Z}} f_j(x), \quad f_j(x) = f(x)\mathbb{1}\{A2^{(j-1)\alpha} < f(x) \leqslant A2^{j\alpha}\}.$$

Then $\|f_j\|_{L^\infty} \leqslant A2^{j\alpha}$ and

$$\|f_j\|_{L^r} \leqslant A2^{j\alpha}|\{x : f(x) > A2^{(j-1)\alpha}\}|^{1/r}$$
$$\leqslant A2^{j\alpha}(CA^{-1}2^{-(j-1)\alpha})^{p/r} = B2^{-j\beta}$$

with $B \approx C^{1/\theta} A^{1-1/\theta}$. The relations between the constants A, B, and C are obvious from the above arguments. $\qquad\square$

5. Proof of Corollary 1.4

It is known that there exist bijective mappings $\sigma \mapsto \sigma^{*1}$ and $\sigma \mapsto \sigma^{*2}$ of $BS^m_{\rho,\rho}$, $0 \leqslant \rho < 1$, onto itself such that

$$(5.1) \quad \int T_\sigma(f,g)(x)h(x)\mathrm{d}x = \int T_{\sigma^{*1}}(h,g)(x)f(x)\mathrm{d}x = \int T_{\sigma^{*2}}(f,h)(x)g(x)\mathrm{d}x$$

for all $f, g, h \in \mathcal{S}$ (see [2, Theorem 2.1]). By duality,

$$\|T_\sigma\|_{L^2 \times L^\infty \to L^2} = \|T_{\sigma^{*2}}\|_{L^2 \times L^2 \to L^1} = \|T_{(\sigma^{*2})^{*1}}\|_{L^\infty \times L^2 \to L^2}.$$

In particular, if one of the above is finite, then the other two are also finite. Thus the desired result for $(p,q) = (2,\infty), (2,2), (\infty,2)$ follows from Theorem 1.2. Similarly, by the duality between H^1 and BMO,

$$\|T_\sigma\|_{L^\infty \times L^\infty \to \mathrm{BMO}} \approx \|T_{\sigma^{*1}}\|_{H^1 \times L^\infty \to L^1} \approx \|T_{\sigma^{*2}}\|_{L^\infty \times H^1 \to L^1}.$$

Hence the desired result for $(p,q) = (\infty,\infty), (1,\infty), (\infty,1)$ follows from Theorem 1.3. Other cases can be obtained from interpolation. As for the interpolation argument, see for example [1, Proof of Theorem 2.2].

Appendix A.

In this appendix, we shall prove Proposition 1.1. Let $0 < p, q, r \leqslant \infty$ and $1/p + 1/q = 1/r$. We write $m_0 = m_0(p,q)$. Recall that $m_\rho(p,q) = (1-\rho)m_0$. For simplicity of notation, we only consider the case $r < \infty$, but the argument below works in the case $r = \infty$ as well. In fact, in the case $r = \infty$, all we need is to rewrite L^r by BMO.

In [10, Theorem A.2], it is already proved that if $T_\sigma : H^p \times H^q \to L^r$ for all $m \in BS^m_{\rho,\rho}$ then $m \leqslant (1-\rho)m_0$. Hence, in order to complete the proof of Proposition 1.1, it is sufficient to show that if $m < (1-\rho)m_0$ then $T_\sigma : H^p \times H^q \to L^r$ for all $\sigma \in BS^m_{\rho,\rho}$. As we mentioned in Introduction, this has been proved in [9] and [1] in the range $1/p + 1/q \leqslant 1$. Here we shall give a proof that is valid for all $0 < p, q \leqslant \infty$.

We use the fact that the case $\rho = 0$ is already known. To be precise, it is known that $T_\sigma : H^p \times H^q \to L^r$ for all $\sigma \in BS^{m_0}_{0,0}$ (see [10, Theorem 1.1]).

By virtue of the closed graph theorem, this boundedness is equivalent to the claim that there exists a positive integer N and a constant c such that

$$(A.1) \quad \|T_\sigma\|_{H^p \times H^q \to L^r}$$

$$\leqslant c \max_{|\alpha|,|\beta|,|\gamma| \leqslant N} \left(\sup_{x,\xi,\eta \in \mathbb{R}^n} (1+|\xi|+|\eta|)^{-m_0} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \right)$$

for all $\sigma \in BS_{0,0}^{m_0}$ (see [1, Lemma 2.6]).

