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VIRTUAL BRAIDS AND PERMUTATIONS

by Paolo BELLINGERI & Luis PARIS

Abstract. — Let VB_n be the virtual braid group on n strands and let $\mathfrak{S}_n$ be the symmetric group on n letters. Let $n, m \in \mathbb{N}$ such that $n \geqslant 5$, $m \geqslant 2$ and $n \geqslant m$. We determine all possible homomorphisms from VB_n to $\mathfrak{S}_m$, from $\mathfrak{S}_n$ to VB_m and from VB_n to VB_m. As corollaries we get that $\mathrm{Out}(\mathrm{VB}_n)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and that VB_n is both, Hopfian and co-Hofpian.

Résumé. — Soient VB_n le groupe de tresses virtuelles à n brins et $\mathfrak{S}_n$ le groupe symétrique de l'ensemble à n éléments. Soient $n, m \in \mathbb{N}$ tels que $n \geqslant 5$, $m \geqslant 2$ et $n \geqslant m$. Nous déterminons tous les homomorphismes de VB_n dans $\mathfrak{S}_m$, de $\mathfrak{S}_n$ dans VB_m et de VB_n dans VB_m. Comme corollaires nous obtenons que $\mathrm{Out}(\mathrm{VB}_n)$ est isomorphe à $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ et que VB_n est à la fois hopfien et co-hofpien.

1. Introduction

The study of the homomorphisms from the braid group B_n on n strands to the symmetric group $\mathfrak{S}_n$ goes back to Artin [1] himself. He was right thinking that this would be an important step toward the determination of the automorphism group of B_n. As pointed out by Lin [18, 19], all the homomorphisms from B_n to $\mathfrak{S}_m$ for $n \geqslant m$ are easily deduced from the ideas of Artin [1]. The automorphism group of B_n was then determined by Dyer–Grossman [13] using in particular Artin's results in [1]. The homomorphisms from B_n to B_m were determined by Lin [18, 19] for $n > m$ and by Castel [9] for $n = m$ and $n \geqslant 6$.

Virtual braids were introduced by Kauffman [16] together with virtual knots and links. They have interpretations in terms of diagrams (see Kauffman [16], Kamada [15] and Vershinin [21]) and also in terms of braids in

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thickened surfaces (see Cisneros de la Cruz [12]) and now there is a quite extensive literature on them. Despite their interest in low-dimensional topology, the virtual braid groups are poorly understood from a combinatorial point of view. Among the known results for these groups there are solutions to the word problem in Bellingeri–Cisneros de la Cruz–Paris [8] and in Chterental [10] and the calculation of some terms of its lower central series in Bardakov–Bellingeri [4], and some other results but not many more. For example, it is not known whether these groups have a solution to the conjugacy problem or whether they are linear.

In the present paper we prove results on virtual braid groups in the same style as those previously mentioned for the braid groups. More precisely, we take $n, m \in \mathbb{N}$ such that $n \geqslant 5$, $m \geqslant 2$ and $n \geqslant m$ and

(1) we determine all the homomorphisms from VB_n to $\mathfrak{S}_m$,
(2) we determine all the homomorphisms from $\mathfrak{S}_n$ to VB_m,
(3) we determine all the homomorphisms from VB_n to VB_m.

From these classifications it will follow that $\mathrm{Out}(\mathrm{VB}_n)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and that VB_n is both, Hopfian and co-Hopfian.

Our study is totally independent from the works of Dyer–Grossman [13], Lin [18, 19] and Castel [9] cited above and all similar works on braid groups. Our viewpoint/strategy is also new for virtual braid groups, although some aspects are already present in Bellingeri–Cisneros de la Cruz–Paris [8]. Our idea/main-contribution consists on first observing that the virtual braid group VB_n decomposes as a semi-direct product $\mathrm{VB}_n = \mathrm{KB}_n \rtimes \mathfrak{S}_n$ of an Artin group KB_n by the symmetric group $\mathfrak{S}_n$, and secondly and mainly on making a deep study of the action of $\mathfrak{S}_n$ on KB_n. From this study we deduce that there exists a unique embedding of the symmetric group $\mathfrak{S}_n$ into the virtual braid group VB_n up to conjugation (see Lemma 5.1). From there the classifications are deduced with some extra work.

The group VB_2 is isomorphic to $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. On the other hand, fairly precise studies of the combinatorial structure of VB_3 can be found in Bardakov–Bellingeri [4], Bardakov–Mikhailov–Vershinin–Wu [5] and Bellingeri–Cisneros de la Cruz–Paris [8]. From this one can probably determine (with some difficulties) all the homomorphisms from VB_n to $\mathfrak{S}_m$, from $\mathfrak{S}_n$ to VB_m, and from VB_n to VB_m, for $n \geqslant m$ and $n = 2, 3$. Our guess is that the case $n = 4$ will be much more tricky.

Our paper is organized as follows. In Section 2 we give the definitions of the homomorphisms involved in the paper, we state our three main theorems, and we prove their corollaries. Section 3 is devoted to the study of the action of the symmetric group $\mathfrak{S}_n$ on the above mentioned group KB_n. The results of this section are technical but they provide key informations on the structure of VB_n that are essential in the proofs of our main theorems. We also think that they are interesting by themselves and may be used for other purposes. Our main theorems are proved in Section 4, Section 5 and Section 6, respectively.

2. Definitions and statements

In the paper we are interested in the algebraic and combinatorial aspects of virtual braid groups, hence we will adopt their standard definition in terms of generators and relations. So, the *virtual braid group* VB_n on n strands is the group defined by the presentation with generators $\sigma_1, \dots, \sigma_{n-1}, \tau_1, \dots, \tau_{n-1}$ and relations

$$\tau_i^2 = 1 \text{ for } 1 \leqslant i \leqslant n-1\,,$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i\,,\ \ \tau_i\tau_j = \tau_j\tau_i\,,\ \ \tau_i\sigma_j = \sigma_j\tau_i \text{ for } |i-j| \geqslant 2\,,$$
$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j\,,\ \ \tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j\,,\ \ \tau_i\tau_j\sigma_i = \sigma_j\tau_i\tau_j \text{ for } |i-j| = 1\,.$$

For each $i \in \{1, \dots, n-1\}$ we set $s_i = (i, i+1)$. It is well-known that the symmetric group $\mathfrak{S}_n$ on n letters has a presentation with generators $s_1, \dots, s_{n-1}$ and relations

$$s_i^2 = 1 \text{ for } 1 \leqslant i \leqslant n-1\,,\ \ s_is_j = s_js_i \text{ for } |i-j| \geqslant 2\,,$$
$$s_is_js_i = s_js_is_j \text{ for } |i-j| = 1\,.$$

Now, we define the homomorphisms that are involved in the paper. Let G, H be two groups. We say that a homomorphism $\psi : G \to H$ is *Abelian* if the image of ψ is an Abelian subgroup of H. Note that the abelianization of $\mathfrak{S}_n$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, hence the image of any Abelian homomorphism $\varphi : \mathfrak{S}_n \to H$ is either trivial or a cyclic group of order 2. On the other hand, the abelianization of VB_n is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the copy of $\mathbb{Z}$ is generated by the class of σ_1 and the copy of $\mathbb{Z}/2\mathbb{Z}$ is generated by the class of τ_1. Thus, the image of any Abelian homomorphism $\psi : \mathrm{VB}_n \to H$ is a quotient of $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

From the presentations of VB_n and $\mathfrak{S}_n$ given above we see that there are epimorphisms $\pi_P : \mathrm{VB}_n \to \mathfrak{S}_n$ and $\pi_K : \mathrm{VB}_n \to \mathfrak{S}_n$ defined by $\pi_P(\sigma_i) = \pi_P(\tau_i) = s_i$ for all $1 \leqslant i \leqslant n-1$ and by $\pi_K(\sigma_i) = 1$ and $\pi_K(\tau_i) = s_i$ for all $1 \leqslant i \leqslant n-1$, respectively. The kernel of π_P is called the *virtual pure braid group* and is denoted by VP_n. A presentation of this group can be found in Bardakov [3]. It is isomorphic to the group of the

Yang–Baxter equation studied in Bartholdi–Enriquez–Etingof–Rains [6]. The kernel of π_K does not have any particular name. It is denoted by KB_n. It is an Artin group, hence we can use tools from the theory of Artin groups to study it and get results on VB_n itself (see for example Godelle–Paris [14] or Bellingeri–Cisneros de la Cruz–Paris [8]). This group will play a prominent role in our study. It will be described and studied in Section 3.

Using again the presentations of $\mathfrak{S}_n$ and VB_n we see that there is a homomorphism $\iota : \mathfrak{S}_n \to \mathrm{VB}_n$ that sends s_i to τ_i for all $1 \leqslant i \leqslant n-1$. Observe that ι is a section of both, π_P and π_K, hence ι is injective and we have the decompositions $\mathrm{VB}_n = \mathrm{VP}_n \rtimes \mathfrak{S}_n$ and $\mathrm{VB}_n = \mathrm{KB}_n \rtimes \mathfrak{S}_n$.

The two main automorphisms of VB_n that are involved in the paper are the automorphisms $\zeta_1, \zeta_2 : \mathrm{VB}_n \to \mathrm{VB}_n$ defined by $\zeta_1(\sigma_i) = \tau_i\sigma_i\tau_i$ and $\zeta_1(\tau_i) = \tau_i$ for all $1 \leqslant i \leqslant n-1$, and by $\zeta_2(\sigma_i) = \sigma_i^{-1}$ and $\zeta_2(\tau_i) = \tau_i$ for all $1 \leqslant i \leqslant n-1$, respectively. It is easily checked that ζ_1 and ζ_2 are of order two and commute, hence they generate a subgroup of $\mathrm{Aut}(\mathrm{VB}_n)$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The last homomorphism (automorphism) concerned by the paper appears only for $n = 6$. This is the automorphism $\nu_6 : \mathfrak{S}_6 \to \mathfrak{S}_6$ defined by

$$\begin{gathered}
\nu_6(s_1) = (1,2)(3,4)(5,6)\,,\ \ \nu_6(s_2) = (2,3)(1,5)(4,6)\,,\\
\nu_6(s_3) = (1,3)(2,4)(5,6)\,,\ \ \nu_6(s_4) = (1,2)(3,5)(4,6)\,,\\
\nu_6(s_5) = (2,3)(1,4)(5,6)\,.
\end{gathered}$$

It is well-known that $\mathrm{Out}(\mathfrak{S}_n)$ is trivial for $n \neq 6$ and that $\mathrm{Out}(\mathfrak{S}_6)$ is a cyclic group of order 2 generated by the class of ν_6. Note also that ν_6^2 is the conjugation by $w_0 = (1,6,2,5,3)$.

We give a last definition before passing to the statements. Let G, H be two groups. For each $\alpha \in H$ we denote by $c_\alpha : H \to H$, $\beta \mapsto \alpha\beta\alpha^{-1}$, the conjugation by α. We say that two homomorphisms $\psi_1, \psi_2 : G \to H$ are *conjugate* and we write $\psi_1 \sim_c \psi_2$ if there exists $\alpha \in H$ such that $\psi_2 = c_\alpha \circ \psi_1$.

In Section 4 we determine the homomorphisms from VB_n to $\mathfrak{S}_m$ up to conjugation. More precisely we prove the following.

THEOREM 2.1. — *Let $n, m \in \mathbb{N}$ such that $n \geqslant 5$, $m \geqslant 2$ and $n \geqslant m$, and let $\psi : \mathrm{VB}_n \to \mathfrak{S}_m$ be a homomorphism. Then, up to conjugation, one of the following possibilities holds.*

(1) *ψ is Abelian,*
(2) *$n = m$ and $\psi \in \{\pi_K, \pi_P\}$,*
(3) *$n = m = 6$ and $\psi \in \{\nu_6 \circ \pi_K, \nu_6 \circ \pi_P\}$.*

In Section 5 we determine the homomorphisms from $\mathfrak{S}_n$ to VB_m up to conjugation. More precisely we prove the following.

Theorem 2.2. — *Let $n, m \in \mathbb{N}$ such that $n \geqslant 5$, $m \geqslant 2$ and $n \geqslant m$, and let $\varphi : \mathfrak{S}_n \to \mathrm{VB}_m$ be a homomorphism. Then, up to conjugation, one of the following possibilities holds.*

(1) *φ is Abelian,*
(2) *$n = m$ and $\varphi = \iota$,*
(3) *$n = m = 6$ and $\varphi = \iota \circ \nu_6$.*

Finally, in Section 6 we determine the homomorphisms from VB_n to VB_m up to conjugation. More precisely we prove the following.

