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Logarithmic forms and singular projective foliations

LOGARITHMIC FORMS AND SINGULAR PROJECTIVE FOLIATIONS

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ABSTRACT. — In this article we study polynomial logarithmic q-forms on a projective space and characterize those that define singular foliations of codimension q. Our main result is the algebraic proof of their infinitesimal stability when $q = 2$ with some extra degree assumptions. We determine new irreducible components of the moduli space of codimension two singular projective foliations of any degree, and we show that they are generically reduced in their natural scheme structure. Our method is based on an explicit description of the Zariski tangent space of the corresponding moduli space at a given generic logarithmic form. Furthermore, we lay the groundwork for an extension of our stability results to the general case $q \geqslant 2$.

RÉSUMÉ. — Dans cet article nous étudions des q-formes logaritmiques polynomiales sur un espace projectif et nous caractérisons celles qui définissent des feuilletages singuliers de codimension q. Notre principal résultat est la preuve algébrique de leur stabilité infinitésimale lorsque $q = 2$ avec quelques hypothèses supplémentaires sur leurs degrés. Nous donnons des nouvelles composantes irréductibles des espaces de modules des feuilletages projectifs de codimension deux et de degré quelconque, et nous montrons que ces composantes sont génériquement réduites selon leur structure naturelle de schéma. Notre méthode est basée sur le calcul explicite de l'espace tangent de Zariski de l'espace de modules en une forme logarithmique générique. Nous posons aussi les bases pour l'extension de nos résultats de stabilité au cas général $q \geqslant 2$

1. Introduction

This article is concerned with the study of complex projective logarithmic forms of arbitrary degrees in the setting of the theory of algebraic foliations. The problem which motivates this paper is the analysis of the irreducible components of moduli spaces of algebraic singular projective foliations, and the description of their geometry. For some classical results on this

classification problem for codimension one foliations we refer to e.g. [1], [2] and [11]. More in general, we recommend [6] and [7] for some remarkable facts about the higher codimensional case.

Fix $q \in \mathbb{N}$, $n, d \in \mathbb{N}_{>q}$ and $m \in \mathbb{N}_{\geqslant q+1}$. Also, set $\mathbf{d} = (d_1, \ldots, d_m)$ such that $d = \sum_{i=1}^{m} d_i$. A polynomial logarithmic q-form of type $\mathbf{d}$ is a projective twisted differential form

$$(1.1) \qquad \omega = \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I|=q}} \lambda_I \hat{F}_I \mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_q},$$

where $\lambda \in \bigwedge^q \mathbb{C}^m$ satisfies that its interior product by $\mathbf{d}$ vanishes (i.e., $i_{\mathbf{d}}(\lambda) = 0 \in \bigwedge^{q-1} \mathbb{C}^m$), $F_1, \ldots, F_m$ are homogeneous polynomials in $n + 1$ variables of degrees $d_1, \ldots, d_m$ and $\hat{F}_I = \prod_{j \notin I} F_j$ for every $I \subset \{1, \ldots, m\}$. If we assume that λ is totally decomposable (i.e., $\lambda = \lambda^1 \wedge \cdots \wedge \lambda^q$), then ω is an element of $H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))$ that also satisfies the Plücker's decomposability equation (2.1) and the so called Frobenius's integrability equation (2.2) (see Section 2.1 for more details). These conditions ensure that the distribution associated to its kernel at each point determines a singular complex foliation of codimension q on $\mathbb{P}^n$.

In addition, the forms as in (1.1) are related to the classical sheaf of logarithmic forms in the following sense. With the notation above, if we set $\mathcal{D}_F$ as the divisor defined by the zero locus of $F = \prod_{i=1}^{m} F_i$, and assume it has simple normal crossings, then all the elements $\eta \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(\log \mathcal{D}_F))$ can be described in homogeneous coordinates by the formula

$$\eta = \frac{\omega}{F} = \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I|=q}} \lambda_I \frac{\mathrm{d}F_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}F_{i_q}}{F_{i_q}}.$$

Remember that the sheaf $\Omega^\bullet_{\mathbb{P}^n}(\log \mathcal{D}_F)$ is defined by all the rational forms η such that η and $\mathrm{d}\eta$ have at most simple poles along $\mathcal{D}_F$. In other words, the global sections of the sheaf of logarithmic forms over $\mathcal{D}_F$ are in 1-1 correspondence with the vectors $\lambda \in \bigwedge^q \mathbb{C}^m$ such that $i_{\mathbf{d}}(\lambda) = 0$. See Proposition 2.6 for more details.

On the other hand, denote by $\mathcal{F}_q(d, \mathbb{P}^n) \subset \mathbb{P}H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))$ the moduli space of algebraic singular projective foliations of degree d on $\mathbb{P}^n$, which corresponds to the algebraic space of twisted q-forms of total degree d satisfying the announced equations (2.1) and (2.2).

We consider ρ as a rational map parametrizing those logarithmic q-forms of type $\mathbf{d}$ that define foliations (see Definition 2.21), and we set $\mathcal{L}_q(\mathbf{d}, n) \subset \mathcal{F}_q(d, \mathbb{P}^n)$ as the Zariski closure of its image. We refer to this projective irreducible variety as the logarithmic variety associated to the

partition $\mathbf{d}$. Our purpose is to show that these logarithmic varieties are irreducible components of the corresponding moduli spaces of foliations. In this paper, we solve this problem for $q = 2$ and with some condition among the vector of degrees $\mathbf{d}$ (see the 2-balanced assumption from Definition 3.18). Furthermore, we expect that, up to some technical details, our results and proofs can be adapted for any $q \leqslant n - 2$ at least assuming an extended q-balanced condition. When $q = n - 1$ there is no hope to establish a stability result even for a generic logarithmic form. In this case the corresponding moduli space coincides with $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{T}\mathbb{P}^n)$.

In order to put our work in context, we shall explain some related articles. It was proved in [1] by Omegar Calvo Andrade that the logarithmic varieties $\mathcal{L}_1(\mathbf{d}, n)$ are irreducible components of $\mathcal{F}_1(d, \mathbb{P}^n)$, and so this type of forms satisfy a sort of stability condition among all the integrable forms (see integrability condition (2.1) for $q = 1$). Actually, his article is essentially based on analytical and topological methods, and the main results are proved for logarithmic foliations on a general complex manifold M with the assumption $H^1(M, \mathbb{C}) = 0$. Later, in [5] it was developed a new proof of the stability result for polynomial logarithmic one-forms in projective spaces. Here the methods are completely algebraic and provide further information about the irreducible components $\mathcal{L}_1(\mathbf{d}, n)$. For example, it is deduced that the scheme $\mathcal{F}_1(d, \mathbb{P}^n)$ is generically reduced along these components.

Moreover, in [7] the authors proved that the varieties $\mathcal{L}_q(\mathbf{d}, n)$ are irreducible components when $m = q + 1$ (lower possible value for m). In this case, the corresponding differential q-forms determine projective foliations which are tangent to the fibers of quasi-homogeneous rational maps. From now, we refer to this type of forms as rational q-forms of type $\mathbf{d}$. It is also remarkable that there are not many known irreducible components of the moduli space $\mathcal{F}_q(d, \mathbb{P}^n)$ for general q. See for instance the introductions of [6] and [7].

In some sense, this article is concerned with obtaining a common generalization of the definitions and results from [1] and [7], using the same algebraic methods as in [5]. Notice that formula (1.1) coincides with that given for a general rational q-form of type $\mathbf{d}$ in [7] when $q = m - 1$, and with that given for a polynomial logarithmic 1-form in [1] when $q = 1$.

The article is organized as follows. In Section 2 we present a brief summary of the objects involved in the definitions of $\mathcal{F}_q(d, \mathbb{P}^n)$, $\mathcal{L}_q(\mathbf{d}, n)$ and the corresponding parametrization map, with a particular attention to the case $q = 2$. We suggest to consult Definitions 2.12 and 2.21, and also Propositions 2.14 and 2.17 that contain our definition of logarithmic q-forms of

type $\mathbf{d}$, and a characterization of those that determine singular projective foliations of codimension q.

Since our methods are based on Zariski tangent space calculations, in Section 2 we also present a description of these spaces for $\mathcal{F}_2(\mathbf{d}, \mathbb{P}^n)$ and the space of parameters $\mathcal{P}_2(\mathbf{d})$ of the natural parametrization ρ. In addition, a formula for the derivative of the parametrization map is given at the end of the section. Our main tool is based on the comparison of the image of this derivative with the tangent space of $\mathcal{F}_2(d, \mathbb{P}^n)$ at a generic logarithmic form of type $\mathbf{d}$. See our Subsections 2.3 and 2.4 for a complete treatment of these tasks.

Our Section 3 contains a development of the main result: Theorem 3.1, which establishes that the logarithmic varieties $\mathcal{L}_2(\mathbf{d}, \mathbb{P}^n)$ are irreducible components of the moduli space $\mathcal{F}_2(d, \mathbb{P}^n)$. In addition, we deduce that these components are generically reduced according to its induced scheme structure. Our result assumes that $m, n > 3$ and that the vector of degrees $\mathbf{d}$ is 2-balanced, that is

$$d_i + d_j < \sum_{k \neq i, \neq j} d_k \quad \forall\, i, j \in \{1, \ldots, m\}.$$

The formal statement of the theorem is the following:

THEOREM. — *Fix natural numbers $n, m \in \mathbb{N}_{>3}$, and a 2-balanced vector of degrees $\mathbf{d} = (d_1, \ldots, d_m)$ with $d = \sum_i d_i$. The variety $\mathcal{L}_2(\mathbf{d}, n)$ is an irreducible component of the moduli space $\mathcal{F}_2(d, \mathbb{P}^n)$. Furthermore, the scheme $\mathcal{F}_2(d, \mathbb{P}^n)$ is generically reduced along this component, in particular at the points of $\rho(\mathcal{U}_2(\mathbf{d}))$.*

The proof of Theorem 3.1 is supported on Proposition 3.2 (surjectivity of the derivative of the natural parametrization). This method had been originally used in [6] and [7], and was also applied in [5]. However, the proof of this proposition is quite technical, and it is obtained through various steps and lemmas, including certain results of independent interest. See for instance Steps 1 to 7.

Finally, we would like to observe the followings facts about the general case $q \geqslant 2$. The definition of our natural parametrization ρ, the varieties $\mathcal{L}_q(\mathbf{d}, n)$ and some of our lemmas are still valid for larger q (see Subsection 3.2.1). However, most of the steps of the proof of Proposition 3.2 are quite technical, and the combinatorics behind them is hard to extend for $q > 2$. See also Remark 3.28 for more details.

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Remark 1.1. — After completing and having posted this article we learned about the article [3] by D. Cerveau and A. Lins Neto. Some of our results for logarithmic q-forms turn out to be contained in this paper, but the methods of proof are completely different. In particular, the authors obtain a stability result for the general case $1 \leqslant q \leqslant n - 2$, after reducing the problem to the case of foliations of dimension two (see [3, Theorems 5 and 6]). We would like to observe that our methods, based on computing the Zariski tangent space at a generic point, are completely algebraic and provide further information in the particular case of logarithmic foliations of codimension two. Especially, the fact that the moduli space of singular projective foliations results generically reduced along the logarithmic irreducible components that we obtained.

