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GEOMETRIC INEQUALITIES FOR MANIFOLDS WITH RICCI CURVATURE IN THE KATO CLASS

by Gilles CARRON

Abstract. — We obtain Euclidean volume growth results for complete Riemannian manifolds satisfying a Euclidean Sobolev inequality and a spectral type condition on the Ricci curvature. We also obtain eigenvalue estimates, heat kernel estimates, and Betti number estimates for closed manifolds whose Ricci curvature is controlled in the Kato class.

Résumé. — On démontre qu'une variété riemannienne complète vérifiant une inégalité de Sobolev euclidienne et dont la courbure de Ricci est petite dans une classe de Kato et à croissance euclidienne du volume. On obtient aussi des estimations spectrales, du noyau de la chaleur et du premier nombre de Betti des variétés riemanniennes compactes dont la courbure de Ricci est controlée dans une classe de Kato.

1. Introduction

1.1. Volume growth

1.1.1. Motivation

One of our motivations was a quest for a higher dimensional analogue of the following beautiful result of P. Castillon ([11]):

Theorem 1.1. — *Let (M^2, g) be a complete noncompact Riemannian surface with nonnegative Laplacian Δ, Gaussian curvature K_g and Riemannian measure dA_g. Assume that there is some $\lambda > \frac{1}{4}$ such that the Schrödinger operator*

$$\Delta + \lambda K_g$$

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is nonnegative, i.e.,

$$\forall\, \psi \in \mathcal{C}_0^\infty(M)\colon\ \int_M \left(|\mathrm{d}\psi|_g^2 + \lambda K_g \psi^2\right) \mathrm{d}A_g \geqslant 0.$$

Then, for all $R \geqslant 0$ and all $x \in M$,

$$\mathrm{Area}_g\left(B(x,R)\right) \leqslant c(\lambda)\, R^2,$$

where $c(\lambda)$ is a constant which only depends on λ.

In fact, P. Castillon has shown that such a surface is either conformally equivalent to the plane or to the cylinder:

$$(M^2, g) \underset{\text{conf.}}{\simeq} \mathbb{C} \quad \text{or} \quad (M^2, g) \underset{\text{conf.}}{\simeq} \mathbb{C}^*.$$

More generally, when the operator $\Delta + \lambda K_g$ is only assumed to have finitely many negative eigenvalues for some $\lambda > 1/4$, then the same conclusion holds but in this case $\sup_R \frac{\mathrm{Area}_g(B(x,R))}{R^2}$ depends on M and λ.

The purpose of this paper is to investigate a possible higher dimensional analogue of Theorem 1.1. Indeed, it is sometimes crucial to get a Euclidean volume growth estimate, recall that we say that a complete Riemannian manifold (M^n, g) has Euclidean volume growth, if

$$\text{(EVG)} \qquad \forall\, R > 0 \ :\ \ \mathrm{vol}\, B(x,R) \leqslant C R^n$$

where the constant C may depend on the point x or not ; for instance this kind of estimate was one of the difficult results obtained by G. Tian and J. Viaclovsky ([53]), and this was a key point toward the description of the moduli spaces of critical Riemannian metrics in dimension four ([54]).

1.1.2. Definitions

According to the Bishop–Gromov comparison theorem, a complete Riemannian manifold (M^n, g) with nonnegative Ricci curvature has at most Euclidean volume growth,

$$\forall\, x \in M,\ R > 0 :\ \mathrm{vol}\, B(x,R) \leqslant \omega_n R^n,$$

where ω_n is the Euclidean volume of the unit Euclidean $n-$ball.

If (M, g) is a Riemannian manifold, we introduce the function Ric_- defined by

$$\mathrm{Ric}_-(x) := \max\{-\kappa(x), 0\}$$

where

$$\kappa(x) := \inf_{\vec{v} \in T_x M,\, g_x(\vec{v},\vec{v})=1} \mathrm{Ricci}_x(\vec{v},\vec{v});$$

so that we have $\mathrm{Ricci}_x \geqslant -\mathrm{Ric}_-(x) g_x$.

We are looking for a spectral condition on the Schrödinger operator

$$L_\lambda = \Delta - \lambda \,\mathrm{Ric}_-$$

that would imply a Euclidean volume growth. We do not think that it is possible to prove a result similar to Theorem 1.1 in higher dimension, under the sole assumption that L_λ is nonnegative in L^2.

Recall that L_λ is nonnegative on L^2 if and only if there is a positive function h on M such that $L_\lambda h = 0$. We introduce a stronger condition:

Definition 1.2. — *The Schrödinger operator L_λ is said to be* gaugeable *if there is a function $h\colon M \to \mathbb{R}$ and a constant γ such that $1 \leqslant h \leqslant \gamma$ and $L_\lambda h = 0$. The constant γ is called the gaugeability constant of L_λ.*

The behavior of the heat semigroup associated with the Schrödinger operator L_λ can be quite different on L^2 and on L^p. For instance the fact that the heat semigroup is uniformly bounded on L^∞:

$$\sup_{t>0} \left\| e^{-tL_\lambda} \right\|_{L^\infty \to L^\infty} < +\infty$$

implies the nonnegativity of the Schrödinger operator L_λ on L^2. It may happen that a Schrödinger operator is nonnegative on L^2 but the associated semigroup is not uniformly bounded on L^∞ ([23]). The uniform boundedness of the semigroup $\left(e^{-tL_\lambda}\right)_{t>0}$ on L^∞ is strongly related to the fact that the Schrödinger operator L_λ is gaugeable (see [57] and Theorem 2.14). Hence the gaugeability condition could be interpreted as an L^∞ spectral condition.

It is well known that a Sobolev inequality is useful in order to control the behavior of the heat semigroup $e^{-t\Delta}$.

Definition 1.3. — *We say that a complete Riemannian manifold (M^n, g) satisfies the Euclidean Sobolev inequality with Sobolev constant μ if*

$$\text{(Sob)} \qquad \forall\, \psi \in \mathcal{C}_0^\infty(M)\colon\ \mu \left(\int_M \psi^{\frac{2n}{n-2}} \,\mathrm{dv}_g \right)^{1-\frac{2}{n}} \leqslant \int_M |\mathrm{d}\psi|_g^2 \,\mathrm{dv}_g$$

According to a celebrated result of J. Nash and N. Varopoulos ([45, 55]), the Euclidean Sobolev inequality is equivalent to a Euclidean type upper bound on the heat kernel:

$$\text{(EUB)} \qquad \forall\, t > 0, x, y \in M\colon\ H(t,x,y) \leqslant C\, t^{-\frac{n}{2}} e^{-\frac{d(x,y)^2}{Ct}}.$$

That is to say, given the Sobolev constant μ and the dimension n, there is a constant $C = C(n,\mu)$ such that the Euclidean type upper bound on the heat kernel (EUB) holds. Conversely, if the Euclidean type upper bound on the heat kernel (EUB) holds for some constant C, then (M^n, g) satisfies the Sobolev inequality (Sob) with some constant $\mu = \mu(n, C)$.

1.1.3. Main result

Theorem 1.4. — *Let (M^n, g) be a complete Riemannian manifold of dimension $n > 2$. Assume that*

- *(M^n, g) satisfies the Euclidean Sobolev inequality* (Sob) *with Sobolev constant μ.*
- *There is a $\delta > 0$ such that the Schrödinger operator $\Delta - (n-2)\times(1+\delta)\operatorname{Ric}_-$ is gaugeable with gaugeability constant γ.*

Then, there is a constant θ depending only on n, δ, γ and μ, such that for all $x \in M$ and $R \geqslant 0$:

$$\frac{1}{\theta} R^n \leqslant \operatorname{vol} B(x,R) \leqslant \theta\, R^n.$$

In fact, we already know that the Sobolev inequality (Sob) implies a lower bound on the volume of geodesic balls ([2, 7]): there is a constant c_n such that for all $x \in M$ and $R > 0$,

$$c_n \mu^{\frac{n}{2}} R^n \leqslant \operatorname{vol} B(x,R).$$

The log-Sobolev inequality ([46, Proposition 5.1]) also yields the same conclusion. Hence, the crucial point in the proof of Theorem 1.4 is to get the upper bound.

Remark 1.5. — If (M^n, g) satisfies the Euclidean inequality (Sob) with constant μ then any of the following conditions implies that L_λ is gaugeable for some $\lambda > (n-2)$:

(1) There is some $\epsilon \in (0,1)$ such that $\operatorname{Ric}_- \in L^{\frac{n}{2}(1-\epsilon)} \cap L^{\frac{n}{2}(1+\epsilon)}$ and the Schrödinger operator $\Delta - (n-2)(1+\epsilon)\operatorname{Ric}_-$ is nonnegative in L^2.

(2) There is some $\epsilon \in (0,1)$ such that $\operatorname{Ric}_- \in L^{\frac{n}{2}(1-\epsilon)} \cap L^{\frac{n}{2}(1+\epsilon)}$ and

$$\int_M \operatorname{Ric}_-^{\frac{n}{2}} \mathrm{dv}_g \leqslant \left(\frac{\mu}{n-2}\right)^{\frac{n}{2}} (1-\epsilon).$$

(3) $$\sup_{x\in M} \int_0^\infty \frac{1}{r^{n-1}} \left(\int_{B(x,r)} \operatorname{Ric}_-(y)\, \mathrm{dv}_g(y) \right) \mathrm{d}r < \epsilon_n \mu^{\frac{n}{2}},$$
where ϵ_n is a computable constant depending only on n.

The first two conditions are due to B. Devyver ([25]). The last one is an easy consequence of Green kernel estimates (see for instance [20, Theorem 3.1]).

1.1.4. Overview of the proof (see Section 4 for more details)

Following T. Colding ([14]), when (M^n, g) is a nonparabolic manifold, we introduce $b_o(x) = G(o,x)^{-\frac{1}{n-2}}$ where $G(o,\cdot\,)$ is the Green kernel with pole at $o \in M$, normalized so that $b(x) \simeq_{x \to o} d(o,x)$. When (M^n, g) has nonnegative Ricci curvature, T. Colding has shown that

$$|\mathrm{d}b_o| \leqslant 1.$$

The crucial point in the proof of Theorem 1.4 is to prove a uniform bound on the gradient of b_o:

$$|\mathrm{d}b_o| \leqslant \Gamma.$$

Hence $B(o,R) \subset \{x \in M, b_o(x) \leqslant \Gamma R\}$, and using [7, Proposition 1.14], we know that

$$\mathrm{vol}_g\{x \in M, b_o(x) \leqslant \Gamma R\} \leqslant C(\mu, n)\Gamma^n R^n.$$

1.1.5. Other definitions

Definition 1.6. — *A complete Riemannian manifold (M^n, g) is said to be doubling if there is a constant θ such that*

$$\text{(D)} \qquad \forall\, x \in M,\ R > 0 :\ \mathrm{vol}\, B(x, 2R) \leqslant \theta\, \mathrm{vol}\, B(x,R).$$

Definition 1.7. — *A complete Riemannian manifold (M^n, g) satisfies the Poincaré inequality if there is a constant γ such that for any geodesic ball B of radius r, we have*

$$\text{(PI)} \qquad \forall\, \psi \in \mathcal{C}^1(B):\ \int_B (\psi - \psi_B)^2 \,\mathrm{dv}_g \leqslant \gamma r^2 \int_B |\mathrm{d}\psi|_g^2 \,\mathrm{dv}_g\,.$$

Here and thereafter, for an arbitrary $\mathcal{O} \subset M$ with $0 < \mathrm{vol}_g \mathcal{O} < +\infty$,

$$\psi_{\mathcal{O}} = \frac{1}{\mathrm{vol}\,\mathcal{O}} \int_{\mathcal{O}} \psi \,\mathrm{dv}_g\,.$$

Recall that the heat kernel of (M, g) is the Schwartz kernel of the heat operator $e^{-t\Delta}$. It is the positive function $H\colon (0, +\infty) \times M \times M \to \mathbb{R}$ such that for any $f \in L^2(M, \mathrm{dv}_g)$:

$$\left(e^{-t\Delta} f\right)(x) = \int_M H(t,x,y) f(y) \,\mathrm{dv}_g(y).$$

Definition 1.8. — *We say that the heat kernel of (M^n, g) satisfies the Li–Yau estimates if there are positive constants c, C such that*

$$\text{(LY)} \qquad \frac{c}{\mathrm{vol}\, B(x, \sqrt{t})} e^{-\frac{d(x,y)^2}{ct}} \leqslant H(t,x,y) \leqslant \frac{C}{\mathrm{vol}\, B(x,\sqrt{t})} e^{-\frac{d(x,y)^2}{Ct}}.$$

Definition 1.9. — *We say that the heat kernel of* (M^n, g) *admits a Gaussian upper bound if there is a positive constant* C *such that*

$$\text{(GUB)}\qquad H(t,x,y) \leqslant \frac{C}{\operatorname{vol} B(x,\sqrt{t})} e^{-\frac{d(x,y)^2}{Ct}}.$$

Remark that if (M, g) has nonnegative Ricci curvature, then the Bishop–Gromov comparison theorem implies that it is doubling. According to P. Buser ([6]), (M, g) satisfies the Poincaré inequality (PI) and, according to a famous result of P. Li and S-T. Yau [42], its heat kernel satisfies the Li–Yau estimates. Furthermore, the estimates (LY) are equivalent to the conditions (D and PI) ([29, 51]). Moreover, according to a nice observation by T. Coulhon ([16]), we note that the lower bound

$$\forall\, t > 0,\ x, y \in M\colon\ H(t,x,y) \geqslant c\, t^{-\frac{n}{2}} e^{-\frac{d(x,y)^2}{ct}}$$

yields a Euclidean upper bound on the volume of geodesic balls.

1.1.6. Consequences of Theorem 1.4

Theorem 1.10. — *If* (M, g) *is a complete Riemannian manifold which satisfies the conditions of Theorem 1.4, then:*

- (M, g) *is doubling and satisfies the Poincaré inequality* (PI).
- *The heat kernel of* (M, g) *satisfies the Li–Yau estimates* (LY).
- *For any* $p\in(1,+\infty)$, *the Riesz transform* $d\Delta^{-\frac{1}{2}}: L^p(M)\to L^p(T^*M)$ *is a bounded operator.*

Remark 1.11. — According to D. Bakry ([4]), on a complete Riemannian manifold with nonnegative Ricci curvature, the Riesz transform is a bounded operator on L^p for any $p \in (1, +\infty)$.

1.1.7. Gaugeability and the Kato constant

The gaugeability of the Schrödinger operator $\Delta - \lambda \operatorname{Ric}_-$ is strongly related to Kato constants. These constants measure the size of the potential Ric_- relative to Δ. For a nice introduction to Kato constants, we recommend [34, Chapter 6].

Definition 1.12. — *Let* $G(\,\cdot\,,\cdot\,)$ *be the positive minimal Green kernel of* (M^n, g). *The Kato constant of* Ric_- *is defined by*

$$\mathrm{K}(\operatorname{Ric}_-) := \sup_{x\in M} \int_M G(x,y) \operatorname{Ric}_-(y)\, \mathrm{dv}_g(y)\,.$$

Definition 1.13. — *Let $\{H(t,x,y)\}_{(t,x,y)\in\mathbb{R}_+\times M\times M}$ be the heat kernel of (M^n,g). The parabolic Kato constant of* Ric_- *at time T is defined by*

$$\mathrm{k}_T(\mathrm{Ric}_-)=\sup_{x\in M}\int_0^T\int_M H(t,x,y)\,\mathrm{Ric}_-(y)\,\mathrm{dv}_g(y)\mathrm{d}t.$$

As we have $G(x,y)=\int_0^\infty H(t,x,y)\mathrm{d}t$, we easily deduce that

$$\lim_{T\to+\infty}\mathrm{k}_T(\mathrm{Ric}_-)=\mathrm{K}(\mathrm{Ric}_-).$$

The observation is as follows.

Lemma 1.14. — *Assume that* $\mathrm{K}(\mathrm{Ric}_-)<\frac{1}{n-2}$, *and that $\lambda>n-2$ is such that* $\lambda\,\mathrm{K}(\mathrm{Ric}_-)<1$. *Then, the Schrödinger operator* $\Delta-\lambda\,\mathrm{Ric}_-$ *is gaugeable with gaugeability constant*

$$\gamma=\frac{\lambda\,\mathrm{K}(\mathrm{Ric}_-)}{1-\lambda\,\mathrm{K}(\mathrm{Ric}_-)}.$$

The conditions for gaugeabily given in Remark 1.5 imply an estimate of the Kato constant of Ric_-.

1.1.8. Localization at infinity

It is possible to get a slightly weaker result, if we only get a control of the Ricci curvature outside a compact set.

Theorem 1.15. — *Let (M,g) be a complete Riemannian manifold which satisfies the Euclidean Sobolev inequality* (Sob). *Assume that there is a compact subset $K\subset M$ such that*

$$\sup_{x\in M\setminus K}\int_{M\setminus K}G(x,y)\,\mathrm{Ric}_-(y)\,\mathrm{dv}_g(y)\;<\frac{1}{16n}.$$

Then,

- *there is a constant θ such that, for all $x\in M,\ R\geqslant 0$,*
$$\frac{1}{\theta}\,R^n\leqslant\operatorname{vol}B(x,R)\leqslant\theta\,R^n.$$
- *(M^n,g) is doubling,*
- *its heat kernel satisfies* (GUB),
- *for $n\geqslant 4$ and $p\in(1,n)$, the Riesz transform $d\Delta^{-\frac12}\colon L^p(M)\to L^p(T^*M)$ is a bounded operator.*

Remarks 1.16.

- The value $\frac{1}{16n}$ is not optimal but quite explicit.
- The constant θ depends on (M,g). It cannot be estimated from the dimension, the Sobolev constant. Indeed the geometry of (M,g) on a neighborhood of the compact set K has some influence on this constant θ.

According to [25], the assumptions of Theorem 1.15 are satisfied by complete Riemannian manifolds satisfying a Euclidean Sobolev inequality, and such that for some $\epsilon \in (0,1)$, $\mathrm{Ric}_- \in L^{\frac{n}{2}(1-\epsilon)} \cap L^{\frac{n}{2}(1+\epsilon)}$.

1.2. The case of closed manifolds

Recents papers have emphasized how a control on the Kato constant of the Ricci curvature can be useful in order to control some geometrical quantities for closed or complete Riemannian manifolds ([5, 17, 20, 25, 47, 48, 49, 58, 59]). For a closed Riemannian manifold (M,g), we will explain how the works of Qi S. Zhang and M. Zhu [59], together with some classical ideas, can be used in order to obtain geometric and topological estimates based on the Kato constant of the Ricci curvature. Recently C. Rose has also obtained similar results based on this idea ([48]).

Definition 1.17. — *Let (M^n,g) be a closed Riemannian manifold of diameter D. The scale invariant geometric quantity $\xi(M,g)$ is the smallest positive real number ξ such that, for all $x \in M$,*

$$\int_0^{\frac{D^2}{\xi^2}} \int_M H(t,x,y)\,\mathrm{Ric}_-(y)\,\mathrm{dv}_g(y)\mathrm{d}t \leqslant \frac{1}{16n}.$$

If $T(M,g) > 0$ is the largest time, such that $\mathrm{k}_T(\mathrm{Ric}_-) \leqslant \frac{1}{16n}$ then, we have $\xi(M,g)\sqrt{T(M,g)} = D$.

For instance if the Ricci curvature is bounded from below, $\mathrm{Ricci} \geqslant -(n-1)\kappa^2 g$, then $\xi(M,g) \leqslant 4\kappa D$. In this case, it is well known that the geometry of (M^n,g) is well controlled by the geometrical quantity κD. We obtain almost the same results in terms of the new quantity $\xi(M,g)$.

Theorem 1.18. — *Let (M,g) be a closed Riemannian manifold of dimension n and diameter D. Then there is a constant γ_n, which depends only on n, such that*

(1) *the first nonzero eigenvalue* λ_1 *of the Laplacian satisfies*

$$\lambda_1 \geqslant \frac{\gamma_n^{-1-\xi(M,g)}}{D^2};$$

(2) *the first Betti number of* M *satisfies*

$$b_1(M) \leqslant n + \frac{1}{4} + \xi(M,g)\gamma_n^{1+\xi(M,g)}.$$

In particular, there exists $\epsilon_n > 0$ *such that if* $\xi(M,g) < \epsilon_n$ *then*

$$b_1(M) \leqslant n.$$

(3) (M,g) *is doubling: for any* $x \in M$ *and* $0 \leqslant R \leqslant D/2$,

$$\operatorname{vol} B(x,2R) \leqslant \gamma_n^{1+\xi(M,g)} \operatorname{vol} B(x,R);$$

(4) *for all* $t > 0$ *and* $x \in M$,

$$H(t,x,x) \leqslant \frac{\gamma_n^{1+\xi(M,g)}}{\operatorname{vol} B(x,\sqrt{t})}.$$

We can get a slight improvement of the previous theorem with a stronger control on the Ricci curvature.

Proposition 1.19. — *Let* (M,g) *be a closed Riemannian manifold of dimension* n *and diameter* D*. Let* $p > 1$*, and assume that for some* $T, \Lambda > 0$,

$$D^{2p-2} \sup_{x\in M} \int_0^T H(s,x,y) \operatorname{Ric}_-^p(y) \, \mathrm{dv}_g \, \mathrm{d}s \leqslant \Lambda^p.$$

Then (with q *the exponent dual to* p*:* $pq = p + q$*),*

$$\xi(M,g) \leqslant \alpha(n,D,\Lambda,T,p) := \max\left\{ \frac{D}{\sqrt{T}}, (16n\Lambda)^{q/2} \right\}.$$

Moreover, there is a constant θ*, depending only on* $\alpha(n,D,\Lambda,T,p)$ *and* n*, such that for any* $x \in M$ *and* $0 \leqslant r \leqslant R \leqslant D$,

$$\frac{\operatorname{vol} B(x,R)}{R^n} \leqslant \theta \frac{\operatorname{vol} B(x,r)}{r^n} \leqslant \theta^2.$$

A quick comparison between Theorem 1.4 and Theorem 1.18 naturally leads to the question whether the Euclidean Sobolev inequality is necessary in Theorem 1.4. According to Qi S. Zhang and M. Zhu in [59], the results obtained in Theorem 1.18 could be generalized to complete Riemannian manifold provided one has good approximations of the distance function: there exists $c > 0$ such that, for all $o \in M$, there exists $\chi_o \colon M \to \mathbb{R}_+$ satisfying

$$d(o,x)/c \leqslant \chi_o(x) \leqslant c d(o,x),$$
$$|\mathrm{d}\chi_o|^2 + |\Delta \chi_o| \leqslant c.$$

This is a very strong hypothesis. Our proof of Theorem 1.4 and Theorem 1.15 yields a comparison between the level sets of the Green kernel and geodesic spheres. As a consequence, we prove the existence of such an approximation of the distance function.

Our estimates on the first Betti number is a generalization of the one obtained by M. Gromov under a lower bound on the Ricci curvature. According to T. Colding [12, 13], one knows that there exists an $\epsilon(n) > 0$ such that if (M^n, g) is a closed n-dimensional manifold with $\mathrm{Ric}_- \operatorname{diam}^2(M) < \epsilon(n)$ and $b_1(M) = n$, then M is diffeomorphic to a torus $\mathbb{T}^n$. A natural question is then to ask what can be said on a closed Riemannian manifold satisfying $\xi(M, g) << 1$ and $b_1(M) = n$. We believe that such a manifold should be close to a torus $\mathbb{T}^n$ in the Gromov–Hausdorff topology. In order to say more, it would need to understand spaces which are Gromov–Hausdorff limits of Riemannian manifolds (M^n, g), with $\xi(M, g) \leqslant \Xi$ and $\operatorname{diam} M \leqslant D$. Note that our results yield a precompactness result in the Gromov–Hausdorff topology for these class of spaces. We hope that the results of this paper will turn out useful to give some answers to a question of G. Tian and J. Viaclosky about critical metrics in higher dimension (see [54, Section 8.2]).