Theorem 2.3. — *Let $n, m \in \mathbb{N}$ such that $n \geqslant 5$, $m \geqslant 2$ and $n \geqslant m$, and let $\psi : \mathrm{VB}_n \to \mathrm{VB}_m$ be a homomorphism. Then, up to conjugation, one of the following possibilities holds.*

(1) *ψ is Abelian,*
(2) *$n = m$ and $\psi \in \{\iota \circ \pi_K, \iota \circ \pi_P\}$,*
(3) *$n = m = 6$ and $\psi \in \{\iota \circ \nu_6 \circ \pi_K, \iota \circ \nu_6 \circ \pi_P\}$,*
(4) *$n = m$ and $\psi \in \{\mathrm{id}, \zeta_1, \zeta_2, \zeta_1 \circ \zeta_2\} = \langle \zeta_1, \zeta_2 \rangle$.*

A group G is called *Hopfian* if every surjective homomorphism $\psi : G \to G$ is also injective. On the other hand, it is called *co-Hopfian* if every injective homomorphism $\psi : G \to G$ is also surjective. It is known that the braid group B_n is Hopfian but not co-Hopfian. However, it is quasi-co-Hopfian by Bell–Margalit [7]. The property of hopfianity for the braid group B_n follows from the fact that it can be embedded in the automorphism group of the free group F_n (see Artin [2]). We cannot apply such an argument for the virtual braid group VB_n since we do not know if it can be embedded into the automorphism group of a finitely generated free group. A first consequence of Theorem 2.3 is the following.

Corollary 2.4. — *Let $n \in \mathbb{N}$, $n \geqslant 5$. Then VB_n is Hopfian and co-Hopfian.*

Proof. — We see in Theorem 2.3 that, up to conjugation, the only surjective homomorphisms from VB_n to VB_n are the elements of $\{\mathrm{id}, \zeta_1, \zeta_2, \zeta_1 \circ \zeta_2\} = \langle \zeta_1, \zeta_2 \rangle$, and they are all automorphisms, hence VB_n is Hopfian. We show in the same way that VB_n is co-Hopfian. □

Recall that, by Dyer–Grossman [13], the group $\mathrm{Out}(B_n)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Here we show that $\mathrm{Out}(\mathrm{VB}_n)$ is a little bigger, that is:

Corollary 2.5. — *Let $n \in \mathbb{N}$, $n \geqslant 5$. Then $\mathrm{Out}(\mathrm{VB}_n)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and is generated by the classes of ζ_1 and ζ_2.*

The proof of Corollary 2.5 relies on the following lemma whose proof is given in Section 3.

Lemma 2.6. — *Let $n \in \mathbb{N}$, $n \geqslant 5$. Then ζ_1 is not an inner automorphism of* VB_n.

Proof of Corollary 2.5. — It follows from Theorem 2.3 that $\mathrm{Out}(\mathrm{VB}_n)$ is generated by the classes of ζ_1 and ζ_2. We also know that these two automorphisms are of order two and commute. So, it suffices to show that none of the elements $\zeta_1, \zeta_2, \zeta_1 \circ \zeta_2$ is an inner automorphism. It is easily seen from its presentation that the abelianization of VB_n is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the copy of $\mathbb{Z}$ is generated by the class of σ_1 and the copy of $\mathbb{Z}/2\mathbb{Z}$ is generated by the class of τ_1. Since $\zeta_2(\sigma_1) = \sigma_1^{-1}$, ζ_2 acts non-trivially on the abelianization of VB_n, hence ζ_2 is not an inner automorphism. The transformation $\zeta_1 \circ \zeta_2$ is not an inner automorphism for the same reason, and ζ_1 is not an inner automorphism by Lemma 2.6. □

In order to determine $\mathrm{Out}(B_n)$, Dyer and Grossman [13] use also another result of Artin [1] which says that the pure braid group on n strands is a characteristic subgroup of B_n. From Theorem 2.3 it immediately follows that the equivalent statement for virtual braid groups holds. More precisely, we have the following.

Corollary 2.7. — *Let $n \in \mathbb{N}$, $n \geqslant 5$. Then the groups VP_n and KB_n are both characteristic subgroups of* VB_n.

3. Preliminaries

Let $n \geqslant 3$. Recall that we have an epimorphism $\pi_K : \mathrm{VB}_n \to \mathfrak{S}_n$ which sends σ_i to 1 and τ_i to s_i for all $1 \leqslant i \leqslant n-1$. Recall also that KB_n denotes the kernel of π_K and that we have the decomposition $\mathrm{VB}_n = \mathrm{KB}_n \rtimes \mathfrak{S}_n$. The aim of the present section is to prove three technical results on the action of $\mathfrak{S}_n$ on KB_n (Lemma 3.7, Lemma 3.10 and Lemma 3.12). These three lemmas are key points in the proofs of Theorem 2.2 and Theorem 2.3. We think also that they are interesting by themselves and may be used in the future for other purposes. The two main tools in the proofs of these lemmas are the Artin groups and the amalgamated products.

The following is proved in Rabenda's master thesis at the Université de Bourgogne in Dijon in 2003. This thesis is actually unavailable but the proof of the proposition can also be found in Bardakov–Bellingeri [4].

Proposition 3.1. — *For* $1 \leqslant i < j \leqslant n$ *we set*

$$\delta_{i,j} = \tau_i \tau_{i+1} \cdots \tau_{j-2} \sigma_{j-1} \tau_{j-2} \cdots \tau_{i+1} \tau_i \,,$$
$$\delta_{j,i} = \tau_i \tau_{i+1} \cdots \tau_{j-2} \tau_{j-1} \sigma_{j-1} \tau_{j-1} \tau_{j-2} \cdots \tau_{i+1} \tau_i \,.$$

Then KB_n *has a presentation with generating set*

$$\mathcal{S} = \{\delta_{i,j} \mid 1 \leqslant i \neq j \leqslant n\}\,,$$

and relations

$$\delta_{i,j}\delta_{k,\ell} = \delta_{k,\ell}\delta_{i,j} \qquad \text{for } i,j,k,\ell \text{ pairwise distinct,}$$
$$\delta_{i,j}\delta_{j,k}\delta_{i,j} = \delta_{j,k}\delta_{i,j}\delta_{j,k} \qquad \text{for } i,j,k \text{ pairwise distinct.}$$

Moreover, the action of each $w \in \mathfrak{S}_n$ *on a generator* $\delta_{i,j}$ *is by permutation of the indices, that is,* $w(\delta_{i,j}) = \delta_{w(i),w(j)}$.

Let $\mathcal{S}$ be a finite set. A *Coxeter matrix* over $\mathcal{S}$ is a square matrix $M = (m_{s,t})_{s,t\in\mathcal{S}}$ indexed by the elements of $\mathcal{S}$ such that $m_{s,s} = 1$ for all $s \in \mathcal{S}$ and $m_{s,t} = m_{t,s} \in \{2,3,4,\dots\} \cup \{\infty\}$ for all $s,t \in \mathcal{S}$, $s \neq t$. For $s,t \in \mathcal{S}$, $s \neq t$, and $m \geqslant 2$, we denote by $\Pi(s,t,m)$ the word $sts\cdots$ of length m. In other words, we have $\Pi(s,t,m) = (st)^{\frac{m}{2}}$ if m is even and $\Pi(s,t,m) = (st)^{\frac{m-1}{2}}s$ if m is odd. The *Artin group* associated with the Coxeter matrix $M = (m_{s,t})_{s,t\in\mathcal{S}}$ is the group $A = A_M$ defined by the presentation

$$A = \langle \mathcal{S} \mid \Pi(s,t,m_{s,t}) = \Pi(t,s,m_{s,t}) \text{ for } s,t \in \mathcal{S},\ s \neq t,\ m_{s,t} \neq \infty \rangle\,.$$

Example. — Let $\mathcal{S} = \{\delta_{i,j} \mid 1 \leqslant i \neq j \leqslant n\}$. Define a Coxeter matrix $M = (m_{s,t})_{s,t\in\mathcal{S}}$ as follows. Set $m_{s,s} = 1$ for all $s \in \mathcal{S}$. Let $s,t \in \mathcal{S}$, $s \neq t$. If $s = \delta_{i,j}$ and $t = \delta_{k,\ell}$, where i,j,k,ℓ are pairwise distinct, then set $m_{s,t} = 2$. If $s = \delta_{i,j}$ and $t = \delta_{j,k}$, where i,j,k are pairwise distinct, then set $m_{s,t} = m_{t,s} = 3$. Set $m_{s,t} = \infty$ for all other cases. Then, by Proposition 3.1, KB_n is the Artin group associated with $M = (m_{s,t})_{s,t\in\mathcal{S}}$.

If $\mathcal{X}$ is a subset of $\mathcal{S}$, then we set $M[\mathcal{X}] = (m_{s,t})_{s,t\in\mathcal{X}}$ and we denote by $A[\mathcal{X}]$ the subgroup of A generated by $\mathcal{X}$.

Theorem 3.2 (van der Lek [17]). — *Let* A *be the Artin group associated with a Coxeter matrix* $M = (m_{s,t})_{s,t\in\mathcal{S}}$, *and let* $\mathcal{X}$ *be a subset of* $\mathcal{S}$. *Then* $A[\mathcal{X}]$ *is the Artin group associated with* $M[\mathcal{X}]$. *Moreover, if* $\mathcal{X}$ *and* $\mathcal{Y}$ *are two subsets of* $\mathcal{S}$, *then* $A[\mathcal{X}] \cap A[\mathcal{Y}] = A[\mathcal{X} \cap \mathcal{Y}]$.

In our example, for $\mathcal{X} \subset \mathcal{S}$ we denote by $\mathrm{KB}_n[\mathcal{X}]$ the subgroup of KB_n generated by $\mathcal{X}$. So, by Theorem 3.2, $\mathrm{KB}_n[\mathcal{X}]$ is still an Artin group and it has a presentation with generating set $\mathcal{X}$ and only two types of relations:

- $st = ts$ if $m_{s,t} = 2$ in $M[\mathcal{X}]$,
- $sts = tst$ if $m_{s,t} = 3$ in $M[\mathcal{X}]$.

The following result can be easily proved from Theorem 3.2 using presentations, but it is important to highlight it since it will be often used throughout the paper.

Lemma 3.3. — *Let A be an Artin group associated with a Coxeter matrix $M = (m_{s,t})_{s,t\in\mathcal{S}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be two subsets of $\mathcal{S}$ such that $\mathcal{X} \cup \mathcal{Y} = \mathcal{S}$ and $m_{s,t} = \infty$ for all $s \in \mathcal{X} \setminus (\mathcal{X} \cap \mathcal{Y})$ and $t \in \mathcal{Y} \setminus (\mathcal{X} \cap \mathcal{Y})$. Then $A = A[\mathcal{X}] *_{A[\mathcal{X}\cap\mathcal{Y}]} A[\mathcal{Y}]$.*

Let G be a group and let H be a subgroup of G. A *transversal* of H in G is a subset T of G such that for each $\alpha \in G$ there exists a unique $\theta \in T$ such that $\alpha H = \theta H$. For convenience we will always suppose that a transversal T contains 1 and we set $T^* = T \setminus \{1\}$. The following is classical in the theory and is proved in Serre [20, Section 1.1, Theorem 1].

Theorem 3.4 (Serre [20]). — *Let $G_1, \dots, G_p, H$ be a collection of groups. We suppose that H is a subgroup of G_j for all $j \in \{1, \dots, p\}$ and we consider the amalgamated product $G = G_1 *_H G_2 *_H \cdots *_H G_p$. For each $j \in \{1, \dots, p\}$ we choose a transversal T_j of H in G_j. Then each element $\alpha \in G$ can be written in a unique way in the form $\alpha = \theta_1\theta_2 \cdots \theta_\ell \beta$ such that:*

(1) *$\beta \in H$ and, for each $i \in \{1, \dots, \ell\}$, there exists $j = j(i) \in \{1, \dots, p\}$ such that $\theta_i \in T_j^* = T_j \setminus \{1\}$,*
(2) *$j(i) \neq j(i+1)$ for all $i \in \{1, \dots, \ell - 1\}$.*

In particular, we have $\alpha \in H$ if and only if $\ell = 0$ and $\alpha = \beta$.

The expression of α given in the above theorem is called the *normal form* of α. It depends on the amalgamated product and on the choice of the transversal of H in G_j for all j.

Consider the notation introduced in Theorem 3.4. So, $G = G_1 *_H \cdots *_H G_p$ and T_j is a transversal of H in G_j for all $j \in \{1, \dots, p\}$. Let $\alpha \in G$. We suppose that α is written in the form $\alpha = \alpha_1 \cdots \alpha_\ell$ such that $\ell \geqslant 1$,

(1) for each $i \in \{1, \dots, \ell\}$ there exists $j = j(i) \in \{1, \dots, p\}$ such that $\alpha_i \in G_j \setminus H$,
(2) $j(i) \neq j(i+1)$ for all $i \in \{1, \dots, \ell - 1\}$.