2. Definitions and constructions

In this article we shall use the following basic notation concerning polynomials and forms.

$S^{(n)} = \mathbb{C}[x_0, \ldots, x_n]$ for the graded ring of polynomials with complex coefficients in $n + 1$ variables. When n is understood we write $S^{(n)} = S$.

$S_d^{(n)} = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ for the space of homogeneous polynomials of degree d in $n + 1$ variables. When n is understood we denote $S_d^{(n)} = S_d$.

$\Omega_X^q(\mathcal{L}) = \Omega_X^q \otimes \mathcal{O}(\mathcal{L})$ for the sheaf of differential forms twisted by a line bundle $\mathcal{L}$ on X.

$i_v : \Omega_X^\bullet \to \Omega_X^{\bullet-1}$ for the interior product or contraction with a vector field v on a variety X. In addition, if $m, q \in \mathbb{N}$, $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{C}^m$ and $\lambda \in \bigwedge^q \mathbb{C}^m$, we also denote by $i_{\mathbf{d}}(\lambda) \in \bigwedge^{q-1} \mathbb{C}^m$ the interior product of λ by $\mathbf{d}$, that is

$$i_{(d_1,\ldots,d_m)}\left(\sum_{I:|I|=q} \lambda_I e_{i_1} \wedge \cdots \wedge e_{i_q} \right)$$
$$= \sum_{I:|I|=q} \sum_{j=1}^{q} \lambda_I (-1)^{j+1} d_{i_j} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_q},$$

for $\{e_1, \ldots, e_m\}$ the canonical base of $\mathbb{C}^m$.

2.1. Singular projective foliations and logarithmic forms

We write $\mathcal{F}_q(d, \mathbb{P}^n)$ for the moduli space of singular projective foliations of codimension q and degree d. Recall from [13] (or see also [7]) that this space is naturally described by projective classes of twisted projective forms $\omega \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(d))$ which satisfy both the Plücker's decomposability condition

$$(2.1) \qquad i_v(\omega) \wedge \omega = 0 \qquad \forall\, v \in \bigwedge^{q-1} \mathbb{C}^{n+1}$$

and the so called Frobenius's integrability condition

$$(2.2) \qquad i_v(\omega) \wedge \mathrm{d}\omega = 0 \qquad \forall\, v \in \bigwedge^{q-1} \mathbb{C}^{n+1}.$$

The elements like v can be considered as local frames on the affine cone over $\mathbb{P}^n$, or alternatively as rational multi-vector fields.

The first equation ensures that ω is locally decomposable outside its singular set, so this section of $\Omega_{\mathbb{P}^n}^q(d)$ belongs to the corresponding Grassmannian space at that points. The second equation is a condition to guarantee that the singular distribution associated to the kernel of ω is also integrable. See for instance [13, Proposition 1.2.2]. We also recall that the singular set of the foliation induced by ω corresponds to its vanishing points $S_\omega = \{p \in \mathbb{P}^n : \omega(p) = 0\}$.

Remark 2.1. — Each element $\omega \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(d))$ can be represented in homogeneous coordinates by homogeneous affine q-forms of total degree d like

$$\omega = \sum_{I=\{i_1 \ldots, i_q\}} A_I(z)\mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_q},$$

where $\{A_I\}_I \subset S_{d-q}$ are selected in order to satisfy the so called *descend* condition

$$(2.3) \qquad i_R(\omega) = 0 \in H^0(\mathbb{C}^{n+1}, \Omega_{\mathbb{C}^{n+1}}^{q-1}).$$

Here R denotes the radial Euler field $\sum_i z_i \frac{\partial}{\partial z_i}$.

Remark 2.2. — For $q = 2$ the decomposability equation (2.1) is slightly simpler than in the general case because is equivalent to $i_v(\omega \wedge \omega) = 0$ $\forall\, v \in \mathbb{C}^{n+1}$, so it can be replaced by

$$(2.4) \qquad \omega \wedge \omega = 0.$$

In conclusion, we define $\mathcal{F}_q(d, \mathbb{P}^n)$, the algebraic space of codimension q singular foliations on $\mathbb{P}^n$ of degree $d \geqslant q$, as the following set

$$\left\{ [\omega] \in \mathbb{P}H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d)) : \omega \text{ satisfies } (2.1), (2.2) \text{ and } \mathrm{codim}(S_\omega) \geqslant 2 \right\}.$$

Thus each $[\omega] \in \mathcal{F}_q(d, \mathbb{P}^n)$ determines a singular holomorphic foliation on $\mathbb{P}^n$ whose leaves are of codimension q, i.e., a regular holomorphic foliation outside the singular set S_ω. Moreover, the degree of the divisor of tangencies of such leaves with a generic linearly embedded $\mathbb{P}^q$ in $\mathbb{P}^n$ is in this case $d - q - 1$ (sometimes also used to refer to the degree of the corresponding foliation).

On the other hand, we want to define correct formulas for those polynomial logarithmic q-forms in $H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))$ that define foliations according to the above-mentionned equations.

First, we recall the principal definitions and properties of the well-known sheaf of meromorphic logarithmic forms and its classical residues in our particular case of interest. For a more extensive development of this task see [8].

Let $\mathcal{D} = \sum_{i=1}^m (F_i = 0)$ be a projective divisor defined by homogeneous polynomials $F_i \in S_{d_i}$ for $i = 1, \ldots, m$. The sheaf $\Omega^q_{\mathbb{P}^n}(\log \mathcal{D})$ of meromorphic logarithmic q-forms is defined as a subsheaf of the sheaf $\Omega^q_{\mathbb{P}^n}(*\mathcal{D})$ of meromorphic forms with arbitrary poles along $\mathcal{D}$ according to

$$\Omega^q_{\mathbb{P}^n}(\log \mathcal{D}) = \{\alpha \in \Omega^q_{\mathbb{P}^n}(*\mathcal{D}) : \alpha \text{ and } d\alpha \text{ have simple poles along } \mathcal{D}\}.$$

If $\mathcal{D}$ has simple normal crossings, it is a well known fact that

$$(\Omega^\bullet_{\mathbb{P}^n}(\log \mathcal{D}), d) \hookrightarrow (\Omega^\bullet_{\mathbb{P}^n}(*\mathcal{D}), d)$$

determines a subcomplex and a quasi-isomorphism (see [14, Proposition 4.3]).

We consider the following usual filtration

$$(2.5) \qquad W_k(\Omega^q_{\mathbb{P}^n}(\log \mathcal{D})) = \begin{cases} 0 & \text{if } k < 0 \\ \Omega^{q-k}_{\mathbb{P}^n} \wedge \Omega^k_{\mathbb{P}^n}(\log \mathcal{D}) & \text{if } 0 \leqslant k \leqslant q \\ \Omega^q_{\mathbb{P}^n}(\log \mathcal{D}) & \text{if } k \geqslant q, \end{cases}$$

and use the notation

- $X_i = (F_i = 0)$ for $i = 1, \ldots, m$,
- $X_I = X_{i_1} \cap \cdots \cap X_{i_k}$ for $I = \{i_1 \ldots i_k\} \subset \{1, \ldots, m\}$,
- $\mathcal{D}(I) = \sum_{j \notin I} X_I \cap X_j$ as a divisor on X_I,
- $j_I = X_I \hookrightarrow \mathbb{P}^n$,
- $X^k_\mathcal{D} = \coprod_{I:|I|=k} X_I$,
- $j_k = X^k_\mathcal{D} \hookrightarrow \mathbb{P}^n$.

The desired formula for polynomial logarithmic forms will be supported on the characterization of the global sections of the sheaf $\Omega^q_{\mathbb{P}^n}(\log\mathcal{D})$. Our arguments will rely on the following known constructions and results regarding this classical sheaf of forms.

PROPOSITION 2.3. — *Assume that $\mathcal{D}$ has simple normal crossings.*

(1) *There is a well defined residue map*

$$\mathrm{Res}_I : \Omega^{\bullet}_{\mathbb{P}^n}(\log\mathcal{D}) \longrightarrow \Omega^{\bullet}_{X_I}(\log\mathcal{D}(I))[-k]$$

for each multi-index I with $|I| = k$.

(2) *The residue map restricts to*

$$\mathrm{Res}_I : W_k(\Omega^{\bullet}_{\mathbb{P}^n}(\log\mathcal{D}) \longrightarrow (j_I)_*(\Omega^{\bullet}_{X_I}[-k]),$$

is surjective and also well defined on the quotient $Gr^W_k(\Omega^{\bullet}_{\mathbb{P}^n}(\log\mathcal{D}))$.

(3) *The total residue map*

$$\mathrm{Res}_k := \bigoplus_{I:|I|=k} \mathrm{Res}_I : Gr^W_k(\Omega^{\bullet}_{\mathbb{P}^n}(\log\mathcal{D})) \longrightarrow (j_k)_*(\Omega^{\bullet}_{X^k_{\mathcal{D}}}[-k])$$

is an isomorphism.

Proof. — See [14, Lemma 4.6], or [8] for an extended overview. $\qquad\square$

Now we are able to compute $H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(\log\mathcal{D}))$.

LEMMA 2.4. — *With the notation above, assume $q < \min\{m, n-1\}$. Then the following computations hold:*

(1) $H^0(\mathbb{P}^n, W_k(\Omega^q_{\mathbb{P}^n}(\log\mathcal{D}))) = 0 \quad \forall\, k = 0,\dots, q-1,$

(2) $H^1(\mathbb{P}^n, W_k(\Omega^q_{\mathbb{P}^n}(\log\mathcal{D}))) = 0 \quad \forall\, k = 0,\dots, q-2,$

(3) $H^1(\mathbb{P}^n, W_{q-1}(\Omega^q_{\mathbb{P}^n}(\log\mathcal{D})))$ *has a natural injective map to $\bigwedge^{q-1}\mathbb{C}^m$.*

Proof. — First, recall that the Hodge numbers associated to the classical projective space are $h^{p,q}_{\mathbb{P}^n} = \delta_{pq}$ (Kronecker delta). Due to the normal crossings condition and the Lefschetz hyperplane theorem (see for instance [12, Section 3.1]) it is also true that

$$h^{p,q}_{X_I} = \delta_{pq} \quad \forall\, p, q, I : p + q < n - |I|,$$

for every complete intersection subvariety X_I. Now, we proceed by induction on the number $k \geqslant 0$. The base case is trivial. Then, the results (1) and (2) are immediate consequences of considering the long exact sequence in cohomology associated to the exact sequence

$$(2.6)\ \ 0 \longrightarrow W_{k-1}(\Omega^q_{\mathbb{P}^n}(\log\mathcal{D})) \longrightarrow W_k(\Omega^q_{\mathbb{P}^n}(\log\mathcal{D})) \longrightarrow (j_k)_*(\Omega^{q-k}_{X^k_{\mathcal{D}}}) \longrightarrow 0,$$

where the exactness follows from Proposition 2.3. Finally, consider the long sequence on cohomology for $k = q - 1$, that is

$$0 \to H^1(\mathbb{P}^n, W_{q-1}(\Omega_{\mathbb{P}^n}^q)) \to \bigoplus_{I:|I|=q-1} H^1(X_I, \Omega_{X_I}^1)$$

$$\to H^2(\mathbb{P}^n, W_{q-2}(\Omega_{\mathbb{P}^n}^q(\log \mathcal{D}))) \dots \ ,$$

and use the knowledge of the Hodge number $h_{X_I}^{1,1}$ for each I of size $q - 1$ to show our assertion (3). $\qquad\square$

With a similar proof and the same inductive argument, we can also state the following result.