A lower bound on the Ricci curvature also yields some isoperimetric inequalities. In [10], we have shown that a bound on the Kato constant of the Ricci curvature yields some isoperimetric inequality.

In the pioneering paper ([27]), S. Gallot proved isoperimetric inequalities, eigenvalue and heat kernel estimates for closed Riemannian manifold (M^n, g), under a control of Ric_- in L^p (for $p > n/2$). It would be interesting to know whether one can directly get a control of the Ricci curvature in some Kato class from a control of Ric_- in L^p (for $p > n/2$), without using Gallot's isoperimetric inequality.

1.3. Localization in a geodesic ball

Our ideas can be adapted to understand the geometry of a geodesic ball under some stronger control on the Ricci curvature.

Theorem 1.20. — *Let (M^n, g) be a Riemannian manifold. Assume that $B(o, 3R) \subset M$ is a relatively compact geodesic ball. Let $p > 1$ and let $q := p/(p-1)$. The Green kernel for the Laplacian Δ on $B(o, 3R)$ for the Dirichlet boundary condition is denoted G_{3R}. Define the constant Λ by*

$$\Lambda^p := R^{2p-2} \sup_{x \in B(o,3R)} \int_{B(o,3R)} G_{3R}(x, y) \,\mathrm{Ric}_-(y)^p \,\mathrm{dv}_g(y).$$

Assume that

(1) *for some* $\delta > \frac{(q(n-2)-2)^2}{8q(n-2)}$, *the operator* $\Delta - (1+\delta)(n-2)\operatorname{Ric}_-$ *is nonnegative on* $B(o,3R)$,

(2) *the ball* $B(o,3R)$ *satisfies the Sobolev inequality* (Sob) *with constant* μ.

Then, there exist constants θ *and* γ, *which only depend on* $n, p, \delta, \Lambda, \mu$ *and on the volume density* $\frac{\operatorname{vol} B(o,3R)}{R^n}$, *such that for any* $x \in B(o,R)$ *and any* $r \in (0,R)$,

- $\frac{1}{\theta} r^n \leqslant \operatorname{vol} B(x,r) \leqslant \theta r^n$,
- $\forall\, \psi \in \mathcal{C}^1(B(x,r))\colon \int_{B(x,r)} (\psi - \psi_{B(x,r)})^2 \,\mathrm{dv}_g \leqslant \gamma r^2 \int_{B(x,r)} |\mathrm{d}\psi|_g^2 \,\mathrm{dv}_g\,.$

1.4. Organization of the paper

In the next section, we review and collect some analytical tools which will be used in the paper. For instance, we describe Agmon's type volume estimate which are mainly due to P. Li and J. Wang ([40, 41]). These estimates will be crucial in the proof of Theorem 1.4. We also prove a new elliptic estimate based on a variation of the De Giorgi–Nash–Moser iteration scheme which will be useful in the proof of Theorem 1.20. The third section is devoted to the proof of Theorem 1.18. Theorem 1.4 and the first part of Theorem 1.15 are proved in the fourth section. The last sections are devoted to the end of the proofs of Theorem 1.15 and Theorem 1.20.

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2. Preliminaries

In this section we review some classical results which will be used throughout the paper. We consider (M,g) a Riemannian manifold, and a measure $\mathrm{dm} = \Phi\,\mathrm{dv}_g$ where dv_g is the Riemannian measure and Φ a positive Lipschitz function; the L^p norm associated with this measure will be noted $\|\cdot\|_p$ or $\|\cdot\|_{\mathrm{m},p}$.

2.1. Laplacians

2.1.1.

The Laplacian Δ_{m} or Δ_Φ is the differential operator defined by the Green formula:

$$\forall\, \psi \in \mathcal{C}_0^\infty(M)\colon \int_M |\mathrm{d}\psi|_g^2\,\mathrm{dm} = \int_M (\Delta_{\mathrm{m}}\psi)\,\psi\,\mathrm{dm}\,.$$

It is associated with the quadratic form:

$$\psi \in \mathcal{C}_0^\infty(M) \longmapsto q(\psi) := \int_M |\mathrm{d}\psi|_g^2\,\mathrm{dm}\,. \tag{QF}$$

The geometric Laplacian will be noted $\Delta = \Delta_1$ and we have the formula

$$\Delta_{\mathrm{m}}\psi = \Delta\psi - \langle \mathrm{d}\log\Phi, \mathrm{d}\psi\rangle_g.$$

The Friedrichs realization of the operator Δ_{m} is associated with the minimal extension of the above quadratic form. We introduce $\mathcal{D}(q)$ to be the completion of $\mathcal{C}_0^\infty(M)$ with respect to the norm,

$$\psi \mapsto \sqrt{q(\psi) + \|\psi\|_2^2}.$$

The domain of the operator Δ_{m} is given by

$$\mathcal{D}\left(\Delta_{\mathrm{m}}\right) = \{v \in \mathcal{D}(q),\ \exists\, C \text{ such that } \forall\, \varphi \in \mathcal{C}_0^\infty(M)\colon\ |\langle v, \Delta_{\mathrm{m}}\varphi\rangle| \leqslant C\|\varphi\|_2\}.$$

Remarks 2.1.

- If (M,g) is geodesically complete, then
$$\Delta_{\mathrm{m}}\colon \mathcal{C}_0^\infty(M) \longrightarrow L^2(M,\mathrm{dm})$$
has a unique selfadjoint extension.
- If M is the interior of a compact manifold with boundary $M = X \setminus \partial X$ and if g and Φ have Lipschitz extensions to X then the Friedrichs realization of the operator Δ_{m} is the Laplacian associated with the Dirichlet boundary condition.

2.1.2. Chain rule

When $v \in \mathcal{C}^\infty(M)$ and $f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, by a direct computation, we have

$$\Delta_{\mathrm{m}} f(v) = f'(v)\,\Delta_{\mathrm{m}} v - f''(v)|\mathrm{d}v|_g^2.$$

In particular if f is nondecreasing, convex and if $\Delta_{\mathrm{m}} v \leqslant V$ where V is a nonnegative function then $\Delta_{\mathrm{m}} f(v) \leqslant f'(v)\,V$. By approximation, this can be generalized to weak solutions and nonsmooth convex functions. Recall that if $v \in L^1_{\mathrm{loc}}$ we say that

$$\Delta_{\mathrm{m}} v \leqslant V \text{ weakly}$$

if for any nonnegative $\varphi \in C_0^\infty(M)$:

$$\int_M v\Delta_{\mathrm{m}}\varphi\,\mathrm{dm} \leqslant \int_M V\varphi\,\mathrm{dm}\,.$$

When $v \in W^{1,2}_{\mathrm{loc}}$ then

$$\Delta_{\mathrm{m}} v \leqslant V \text{ weakly}$$

if and only if for any nonnegative $\varphi \in C_0^\infty(M)$ (or $\varphi \in \mathcal{D}(q)$):

$$\int_M \langle \mathrm{d}v, \mathrm{d}\varphi\rangle_g\,\mathrm{dm} \leqslant \int_M V\varphi\,\mathrm{dm}\,.$$

Then it is classical to get the following (where for $x \in \mathbb{R}$: $x_+ = \max\{x, 0\}$)

Lemma 2.2. — *Let $v \in W^{1,2}_{\mathrm{loc}} \cap \mathcal{C}^0$ and let $V \in L^1_{\mathrm{loc}}$ be such that $V \geqslant 0$ and $\Delta_{\mathrm{m}} v \leqslant V$. Then for any $\alpha \geqslant 1$, we get:*

$$\Delta_{\mathrm{m}} v_+^\alpha \leqslant \alpha V v_+^{\alpha-1}.$$
$$\Delta_{\mathrm{m}} (v-1)_+^\alpha \leqslant \alpha V (v-1)_+^{\alpha-1}.$$

2.1.3. Integration by parts formula

The formula

$$|\mathrm{d}(\chi v)|_g^2 = |\mathrm{d}\chi|_g^2 v^2 + \langle \mathrm{d}v, \mathrm{d}(\chi^2 v)\rangle_g, \tag{2.1}$$

implies the following integration by parts inequality.

Lemma 2.3. — *Let $v \in W^{1,2}_{\mathrm{loc}}$, and let $V \in L^1_{\mathrm{loc}}$ be a nonnegative function such that:*

$$\Delta_{\mathrm{m}} v \leqslant V \textit{ weakly.}$$

Then, for every Lipschitz function χ with compact support,

$$\int_M |\mathrm{d}(\chi v)|_g^2\,\mathrm{dm} \leqslant \int_M |\mathrm{d}\chi|_g^2 v^2\,\mathrm{dm} + \int_M \chi^2 v V\,\mathrm{dm}\,.$$

We would like to make sure that this inequality is also valid for Lipschitz functions which are constant at infinity. The notion of parabolicity is precisely what is needed.

Definition 2.4. — *A Borel measure* $\mathrm{d}\mu$ *on a Riemannian manifold is called parabolic if there is a sequence* (χ_k) *of smooth functions with compact support such that,*

- $0 \leqslant \chi_k \leqslant 1$,
- $\chi_k \to 1$ *uniformly on compact sets,*
- $\lim_{k\to\infty} \int_M |\mathrm{d}\chi_k|_g^2 \mathrm{d}\mu = 0.$

Then we have the following refinement of Lemma 2.3.

Lemma 2.5. — *Let* $v \in W^{1,2}_{\mathrm{loc}}$, *and let* $V \in L^1_{\mathrm{loc}}$ *be a nonnegative function such that*

$$\Delta_{\mathrm{m}} v \leqslant V \text{ weakly.}$$

If the measure $v^2\mathrm{dm}$ *is parabolic then, for every Lipschitz function* χ *which is constant outside a compact set and such that* $\int_M \chi^2 vV\,\mathrm{dm} < \infty$,

$$\int_M |\mathrm{d}(\chi v)|_g^2\,\mathrm{dm} \leqslant \int_M |\mathrm{d}\chi|_g^2 v^2\,\mathrm{dm} + \int_M \chi^2 vV\,\mathrm{dm}\,.$$

Remark 2.6. — If (M,g) is geodesically complete and if $M(r) := \int_{B(o,r)} v^2\,\mathrm{dm}$ satisfies

$$M(r) = \mathcal{O}\left(r^2\right) \text{ or } \int_1^\infty \frac{r\mathrm{d}r}{M(r)} = +\infty$$

then the measure $v^2\,\mathrm{dm}$ is parabolic (see [31]).

2.2. Sobolev inequalities

Definition 2.7. — *We say that a weighted complete Riemannian manifold* (M^n, g, m) *satisfies the Euclidean Sobolev inequality with Sobolev constant* μ *if*

$$(\mathrm{Sob_m}) \qquad \forall\,\psi \in \mathcal{C}_0^\infty(M):\ \mu\left(\int_M \psi^{\frac{2n}{n-2}}\,\mathrm{dm}\right)^{1-\frac{2}{n}} \leqslant \int_M |\mathrm{d}\psi|_g^2\,\mathrm{dm}$$

We recall here some classical results which hold in the presence of the Sobolev inequality.

Theorem 2.8. — *Let* (M,g,m) *be a weighted Riemanian manifold. Assume it satisfies the Euclidean Sobolev inequality* $(\mathrm{Sob_m})$ *with Sobolev constant* μ. *Then there is a positive constant* c_n, *such that the following properties hold.*

(1) *The heat kernel associated with the Laplacian* Δ_{m} *satisfies:*

$$\forall\, x \in M, \forall\, t > 0\colon\ H_{\mathrm{m}}(t,x,x) \leqslant \frac{c_n}{(\mu t)^{\frac{n}{2}}}.$$

(2) *The associated positive minimal Green kernel satisfies:*

(a) $\forall\, x, y \in M\colon\ G_{\mathrm{m}}(x,y) \leqslant \frac{c_n}{\mu^{\frac{n}{2}}} \frac{1}{d^{n-2}(x,y)}$.

(b) $\forall\, x \in M, \forall\, t > 0\colon\ \mathrm{m}\left(\{y \in M; G_{\mathrm{m}}(x,y) > t\}\right) \leqslant (\mu t)^{-\frac{n}{n-2}}$.

(c) *If* $\alpha \in (0, n/(n-2))$ *and if* $\Omega \subset M$ *has finite* m*-measure, then*

$$\left(\int_\Omega G^\alpha_{\mathrm{m}}(x,y)\,\mathrm{dm}(y)\right)^{\frac{1}{\alpha}} \leqslant \left(\frac{n}{(n-2)\alpha - n}\right)^{\frac{1}{\alpha}} \frac{1}{\mu} \left(\mathrm{m}(\Omega)\right)^{\frac{1}{\alpha}-1+\frac{2}{n}}.$$

(3) *If* $B(x,r) \subset M$ *is a relatively compact geodesic ball in* M*, and if* $v \in W^{1,2}_{\mathrm{loc}}(B(x,r))$ *satisfies*

$$\Delta_{\mathrm{m}} v \leqslant 0$$

then, for $p \geqslant 2$*, there is a positive constant* $c_{n,p}$ *such that:*

$$|v(x)|^p \leqslant \frac{c_{n,p}}{\left(\sqrt{\mu}\, r\right)^n} \int_{B(x,r)} |v|^p(y)\,\mathrm{dm}(y).$$

(4) *If* $B(x,r) \subset M$ *is a relatively compact geodesic ball in* M*, then*

$$\frac{1}{c(n)} \mu^{\frac{n}{2}} r^n \leqslant \mathrm{m}\left(B(x,r)\right).$$

Remarks 2.9.

- The upper bound on the heat kernel comes essentially from an adaptation in this setting of old ideas of J. Nash ([45]). In fact both properties are equivalent ([55]).
- The estimate on the heat kernel implies a Gaussian upper bound for the heat kernel ([21]):

$$\forall\, x, y \in M,\ \forall\, t > 0\colon\ H_{\mathrm{m}}(t,x,y) \leqslant \frac{c_n}{(\mu t)^{\frac{n}{2}}} e^{-\frac{d^2(x,y)}{5t}},$$

  and the formula

$$G_{\mathrm{m}}(x,y) = \int_0^{+\infty} H_{\mathrm{m}}(t,x,y)\mathrm{d}t$$

  yields the estimate (2a) on the Green kernel.
- The property (2b) is equivalent to the Sobolev inequality ([7]).
- The elliptic estimate (3) is proved by a classical De Giorgi–Nash–Moser iteration method. The lower bound (4) on the volume is a consequence of this elliptic estimate applied to the constant function 1 (see [2, 7]).

2.3. Schrödinger operators and the Doob transform

2.3.1. Schrödinger operators

Assume that $V \in L^\infty_{\mathrm{loc}}$ is a nonnegative function such that the quadratic form

$$\psi \in \mathcal{C}_0^\infty(M) \longmapsto q_V(\psi) := \int_M \left[|\mathrm{d}\psi|_g^2 - V\psi^2\right] \mathrm{dm},$$

is bounded from below; i.e., there is a constant Λ such that

$$\forall\, \psi \in \mathcal{C}_0^\infty(M) : q_V(\psi) \geqslant -\Lambda \int_M \psi^2 \,\mathrm{dm}\,.$$

With the Friedrichs extension procedure, we get a self-adjoint operator which will be denoted:

$$L := \Delta_{\mathrm{m}} - V.$$

An easy consequence of the maximum principle, or of its weak formulation, is that

$$\forall\, x, y \in M,\ \forall\, t > 0\colon\ H_m(t,x,y) \leqslant H_L(t,x,y),$$

where H_L denotes the heat kernel of the operator L.

Definition 2.10. — *The operator L is said to be subcritical, if L has a positive minimal Green kernel G_L.*

Remark 2.11 ([31]*).* — The weighted Laplacian is subcritical if and only if the measure dm is not parabolic in the sense of Definition 2.4. In that case, we say that the weighted Riemannian manifold (M, g, m) is nonparabolic.

When L is subcritical, we have

$$\forall\, x, y \in M\colon\ G_m(x,y) \leqslant G_L(x,y).$$

2.3.2. The Doob Transform

Assume that (M, g) is complete noncompact, and that the operator L is nonnegative,

$$\forall\, \psi \in \mathcal{C}_0^\infty(M) : \int_M \left[|\mathrm{d}\psi|_g^2 - V\psi^2\right] \mathrm{dm} \geqslant 0.$$

Then, the Agmon–Allegretto–Piepenbrink theorem ([1, 26, 44]) implies that there is a positive function $h \in W^{1,2}_{\mathrm{loc}}$ such that

$$Lh = 0.$$

Remark that because V is assumed to be locally bounded, we also have $h \in W^{2,p}_{\mathrm{loc}}$ for any $p < \infty$.

Using the formula (2.1) and integrating by parts, we get that for any $\psi \in \mathcal{C}^\infty_0(M)$:

$$\int_M \left[|\mathrm{d}(h\psi)|^2_g - V h^2 \psi^2 \right] \mathrm{dm} = \int_M |\mathrm{d}\psi|^2_g h^2 \,\mathrm{dm}\,. \tag{2.2}$$

Hence the Schrödinger operator L and the Laplacian $\Delta_{h^2 m}$ are conjugate :

$$L(h\psi) = h\Delta_{h^2 m}\psi$$

and we have:

$$H_L(t,x,y) = h(x)h(y)H_{h^2 m}(t,x,y).$$

This conjugacy is called the Doob transform (or Doob h-transform) and it is the key point in order to get estimates on the Green and heat kernels of the Schrödinger operator L.

2.3.3. The Kato condition and uniform boundedness in L^∞

The Laplacian Δ_{m} is sub-Markovian, that is to say,

$$\forall\, t > 0, \forall\, x \in M\colon \int_M H_{\mathrm{m}}(t,x,y)\,\mathrm{dm}(y) \leqslant 1. \tag{2.3}$$

An equivalent formulation is that

$$\|e^{-t\Delta_{\mathrm{m}}}\|_{L^\infty \to L^\infty} \leqslant 1.$$

We are interested in similar properties for Schrödinger operators. The nonnegativity of L implies that the semigroup $\left\{e^{-tL}\right\}_t$ is uniformly bounded on L^2; but it is not necessarily uniformly bounded on L^∞. However the above Doob transform guarantees that if the Schrödinger operator L has a zero eigenfunction h satisfying

$$1 \leqslant h \leqslant \gamma,$$

then the semigroup $\left\{e^{-tL}\right\}_t$ is uniformly bounded on L^∞.

Definition 2.12. — *We say that the Schrödinger operator $L = \Delta_{\mathrm{m}} - V$, with nonnegative potential V, is uniformly stable if any of the following equivalent conditions is satisfied.*

(1) $\sup_{t>0} \left\|e^{-tL}\right\|_{L^\infty \to L^\infty} < \infty$.
(2) $\sup_{t>0} \left\|e^{-tL}\right\|_{L^1 \to L^1} < \infty$.
(3) *There is a constant γ such that, for all $t > 0$ and all $x \in M$,*

$$\left(e^{-tL}1\right)(x) = \int_M H_L(t,x,y)\,\mathrm{dm}(y) \leqslant \gamma.$$

The equivalences follow from the study of Qi S. Zhang and Z. Zhao ([57], see also [60]).

Definition 2.13. — *Assume that $V \geqslant 0$ is not identically zero. We say that the Schrödinger operator $L = \Delta_{\mathrm{m}} - V$ is gaugeable with gaugeability constant $\gamma \geqslant 1$ if any of the following equivalent conditions is satisfied.*

(a) *There is an $h \in W^{1,2}_{\mathrm{loc}}$ such that*
$$Lh = 0 \text{ and } 1 \leqslant h \leqslant \gamma.$$

(b) *L is subcritical, i.e., it has a positive minimal Green kernel G_L, and there is a constant γ such that*
$$\forall\, x \in M\colon \int_M G_L(x,y)V(y)\,\mathrm{dm}(y) \leqslant \gamma - 1.$$

Proof of the equivalences in Definition 2.13. — If we assume that property (b) holds, then
$$h(x) = 1 + \int_M G_L(x,y)V(y)\,\mathrm{dm}(y)$$
is a bounded solution of the equation $Lh = 0$ such that $h \geqslant 1$, hence the property (a) holds.

If we assume that property (a) holds, the Doob transform implies that L is nonnegative. We have assumed that V is not identically zero, hence the nonnegativity of L implies that Δ_{m} is subcritical, now the Doob transform and the fact that h is bounded insure that the operator L is subcritical. For a relatively compact domain $\Omega \subset M$, we consider the solution of the Dirichlet boundary problem:
$$\begin{cases} \Delta_{\mathrm{m}} h_\Omega = V h_\Omega & \text{on } \Omega, \\ h_\Omega = 1 & \text{on } \partial\Omega. \end{cases}$$
Let $G_L(\,\cdot\,,\cdot\,;\Omega)$ denote the Green function of the operator L on Ω, with the Dirichlet boundary condition. Then we have $h_\Omega = 1 + v_\Omega$ where
$$v_\Omega = \int_M G_L(x,y;\Omega)V(y)\,\mathrm{dm}(y).$$
The maximum principle implies that
$$\frac{h}{\gamma} \leqslant h_\Omega \leqslant h,$$
and that $\Omega \mapsto h_\Omega$ is increasing, hence we can define
$$\widetilde{h}(x) = \lim_{\Omega \to M} h_\Omega(x)$$

and we have

$$\widetilde{h}(x) \leqslant \gamma,$$

and

$$\widetilde{h}(x) = 1 + \int_M G_L(x,y)V(y)\,\mathrm{dm}(y).$$

Hence the property (b) holds. □

Theorem 2.14 ([57], see also [60]). — *Let L be a Schrödinger operator with nonnegative potential V. The following relations hold.*

(a) *The gaugeability condition implies the uniform stability condition.*
(b) *If (M,g,m) is stochastically complete, i.e., $\forall\, t>0\colon\ \left(e^{-t\Delta_{\mathrm{m}}}1\right)=1$, then the gaugeability condition is equivalent to the uniform stability condition.*
(c) *If the operator Δ_{m} is subcritical and if the Kato constant of V is smaller than* 1*,*

$$\mathrm{K}(V) := \sup_{x\in M}\int_M G_{\mathrm{m}}(x,y)V(y)\,\mathrm{dm}(y) < 1,$$

then $L=\Delta_{\mathrm{m}}-V$ is gaugeable

Remark 2.15. — According to [31], (M,g,m) is stochastically complete provided that there is some $o\in M$ and some positive constant c such that for any $R>0$:

$$\mathrm{m}\left(B(o,R)\right) \leqslant ce^{cR^2}.$$

Proof of Theorem 2.14. — Let's explain why under the stochastically completeness assumption, the uniform stability implies the gaugeability.

The stochastic completeness condition implies that for all $x\in M$, the function $t\mapsto\int_M H_L(t,x,y)\,\mathrm{dm}(y)$ is nondecreasing. Indeed, the semigroup property implies that if $t,\tau>0$ then

$$\int_M H_L(t+\tau,x,y)\,\mathrm{dm}(y) = \int_{M\times M} H_L(t,x,z)H_L(\tau,z,y)\,\mathrm{dm}(z)\,\mathrm{dm}(y).$$

Using $H_L(\tau,z,y)\geqslant H_{\mathrm{m}}(\tau,z,y)$ and $\int_M H_{\mathrm{m}}(\tau,z,y)\,\mathrm{dm}(y)=1$, one gets:

$$\int_M H_L(t+\tau,x,y)\,\mathrm{dm}(y) \geqslant \int_M H_L(t,x,y)\,\mathrm{dm}(y).$$

Hence if the condition (c) is satisfied then we can define

$$h(x) = \sup_{t>0}\int_M H_L(t,x,y)\,\mathrm{dm}(y) = \lim_{t\to+\infty}\int_M H_L(t,x,y)\,\mathrm{dm}(y).$$

We have $1\leqslant h\leqslant\gamma$ and for all $\tau>0$:

$$\int_M H_L(\tau,x,y)h(y)\,\mathrm{dm}(y) = h(x).$$

Hence $Lh = 0$. □

Remark 2.16. — The set

$$\{\lambda \geqslant 0, \Delta_{\mathrm{m}} - \lambda V \text{ is gaugeable (resp. uniformly stable)}\}$$

is an interval of the type $[0, \omega)$ or $[0, \omega]$.