We define $\beta_i \in H$ and $\theta_i \in T^*_{j(i)}$ for $i \in \{1, \dots, \ell\}$ by induction on i as follows. First, θ_1 is the element of $T^*_{j(1)}$ such that $\alpha_1 H = \theta_1 H$ and β_1 is the element of H such that $\alpha_1 = \theta_1 \beta_1$. We suppose that $i \geqslant 2$ and that $\beta_{i-1} \in H$ is defined. Then θ_i is the element of $T^*_{j(i)}$ such that $\beta_{i-1}\alpha_i H = \theta_i H$ and β_i is the element of H such that $\beta_{i-1}\alpha_i = \theta_i\beta_i$. The following result is quite obvious and its proof is left to the reader.

Proposition 3.5. — *Under the above notations and hypothesis $\alpha = \theta_1 \cdots \theta_\ell \beta_\ell$ is the normal form of α. In particular, $\alpha \notin H$ (since $\ell \geqslant 1$).*

We turn now to the proofs of our technical lemmas.

Lemma 3.6. — *Let G_1, G_2, H be three groups. We suppose that H is a common subgroup of G_1 and G_2 and we set $G = G_1 *_H G_2$. Let $\tau : G \to G$ be an automorphism of order 2 such that $\tau(G_1) = G_2$ and $\tau(G_2) = G_1$. Let $G^\tau = \{\alpha \in G \mid \tau(\alpha) = \alpha\}$. Then $G^\tau \subset H$.*

Proof. — Since $\tau(G_1) = G_2$, $\tau(G_2) = G_1$ and $H = G_1 \cap G_2$, we have $\tau(H) = H$. Let T_1 be a transversal of H in G_1. Set $T_2 = \tau(T_1)$. Then T_2 is a transversal of H in G_2. Let $\alpha \in G$ and let $\alpha = \theta_1\theta_2 \cdots \theta_\ell \beta$ be the normal form of α. Notice that the normal form of $\tau(\alpha)$ is $\tau(\theta_1)\,\tau(\theta_2) \cdots \tau(\theta_\ell)\,\tau(\beta)$. Suppose that $\ell \geqslant 1$. Without loss of generality we can assume that $\theta_1 \in T_1^*$. Then $\tau(\theta_1) \in T_2^*$, hence $\theta_1 \neq \tau(\theta_1)$, thus $\alpha \neq \tau(\alpha)$, and therefore $\alpha \notin G^\tau$. So, if $\alpha \in G^\tau$, then $\ell = 0$ and $\alpha = \beta \in H$. □

Recall from Proposition 3.1 that $\mathrm{VB}_n = \mathrm{KB}_n \rtimes \mathfrak{S}_n$ and that the action of $\mathfrak{S}_n$ on KB_n is defined by $w(\delta_{i,j}) = \delta_{w(i),w(j)}$ for all $w \in \mathfrak{S}_n$ and all $\delta_{i,j} \in \mathcal{S}$. Recall also that for each generator s_i of $\mathfrak{S}_n$ and each $\alpha \in \mathrm{KB}_n$ we have $s_i(\alpha) = \tau_i \alpha \tau_i$. Throughout the paper we will use both interpretations, as action of s_i and as conjugation by τ_i. For each $1 \leqslant k \leqslant n$ we set $\mathcal{S}_k = \{\delta_{i,j} \mid k \leqslant i \neq j \leqslant n\}$.

Lemma 3.7. — *Let $\mathcal{X}$ be a subset of $\mathcal{S}$ invariant under the action of s_1. Then $\mathrm{KB}_n[\mathcal{X}]^{s_1} = \mathrm{KB}_n[\mathcal{X} \cap \mathcal{S}_3]$.*

Proof. — Set $\mathcal{U} = \{\delta_{i,j} \in \mathcal{X} \mid (i,j) \notin \{(1,2),(2,1)\}\}$. We first prove that $\mathrm{KB}_n[\mathcal{X}]^{s_1} \subset \mathrm{KB}_n[\mathcal{U}]$. If $\mathcal{X} \subset \mathcal{U}$ there is nothing to prove. We can therefore suppose that $\mathcal{X} \not\subset \mathcal{U}$. Since $\mathcal{X}$ is invariant under the action of s_1, we have $\mathcal{X} = \mathcal{U} \cup \{\delta_{1,2}, \delta_{2,1}\}$. Set $\mathcal{U}' = \mathcal{U} \cup \{\delta_{1,2}\}$ and $\mathcal{U}'' = \mathcal{U} \cup \{\delta_{2,1}\}$. By Lemma 3.3, $\mathrm{KB}_n[\mathcal{X}] = \mathrm{KB}_n[\mathcal{U}'] *_{\mathrm{KB}_n[\mathcal{U}]} \mathrm{KB}_n[\mathcal{U}'']$. Moreover, $s_1(\mathrm{KB}_n[\mathcal{U}']) = \mathrm{KB}_n[\mathcal{U}'']$ and $s_1(\mathrm{KB}_n[\mathcal{U}'']) = \mathrm{KB}_n[\mathcal{U}']$. So, by Lemma 3.6, $\mathrm{KB}_n[\mathcal{X}]^{s_1} \subset \mathrm{KB}_n[\mathcal{U}]$.

For $2 \leqslant k \leqslant n$ we set $\mathcal{V}_k = \{\delta_{i,j} \in \mathcal{X} \mid (i,j) \notin (\{1,2\} \times \{1,2,\dots,k\})\}$. We show by induction on k that $\mathrm{KB}_n[\mathcal{X}]^{s_1} \subset \mathrm{KB}_n[\mathcal{V}_k]$. Since $\mathcal{V}_2 = \mathcal{U}$ the case $k = 2$ holds. Suppose that $k \geqslant 3$ and that the inductive hypothesis holds, that is, $\mathrm{KB}_n[\mathcal{X}]^{s_1} \subset \mathrm{KB}_n[\mathcal{V}_{k-1}]$. If $\mathcal{V}_k = \mathcal{V}_{k-1}$ there is nothing to prove. We can therefore suppose that $\mathcal{V}_k \neq \mathcal{V}_{k-1}$. Since $\mathcal{V}_{k-1}$ is invariant under the action of s_1 we have $\mathcal{V}_{k-1} = \mathcal{V}_k \cup \{\delta_{1,k}, \delta_{2,k}\}$. Set $\mathcal{V}'_k = \mathcal{V}_k \cup \{\delta_{1,k}\}$ and $\mathcal{V}''_k = \mathcal{V}_k \cup \{\delta_{2,k}\}$. By Lemma 3.3, $\mathrm{KB}_n[\mathcal{V}_{k-1}] = \mathrm{KB}_n[\mathcal{V}'_k] *_{\mathrm{KB}_n[\mathcal{V}_k]} \mathrm{KB}_n[\mathcal{V}''_k]$. Moreover, $s_1(\mathrm{KB}_n[\mathcal{V}'_k]) = \mathrm{KB}_n[\mathcal{V}''_k]$ and $s_1(\mathrm{KB}_n[\mathcal{V}''_k]) = \mathrm{KB}_n[\mathcal{V}'_k]$. So, by Lemma 3.6, $\mathrm{KB}_n[\mathcal{X}]^{s_1} \subset \mathrm{KB}_n[\mathcal{V}_k]$.

For $2 \leqslant k \leqslant n$ we set $\mathcal{W}_k = \{\delta_{i,j} \in \mathcal{X} \mid (i,j) \notin (\{1,2\}\times\{1,2,\dots,n\})$ and $(i,j) \notin (\{1,2,\dots,k\}\times\{1,2\})\}$. We show by induction on k that $\mathrm{KB}_n[\mathcal{X}]^{s_1} \subset \mathrm{KB}_n[\mathcal{W}_k]$. Since $\mathcal{W}_2 = \mathcal{V}_n$ the case $k=2$ holds. Suppose that $k \geqslant 3$ and that the inductive hypothesis holds, that is, $\mathrm{KB}_n[\mathcal{X}]^{s_1} \subset \mathrm{KB}_n[\mathcal{W}_{k-1}]$. If $\mathcal{W}_k = \mathcal{W}_{k-1}$ there is nothing to prove. We can therefore suppose that $\mathcal{W}_k \neq \mathcal{W}_{k-1}$. Since $\mathcal{W}_{k-1}$ is invariant under the action of s_1 we have $\mathcal{W}_{k-1} = \mathcal{W}_k \cup \{\delta_{k,1}, \delta_{k,2}\}$. Set $\mathcal{W}'_k = \mathcal{W}_k \cup \{\delta_{k,1}\}$ and $\mathcal{W}''_k = \mathcal{W}_k \cup \{\delta_{k,2}\}$. By Lemma 3.3, $\mathrm{KB}_n[\mathcal{W}_{k-1}] = \mathrm{KB}_n[\mathcal{W}'_k] *_{\mathrm{KB}_n[\mathcal{W}_k]} \mathrm{KB}_n[\mathcal{W}''_k]$. Moreover, $s_1(\mathrm{KB}_n[\mathcal{W}'_k]) = \mathrm{KB}_n[\mathcal{W}''_k]$ and $s_1(\mathrm{KB}_n[\mathcal{W}''_k]) = \mathrm{KB}_n[\mathcal{W}'_k]$. So, by Lemma 3.6, $\mathrm{KB}_n[\mathcal{X}]^{s_1} \subset \mathrm{KB}_n[\mathcal{W}_k]$. The inclusion $\mathrm{KB}_n[\mathcal{X}]^{s_1} \subset \mathrm{KB}_n[\mathcal{X}\cap\mathcal{S}_3]$ therefore follows from the fact that $\mathcal{W}_n = \mathcal{X} \cap \mathcal{S}_3$. The reverse inclusion $\mathrm{KB}_n[\mathcal{X} \cap \mathcal{S}_3] \subset \mathrm{KB}_n[\mathcal{X}]^{s_1}$ is obvious. □

By applying the action of the symmetric group, from Lemma 3.7 it immediately follows:

Corollary 3.8. — *Let $k \in \{1,\dots,n-1\}$, let $\mathcal{X}$ be a subset of $\mathcal{S}$ invariant under the action of s_k, and let $\mathcal{U}_k = \{\delta_{i,j} \in \mathcal{S} \mid i,j \notin \{k,k+1\}\}$. Then $\mathrm{KB}_n[\mathcal{X}]^{s_k} = \mathrm{KB}_n[\mathcal{X} \cap \mathcal{U}_k]$.*

Lemma 3.9. — *Let G_1, G_2, H be three groups. We suppose that H is a common subgroup of G_1 and G_2 and we set $G = G_1 *_H G_2$. Let $\tau : G \to G$ be an automorphism of order 2 such that $\tau(G_1) = G_2$ and $\tau(G_2) = G_1$. Let $\alpha \in G$ such that $\tau(\alpha) = \alpha^{-1}$. Then there exist $\alpha' \in G$ and $\beta' \in H$ such that $\tau(\beta') = \beta'^{-1}$ and $\alpha = \alpha'\beta'\,\tau(\alpha'^{-1})$.*

Proof. — Since $\tau(G_1) = G_2$ and $\tau(G_2) = G_1$, we have $\tau(H) = H$. Let T_1 be a transversal of H in G_1. Then $T_2 = \tau(T_1)$ is a transversal of H in G_2. Let $\alpha \in G$ such that $\tau(\alpha) = \alpha^{-1}$. Let $\alpha = \theta_1\theta_2\cdots\theta_\ell\beta$ be the normal form of α. We prove by induction on ℓ that there exist $\alpha' \in G$ and $\beta' \in H$ such that $\tau(\beta') = \beta'^{-1}$ and $\alpha = \alpha'\beta'\,\tau(\alpha'^{-1})$. The case $\ell = 0$ being trivial we can suppose that $\ell \geqslant 1$ and that the inductive hypothesis holds. We have $1 = \alpha\,\tau(\alpha) = \theta_1\cdots\theta_\ell\beta\,\tau(\theta_1)\cdots\tau(\theta_\ell)\,\tau(\beta)$. By Proposition 3.5 we must have $\theta_\ell\beta\,\tau(\theta_1) \in H$, namely, there exists $\beta_1 \in H$ such that $\theta_\ell\beta = \beta_1\,\tau(\theta_1)^{-1}$. Note that this inclusion implies that ℓ is even (and therefore $\ell \geqslant 2$), since we should have either $\theta_\ell, \tau(\theta_1) \in G_1 \setminus H$ or $\theta_\ell, \tau(\theta_1) \in G_2 \setminus H$.