COROLLARY 2.5. — *If we assume that $\mathcal{D}$ has simple normal crossings and $q < \min\{m, n - j\}$, then*

$$H^j(\mathbb{P}^n, W_k(\Omega_{\mathbb{P}^n}^q(\log \mathcal{D}))) = 0 \quad \forall\, k = 1, \dots, q - j - 1.$$

PROPOSITION 2.6. — *Assume $q < \min\{m, n - 1\}$. Let $F_1, \dots, F_m$ be homogeneous polynomials with simple normal crossings and respective degrees $d_1, \dots, d_m$, defying a divisor $\mathcal{D}$ as before. For every global section $\eta \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(\log \mathcal{D}))$, there exist unique constants $\lambda = \{\lambda_I\}_{I:|I|=q} \in \bigwedge^q \mathbb{C}^m$ such that η can be written in homogeneous coordinates as*

$$\eta = \sum_{\substack{I \subset \{1,\dots,m\} \\ |I|=q}} \lambda_I \frac{\mathrm{d}F_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}F_{i_q}}{F_{i_q}},$$

where also λ satisfies $i_{\mathbf{d}}(\lambda) = 0$ to ensure that $i_R(\eta) = 0$. Recall that $i_{\mathbf{d}}$ denotes the interior product by the constant vector $\mathbf{d} = (d_1, \dots, d_m)$.

Proof. — We use once more the exact sequence of sheaves (2.6) considered at the proof of Lemma 2.4, but now in the particular case $k = q$, and we get

$$0 \longrightarrow W_{q-1}(\Omega_{\mathbb{P}^n}^q(\log \mathcal{D})) \longrightarrow \Omega_{\mathbb{P}^n}^q(\log \mathcal{D}) \xrightarrow{\mathrm{Res}_q} (j_q)_*(\mathcal{O}_{X_{\mathcal{D}}^q}) \longrightarrow 0.$$

The result follows from considering again the associated long exact sequence in cohomology, and applying the previous lemma to deduce the injectivity of the total residue map Res_q on global sections. Notice that the normal crossing assumption ensures that the image of the total residue map described in homogeneous coordinates can be considered as a subset of $\mathbb{C}^{\binom{m}{q}} \simeq \bigwedge^q \mathbb{C}^m$. Hence, the residues of the form $\sum \lambda_I \frac{\mathrm{d}F_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}F_{i_q}}{F_{i_q}}$ are the numbers (λ_I). In addition, the kernel of the first connection morphism of the long exact sequence is exactly determined by the descend condition

among the previous form, that is $i_{\mathbf{d}}(\lambda) = 0$. This equation characterizes the image of Res_q and implies our statement. $\qquad\square$

Remark 2.7. — In fact, last proposition can be proved for a general complex projective algebraic variety X of dimension n with the assumption

$$h_X^{p,0} = \dim(H^0(X, \Omega_X^p)) = 0,$$

for $1 \leqslant p \leqslant n - 1$.

COROLLARY 2.8. — *Select $\lambda \in \bigwedge^q \mathbb{C}^m$ and homogeneous polynomials $F_i \in S_{d_i}$ having simple normal crossings. Then $\sum \lambda_I \frac{\mathrm{d}F_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}F_{i_q}}{F_{i_q}} = 0$ if and only if $\lambda = 0$.*

The last statement is a sort of extension of [11, Lemma 3.3.1] to the case of higher degree logarithmic forms, with a more restrictive hypothesis.

COROLLARY 2.9. — *For every simple normal crossings divisor $\mathcal{D}$ set as before we have*

$$H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(\log \mathcal{D})) \simeq \left\{ \lambda \in \bigwedge^q \mathbb{C}^m : i_{\mathbf{d}}(\lambda) = 0 \right\}.$$

Moreover, notice that for each $q \in \mathbb{N}_{<m}$ the right set corresponds to the degree q cycles of the Koszul complex associated to the vector $\mathbf{d} \in \mathbb{C}^m$, i.e., the exact complex

$$\mathcal{K}(\mathbf{d}) : 0 \to \bigwedge^m \mathbb{C}^m \xrightarrow{i_{\mathbf{d}}} \bigwedge^{m-1} \mathbb{C}^m \xrightarrow{i_{\mathbf{d}}} \cdots \xrightarrow{i_{\mathbf{d}}} \bigwedge \mathbb{C}^m \xrightarrow{i_{\mathbf{d}}} \mathbb{C} \to 0.$$

DEFINITION 2.10. — *Fix natural numbers d and m. A partition of d into m parts is an m-tuple of degrees $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{N}^m$ satisfying $\sum_{i=1}^m d_i = d$.*

For future convenience, we will use the following notation.

DEFINITION 2.11. — *For $F_i \in S_{d_i}$ as above we denote*

$$\underline{F} = (F_1, \ldots, F_m),$$

$$F = \prod_{j=1}^m F_j, \qquad \hat{F}_i = \prod_{j \neq i} F_j, \qquad \hat{F}_{ij} = \prod_{k \neq i, k \neq j} F_k, \ (i \neq j),$$

or, more generally, for $I \subset \{1, \ldots, m\}$ we write $\hat{F}_I = \prod_{j \notin I} F_j$.

DEFINITION 2.12. — *Fix* **d** *as a partition of* $d \in \mathbb{N}_{>q}$ *into* m *parts, and assume* $m \geqslant q + 1$. *A twisted projective differential* q-*form* $\omega \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))$ *is a polynomial logarithmic* q-*form of type* **d** *if it is defined by the following formula*

$$(2.7) \qquad \omega = F \left(\sum_{I:|I|=q} \lambda_I \frac{\mathrm{d}F_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}Fi_q}{F_{i_q}} \right) = \sum_{I:|I|=q} \lambda_I \hat{F}_I \mathrm{d}F_I,$$

where $F_i \in S_{d_i}$ *and* $\lambda \in \bigwedge^q \mathbb{C}^m$ *satisfies* $i_{\mathbf{d}}(\lambda) = 0$.

Remark 2.13. — From Corollary 2.9 we deduce that for every logarithmic q-form ω as in (2.7), there exist $\gamma \in \bigwedge^{q+1} \mathbb{C}^m$ such that $i_{\mathbf{d}}(\gamma) = \lambda$ and

$$\omega = i_R \left(\sum_{J:|J|=q+1} \gamma_J \hat{F}_J \mathrm{d}F_J \right).$$

Hence the above formula could be an alternative definition for logarithmic q-forms of type **d**. See also [7] to compare our formulas with the definition presented there for rational q-forms.

At first, we want to determine when this type of forms define foliations of codimension q, i.e., when the forms as in (2.7) satisfy the equations of $\mathcal{F}_q(d, \mathbb{P}^n)$. For simplicity, we start with the Plücker's decomposability condition (2.1) (or equivalently (2.4)) for $q = 2$.

PROPOSITION 2.14. — *If* ω *is a logarithmic* 2-*form of type* **d** *and the polynomials involved have simple normal crossings, then the following conditions are equivalent:*

(1) $\omega \wedge \omega = 0$
(2) $\lambda \wedge \lambda = 0$.

Proof. — Notice that

$$\omega \wedge \omega = F^2 \left(\sum_{i \neq j \neq k \neq l} (\lambda \wedge \lambda)_{ijkl} \frac{\mathrm{d}F_i}{F_i} \wedge \frac{\mathrm{d}F_j}{F_j} \wedge \frac{\mathrm{d}F_k}{F_k} \wedge \frac{\mathrm{d}F_l}{F_l} \right).$$

The result then follows from Corollary 2.8. $\qquad\square$

Remark 2.15. — For ω as above, the condition $\lambda \wedge \lambda = 0$ implies that there exist $\lambda^1, \lambda^2 \in \mathbb{C}^m$ such that $\lambda = \lambda^1 \wedge \lambda^2$. In this case, if we consider the meromorphic logarithmic forms defined by

$$\eta_j = \sum_{i=1}^m (\lambda^j)_i \frac{\mathrm{d}F_i}{F_i} \quad j = 1, 2,$$

then the meromorphic logarithmic 2-form $\eta = \frac{\omega}{F} = \eta_1 \wedge \eta_2$ is globally decomposable.

Remark 2.16. — If we consider a totally decomposable q-vector $\lambda = \lambda^1 \wedge \cdots \wedge \lambda^q \in \bigwedge^q \mathbb{C}^m$ with $i_{\mathbf{d}}(\lambda) = 0$, then for every selection of homogeneous polynomials, the induced logarithmic q-form of type $\mathbf{d}$ as in (2.7) satisfies the Plücker's decomposability equation (2.1). Using the above notation, the reason is that the meromorphic q-form $\eta = \frac{\omega}{F} = \eta_1 \wedge \cdots \wedge \eta_q$ is globally decomposable.

Let us recall that the grassmannian space $Gr(q, \mathbb{C}^m)$ of q-dimensional subspaces of $\mathbb{C}^m$ can be considered as a projective algebraic subset of $\mathbb{P}(\bigwedge^q \mathbb{C}^m)$ via the Plücker embedding

$$\iota : Gr(q, \mathbb{C}^m) \longrightarrow \mathbb{P}\left(\bigwedge^q \mathbb{C}^m \right)$$

$$\mathrm{span}(\lambda^1, \ldots, \lambda^q) \longmapsto [\lambda^1 \wedge \cdots \wedge \lambda^q].$$

With a slight abuse of notation, we only write λ or $\lambda^1 \wedge \cdots \wedge \lambda^q$ for the elements of $Gr(q, \mathbb{C}^m)$ and (λ_I) for its corresponding antisymmetric coordinates.

PROPOSITION 2.17. — *For $\mathbf{d}$ as above, $\lambda \in Gr(q, \mathbb{C}^m)$ such that $i_{\mathbf{d}}(\lambda) = 0$ and $(F_i)_{i=1}^m \in \prod_{i=1}^m S_{d_i}$, the logarithmic q-form of type $\mathbf{d}$*

$$\omega = \sum_{I : |I| = q} \lambda_I \hat{F}_I \mathrm{d}F_I$$

is a twisted projective form of total degree d, i.e., $\omega \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(d)$, and satisfies (2.1) and (2.2). Hence $[\omega] \in \mathcal{F}_q(d, \mathbb{P}^n)$.