Now assume that $0 < \rho < 1$ and $\sigma \in BS_{\rho,\rho}^m$ with $m < (1-\rho)m_0$. In the same way as in Section 4, we write $\sigma = \sum_{j=0}^\infty \sigma_j$ as in (4.1) and (4.2), and define $\widetilde{\sigma}_j$ by (4.12). Then (4.13) holds and this, together with the relation $1/p + 1/q = 1/r$, implies

$$(A.2) \qquad \|T_{\sigma_j}\|_{H^p \times H^q \to L^r} = \|T_{\widetilde{\sigma}_j}\|_{H^p \times H^q \to L^r}.$$

Also, from the same argument as in (4.14), we see that $\widetilde{\sigma}_j$ satisfies the estimate

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \widetilde{\sigma}_j(x,\xi,\eta)| \leqslant C_{\alpha,\beta,\gamma} \, 2^{j(m-(1-\rho)m_0)}(1+|\xi|+|\eta|)^{m_0}.$$

Combining this with (A.2) and (A.1), we have

$$\|T_{\sigma_j}\|_{H^p \times H^q \to L^r} = \|T_{\widetilde{\sigma}_j}\|_{H^p \times H^q \to L^r} \lesssim 2^{j(m-(1-\rho)m_0)}.$$

Since $m < (1-\rho)m_0$, the above inequality implies that $T_\sigma = \sum_{j=0}^\infty T_{\sigma_j}$ is bounded from $H^p \times H^q \to L^r$. This completes the proof of Proposition 1.1.

BIBLIOGRAPHY

[1] Á. Bényi, F. Bernicot, D. Maldonado, V. Naibo & R. H. Torres, "On the Hörmander classes of bilinear pseudodifferential operators II", *Indiana Univ. Math. J.* **62** (2013), no. 6, p. 1733-1764.

[2] Á. Bényi, D. Maldonado, V. Naibo & R. H. Torres, "On the Hörmander classes of bilinear pseudodifferential operators", *Integral Equations Oper. Theory* **67** (2010), no. 3, p. 341-364.

[3] Á. Bényi & R. H. Torres, "Symbolic calculus and the transposes of bilinear pseudodifferential operators", *Commun. Partial Differ. Equations* **28** (2003), no. 5-6, p. 1161-1181.

[4] ———, "Almost orthogonality and a class of bounded bilinear pseudodifferential operators", *Math. Res. Lett.* **11** (2004), no. 1, p. 1-11.

[5] A.-P. Calderón & R. Vaillancourt, "A class of bounded pseudo-differential operators", *Proc. Natl. Acad. Sci. USA* **69** (1972), p. 1185-1187.

[6] R. R. Coifman & Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque, vol. 57, Société Mathématique de France, 1978, with an English summary, i+185 pages.

[7] L. Grafakos, *Classical Fourier analysis*, second ed., Graduate Texts in Mathematics, vol. 249, Springer, 2008, xvi+489 pages.

[8] L. GRAFAKOS & R. H. TORRES, "Multilinear Calderón-Zygmund theory", *Adv. Math.* **165** (2002), no. 1, p. 124-164.

[9] N. MICHALOWSKI, D. RULE & W. STAUBACH, "Multilinear pseudodifferential operators beyond Calderón-Zygmund theory", *J. Math. Anal. Appl.* **414** (2014), no. 1, p. 149-165.

[10] A. MIYACHI & N. TOMITA, "Calderón-Vaillancourt-type theorem for bilinear operators", *Indiana Univ. Math. J.* **62** (2013), no. 4, p. 1165-1201.

[11] V. NAIBO, "On the $L^\infty \times L^\infty \to BMO$ mapping property for certain bilinear pseudodifferential operators", *Proc. Am. Math. Soc.* **143** (2015), no. 12, p. 5323-5336.

[12] E. M. STEIN, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, 1993, with the assistance of Timothy S. Murphy, xiv+695 pages.

Akihiko MIYACHI
Department of Mathematics
Tokyo Woman's Christian University
Zempukuji, Suginami-ku, Tokyo 167-8585, Japan
miyachi@lab.twcu.ac.jp

Naohito TOMITA
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043, Japan
tomita@math.sci.osaka-u.ac.jp