Thus, $\alpha = \theta_1\theta_2 \cdots \theta_{\ell-1}\beta_1\, \tau(\theta_1^{-1}) = \theta_1\alpha_1\, \tau(\theta_1^{-1})$, where $\alpha_1 = \theta_2 \cdots \theta_{\ell-1}\beta_1$. We have $\tau(\alpha_1) = \alpha_1^{-1}$ since $\tau(\alpha) = \alpha^{-1}$, so, by induction, there exist $\alpha_1' \in G$ and $\beta' \in H$ such that $\tau(\beta') = \beta'^{-1}$ and $\alpha_1 = \alpha_1'\beta'\,\tau(\alpha_1'^{-1})$. We set $\alpha' = \theta_1\alpha_1'$. Then $\tau(\beta') = \beta'^{-1}$ and $\alpha = \alpha'\beta'\,\tau(\alpha'^{-1})$. □

Lemma 3.10. — *Let $\mathcal{X}$ be a subset of $\mathcal{S}$ invariant under the action of s_1. Let $\alpha \in \mathrm{KB}_n[\mathcal{X}]$ such that $s_1(\alpha) = \alpha^{-1}$. Then there exists $\alpha' \in \mathrm{KB}_n[\mathcal{X}]$ such that $\alpha = \alpha'\, s_1(\alpha'^{-1})$.*

Proof. — Let $\alpha \in \mathrm{KB}_n[\mathcal{X}]$ such that $s_1(\alpha) = \alpha^{-1}$. Let $\mathcal{U} = \{\delta_{i,j} \in \mathcal{X} \mid (i,j) \notin \{(1,2),(2,1)\}\}$. We show that there exist $\alpha' \in \mathrm{KB}_n[\mathcal{X}]$ and $\beta' \in \mathrm{KB}_n[\mathcal{U}]$ such that $s_1(\beta') = \beta'^{-1}$ and $\alpha = \alpha'\beta'\, s_1(\alpha'^{-1})$. If $\mathcal{X} = \mathcal{U}$ there is nothing to prove. We can therefore suppose that $\mathcal{X} \neq \mathcal{U}$. Since $\mathcal{X}$ is invariant under the action of s_1 we have $\mathcal{X} = \mathcal{U} \cup \{\delta_{1,2}, \delta_{2,1}\}$. Set $\mathcal{U}' = \mathcal{U} \cup \{\delta_{1,2}\}$ and $\mathcal{U}'' = \mathcal{U} \cup \{\delta_{2,1}\}$. By Lemma 3.3, $KB_n[\mathcal{X}] = \mathrm{KB}_n[\mathcal{U}'] *_{\mathrm{KB}_n[\mathcal{U}]} \mathrm{KB}_n[\mathcal{U}'']$. Moreover, $s_1(\mathrm{KB}_n[\mathcal{U}']) = \mathrm{KB}_n[\mathcal{U}'']$ and $s_1(\mathrm{KB}_n[\mathcal{U}'']) = \mathrm{KB}_n[\mathcal{U}']$. So, by Lemma 3.9, there exist $\alpha' \in \mathrm{KB}_n[\mathcal{X}]$ and $\beta' \in \mathrm{KB}_n[\mathcal{U}]$ such that $s_1(\beta') = \beta'^{-1}$ and $\alpha = \alpha'\beta'\, s_1(\alpha'^{-1})$.

For $2 \leqslant k \leqslant n$ we set $\mathcal{V}_k = \{\delta_{i,j} \in \mathcal{X} \mid (i,j) \notin (\{1,2\} \times \{1,2,\dots,k\})\}$. We show by induction on k that there exist $\alpha' \in \mathrm{KB}_n[\mathcal{X}]$ and $\beta' \in \mathrm{KB}_n[\mathcal{V}_k]$ such that $s_1(\beta') = \beta'^{-1}$ and $\alpha = \alpha'\beta'\, s_1(\alpha'^{-1})$. Since $\mathcal{V}_2 = \mathcal{U}$ the case $k = 2$ follows from the previous paragraph. Suppose that $k \geqslant 3$ and that the inductive hypothesis holds. So, there exist $\alpha_1' \in \mathrm{KB}_n[\mathcal{X}]$ and $\beta_1' \in \mathrm{KB}_n[\mathcal{V}_{k-1}]$ such that $s_1(\beta_1') = \beta_1'^{-1}$ and $\alpha = \alpha_1'\beta_1'\, s_1(\alpha_1'^{-1})$. If $\mathcal{V}_k = \mathcal{V}_{k-1}$ there is nothing to prove. Thus, we can suppose that $\mathcal{V}_k \neq \mathcal{V}_{k-1}$. Since $\mathcal{V}_{k-1}$ is invariant under the action of s_1 we have $\mathcal{V}_{k-1} = \mathcal{V}_k \cup \{\delta_{1,k}, \delta_{2,k}\}$. Set $\mathcal{V}_k' = \mathcal{V}_k \cup \{\delta_{1,k}\}$ and $\mathcal{V}_k'' = \mathcal{V}_k \cup \{\delta_{2,k}\}$. By Lemma 3.3, $\mathrm{KB}_n[\mathcal{V}_{k-1}] = \mathrm{KB}_n[\mathcal{V}_k'] *_{\mathrm{KB}_n[\mathcal{V}_k]} \mathrm{KB}_n[\mathcal{V}_k'']$. Moreover, $s_1(\mathrm{KB}_n[\mathcal{V}_k']) = \mathrm{KB}_n[\mathcal{V}_k'']$ and $s_1(\mathrm{KB}_n[\mathcal{V}_k'']) = \mathrm{KB}_n[\mathcal{V}_k']$. By Lemma 3.9 it follows that there exist $\alpha_2' \in \mathrm{KB}_n[\mathcal{V}_{k-1}]$ and $\beta' \in \mathrm{KB}_n[\mathcal{V}_k]$ such that $s_1(\beta') = \beta'^{-1}$ and $\beta_1' = \alpha_2'\beta'\, s_1(\alpha_2'^{-1})$. Set $\alpha' = \alpha_1'\alpha_2'$. Then $\beta' \in \mathrm{KB}_n[\mathcal{V}_k]$, $s_1(\beta') = \beta'^{-1}$ and $\alpha = \alpha'\beta'\, s_1(\alpha'^{-1})$.

For $2 \leqslant k \leqslant n$ we set $\mathcal{W}_k = \{\delta_{i,j} \in \mathcal{X} \mid (i,j) \notin (\{1,2\} \times \{1,2,\dots,n\})$ and $(i,j) \notin (\{1,2,\dots,k\} \times \{1,2\})\}$. We show by induction on k that there exist $\alpha' \in \mathrm{KB}_n[\mathcal{X}]$ and $\beta' \in \mathrm{KB}_n[\mathcal{W}_k]$ such that $s_1(\beta') = \beta'^{-1}$ and $\alpha = \alpha'\beta'\, s_1(\alpha'^{-1})$. Since $\mathcal{W}_2 = \mathcal{V}_n$ the case $k = 2$ follows from the previous paragraph. Suppose that $k \geqslant 3$ and that the inductive hypothesis holds. So, there exist $\alpha_1' \in \mathrm{KB}_n[\mathcal{X}]$ and $\beta_1' \in \mathrm{KB}_n[\mathcal{W}_{k-1}]$ such that $s_1(\beta_1') = \beta_1'^{-1}$ and $\alpha = \alpha_1'\beta_1'\, s_1(\alpha_1'^{-1})$. If $\mathcal{W}_k = \mathcal{W}_{k-1}$ there is nothing to prove. We can therefore suppose that $\mathcal{W}_k \neq \mathcal{W}_{k-1}$. Since $\mathcal{W}_{k-1}$ is invariant under the action of s_1 we have $\mathcal{W}_{k-1} = \mathcal{W}_k \cup \{\delta_{k,1}, \delta_{k,2}\}$. Set $\mathcal{W}_k' = \mathcal{W}_k \cup \{\delta_{k,1}\}$ and $\mathcal{W}_k'' = \mathcal{W}_k \cup \{\delta_{k,2}\}$. By Lemma 3.3, $\mathrm{KB}_n[\mathcal{W}_{k-1}] = \mathrm{KB}_n[\mathcal{W}_k'] *_{\mathrm{KB}_n[\mathcal{W}_k]} \mathrm{KB}_n[\mathcal{W}_k'']$.

Moreover, $s_1(\mathrm{KB}_n[\mathcal{W}_k']) = \mathrm{KB}_n[\mathcal{W}_k'']$ and $s_1(\mathrm{KB}_n[\mathcal{W}_k'']) = \mathrm{KB}_n[\mathcal{W}_k']$. By Lemma 3.9 it follows that there exist $\alpha_2' \in \mathrm{KB}_n[\mathcal{W}_{k-1}]$ and $\beta' \in \mathrm{KB}_n[\mathcal{W}_k]$ such that $s_1(\beta') = \beta'^{-1}$ and $\beta_1' = \alpha_2'\beta'\, s_1(\alpha_2'^{-1})$. Set $\alpha' = \alpha_1'\alpha_2'$. Then $\beta' \in \mathrm{KB}_n[\mathcal{W}_k]$, $s_1(\beta') = \beta'^{-1}$ and $\alpha = \alpha'\beta'\, s_1(\alpha'^{-1})$.

Notice that $\mathcal{W}_n = \mathcal{X} \cap \mathcal{S}_3$. Recall that $s_1(\beta') = \beta'$ for all $\beta' \in \mathrm{KB}_n[\mathcal{X} \cap \mathcal{S}_3] = \mathrm{KB}_n[\mathcal{W}_n]$ (see Lemma 3.7). By the above there exist $\alpha' \in \mathrm{KB}_n[\mathcal{X}]$ and $\beta' \in \mathrm{KB}_n[\mathcal{W}_n]$ such that $s_1(\beta') = \beta'^{-1}$ and $\alpha = \alpha'\beta'\, s_1(\alpha'^{-1})$. So, $\beta' = s_1(\beta') = \beta'^{-1}$, hence $\beta'^2 = 1$, and therefore $\beta' = 1$, since, by Godelle–Paris [14], KB_n is torsion free. So, $\alpha = \alpha'\, s_1(\alpha'^{-1})$. □

Lemma 3.11. — *Let $G_1, G_2, \dots, G_p, H$ be a collection of groups. We suppose that H is a subgroup of G_j for all $j \in \{1, \dots, p\}$ and we consider the amalgamated product $G = G_1 *_H G_2 *_H \cdots *_H G_p$. Let $\tau_1, \tau_2 : G \to G$ be two automorphisms satisfying the following properties:*

(1) *$\tau_1^2 = \tau_2^2 = 1$ and $\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$.*
(2) *For all $i \in \{1, 2\}$ and $j \in \{1, \dots, p\}$ there exists $k \in \{1, \dots, p\}$ such that $\tau_i(G_j) = G_k$.*
(3) *For all $j \in \{1, \dots, p\}$ there exists $i \in \{1, 2\}$ such that $\tau_i(G_j) \neq G_j$.*
(4) *For all $i \in \{1, 2\}$ we have $\tau_i(H) = H$.*
(5) *For all $i \in \{1, 2\}$ and $\gamma \in H$ such that $\tau_i(\gamma) = \gamma^{-1}$ there exists $\delta \in H$ such that $\gamma = \delta\, \tau_i(\delta^{-1})$.*

Let $\alpha \in G$ satisfying the following equation:

$$\alpha\, \tau_2(\alpha^{-1})\, (\tau_2\tau_1)(\alpha)\, (\tau_2\tau_1\tau_2)(\alpha^{-1})\, (\tau_1\tau_2)(\alpha)\, \tau_1(\alpha^{-1}) = 1\,. \tag{3.1}$$

Then there exist $\alpha', \alpha'' \in G$ and $\beta \in H$ such that $\alpha = \alpha'\beta\alpha''$, $\tau_1(\alpha') = \alpha'$, $\tau_2(\alpha'') = \alpha''$ and β satisfies (3.1).

Proof. — Let $\alpha \in G$ satisfying (3.1). It is easily checked that, if α is written $\alpha = \alpha'\beta\alpha''$, where $\tau_1(\alpha') = \alpha'$ and $\tau_2(\alpha'') = \alpha''$, then β also satisfies (3.1). So, it suffices to show that there exist $\alpha', \alpha'' \in G$ and $\beta \in H$ such that $\alpha = \alpha'\beta\alpha''$, $\tau_1(\alpha') = \alpha'$ and $\tau_2(\alpha'') = \alpha''$. If $\alpha \in H$ there is nothing to prove. We can therefore suppose that $\alpha \notin H$. We write α in the form $\alpha = \alpha_1\alpha_2 \cdots \alpha_\ell$ such that

(a) for all $i \in \{1, \dots, \ell\}$ there exists $j = j(i) \in \{1, \dots, p\}$ such that $\alpha_i \in G_j \setminus H$,
(b) $j(i) \neq j(i+1)$ for all $i \in \{1, \dots, \ell - 1\}$.

We argue by induction on ℓ.