Proof. — If we take into consideration Remark 2.16, then it only remains to prove that ω satisfies the integrability equation $i_v(\omega) \wedge \mathrm{d}\omega = 0$, for all $v \in \bigwedge^{q-1} \mathbb{C}^{n+1}$. This follows by a straight forward calculation using that $\mathrm{d}\omega = \frac{\mathrm{d}F}{F} \wedge \omega$ and $i_v(\omega) \wedge \omega = 0$. $\qquad\square$

The next result characterizes in an useful way the condition $i_{\mathbf{d}}(\lambda) = 0$ for $\lambda \in Gr(q, \mathbb{C}^m)$.

LEMMA 2.18. — *For $\lambda = \lambda^1 \wedge \cdots \wedge \lambda^q \in \bigwedge^q(\mathbb{C}^m) - \{0\}$, the following equations are equivalent:*

   (1) $i_{\mathbf{d}}(\lambda) = 0$
   (2) $i_{\mathbf{d}}(\lambda^i) = 0, \quad \forall\, i = 1, \ldots, q.$

Proof. — The equivalence follows from

$$i_{\mathbf{d}}(\lambda) = \sum_{j=1}^{m} (-1)^j \, i_{\mathbf{d}}(\lambda^j) \, \lambda^1 \wedge \cdots \wedge \hat{\lambda}^j \wedge \cdots \wedge \lambda^q,$$

and the fact that $\{\lambda^1 \wedge \cdots \wedge \hat{\lambda}^j \wedge \cdots \wedge \lambda^q\}_j$ are linearly independent. $\square$

COROLLARY 2.19. — *The following equality holds:*

$$\{\lambda \in Gr(q, \mathbb{C}^m) : i_{\mathbf{d}}(\lambda) = 0\} = Gr(q, \mathbb{C}_{\mathbf{d}}^m),$$

where $\mathbb{C}_{\mathbf{d}}^m$ denotes the linear space of vectors $\mu \in \mathbb{C}^m$ such that $i_{\mathbf{d}}\mu = \mu \cdot \mathbf{d} = \sum_{i=1}^{m} \mu_i d_i = 0$.

Remark 2.20. — Notice that we have not proved that if ω is a logarithmic q-form of type $\mathbf{d}$ as in Definition 2.12, then the moduli equations (2.1) and (2.2) are equivalent to the condition $\lambda \in Gr(q, \mathbb{C}_{\mathbf{d}}^m)$, i.e., ω globally decomposable outside $\mathcal{D}_F = (F = 0)$. Proposition 2.17 only states a partial answer to that problem. However, when $q = 2$ the equivalence is in fact true according to Proposition 2.14. It is also remarkable that the case $q > 2$ is an open question in [3, Problem 1].

2.2. The logarithmic varieties and the natural parametrization

Consider again natural numbers $q \in \mathbb{N}$, $n \in \mathbb{N}_{>q+1}$, $d, m \in \mathbb{N}_{\geqslant q+1}$ and a partition $\mathbf{d}$ of d into m parts. According to Proposition 2.17 we denote by $l_q(\mathbf{d}, n)$ the algebraic set of logarithmic q-forms of type $\mathbf{d}$ which are globally decomposable outside their corresponding divisor $\mathcal{D}_F = (F = 0)$, and whose projective classes determine codimension q foliations. Actually, this set also coincides with the image of the multi-linear map

$$\phi : (\mathbb{C}_{\mathbf{d}}^m)^q \times \prod_{i=1}^{m} S_{d_i} \longrightarrow H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(d))$$

$$((\lambda^1, \ldots, \lambda^q), (F_1, \ldots, F_m)) \longmapsto \sum_{I : |I| = q} (\lambda^1 \wedge \cdots \wedge \lambda^q)_I \hat{F}_I \mathrm{d}F_I.$$

DEFINITION 2.21. — *We introduce the following map as our natural parametrization:*

(2.8)

$$\rho : \mathcal{P}_q(\mathbf{d}) := Gr(q, \mathbb{C}_{\mathbf{d}}^m) \times \prod_{i=1}^m \mathbb{P}(S_{d_i}) - \to \mathcal{F}_q(d, \mathbb{P}^n) \subset \mathbb{P}H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(d))$$

$$(\lambda = [(\lambda^1 \wedge \cdots \wedge \lambda^q)], \underline{F} = ([F_1], \ldots, [F_m])) \longmapsto [\omega] = \left[\sum_{I:|I|=q} \lambda_I \hat{F}_I dF_I \right].$$

From now on and when there is no confusion, we avoid the notation $[\cdot]$ for the corresponding projective classes of the elements involved in the definition of ρ.

DEFINITION 2.22. — *We define the logarithmic variety $\mathcal{L}_q(\mathbf{d}, n)$ as the Zariski closure of the image of ρ, i.e., $\mathcal{L}_q(\mathbf{d}, n) = \overline{\operatorname{im} \rho} \subset \mathcal{F}_q(d, \mathbb{P}^n)$.*

Remarks 2.23.

(1) ρ is only a rational map and so it is not well defined on the whole space of parameters $\mathcal{P}_q(\mathbf{d})$. In particular, on the parameters which give rise to forms that vanish completely (base locus of the parametrization).
(2) The space $\mathcal{P}_q(\mathbf{d})$ is an algebraic irreducible projective variety of dimension $\sum_{i=1}^m \binom{n+d_i}{d_i} - m + q(m-1-q)$.
(3) Notice that the image of ρ also coincides with $\mathbb{P}l_q(\mathbf{d}, n)$.
(4) The space $\mathcal{L}_q(\mathbf{d}, n)$ is a projective irreducible subvariety of $\mathcal{F}_q(d, \mathbb{P}^n)$.

With some conditions among $\mathbf{d}$, we will prove that $\mathcal{L}_2(\mathbf{d}, n)$ is an irreducible component of the space $\mathcal{F}_2(d, \mathbb{P}^n)$. From now on, in general, we assume $q = 2$.

For our prompt purposes we need to assume some generic conditions on the parameters in $\mathcal{P}_2(\mathbf{d})$. The polynomials $\{F_i\}$ and the constants $\{\lambda_{ij}\}$ are general according to the following definition.

DEFINITION 2.24. — *We take the non-empty algebraic open subset of $\mathcal{P}_2(\mathbf{d})$ defined by*

$$\mathcal{U}_2(\mathbf{d}) = \{(\lambda, \underline{F}) \in \mathcal{P}_2(\mathbf{d}) : \lambda_{ij} \neq 0, \ \lambda_{ij} - \lambda_{ik} + \lambda_{jk} \neq 0, \ \lambda_{ij} - \lambda_{ik} - \lambda_{jk} \neq 0$$
$$\text{and } F_1, \ldots, F_m \text{ are smooth irreducible with normal crossings}\}.$$

Remark 2.25. — It follows from Corollary 2.8 that $\mathcal{U}_2(\mathbf{d})$ does not intersect the base locus of ρ.

A more detailed analysis of the base locus of ρ, including a description of its irreducible components and scheme structure will be pursued in [4].

2.3. The Zariski tangent space of $\mathcal{F}_2(d, \mathbb{P}^n)$

For a scheme X and a point $x \in X$ we denote by $\mathcal{T}_x X$ the Zariski tangent space of X at x. Remember that for a classical projective space $X = \mathbb{P}\mathbb{V}$, associated to a finite dimensional vector space $\mathbb{V}$, it is common to identify its Zariski tangent space at a given point $x = \pi(v)$ with

$$\mathcal{T}_x X = \mathbb{V}/_{\langle v \rangle}.$$

Also recall that $\mathcal{F}_2(d, \mathbb{P}^n)$ is the closed subscheme of the projective space $\mathbb{P}(H^0(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}(d)))$ defined by equations (2.1) and (2.4). With a slight abuse of notation, the Zariski tangent space $\mathcal{T}_\omega \mathcal{F}_2(d, \mathbb{P}^n)$ can be represented by the forms $\alpha \in H^0(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}(d))/_{\langle \omega \rangle}$ such that

$$(\omega + \epsilon \alpha) \wedge (\omega + \epsilon \alpha) = 0 \qquad \text{and}$$
$$(i_v \omega + \epsilon i_v \omega) \wedge (d\omega + \epsilon d\alpha) = 0 \qquad \forall\, v \in \mathbb{C}^{n+1}, \text{ with } \epsilon^2 = 0.$$

Remark 2.26. — If we fix an element $\omega \in \mathcal{F}_2(d, \mathbb{P}^n)$, then a simple calculation shows that $\alpha \in H^0(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}(d))/_{\langle \omega \rangle}$ belongs to the Zariski tangent space of $\mathcal{F}_2(d, \mathbb{P}^n)$ at ω if and only if it fulfills the following equations:

$$(2.9) \qquad \alpha \wedge \omega = 0$$

$$(2.10) \qquad (i_v \omega \wedge d\alpha) + (i_v \alpha \wedge d\omega) = 0 \qquad \forall\, v \in \mathbb{C}^{n+1}.$$

We refer to the first equation as the *decomposability perturbation equation* and to the second one as the *integrability perturbation equation*. In conclusion, we have

$$\mathcal{T}_\omega \mathcal{F}_2(d, \mathbb{P}^n) = \{\alpha \in H^0(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}(d))/_{\langle \omega \rangle} : \alpha \text{ satisfies } (2.9) \text{ and } (2.10)\}.$$

2.4. The derivative of the natural parametrization

One of our main purposes is to show that the derivative of ρ is surjective (see for instance Section 3.1). In order to carry out this plan we need to set some notation.

As it was explained in the previous section, for every $e \in \mathbb{N}$ we have

$$\mathcal{T}_{\pi(F)} \mathbb{P}(S_e) = S_e/_{\langle F \rangle}.$$

From now on we will write F' for a general element of this space.

Moreover, let $\lambda = [\lambda^1 \wedge \lambda^2]$ be an element in the grassmannian space $Gr(2, \mathbb{C}_{\mathbf{d}}^m)$. The Zariski tangent space of $Gr(2, \mathbb{C}_{\mathbf{d}}^m)$ at λ has a natural identification with the space of antisymmetric vectors $\lambda' \in \bigwedge^2(\mathbb{C}_{\mathbf{d}}^m)/\langle\lambda\rangle$ such that

$$(2.11) \qquad\qquad\qquad \lambda' \wedge \lambda = 0.$$

Even more, the vectors λ' satisfying the above condition can be written as

$$(2.12) \qquad\qquad \lambda' = (\lambda')^1 \wedge \lambda^2 + \lambda^1 \wedge (\lambda')^2,$$

where $(\lambda')^1, (\lambda')^2 \in \mathbb{C}_{\mathbf{d}}^m/\langle\lambda^1, \lambda^2\rangle$. In general, we will write λ' for the elements of $\mathcal{T}_\lambda Gr(2, \mathbb{C}_{\mathbf{d}}^m)$, i.e., for the vectors $\lambda' \in \bigwedge^2(\mathbb{C}_{\mathbf{d}}^m)/\langle\lambda\rangle$ fulfilling equation (2.11) or equivalently of the type (2.12).