2.3.4. Subcriticality, Green kernel and parabolicity

The subcriticality of a Schrödinger operator L is a strengthening of the nonnegativity property.

PROPOSITION 2.17 ([60]). — *For a Schrödinger operator L with nonnegative potential V, we have the following equivalent properties:*

(1) *L is subcritical.*
(2) *There is a non-empty open set $\Omega \subset M$ and a positive constant κ such that*
$$\forall\, \psi \in \mathcal{C}_0^\infty(M) \colon \kappa \int_\Omega \psi^2 \,\mathrm{dm} \leqslant \int_M \left[|\mathrm{d}\psi|_g^2 - V\psi^2\right] \mathrm{dm}\,.$$
(3) *For all relatively compact open subset $\Omega \subset M$, there is a positive constant κ such that*
$$\forall\, \psi \in \mathcal{C}_0^\infty(M) \colon \kappa \int_\Omega \psi^2 \,\mathrm{dm} \leqslant \int_M \left[|\mathrm{d}\psi|_g^2 - V\psi^2\right] \mathrm{dm}\,.$$
(4) *There is a positive Green kernel for L.*
(5) *If $h \in W^{1,2}_{\mathrm{loc}}$ is a positive solution of the equation $Lh = 0$, then $(M, g, h^2 m)$ is nonparabolic (see Remark 2.11).*
(6) *If $h \in W^{1,2}_{\mathrm{loc}}$ is a positive solution of the equation $Lh = 0$, then the operator $\Delta_{h^2 m}$ has a positive Green kernel.*

2.3.5. Elliptic estimates for Schrödinger operators

The Euclidean Sobolev inequality and the gaugeability property imply good estimates on the Green kernel of the operator L.

THEOREM 2.18. — *Let (M, g, m) be a weighted Riemannian manifold and assume that it satisfies the Euclidean Sobolev inequality $(\mathrm{Sob}_{\mathrm{m}})$ with constant μ. Let $L = \Delta_{\mathrm{m}} - V$ be a Schrödinger operator with nonnegative potential V and assume that L is gaugeable with gaugeability constant γ. Then there is a positive constant c_n such that the following properties hold:*

(1) *The heat kernel associated with the Laplacian* Δ_L *satisfies*

$$\forall\, x \in M, \forall\, t > 0\colon\ H_L(t,x,x) \leqslant \frac{c_n\gamma^n}{(\mu t)^{\frac{n}{2}}}.$$

(2) *The associated positive minimal Green kernel satisfies:*

$$\forall\, x, y \in M\colon\ G_L(x,y) \leqslant \frac{c_n}{\mu^{\frac{n}{2}}} \frac{\gamma^n}{d^{n-2}(x,y)}.$$

(3) *Let* $B(x,r) \subset M$ *be a relatively compact geodesic balls in* M*, and Assume that* $v \in W^{1,2}_{\mathrm{loc}}(B(x,r))$ *satisfies*

$$Lv \leqslant 0.$$

Then, for $p \geqslant 2$ *there is a positive constant* $C(n,p)$ *such that*

$$|v(x)|^p \leqslant \frac{C(n,p)}{\left(\sqrt{\mu}\, r\right)^n} \gamma^{n-2+p} \int_{B(x,r)} |v|^p(y)\, \mathrm{dm}(y).$$

All these results follow from the Doob transform and the fact that the new measure $d\widetilde{\mathrm{m}} = h^2\,\mathrm{dm}$ satisfies the Sobolev inequality:

$$\forall\, \psi \in \mathcal{C}^\infty_0(M):\ \mu\gamma^{-\frac{2}{n}(n-2)} \left(\int_M |\psi|^{\frac{2n}{n-2}} \mathrm{d}\widetilde{\mathrm{m}}\right)^{1-\frac{2}{n}} \leqslant \int_M |\mathrm{d}\psi|^2 \mathrm{d}\widetilde{\mathrm{m}}.$$

2.3.6. Estimate on the gaugeability constant

In [25], B. Devyver has studied conditions under which a nonnegative Schrödinger operator is gaugeable. One of his results is the following.

Theorem 2.19. — *Let* (M^n, g, m) *be a complete weighted Riemannian manifold and* $V \in L^\infty_{\mathrm{loc}}$ *a nonnegative function. Assume that the Schrödinger operator* $L = \Delta_{\mathrm{m}} - V$ *is strongly positive: there is some* $\delta > 0$ *such that the operator* $\Delta_{\mathrm{m}} - (1+\delta)V$ *is nonnegative:*

$$\forall\, \psi \in \mathcal{C}^\infty_0(M) : (1+\delta) \int_M V\psi^2\, \mathrm{dm} \leqslant \int_M |\mathrm{d}\psi|^2_g\, \mathrm{dm}\,.$$

Assume moreover that the Kato constant of V *is small at infinity: there is a compact subset* $K \subset M$ *and some* $\varepsilon \in (0,1)$ *such that*

$$\forall\, x \notin K\colon\ \int_{M\setminus K} G_{\mathrm{m}}(x,y)V(y)\, \mathrm{dm}(y) \leqslant 1-\varepsilon,$$

then L *is gaugeable.*

Moreover [24], B. Devyver has shown:

Theorem 2.20. — *Let* (M, g, m) *be a weighted Riemannian manifold and assume that it satisfies the Euclidean Sobolev inequality* $(\mathrm{Sob_m})$*. Let* $V \in L^\infty_{\mathrm{loc}}$ *be a nonnegative function such that*

- *for some* $\varepsilon \in (0,1)$, $V \in L^{(1\pm\varepsilon)\frac{n}{2}}$,
- $\ker_{L^{\frac{2n}{n-2}}} L = \left\{v \in L^{\frac{2n}{n-2}}(M, \mathrm{dm}) : Lv = 0\right\} = \{0\}$.

Then, L is gaugeable.

For geometric applications, it is sometimes useful to obtain explicit bounds on the function h used in the Doob transform. The second hypothesis in Theorem 2.20 is satisfied whenever

$$\int_M V^{\frac{n}{2}}\,\mathrm{dm} \leqslant (1-\varepsilon)\mu,$$

and in this case we can follow the argument given in [24] in order to get an estimate of $\|h\|_\infty$ which only depends on n, μ, ε, $\int_M V^{(1-\epsilon)\frac{n}{2}}\,\mathrm{dm}$ and $\int_M V^{(1+\epsilon)\frac{n}{2}}\,\mathrm{dm}$. The next proposition gives such a local estimate.

Proposition 2.21. — *Let* (M^n, g, m) *be a weighted Riemannian manifold and* $B(x, 2R) \subset M$ *be a relatively compact geodesic ball. Let* $V \in L^\infty_{\mathrm{loc}}$ *be a nonnegative function. Assume that for some constant* $\mu, \delta > 0$, $p > 1$ *and* $\Lambda > 0$, *the following conditions hold.*

- *The ball* $B(x, 2R)$ *satisfies the Euclidean Sobolev inequality* $(\mathrm{Sob_m})$ *with Sobolev constant* μ.
- *The Schrödinger operator L is strongly positive:*

$$\forall\, \psi \in \mathcal{C}^\infty_0\left(B(x,2R)\right) : (1+\delta)\int_{B(x,2R)} V\psi^2\,\mathrm{dm} \leqslant \int_{B(x,2R)} |\mathrm{d}\psi|^2_g\,\mathrm{dm}\,.$$

- *If* $G_{\mathrm{m}}(z,y)$ *is the Dirichlet Green kernel of the Laplacien* Δ_{m} *on* $B(x,R)$, *then :*

$$R^{2(p-1)} \sup_{z\in B(x,R)} \int_{B(x,R)} G_{\mathrm{m}}(z,y) V^p(y)\,\mathrm{dm}(y) \leqslant \Lambda^p.$$

Then there is a constant γ depending only on $n, p, \Lambda, \delta, \frac{m(B(x,2R))}{\mu^{\frac{n}{2}} R^n}$ *such that the solution of the Dirichlet boundary problem:*

$$\begin{cases} \Delta_{\mathrm{m}} h - Vh = 0 & \text{on } B(x,R) \\ h = 1 & \text{on } \partial B(x,R) \end{cases}$$

satisfies

$$1 \leqslant h \leqslant \gamma.$$

Proof. — By scaling, we can suppose that $R=1$ and let $B:=B(x,1)$ and $2B:=B(x,2)$.

We first get an integral estimate on $v:=h-1$. If $W_0^{1,2}(B)$ is the closure of $\mathcal{C}_0^\infty(B)$ for the norm $\psi\mapsto \|\mathrm{d}\psi\|_2+\|\psi\|_2$, we have $v\in W_0^{1,2}(B)$ and

$$\Delta_{\mathrm{m}} v - Vv = V,$$

hence

$$\int_B \left[|\mathrm{d}v|_g^2 - Vv^2\right]\mathrm{dm} = \int_B Vv\,\mathrm{dm} \leqslant \|v\|_\infty \int_B V\,\mathrm{dm}\,.$$

We let

$$L := \|v\|_\infty.$$

Using the strong positivity and the function

$$\xi(y) = \min\left\{2-d(x,y),1\right\},$$

we get

$$\int_B V\,\mathrm{dm} \leqslant \int_{2B} V\xi^2\,\mathrm{dm} \leqslant \frac{1}{1+\delta}\int_{2B} |\mathrm{d}\xi|_g^2\,\mathrm{dm} \leqslant m(2B).$$

Using again the strong positivity and the Sobolev inequality we get:

$$\frac{\mu\delta}{1+\delta}\left(\int_B v^{\frac{2n}{n-2}}\,\mathrm{dm}\right)^{1-\frac{2}{n}} \leqslant \frac{\delta}{1+\delta}\int_B |\mathrm{d}v|_g^2\,\mathrm{dm} \leqslant \int_B \left[|\mathrm{d}v|_g^2 - Vv^2\right]\mathrm{dm}\,.$$

So that we get

$$\left(\int_B v^{\frac{2n}{n-2}}\,\mathrm{dm}\right)^{1-\frac{2}{n}} \leqslant L\,\frac{m(2B)}{\mu\delta}. \tag{2.4}$$

The function v is a solution of the integral equation:

$$v(z) = \int_B G_{\mathrm{m}}(z,y)V(y)\,\mathrm{dm}(y) + \int_B G_{\mathrm{m}}(z,y)V(y)v(y)\,\mathrm{dm}(y). \tag{2.5}$$

Let $q=p/(p-1)$, using Hölder inequality and the integral estimate (2c) in Theorem 2.8, we estimate the first term in the RHS of (2.5)

$$\int_B G_{\mathrm{m}}(z,y)V(y)\,\mathrm{dm}(y) \leqslant \Lambda\left(\int_B G(z,y)\,\mathrm{dm}(y)\right)^{\frac{1}{q}} \leqslant \Lambda\left(\frac{m(B)}{\mu^{\frac{n}{2}}}\right)^{\frac{2}{nq}}.$$

Introducing

$$\mathbf{I} = \Lambda\left(\frac{m(2B)}{\mu^{\frac{n}{2}}}\right)^{\frac{2}{nq}},$$

we get

$$\int_B G_{\mathrm{m}}(z,y)V(y)\,\mathrm{dm}(y) \leqslant \mathbf{I}. \tag{2.6}$$

For the second term in the RHS of (2.5), we introduce:

$$\psi(z) := \int_B G_{\mathrm{m}}(z,y)v(y)\,\mathrm{dm}(y).$$

Then using again the Hölder inequality, we get:

$$\begin{aligned}(2.7)\quad \int_B G_{\mathrm{m}}(z,y)V(y)v(y)\,\mathrm{dm}(y)&\\ &\leqslant \left(\int_B G_{\mathrm{m}}(z,y)V^p(y)v(y)\,\mathrm{dm}(y)\right)^{\frac{1}{p}}\psi^{\frac{1}{q}}(z)\\ &\leqslant \Lambda L^{\frac{1}{p}}\psi^{\frac{1}{q}}(z).\end{aligned}$$

If β is such that

$$\beta > \frac{n}{2} \text{ and } \beta \geqslant \frac{2n}{n-2}$$

then with $\alpha = \beta/(\beta-1)$ and the integral estimate (2c) in Theorem 2.8 we get:

$$\begin{aligned}\psi(z) &\leqslant \left(\frac{n}{(n-2)\alpha-n}\right)^{\frac{1}{\alpha}}\frac{1}{\mu}\,(m(B))^{\frac{1}{\alpha}-1+\frac{2}{n}}\,\|v\|_\beta\\ &\leqslant \left(\frac{n}{(n-2)\alpha-n}\right)^{\frac{1}{\alpha}}\frac{1}{\mu}\,(m(2B))^{\frac{1}{\alpha}-1+\frac{2}{n}}\,\|v\|_\beta.\end{aligned}$$

The estimate (2.4) implies that:

$$\|v\|_\beta \leqslant L^{1-\frac{n}{n-2}\frac{1}{\beta}}\left(\frac{m(2B)}{\delta\mu}\right)^{\frac{n}{n-2}\frac{1}{\beta}}.$$

After a bit of arithmetic, we get that:

$$\begin{aligned}(2.8)\quad \int_B G_{\mathrm{m}}(z,y)V(y)v(y)\,\mathrm{dm}(y)&\\ &\leqslant \mathbf{I}\,L^{1-\frac{n}{n-2}\frac{1}{q\beta}}\left(\frac{m(2B)}{(\delta\mu)^{\frac{n}{2}}}\right)^{\frac{2}{n-2}\frac{1}{q\beta}}\left(\frac{n}{(n-2)\alpha-n}\right)^{\frac{1}{q\alpha}}.\end{aligned}$$

With (2.4) and (2.8), we get

$$L \leqslant \mathbf{I} + \mathbf{C}^\kappa L^{1-\kappa}$$

where $\kappa = \frac{n}{n-2}\frac{1}{q\beta}$ and

$$\mathbf{C}^\kappa = \mathbf{I}\left(\frac{m(2B)}{(\delta\mu)^{\frac{n}{2}}}\right)^{\frac{2}{n-2}\frac{1}{q\beta}}\left(\frac{n}{(n-2)\alpha-n}\right)^{\frac{1}{q\alpha}}.$$

In order to conclude, we distinguish two cases:

- The first one being when $\mathbf{I} \leqslant \frac{1}{2}$. Because the Kato constant of V is smaller than $\mathbf{I}$, we know that $1 \leqslant h \leqslant 2$.
- The second case is when $\mathbf{I} \geqslant \frac{1}{2}$. The above inequality implies that
$$L \leqslant \max\{2\mathbf{I}, 2^{\frac{1}{\kappa}}\mathbf{C}\}.$$
But
$$2^{\frac{1}{\kappa}}\mathbf{C} = c(n,p,\beta)\mathbf{I}^{q\beta\frac{n-2}{n}}\frac{m^{\frac{2}{n}}(2B)}{\delta\mu},$$
and recall that by Theorem 2.8 (4), the Sobolev inequality implies that $\frac{m^{\frac{2}{n}}(2B)}{\mu}$ is bounded from below by a constant which depends only of n and that $q\beta\frac{n-2}{n} > 1$, hence there is a constant c such that $\mathbf{C} \geqslant c\mathbf{I}$ and we get:
$$L \leqslant c(n,p,\beta)\mathbf{I}^{q\beta\frac{n-2}{n}}\frac{m^{\frac{2}{n}}(2B)}{\min\{\delta,1\}\mu}. \qquad \square$$

2.4. Kato constants

In this section, we compare the parabolic and elliptic Kato constants. This comparison already appears in [33]. Let (M^n, g, m) be a weighted Riemannian manifold and $V \in L^\infty_{\mathrm{loc}}$ be a nonnegative function. For $\lambda \geqslant 0$, we define the elliptic Kato constant

$$\mathrm{K}_\lambda(V) = \left\|(\Delta_{\mathrm{m}} + \lambda)^{-1} V\right\|_\infty. \tag{2.9}$$

If $G_{\mathrm{m},\lambda}(x,y)$ is the Green kernel of the operator $(\Delta_{\mathrm{m}} + \lambda)$, then

$$\mathrm{K}_\lambda(V) = \sup_{x\in M}\int_M G_{\mathrm{m},\lambda}(x,y)V(y)\,\mathrm{dm}\,.$$

If $T > 0$, the parabolic Kato constant of V is defined by

$$\begin{aligned}\mathrm{k}_T(V) &= \sup_{x\in M}\int_0^T\int_M H_{\mathrm{m}}(t,x,y)V(y)\,\mathrm{dv}_g(y)\mathrm{d}t \\ &= \left\|\int_0^T e^{-t\Delta_{\mathrm{m}}}V\mathrm{d}t\right\|_\infty.\end{aligned} \tag{2.10}$$

We always have

$$\mathrm{K}_0(V) = \mathrm{k}_{+\infty}(V)$$

and

Lemma 2.22. — *For any $T > 0$:*

$$e^{-1}\,\mathrm{k}_T(V) \leqslant \mathrm{K}_{\frac{1}{T}}(V) \leqslant \frac{e}{e-1}\,\mathrm{k}_T(V).$$

Proof. — We have the relationship:

$$(\Delta_{\mathrm{m}} + \lambda)^{-1}\,V = \int_0^{+\infty} e^{-\lambda t} e^{-t\Delta_{\mathrm{m}}} V \mathrm{d}t.$$

Hence

$$(\Delta_{\mathrm{m}} + \lambda)^{-1}\,V \geqslant \int_0^{T} e^{-\lambda T} e^{-t\Delta_{\mathrm{m}}} V \mathrm{d}t,$$

it is then easy to deduce the lower bound:

$$e^{-1}\,\mathrm{k}_T(V) \leqslant \mathrm{K}_{\frac{1}{T}}(V).$$

We also have

$$\begin{aligned}(\Delta_{\mathrm{m}} + \lambda)^{-1}\,V &= \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} e^{-\lambda t} e^{-t\Delta_{\mathrm{m}}} V \mathrm{d}t \\ &= \sum_{k=0}^{\infty} e^{-\lambda kT} e^{-kT\Delta_{\mathrm{m}}} \int_0^{T} e^{-\lambda t} e^{-t\Delta_{\mathrm{m}}} V \mathrm{d}t.\end{aligned}$$

Recall that the heat semigroup is sub-Markovian:

$$\left\| e^{-kT\Delta_{\mathrm{m}}} \right\|_{L^\infty \to L^\infty} \leqslant 1,$$

hence

$$\begin{aligned}\left\|(\Delta_{\mathrm{m}} + \lambda)^{-1}\,V\right\|_\infty &\leqslant \sum_{k=0}^{\infty} e^{-\lambda kT} \left\| \int_0^{T} e^{-t\Delta_{\mathrm{m}}} V \mathrm{d}t \right\|_\infty \\ &\leqslant \sum_{k=0}^{\infty} e^{-\lambda kT}\,\mathrm{k}_T(V) = \frac{1}{1-e^{-\lambda T}}\,\mathrm{k}_T(V).\end{aligned}$$ □

2.5. Agmon's volume estimate

In this subsection, we review Agmon's volume estimates. These estimates are due to S. Agmon [1] and P. Li and J. Wang [40, 41]). We give a slightly different proof, as well as a new Hardy type estimate (Proposition 2.26). This new result will be crucial in the proof of Theorem 1.4. The starting result is the following.

PROPOSITION 2.23. — *Let* m *be a locally finite positive measure on* $\mathbb{R}_+$. *Assume that it satisfies the following spectral gap estimate,*

$$\text{(SG)} \qquad \forall\, \psi \in \mathcal{C}_0^\infty\left(\mathbb{R}_+^*\right) : \ \frac{h^2}{4}\int_{\mathbb{R}_+} \psi^2(t)\,\mathrm{dm}(t) \leqslant \int_{\mathbb{R}_+} \psi'^2(t)\,\mathrm{dm}(t).$$

Then, we have the dichotomy:

(1) *Either* $m(\mathbb{R}_+) < +\infty$, *and* $m\left([R,+\infty)\right) = \mathcal{O}\left(e^{-hR}\right)$.
(2) *Or* $m(\mathbb{R}_+) = +\infty$, *and there is some positive constant* C *such that, for all* $R \geqslant 1$,
$$m\left([0,R]\right) \geqslant Ce^{hR}.$$

Remark 2.24. — In the proof, we can always assume that m is a smooth measure
$$\mathrm{dm} = L(t)\mathrm{d}t.$$
Indeed, let ρ be a smooth nonnegative function with compact support in $(0,1)$ which satisfies $\int_0^1 \rho(t)dt = 1$. Define the family of smooth measures m_ε by
$$\mathrm{m}_\varepsilon(f) = \int_{\mathbb{R}_+\times\mathbb{R}} f(\tau-\varepsilon t)\rho(t)\,\mathrm{dm}(\tau)\mathrm{d}t.$$
We have that m_ε converges weakly to m when $\epsilon \to 0+$, and each m_ε satisfies the spectral gap inequality (SG) with the same constant.

Proof. — Let's first consider the case where the measure m is parabolic. This implies that the spectral gap estimate (SG) is valid for any smooth function with support in $\mathbb{R}_+^*$ and constant outside some compact set, in particular:
$$\mathrm{m}(\mathbb{R}_+) < +\infty.$$
We introduce the cut-off function:
$$\xi(t) = \begin{cases} 0 & \text{if } t \leqslant 1,\\ t-1 & \text{if } 1 \leqslant t \leqslant 2,\\ 1 & \text{if } 2 \leqslant t. \end{cases}$$
We test the spectral gap estimate (SG) on the function
$$\psi_R(t) = \xi(t)\, e^{h\frac{\min\{t,R\}}{2}},$$
and when $R \geqslant 2$, we get the estimate
$$\frac{h^2}{4}\int_1^2 \psi_R^2(t)\,\mathrm{dm}(t) + \frac{h^2}{4}e^{hR}m\left([R,+\infty)\right) \leqslant \int_1^2 \psi_R'^2(t)\,\mathrm{dm}(t).$$
Hence
$$\frac{h^2}{4}e^{hR}m\left([R,+\infty)\right) \leqslant (1+h)^2e^{2h}\, m\left([1,2]\right).$$

In the second case, the measure m is nonparabolic and necessarily

$$m\left(\mathbb{R}_+\right)=+\infty.$$

According to the Remark 2.24, we can always assume that m is smooth: $\mathrm{dm}=L(t)\mathrm{d}t$. We introduce the function

$$g(t)=\int_t^\infty \frac{\mathrm{d}s}{L(s)}.$$

The measure m being nonparabolic, we know that g is well defined. Moreover g is a harmonic function for the Laplacian

$$\Delta_{\mathrm{m}}=-\frac{1}{L(t)}\frac{\mathrm{d}}{\mathrm{d}t}L(t)\frac{\mathrm{d}}{\mathrm{d}t}.$$

Hence the spectral gap estimate (SG) implies that

$$\int_{\mathbb{R}_+} g^2(t)\,\mathrm{dm}(t)<+\infty$$

Indeed if we test the spectral gap estimate (SG) on the function

$$g_R(t)=\begin{cases}\xi(t)\ \int_t^R \frac{\mathrm{d}s}{L(s)} & \text{if } t\leqslant R\\ 0 & \text{if } t\geqslant R\end{cases}$$

then we get that:

$$\frac{h^2}{4}\int_2^R g_R^2(t)\,\mathrm{dm}(t)\leqslant c+g_R(2),$$

for some constant c independent of R.