Suppose first that $\ell = 1$. We can assume without loss of generality that $\alpha \in G_1 \setminus H$. We set $\beta_1 = \alpha$, $\beta_2 = \tau_2(\alpha^{-1})$, $\beta_3 = (\tau_2\tau_1)(\alpha)$, $\beta_4 = (\tau_2\tau_1\tau_2)(\alpha^{-1})$, $\beta_5 = (\tau_1\tau_2)(\alpha)$, and $\beta_6 = \tau_1(\alpha^{-1})$. By hypothesis we have

$\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6 = 1$ and, for each $i \in \{1,\dots,6\}$, there exists $j = j(i) \in \{1,\dots,p\}$ such that $\beta_i \in G_j \setminus H$. If we had $\tau_1(G_1) \neq G_1$ and $\tau_2(G_1) \neq G_1$, then we would have $j(i) \neq j(i+1)$ for all $i \in \{1,\dots,5\}$, thus, by Proposition 3.5, we would have $\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6 \neq 1$: contradiction. So, either $\tau_1(G_1) = G_1$ or $\tau_2(G_1) = G_1$. On the other hand, by Condition (3) in the statement of the lemma, either $\tau_1(G_1) \neq G_1$ or $\tau_2(G_1) \neq G_1$. Thus, either $\tau_1(G_1) = G_1$ and $\tau_2(G_1) \neq G_1$, or $\tau_1(G_1) \neq G_1$ and $\tau_2(G_1) = G_1$. We assume that $\tau_1(G_1) = G_1$ and $\tau_2(G_1) \neq G_1$. The case $\tau_1(G_1) \neq G_1$ and $\tau_2(G_1) = G_1$ is proved in a similar way. So, $j(1) \neq j(2)$, $j(2) = j(3)$, $j(3) \neq j(4)$, $j(4) = j(5)$, and $j(5) \neq j(6)$. Since $\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6 = 1$, by Proposition 3.5, either $\beta_2\beta_3 \in H$, or $\beta_4\beta_5 \in H$, that is, $\alpha^{-1}\,\tau_1(\alpha) = \gamma \in H$. Now, we have $\tau_1(\gamma) = \tau_1(\alpha^{-1})\alpha = \gamma^{-1}$, hence, by Condition (5) in the statement of the lemma, there exists $\delta \in H$ such that $\gamma = \delta\,\tau_1(\delta^{-1})$. We set $\alpha' = \alpha\delta$, $\alpha'' = 1$ and $\beta = \delta^{-1}$. Then $\alpha = \alpha'\beta\alpha''$, $\tau_1(\alpha') = \alpha'$, $\tau_2(\alpha'') = \alpha''$ and $\beta \in H$.

Suppose that $\ell \geqslant 2$ and that the inductive hypothesis holds. It follows from Proposition 3.5 and (3.1) that either $\alpha_\ell\,\tau_2(\alpha_\ell^{-1}) \in H$ or $\tau_1(\alpha_1^{-1})\,\alpha_1 \in H$. Suppose that $\alpha_\ell\,\tau_2(\alpha_\ell^{-1}) \in H$. Then there exists $\gamma \in H$ such that $\alpha_\ell = \gamma\,\tau_2(\alpha_\ell)$. By applying τ_2 to this equality we obtain $\tau_2(\alpha_\ell) = \tau_2(\gamma)\,\alpha_\ell$, and therefore $\tau_2(\gamma) = \gamma^{-1}$. By hypothesis there exists $\delta \in H$ such that $\gamma = \delta^{-1}\,\tau_2(\delta)$. We set $\alpha'_\ell = 1$, $\alpha''_\ell = \delta\alpha_\ell$, $\alpha''_{\ell-1} = \alpha_{\ell-1}\delta^{-1}$ and $\beta_1 = \alpha_1\cdots\alpha_{\ell-2}\alpha''_{\ell-1}$. We have $\alpha = \alpha'_\ell\beta_1\alpha''_\ell$, $\tau_1(\alpha'_\ell) = \alpha'_\ell$ and $\tau_2(\alpha''_\ell) = \alpha''_\ell$, hence β_1 satisfies (3.1). By induction, there exists $\gamma', \gamma'' \in G$ and $\beta \in H$ such that $\beta_1 = \gamma'\beta\gamma''$, $\tau_1(\gamma') = \gamma'$ and $\tau_2(\gamma'') = \gamma''$. Set $\alpha' = \alpha'_\ell\gamma'$ and $\alpha'' = \gamma''\alpha''_\ell$. Then $\alpha = \alpha'\beta\alpha''$, $\tau_1(\alpha') = \alpha'$ and $\tau_2(\alpha'') = \alpha''$. The case $\tau_1(\alpha_1^{-1})\,\alpha_1 \in H$ can be proved in a similar way. ☐

Lemma 3.12. — *Let $\mathcal{X}$ be a subset of $\mathcal{S}$ invariant under the action of s_1 and under the action of s_2. Let $\alpha \in \mathrm{KB}_n[\mathcal{X}]$ such that*

$$\alpha\, s_2(\alpha^{-1})\,(s_2s_1)(\alpha)\,(s_2s_1s_2)(\alpha^{-1})\,(s_1s_2)(\alpha)\,s_1(\alpha^{-1}) = 1\,. \tag{3.2}$$

Then there exist $\alpha', \alpha'' \in \mathrm{KB}_n[\mathcal{X}]$ such that $s_1(\alpha') = \alpha'$, $s_2(\alpha'') = \alpha''$ and $\alpha = \alpha'\alpha''$.

Proof. — For $4 \leqslant k \leqslant n$ we set $\mathcal{U}_k = \{\delta_{i,j} \in \mathcal{X} \mid (i,j) \notin (\{1,2,3\} \times \{4,\dots,k\})\}$. We set also $\mathcal{U}_3 = \mathcal{X}$. We show by induction on k that there exist $\alpha', \alpha'' \in \mathrm{KB}_n[\mathcal{X}]$ and $\beta \in \mathrm{KB}_n[\mathcal{U}_k]$ such that $\alpha = \alpha'\beta\alpha''$, $s_1(\alpha') = \alpha'$, $s_2(\alpha'') = \alpha''$ and β satisfies (3.2). The case $k = 3$ is true by hypothesis. Suppose that $4 \leqslant k \leqslant n$ and that the inductive hypothesis holds. So, there exist $\alpha'_1, \alpha''_1 \in \mathrm{KB}_n[\mathcal{X}]$ and $\beta_1 \in \mathrm{KB}_n[\mathcal{U}_{k-1}]$ such that $\alpha = \alpha'_1\beta_1\alpha''_1$, $s_1(\alpha'_1) = \alpha'_1$, $s_2(\alpha''_1) = \alpha''_1$ and β_1 satisfies (3.2). If $\mathcal{U}_{k-1} = \mathcal{U}_k$ there is nothing to prove.

Suppose that $\mathcal{U}_{k-1} \neq \mathcal{U}_k$. Since $\mathcal{U}_{k-1}$ is invariant under the action of s_1 and under the action of s_2, we have $\mathcal{U}_{k-1} = \mathcal{U}_k \cup \{\delta_{1,k}, \delta_{2,k}, \delta_{3,k}\}$. Set $G_j = \mathrm{KB}_n[\mathcal{U}_k \cup \{\delta_{j,k}\}]$ for $j \in \{1,2,3\}$ and $H = \mathrm{KB}_n[\mathcal{U}_k]$. By Lemma 3.3, $\mathrm{KB}_n[\mathcal{U}_{k-1}] = G_1 *_H G_2 *_H G_3$. Moreover, we have the following properties.

- For each $i \in \{1,2\}$ and each $j \in \{1,2,3\}$ there exists $k \in \{1,2,3\}$ such that $s_i(G_j) = G_k$.
- For each $j \in \{1,2,3\}$ there exists $i \in \{1,2\}$ such that $s_i(G_j) \neq G_j$.
- For each $i \in \{1,2\}$ we have $s_i(H) = H$.
- By Lemma 3.10, for each $i \in \{1,2\}$ and each $\gamma \in H$ such that $s_i(\gamma) = \gamma^{-1}$, there exists $\delta \in H$ such that $\gamma = \delta\, s_i(\delta^{-1})$.

By Lemma 3.11 it follows that there exist $\alpha_2', \alpha_2'' \in \mathrm{KB}_n[\mathcal{U}_{k-1}]$ and $\beta \in \mathrm{KB}_n[\mathcal{U}_k]$ such that $\beta_1 = \alpha_2'\beta\alpha_2''$, $s_1(\alpha_2') = \alpha_2'$, $s_2(\alpha_2'') = \alpha_2''$ and β satisfies (3.2). We set $\alpha' = \alpha_1'\alpha_2'$ and $\alpha'' = \alpha_2''\alpha_1''$. Then $\alpha = \alpha'\beta\alpha''$, $s_1(\alpha') = \alpha'$ and $s_2(\alpha'') = \alpha''$.

For $4 \leqslant k \leqslant n$ we set $\mathcal{V}_k = \{\delta_{i,j} \in \mathcal{U}_n \mid (i,j) \notin (\{4,\dots,k\} \times \{1,2,3\})\}$. We set also $\mathcal{V}_3 = \mathcal{U}_n$. We show by induction on k that there exist $\alpha', \alpha'' \in \mathrm{KB}_n[\mathcal{X}]$ and $\beta \in \mathrm{KB}_n[\mathcal{V}_k]$ such that $\alpha = \alpha'\beta\alpha''$, $s_1(\alpha') = \alpha'$, $s_2(\alpha'') = \alpha''$ and β satisfies (3.2). The case $k = 3$ is true by the above. Suppose that $4 \leqslant k \leqslant n$ and that the inductive hypothesis holds. So, there exist $\alpha_1', \alpha_1'' \in \mathrm{KB}_n[\mathcal{X}]$ and $\beta_1 \in \mathrm{KB}_n[\mathcal{V}_{k-1}]$ such that $\alpha = \alpha_1'\beta_1\alpha_1''$, $s_1(\alpha_1') = \alpha_1'$, $s_2(\alpha_1'') = \alpha_1''$ and β_1 satisfies (3.2). If $\mathcal{V}_{k-1} = \mathcal{V}_k$ there is nothing to prove. Suppose that $\mathcal{V}_{k-1} \neq \mathcal{V}_k$. Since $\mathcal{V}_{k-1}$ is invariant under the action of s_1 and under the action of s_2, we have $\mathcal{V}_{k-1} = \mathcal{V}_k \cup \{\delta_{k,1}, \delta_{k,2}, \delta_{k,3}\}$. Set $G_j = \mathrm{KB}_n[\mathcal{V}_k \cup \{\delta_{k,j}\}]$ for $j \in \{1,2,3\}$ and $H = \mathrm{KB}_n[\mathcal{V}_k]$. By Lemma 3.3, $\mathrm{KB}_n[\mathcal{V}_{k-1}] = G_1 *_H G_2 *_H G_3$. Moreover, we have the following properties.

- For each $i \in \{1,2\}$ and each $j \in \{1,2,3\}$ there exists $k \in \{1,2,3\}$ such that $s_i(G_j) = G_k$.
- For each $j \in \{1,2,3\}$ there exists $i \in \{1,2\}$ such that $s_i(G_j) \neq G_j$.
- For each $i \in \{1,2\}$ we have $s_i(H) = H$.
- By Lemma 3.10, for each $i \in \{1,2\}$ and each $\gamma \in H$ such that $s_i(\gamma) = \gamma^{-1}$, there exists $\delta \in H$ such that $\gamma = \delta\, s_i(\delta^{-1})$.

By Lemma 3.11 it follows that there exist $\alpha_2', \alpha_2'' \in \mathrm{KB}_n[\mathcal{V}_{k-1}]$ and $\beta \in \mathrm{KB}_n[\mathcal{V}_k]$ such that $\beta_1 = \alpha_2'\beta\alpha_2''$, $s_1(\alpha_2') = \alpha_2'$, $s_2(\alpha_2'') = \alpha_2''$ and β satisfies (3.2). We set $\alpha' = \alpha_1'\alpha_2'$ and $\alpha'' = \alpha_2''\alpha_1''$. Then $\alpha = \alpha'\beta\alpha''$, $s_1(\alpha') = \alpha'$ and $s_2(\alpha'') = \alpha''$.