Remark 2.27. — Let $(\lambda^1 \wedge \lambda^2, (F_i)_{i=1}^m) = (\lambda, \underline{F}) \in \mathcal{P}_2(\mathbf{d})$ be a fixed parameter in the domain of ρ and write $\omega = \rho((\lambda, \underline{F})) = \sum_{i \neq j} \lambda_{ij} \hat{F}_{ij} dF_i \wedge dF_j$. From the multilinearity of ϕ and the definition of ρ, we obtain the following formula for the derivative of ρ:

$$(2.13) \quad d\rho(\lambda, \underline{F}) : \mathcal{T}_\lambda Gr(2, \mathbb{C}_{\mathbf{d}}^m) \times \prod_{i=1}^m \mathcal{T}_{F_i} \mathbb{P}S_{d_i} \longrightarrow \mathcal{T}_\omega \mathcal{F}_2(d, \mathbb{P}^n)$$

$$(\lambda', (F_1', \ldots, F_m')) \longmapsto \sum_{i \neq j} \lambda'_{ij} \hat{F}_{ij} dF_i \wedge dF_j$$

$$+ \sum_{i \neq j \neq k} \lambda_{ij} \hat{F}_{ijk} F_k' dF_i \wedge dF_j + 2 \sum_{i \neq j} \lambda_{ij} \hat{F}_{ij} dF_i' \wedge dF_j.$$

We will use the following notation for the forms in the image of the partial derivatives of ρ:

$$\alpha_1 = d\rho(\lambda, \underline{F})(\lambda', (0, \ldots, 0)) \qquad \text{with } \lambda' \in \mathcal{T}_\lambda Gr(2, \mathbb{C}_{\mathbf{d}}^m),$$
$$\alpha_2 = d\rho(\lambda, \underline{F})(0, (F_1', \ldots, F_m')) \qquad \text{with } F_i' \in S_{d_i}/\langle F_i\rangle.$$

Remark 2.28. — For each $(\lambda, \underline{F}) \in \mathcal{P}_2(\mathbf{d})$ we have an inclusion of vector spaces $\text{im}(d\rho(\lambda, \underline{F})) \subset \mathcal{T}_\omega \mathcal{F}_2(d, \mathbb{P}^n)$. In particular, the forms like α_1 and α_2 satisfy the perturbation equations (2.9) and (2.10).

Now, we want to distinguish the two introduced partial derivatives. With the notation of Section 2.1 (see our definitions in (2.5)), we will see they vanish on different strata $X_{\mathcal{D}_F}^k$ associated to the divisor $\mathcal{D}_F = (F = 0) = \bigcup_{i=1}^m X_i = (F_i = 0)$. Under the assumptions of Definition 2.24, each X_I is a smooth complete intersection of codimension $|I|$. Thus, the stratum $X_{\mathcal{D}_F}^k$ has codimension k and is singular along $X_{\mathcal{D}_F}^{k+1}$. We have the following results concerning these spaces.

PROPOSITION 2.29. — *For every $m > 1$ and $k \in \{1, \ldots, m\}$, the satura-ted homogeneous ideal $I^{(k)}$ associated to $X_{\mathcal{D}_F}^k$ is generated by $\{\hat{F}_J\}_{|J|=k-1}$. Hence we have a surjective map $\bigoplus_{|J|=k-1} \mathcal{O}_{\mathbb{P}^n}(-\hat{d}_J) \twoheadrightarrow \mathcal{I}_{X_{\mathcal{D}_F}^k}$, where $\mathcal{I}_{X_{\mathcal{D}_F}^k} \subset \mathcal{O}_{\mathbb{P}^n}$ denotes the ideal sheaf of regular functions vanishing on each stratum, and $\hat{d}_J = \sum_{j \notin J} d_j$.*

Proof. — We will proceed by induction on the total number of polynomi-als m, where $m \geqslant k$ is required. The base case is trivial, left to the reader. It is also clear that $I^{(k)} = \bigcap_{J:|J|=k} \langle F_{j_1}, \ldots, F_{j_k} \rangle$, and then we are reduced to prove that

$$\bigcap_{J \subset \{1,\ldots,m\}:|J|=k} \langle F_{j_1}, \ldots, F_{j_k} \rangle = \langle \hat{F}_J \rangle_{\substack{J \subset \{1,\ldots,m\}: \\ |J|=k-1}}.$$

One inclusion is always clear and does not require the inductive argument. Next, if we separate in the left term the multi-indexes of the intersection which do not contain m and use the inductive hypothesis, then we obtain

$$(2.14) \qquad I^{(k)} = \bigcap_{\substack{J \subset \{1,\ldots,m-1\}: \\ |J|=k-1}} \langle \hat{F}_{J \cup \{m\}} \rangle \cap \langle F_{j_1}, \ldots, F_{j_{k-1}}, F_m \rangle.$$

As a consequence, for every $P \in I^{(k)}$ there exists a family of polynomials $\{H_J\}_{J:|J|=k-1}$ such that

$$(2.15) \qquad P = \sum_{J:|J|=k-1} H_J \hat{F}_{J \cup \{m\}}.$$

For each $J_0 = \{j_1, \ldots, j_{k-1}\} \subset \{1, \ldots, m-1\}$ the class $[P]$ in the quotient ring $\mathbb{C}[z_0, \ldots, z_n] \big/ \langle F_{j_1}, \ldots, F_{j_{k-1}}, F_m \rangle$ is equal to zero. This is a conse-quence of formula (2.14). Hence we have $[H_{J_0}][\hat{F}_{J_0 \cup \{m\}}] = 0$. Finally, since the ring is integral and $F_l \notin \langle F_{j_1}, \ldots, F_{j_{k-1}}, F_m \rangle$, for every index $l \notin J_0$ distinct of m, we get $[H_{J_0}] = 0$. This fact applied to equality (2.15) allows us to show the other needed inclusion. $\qquad \square$

Remark 2.30. — If $\omega = \rho(\lambda, \underline{F})$ is a projective logarithmic q-form of type **d**, then $X_{\mathcal{D}_F}^{q+1}$ is contained in its singular set.

Remark 2.31. — Assume $m > 3$. With notation as in Remark 2.27, notice that α_1 vanishes on $X_{\mathcal{D}_F}^3$ and α_2 on $X_{\mathcal{D}_F}^4$, and so every element α in the image of $d\rho$ vanishes on $X_{\mathcal{D}_F}^4$.

3. Infinitesimal stability of a generic logarithmic 2-form

3.1. Main results

Our desired result can be summarized in the next theorem. For some technical reasons that will be explained later, the vector of degrees $\mathbf{d}$ is assumed to be 2-balanced (see Definition 3.18).

THEOREM 3.1. — *Fix natural numbers $n, m \in \mathbb{N}_{>3}$, and a 2-balanced vector of degrees $\mathbf{d} = (d_1, \ldots, d_m)$ with $d = \sum_i d_i$. The variety $\mathcal{L}_2(\mathbf{d}, n)$ is an irreducible component of the moduli space $\mathcal{F}_2(d, \mathbb{P}^n)$. Furthermore, the scheme $\mathcal{F}_2(d, \mathbb{P}^n)$ is generically reduced along this component, in particular at the points of $\rho(\mathcal{U}_2(\mathbf{d}))$.*

The proof of the above theorem is implied by the surjectivity of the derivative of the natural parametrization ρ (Proposition 3.2 below), combined with some arguments of scheme theory. This method is the same as the one used to prove [7, Theorem 2.1] and [6, Theorem 1]. Moreover, it was also used in [5, Theorem 8.2] for an alternative algebraic proof of the stability of projective logarithmic one forms. As a consequence, we will only present a proof of the corresponding proposition.

PROPOSITION 3.2. — *With the notation of Theorem 3.1, let $(\lambda, \underline{F}) \in \mathcal{U}_2(\mathbf{d})$ and $\omega = \rho(\lambda, \underline{F})$. Then the derivative*

$$\mathrm{d}\rho(\lambda, \underline{F}) : \mathcal{T}_\lambda Gr(2, \mathbb{C}^m_{\mathbf{d}}) \times \prod_{i=1}^m \mathcal{T}_{F_i} \mathbb{P} S_{d_i} \longrightarrow \mathcal{T}_\omega \mathcal{F}_2(d, \mathbb{P}^n)$$

is surjective.

Proof. — It will be attained in the following section.   □

Remark 3.3. — Most of our definitions and constructions concerning logarithmic q-forms of type $\mathbf{d} = (d_1, \ldots, d_m)$ assume $m \geqslant q + 1$. However, the techniques applied in the proof of Proposition 3.2 (and hence of Theorem 3.1) require $q = 2$ and $m > 3$. Notice that the case $m = q + 1$ corresponds to rational q-forms, and it is already known that they determine irreducible and generically reduced components of $\mathcal{F}_q(d, \mathbb{P}^n)$ (see [7]).

3.2. Surjectivity of the derivative of the natural parametrization

Let us now start with several steps towards the proof of Proposition 3.2.

3.2.1. Lemmas

We keep the generic conditions assumed in Definition 2.24. In particular, we fix homogeneous irreducible polynomials $F_1, \ldots, F_m$ with smooth normal crossings.

Remark 3.4. — With the normal crossing assumption, the next property is satisfied:

($*$) for $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$ and every point $x \in \mathcal{C}(X_I) - \{0\} \subset \mathbb{C}^{n+1}$, we have

$$\mathrm{d}_x F_{i_1} \wedge \cdots \wedge \mathrm{d}_x F_{i_k} \neq 0,$$

where $\mathcal{C}(X_I)$ denotes the affine cone over X_I and $\mathrm{d}_x F_j$ is the differential of F_j at x.

The following lemma is similar to that described in [7, Lemma 2.2], and also it is a consequence of Saito's Lemma [15] turned up for our purposes.