Using the Doob transform with the function g (cf the formula 2.2), we get for any function $\psi\in\mathcal{C}_0^\infty\left(\mathbb{R}_+^*\right)$:

$$\begin{aligned}\int_0^\infty\left((\psi g)'\right)^2\mathrm{dm}&=\int_0^\infty\left(\psi'\right)^2 g^2\,\mathrm{dm}+\int_0^\infty \psi^2 g(Lg)\,\mathrm{dm}\\&=\int_0^\infty\left(\psi'\right)^2 g^2\,\mathrm{dm}\,.\end{aligned}$$

Hence we get the spectral gap estimate (SG):

$$\forall\,\psi\in\mathcal{C}_0^\infty\left(\mathbb{R}_+^*\right):\ \frac{h^2}{4}\int_{\mathbb{R}_+}\psi^2(t)g^2(t)\,\mathrm{dm}(t)\leqslant\int_{\mathbb{R}_+}\psi'^2(t)g^2(t)\,\mathrm{dm}(t).$$

The new measure $g^2\,\mathrm{dm}$ is finite hence we already know that there is a constant c such that for all $R\geqslant 0$ then,

$$\int_R^\infty g^2(t)\,\mathrm{dm}(t)\leqslant ce^{-hR}.$$

Choosing $\psi(t)=\xi(t-R+1)$, we get

$$\int_R^{R+1}\left(g'\right)^2(t)\,\mathrm{dm}(t)\leqslant\int_{R-1}^{\infty}g^2(t)\,\mathrm{dm}(t)\leqslant ce^{-hR}.$$

It follows that

$$\int_R^{R+1}\frac{\mathrm{d}s}{L(s)}\leqslant ce^{-hR},$$

and the Cauchy–Schwarz inequality yields

$$1\leqslant\left(\int_R^{R+1}\frac{\mathrm{d}s}{L(s)}\right)\left(\int_R^{R+1}L(s)\mathrm{d}s\right)\leqslant ce^{-hR}m\left([R,R+1]\right). \qquad \square$$

Let us now give some consequences of Proposition 2.23. The following corollary is borrowed from [40].

Corollary 2.25. — *Let (M,g,m) be a complete weighted Riemannian manifold. Let $K\subset M$ be a compact set, $\mathcal{U}\subset M\setminus K$ be an unbounded connected component of $M\setminus K$ satisfying the spectral gap condition,*

$$\forall\,\psi\in\mathcal{C}_0^\infty\left(\mathcal{U}\right):\ \frac{\lambda_0^2}{4}\int_{\mathcal{U}}\psi^2\,\mathrm{dm}\leqslant\int_M|\mathrm{d}\psi|^2\,\mathrm{dm}\,.$$

Assume that $f\colon\mathcal{U}\to\mathbb{R}$ and $\lambda<\lambda_0$ satisfy:

$$\Delta_{\mathrm{m}}f\leqslant\lambda f.$$

Let $h=2\sqrt{\lambda-\lambda_0}$. We have the dichotomy:

(1) *Either $f\in L^2$ and $\int_{\mathcal{U}\setminus B(o,R)}f^2\,\mathrm{dm}=\mathcal{O}\left(e^{-hR}\right)$, when $R\to+\infty$.*
(2) *Or there is some positive constant C such that for all $R\geqslant1$:*

$$\int_{\mathcal{U}\cap B(o,R)}f^2\,\mathrm{dm}\geqslant Ce^{hR}.$$

Proof. — We test the above spectral gap for radial function

$$\psi(x)=f(d(K,x))$$

and with the measure

$$\mu([0,R])=\mathrm{m}\left(\{x\in\mathcal{U},d(x,K)<R\}\right),$$

we get

$$\forall\,\psi\in\mathcal{C}_0^\infty\left(\mathbb{R}_+^*\right):\ \frac{h^2}{4}\int_{\mathbb{R}_+}f^2(t)\mathrm{d}\mu\leqslant\int_{\mathbb{R}_+}f'(t)^2\mathrm{d}\mu$$

The corollary is then a direct consequence of Proposition 2.23. $\square$

A logarithmic change of variables yields the following consequence for a Hardy type inequality.

Proposition 2.26. — *Let m be a locally finite positive measure on* $[1,\infty)$. *Assume that the following Hardy type inequality holds,*

$$\forall\, \psi \in \mathcal{C}_0^\infty\left((1,\infty)\right): \ \frac{(\nu-2)^2}{4}\int_1^{+\infty} \frac{\psi^2(t)}{t^2}\,\mathrm{dm}(t) \leqslant \int_1^{+\infty} \psi'^2(t)\,\mathrm{dm}(t).$$

Then, we have the dichotomy:

(1) *Either* $\int_1^{+\infty} \frac{\mathrm{dm}(t)}{t^2} < +\infty$ *and*

$$\int_R^{+\infty} \frac{\mathrm{dm}(t)}{t^2} = \mathcal{O}\left(\frac{1}{R^{\nu-2}}\right)$$

when $R \to +\infty$,

(2) *or there is some positive constant C such that for all* $R \geqslant 1$:

$$\int_{[1,R]} \mathrm{dm}(t) \geqslant CR^\nu.$$

2.6. Asymptotics of the Green kernel

2.6.1. Near the pole

Let $L = \Delta_{\mathrm{m}} - V$ be a Schrödinger operator on a weighted smooth Riemannian manifold (M, g, m) of dimension $n > 2$, with a smooth potential. If G is a positive solution of the equation

$$LG = \delta_o,$$

then according to [37, Section 17.4] G has a polyhomogeneous expansion near o whose first term is

$$G(x) \simeq \frac{c_n}{d^{n-2}(x,o)},$$

where $c_n = \left((n-2)\operatorname{vol}\mathbb{S}^{n-1}\right)^{-1}$. More precisely, letting $r(x) := d(x,o)$, there is some function $\psi \in C^1(M)$ such that

$$G = \frac{\psi}{r^{n-2}} \ \text{ and } \ \psi(o) = c_n.$$

In particular, defining $b\colon M \to \mathbb{R}_+$ by $G = c_n b^{2-n}$, then $b(o) = 0$ and

$$|\mathrm{d}b|(x) = 1 + \mathcal{O}\left(d(o,x)\right).$$

In general if V is not smooth but only locally bounded, we get that

$$G(x) \simeq \frac{c(n)}{d^{n-2}(x,o)}.$$

2.6.2. Near infinity

Let (M, g, m) be a nonparabolic weighted Riemannian manifold, with minimal positive Green kernel G_{m}. Let $o \in M$ and let K be a compact subset of M containing o in its interior. Then,

$$\int_{M\setminus K} |\mathrm{d}_x G_{\mathrm{m}}(o,x)|^2 \,\mathrm{dm}(x) < \infty.$$

Indeed, we can always assume that the boundary of K is smooth. When Ω is a relatively compact open subset of M, containing K, we let G_{m}^{Ω} denote the minimal positive Green kernel of (Ω, g, m). We know that

$$\lim_{\Omega\to M} G_{\mathrm{m}}^{\Omega}(o,x) = G_{\mathrm{m}}(o,x)$$

where the convergence is in $\mathcal{C}^\infty(M \setminus \{o\})$. But the Green formula yields that

$$\int_{M\setminus K} |\mathrm{d}_x G_{\mathrm{m}}^{\Omega}(o,x)|^2 \,\mathrm{dm}(x) = \int_{\partial K} G_{\mathrm{m}}^{\Omega}(o,x) \frac{\partial G_{\mathrm{m}}^{\Omega}}{\partial \vec{\nu}_x}(o,x)\mathrm{d}\sigma(x),$$

where $\vec{\nu}\colon \partial K \to TM$ is the unit normal inward normal to K. Hence

$$\int_{M\setminus K} |\mathrm{d}_x G_{\mathrm{m}}(o,x)|^2 \,\mathrm{dm}(x) \leqslant \int_{\partial K} G_{\mathrm{m}}(o,x) \frac{\partial G_{\mathrm{m}}}{\partial \vec{\nu}_x}(o,x)\mathrm{d}\sigma(x).$$

We are now interested in the equality case in the above inequality.

Proposition 2.27. — *Assume that* $\lim_{x\to\infty} G(o,x) = 0$ *then*

$$\int_{M\setminus K} |\mathrm{d}_x G_{\mathrm{m}}(o,x)|^2 \,\mathrm{dm}(x) = \int_{\partial K} G_{\mathrm{m}}(o,x) \frac{\partial G_{\mathrm{m}}}{\partial \vec{\nu}_x}(o,x)\mathrm{d}\sigma(x). \tag{2.11}$$

Moreover the measure $G_{\mathrm{m}}(o,x)^2\,\mathrm{dm}(x)$ *is parabolic on* $M \setminus K$.

Proof. — Let $\ell > 0$, our hypothesis implies that the set $\{x \in M, G_{\mathrm{m}}(o,x) \geqslant \frac{1}{\ell}\} \cup \{o\}$ is compact. Let u be a smooth function on $\mathbb{R}_+$ such that $|u'| \leqslant 2$, $u = 0$ on $[0,1]$ and $u = 1$ on $[2,+\infty)$. We introduce the cut-off function defined by:

$$\xi_\ell(x) = u\left(\ell G_{\mathrm{m}}(o,x)\right)$$

Let $\epsilon := \inf_{x\in\partial K} G_{\mathrm{m}}(o,x)$. If $\ell\epsilon > 1$ then the maximum principle guarantees the inclusion:

$$\left\{x, G_{\mathrm{m}}(o,x) \leqslant \frac{1}{\ell}\right\} \subset M \setminus K.$$

The Green formula yields:

$$\int_{M\setminus K} \left|\mathrm{d}_x \left(\xi_\ell(x) G_{\mathrm{m}}(o,x)\right)\right|^2 \mathrm{dm}(x) = \int_{M\setminus K} |\mathrm{d}\xi_\ell|^2\, G_{\mathrm{m}}(o,x)^2 \,\mathrm{dm}(x).$$

If we introduce

$$\Omega_\ell=\left\{x\in M,\ \frac{1}{\ell}\leqslant G_{\mathrm{m}}(o,x)\leqslant\frac{2}{\ell}\right\}$$

then we get

$$\begin{aligned}\int_{M\setminus K}\left|\mathrm{d}\xi_\ell\right|^2 G_{\mathrm{m}}(o,x)^2\,\mathrm{dm}(x)&\leqslant 4\ell^2\int_{\Omega_\ell}\left|\mathrm{d}_xG_{\mathrm{m}}(o,x)\right|^2 G_{\mathrm{m}}(o,x)^2\,\mathrm{dm}(x)\\&\leqslant 16\int_{\Omega_\ell}\left|\mathrm{d}_xG_{\mathrm{m}}(o,x)\right|^2\mathrm{dm}(x).\end{aligned}$$

Hence

$$\lim_{\ell\to+\infty}\int_{M\setminus K}\left|\mathrm{d}\xi_\ell\right|^2 G_{\mathrm{m}}(o,x)^2\,\mathrm{dm}(x)=0,$$

and the equality (2.11) holds. Moreover the sequence (ξ_ℓ) satisfies the required properties of Definition 2.4 which show the parabolicity of the measure $G_{\mathrm{m}}(o,x)^2\,\mathrm{dm}(x)$ on $M\setminus K$. □

With the Doob transform, we have a similar result for Schrödinger operators.

Proposition 2.28. — *Let (M,g,m) be a nonparabolic weighted Riemannian manifold, and V a locally bounded nonnegative function. Assume that the Schrödinger operator $L=\Delta_{\mathrm{m}}-V$ is gaugeable and that for some $p\in M$, the Green kernel of L satisfies $\lim_{x\to\infty}G_L(p,x)=0$. Then, the measure $G_L^2(p,y)\,\mathrm{dv}_g(y)$ is parabolic on $M\setminus B(o,2)$.*

A last but useful property of the Green kernel, is the following very general Hardy type inequality [8].

Proposition 2.29. — *Let (M,g,m) be a nonparabolic weighted Riemannian manifold of dimension $n>2$. If $o\in M$, we let $b(x)=G_{\mathrm{m}}(o,x)^{-\frac{1}{n-2}}$, then*

$$\forall\,\psi\in\mathcal{C}_0^\infty(M)\colon\ \frac{(n-2)^2}{4}\int_M\frac{|\mathrm{d}b|^2}{b^2}\psi^2\,\mathrm{dm}\leqslant\int_M|\mathrm{d}\psi|^2\,\mathrm{dm}\,.$$

In fact when G is any positive harmonic function, the above inequality holds for $b=G^{-\frac{1}{n-2}}$.

2.7. Some formulas for the gradient of the Green kernel

The inequality (2.12) below is due to T. Colding and W. Minicozzi ([15, Corollary 2.13]).

Let G be a positive harmonic function on a Riemannian manifold (M^n, g) and $b := G^{-\frac{1}{n-2}}$. Then, for all $p \geqslant \frac{n-2}{n-1}$,

$$\Delta\left(G|\mathrm{d}b|^p\right) \leqslant p\,\mathrm{Ric}_-\left(G|\mathrm{d}b|^p\right). \tag{2.12}$$

This inequality can also be proved from Yau's inequality [56, Lemma 2]

$$\left|\nabla|\mathrm{d}G|\right|^2 \leqslant \frac{n-1}{n}\left|\nabla \mathrm{d}G\right|^2.$$

This inequality classically implies that

$$\Delta|\mathrm{d}G|^{\frac{n-2}{n-1}} - \frac{n-2}{n-1}\,\mathrm{Ric}_-\,|\mathrm{d}G|^{\frac{n-2}{n-1}} \leqslant 0. \tag{2.13}$$

Define b by $G = b^{2-n}$. Then,

$$|\mathrm{d}G|^{\frac{n-2}{n-1}} = (n-2)^{\frac{n-2}{n-1}}G\,|\mathrm{d}b|^{\frac{n-2}{n-1}}.$$

The Doob transform yields that $u := |\mathrm{d}b|^{\frac{n-2}{n-1}}$ satisfies

$$\Delta_{G^2}u \leqslant \frac{n-2}{n-1}\,\mathrm{Ric}_-\,u.$$

According to (2.1.2), we get that for all $\alpha \geqslant 1$:

$$\Delta_{G^2}u^\alpha \leqslant \alpha\frac{n-2}{n-1}\,\mathrm{Ric}_-\,u^\alpha.$$

Using the Doob transform again, we get $\Delta_{G^2}u^\alpha = G^{-1}\Delta\left(G\,|\mathrm{d}b|^{\alpha\frac{n-2}{n-1}}\right)$ and the inequality (2.12) follows.

2.8. An elliptic estimate

In this subsection, we obtain a new estimate for the gradient of a positive harmonic function; our result is based on a new variation on the De Giorgi–Nash–Moser iteration scheme.

Proposition 2.30. — *Let (M^n, g) be a Riemannian manifold which satisfies the Euclidean Sobolev inequality* (Sob) *with Sobolev constant μ, and assume that the Schrödinger operator $L = \Delta - \frac{n-2}{n-1}\,\mathrm{Ric}_-$ is gaugeable, with gaugeability constant γ. Let $G\colon M \longrightarrow \mathbb{R}_+^*$ be a positive harmonic function and define $b\colon M \longrightarrow \mathbb{R}_+^*$ by*

$$G = \frac{1}{b^{n-2}}.$$

Assume moreover that $R > 0$ is such that the set $\Omega_R^\# = \{x \in M; \frac{R}{2} \leqslant b(x) \leqslant \frac{5}{2}R\}$ is compact and let $\Omega_R = \{x \in M; R \leqslant b(x) \leqslant 2R\}$. Then, for any $p > n$,

$$\sup_{\Omega_R} |\mathrm{d}b|^{p-n} \leqslant \frac{C_n^{1+p}\gamma^{p\frac{n-1}{n-2}+n-2}}{\mu^{\frac{n}{2}} R^n} \int_{\Omega_R^\#} |\mathrm{d}b|^p \,\mathrm{dv}_g\,.$$

Remark 2.31. — The second hypothesis is satisfied when we have the following bound on the Kato constant of the Ricci curvature:

$$\mathrm{K}\left(\mathrm{Ric}_-\right) \leqslant \frac{n-1}{n-2}\left(1-\frac{1}{\gamma}\right).$$

Proof. — We let

$$f := G\,|\mathrm{d}b|^{\frac{n-2}{n-1}}\,.$$

Yau's inequality (2.13) implies that

$$\Delta f - \frac{n-2}{n-1}\,\mathrm{Ric}_-\, f \leqslant 0.$$

Let $h\colon M \longrightarrow [1,\gamma]$ be such that $Lh = 0$. Then the function $F = f/h$ satisfies $\Delta_{h^2} F \leqslant 0$ and for all $\alpha \geqslant 1$ we have:

$$\Delta_{h^2} F^\alpha \leqslant 0.$$

So that if $\xi \in \mathcal{C}_0^\infty(\Omega_R^\#)$, we have:

$$\int_M |\mathrm{d}(\xi F^\alpha)|^2\, h^2 \,\mathrm{dv}_g \leqslant \int_M |\mathrm{d}\xi|^2\, F^{2\alpha} h^2 \,\mathrm{dv}_g\,. \tag{2.14}$$

Moreover the Sobolev inequality and the fact that $1 \leqslant h \leqslant \gamma$ imply that for $\widehat{\mu} := \mu\gamma^{\frac{4}{n}-2}$, we have:

$$\widehat{\mu}\left(\int_M (\xi F^\alpha)^{\frac{2n}{n-2}}\, h^2 \,\mathrm{dv}_g\right)^{1-\frac{2}{n}} \leqslant \int_M |\mathrm{d}(\xi F^\alpha)|^2\, h^2 \,\mathrm{dv}_g\,. \tag{2.15}$$

We now define $\mathrm{dm} = h^2 \,\mathrm{dv}_g$, $\kappa := \frac{n}{n-2}$,

$$R_k = 2R + \sum_{\ell=k}^\infty \frac{R}{2^{\ell+2}} \text{ and } r_k = R - \sum_{\ell=k}^\infty \frac{R}{2^{\ell+2}} \text{ and } \Omega_k = \{b \in [r_k, R_k]\}.$$

We are going to use the inequalities (2.14), (2.15) with

$$\xi_k = \rho_k(b)$$

where

$$\rho_k = \begin{cases} 1 & \text{on } [r_{k+1}, R_{k+1}] \\ 0 & \text{outside } [r_k, R_k] \end{cases}$$

and

$$|\rho_k'| \leqslant \frac{2^{k+2}}{R}.$$

We get

$$\widehat{\mu}\left(\int_{\Omega_{k+1}} (F^{\alpha})^{\frac{2n}{n-2}}\,\mathrm{dm}\right)^{1-\frac{2}{n}} \leqslant \frac{4^{k+2}}{R^2}\int_{\Omega_k} |\mathrm{d}b|^2 F^{2\alpha}\,\mathrm{dm}\,.$$

But

$$|\mathrm{d}b|^2 F^{2\alpha} = F^{2\alpha+2\frac{n-1}{n-2}}\left(\frac{g}{h}\right)^{-2\frac{n-1}{n-2}}.$$

On Ω_k we have:

$$\left(\frac{g}{h}\right)^{-2\frac{n-1}{n-2}} \leqslant \gamma^{2\frac{n-1}{n-2}}\left(\frac{5R}{2}\right)^{2n-2}.$$

We now introduce

$$\beta_{k+1} = \kappa 2\alpha_k \text{ and } \beta_k = 2\alpha_k + 2\frac{n-1}{n-2},$$

where

$$\beta_k = \kappa^k\left(\beta_0 - n\frac{n-1}{n-2}\right) + n\frac{n-1}{n-2}.$$

We have proved the estimate:

$$\left(\int_{\Omega_{k+1}} F^{\beta_{k+1}}\,\mathrm{dm}\right)^{1-\frac{2}{n}} \leqslant \frac{\gamma^{2\frac{n-1}{n-2}}4^{k+2}}{\widehat{\mu}R^2}\left(\frac{5R}{2}\right)^{2n-2}\int_{\Omega_k} F^{\beta_k}\,\mathrm{dm}\,. \tag{2.16}$$

Hence for $p = \frac{n-2}{n-1}\beta_0$, by iteration we get

$$\lim_{k\to\infty}\left(\int_{\Omega_{k+1}} F^{\beta_{k+1}}\,\mathrm{dm}\right)^{\kappa^{-n}} \leqslant \Gamma\int_{\Omega_R^{\#}} F^{2\alpha_0}\,\mathrm{dm},$$

where

$$\Gamma = \left(\frac{16\gamma^{2\frac{n-1}{n-2}}}{\widehat{\mu}R^2}\left(\frac{5R}{2}\right)^{2n-2}\right)^{\sum_{\ell=0}^{\infty}\kappa^{-\ell}} 4^{\sum_{\ell=0}^{\infty}\ell\kappa^{-\ell}}.$$

But

$$\begin{aligned}\lim_{k\to\infty}\left(\int_{\Omega_{k+1}} F^{\beta_{k+1}}\,\mathrm{dm}\right)^{\kappa^{-n}} &= \sup_{\Omega_R} F^{\beta_0 - n\frac{n-1}{n-2}}\\ &= \sup_{\Omega_R} h^{-(p-n)\frac{n-1}{n-2}} b^{-(p-n)(n-1)}|\mathrm{d}b|^{p-n}\\ &\geqslant \gamma^{-(p-n)\frac{n-1}{n-2}}(2R)^{-(p-n)(n-1)}\sup_{\Omega_R}|\mathrm{d}b|^{p-n}\end{aligned}$$

and

$$\Gamma = c(n)\gamma^{n\frac{n-1}{n-2}}\widehat{\mu}^{-\frac{n}{2}}R^{n(n-2)}.$$

Moreover

$$\int_{\Omega_R^{\#}} F^{2\alpha_0}\,\mathrm{dm} = \int_{\Omega_R^{\#}} h^{-p\frac{n-1}{n-2}} b^{-p(n-1)} |\mathrm{d}b|^p h^2\,\mathrm{dv}_g$$
$$\leqslant \left(\frac{2}{R}\right)^{p(n-1)} \int_{\Omega_R^{\#}} |\mathrm{d}b|^p\,\mathrm{dv}_g\,.$$

Hence after a bit of arithmetic, we obtain:

$$\sup_{\Omega_R} |\mathrm{d}b|^{p-n} \leqslant \frac{c(n)4^p\gamma^{p\frac{n-1}{n-2}+n-2}}{\mu^{\frac{n}{2}} R^n} \int_{\Omega_R^{\#}} |\mathrm{d}b|^p\,\mathrm{dv}_g\,. \qquad \square$$

3. Case of closed manifolds

In this section, we elaborate on a recent result of Qi S. Zhang and M. Zhu [59] in order to obtain geometric and topological estimates based on the Kato constant of the Ricci curvature.