Set $\mathcal{W}_1 = \{\delta_{i,j} \in \mathcal{X} \mid (i,j) \in (\{1,2,3\} \times \{1,2,3\})\}$ and $\mathcal{W}_2 = \{\delta_{i,j} \in \mathcal{X} \mid (i,j) \in (\{4,\dots,n\} \times \{4,\dots,n\})\}$. Notice that $\mathcal{V}_n = \mathcal{W}_1 \sqcup \mathcal{W}_2$, $\mathrm{KB}_n[\mathcal{V}_n] = \mathrm{KB}_n[\mathcal{W}_1] \times \mathrm{KB}_n[\mathcal{W}_2]$, and $s_1(\gamma) = s_2(\gamma) = \gamma$ for all $\gamma \in \mathrm{KB}_n[\mathcal{W}_2]$. By the above there exist $\alpha_1', \alpha_1'' \in \mathrm{KB}_n[\mathcal{X}]$ and $\beta_1 \in \mathrm{KB}_n[\mathcal{V}_n]$ such that $\alpha =$

$\alpha'_1\beta_1\alpha''_1$, $s_1(\alpha'_1)=\alpha'_1$, $s_2(\alpha''_1)=\alpha''_1$ and β_1 satisfies (3.2). Let $\beta\in \mathrm{KB}_n[\mathcal{W}_1]$ and $\alpha''_2\in \mathrm{KB}_n[\mathcal{W}_2]$ such that $\beta_1=\beta\alpha''_2$. Set $\alpha'=\alpha'_1$ and $\alpha''=\alpha''_2\alpha''_1$. Then $\alpha=\alpha'\beta\alpha''$, $s_1(\alpha')=\alpha'$, $s_2(\alpha'')=\alpha''$ and β satisfies (3.2).

Let $\mathcal{W}_{1,1}=\{\delta_{i,j}\in\mathcal{X}\mid(i,j)\in\{(1,2),(2,3),(3,1)\}\}$ and $\mathcal{W}_{1,2}=\{\delta_{i,j}\in\mathcal{X}\mid(i,j)\in\{(2,1),(3,2),(1,3)\}\}$. Set $G_1=\mathrm{KB}_n[\mathcal{W}_{1,1}]$ and $G_2=\mathrm{KB}_n[\mathcal{W}_{1,2}]$. By Lemma 3.3, $\mathrm{KB}_n[\mathcal{W}_1]=G_1*G_2$. Moreover, $s_1(G_1)=s_2(G_1)=G_2$ and $s_1(G_2)=s_2(G_2)=G_1$. From Lemma 3.11 applied with $H=\{1\}$ it follows that there exist $\beta',\beta''\in\mathrm{KB}_n[\mathcal{W}_1]$ such that $\beta=\beta'\beta''$, $s_1(\beta')=\beta'$ and $s_2(\beta'')=\beta''$. Actually, by Lemma 3.7, $\beta'=\beta''=1$, hence $\beta=1$. So, $\alpha=\alpha'\alpha''$, $s_1(\alpha')=\alpha'$ and $s_2(\alpha'')=\alpha''$. ☐

As announced in Section 2, we take advantage of the results of the present section to prove Lemma 2.6.

Proof of Lemma 2.6. — Suppose instead that ζ_1 is an inner automorphism, that is, $\zeta_1=c_\gamma:\mathrm{VB}_n\to\mathrm{VB}_n$, $\delta\mapsto\gamma\delta\gamma^{-1}$, for some $\gamma\in\mathrm{VB}_n$. We have $\gamma\neq 1$ since $\zeta_1\neq\mathrm{id}$. We write γ in the form $\gamma=\alpha\,\iota(w)$ with $\alpha\in\mathrm{KB}_n$ and $w\in\mathfrak{S}_n$. For each $i\in\{1,\dots,n-1\}$ we have $s_i=\pi_K(\tau_i)=\pi_K(\zeta_1(\tau_i))=\pi_K(\gamma\tau_i\gamma^{-1})=ws_iw^{-1}$, hence w lies in the center of $\mathfrak{S}_n$ which is trivial, and therefore $w=1$ and $\gamma=\alpha\in\mathrm{KB}_n$.

Note that $\zeta_1(\delta_{i,j})=\delta_{j,i}$ for all $i,j\in\{1,\dots,n\}$, $i\neq j$. Take $i,j\in\{1,\dots,n\}$, $i<j$, and set $\mathcal{U}_{i,j}=\mathcal{S}\setminus\{\delta_{i,j},\delta_{j,i}\}$, $\mathcal{U}'_{i,j}=\mathcal{U}_{i,j}\cup\{\delta_{i,j}\}$ and $\mathcal{U}''_{i,j}=\mathcal{U}_{i,j}\cup\{\delta_{j,i}\}$. By Lemma 3.3, $\mathrm{KB}_n=\mathrm{KB}_n[\mathcal{U}'_{i,j}]*_{\mathrm{KB}_n[\mathcal{U}_{i,j}]}\mathrm{KB}_n[\mathcal{U}''_{i,j}]$. Moreover, $\zeta_1(\mathrm{KB}_n[\mathcal{U}'_{i,j}])=\mathrm{KB}_n[\mathcal{U}''_{i,j}]$ and $\zeta_1(\mathrm{KB}_n[\mathcal{U}''_{i,j}])=\mathrm{KB}_n[\mathcal{U}'_{i,j}]$. From Lemma 3.6 it follows that $\mathrm{KB}_n^{\zeta_1}\subset\mathrm{KB}_n[\mathcal{U}_{i,j}]$. Since $\bigcap_{1\leqslant i<j\leqslant n}\mathcal{U}_{i,j}=\emptyset$, by Theorem 3.2, $\bigcap_{1\leqslant i<j\leqslant n}\mathrm{KB}_n[\mathcal{U}_{i,j}]=\mathrm{KB}_n[\emptyset]=\{1\}$, thus $\mathrm{KB}_n^{\zeta_1}=\{1\}$. But $\alpha\in\mathrm{KB}_n^{\zeta_1}$ and $\alpha=\gamma\neq 1$, which is a contradiction. So, ζ_1 is not an inner automorphism. ☐

4. From virtual braid groups to symmetric groups

The following is well-known and can be easily deduced from Artin [1] and Lin [18, 19]. It is a preliminary to the proof of Theorem 2.1.

Proposition 4.1. — *Let $n,m\in\mathbb{N}$ such that $n\geqslant 5$, $m\geqslant 2$ and $n\geqslant m$, and let $\varphi:\mathfrak{S}_n\to\mathfrak{S}_m$ be a homomorphism. Then, up to conjugation, one of the following possibilities holds.*

(1) *φ is Abelian,*
(2) *$n=m$ and $\varphi=\mathrm{id}$,*
(3) *$n=m=6$ and $\varphi=\nu_6$.*

Proof of Theorem 2.1. — Let $n, m \in \mathbb{N}$ such that $n \geqslant 5$, $m \geqslant 2$ and $n \geqslant m$, and let $\psi : \mathrm{VB}_n \to \mathfrak{S}_m$ be a homomorphism. By Proposition 4.1 one of the following possibilities holds up to conjugation.

- $\psi \circ \iota$ is Abelian,
- $n = m$ and $\psi \circ \iota = \mathrm{id}$,
- $n = m = 6$ and $\psi \circ \iota = \nu_6$.

Suppose that $\psi \circ \iota$ is Abelian. Then there exists $w_1 \in \mathfrak{S}_m$ such that $w_1 = (\psi \circ \iota)(s_i) = \psi(\tau_i)$ for all $i \in \{1, \dots, n-1\}$. Since $s_i^2 = 1$, we have $w_1^2 = 1$. Set $w_2 = \psi(\sigma_1)$. From the relation $\tau_i \tau_{i+1} \sigma_i = \sigma_{i+1} \tau_i \tau_{i+1}$ it follows that $\psi(\sigma_i) = w_1^2 \psi(\sigma_i) = \psi(\sigma_{i+1}) w_1^2 = \psi(\sigma_{i+1})$ for all $i \in \{1, \dots, n-2\}$, hence $\psi(\sigma_i) = w_2$ for all $i \in \{1, \dots, n-1\}$. Finally, from the relation $\tau_1 \sigma_3 = \sigma_3 \tau_1$ it follows that $w_1 w_2 = w_2 w_1$. So, ψ is Abelian.

Suppose that $n = m$ and $\psi \circ \iota = \mathrm{id}$. Then $\psi(\tau_i) = s_i$ for all $i \in \{1, \dots, n-1\}$. From the relations $\sigma_1 \tau_i = \tau_i \sigma_1$, $3 \leqslant i \leqslant n-1$, it follows that $\psi(\sigma_1)$ lies in the centralizer of $\langle s_3, \dots, s_{n-1} \rangle$ in $\mathfrak{S}_n$, which is equal to $\langle s_1 \rangle = \{1, s_1\}$, hence either $\psi(\sigma_1) = 1$ or $\psi(\sigma_1) = s_1$. If $\psi(\sigma_1) = 1$, then $\psi(\sigma_i) = 1$, since σ_i is conjugate to σ_1 in VB_n, for all $i \in \{1, \dots, n-1\}$, hence $\psi = \pi_K$. Assume that $\psi(\sigma_1) = s_1$. We show by induction on i that $\psi(\sigma_i) = s_i$ for all $i \in \{1, \dots, n-1\}$. The case $i = 1$ is true by hypothesis. Suppose that $i \geqslant 2$ and $\psi(\sigma_{i-1}) = s_{i-1}$. Then, since $\tau_{i-1} \tau_i \sigma_{i-1} = \sigma_i \tau_{i-1} \tau_i$, we have $\psi(\sigma_i) = s_{i-1} s_i s_{i-1} s_i s_{i-1} = s_i$. So, $\psi(\tau_i) = \psi(\sigma_i) = s_i$ for all $i \in \{1, \dots, n-1\}$, that is, $\psi = \pi_P$.

Suppose that $n = m = 6$ and $\psi \circ \iota = \nu_6$. Then $\nu_6^{-1} \circ \psi \circ \iota = \mathrm{id}$, hence, by the above, either $\nu_6^{-1} \circ \psi = \pi_K$ or $\nu_6^{-1} \circ \psi = \pi_P$, and therefore either $\psi = \nu_6 \circ \pi_K$ or $\psi = \nu_6 \circ \pi_P$. ☐

5. From symmetric groups to virtual braid groups

The core of the proof of Theorem 2.2 lies in the following lemma.

LEMMA 5.1. — *Let $n \in \mathbb{N}$, $n \geqslant 3$, and let $\varphi : \mathfrak{S}_n \to \mathrm{VB}_n$ be a homomorphism such that $\pi_K \circ \varphi = \mathrm{id}$. Then φ is conjugate to ι.*

Proof. — Since $\pi_K \circ \varphi = \mathrm{id}$, for each $i \in \{1, \dots, n-1\}$ there exists $\alpha_i \in \mathrm{KB}_n$ such that $\varphi(s_i) = \alpha_i \tau_i$. We prove by induction on k that there exists a homomorphism $\varphi' : \mathfrak{S}_n \to \mathrm{VB}_n$ conjugate to φ such that $\pi_K \circ \varphi' = \mathrm{id}$ and $\varphi'(s_i) = \tau_i$ for all $i \in \{1, \dots, k\}$. The case $k = n-1$ ends the proof of the lemma.

Suppose that $k = 1$. We have $1 = \varphi(s_1)^2 = \alpha_1\, s_1(\alpha_1)\, \tau_1^2 = \alpha_1\, s_1(\alpha_1)$, hence $s_1(\alpha_1) = \alpha_1^{-1}$. By Lemma 3.10 there exists $\beta_1 \in \mathrm{KB}_n$ such that

$\alpha_1 = \beta_1\, s_1(\beta_1^{-1})$. Thus, $\varphi(s_1) = \alpha_1\tau_1 = \beta_1\, s_1(\beta_1^{-1})\,\tau_1 = \beta_1\tau_1\beta_1^{-1}$. Set $\varphi' = c_{\beta_1^{-1}} \circ \varphi$. Then φ' is conjugate to φ, $\pi_K \circ \varphi' = \mathrm{id}$, since $\beta_1 \in \mathrm{KB}_n$, and $\varphi'(s_1) = \tau_1$.

We assume that $k = 2$ and $\varphi(s_1) = \tau_1$. Since $\varphi(s_1s_2s_1) = \varphi(s_2s_1s_2)$, we have $\tau_1\alpha_2\tau_2\tau_1 = \alpha_2\tau_2\tau_1\alpha_2\tau_2$, hence $s_1(\alpha_2) = \alpha_2\,(s_2s_1)(\alpha_2)$. On the other hand, as in the previous paragraph, from the equality $\varphi(s_2)^2 = 1$ it follows that there exists $\beta_2 \in \mathrm{KB}_n$ such that $\alpha_2 = \beta_2\, s_2(\beta_2^{-1})$. Thus,

$$\begin{aligned} s_1\big(\beta_2\, s_2(\beta_2^{-1})\big) &= \big(\beta_2\, s_2(\beta_2^{-1})\big)\,(s_2s_1)\big(\beta_2\, s_2(\beta_2^{-1})\big) \\ \Rightarrow\quad & \beta_2\, s_2(\beta_2^{-1})\,(s_2s_1)(\beta_2)\,(s_2s_1s_2)(\beta_2^{-1})\,(s_1s_2)(\beta_2)\,s_1(\beta_2^{-1}) = 1\,. \end{aligned}$$

By Lemma 3.12 there exist $\beta_2', \beta_2'' \in \mathrm{KB}_n$ such that $\beta_2 = \beta_2'\beta_2''$, $s_1(\beta_2') = \beta_2'$ and $s_2(\beta_2'') = \beta_2''$. Thus, $\varphi(s_2) = \alpha_2\tau_2 = \beta_2\tau_2\beta_2^{-1} = \beta_2'\tau_2\beta_2'^{-1}$. Set $\varphi' = c_{\beta_2'^{-1}} \circ \varphi$. Then φ' is conjugate to φ, $\pi_K \circ \varphi' = \mathrm{id}$, since $\beta_2' \in \mathrm{KB}_n$, $\varphi'(s_1) = \tau_1$, since $s_1(\beta_2') = \beta_2'$, and $\varphi'(s_2) = \tau_2$ by construction.