LEMMA 3.5 (Division Lemma). — *Assume* $n, m \in \mathbb{N}_{\geqslant 3}$, *and fix* $k \in \{1, 2\}$, *integers* j *and* q *with* $1 \leqslant j \leqslant q \leqslant n - 2$, *and* $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, m\}$. *If* $\mu \in H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d))$ *is a twisted* k-*form of total degree* d *such that in homogeneous coordinates satisfies*

$$(3.1) \qquad \left(\mu \wedge \mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_j} \right)|_{\mathcal{C}(X_I)} = 0,$$

then there exists forms $\{\gamma_r\}_{r=1}^j \subset H^0(\mathbb{C}^{n+1}, \Omega^{k-1}_{\mathbb{C}^{n+1}})$, *where each* γ_r *is a homogeneous affine form of total degree* $d - d_{i_r}$, *such that*

$$\mu|_{X_I} = \left(\sum_{r=1}^j \gamma_r \wedge \mathrm{d}F_{i_r} \right) \Big|_{X_I}.$$

Proof. — Notice that the restricted sheaf $\Omega^1_{\mathbb{C}^{n+1}}|_{\mathcal{C}(X_I)}$ is an $\mathcal{O}_{\mathcal{C}(X_I)}$-module freely generated on global sections according to: $\Omega^1_{\mathbb{C}^{n+1}}|_{\mathcal{C}(X_I)} = \mathcal{O}_{\mathcal{C}(X_I)} \cdot \mathrm{d}z_0|_{\mathcal{C}(X_I)} \oplus \cdots \oplus \mathcal{O}_{\mathcal{C}(X_I)} \cdot \mathrm{d}z_n|_{\mathcal{C}(X_I)}$. Moreover, the property ($*$) of the previous remark implies that the unique singularity of the j-form $\mathrm{d}F_{i_1}|_{\mathcal{C}(X_I)} \wedge \cdots \wedge \mathrm{d}F_{i_j}|_{\mathcal{C}(X_I)}$ is the point zero. Now, consider the ideal $\mathcal{A}$ generated by the coefficients $\{a_{l_1 \ldots l_j}\}$ determined by the decomposition

$$(\mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_j})|_{\mathcal{C}(X_I)} = \sum_{0 \leqslant l_1 < \cdots < l_j \leqslant n} a_{l_1 \ldots l_j} \mathrm{d}z_{l_1}|_{\mathcal{C}(X_I)} \wedge \cdots \wedge (\mathrm{d}z_{l_j})|_{\mathcal{C}(X_I)}.$$

The depth of $\mathcal{A}$ is greater or equal than three because of the normal crossings hypothesis. Hence, we are able to apply Saito's lemma (see [15]) to divide 1-forms and 2-forms restricted to $\mathcal{C}(X_I)$. From this lemma

and equation (3.1), we deduce the existence of homogeneous forms $\tilde{\gamma}_r$ in $H^0(\mathcal{C}(X_I), \Omega^{k-1}_{\mathbb{C}^{n+1}}|_{\mathcal{C}(X_I)})$ of degrees $d - d_{i_r}$ for $r = 1, \ldots, j$ such that

$$\mu|_{\mathcal{C}(X_I)} = \sum_{r=1}^{j} \tilde{\gamma}_r \wedge (\mathrm{d}F_{i_r}|_{\mathcal{C}(X_I)}).$$

Next, we need to construct global homogeneous forms $\{\gamma_r\}$ whose restriction to $\mathcal{C}(X_I)$ coincides with $\{\tilde{\gamma}_r\}$. When $k = 1$, each function $\tilde{\gamma}_r$ can be considered as an element of $H^0(X_I, \mathcal{O}_{X_I}(d - d_{i_r}))$. Since each X_I is a smooth complete intersection, the corresponding restriction map is surjective (see for instance [10, Exercise 5.5]). Using this fact, we deduce the existence of $\gamma_k \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d - d_{i_k}))$ as wanted. When $k = 2$, we have $\tilde{\gamma}_r = \sum_{s=0}^{n} A_s^r \mathrm{d}z_s|_{\mathcal{C}(X_I)}$, where $\{A_s^r\}_{r,s}$ are elements in $H^0(X_I, \mathcal{O}_{X_I}(d - d_{i_r} - 1))$. Finally, we may apply the previous procedure to each function A_s^r, and hence construct homogeneous global affine 1-forms and as claimed. $\square$

Remarks 3.6.

(1) The above proof does not use the fact that the form μ descends to the projective space, that is $i_R(\mu) = 0$. So the conclusion of Lemma 3.5 holds for every homogeneous affine form $\mu \in H^0(\mathbb{C}^{n+1}, \Omega^k_{\mathbb{C}^{n+1}})$ satisfying equation (3.1).

(2) The depth of the ideal $\mathcal{A}$ defined in the previous proof is greater or equal than the dimension of $\mathcal{C}(X_I)$, which is exactly $n + 1 - q$. Hence by an inductive argument, the previous result is in fact true for $k \leqslant n - q$.

Remark 3.7. — Let X be an algebraic variety and $Y \overset{i}{\hookrightarrow} X$ a subvariety whose sheaf of ideals is denoted by $\mathcal{I}_Y \subset \mathcal{O}_X$. Also let $\mathcal{E}$ be a locally free sheaf on X. Taking global sections on the exact sequence

$$0 \to \mathcal{E} \otimes \mathcal{I}_Y \to \mathcal{E} \to i_*(\mathcal{E} \otimes \mathcal{O}_Y) := \mathcal{E}|_Y \to 0,$$

we can identify the global sections of $\mathcal{E}$ vanishing on Y with the elements of $H^0(X, \mathcal{E} \otimes \mathcal{I}_Y)$.

LEMMA 3.8 (Vanishing Lemma). — *Let $\mu \in H^0(\mathbb{C}^{n+1}, \Omega^2_{\mathbb{C}^{n+1}})$ be a homogeneous affine 2-form of total degree d. If μ vanishes on the affine cone $\mathcal{C}(X^k_{\mathcal{D}_F})$ for some $k \in \mathbb{N}_{\geqslant 2}$, then*

$$\mu = \sum_{I:|I|=k-1} \hat{F}_I \, \mu_I$$

for some homogeneous affine 2-forms μ_I of degree $\sum_{i \in I} d_i$.

Proof. — The result is a consequence of Proposition 2.29 and the previous remark applied to $X = \mathbb{C}^{n+1}$, $Y = \mathcal{C}(X_{\mathcal{D}_F}^k)$ and $\mathcal{E} = \Omega_{\mathbb{C}^{n+1}}^2$. $\qquad\square$

LEMMA 3.9 (Fundamental Lemma). — *Assume $m \in \mathbb{N}_{>3}$. Consider a family of homogeneous polynomials $\{B_{ijk}\}$ whose indexes are selected on the set $\{1, \ldots, m\}$. Suppose $B_{ijk} = B_{jik}$ and $\deg(B_{ijk}) = d_k$. If these polynomials also satisfy the relations*

$$(3.2) \qquad B_{ijl}(x) = B_{ikl}(x) = B_{jkl}(x) \qquad \forall\, x \in X_{ijkl},$$

then for each $k \in \{1, \ldots, m\}$ there exists a polynomial $F_k' \in S_{d_k}$ such that

$$B_{ijk}(x) = F_k'(x) \qquad \forall\, x \in X_{ijk}.$$

Proof. — Fix two indexes $j, k \in \{1, \ldots, m\}$, write $X = X_{jk}$ and $\mathcal{D}_X$ for the restriction of the divisor $\mathcal{D}_F$ to X. The family $\{B_{ijk}\}_{i \neq j, k}$ determines a well defined object in $H^0(\mathcal{D}_X, \mathcal{O}_{\mathcal{D}_X}(d_k))$, by (3.2). Now, take into consideration the exact sequence

$$0 \to \mathcal{I}_{\mathcal{D}_X}(d_k) \to \mathcal{O}_X(d_k) \to i_*(\mathcal{O}_{\mathcal{D}_X})(d_k) \to 0,$$

and the associated long exact sequence in cohomology

$$0 \to H^0(X, \mathcal{I}_{\mathcal{D}_X}(d_k)) \to H^0(X, \mathcal{O}_X(d_k)) \to H^0(\mathcal{D}_X, \mathcal{O}_{\mathcal{D}_X}(d_k))$$
$$\to H^1(X, \mathcal{I}_{\mathcal{D}_X}(d_k)) \to \cdots$$

Observe that $\mathcal{D}_X = (\hat{F}_{jk}|_X = 0)$, and so $\mathcal{I}_{\mathcal{D}_X}(d_k) \simeq \mathcal{O}_X(2d_k + d_j - d)$. Moreover, since $H^1(X, \mathcal{O}_X(2d_k + d_j - d)) = 0$ (see again [10, Exercise 5.5]), the second morphism of the long sequence is surjective. Summarily, there exists an element $\tilde{H}_{jk} \in H^0(X, \mathcal{O}_X(d_k))$ whose restriction to $\mathcal{D}_X$ coincides with $\{B_{ijk}\}_{i \neq j, k}$.

With a similar argument, the restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_k)) \xrightarrow{\,|_X\,} H^0(X, \mathcal{O}_X(d_k))$$

is also surjective. Then, we can choose a global homogeneous polynomial H_{jk} such that $H_{jk} = B_{ijk}$ on X_{ijk} for all $i \neq j, k$. In addition, we have $H_{jk} = B_{ijk} = B_{jik} = H_{ik}$ on X_{ijk}. Hence, we have construct a family of homogeneous polynomials $\{H_{jk}\}_{j,k}$ with $\deg(H_{jk}) = d_k$ such that $H_{ik} = H_{jk}$ on every triple intersection X_{ijk}.

Finally, we need to apply again the previous procedure but now over the family $\{H_{jk}\}$ on the variety $X = X_k$ (see also [5, Proposition 8.9]). From this result we deduce the existence of $F_k' \in S_{d_k}$ such that $H_{jk} = F_k'$ on X_{jk} as claimed. $\qquad\square$

Remark 3.10. — The previous lemma is also true for families of polynomials supported on an arbitrary number of indexes, and the proof can be preformed by induction.

We end this section with the construction of certain rational vector fields dual, in some sense, to the homogeneous affine 1-forms $\{dF_i\}_{i=1}^m$.

Remark 3.11 (Fields with the δ-property). — Fix an integer $r < n$ and select $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}$. For each $j \in \{1, \ldots, r\}$ we can select, locally on the points of X_I, a rational vector field V_j^I such that

$$(3.3) \qquad i_{V_j^I}(dF_{i_k}) = \delta_{jk} \text{ on } X_I \quad \forall\, k \in \{1, \ldots, r\}.$$

A similar construction was used in [7, Proof of Proposition 3.1].

The correct objects to consider are the so called *logarithmic vector fields*. The formal definition of a logarithmic derivation along an hypersurface $\mathcal{D}$ of a complex algebraic variety X (with associated ideal sheaf $\mathcal{I}$) is the following:

$$\mathrm{Der}_X(\log \mathcal{D})_p = \{\chi \in (\mathrm{Der}_X)_p : \chi(\mathcal{I}_p) \subset \mathcal{I}_p\}.$$

We refer to [16] for more details and properties. In particular, we shall use that

$$\Omega^1_X(\log \mathcal{D})_p \text{ and } \mathrm{Der}_X(\log \mathcal{D})_p$$

are reflexive $\mathcal{O}_{X,p}$-modules, dual to each other. Also, when the divisor has simple normal crossings we are able to apply Saito's criterion ([16] p. 270). If we fix local coordinates $f_1, \ldots, f_n$ and assume that the divisor is defined by the zero locus of $f_1 \ldots f_s$, then $\frac{df_1}{f_1}, \ldots, \frac{df_s}{f_s}, df_{s+1}, \ldots, df_n$ is a free system of generators of $\Omega^1_X(\log \mathcal{D})_p$. And the local fields $f_1 \cdot \frac{\partial}{\partial f_1}, \ldots, f_s \cdot \frac{\partial}{\partial f_s}, \frac{\partial}{\partial f_{s+1}}, \ldots, \frac{\partial}{\partial f_n}$ determines a dual basis of $\mathrm{Der}_X(\log \mathcal{D})_p$.

In conclusion, for the simple normal crossings projective divisor $\mathcal{D}_F$ determined by the zero locus of $\prod_{i=1}^m F_i$, it is possible to construct vector fields V_j^I with the property (3.3) as claimed.

3.2.2. Beginning of the proof of Proposition 3.2. Steps 1 to 4

Note. — The hypothesis of $\mathbf{d}$ being 2-balanced will be introduced only when necessary.