3.1. A differential inequality

In [59, p. 486], the authors prove the following:

Proposition 3.1. — *Let* (M^n, g) *be a* complete *Riemannian manifold, let* $u\colon [0,T]\times M \to \mathbb{R}$ *be a positive solution of the heat equation,*

$$\frac{\partial u}{\partial t} + \Delta u = 0,$$

and $J\colon [0,T]\times M \to \mathbb{R}$ *an auxiliary positive function. The function*

$$Q := \alpha J |\mathrm{d}\log u|^2 - \frac{\partial}{\partial t}\log u$$

satisfies

$$\left(\frac{\partial}{\partial t} + \Delta\right) Q - 2\langle \mathrm{d}\log u, \mathrm{d}\, Q\rangle$$
$$\leqslant \alpha |\mathrm{d}\log u|^2 \left(\frac{\partial J}{\partial t} + \Delta J + \frac{5}{\delta}\frac{|\mathrm{d}J|^2}{J} - 2\operatorname{Ric}_- J\right)$$
$$- (2-\delta)\alpha J |\nabla \mathrm{d}\log u|^2 + \delta\alpha J |\mathrm{d}\log u|^4\,.$$

3.2. Finding a good gauge function

We are now looking for a solution of the equation

$$\begin{cases} \frac{\partial}{\partial t} J + \Delta J + \frac{5}{\delta}\frac{|\mathrm{d}J|^2}{J} - 2\operatorname{Ric}_- J = 0, \\ J(0,x) = 1. \end{cases} \tag{3.1}$$

If we let

$$I := J^{-\frac{5-\delta}{\delta}} \text{ or } J = I^{-\frac{\delta}{5-\delta}},$$

the equation is equivalent to

$$\begin{cases} \frac{\partial}{\partial t} I + \Delta I = 2\frac{5-\delta}{\delta}\operatorname{Ric}_- I, \\ I(0,x) = 1. \end{cases}$$

Using Duhamel's formula, this equation can be converted into the integral equation

$$I(t,x) = 1 + 2\frac{5-\delta}{\delta}\int_0^t \int_M H(t-s,x,y)\operatorname{Ric}_-(y) I(s,y)\,\mathrm{dv}_g(y)\mathrm{d}s.$$

Recall the definition of the (parabolic) Kato constant of the function Ric_-:

$$\begin{aligned} \mathrm{k}_T(\operatorname{Ric}_-) &= \sup_{x\in M}\int_0^T \int_M H(t,x,y)\operatorname{Ric}_-(y)\,\mathrm{dv}_g(y)\mathrm{d}t \\ &= \left\| \int_0^T e^{-t\Delta}\operatorname{Ric}_-\,\mathrm{d}t \right\|_{\infty}. \end{aligned}$$

An easy application of the fixed point theorem in $L^\infty([0,T]\times M)$ yields that if $\delta\in(0,1)$ and $\mathrm{k}_T(\operatorname{Ric}_-)\leqslant\frac{\delta}{16}$ then the above integral equation has a unique solution $I\in L^\infty([0,T]\times M)$ with

$$1\leqslant I(t,x)\leqslant 1+4\frac{5-\delta}{\delta}\mathrm{k}_T(\operatorname{Ric}_-)\leqslant e^{4\frac{5-\delta}{\delta}\mathrm{k}_T(\operatorname{Ric}_-)}.$$

Hence, we have proved

Lemma 3.2. — *If $\delta\in(0,1)$ and $\mathrm{k}_T(\operatorname{Ric}_-)\leqslant\frac{\delta}{16}$, then the equation* (3.1) *has a unique positive solution J and this solution satisfies:*

$$e^{-4\,\mathrm{k}_T(\operatorname{Ric}_-)}\leqslant J(t,x)\leqslant 1.$$

3.3. The gradient estimate of Li and Yau

We now assume that M is closed, and we use the gauging function J given by Lemma 3.2. Let (t_0, x_0) be a point where the function tQ reaches its maximum on $[0,T]\times M$. We get:

$$\begin{aligned}\frac{Q(t_0,x_0)}{t_0} &\leqslant \left(\frac{\partial}{\partial t}+\Delta\right)Q - 2\langle \mathrm{d}\log u, dQ\rangle\\ &\leqslant -(2-\delta)\alpha J|\nabla \mathrm{d}\log u|^2+\delta\alpha J|\mathrm{d}\log u|^4.\end{aligned}$$

Let $\alpha\in(0,1)$ and assume that $Q(t_0,x_0)\geqslant 0$. We get at (t_0,x_0):

$$\begin{aligned}|\nabla \mathrm{d}\log u|^2 \geqslant \frac{1}{n}\left(\Delta\log u\right)^2 &= \frac{1}{n}\left(|\mathrm{d}\log u|^2-\frac{\partial}{\partial t}\log u\right)^2,\\ &= \frac{1}{n}\left(Q+(1-\alpha J)|\mathrm{d}\log u|^2\right)^2,\\ &\geqslant \frac{1}{n}Q^2+\frac{(1-\alpha J)^2}{n}|\mathrm{d}\log u|^4.\end{aligned}\tag{3.2}$$

So that

$$\begin{aligned}0\leqslant \frac{Q(t_0,x_0)}{t_0}&\left(1-\frac{(2-\delta)\alpha J}{n}t_0Q(t_0,x_0)\right)\\ &+\left(\delta-(2-\delta)\frac{(1-\alpha J)^2}{n}\right)\alpha J|\mathrm{d}\log u|^4.\end{aligned}$$

Because $0\leqslant J\leqslant 1$, we have

$$-(1-\alpha J)^2\leqslant -(1-\alpha)^2.$$

Assuming that

$$\delta<\frac{2}{n+1},$$

we choose $\alpha = 1-\sqrt{\frac{n\delta}{2-\delta}}$, and get

$$t_0Q(t_0,x_0)\leqslant \frac{n}{(2-\delta)\alpha J}\,.$$

We now make several choices: Assuming that $\mathrm{k}_T(\mathrm{Ric}_-)\leqslant\frac{1}{16n}$, we let $\delta = 16\,\mathrm{k}_T(\mathrm{Ric}_-)$ and $\alpha = 1-\sqrt{\frac{n\delta}{2-\delta}}$. With these choices, we have

$$\alpha J\geqslant e^{-4\,\mathrm{k}_T(\mathrm{Ric}_-)}\left(1-4\sqrt{n\,\mathrm{k}_T(\mathrm{Ric}_-)}\right)\geqslant e^{-8\sqrt{n\,\mathrm{k}_T(\mathrm{Ric}_-)}},$$

and

$$(2-\delta)\alpha J\geqslant 2\left(1-\frac{\delta}{2}\right)\alpha J\geqslant 2e^{-12\,\mathrm{k}_T\,-4\sqrt{n\,\mathrm{k}_T(\mathrm{Ric}_-)}}.$$

Since

$$12\,\mathrm{k}_T(\mathrm{Ric}_-)+4\sqrt{n\,\mathrm{k}_T(\mathrm{Ric}_-)}\leqslant 4\sqrt{\mathrm{k}_T(\mathrm{Ric}_-)}\left(\frac{3}{4\sqrt{n}}+\sqrt{n}\right)\\ \leqslant 8\sqrt{n\,\mathrm{k}_T(\mathrm{Ric}_-)},$$

we have shown the following proposition.

Proposition 3.3. — *Let (M^n,g) be a closed Riemannian manifold. Assume that, for some $T>0$,*

$$\mathrm{k}_T(\mathrm{Ric}_-)=\sup_{x\in M}\int_0^T\int_M H(t,x,y)\,\mathrm{Ric}_-(y)\,\mathrm{dv}_g(y)\mathrm{d}t\leqslant\frac{1}{16n}.$$

Let $u\colon[0,T]\times M\to\mathbb{R}$ be a positive solution of the heat equation

$$\frac{\partial u}{\partial t}+\Delta u=0.$$

Then, on $[0,T]\times M$, we have

$$e^{-8\sqrt{n\,\mathrm{k}_T(\mathrm{Ric}_-)}}\,\frac{|\mathrm{d}u|^2}{u^2}-\frac{1}{u}\frac{\partial u}{\partial t}\leqslant\frac{n}{2t}e^{8\sqrt{n\,\mathrm{k}_T(\mathrm{Ric}_-)}},$$

and

$$e^{-2}\,\frac{|\mathrm{d}u|^2}{u^2}-\frac{1}{u}\frac{\partial u}{\partial t}\leqslant\frac{n}{2t}e^{2}.$$

3.4. Heat kernel estimate

Throughout the remaining part of this section, we make the following general assumptions

$(\star)$ (M^n,g) is a closed Riemannian manifold

with diameter $D:=\mathrm{diam}(M,g)$.

We define

- $T(M,g)$ to be the largest time T such that
$$\mathrm{k}_T(\mathrm{Ric}_-)\leqslant\frac{1}{16n},$$
- the scale invariant geometric quantity $\xi(M,g)$ by $\xi^2(M,g)\times T(M,g)=D^2$,
- $\nu:=e^2n$.

For instance, if we have $\mathrm{Ricci}_g\geqslant-(n-1)\kappa^2 g$, then $\xi(M,g)\leqslant 4n\kappa D$.
Following the arguments of P. Li and S-T. Yau [42], we easily get:

Lemma 3.4. — *Assume* $(\star)$. *Let* $u\colon [0,T(M,g)]\times M\to\mathbb{R}$ *be a positive solution of the heat equation. For* $0\leqslant s\leqslant t\leqslant T(M,g)$ *and* $x,y\in M$,

$$u(s,x)\leqslant\left(\frac{t}{s}\right)^{\frac{\nu}{2}}u(t,x),$$

and

$$u(s,x)\leqslant\left(\frac{t}{s}\right)^{\frac{\nu}{2}}e^{2\frac{d^2(x,y)}{t-s}}\,u(t,y).$$

Proof. — The first assertion is a direct consequence of Proposition 3.3, indeed we have

$$e^{-2}\,\frac{|\mathrm{d}u|^2}{u}\leqslant\frac{\partial u}{\partial t}+\frac{\nu}{2t}u=t^{-\frac{\nu}{2}}\frac{\partial}{\partial t}\left(t^{\frac{\nu}{2}}u\right).$$

Concerning the second statement, we introduce $\gamma\colon[0,t-s]\to M$ a minimizing geodesic joining y to x and we define

$$\phi(\tau)=\log u\left(t-\tau,\gamma(\tau)\right),$$

so that $\phi(0)=\log u\,(t,y)$ and $\phi(t-s)=\log u\,(s,x)$. We have

$$\begin{aligned}\dot\phi(\tau)&=-\frac{1}{u}\frac{\partial u}{\partial t}+\langle\dot\gamma,\mathrm{d}u\rangle\\&\leqslant\frac{\nu}{2(t-\tau)}-e^{-2}\frac{|\mathrm{d}u|^2}{u^2}+\langle\dot\gamma,\mathrm{d}u\rangle\\&\leqslant\frac{\nu}{2(t-\tau)}+\frac{e^2}{4}|\dot\gamma|^2=\frac{\nu}{2(t-\tau)}+\frac{e^2\mathrm{d}^2(x,y)}{4(t-s)^2}\\&\leqslant\frac{\nu}{2(t-\tau)}+2\frac{\mathrm{d}^2(x,y)}{(t-s)^2}.\end{aligned}\tag{3.3}$$

Integrating this, we get

$$\frac{u(s,x)}{u(t,y)}\leqslant\left(\frac{t}{s}\right)^{\frac{\nu}{2}}e^{2\frac{\mathrm{d}^2(x,y)}{t-s}}.$$

□

This result leads to heat kernel estimates.

Theorem 3.5. — *Assume* $(\star)$. *There is a constant* c_n *such that, for* $0\leqslant s\leqslant t\leqslant T(M,g)/2$ *and* $y\in B(x,\sqrt{t})$,

$$H(s,y,y)\leqslant\left(\frac{t}{s}\right)^{\frac{\nu}{2}}\frac{c_n}{\operatorname{vol}B(x,\sqrt{t})}.$$

Moreover for any $s\geqslant T(M,g)/2$ *and* $x,y\in M$, *we have:*

$$H(s,x,y)\leqslant\frac{c_n^{1+\xi(M,g)}}{\operatorname{vol}M}.$$

Proof. — We let $T = T((M,g)$ and $\xi = \xi(M,g)$. Let $\gamma_n = 2^{\frac{\nu}{2}} e^2$. Using Lemma 3.4, we know that if $d(x,y) \leqslant \sqrt{t}$ and $t \leqslant T/2$, then

$$H(t,x,x) \leqslant \gamma_n H(2t,x,y) \leqslant \gamma_n^2 H(3t,x,x).$$

The function $t \mapsto H(t,x,x)$ is nonincreasing, hence

$$\gamma_n^{-1} H(t,x,x) \leqslant H(2t,x,y) \leqslant \gamma_n H(t,x,x),$$

and

$$\gamma_n^{-2} H(t,x,x) \leqslant H(t,y,y) \leqslant \gamma_n^2 H(t,x,x).$$

Integrating the inequality, $H(t,x,x) \leqslant \gamma_n H(2t,x,y)$, over $y \in B(x,\sqrt{t})$ and using that

$$\int_{B(x,\sqrt{t})} H(2t,x,y)\,\mathrm{dv}_g(y) \leqslant \int_M H(2t,x,y)\,\mathrm{dv}_g(y) = 1,$$

we get

$$H(t,x,x) \leqslant \frac{\gamma_n}{\operatorname{vol} B(x,\sqrt{t})}\,,$$

and, for $y \in B(x,\sqrt{t})$,

$$H(t,y,y) \leqslant \frac{\gamma_n^3}{\operatorname{vol} B(x,\sqrt{t})}.$$

The first part of the Lemma 3.4 implies the first assertion.

Concerning the second assertion: let $t \leqslant T/2$ and let $y,z \in M$ be such that $d(z,y) \leqslant \sqrt{t}$. Then, for any $\sigma \geqslant 0$,

$$H(\sigma+t,x,y) \leqslant \gamma_n H(\sigma+2t,x,z).$$

Assume now $s \geqslant T/2$.

If $D \leqslant \sqrt{T/2}$, using $t = D^2$ and $\sigma = s-t$, we get for all $y,z \in M$:

$$H(s,x,y) \leqslant \gamma_n H(s+t,x,z).$$

Integrating this inequality over $z \in M$, we get

$$H(s,x,y) \leqslant \frac{\gamma_n}{\operatorname{vol} M}.$$

Assume now that $\sqrt{T/2} \leqslant D$, and let $N \in \mathbb{N}$ be such $(N-1)\sqrt{T/2} \leqslant D \leqslant N\sqrt{T/2}$, that is to say

$$(N-1) \leqslant \sqrt{2}\xi \leqslant N.$$

Then we can find $y_0 = y, y_1, \ldots, y_N = z$ with

$$d(y_i,y_{i+1}) \leqslant \sqrt{T/2}.$$

The inequalities

$$H(s+iT/2,x,y_i) \leqslant \gamma_n H(s+(i+1)T/2,x,y_{i+1}),\ i \in \{0,\ldots,N-1\}$$

yield
$$H(s,x,y) \leqslant \gamma_n^N H(s+NT/2,x,z).$$
Integrating over $z \in M$, we get
$$H(s,x,y) \leqslant \frac{\gamma_n^N}{\operatorname{vol} M}.$$
Recall that $N \leqslant 1+\sqrt{2}\xi(M,g)$, the second assertion of Theorem 3.5 follows. □

3.5. Geometric consequences

3.5.1. Eigenvalue estimate

Proposition 3.6. — *Assume* $(\star)$*. There is a positive constant* c_n *such that the first nonzero eigenvalue of the Laplacian on* (M,g) *satisfies*
$$\lambda_1 \geqslant \frac{c_n^{-1-\xi(M,g)}}{D^2}.$$

Proof. — We still let $\xi=\xi(M,g)$. Let $f\colon M\to\mathbb{R}$ be a L^2 normalized eigenfunction associated with λ_1:
$$\Delta f=\lambda_1 f \text{ and } \|f\|_2=1.$$
Then $(t,x)\mapsto e^{-\lambda_1 t}f(x)$ is a solution of the heat equation and according to the Bochner formula, the function
$$u(t,x):=e^{-\lambda_1 t}|\mathrm{d}f|(x)$$
satisfies:
$$\frac{\partial u}{\partial t}+\Delta u \leqslant \operatorname{Ric}_- u.$$
Let $\tau=\min\left\{\frac{T(M,g)}{2},D^2\right\}$. The function
$$U(s,x)=\int_M H(\tau-s,x,y)u(s,y)\,\mathrm{dv}_g(y)$$
satisfies
$$\begin{aligned}\frac{\partial U}{\partial s}(s,x)&=\int_M H(\tau-s,x,y)\left(\frac{\partial u}{\partial s}+\Delta u\right)(s,y)\,\mathrm{dv}_g(y)\\ &\leqslant \int_M H(\tau-s,x,y)\operatorname{Ric}_-(y)u(s,y)\,\mathrm{dv}_g(y).\end{aligned}$$

Hence integrating this inequality, we get:

$$\begin{aligned}U(\tau,x)-U(0,x)&=u(\tau,x)-\int_M H(\tau,x,y)u(0,y)\,\mathrm{dv}_g(y)\\&\leqslant\int_{[0,\tau]\times M}H(\tau-s,x,y)\operatorname{Ric}_-(y)u(s,y)\,\mathrm{dv}_g(y)\mathrm{d}s.\end{aligned}$$

If we let

$$L:=\|\mathrm{d}f\|_\infty,$$

then using the estimate on the heat kernel Theorem 3.5, we have

$$\begin{aligned}Le^{-\tau\lambda_1}&\leqslant\frac{1}{16n}L+\int_M H(\tau,x,y)u(0,y)\,\mathrm{dv}_g(y)\\&\leqslant\frac{1}{16n}L+\frac{c_ne^{c_n\xi(M,g)}}{\operatorname{vol}M}\int_M u(0,y)\,\mathrm{dv}_g(y)\\&\leqslant\frac{1}{16n}L+\frac{c_ne^{c_n\xi(M,g)}}{\sqrt{\operatorname{vol}M}}\|u\|_2\\&\leqslant\frac{1}{16n}L+\frac{c_ne^{c_n\xi(M,g)}}{\sqrt{\operatorname{vol}M}}\sqrt{\lambda_1}.\end{aligned}$$

We have to distinguish two cases.

- First case: $e^{-\tau\lambda_1}\leqslant\frac{1}{8n}$, i.e.,

$$\lambda_1\geqslant\frac{\log(8n)}{\tau}\geqslant\frac{\log(8n)}{D^2}.$$

- Second case: $e^{-\tau\lambda_1}\geqslant\frac{1}{8n}$, then we get that

$$L\leqslant 16n\frac{c_ne^{c_n\xi(M,g)}}{\sqrt{\operatorname{vol}M}}\sqrt{\lambda_1}.$$

As $\int_M f(y)\,\mathrm{dv}_g(y)=0$, we can find $o\in M$ such that $f(o)=0$ and then we have for any $x\in M$:

$$|f(x)|^2\leqslant|f(x)-f(o)|^2\leqslant L^2\,D^2.$$

Hence

$$1\leqslant L^2\,D^2\operatorname{vol}(M)\leqslant 256n^2c_n^2e^{2c_n\xi(M,g)}D^2\lambda_1.\qquad\square$$

3.5.2. Sobolev inequality

Proposition 3.7. — *Assume* $(\star)$. *There is a constant* c_n *such that we have the following Sobolev inequality:* $\forall\,\psi\in\mathcal{C}^\infty(M)$,

$$\operatorname{vol}^{\frac{2}{\nu}}(M)\|\psi\|^2_{\frac{2\nu}{\nu-2}}\leqslant c_n^{1+\xi(M,g)}D^2\|\mathrm{d}\psi\|_2^2+\|\psi\|_2^2.$$

Proof. — We have already shown that if $x \in M$ and $t \in [0, \tau]$, then
$$H(t,x,x) \leqslant \left(\frac{\tau}{t}\right)^{\frac{\nu}{2}} \frac{c_n e^{c_n \xi(M,g)}}{\operatorname{vol} M}.$$
Defining
$$\Gamma := \left(\frac{\nu}{2e}\right)^{\nu} \frac{\tau^{\frac{\nu}{2}} c_n e^{c_n \xi(M,g)}}{\operatorname{vol} M},$$
we can conclude that, for all $t > 0$ and $x \in M$,
$$H(t,x,x) e^{-\frac{t}{\tau}} \leqslant \Gamma\, t^{-\frac{\nu}{2}}.$$
According to N. Varopoulos ([55]), there is a constant c which depends only on n, such that we have the Sobolev inequality: $\forall\, \psi \in \mathcal{C}^\infty(M)$,
$$\|\psi\|^2_{\frac{2\nu}{\nu-2}} \leqslant c\Gamma^{\frac{2}{\nu}} \left(\|\mathrm{d}\psi\|_2^2 + \frac{1}{\tau}\|\psi\|_2^2 \right).$$
If $\psi \in \mathcal{C}^\infty(M)$ is such that $\int_M \psi(y)\, \mathrm{dv}_g(y) = 0$ then we have
$$\|\psi\|_2^2 \leqslant \lambda_1^{-1} \|\mathrm{d}\psi\|_2^2.$$
But
$$c\Gamma^{\frac{2}{\nu}} \frac{1}{\tau} \lambda_1^{-1} \leqslant c_n \frac{D^2 e^{c_n \xi(M,g)}}{\operatorname{vol}^{\frac{2}{\nu}} M},$$
hence
$$\max\{c\Gamma^{\frac{2}{\nu}}, c\Gamma^{\frac{2}{\nu}} \frac{1}{\tau} \lambda_1^{-1}\} \leqslant c_n \frac{D^2 e^{c_n \xi(M,g)}}{\operatorname{vol}^{\frac{2}{\nu}} M}.$$
We finally get that, for all $\psi \in \mathcal{C}^\infty(M)$,
$$\left\| \psi - \frac{1}{\operatorname{vol} M} \int_M \psi \right\|^2_{\frac{2\nu}{\nu-2}} \leqslant c_n \frac{D^2 e^{c_n \xi(M,g)}}{\operatorname{vol}^{\frac{2}{\nu}} M} \|\mathrm{d}\psi\|_2^2. \qquad \square$$

3.5.3. The doubling condition

Proposition 3.8. — *Assume* $(\star)$. *There is a constant* c_n *such that if* $x \in M$ *and* $0 < r < R \leqslant D$, *then*
$$\frac{\operatorname{vol} B(x,R)}{R^\nu} \leqslant c_n^{1+\xi(M,g)} \frac{\operatorname{vol} B(x,r)}{r^\nu}.$$

Proof. — When $0 < s \leqslant t \leqslant \tau$ and $y \in B(x, \sqrt{t})$, we have already shown that
$$H(s,y,y) \leqslant \left(\frac{t}{s}\right)^{\frac{\nu}{2}} \frac{\gamma_n}{\operatorname{vol} B(x,\sqrt{t})}.$$
Hence, when $\Omega \subset B(x,\sqrt{t})$, we have
$$e^{-\lambda_1(\Omega)s} \leqslant \int_\Omega H_\Omega(s,y,y)\mathrm{d}y \leqslant \int_\Omega H(s,y,y)\mathrm{d}y \leqslant \left(\frac{t}{s}\right)^{\frac{\nu}{2}} \frac{\gamma_n \operatorname{vol} \Omega}{\operatorname{vol} B(x,\sqrt{t})}.$$

Assuming that

$$\operatorname{vol}\Omega \leqslant \frac{1}{2\gamma_n} \operatorname{vol} B(x, \sqrt{t}),$$

one gets

$$e^{-\lambda_1(\Omega)t} \leqslant \frac{1}{2}.$$

Choosing $s = \ln(2)/\lambda_1(\Omega) \leqslant t$, we obtain[(1)]

$$\frac{1}{2} \leqslant \gamma_n \left(t\, \lambda_1(\Omega)\right)^{\frac{2}{\nu}} \frac{\operatorname{vol}\Omega}{\operatorname{vol} B(x, \sqrt{t})}.$$

Let $0 < r \leqslant R \leqslant \sqrt{\tau}$, we distinguish two cases.