We assume that $k \geqslant 3$ and $\varphi(s_i) = \tau_i$ for all $i \in \{1, \dots, k-1\}$. Let $\ell \in \{1, \dots, k-2\}$. Since $\varphi(s_k s_\ell) = \varphi(s_\ell s_k)$, we have $\alpha_k\tau_k\tau_\ell = \tau_\ell\alpha_k\tau_k$, hence $s_\ell(\alpha_k) = \alpha_k$. By Corollary 3.8 it follows that $\alpha_k \in \mathrm{KB}_n[\mathcal{U}_\ell]$, where $\mathcal{U}_\ell = \{\delta_{i,j} \in \mathcal{S} \mid i,j \notin \{\ell, \ell{+}1\}\}$. Recall that $\mathcal{S}_k = \{\delta_{i,j} \in \mathcal{S} \mid k \leqslant i \neq j \leqslant n\}$. We have $\bigcap_{1\leqslant\ell\leqslant k-2} \mathcal{U}_\ell = \mathcal{S}_k$ hence, by Theorem 3.2, $\alpha_k \in \mathrm{KB}_n[\mathcal{S}_k]$. Since $\varphi(s_{k-1}s_ks_{k-1}) = \varphi(s_ks_{k-1}s_k)$, we have $\tau_{k-1}\alpha_k\tau_k\tau_{k-1} = \alpha_k\tau_k\tau_{k-1}\alpha_k\tau_k$, hence $s_{k-1}(\alpha_k) = \alpha_k\,(s_ks_{k-1})(\alpha_k)$. On the other hand, as in the two previous paragraphs, from the equality $\varphi(s_k)^2 = 1$ it follows that there exists $\beta_k \in \mathrm{KB}_n[\mathcal{S}_k]$ such that $\alpha_k = \beta_k\, s_k(\beta_k^{-1})$. So,

$$\begin{aligned} & s_{k-1}\big(\beta_k\, s_k(\beta_k^{-1})\big) = \big(\beta_k\, s_k(\beta_k^{-1})\big)\,(s_ks_{k-1})\big(\beta_k\, s_k(\beta_k^{-1})\big) \\ & \Rightarrow \beta_k\, s_k(\beta_k^{-1})\,(s_ks_{k-1})(\beta_k)\,(s_ks_{k-1}s_k)(\beta_k^{-1})\,(s_{k-1}s_k)(\beta_k)\,s_{k-1}(\beta_k^{-1}) = 1\,. \end{aligned}$$

Note that $\mathcal{S}_k$ is not invariant under the action of s_{k-1}, but $\mathcal{S}_{k-1}$ is and $\mathcal{S}_k \subset \mathcal{S}_{k-1}$. By Lemma 3.12 there exist $\beta_k', \beta_k'' \in \mathrm{KB}_n[\mathcal{S}_{k-1}]$ such that $\beta_k = \beta_k'\beta_k''$, $s_{k-1}(\beta_k') = \beta_k'$ and $s_k(\beta_k'') = \beta_k''$. So, $\varphi(s_k) = \alpha_k\tau_k = \beta_k\tau_k\beta_k^{-1} = \beta_k'\tau_k\beta_k'^{-1}$. Since $s_{k-1}(\beta_k') = \beta_k'$, by Corollary 3.8, $\beta_k' \in \mathrm{KB}_n[\mathcal{U}_{k-1}]$, where $\mathcal{U}_{k-1} = \{\delta_{i,j} \in \mathcal{S} \mid i,j \notin \{k-1, k\}\}$. Since $\mathcal{S}_{k-1} \cap \mathcal{U}_{k-1} = \mathcal{S}_{k+1}$, by Theorem 3.2 it follows that $\beta_k' \in \mathrm{KB}_n[\mathcal{S}_{k+1}]$, hence $s_i(\beta_k') = \beta_k'$ for all $i \in \{1, \dots, k-1\}$. Set $\varphi' = c_{\beta_k'^{-1}} \circ \varphi$. Then φ' is conjugate to φ, $\pi_K \circ \varphi' = \mathrm{id}$, and $\varphi'(s_i) = \tau_i$ for all $i \in \{1, \dots, k\}$. □

Proof of Theorem 2.2. — Let $n, m \in \mathbb{N}$ such that $n \geqslant 5$, $m \geqslant 2$, and $n \geqslant m$, and let $\varphi : \mathfrak{S}_n \to \mathrm{VB}_m$ be a homomorphism. By Proposition 4.1 one of the following possibilities holds up to conjugation.

- $\pi_K \circ \varphi$ is Abelian,
- $n = m$ and $\pi_K \circ \varphi = \mathrm{id}$,

- $n = m = 6$ and $\pi_K \circ \varphi = \nu_6$.

Assume that $\pi_K \circ \varphi$ is Abelian. There exists $w \in \mathfrak{S}_m$ such that $(\pi_K \circ \varphi)(s_i) = w$ for all $i \in \{1, \dots, n-1\}$. Set $\beta_0 = \iota(w) \in \mathrm{VB}_m$. For each $i \in \{1, \dots, n-1\}$ there exists $\alpha_i \in \mathrm{KB}_m$ such that $\varphi(s_i) = \alpha_i \beta_0$. Since $s_1^2 = 1$, we have $w^2 = 1$, hence $\beta_0^2 = 1$. On the other hand, for each $i \in \{1, \dots, n-1\}$, we have $1 = \varphi(s_i)^2 = \alpha_i \beta_0 \alpha_i \beta_0$, hence $\alpha_i \beta_0 = \beta_0 \alpha_i^{-1}$. Let $i \in \{2, \dots, n-1\}$. We have $\varphi(s_1 s_i) = \alpha_1 \beta_0 \beta_0 \alpha_i^{-1} = \alpha_1 \alpha_i^{-1} \in \mathrm{KB}_m$, this element is of finite order, since $s_1 s_i$ is of finite order, and, by Godelle–Paris [14], KB_m is torsion free, hence $\alpha_1 \alpha_i^{-1} = 1$, that is, $\alpha_i = \alpha_1$. So, $\varphi(s_i) = \alpha_1 \beta_0$ for all $i \in \{1, \dots, n-1\}$, hence φ is Abelian.

Suppose that $n = m$ and $\pi_K \circ \varphi = \mathrm{id}$. Then, by Lemma 5.1, φ is conjugate to ι.

Suppose that $n = m = 6$ and $\pi_K \circ \varphi = \nu_6$. We have $\pi_K \circ \varphi \circ \nu_6^{-1} = \mathrm{id}$ hence, by Lemma 5.1, $\varphi \circ \nu_6^{-1}$ is conjugate to ι, that is, there exists $\alpha \in \mathrm{VB}_6$ such that $\varphi \circ \nu_6^{-1} = c_\alpha \circ \iota$. Then $\varphi = c_\alpha \circ \iota \circ \nu_6$, hence φ is conjugate to $\iota \circ \nu_6$. ☐

6. From virtual braid groups to virtual braid groups

Lemma 6.1. — *Let $n \geqslant 3$, let $i, j, k \in \{1, \dots, n\}$ pairwise distinct, and let $\ell_1, \ell_2 \in \mathbb{Z}$. Then $\delta_{i,j}^{\ell_1} \delta_{j,k}^{\ell_2} = 1$ if and only if $\ell_1 = \ell_2 = 0$. Similarly, we have $\delta_{j,i}^{\ell_1} \delta_{k,j}^{\ell_2} = 1$ if and only if $\ell_1 = \ell_2 = 0$.*

Proof. — Suppose that $\delta_{i,j}^{\ell_1} \delta_{j,k}^{\ell_2} = 1$. Set $\ell_1 = 2t_1 + \varepsilon_1$ and $\ell_2 = 2t_2 + \varepsilon_2$ where $t_1, t_2 \in \mathbb{Z}$ and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$. We have $\pi_P(\delta_{i,j}^{\ell_1} \delta_{j,k}^{\ell_2}) = (i,j)^{\varepsilon_1} (j,k)^{\varepsilon_2} = 1$, hence $\varepsilon_1 = \varepsilon_2 = 0$. So, $(\delta_{i,j}^2)^{t_1} (\delta_{j,k}^2)^{t_2} = 1$. By Crisp–Paris [11] the subgroup of $\mathrm{KB}_n[\{\delta_{i,j}, \delta_{j,k}\}]$ generated by $\{\delta_{i,j}^2, \delta_{j,k}^2\}$ is a free group freely generated by $\{\delta_{i,j}^2, \delta_{j,k}^2\}$, hence $t_1 = t_2 = 0$. We show in the same way that, if $\delta_{j,i}^{\ell_1} \delta_{k,j}^{\ell_2} = 1$, then $\ell_1 = \ell_2 = 0$. It is clear that, if $\ell_1 = \ell_2 = 0$, then $\delta_{i,j}^{\ell_1} \delta_{j,k}^{\ell_2} = \delta_{j,i}^{\ell_1} \delta_{k,j}^{\ell_2} = 1$. ☐

Lemma 6.2. — *Let $n = 6$. Set $u_i = \nu_6(s_i)$ for all $i \in \{1, 2, 3, 4, 5\}$. Let H be the subgroup of $\mathfrak{S}_6$ generated by $\{u_3, u_4, u_5\}$. Then $\mathrm{KB}_6^H = \{1\}$, where $\mathrm{KB}_6^H = \{\alpha \in \mathrm{KB}_6 \mid w(\alpha) = \alpha$ for all $w \in H\}$.*

Proof. — Let $U = \{u_3, u_4, u_5, u_3u_4u_3, u_4u_5u_4, u_3u_4u_5u_4u_3\}$. We have

$$u_3 = (1,3)(2,4)(5,6)\,,\ u_4 = (1,2)(3,5)(4,6)\,,\ u_5 = (2,3)(1,4)(5,6)\,,$$
$$u_3u_4u_3 = (1,6)(2,5)(3,4)\,,\ u_4u_5u_4 = (1,5)(2,6)(3,4)\,,$$
$$u_3u_4u_5u_4u_3 = (1,2)(3,6)(4,5)\,.$$

Let $1 \leqslant i < j \leqslant 6$. Set $\mathcal{U}_{i,j} = \mathcal{S} \setminus \{\delta_{i,j}, \delta_{j,i}\}$, $\mathcal{U}'_{i,j} = \mathcal{U}_{i,j} \cup \{\delta_{i,j}\}$ and $\mathcal{U}''_{i,j} = \mathcal{U}_{i,j} \cup \{\delta_{j,i}\}$. We have $\mathrm{KB}_6 = \mathrm{KB}_6[\mathcal{U}'_{i,j}] *_{\mathrm{KB}_6[\mathcal{U}_{i,j}]} \mathrm{KB}_6[\mathcal{U}''_{i,j}]$ by Lemma 3.3. On the other hand, it is easily observed that there exists $v \in U$ such that (i,j) is a cycle in the decomposition of v as a product of disjoint cycles. For that v we have $v(\mathrm{KB}_6[\mathcal{U}'_{i,j}]) = \mathrm{KB}_6[\mathcal{U}''_{i,j}]$ and $v(\mathrm{KB}_6[\mathcal{U}''_{i,j}]) = \mathrm{KB}_6[\mathcal{U}'_{i,j}]$. By Lemma 3.6 we deduce that $\mathrm{KB}_6^H \subset \mathrm{KB}_6^v \subset \mathrm{KB}_6[\mathcal{U}_{i,j}]$. We have $\bigcap_{1\leqslant i<j\leqslant 6} \mathcal{U}_{i,j} = \emptyset$, hence, by Theorem 3.2, $\mathrm{KB}_6^H \subset \mathrm{KB}_6[\emptyset] = \{1\}$. □

We will use the following notation in the proof of Theorem 2.3. Let $F_2 = F_2(x,y)$ be the free group of rank 2 freely generated by $\{x,y\}$. If G is a group, $\alpha, \beta \in G$, and $\omega \in F_2$, then $\omega(\alpha,\beta)$ denotes the image of ω under the homomorphism $F_2 \to G$ which sends x to α and y to β.