As before, fix $n, m \in \mathbb{N}_{>3}$ and a partition $\mathbf{d}$ of d into m parts. Let $(\lambda^1 \wedge \lambda^2, (F_i)_{i=1}^m) = (\lambda, \underline{F}) \in \mathcal{U}_2(\mathbf{d})$ and $\omega = \rho(\lambda, \underline{F}) = \sum_{i \neq j} \lambda_{ij} \hat{F}_{ij} dF_i \wedge dF_j$. Consider $\alpha \in \mathcal{T}_\omega(\mathcal{F}_2(d, \mathbb{P}^n))$, i.e., a projective form $\alpha \in H^0(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}(d))/\langle \omega \rangle$ satisfying equations (2.9) and (2.10). We want to prove that there exists

an element $(\lambda', (F_i')_{i=1}^m) \in \mathcal{T}_{(\lambda,\underline{F})}\mathcal{U}_2(\mathbf{d})$ such that $\mathrm{d}\rho(\lambda, \underline{F})(\lambda', (F_i')_{i=1}^m) = \alpha$. Using formula (2.13), the problem is equivalent to show that

$$\alpha = \sum_{i \neq j} \lambda'_{ij} \hat{F}_{ij} \mathrm{d}F_i \wedge \mathrm{d}F_j + \sum_{i \neq j \neq k} \lambda_{ij} \hat{F}_{ijk} F'_k \mathrm{d}F_i \wedge \mathrm{d}F_j + 2 \sum_{i \neq j} \lambda_{ij} \hat{F}_{ij} \mathrm{d}F'_i \wedge \mathrm{d}F_j.$$

In the sake of clarity, we will separate the proof in several steps related to the possible vanishing of α on each stratum $X^k_{\mathcal{D}_F}$. First, we shall prove that α vanishes on $X^4_{\mathcal{D}_F}$, and describe its restriction to $X^3_{\mathcal{D}_F}$.

PROPOSITION 3.12 (Step 1). — *If α is a Zariski tangent vector of $\mathcal{F}_2(d, \mathbb{P}^n)$ at ω, then $\alpha|_{X^4_{\mathcal{D}_F}} = 0$. Moreover, the following decomposition holds:*

$$\alpha = \sum_{i \neq j \neq k} \hat{F}_{ijk} A_{ijk} \mathrm{d}F_i \wedge \mathrm{d}F_j + \varepsilon,$$

for some homogeneous polynomials $\{A_{ijk}\}$ with $A_{ijk} = -A_{jik}$, and $\varepsilon \in H^0(\mathbb{C}^{n+1}, \Omega^2_{\mathbb{C}^{n+1}})$ a homogeneous form of total degree d satisfying $\varepsilon|_{X^3_{\mathcal{D}_F}} = 0$.

Proof. — Since $X^3_{\mathcal{D}_F}$ is contained in the singular set of ω, we have $i_v(\omega)|_{X_{ijk}} = 0$ for each vector v and every piece X_{ijk} of $X^3_{\mathcal{D}_F}$. Also, for every i, j and k notice that

$$\mathrm{d}\omega|_{X_{ijk}} = ((\lambda_{ij} - \lambda_{ik} + \lambda_{jk})\hat{F}_{ijk}\mathrm{d}F_i \wedge \mathrm{d}F_j \wedge \mathrm{d}F_k)|_{X_{ijk}}.$$

Then, the integrability perturbation equation $i_v(\alpha) \wedge \mathrm{d}\omega + i_v(\omega) \wedge \mathrm{d}\alpha = 0$ reduces to $i_v\alpha \wedge ((\lambda_{ij} - \lambda_{ik} + \lambda_{jk})\hat{F}_{ijk}\mathrm{d}F_i \wedge \mathrm{d}F_j \wedge \mathrm{d}F_k) = 0$ on X_{ijk}. Taking into consideration the definition of $\mathcal{U}_2(\mathbf{d})$ we get

$$(3.4) \qquad i_v\alpha \wedge \mathrm{d}F_i \wedge \mathrm{d}F_j \wedge \mathrm{d}F_k = 0 \quad \text{on } X_{ijk} - X^4_{\mathcal{D}_F}.$$

If we select certain local bases at the points of X_{ijk} as in Remark 3.11 turned up for this particular case, then after some straightforward calculations we obtain

$$\alpha \wedge \mathrm{d}F_i \wedge \mathrm{d}F_j \wedge \mathrm{d}F_k = 0 \quad \text{on } X_{ijk}.$$

Now, we are able to apply Lemma 3.5 (division lemma), and deduce that

$$(3.5) \qquad \alpha = \gamma_i \wedge \mathrm{d}F_i + \gamma_j \wedge \mathrm{d}F_j + \gamma_k \wedge \mathrm{d}F_k \quad \text{on } X_{ijk},$$

for some homogeneous forms $\gamma_l \in H^0(\mathbb{C}^{n+1}, \Omega^1_{\mathbb{C}^{n+1}})$ of total degree $d - d_l$. Again, we need to use fields with the δ-property developed in Remark 3.11. In particular, we choose a rational vector field Y^j_{ijk} such that

$$(i_{Y^j_{ijk}}(\mathrm{d}F_j))|_{X_{ijk}} = 1 \quad \text{and} \quad (i_{Y^j_{ijk}}(\mathrm{d}F_i))|_{X_{ijk}} = (i_{Y^j_{ijk}}(\mathrm{d}F_k))|_{X_{ijk}} = 0.$$

Therefore, equation (3.4) combined with decomposition (3.5) implies that

$$\gamma_j \wedge \mathrm{d}F_i \wedge \mathrm{d}F_j \wedge \mathrm{d}F_k = 0,$$

at the points of X_{ijk}. We can also permute indexes to deduce the same condition for γ_i and γ_k. Applying again the division lemma, we finally achieve the following formula:

$$(3.6) \quad \alpha = G_{ijk}\mathrm{d}F_i \wedge \mathrm{d}F_j + G_{jki}\mathrm{d}F_j \wedge \mathrm{d}F_k + G_{ikj}\mathrm{d}F_i \wedge \mathrm{d}F_k \quad \text{on } X_{ijk},$$

where $\{G_{rst}\}_{r,s,t}$ are homogeneous polynomials of the correct degree.

Now, if we fix another index l and compare the previous decompositions on their restriction to $X_{ijkl} = X_{ijk} \cap X_{jkl} \cap X_{ikl}$, then we get

$$\alpha|_{X_{ijkl}} = (G_{ijk}\mathrm{d}F_i \wedge \mathrm{d}F_j + G_{jki}\mathrm{d}F_j \wedge \mathrm{d}F_k + G_{ikj}\mathrm{d}F_i \wedge \mathrm{d}F_k)|_{X_{ijkl}}$$
$$= (G_{jkl}\mathrm{d}F_j \wedge \mathrm{d}F_k + G_{klj}\mathrm{d}F_k \wedge \mathrm{d}F_l + G_{jlk}\mathrm{d}F_j \wedge \mathrm{d}F_l)|_{X_{ijkl}}.$$

Due to the normal crossings hypothesis, the restricted forms $\mathrm{d}F_i|_{X_{ijkl}}$, $\mathrm{d}F_j|_{X_{ijkl}}$, $\mathrm{d}F_k|_{X_{ijkl}}$ and $\mathrm{d}F_l|_{X_{ijkl}}$ are linearly independent at every point of X_{ijkl}. Hence, we have $G_{ijk}|_{X_{ijkl}} = 0$ for all $l \neq i, j, k$. As a consequence, for every selection of indexes i, j and k there exists a homogeneous polynomial A_{ijk} such that $G_{ijk} = \hat{F}_{ijk}A_{ijk}$ on X_{ijk}. It is easy to check that both families $\{G_{ijk}\}$ and $\{A_{ijk}\}$ can be selected antisymmetric in the first two indexes.

Finally, notice that α and $\sum \hat{F}_{ijk}A_{ijk}\mathrm{d}F_i \wedge \mathrm{d}F_j$ have the same restriction to each X_{ijk}. Hence the form $\varepsilon = \alpha - \sum \hat{F}_{ijk}A_{ijk}\mathrm{d}F_i \wedge \mathrm{d}F_j$ is a homogeneous affine form of total degree d that vanishes on $X^3_{\mathcal{D}_F}$, as claimed. □

COROLLARY 3.13. — *Any Zariski tangent vector $\alpha \in \mathcal{T}_\omega \mathcal{F}_2(d, \mathbb{P}^n)$ may be written as*

$$\alpha = \sum_{i \neq j \neq k} \hat{F}_{ijk}A_{ijk}\mathrm{d}F_i \wedge \mathrm{d}F_j + \sum_{i < j} \hat{F}_{ij}\varepsilon_{ij}$$

for some homogeneous affine 2-forms $\{\varepsilon_{ij}\}$.

Proof. — It follows from Proposition 3.12 (step 1) and Lemma 3.8 (vanishing lemma). □

The next step deals with the existence of the expected polynomials "F_i'". We would like to obtain some equations for the polynomials $\{A_{ijk}\}$ of step 1 to deduce that each A_{ijk}/λ_{ij} only depends on the index k.

PROPOSITION 3.14 (Step 2). — *With the notation of Proposition 3.12, define $B_{ijk} = A_{ijk}/\lambda_{ij}$ for $i \neq j \neq k$. Then, these new polynomials necessarily satisfy the relations*

$$B_{ijl}(x) = B_{jkl}(x) = B_{ikl}(x) \quad \forall\, x \in X_{ijkl}.$$

Proof. — We will fix in some order four indexes i_0, j_0, k_0 and l_0 to deduce the desired conditions on $X_{i_0j_0k_0l_0}$.