- First case: $\operatorname{vol} B(x, r) \geqslant \frac{1}{2\gamma_n} \operatorname{vol} B(x, R)$.
- Second case: $\operatorname{vol} B(x, r) \leqslant \frac{1}{2\gamma_n} \operatorname{vol} B(x, R)$. In this case, we have shown that, for all $\Omega \subset B(x, r)$,

$$\lambda_1(\Omega) \geqslant \frac{1}{R^2} \left(\frac{\operatorname{vol}\Omega}{2\gamma_n \operatorname{vol} B(x, R)} \right)^{-\frac{2}{\nu}}.$$

From [7], we know that the following Sobolev inequality holds,

$$\forall\, \psi \in \mathcal{C}_0^\infty(B(x, r))\colon \|\psi\|^2_{\frac{2\nu}{\nu-2}} \leqslant c_n \frac{R^2}{\left(\operatorname{vol} B(x, R)\right)^{\frac{2}{\nu}}} \|\mathrm{d}\psi\|_2^2,$$

and according to Theorem 2.8, we get the lower bound

$$\operatorname{vol} B(x, r) \geqslant c_n \frac{\operatorname{vol} B(x, R)}{R^\nu} r^\nu.$$

We have shown that if $\rho = \sqrt{\tau}$, then for all $0 < r < R \leqslant \rho$,

$$\frac{\operatorname{vol} B(x, r)}{\operatorname{vol} B(x, R)} \geqslant \min\left\{ \frac{1}{2\gamma_n}, c_n \frac{r^\nu}{R^\nu} \right\} \geqslant \min\left\{ c_n, \frac{1}{2\gamma_n} \right\} \frac{r^\nu}{R^\nu}.$$

In fact this local doubling condition implies a global one: we claim that there is a constant c_n such that if $\theta \geqslant 1$ and $r \in (0, \rho)$, then

$$\operatorname{vol} B(x, \theta r) \leqslant c_n^{1+\theta} \operatorname{vol} B(x, r)$$

This follows for instance from [36, Subsection 2.3] (however look for instance at the hypothesis of [3, Theorem 1.5]). Because we need an explicit estimate, we explain the proof of that fact.

(1) where γ_n is a constant which only depends on the dimension and whose value can change from one line to another.

Lemma 3.9. — *Assume that (X, d, μ) is a measure metric space which satisfies the local doubling condition: for some $r_0 > 0$, $\gamma > 0$ and for all $r \in [0, r_0]$,*

$$\mu(B(x, 2r)) \leqslant \gamma\mu(B(x, r)).$$

Then for every $\theta \geqslant 1$ and $r \in [0, r_0]$:

$$\mu(B(x, \theta r)) \leqslant \gamma^{50+50\theta}\mu(B(x, r))$$

Proof of Lemma 3.9. — By scaling we can assume that $r_0 = 1$. Let $R \geqslant 0$, we have

$$B\left(x, R + \frac{1}{20}\right) = \bigcup_{p \in B(x,R)} B\left(p, \frac{1}{20}\right).$$

Using Vitali's covering lemma, we can find a family of pairwise disjoint balls $B(p_\alpha, 1/20)$ such that

$$B\left(x, R + 1/20\right) \subset \bigcup_\alpha B\left(p_\alpha, 1/4\right).$$

Hence, using the doubling condition,

$$\begin{aligned}\mu((B\left(x, R + 1/20\right)) &\leqslant \sum_\alpha \mu((B\left(p_\alpha, 1/4\right)) \\ &\leqslant \gamma^3 \sum_\alpha \mu((B\left(p_\alpha, 1/32\right)).\end{aligned}$$

But the balls $B\left(p_\alpha, 1/32\right)$ are disjoints and included in $B\left(x, R + \frac{1}{32}\right)$; recall that each $p_\alpha \in B\left(x, R\right)$, hence

$$\mu((B\left(x, R + 1/20\right)) \leqslant \gamma^3\mu((B\left(x, R + 1/32\right)).$$

So that if $N \in \mathbb{N} \setminus \{0\}$:

$$\begin{aligned}\mu((B\left(x, N\right)) &\leqslant \gamma^3\mu((B\left(x, N - 1/10\right)) \\ &\leqslant \gamma^{30}\mu((B\left(x, N - 1\right)) \\ &\leqslant \gamma^{30N-30}\mu((B\left(x, 1\right)).\end{aligned}$$ □

End of the proof of Proposition 3.8. —

$$\frac{\operatorname{vol} M}{D^\nu} \leqslant c_n^{1+\frac{D}{\rho}} \frac{\rho^\nu}{D^\nu} \frac{\operatorname{vol} B(x, \rho)}{\rho^\nu} \leqslant c_n^{1+2\xi(M,g)} \frac{\operatorname{vol} B(x, \rho)}{\rho^\nu}.$$ □

Note that Proposition 3.8 yields the following global bound on the heat kernel.

Corollary 3.10. — *Assume* $(\star)$. *There is a constant* c_n *such that, if* $x \in M$ *and* $t > 0$, *then*

$$H(t,x,x) \leqslant \frac{c_n^{1+\xi(M,g)}}{\operatorname{vol} B(x,\sqrt{t})}.$$

3.5.4. Poincaré inequality

Proposition 3.11. — *Assume* $(\star)$. *There is a constant* c_n *such that any ball* B *of radius* $r \leqslant \min\{D, (T(M,g)/2)^2\}$ *satisfies the Poincaré inequality:*

$$\forall\, \psi \in \mathcal{C}^\infty(B)\,,\ \left\|\psi - \psi_B\right\|_2 \leqslant c_n r \left\|\mathrm{d}\psi\right\|_2.$$

Furthermore, for any ball B *of radius* r:

$$\forall\, \psi \in \mathcal{C}^\infty(B)\,,\ \left\|\psi - \int_B \psi \frac{\mathrm{dv}_g}{\operatorname{vol} B}\right\|_2 \leqslant c_n^{1+\xi(M,g)} r \left\|\mathrm{d}\psi\right\|_2.$$

Proof. — According to the results of L. Saloff-Coste and A. Grigor'yan [29, 51, 52], we only need to find a positive constant ϵ_n such that, for $t \leqslant \tau$ and all $y \in B(x,\sqrt{t})$,

$$\frac{\epsilon_n}{\operatorname{vol} B(x,\sqrt{t})} \leqslant H(t,x,y).$$

But we already know that if $t \leqslant \tau$ and $y \in B(x,\sqrt{t})$, then

$$c_n^{-1} H(t,x,y) \leqslant H(t,x,x) \leqslant c_n H(t,x,y).$$

Hence for all $\delta \in (0,1)$:

$$\begin{aligned} c_n \operatorname{vol} B(x,\sqrt{t}) H(t,x,y) &\geqslant \operatorname{vol} B(x,\sqrt{t}) H(t,x,x) \\ &\geqslant \int_{B(x,\sqrt{t})} H(t,x,z)\,\mathrm{dv}_g(z) \\ &\geqslant \left(\frac{\delta}{2}\right)^{\nu} \int_{B(x,\sqrt{t})} H\left(\delta t/2, x, z\right)\mathrm{dv}_g(z) \\ &= \left(\frac{\delta}{2}\right)^{\nu} \left(1 - \int_{M\setminus B(x,\sqrt{t})} H(\delta t/2,x,z)\,\mathrm{dv}_g(z)\right). \end{aligned}$$

Our Harnack type estimate (Lemma 3.4) yields that if $\xi \in B(x,\sqrt{\delta t})$ then

$$H(\delta t/2, x, z) \leqslant C_n H(\delta t, \xi, z),$$

so that we get the estimate:

$$\int_{M\setminus B(x,\sqrt{t})} H(\delta t/2,x,z)\,\mathrm{dv}_g(z) \leqslant C\int_{B(x,\sqrt{\delta t})\times\left(M\setminus B(x,\sqrt{t})\right)} H(\delta t,\xi,z)\frac{\mathrm{dv}_g(\xi)\,\mathrm{dv}_g(z)}{\operatorname{vol}B(x,\sqrt{\delta t})}.$$

Moreover

$$\int_{B(x,\sqrt{\delta t})\times\left(M\setminus B(x,\sqrt{t})\right)} H(\delta t,\xi,z)\,\mathrm{dv}_g(\xi)\,\mathrm{dv}_g(z) = \sum_{k=1}^{\infty}\int_{B(x,\sqrt{\delta t})\times\left(B(x,(k+1)\sqrt{t})\setminus B(x,k\sqrt{t})\right)} H(\delta t,\xi,z)\,\mathrm{dv}_g(\xi)\,\mathrm{dv}_g(z).$$

Assume that A and B are two Borel sets in M, with finite volume, and such that, for some $R>0$ and all $(x,y)\in A\times B$, $d(x,y)\geqslant R>0$. Then, Davies–Gaffney estimate [22] yields that

$$\int_{A\times B} H(t,\xi,z)\,\mathrm{dv}_g(\xi)\,\mathrm{dv}_g(z)\leqslant\sqrt{\operatorname{vol}_g A\ \operatorname{vol}_g B}\ e^{-\frac{R^2}{4t}}.$$

So that when $k^2>\delta$, we get

$$\int_{B(x,\sqrt{\delta t})\times\left(B(x,(k+1)\sqrt{t})\setminus B(x,k\sqrt{t})\right)} H(\delta t,\xi,z)\,\mathrm{dv}_g(\xi)\,\mathrm{dv}_g(z) \leqslant e^{-\frac{(k-\sqrt{\delta})^2}{4\delta}}\left(\operatorname{vol}B(x,\sqrt{\delta t})\times\operatorname{vol}B(x,(k+1)\sqrt{t})\right)^{\frac12}.$$

Lemma 3.9 implies that

$$\operatorname{vol}B(x,(k+1)\sqrt{t})\leqslant e^{c_n+c_n\frac{k+1}{\sqrt{\delta}}}\operatorname{vol}B(x,\sqrt{\delta t}),$$

and, one gets

$$\int_{B(x,\sqrt{\delta t})\times\left(M\setminus B(x,\sqrt{t})\right)} H(\delta t,\xi,z)\,\mathrm{dv}_g(\xi)\,\mathrm{dv}_g(z)\leqslant\sum_{k=1}^{\infty}e^{c_n+c_n\frac{k+1}{\sqrt{\delta}}-\frac{(k-\sqrt{\delta})^2}{4\delta}}.$$

We can choose $\delta=\delta_n$ to be small enough so that this sum is small enough, and then we get

$$c_n\operatorname{vol}B(x,\sqrt{t})H(t,x,y)\geqslant\left(\frac{\delta_n}{2}\right)^{\nu}. \qquad \square$$

3.5.5. First Betti number

Proposition 3.12. — *Assume* $(\star)$. *There is a constant* c_n *such that* $b_1(M) \leqslant n + \frac{1}{4} + \xi(M,g) c_n^{\xi(M,g)}$. *Moreover there is a constant* $\epsilon_n > 0$ *such that if* $\xi(M,g) < \epsilon_n$, *then* $b_1(M) \leqslant n$.

Proof. — This result relies on an improvement of the upper bound on the heat kernel. We have shown that

$$\left\| e^{-\frac{T(M,g)}{2}\Delta} \right\|_{L^2 \to L^\infty} \leqslant \frac{c_n^{1+\xi(M,g)}}{\sqrt{\operatorname{vol} M}}.$$

Hence if P is the L^2-projection on the vector space of constant function then

$$\begin{aligned}
\left\| e^{-T(M,g)\Delta} - P \right\|_{L^2 \to L^\infty} &= \left\| e^{-\frac{T(M,g)}{2}\Delta} \left(e^{-\frac{T(M,g)}{2}\Delta} - P \right) \right\|_{L^2 \to L^\infty} \\
&\leqslant \left\| e^{-\frac{T(M,g)}{2}\Delta} \right\|_{L^2 \to L^\infty} \left\| \left(e^{-\frac{T(M,g)}{2}\Delta} - P \right) \right\|_{L^2 \to L^2} \\
&\leqslant \frac{c_n^{1+\xi(M,g)}}{\sqrt{\operatorname{vol} M}} \lambda_1^{-\frac{1}{2}} \left\| \mathrm{d} \left(e^{-\frac{T(M,g)}{2}\Delta} \right) \right\|_{L^2 \to L^2} \\
&\leqslant \frac{c_n^{1+\xi(M,g)}}{\sqrt{\operatorname{vol} M}} \lambda_1^{-\frac{1}{2}} \frac{1}{\sqrt{T(M,g)}}
\end{aligned}$$

Recall our lower bound on λ_1, that $T(M,g)\xi^2 = D^2$ and that

$$\|P\|_{L^2 \to L^\infty} = \frac{1}{\sqrt{\operatorname{vol} M}}.$$

Then

$$\begin{aligned}
\left\| e^{-T(M,g)\Delta} \right\|_{L^2 \to L^\infty} &\leqslant \left\| e^{-T(M,g)\Delta} - P \right\|_{L^2 \to L^\infty} + \|P\|_{L^2 \to L^\infty} \\
&\leqslant \frac{1 + \xi c_n^{1+\xi(M,g)}}{\sqrt{\operatorname{vol} M}}.
\end{aligned}$$

If $\alpha \in \mathcal{C}^\infty(T^*M)$ satisfies $d\alpha = d^*\alpha = 0$, then the Bochner formula implies that

$$\Delta|\alpha| \leqslant \operatorname{Ric}_- |\alpha|.$$

Hence

$$\begin{aligned}
|\alpha|(x) &\leqslant \left(e^{-T(M,g)\Delta} |\alpha| \right)(x) + \int_0^{T(M,g)} \left(e^{-s\Delta} \operatorname{Ric}_- |\alpha| \right)(x) \mathrm{d}s \\
&\leqslant \frac{1 + \xi c_n^{1+\xi(M,g)}}{\sqrt{\operatorname{vol} M}} \|\alpha\|_2 + \mathrm{k}_{T(M,g)}(\operatorname{Ric}_-) \|\alpha\|_\infty \\
&\leqslant \frac{1 + \xi c_n^{1+\xi(M,g)}}{\sqrt{\operatorname{vol} M}} \|\alpha\|_2 + \frac{1}{16n} \|\alpha\|_\infty.
\end{aligned}$$

Finally we obtain that for any $\alpha \in \mathcal{H}^1(M,g) = \{\alpha \in \mathcal{C}^\infty(T^*M)\colon\ \mathrm{d}\alpha = \mathrm{d}^*\alpha = 0\}$:

$$\|\alpha\|_\infty \leqslant \frac{1}{1-\frac{1}{16n}} \frac{1+\xi(M,g)c_n^{1+\xi(M,g)}}{\sqrt{\operatorname{vol} M}} \|\alpha\|_2.$$

The Grothendieck theorem [50, Theorem 5.1] (see also [39, Lemma 11] or [28, Théorème 4]) yields that

$$b_1(M) = \dim \mathcal{H}^1(M,g) \leqslant n \left(\frac{1+\xi(M,g)c_n^{1+\xi(M,g)}}{1-\frac{1}{16n}} \right)^2.$$

Then a bit of arithmetic implies the proposition. □

3.6. Euclidean type estimate

3.6.1. Improvement

We now assume that

$$\mathrm{k}_T(\mathrm{Ric}_-) \leqslant \frac{1}{16n} \text{ and } \int_0^T \frac{\sqrt{\mathrm{k}_s(\mathrm{Ric}_-)}}{s}\,\mathrm{d}s \leqslant \Lambda. \tag{3.4}$$

According to Proposition 3.3, if $u\colon [0,T]\times M \to \mathbb{R}_+$ is a positive solution of the heat equation, then

$$-\frac{1}{u}\frac{\partial u}{\partial t} \leqslant \frac{n}{2t} + C_n \frac{\sqrt{\mathrm{k}_t(\mathrm{Ric}_-)}}{t}.$$

Hence if $0 < s < t \leqslant T$, then

$$u(s,x) \leqslant \left(\frac{t}{s}\right)^{\frac{n}{2}} e^{\Lambda C_n} u(t,x).$$

In particular, if $0 < s < t \leqslant T$ and $x \in M$, then the heat kernel satisfies

$$s^{\frac{n}{2}} H(s,x,x) \leqslant e^{\Lambda C_n} t^{\frac{n}{2}} H(t,x,x).$$

And looking at the behavior when $s \to 0+$, we get

$$\frac{e^{-\Lambda C_n}}{t^{\frac{n}{2}}} \leqslant H(t,x,x).$$

Using the upper bound of (3.5), we get that, for $0 < t \leqslant \tau$ and $x \in M$,

$$H(s,x,x) \leqslant \frac{e^{\Lambda C_n}}{t^{\frac{n}{2}}} \frac{c_n e^{c_n \xi(M,g)}}{\operatorname{vol} M},$$

and

$$\operatorname{vol} B\left(x,\sqrt{t}\right) \leqslant c_n e^{\Lambda C_n} t^{\frac{n}{2}}.$$

As a consequence we can improve Propositions (3.7) and (3.8).

Proposition 3.13. — *Let (M^n, g) be a closed Riemannian manifold, such that, for some $T > 0$, the parabolic Kato constant of* Ricci_- *satisfies* (3.4)*. Then,*

- *The Euclidean Sobolev inequality holds,*

$$\forall\, \psi \in \mathcal{C}^\infty(M)\colon\ \mathrm{vol}^{\frac{2}{n}}(M)\|\psi\|^2_{\frac{2n}{n-2}} \leqslant c_n^{1+\xi(M,g)+\Lambda} D^2 \|\mathrm{d}\psi\|_2^2 + \|\psi\|_2^2.$$

- *There is a constant c_n such that, for $x \in M$ and $0 < r \leqslant D$,*

$$c_n^{1+\xi(M,g)+\Lambda}\ \frac{\mathrm{vol}\, M}{D^n} \leqslant \frac{\mathrm{vol}\, B(x,r)}{r^n} \leqslant c_n^{2+\xi(M,g)+\Lambda}.$$

3.6.2. Conditions insuring the estimate (3.4)

If $p \geqslant 1$ and $T > 0$, we assume that $\mathbf{I}$ is such that

$$(\mathrm{diam}\, M)^{2p-2} \sup_{x\in M} \int_0^T H(s,x,y)\,\mathrm{Ric}_-^p(y)\,\mathrm{dv}_g\,\mathrm{d}s \leqslant \mathbf{I}^p. \tag{3.5}$$

Let $q = p/(p-1)$. For $0 \leqslant \underline{T} \leqslant T$ and $x \in M$, using Hölder's inequality, we get

$$\begin{aligned}\int_0^{\underline{T}} H(s,x,y)\,\mathrm{Ric}_-(y)\,\mathrm{dv}_g\,\mathrm{d}s &\leqslant \underline{T}^{\frac{1}{q}} \left(\int_0^{\underline{T}} H(s,x,y)\,\mathrm{Ric}_-^p\,\mathrm{dv}_g\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &\leqslant \mathbf{I}\left(\frac{\underline{T}}{D^2}\right)^{\frac{1}{q}}.\end{aligned}$$

Hence for

$$\underline{T} = \min\left\{T, (16n\mathbf{I})^{-q} D^2\right\},$$

one gets

$$\mathrm{k}_{\underline{T}}(\mathrm{Ric}_-) \leqslant \frac{1}{16n} \quad\text{and}\quad \int_0^{\underline{T}} \frac{\sqrt{\mathrm{k}_s(\mathrm{Ric}_-)}}{s}\mathrm{d}s \leqslant q/(2\sqrt{n}).$$

Hence one gets the following

Theorem 3.14. — *Let (M^n, g) be a closed Riemannian manifold of dimension n and let $p > 1$ and $q = p/(p-1)$. There is a constant γ, which depends only of n, p, such that if* (3.5) *holds for some $\mathbf{I}$, and ξ is defined by*

$$\xi = \max\left\{\frac{D}{\sqrt{T}}, (16n\mathbf{I})^{q/2}\right\},$$

then the following properties hold.

(1) *The first nonzero eigenvalue of the Laplacian satisfies*

$$\lambda_1 \geqslant \frac{\gamma^{-1-\xi}}{D^2},$$

(2) $b_1(M) \leqslant \gamma^{1+\xi}$,
(3) *for any* $0 < r \leqslant R \leqslant D$*:*
$$\frac{\operatorname{vol} B(x,R)}{R^n} \leqslant \gamma^{1+\xi} \frac{\operatorname{vol} B(x,r)}{r^n} \leqslant \gamma^{2+\xi},$$
(4) *the Euclidean Sobolev inequality,*
$$\forall\, \psi \in \mathcal{C}^\infty(M) : \ \operatorname{vol}^{\frac{2}{n}}(M) \|\psi\|^2_{\frac{2n}{n-2}} \leqslant \gamma^{1+\xi} D^2 \|\mathrm{d}\psi\|_2^2 + \|\psi\|_2^2,$$
is satisfied.

3.7. Q-curvature and bound on the Kato constant

It turns out that the Q −curvature gives a natural control on the Kato constant of the Ricci curvature. Recall that when (M, g) is Riemannian manifold of dimension $n \geqslant 4$, the Q −curvature is defined by:
$$\mathrm{Q}_g = \frac{1}{2(n-1)} \Delta \operatorname{Scal}_g - \frac{2}{(n-2)^2} |\operatorname{Ricci}|^2 + c_n \operatorname{Scal}_g^2,$$
where $c_n = \frac{n^3 - 4n^2 + 16n - 16}{8n(n-1)^2(n-2)^2}$. Recall that the Paneitz operator describes the conformal change of the Q-curvature. It is a differential operator of order 4. Recently, M. Gursky and A. Machioldi ([35]) have discovered some new maximum principles for the Paneitz operator when the Q-curvature is nonnegative and the scalar curvature is positive. It turns out that these hypotheses yield a bound on the L^2 Kato constant of Ric_-.

Proposition 3.15. — *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geqslant 4$ such that:*
$$0 \leqslant \mathrm{Q}_g \ \text{ and } 0 \leqslant \operatorname{Scal}_g \leqslant \kappa^2/D^2.$$
Then the conclusion of Theorem 3.14 are satisfied for $\xi = \epsilon_n \kappa$.

Proof. — Indeed if $T > 0$, as $\mathrm{Q}_g \geqslant 0$, we get:
$$\begin{aligned}\frac{2}{(n-2)^2} \int_{[0,T]\times M} & H(s,x,y) |\operatorname{Ricci}|^2(y) \mathrm{d}s \, \mathrm{dv}_g(y) \\ & \leqslant \frac{1}{2(n-1)} \left(\operatorname{Scal}_g(x) - \left(e^{T\Delta} \operatorname{Scal}_g\right)(x)\right) + c_n \frac{\kappa^4 T}{D^4} \\ & \leqslant \frac{1}{2(n-1)} \frac{\kappa^2}{D^2} + c_n \frac{\kappa^4 T}{D^4}.\end{aligned}$$
If we choose $T = \epsilon_n D^2/\kappa^2$, one gets
$$D^2 \sup_{x \in M} \int_0^T H(s,x,y) \operatorname{Ric}_-^p(y) \, \mathrm{dv}_g \, \mathrm{d}s \leqslant \alpha_n \kappa^2$$

and

$$(\mathrm{k}_T(\mathrm{Ric}_-))^2 \leqslant \alpha_n \epsilon_n. \qquad \square$$

3.8. Localization

It follows from ([58, Proof of Theorem 1.1]), that the results of this section can be localized on a geodesic ball provided one gets a good cut-off function:

Proposition 3.16. — *Let (M^n, g) be a closed Riemannian manifold of dimension n and let $B(x,R) \subset \Omega$ be a geodesic ball included in a relatively compact open subset Ω. Assume that there is a smooth function ξ with compact support in Ω such that $\xi = 1$ on $B(x,R)$ and*

$$|\mathrm{d}\xi|^2 + |\Delta\xi| \leqslant c/R^2.$$

Let H_Ω be the heat kernel on Ω with the Dirichlet boundary condition. Consider the assumptions,

(A) *For all $x \in \Omega$:*

$$\int_{[0,\eta R^2]\times\Omega} H_\Omega(s,x,y)\,\mathrm{Ric}_-(y)\,\mathrm{dv}_g(y)\mathrm{d}s \leqslant \frac{1}{16n}.$$

(B) *For some $p > 1$, $\Lambda \in \mathbb{R}_+$ and for all $x \in \Omega$:*

$$\int_{[0,\eta R^2]\times\Omega} H_\Omega(s,x,y)\,\mathrm{Ric}_-^p(y)\,\mathrm{dv}_g(y)\mathrm{d}s \leqslant \Lambda R^{2p-1}.$$

Under the condition (A) *or* (B)*, there is a constant γ which depends only on n, c, η or on n, c, p, Λ such that*

(1) *For any $x \in B(p, R/2)$ and $0 < t \leqslant R^2$ then*

$$H(t,x,x) \leqslant \frac{\gamma}{\mathrm{vol}\,B(x,\sqrt{t})}.$$

(2) *For any $x \in B(p,R/2)$ and $0 \leqslant r \leqslant R/2$:*

$$\mathrm{vol}\,B(x,r) \leqslant \gamma\,\mathrm{vol}\,B(x,2r)$$

(3) *For any $x \in B(p,R/2)$ and $0 \leqslant r \leqslant R/2$, the ball $\mathbf{B} = B(x,r)$ satisfies the Poincaré inequality:*

$$\forall\,\psi \in \mathcal{C}^\infty(B(x,r)),\ \|\psi - \psi_{\mathbf{B}}\|_{L^2(\mathbf{B})} \leqslant \gamma r\,\|\mathrm{d}\psi\|_{L^2(\mathbf{B})}.$$

Moreover if the condition (B) *is satisfied then, for any $x \in B(p,R/2)$ and $0 \leqslant s \leqslant r \leqslant R/2$,*

$$\frac{\mathrm{vol}\,B(x,r)}{r^n} \leqslant \gamma\frac{\mathrm{vol}\,B(x,s)}{s^n} \leqslant \gamma^2\omega_n.$$

4. Volume growth estimate: global results

4.1. The setting

Thourough this section, (M^n, g) is a complete manifold with dimension $n \geqslant 3$. We also assume

(i) that (M^n, g) satisfies the Euclidean Sobolev inequality (Sob) with constant μ,
(ii) that there is some $\delta > 0$ and $\gamma \geqslant 1$ such that the Schrödinger operator $\Delta - (1+\delta)(n-2)\operatorname{Ric}_-$ is gaugeable with gaugeability constant γ.