Proof of Theorem 2.3. — Let $n, m \in \mathbb{N}$ such that $n \geqslant 5$, $m \geqslant 2$ and $n \geqslant m$, and let $\psi : \mathrm{VB}_n \to \mathrm{VB}_m$ be a homomorphism. By Theorem 2.2 one of the following possibilities holds up to conjugation.

- $\psi \circ \iota$ is Abelian,
- $n = m$ and $\psi \circ \iota = \iota$,
- $n = m = 6$ and $\psi \circ \iota = \iota \circ \nu_6$.

Assume that $\psi \circ \iota$ is Abelian. We argue in the same way as in the Abelian case in the proof of Theorem 2.1. There exists $\beta_1 \in \mathrm{VB}_m$ such that $\beta_1 = (\psi \circ \iota)(s_i) = \psi(\tau_i)$ for all $i \in \{1, \dots, n-1\}$. Since $s_1^2 = 1$ we have $\beta_1^2 = 1$. Set $\beta_2 = \psi(\sigma_1)$. From the relation $\tau_i \tau_{i+1} \sigma_i = \sigma_{i+1} \tau_i \tau_{i+1}$ it follows that $\psi(\sigma_i) = \beta_1^2\, \psi(\sigma_i) = \psi(\sigma_{i+1})\, \beta_1^2 = \psi(\sigma_{i+1})$, for all $i \in \{1, \dots, n-2\}$, hence $\psi(\sigma_i) = \beta_2$ for all $i \in \{1, \dots, n-1\}$. Finally, from the relation $\tau_1 \sigma_3 = \sigma_3 \tau_1$ it follows that $\beta_1 \beta_2 = \beta_2 \beta_1$. So, ψ is Abelian.

Assume that $n = m$ and $\psi \circ \iota = \iota$, that is, $\psi(\tau_i) = \tau_i$ for all $i \in \{1, \dots, n-1\}$. For each $i \in \{1, \dots, n-1\}$ we set $\psi(\sigma_i) = \alpha_i\, \iota(w_i)$, where $\alpha_i \in \mathrm{KB}_n$ and $w_i \in \mathfrak{S}_n$. For each $i \in \{3, \dots, n-1\}$ we have $s_i w_1 = (\pi_K \circ \psi)(\tau_i \sigma_1) = (\pi_K \circ \psi)(\sigma_1 \tau_i) = w_1 s_i$, hence w_1 lies in the centralizer of $\langle s_3, \dots, s_{n-1} \rangle$ in $\mathfrak{S}_n$, which is equal to $\langle s_1 \rangle = \{1, s_1\}$, hence either $w_1 = s_1$ or $w_1 = 1$.

Suppose that $w_1 = s_1$, that is, $\psi(\sigma_1) = \alpha_1 \tau_1$. For each $k \in \{3, \dots, n-1\}$ we have $\alpha_1 \tau_1 \tau_k = \psi(\sigma_1 \tau_k) = \psi(\tau_k \sigma_1) = \tau_k \alpha_1 \tau_1$, hence $s_k(\alpha_1) = \alpha_1$. By Corollary 3.8 this implies that $\alpha_1 \in \mathrm{KB}_n[\mathcal{U}_k]$, where $\mathcal{U}_k = \{\delta_{i,j} \mid i,j \notin \{k, k+1\}\}$. We have $\bigcap_{3 \leqslant k \leqslant n-1} \mathcal{U}_k = \{\delta_{1,2}, \delta_{2,1}\}$ hence, by Theorem 3.2, $\alpha_1 \in \mathrm{KB}_n[\{\delta_{1,2}, \delta_{2,1}\}]$. Let $\omega \in F_2$ such that $\alpha_1 = \omega(\delta_{1,2}, \delta_{2,1})$. Note that, by Theorem 3.2, $\mathrm{KB}_n[\{\delta_{1,2}, \delta_{2,1}\}]$ is a free group freely generated by $\{\delta_{1,2}, \delta_{2,1}\}$, hence ω is unique. We have $\sigma_2 = \tau_1 \tau_2 \sigma_1 \tau_2 \tau_1$, hence $\psi(\sigma_2) = \omega(\delta_{2,3}, \delta_{3,2}) \tau_2$.

On the other hand, since $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, we have

$$\omega(\delta_{1,2},\delta_{2,1})\tau_1\omega(\delta_{2,3},\delta_{3,2})\tau_2\omega(\delta_{1,2},\delta_{2,1})\tau_1 = \omega(\delta_{2,3},\delta_{3,2})\tau_2\omega(\delta_{1,2},\delta_{2,1})\tau_1\omega(\delta_{2,3},\delta_{3,2})\tau_2\,,$$

hence

$$\omega(\delta_{1,2},\delta_{2,1})\omega(\delta_{1,3},\delta_{3,1})\omega(\delta_{2,3},\delta_{3,2}) = \omega(\delta_{2,3},\delta_{3,2})\omega(\delta_{1,3},\delta_{3,1})\omega(\delta_{1,2},\delta_{2,1})\,. \tag{6.1}$$

Let $\mathcal{W} = \{\delta_{i,j} \,|\, 1 \leqslant i \neq j \leqslant 3\}$, $\mathcal{W}_1 = \{\delta_{1,2},\delta_{2,3},\delta_{3,1}\}$ and $\mathcal{W}_2 = \{\delta_{2,1},\delta_{3,2},\delta_{1,3}\}$. By Lemma 3.3, $\mathrm{KB}_n[\mathcal{W}] = \mathrm{KB}_n[\mathcal{W}_1] * \mathrm{KB}_n[\mathcal{W}_2]$. From this decomposition and Lemma 6.1 it is easily deduced that the only element of F_2 which satisfies (6.1) is $\omega = 1$, thus $\psi(\sigma_1) = \tau_1$. Then from the equalities $\sigma_i = \tau_{i-1}\tau_i\sigma_{i-1}\tau_i\tau_{i-1}$, $2 \leqslant i \leqslant n-1$, it follows that $\psi(\sigma_i) = \tau_i$ for all $i \in \{2,\dots,n-1\}$, hence $\psi = \iota \circ \pi_P$.

Suppose that $w_1 = 1$, that is, $\psi(\sigma_1) = \alpha_1$. For each $k \in \{3,\dots,n-1\}$ we have $\alpha_1\tau_k = \psi(\sigma_1\tau_k) = \psi(\tau_k\sigma_1) = \tau_k\alpha_1$, hence $s_k(\alpha_1) = \alpha_1$. By Corollary 3.8 this implies that $\alpha_1 \in \mathrm{KB}_n[\mathcal{U}_k]$, where $\mathcal{U}_k = \{\delta_{i,j} \,|\, i,j \notin \{k,k+1\}\}$. As above, $\bigcap_{3\leqslant k\leqslant n-1}\mathcal{U}_k = \{\delta_{1,2},\delta_{2,1}\}$, hence $\alpha_1 \in \mathrm{KB}_n[\{\delta_{1,2},\delta_{2,1}\}]$. Let $\omega \in F_2$ such that $\alpha_1 = \omega(\delta_{1,2},\delta_{2,1})$. Again, since $\mathrm{KB}_n[\{\delta_{1,2},\delta_{2,1}\}]$ is a free group freely generated by $\{\delta_{1,2},\delta_{2,1}\}$, the element ω is unique. We have $\sigma_2 = \tau_1\tau_2\sigma_1\tau_2\tau_1$, hence $\psi(\sigma_2) = (s_1s_2)(\alpha_1) = \omega(\delta_{2,3},\delta_{3,2})$. On the other hand, since $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, we have

$$\omega(\delta_{1,2},\delta_{2,1})\omega(\delta_{2,3},\delta_{3,2})\omega(\delta_{1,2},\delta_{2,1}) = \omega(\delta_{2,3},\delta_{3,2})\omega(\delta_{1,2},\delta_{2,1})\omega(\delta_{2,3},\delta_{3,2})\,. \tag{6.2}$$

Recall that $\mathrm{KB}_n[\mathcal{W}] = \mathrm{KB}_n[\mathcal{W}_1] * \mathrm{KB}_n[\mathcal{W}_2]$, where $\mathcal{W} = \{\delta_{i,j} \,|\, 1 \leqslant i \neq j \leqslant 3\}$, $\mathcal{W}_1 = \{\delta_{1,2},\delta_{2,3},\delta_{3,1}\}$ and $\mathcal{W}_2 = \{\delta_{2,1},\delta_{3,2},\delta_{1,3}\}$. From this decomposition and Lemma 6.1 it is easily seen that, if ω satisfies (6.2), then ω is of the form $\omega = z^k$ with $z \in \{x,y\}$ and $k \in \mathbb{Z}$. By Crisp–Paris [11], if $k \notin \{1,0,-1\}$, then the subgroup of $\mathrm{KB}_n[\mathcal{W}_1]$ generated by $\{\delta_{1,2}^k,\delta_{2,3}^k,\delta_{3,1}^k\}$ is a free group freely generated by $\{\delta_{1,2}^k,\delta_{2,3}^k,\delta_{3,1}^k\}$, hence $\delta_{1,2}^k\delta_{2,3}^k\delta_{1,2}^k \neq \delta_{2,3}^k\delta_{1,2}^k\delta_{2,3}^k$. Similarly, we have $\delta_{2,1}^k\delta_{3,2}^k\delta_{2,1}^k \neq \delta_{3,2}^k\delta_{2,1}^k\delta_{3,2}^k$ if $k \notin \{-1,0,1\}$. So, $\omega \in \{x,x^{-1},y,y^{-1},1\}$. Moreover, from the equalities $\sigma_i = \tau_{i-1}\tau_i\sigma_{i-1}\tau_i\tau_{i-1}$, $2 \leqslant i \leqslant n-1$, it follows that $\psi(\sigma_i) = \omega(\delta_{i,i+1},\delta_{i+1,i})$ for all $i \in \{2,\dots,n-1\}$. If $\omega = x$ then $\psi = \mathrm{id}$, if $\omega = x^{-1}$ then $\psi = \zeta_2$, if $\omega = y$ then $\psi = \zeta_1$, if $\omega = y^{-1}$ then $\psi = \zeta_1 \circ \zeta_2$, and if $\omega = 1$ then $\psi = \iota \circ \pi_K$.

Now, we assume that $n = m = 6$ and $\psi \circ \iota = \iota \circ \nu_6$. For each $i \in \{1,2,3,4,5\}$ we set $u_i = \nu_6(s_i)$ and $\beta_i = \iota(u_i)$. Thus, $\psi(\tau_i) = \beta_i$ for all $i \in \{1,2,3,4,5\}$. For each $i \in \{1,2,3,4,5\}$ we set $\psi(\sigma_i) = \alpha_i\,\iota(w_i)$, where $\alpha_i \in \mathrm{KB}_6$ and $w_i \in \mathfrak{S}_6$. For each $i \in \{3,4,5\}$ we have $u_i w_1 = (\pi_K \circ \psi)(\tau_i \sigma_1) = (\pi_K \circ \psi)(\sigma_1 \tau_i) = w_1 u_i$, hence w_1 lies in the centralizer of $\langle u_3, u_4, u_5 \rangle$ in $\mathfrak{S}_6$, which is equal to $\langle u_1 \rangle = \{1, u_1\}$, thus either $w_1 = u_1$ or $w_1 = 1$. On the other hand, for each $i \in \{3,4,5\}$, we have $\alpha_1\,\iota(w_1)\,\beta_i = \psi(\sigma_1\tau_i) = \psi(\tau_i\sigma_1) = \beta_i\alpha_1\,\iota(w_1)$ and $\iota(w_1)\,\beta_i = \iota(w_1 u_i) = \iota(u_i w_1) = \beta_i\,\iota(w_1)$, hence $u_i(\alpha_1) = \beta_i\alpha_1\beta_i^{-1} = \alpha_1$. By Lemma 6.2 it follows that $\alpha_1 = 1$. So, either $\psi(\sigma_1) = \beta_1$ or $\psi(\sigma_1) = 1$. If $\psi(\sigma_1) = \beta_1$, then, since $\sigma_i = \tau_{i-1}\tau_i\sigma_{i-1}\tau_i\tau_{i-1}$ for all $i \in \{2,3,4,5\}$, we have $\psi(\sigma_i) = \beta_i$ for all $i \in \{1,2,3,4,5\}$, and therefore $\psi = \iota \circ \nu_6 \circ \pi_P$. If $\psi(\sigma_1) = 1$, then, since $\sigma_i = \tau_{i-1}\tau_i\sigma_{i-1}\tau_i\tau_{i-1}$ for all $i \in \{2,3,4,5\}$, we have $\psi(\sigma_i) = 1$ for all $i \in \{1,2,3,4,5\}$, and therefore $\psi = \iota \circ \nu_6 \circ \pi_K$. ☐

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