According to Remark 2.16, we can assume that

$$0 < \delta < \frac{(n-2)^2}{n(3n-4)} \quad \text{and} \quad \delta < \frac{1}{n-2}$$

so that

$$2 \leqslant 2\frac{n-1}{n-2}\frac{1}{1+\delta} \quad \text{and} \quad 2(n-1)\frac{\delta}{1+\delta} < (n-2)\sqrt{\frac{\delta}{1+\delta}}\,.$$

We fix $o \in M$, and consider the Green kernel with pole at o and the function b such that

$$G(o,x) =: \frac{1}{b(x)^{n-2}},$$

where we choose the normalization

$$\Delta_x G(o,x) = (n-2)\operatorname{vol}\mathbb{S}^{n-1}\delta_o,$$

so that $b(x) \simeq d(o,x)$, near o.

For $p \leqslant (1+\delta)(n-2)$, we denote the Green kernel of the Schrödinger operator $\Delta - p\operatorname{Ric}_-$ by G_p.

4.2. A preliminary result

Proposition 4.1. — *With the above notation,*

$$\frac{|\mathrm{d}b|^{(1+\delta)(n-2)}(x)}{b^{n-2}(x)} \leqslant G_{(1+\delta)(n-2)}(o,x).$$

4.3. Proof of Theorem 1.4, assuming Proposition 4.1

Proof. — The gaugeability of the Schrödinger operator $\Delta-(1+\delta)\times(n-2)\operatorname{Ric}_-$ and the Sobolev inequality, together with Theorem 2.18 imply that

$$G_{(1+\delta)(n-2)}(o,x)\leqslant \frac{c_n\gamma^n}{\mu^{\frac{n}{2}}}\frac{1}{d(o,x)^{n-2}}.$$

Let $r(x):=d(o,x)$ so that we have

$$\frac{|\mathrm{d}b|(x)}{b^{\frac{1}{1+\delta}}(x)}\leqslant\left(\frac{c_n\gamma^n}{\mu^{\frac{n}{2}}}\right)^{\frac{1}{(1+\delta)(n-2)}}\frac{1}{r^{\frac{1}{1+\delta}}(x)}. \tag{4.1}$$

Integrating along a minimizing geodesic joining o and x, we get

$$b(x)^{\frac{\delta}{1+\delta}}\leqslant\left(\frac{c_n\gamma^n}{\mu^{\frac{n}{2}}}\right)^{\frac{1}{(1+\delta)(n-2)}} r(x)^{\frac{\delta}{1+\delta}},$$

and

$$b(x)\leqslant\Gamma\, r(x),$$

where

$$\Gamma=\left(\frac{c_n\gamma^n}{\mu^{\frac{n}{2}}}\right)^{\frac{1}{\delta(n-2)}}.$$

Hence the geodesic ball $B(o,R)$ is included in the sub-level set $\{b\leqslant\Gamma R\}$ and from Theorem 2.8 (2b), we have

$$\operatorname{vol}B(o,R)\leqslant\Gamma^n\mu^{-\frac{n}{n-2}}R^n. \qquad\square$$

Remark 4.2. — By the estimate (4.1), we also have $\|\mathrm{d}b\|_\infty\leqslant\Gamma$.

In order to prove Proposition 4.1, we need the following lemma.

LEMMA 4.3. — *Let $\frac{n-2}{n-1}\leqslant p\leqslant(1+\delta)(n-2)$ and $\alpha\geqslant 2$. If*

$$\int_{M\setminus B(o,1)}\left(\frac{|\mathrm{d}b|^p}{b^{n-2}}\right)^{\alpha}\mathrm{dv}_g<\infty$$

then

$$\frac{|\mathrm{d}b|^p(x)}{b^{n-2}(x)}\leqslant G_p(o,x).$$

Proof of Lemma 4.3. — Indeed by (2.12), we know that

$$(\Delta-p\operatorname{Ric}_-)\frac{|\mathrm{d}b|^p}{b^{n-2}}\leqslant 0\quad\text{on } M\setminus\{o\}.$$

The function $x \mapsto \frac{|db|^p(x)}{b^{n-2}(x)}$ is $(\Delta - p\,\mathrm{Ric}_-)$-subharmonic on the ball $B(x, d(o,x)/2)$. The gaugeability of the operator $\Delta - p\,\mathrm{Ric}_-$ and Theorem 2.18 (3) imply that if $x \in M \setminus \{o\}$, then

$$\frac{|db|^p(x)}{b^{n-2}(x)} \leqslant \frac{C}{d(o,x)^{\frac{n}{\alpha}}} \left(\int_{B(x,d(o,x)/2)} \left(\frac{|db|^p}{b^{n-2}} \right)^{\alpha} \mathrm{dv}_g \right)^{\frac{1}{\alpha}}.$$

Hence

$$\lim_{\infty} \frac{|db|^p}{b^{n-2}} = 0.$$

According to what we said in Subsection 2.6.1, there is some $\tau(\epsilon)$ such that $\lim_{\epsilon \to 0} \tau(\epsilon) = 0$, and

$$\frac{|db|^p(x)}{b^{n-2}(x)} \leqslant (1+\tau(\epsilon)) G_p(o,x) \quad \text{on } \partial B(o,\epsilon).$$

The Maximum principle implies then

$$\frac{|db|^p(x)}{b^{n-2}(x)} \leqslant (1+\tau(\epsilon)) G_p(o,x) + \sup_{z \in \partial B(o,R)} \frac{|db|^p(z)}{b^{n-2}(z)} \quad \text{on } B(o,R) \setminus B(o,\epsilon).$$

Letting $\epsilon \to 0$ and $R \to \infty$, the Lemma 4.3 is proved. □

The next step is a bound on the log derivative of the Green kernel.

Lemma 4.4. — *For any $x \in M$:*

$$\frac{|db|^{n-2}(x)}{b^{n-2}(x)} \leqslant G_{n-2}(o,x).$$

Proof of Lemma 4.4. — Let $p_0 = \frac{n-2}{n-1}$. We have already noticed that

$$\int_{M \setminus B(o,1)} |d_x G(o,x)|^2 \, \mathrm{dv}_g(x) < \infty,$$

hence

$$\int_{M \setminus B(o,1)} \left(\frac{|db|^{p_0}}{b^{n-2}} \right)^{2\frac{n-1}{n-2}} \mathrm{dv}_g < \infty$$

According to Lemma 4.3, we have

$$\frac{|db|^{p_0}(x)}{b^{n-2}(x)} \leqslant G_{p_0}(o,x).$$

Our main tool is the very general Hardy inequality (Proposition 2.29): $\forall\, \psi \in \mathcal{C}_0^\infty(M)$,

$$\frac{(n-2)^2}{4} \int_M \frac{|db|^2}{b^2} \psi^2 \, \mathrm{dv}_g \leqslant \int_M |d\psi|^2 \, \mathrm{dv}_g\,.$$

The Schrödinger operator $\Delta - (1+\delta)(n-2)\operatorname{Ric}_-$ is gaugeable hence non-negative, and we have the following Hardy type inequality, $\forall\, \psi \in \mathcal{C}_0^\infty(M)$,

$$\frac{(n-2)^2}{4}\frac{\delta}{1+\delta}\int_M \frac{|\mathrm{d}b|^2}{b^2}\psi^2\,\mathrm{dv}_g \leqslant \int_M \left[|\mathrm{d}\psi|^2 - (n-2)\operatorname{Ric}_-\psi^2\right]\mathrm{dv}_g\,.$$

When $p\in[p_0,n-2]$, using the function $\psi=\xi\frac{|\mathrm{d}b|^p}{b^{n-2}}$ where ξ is a Lipschitz function with compact support in $M\setminus\{o\}$ one gets

$$\frac{(n-2)^2}{4}\frac{\delta}{1+\delta}\int_M \frac{|\mathrm{d}b|^{2+2p}}{b^{2(n-1)}}\xi^2\,\mathrm{dv}_g \leqslant \int_M |\mathrm{d}\xi|^2\frac{|\mathrm{d}b|^{2p}}{b^{2(n-2)}}\,\mathrm{dv}_g\,. \tag{4.2}$$

Assume that for some $p\in[p_0,n-2]$, we have

$$\frac{|\mathrm{d}b|^p(x)}{b^{n-2}(x)}\leqslant G_p(o,x).$$

The operator $\Delta - p\operatorname{Ric}_-$ being gaugeable, we know that $\lim_{x\to\infty}G_p(o,x)=0$. Hence the measure $G_p(o,x)^2\,\mathrm{dv}_g(x)$ is parabolic on $M\setminus B(o,1)$ (see Proposition 2.28), so that the inequality (4.2) is valid for ξ a Lipschitz function that is zero in $B(o,1/2)$ and is equal to 1 outside $B(o,1)$. In particular, one gets

$$\int_{M\setminus B(o,1)}\frac{|\mathrm{d}b|^{2+2p}}{b^{2(n-1)}}\,\mathrm{dv}_g<\infty.$$

If $\bar p=(1+p)\frac{n-2}{n-1}$ one gets

$$\int_{M\setminus B(o,1)}\left(\frac{|\mathrm{d}b|^{\bar p}}{b^{n-2}}\right)^{2\frac{n-1}{n-2}}\mathrm{dv}_g<\infty$$

and with Lemma 4.3, one gets

$$\frac{|\mathrm{d}b|^{\bar p}(x)}{b^{n-2}(x)}\leqslant G_{\bar p}(o,x).$$

If we let $p_k=(n-2)-\left(\frac{n-2}{n-1}\right)^k(n-2-p_0)=(1+p_{k-1})\,\frac{n-2}{n-1}$, our argumentation yields that for all $k\in\mathbb{N}$:

$$\frac{|\mathrm{d}b|^{p_k}(x)}{b^{n-2}(x)}\leqslant G_{p_k}(o,x)\leqslant G_{n-2}(o,x).$$

Hence letting $k\to\infty$, we obtain the following estimate on the log derivative of the Green kernel:

$$\frac{|\mathrm{d}b|^{n-2}(x)}{b^{n-2}(x)}\leqslant G_{n-2}(o,x).$$ □

4.4. Proof of Proposition 4.1

Proof. — We use again the inequality (4.2), for $p = (n-2)$. This inequality is still valid when ξ is a Lipschitz function that is zero in near o and is equal to 1 outside a compact set, we use it with

$$\xi(x) = f(b)$$

where $f\colon (0,\infty) \to \mathbb{R}$ is a smooth function that is 1 outside a compact set and 0 near o:

$$\frac{(n-2)^2}{4}\frac{\delta}{1+\delta}\int_M \frac{|\mathrm{d}b|^{2(n-1)}}{b^{2(n-1)}} f^2(b)\,\mathrm{dv}_g \leqslant \int_M \frac{|\mathrm{d}b|^{2(n-1)}}{b^{2(n-2)}}(f'(b))^2\,\mathrm{dv}_g\,.$$

We introduce the measure m on $[1,\infty]$ defined by

$$\mathrm{m}([1,R]) = \int_{1\leqslant b\leqslant R} \frac{|\mathrm{d}b|^{2(n-1)}}{b^{2(n-2)}}\,\mathrm{dv}_g\,.$$

and we get the Hardy type inequality:

$$\frac{(n-2)^2}{4}\frac{\delta}{1+\delta}\int_1^\infty \frac{1}{t^2} f^2(t)\,\mathrm{dm}(t) \leqslant \int_1^\infty (f'(t))^2\,\mathrm{dm}(t),$$

for any smooth function f that is 0 near 0 and constant outside a compact set. Using Proposition 2.26, we get:

$$\int_R^\infty \frac{1}{t^2}\,\mathrm{dm}(t) = \int_{R\leqslant b} \frac{|\mathrm{d}b|^{2n-2}}{b^{2(n-1)}}\,\mathrm{dv}_g \leqslant \frac{C}{R^{(n-2)\sqrt{\frac{\delta}{1+\delta}}}}.$$

Hence for all $r < (n-2)\sqrt{\frac{\delta}{1+\delta}}$, with $p = \frac{n-2}{1-\frac{r}{2(n-1)}}$, one gets

$$\int_{M\setminus B(o,1)} \left(\frac{|\mathrm{d}b|^p}{b^{n-2}}\right)^{2\frac{n-1}{n-2}-\frac{r}{n-2}} < \infty$$

Our prior restriction on δ allows us to choose $r = (n-1)\frac{2\delta}{1+\delta}$ and $p = (n-2)(1+\delta)$. Then according to the Lemma 4.3, we have proved Proposition 4.1. □

4.5. Proof of Theorem 1.10

In fact, once the Euclidean volume growth has been proved, the properties in Theorem 1.10 immediately follow. Indeed, (M,g) is doubling and satisfies the upper (LY) bound

$$\forall\, t>0, x,y\in M\colon\ H(t,x,y) \leqslant \frac{c e^{-\frac{d^2(x,y)}{5t}}}{\operatorname{vol} B(x,\sqrt{t})}.$$

Moreover, according to (Remark 2.16-b), the operator $L := \Delta - \mathrm{Ric}_-$ is gaugeable and its heat kernel satisfies the same upper (LY) bound. Let $\vec{H}$ be the heat kernel associated with the Hodge-deRham Laplacian $\vec{\Delta}$ acting on 1-forms

$$\vec{\Delta} = \nabla^*\nabla + \mathrm{Ricci}\,.$$

By domination, we know that

$$\forall\, t > 0, x, y \in M\colon\ \left|\vec{H}(t,x,y)\right| \leqslant H_L(t,x,y) \leqslant \frac{ce^{-\frac{d^2(x,y)}{5t}}}{\mathrm{vol}\, B(x,\sqrt{t})}.$$

The results and the proof of [19, Theorem 5.5] imply that

- the heat kernel of (M,g) satisfies the (LY) estimates, hence (M,g) also satisfies the Poincaré inequalities (PI),
- the Riesz transform $d\Delta^{-\frac{1}{2}}$ is $L^p \to L^p$ is bounded for every $p \geqslant 2$.

According to [18], the Riesz transform is also $L^p \to L^p$ bounded for every $p \in (1,2]$.

4.6. Localization at infinity

When the Schrödinger operator $\Delta - (n-2)(1+\delta)\,\mathrm{Ric}_-$ is gaugeable outside a compact set, the arguments of the proof of Theorem 1.4 only lead to an estimate for centered balls.

Proposition 4.5. — *Let (M,g) be a complete Riemannian manifold which satisfies the Euclidean Sobolev inequality* (Sob). *Assume that there is a compact subset $K \subset M$, some $\delta > 0$, and a bounded positive function $h\colon M \setminus K \to \mathbb{R}_+$ such that*

$$\Delta h - (n-2)(1+\delta)\,\mathrm{Ric}_-\, h = 0 \ \ \textit{and}\ \ 1 \leqslant h \leqslant \gamma.$$

Then, for each $o \in M$, there is a constant θ such that for all $R \geqslant 0$:

$$\frac{1}{\theta}\, R^n \leqslant \mathrm{vol}\, B(o,R) \leqslant \theta\, R^n.$$

Remark 4.6. — A priori, the constant θ in the conclusion may depend on the point o. Theorem 1.15 gives conditions under which the constant θ can be chosen uniformly.

Proof. — In the setting of Proposition 4.5, we can always assume that $\delta < (n-2)/(n(3n-4))$.

We can find $W \in C_0^\infty(M)$ nonnegative such that the Schrödinger operator

$$L := \Delta + W - (n-2)(1+\delta)\,\mathrm{Ric}_-$$

is gaugeable. Indeed if $\bar{h}\colon M \to [1/2, 2\gamma]$ is an extension of h, then there is a bounded function q with compact support such that

$$\Delta\bar{h} + q\bar{h} - (n-2)(1+\delta)\operatorname{Ric}_-\bar{h} = 0.$$

Hence the Schrödinger operator $P := \Delta + q - (n-2)(1+\delta)\operatorname{Ric}_-$ is gaugeable and by [25, Theorem 3.2], for any nonnegative function $\mathcal{V}$ with compact support, the operator $P + \mathcal{V}$ is gaugeable.

We will note G_L the Green kernel of the operator L. Let $o \in M$, we still define

$$G(o,x) = \frac{1}{b_o^{n-2}(x)}.$$

Our previous argument can be used to show the following.

Lemma 4.7. — *Let $\rho > 0$ such that $\operatorname{supp} W \subset B(o,\rho)$ and $(n-2)/(n-1) \leqslant p \leqslant (n-2)(1+\delta)$. Assume that*

$$\lim_{\infty} \frac{|\mathrm{d}b_o|^p}{b_o^{n-2}} = 0.$$

Then,

(1) *on $M \setminus B(o,2\rho)$:*

$$\frac{|\mathrm{d}b_o|^p(x)}{b_o^{n-2}(x)} \leqslant \frac{A}{a}\, G_L(o,x)$$

where $A = \sup_{x \in \partial B(o,2\rho)} \frac{|\mathrm{d}b_o(x)|^p}{b_o^{n-2}(x)}$ and $a = \inf_{x \in \partial B(o,2\rho)} G_L(o,x)$.

(2) *If $x \in M \setminus \{o\}$ then*

$$\frac{|\mathrm{d}b_o|^p(x)}{b_o^{n-2}(x)} \leqslant G_L(o,x) + \int_{\operatorname{supp} W} G_L(x,y) W(y) \frac{|\mathrm{d}b_o|^p(y)}{b_o^{n-2}(y)}\, \mathrm{dv}_g(y).$$

The same argumentation yields that the hypothesis of the lemma is satisfied for $p = (n-2)(1+\delta)$.

Proposition 4.8. — *Assume that (M,g) is a complete Riemannian manifold which satisfies the Euclidean Sobolev inequality, and assume that for some $\delta > 0$, the Schrödinger operator $\Delta - (n-2)(1+\delta)\operatorname{Ric}_-$ is gaugeable at infinity. Let $o \in M$. There are positive constants c, ϵ which depend on (M,g) and o such that*

(1) *For all $R > 0$: $\operatorname{vol} B(o,R) \leqslant cR^n$.*

(2) *For any $x \in M$, the Green kernel satisfies*

$$\left(\frac{\epsilon}{d(o,x)}\right)^{n-2} \leqslant G(o,x) \leqslant \frac{1}{(\epsilon d(o,x))^{n-2}}$$

(3) *If b is defined by*
$$G(o,x)=b(x)^{2-n}$$
then
$$|\mathrm{d}b|\leqslant c.$$

5. Volume growth estimate: local results

5.1. Proof of Theorem 1.15

We are going to improve Proposition 4.8 with the result of Proposition 3.16. We assume that (M,g) is a complete Riemannian manifold which satisfies the Euclidean Sobolev inequality and that there is some compact set K such that
$$\sup_{x\in M\setminus K}\int_{M\setminus K} G(x,y)\,\mathrm{Ric}_-(y)\,\mathrm{dv}_g(y)\leqslant\frac{1}{16n}.$$
We know that this estimate on the Kato constant, implies that for every $\lambda<16n$, the Schrödinger operator $\Delta-\lambda\,\mathrm{Ric}_-$ is gaugeable on $M\setminus K$. Hence the conclusion of Proposition 4.8 holds.

Let $o\in M$ be a fixed point and define $r(x):=d(o,x)$ and $b(x):=G(o,x)^{-\frac{1}{n-2}}$. We know that
$$|\mathrm{d}b|\leqslant\frac{1}{\epsilon}\ \text{ and }\ \epsilon r(x)\leqslant b(x)\leqslant\frac{1}{\epsilon}r(x).$$
We already know that geodesics balls centered at o are doubling. Moreover according to the lower Euclidean volume estimate of any geodesic ball, we have
$$\operatorname{vol}B(o,r(x))\leqslant Cr(x)^n\leqslant\operatorname{vol}B\left(x,r(x)/4\right).$$
This property is called *volume comparison* by A. Grigor'yan and L. Saloff-Coste and according to [32, Proposition 4.7], the doubling condition is satisfied provided there is some $\rho>0$ such that, for every $x\in M$ with $r(x)\geqslant\rho$ and any $r\leqslant r(x)/4$,
$$\operatorname{vol}B(x,2r)\leqslant\theta\operatorname{vol}B(x,r).$$
Choose $\rho>0$ such that
$$K\subset B\left(o,\frac{\epsilon^2}{1000}\rho\right).$$
Let $x\in M$ be such that $r(x)\geqslant\rho$. Let $R=r(x)/2$. One can define $\xi_R=u\left(\frac{b}{R}\right)$ where $u\colon\mathbb{R}\to\mathbb{R}$ is a smooth function with compact support in

$[\epsilon/4, 4\epsilon]$ such that $u = 1$ on $[\epsilon/2, 2\epsilon]$. Then we have $\xi_R = 1$ on $B(o, 2R) \setminus B(o, R/2)$ and the support of ξ_R is included in $B\left(o, 4\epsilon^{-2}R\right) \setminus B(o, \frac{1}{4}\epsilon^2 R)$. Since

$$d\xi_R = \frac{1}{R}u'\left(\frac{b}{R}\right) db,$$

and

$$\begin{aligned}\Delta\xi_R &= \frac{1}{R}u'\left(\frac{b}{R}\right)\Delta b - \frac{1}{R^2}u''\left(\frac{b}{R}\right)|\mathrm{d}b|^2\\ &= -(n-1)\frac{1}{R}u'\left(\frac{b}{R}\right)\frac{|\mathrm{d}b|^2}{b} - \frac{1}{R^2}u''\left(\frac{b}{R}\right)|\mathrm{d}b|^2,\end{aligned}$$

there is some constant c (depending only on ϵ and u) such that

$$|\mathrm{d}\xi_R|^2 + |\Delta\xi_R| \leqslant \frac{c}{R^2}.$$

By construction, we have $\operatorname{supp}\xi_R \subset M \setminus K$ and $\xi_R = 1$ on $B(x, r(x)/2)$. Hence we can use Proposition 3.16 and get that there is a constant γ such that, for all $r \in (0, r(x)/4)$,

$$\operatorname{vol} B(x, 2r) \leqslant \gamma \operatorname{vol} B(x, r) \text{ and } H(r^2, x, x) \leqslant \frac{\gamma}{\operatorname{vol} B(x, r)}.$$

We have shown that (M, g) is doubling.

It remains to show the heat kernel estimate. According to [30], the conjunction of the doubling property and of the heat kernel estimates, for all $t > 0$ and all $x, y \in M$,

$$H(t, x, y) \leqslant \frac{Ce^{-\frac{d^2(x,y)}{5t}}}{\operatorname{vol} B(x, \sqrt{t})}$$

is equivalent to the so called relative Faber–Krahn inequality: there are positive constants C, μ such that for any $x \in M$ and $R > 0$ and any open domain[(2)] $\Omega \subset B(x, R)$:

$$\lambda_1^D(\Omega) \geqslant \frac{C}{R^2}\left(\frac{\operatorname{vol}\Omega}{\operatorname{vol} B(x, R)}\right)^{-\frac{2}{\mu}}.$$

But our heat kernel estimates for remote balls imply that the above Faber–Krahn inequality is satisfied for remote balls and the volume estimate and the Sobolev inequality insure that the above Faber–Krahn inequality is

(2) We have denoted by $\lambda_1^D(\Omega)$ the lowest eigenvalue of the Dirichlet Laplacian on Ω:

$$\lambda_1^D(\Omega) = \inf_{\varphi \in \mathcal{C}_0^\infty(\Omega)} \frac{\int_\Omega |\mathrm{d}\varphi|^2}{\int_\Omega |\varphi|^2}.$$

satisfied for balls centered at o. By [9, Proof of Theorem 2.4], the relative Faber–Krahn inequality holds on (M,g).

Once these properties has been shown, the results of [24] imply that when $n \geqslant 4$, then the Riesz transform is L^p bounded for any $p \in (1,n)$. □

5.2. Proof of Theorem 1.20

5.2.1. The setting

Our hypothesis and conclusion being invariant by scaling, we assume $R=1$. We consider (M^n,g) a Riemannian manifold and $B(o,3) \subset M$ a relatively compact geodesic ball. Let $p>1$ and $q=p/(p-1)$. We assume that there are $\mu>0$, $\delta > \frac{(q(n-2)-2)^2}{8q(n-2)}$ and $\Lambda>0$ such that

- the ball $B(o,3)$ satisfies the Euclidean Sobolev inequality (Sob) with Sobolev constant μ,
- the operator $\Delta-(1+\delta)(n-2)\operatorname{Ric}_-$ is nonnegative on $B(o,3)$,
- $\sup_{x\in B(o,3)} \int_{B(o,3)} G(x,y)\operatorname{Ric}_-(y)^p\,\mathrm{dv}_g(y) \leqslant \Lambda^p$, where G is the Green kernel for the Laplacian Δ with the Dirichlet boundary condition of $B(o,3)$.

We are going to prove that there is a constant ϑ which depends only on n,p,δ,Λ, the Sobolev constant μ and $\operatorname{vol}(B(o,3))$, such that, for all $x \in B(o,1)$ and all $r \in (0,1]$,

$$\frac{\operatorname{vol} B(x,r)}{r^n} \leqslant \vartheta.$$

Our objectif is to get an L^∞ bound on the gradient of the Dirichlet Green kernel. Let $p \in B(o,1)$ and consider

$$G(p,\cdot\,) = \frac{1}{b^{n-2}}$$

the Green kernel of the Laplacian on $B(p,1)$ for the Dirichlet boundary conditions and with pole at p. We let $B := B(p,1)$.

5.2.2. L^2-estimate

Let $\nu-2=(n-2)\sqrt{\frac{\delta}{1+\delta}}$, the strong positivity and the very general Hardy inequality yield: $\forall\, \psi \in \mathcal{C}^\infty(B(o,1))$

$$\frac{(\nu-2)^2}{4}\int_B \frac{|\mathrm{d}b|^2}{b^2}\psi^2\,\mathrm{dv}_g \leqslant \int_B \left[|\mathrm{d}\psi|^2-(n-2)\operatorname{Ric}_-\psi^2\right]\mathrm{dv}_g\,.$$

If ξ is a Lipschitz function with compact support in $B \setminus \{p\}$, we use the test function
$$\psi = \xi \frac{|\mathrm{d}b|^\alpha}{b^{n-2}},$$
with $\alpha \leqslant n-2$. Using the integration by parts formula (2.5) and the inequality (2.12),
$$\Delta \frac{|\mathrm{d}b|^\alpha}{b^{n-2}} - (n-2)\operatorname{Ric}_- \frac{|\mathrm{d}b|^\alpha}{b^{n-2}} \leqslant 0$$
we get
$$\frac{(\nu-2)^2}{4} \int_B \frac{|\mathrm{d}b|^{\alpha 2+2}}{b^{2n-2}} \xi^2 \,\mathrm{dv}_g \leqslant \int_B \frac{|\mathrm{d}b|^{\alpha 2}}{b^{2n-4}} |\mathrm{d}\xi|^2 \,\mathrm{dv}_g .$$
Let $\Omega \subset B$ be such that
$$\Omega^r = \{x \in M, d(x,\Omega) < r\} \subset B \setminus \{p\},$$
with $\xi(x) := \max\{1 - d(x,\Omega)/r, 0\}$, we get
$$\frac{(\nu-2)^2}{4} \int_\Omega \frac{|\mathrm{d}b|^{\alpha 2+2}}{b^{2n-2}} \,\mathrm{dv}_g \leqslant \frac{1}{r^2} \int_{\Omega^r} \frac{|\mathrm{d}b|^{\alpha 2}}{b^{2n-4}} \,\mathrm{dv}_g .$$
Using Hölder's inequality, we get
$$\int_\Omega \frac{|\mathrm{d}b|^{\alpha+1\frac{n-2}{n-1}}}{b^{2n-4}} \,\mathrm{dv}_g \leqslant (\operatorname{vol}\Omega)^{\frac{1}{n-1}} \left(\frac{4}{(\nu-2)^2 r^2}\right)^{\frac{n-2}{n-1}} \left(\int_{\Omega^r} \frac{|\mathrm{d}b|^{\alpha 2}}{b^{2n-4}} \,\mathrm{dv}_g\right)^{\frac{n-2}{n-1}} .$$
We are going to iterate this inequality: assume $\Omega^r \subset B \setminus \{p\}$ and $r = r_1 + \dots + r_k$ and let $\kappa = \frac{n-2}{n-1}$
$$\alpha_k = (n-2) + \kappa^k \left(\alpha_0 - (n-2)\right)$$
and
$$v_k = \sum_{i=0}^{k-1} \kappa^i.$$
$$\int_\Omega \frac{|\mathrm{d}b|^{2\alpha_k}}{b^{2n-4}} \,\mathrm{dv}_g$$
$$\leqslant (\operatorname{vol}\Omega^r)^{\frac{v_k}{n-1}} \left(\frac{4}{(\nu-2)^2}\right)^{\frac{n-2}{n-1} v_k} \prod_{i=1}^k \left(\frac{1}{r_i^2}\right)^{\kappa^i} \left(\int_{\Omega^r} \frac{|\mathrm{d}b|^{2\alpha_0}}{b^{2n-4}} \,\mathrm{dv}_g\right)^{\kappa^k} .$$
If we choose $r/2^{i+2} \leqslant r_i \leqslant r/2^i$ and if we let $k \to +\infty$, we get that $\Omega^r \subset B \setminus \{p\}$, then
$$\int_\Omega \frac{|\mathrm{d}b|^{2n-4}}{b^{2n-4}} \,\mathrm{dv}_g \leqslant \frac{c(n)}{(\nu-2)^{2n-4} r^{2n-4}} \operatorname{vol}\Omega^r. \tag{5.1}$$

5.2.3. An integral estimate

We now introduce the function

$$\psi := \frac{(|\mathrm{d}b|-1)_+^{n-2}}{b^{n-2}}.$$

We know that ψ is bounded (see 2.6.1) and satisfies:

$$\Delta\psi-(n-2)\operatorname{Ric}_-\psi \leqslant (n-2)\operatorname{Ric}_- \frac{(|\mathrm{d}b|-1)_+^{n-3}}{b^{n-2}}.$$

Let $\tau\in(1/2,1)$ and let ξ be a Lipschitz function with compact support in B. We have

$$\int_B |\mathrm{d}(\xi\psi^\tau)|^2\,\mathrm{dv}_g = \tau\int_B \Delta\psi\,\psi^{2\tau-1}\xi^2\,\mathrm{dv}_g \\ +\int_B |\mathrm{d}\xi|^2\psi^{2\tau}\,\mathrm{dv}_g + \left(\frac{1}{\tau}-1\right)\int_B |\mathrm{d}\psi^\tau|^2\xi^2\,\mathrm{dv}_g\,.$$

A priori, this holds only if ξ has compact support in $B\backslash\{p\}$, but because ψ is bounded near p, the inequality holds more generally. And for all $\varepsilon\in(0,1)$:

$$\int_B |\mathrm{d}(\xi\psi^\tau)|^2\,\mathrm{dv}_g \geqslant (1-\varepsilon)\int_B \xi^2|\mathrm{d}\psi^\tau|^2\,\mathrm{dv}_g - \left(\frac{1}{\varepsilon}-1\right)\int_B |\mathrm{d}\xi|^2\psi^{2\tau}\,\mathrm{dv}_g,$$

so that

$$\int_B \xi^2|\mathrm{d}\psi^\tau|^2\,\mathrm{dv}_g \leqslant \frac{1}{1-\varepsilon}\int_B |\mathrm{d}(\xi\psi^\tau)|^2\,\mathrm{dv}_g + \frac{1}{\varepsilon}\int_B |\mathrm{d}\xi|^2\psi^{2\tau}\,\mathrm{dv}_g\,.$$

And we get

$$\left(1-\frac{1-\tau}{\tau(1-\varepsilon)}\right)\int_B |\mathrm{d}(\xi\psi^\tau)|^2\,\mathrm{dv}_g \\ \leqslant \tau\int_B \Delta\psi\,\psi^{2\tau-1}\xi^2\,\mathrm{dv}_g + \left(1+\frac{1}{\varepsilon}\left(\frac{1}{\tau}-1\right)\right)\int_B |\mathrm{d}\xi|^2\psi^{2\tau}\,\mathrm{dv}_g\,. \tag{5.2}$$

According to our assumptions on δ, we can choose $\tau\in(1/2,1)$, $\varepsilon\in(0,2-1/\tau)$ such that

$$2\tau-1<\frac{2}{q(n-2)} \quad \text{and} \quad \kappa := \frac{2\tau-1-\varepsilon\tau}{\tau(1-\varepsilon)} - \frac{\tau}{1+\delta} > 0. \tag{5.3}$$

Let $c := \left(1+\frac{1}{\varepsilon}\left(\frac{1}{\tau}-1\right)\right)$, we get:

$$\begin{aligned}
&\kappa \int_B |\mathrm{d}(\xi\psi^\tau)|^2 \,\mathrm{dv}_g \\
&\quad\leqslant \left(1-\frac{1-\tau}{\tau(1-\varepsilon)}\right)\int_B |\mathrm{d}(\xi\psi^\tau)|^2 \,\mathrm{dv}_g - \tau(n-2)\int_B \mathrm{Ric}_-\,\psi^{2\tau}\xi^2 \,\mathrm{dv}_g \\
&\quad\leqslant \tau\int_B \left[\Delta\psi-(n-2)\,\mathrm{Ric}_-\,\psi\right]\psi^{2\tau-1}\xi^2 \,\mathrm{dv}_g + c\int_B |\mathrm{d}\xi|^2\psi^{2\tau}\,\mathrm{dv}_g \\
&\quad\leqslant \tau(n-2)\int_B \mathrm{Ric}_-\,\frac{(|\mathrm{d}b|-1)_+^{2\tau(n-2)-1}}{b^{2\tau(n-2)}}\xi^2\,\mathrm{dv}_g + c\int_B |\mathrm{d}\xi|^2\psi^{2\tau}\,\mathrm{dv}_g\,.
\end{aligned}$$

We now choose

$$\xi(x)=\max\left\{1-2d(p,x),\frac{1}{2}\right\}.$$

Using the L^2 estimate (5.1), we get

$$c\int_B |\mathrm{d}\xi|^2\psi^{2\tau}\,\mathrm{dv}_g \leqslant 4c\int_{B(p,1/2)\setminus B(p,1/4)}\psi^{2\tau}\,\mathrm{dv}_g \leqslant C\,\mathrm{vol}\,B(p,1).$$

Let $Q=2\tau(n-2)$, the Hölder inequality yields

$$\begin{aligned}
&\int_B \mathrm{Ric}_-\,\frac{(|\mathrm{d}b|-1)_+^{2\tau(n-2)-1}}{b^{2\tau(n-2)}}\xi^2\,\mathrm{dv}_g \\
&\qquad\leqslant \left(\int_B \mathrm{Ric}_-\,\psi^{2\tau}\xi^2\,\mathrm{dv}_g\right)^{1-\frac{1}{Q}}\left(\int_B \mathrm{Ric}_-\,\frac{\xi^2}{b^Q}\,\mathrm{dv}_g\right)^{\frac{1}{Q}} \\
&\qquad\leqslant \lambda\int_B \mathrm{Ric}_-\,\psi^{2\tau}\xi^2\,\mathrm{dv}_g + \lambda^{1-Q}\int_B \mathrm{Ric}_-\,\frac{\xi^2}{b^Q}\,\mathrm{dv}_g \\
&\qquad\leqslant \lambda\int_B |\mathrm{d}(\xi\psi^\tau)|^2\,\mathrm{dv}_g + \lambda^{1-Q}\int_B \mathrm{Ric}_-\,\frac{\xi^2}{b^Q}\,\mathrm{dv}_g\,.
\end{aligned}$$

We now choose λ such that $\tau(n-2)\lambda=\kappa/2$, and we get

$$\frac{\kappa}{2}\int_B |\mathrm{d}(\xi\psi^\tau)|^2\,\mathrm{dv}_g \leqslant C\int_B \mathrm{Ric}_-\,\frac{\xi^2}{b^Q}\,\mathrm{dv}_g + C\,\mathrm{vol}\,B(p,1).$$

Using Hölder's inequality again, we have:

$$\begin{aligned}
&\int_B \mathrm{Ric}_-\,\frac{\xi^2}{b^{2\tau(n-2)}}\,\mathrm{dv}_g \\
&\qquad\leqslant \left(\int_B \mathrm{Ric}_-^p\,\frac{1}{b^{n-2}}\,\mathrm{dv}_g\right)^{\frac{1}{p}}\left(\int_B \frac{1}{b^{((2\tau-1)q+1)(n-2)}}\,\mathrm{dv}_g\right)^{\frac{1}{q}} \\
&\qquad\leqslant \Lambda C\left(\frac{1}{\mu}\right)^{2\tau-1+\frac{1}{q}}(\mathrm{vol}\,B)^{1-2\tau+\frac{2}{n}\left(2\tau-1+\frac{1}{q}\right)}\,.
\end{aligned}$$

Defining

$$\mathbf{I} := \Lambda \left(\frac{\operatorname{vol} B(o,3)}{\mu^{\frac{n}{2}}} \right)^{\frac{2}{nq}},$$

we get

$$\int_B \operatorname{Ric}_- \frac{\xi^2}{b^{2\tau(n-2)}} \, \mathrm{dv}_g \leqslant c\mathbf{I} \left(\frac{(\operatorname{vol} B(o,3))^{\frac{2}{n}}}{\mu} \right)^{2\tau-1} (\operatorname{vol} B(o,3))^{1-2\tau}.$$

Recall that according to Theorem 2.8-iv, we have

$$\operatorname{vol} B(o,3) \geqslant c_n \mu^{\frac{n}{2}},$$

hence

$$\int_B \operatorname{Ric}_- \frac{\xi^2}{b^{2\tau(n-2)}} \, \mathrm{dv}_g \leqslant c\mathbf{I} \left(\frac{(\operatorname{vol} B(o,3))^{\frac{2}{n}}}{\mu} \right)^{2\tau-1} \mu^{n(\tau-1/2)}.$$

Using the very general Hardy inequality:

$$\frac{(n-2)^2}{4} \int_B \frac{|\mathrm{d}b|^2}{b^2} (\xi\psi^\tau)^2 \, \mathrm{dv}_g \leqslant \int_B |\mathrm{d}(\xi\psi^\tau)|^2,$$

one gets

$$\int_{B(p,1/4)} \frac{|\mathrm{d}b|^2 (|\mathrm{d}b|-1)_+^{2\tau(n-2)}}{b^{2\tau(n-2)+2}} \, \mathrm{dv}_g \leqslant \Gamma, \tag{5.4}$$

with

$$\Gamma = c \left(\operatorname{vol} B(o,3) + \mathbf{I} \left((\operatorname{vol} B(o,3))^{\frac{2}{n}} / \mu \right)^{2\tau-1} \mu^{n(\tau-1/2)} \right).$$

5.2.4. Bound on the gradient

Recall that

$$\frac{1}{b^{n-2}(x)} \leqslant \frac{c_n}{\mu^{\frac{n}{2}} d(p,x)^{n-2}}$$

Hence, if $R \leqslant \epsilon_n \mu^{\frac{n}{2(n-2)}}$,

$$\Omega_R^{\#} = \left\{ \frac{R}{2} \leqslant b \leqslant \frac{5}{2} R \right\} \subset B(p,1/4).$$

By the coarea formula, we have

$$\int_{\Omega_R^{\#}} |\mathrm{d}b|^2 \, \mathrm{dv}_g = \int_{R/2}^{5R/2} \left(\int_{b=t} |\mathrm{d}b| \right) dt$$

and by the Green formula $\int_{b=t} \frac{|\mathrm{d}b|}{b^{n-1}} = c_n$, hence we obtain

$$\int_{\Omega_R^\#} |\mathrm{d}b|^2 \,\mathrm{dv}_g = c_n R^n.$$

Using the inequality $x^{2\tau(n-2)} \leqslant 2^{2\tau(n-2)} \left(1 + (x-1)_+^{2\tau(n-2)}\right)$, we deduce that

$$\int_{\Omega_R^\#} |\mathrm{d}b|^{2+2\tau(n-2)} \,\mathrm{dv}_g \leqslant cR^n + c\Gamma R^{2+2\tau(n-2)}.$$

Our hypothesis and Proposition 2.21 yield that there is some γ depending only on the constants δ, n, p, I and $(\operatorname{vol} B(o,3))^{\frac{2}{n}}/\mu$, such that the Schrödinger operator $L = \Delta - \frac{n-2}{n-1}\operatorname{Ric}_-$ is gaugeable on $B(p,1)$ with constant γ. We let

$$\rho := \epsilon_n \mu^{\frac{n}{2(n-2)}}$$

and using Proposition 2.30, we get that for $b \leqslant \rho$,

$$|\mathrm{d}b|^{(2\tau-1)(n-2)} \leqslant B^{(2\tau-1)(n-2)}$$

where

$$B^{(2\tau-1)(n-2)} := c\left(1 + \Gamma\rho^{(2\tau-1)(n-2)}\right) \frac{\gamma^{2+2\tau(n-2)\frac{n-1}{n-2}+n-2}}{\mu^{\frac{n}{2}}}.$$

5.2.5. Volume upper bound

If $d(p,x) \leqslant \rho/B$ then we have

$$b(x) \leqslant Bd(p,x)$$

and hence for $r \leqslant \rho$, we get

$$\frac{\operatorname{vol} B(p,r)}{r^n} \leqslant \frac{\operatorname{vol}\{b \leqslant Br\}}{r^n} \leqslant B^n \mu^{-\frac{n}{n-2}}.$$

Whereas for $\rho \leqslant r \leqslant 1$, one gets

$$\frac{\operatorname{vol} B(p,r)}{r^n} \leqslant \frac{\operatorname{vol} B(o,3)}{\rho^n}.$$

5.2.6. Further consequence

It remains to show how one can get the Poincaré inequalities. It is a direct consequence of the following proposition which could have been used in order to prove Theorem 1.10.

Proposition 5.1. — *Assume that $B(x, 2R)$ is a relatively compact geodesic ball in a Riemannian manifold, and assume that*

- *$B(x, 2R)$ satisfies the Euclidean Sobolev inequality* (Sob) *with constant μ,*
- *the Schrödinger operator $\Delta - \mathrm{Ric}_-$ is gaugeable on $B(x, 2R)$ with constant γ.*

Then, letting

$$\lambda := c_n \gamma^n \frac{\operatorname{vol} B(x, 2R)}{\mu^{\frac{n}{2}} R^n},$$

we have the Poincaré type inequality,

$$\forall\, \psi \in \mathcal{C}^1(B(x, 2R))\colon \quad \int_{B(x,R)} (\psi - \psi_{B(x,R)})^2 \,\mathrm{dv}_g \leqslant \lambda R^2 \int_{B(x,2R)} |\mathrm{d}\psi|_g^2 \,\mathrm{dv}_g \,.$$

Proof. — Let $\psi \in \mathcal{C}^1(B(x, 2R))$ and let φ be the harmonic extension of $\psi|_{\partial B(x,2R)}$. The Sobolev inequality implies that

$$\|\psi - \varphi\|_2^2 \leqslant \frac{(\operatorname{vol} B(x, 2R))^{\frac{2}{n}}}{\mu} \|\mathrm{d}\psi - d\varphi\|_2^2. \tag{5.5}$$

The function $|\mathrm{d}\varphi|$ satisfies

$$\Delta |\mathrm{d}\varphi| \leqslant \mathrm{Ric}_- \, |\mathrm{d}\varphi|,$$

hence with Theorem 2.18 (3), one gets

$$\sup_{z \in B(x,R)} |\mathrm{d}\varphi|^2(z) \leqslant \frac{c_n \gamma^n}{\mu^{n/2} R^n} \int_{B(x,2R)} |\mathrm{d}\varphi|^2 \,\mathrm{dv}_g \,.$$

In particular with $c = \varphi(x)$, one gets:

$$\begin{aligned} \|\varphi - c\|_2^2 &\leqslant R^2 \operatorname{vol} B(x, R) \sup_{z \in B(x,R)} |\mathrm{d}\varphi|^2(z) \\ &\leqslant c_n R^2 \gamma^n \frac{\operatorname{vol} B(x, 2R)}{\mu^{n/2} R^n} \int_{B(x,2R)} |\mathrm{d}\varphi|^2 \,\mathrm{dv}_g \,. \end{aligned} \tag{5.6}$$

The conclusion now follows from the inequalities (5.5) and (5.6) and the fact that the ratio

$$\operatorname{vol} B(x, 2R) / (\mu^{n/2} R^n)$$

is bounded from below by a constant which depends only on n. □

In the setting of Subsection (5.2), we have proved that there is a positive constant θ such that for all $x \in B(o, 1)$ and any $r \in (0, 1)$ then

$$\frac{r^n}{\theta} \leqslant \operatorname{vol} B(x, r) \leqslant \theta r^n.$$

Note that by monotonicity of $r \mapsto \operatorname{vol} B(x,r)$, the same kind of inequality is true for all $r \in (0,2)$. Using Proposition 5.1, we get that there is a constant λ such that for any $x \in B(o,1)$ and any $r \in (0,1)$:

$$\forall\, \psi \in \mathcal{C}^1(B(x,2r))\colon \int_{B(x,r)} (\psi - \psi_{B(x,r)})^2 \,\mathrm{dv}_g \leqslant \lambda r^2 \int_{B(x,2r)} |\mathrm{d}\psi|_g^2 \,\mathrm{dv}_g\,.$$

A now classical result of D. Jerison ([38, 43]) implies the announced Poincaré inequalities.

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