First, select j_0 and l_0, and use the construction of Remark 3.11 to select a local rational vector field $Y_{j_0} := Y^{j_0}_{j_0 l_0}$ with the corresponding δ-property. For simplicity, we will work separately on the two terms of the restricted perturbation equation

$$(3.7) \qquad (i_{Y_{j_0}}(\alpha) \wedge \mathrm{d}\omega)|_{X_{j_0 l_0}} + (i_{Y_{j_0}}(\omega) \wedge \mathrm{d}\alpha)|_{X_{j_0 l_0}} = 0.$$

From Corollary 3.13 we have

$$i_{Y_{j_0}}(\alpha) = \sum_{i \neq j \neq k} 2\hat{F}_{ijk} A_{ijk} i_{Y_{j_0}}(\mathrm{d}F_i)\mathrm{d}F_j + \sum_{i<j} \hat{F}_{ij} i_{Y_{j_0}}(\varepsilon_{ij}).$$

Hence its restriction to $X_{j_0 l_0}$ is

$$\sum_k 2\hat{F}_{j_0 l_0 k} A_{l_0 j_0} \mathrm{d}F_{l_0} + \sum_i 2\hat{F}_{i j_0 l_0} A_{i j_0 l_0} i_{Y_{j_0}}(\mathrm{d}F_i)\mathrm{d}F_{j_0}$$

$$+ \sum_i 2\hat{F}_{i j_0 l_0} A_{i l_0 j_0} i_{Y_{j_0}}(\mathrm{d}F_i)\mathrm{d}F_{l_0} + \sum_j 2\hat{F}_{j_0 j l_0} A_{j_0 j l_0} \mathrm{d}F_j$$

$$+ \hat{F}_{j_0 l_0} i_{Y_{j_0}}(\varepsilon_{j_0 l_0}).$$

Also notice that

$$(\mathrm{d}\omega)|_{X_{j_0 l_0}} = \left(\sum_{i \neq j \neq k} \lambda_{ij} \hat{F}_{ijk} \mathrm{d}F_i \wedge \mathrm{d}F_j \wedge \mathrm{d}F_k \right)\Bigg|_{X_{j_0 l_0}}$$

$$= (\chi_{j_0 l_0} \wedge \mathrm{d}F_{j_0} \wedge \mathrm{d}F_{l_0})|_{X_{j_0 l_0}}$$

for some homogeneous affine 1-form $\chi_{j_0 l_0}$. As a consequence, we get the following description of the whole *first term* on its restriction to $X_{j_0 l_0}$:

$$i_{Y_{j_0}}(\alpha) \wedge \mathrm{d}\omega = \sum_{\substack{i \neq j \neq k \neq r \\ r \neq j_0, l_0}} 2\lambda_{ij} A_{j_0 r l_0} \hat{F}_{j_0 r l_0} \hat{F}_{ijk} \mathrm{d}F_r \wedge \mathrm{d}F_i \wedge \mathrm{d}F_j \wedge \mathrm{d}F_k$$

$$+ 2\hat{F}_{j_0 l_0} i_{Y_{j_0}}(\varepsilon_{j_0, l_0}) \wedge \chi_{j_0 l_0} \wedge \mathrm{d}F_{j_0} \wedge \mathrm{d}F_{l_0},$$

where $\hat{F}_{j_0 r l_0} \hat{F}_{ijk}$ can be replaced by $\hat{F}_{j_0 l_0} \hat{F}_{ijkr}$.

Now, we ought to make the same process on the other term of expression (3.7). In this case, observe that

$$(i_{Y_{j_0}}(\omega))|_{X_{j_0 l_0}} = \left(\sum_{i \neq j} 2\lambda_{ij} \hat{F}_{ij} i_{Y_{j_0}}(\mathrm{d}F_i)\mathrm{d}F_j \right)\Bigg|_{X_{j_0 l_0}}$$

$$= (2\lambda_{j_0 l_0} \hat{F}_{j_0 l_0} \mathrm{d}F_{l_0})|_{X_{j_0 l_0}}.$$

After some straight forward calculations we obtain that the restriction of $i_{Y_{j_0}}(\omega) \wedge \mathrm{d}\alpha$ to $X_{j_0 l_0}$ coincides with the restriction of the form

$$2\lambda_{j_0 l_0} \hat{F}_{j_0 l_0} \mathrm{d}F_{l_0} \wedge \left(\sum_{i \neq j \neq k \neq r} A_{ijk} \hat{F}_{ijkr} \mathrm{d}F_i \wedge \mathrm{d}F_j \wedge \mathrm{d}F_r \right.$$

$$\left. + \sum_{i \neq j \neq k} \hat{F}_{ijk} \mathrm{d}A_{ijk} \wedge \mathrm{d}F_i \wedge \mathrm{d}F_j + \sum_{\substack{i \neq j \neq k \\ i < j}} \hat{F}_{ijk} \mathrm{d}F_k \wedge \varepsilon_{ij} + \sum_{i<j} \hat{F}_{ij} \mathrm{d}\varepsilon_{ij} \right).$$

Next, we need to add the two term's expressions obtained. Also, notice that we are allowed to cancel the polynomial factor $\hat{F}_{j_0 l_0}$. Then, select two more distinct indexes i_0 and k_0, and further restrict to $X_{i_0 j_0 k_0 l_0}$ to obtain

$$((\lambda_{j_0 k_0} + \lambda_{k_0 l_0})A_{j_0 i_0 l_0} + (\lambda_{i_0 j_0} + \lambda_{l_0 i_0})A_{j_0 k_0 l_0} + \lambda_{j_0 l_0} A_{i_0 k_0 l_0}) \, \hat{F}_{i_0 j_0 k_0 l_0}$$
$$\mathrm{d}F_{i_0} \wedge \mathrm{d}F_{j_0} \wedge \mathrm{d}F_{k_0} \wedge \mathrm{d}F_{l_0} = 0.$$

Due to the normal crossings hypothesis, the definition of the polynomials $\{B_{ijk}\}$ and the order in which the indexes were selected, we have produced the following equation on $X_{i_0 j_0 k_0 l_0}$:

$$(\lambda_{j_0 k_0} \lambda_{j_0 i_0} + \lambda_{k_0 l_0} \lambda_{j_0 i_0}) B_{j_0 i_0 l_0} + (\lambda_{i_0 j_0} \lambda_{j_0 k_0} + \lambda_{l_0 i_0} \lambda_{j_0 k_0}) B_{j_0 k_0 l_0}$$
$$+ \lambda_{j_0 l_0} \lambda_{i_0 k_0} B_{i_0 k_0 l_0} = 0,$$

from now denoted by $[Eq_{j_0 l_0 i_0 k_0}]$. Since $A_{ijk} = -A_{jik}$, if we consider the sum of equations $[Eq_{i_0 l_0 j_0 k_0}]$, $[2Eq_{j_0 l_0 i_0 k_0}]$ and $[Eq_{k_0 l_0 i_0 j_0}]$, then we have

$$(\lambda_{i_0 j_0} - \lambda_{i_0 k_0} + \lambda_{j_0 k_0})(B_{i_0 j_0 l_0} - B_{j_0 k_0 l_0}) = 0$$

on $X_{i_0 j_0 k_0 l_0}$ as wanted. Finally, the other equality follows from an appropriated permutation of indexes. $\square$

In the next step, we deduce that any tangent vector at ω may be decompose as a sum of a perturbation in the image of $\mathrm{d}\rho$ that vanishes on $X^4_{\mathcal{D}_F}$, with another tangent vector that vanishes on a stratum of lower codimension.

PROPOSITION 3.15 (Step 3). — *For every $\alpha \in \mathcal{T}_\omega \mathcal{F}_2(d, \mathbb{P}^n)$, there exists a family of homogeneous polynomials $F'_1, \ldots, F'_m$ of degrees $d_1, \ldots, d_m$ and another tangent vector $\beta \in \mathcal{T}_\omega \mathcal{F}_2(d, \mathbb{P}^n)$ with $\beta|_{X^3_{\mathcal{D}_F}} = 0$, such that*

$$\alpha = \sum_{i \neq j \neq k} \lambda_{ij} \hat{F}_{ijk} F'_k \mathrm{d}F_i \wedge \mathrm{d}F_j + \sum_{i \neq j} 2\lambda_{ij} \hat{F}_{ij} \mathrm{d}F'_i \wedge \mathrm{d}F_j + \beta$$
$$= \mathrm{d}\rho(\lambda, \underline{F})(0, (F'_i)^m_{i=1}) + \beta.$$

Proof. — First, we use Propositions 3.12 and 3.14 (steps 1 and 2) to get the following decomposition for α:

$$\alpha = \sum_{i \neq j \neq k} \lambda_{ij} \hat{F}_{ijk} B_{ijk} \mathrm{d}F_i \wedge \mathrm{d}F_j + \varepsilon,$$

where $\{B_{ijk}\}$ satisfy the relations $B_{ijl} = B_{jkl} = B_{ikl}$ on X_{ijkl}, and $\varepsilon|_{X^3_{\mathcal{D}_F}} = 0$. By Lemma 3.9 (fundamental lemma), there exist homogeneous polynomials $F'_1, \ldots, F'_m$ of respective degrees $d_1, \ldots, d_m$ such that $B_{ijk} = F'_k$ on X_{ijk}. Then, it is clear that the forms α and $\sum_{i \neq j \neq k} \lambda_{ij} \hat{F}_{ijk} F'_k \mathrm{d}F_i \wedge \mathrm{d}F_j$ have the same restriction to $X^3_{\mathcal{D}_F}$. Moreover, if we add and subtract a suitable term, then we obtain

$$\alpha = \sum_{i \neq j \neq k} \lambda_{ij} \hat{F}_{ijk} F'_k \mathrm{d}F_i \wedge \mathrm{d}F_j + \sum_{i \neq j} 2\lambda_{ij} \hat{F}_{ij} \mathrm{d}F'_i \wedge \mathrm{d}F_j + \beta$$
$$= \mathrm{d}\rho(\lambda, \underline{F})(0, (F'_i)^m_{i=1}) + \beta,$$

where β is a homogeneous affine form such that $\beta|_{X^3_{\mathcal{D}_F}} = 0$. Finally, since $\mathrm{d}\rho(0, (F'_i))$ and α are Zariski tangent vectors at ω, the same condition holds for β. $\qquad\square$

Remark 3.16. — From now on, we are reduced to prove our result for elements $\beta \in \mathcal{T}_\omega \mathcal{F}(d, \mathbb{P}^n)$ with the additional hypothesis $\beta|_{X^3_{\mathcal{D}_F}} = 0$. In advanced, taking into consideration Remark 2.27, these forms are expected to be related to perturbations of the coefficients λ. This fact is going to be true only assuming certain extra condition among $\mathbf{d}$ (balanced case).

Next proposition sets the background to end the proof, and is useful to understand the possible trouble in the non balanced case.

By Lemma 3.8, if $\beta \in \mathcal{T}_\omega \mathcal{F}(d, \mathbb{P}^n)$ vanishes on $X^3_{\mathcal{D}_F}$, then it may be written as

$$\beta = \sum_{i \neq j} \hat{F}_{ij} \beta_{ij},$$

for some homogeneous affine forms $\{\beta_{ij}\}$ such that $\beta_{ij} = \beta_{ji}$. Now, we would like to obtain further information on these new forms β_{ij}.

PROPOSITION 3.17 (Step 4). — *With the notations above, for each selection of i_0, j_0 and k_0 there exist $\lambda'_{i_0 j_0} \in \mathbb{C}$ and homogeneous polynomials $B^{i_0 j_0}_{i_0 k_0}$ and $B^{i_0 j_0}_{j_0 k_0}$ such that*

$$\beta_{i_0 j_0} = \lambda'_{i_0 j_0} \mathrm{d}F_{i_0} \wedge \mathrm{d}F_{j_0} + \hat{F}_{i_0 j_0 k_0} (B^{i_0 j_0}_{i_0 k_0} \mathrm{d}F_{i_0} \wedge \mathrm{d}F_{k_0} + B^{i_0 j_0}_{j_0 k_0} \mathrm{d}F_{j_0} \wedge \mathrm{d}F_{k_0})$$

on $X_{i_0 j_0 k_0}$.

Proof. — We will follow a similar idea to that used in steps 1 and 2. Fix i_0 and j_0, and use Remark 3.11 to select a rational vector field $Y_{j_0} = Y_{j_0}^{i_0 j_0}$ with the corresponding δ-property. From the integrability perturbation equation restricted to $X_{i_0 j_0}$ we have

$$\sum_{i \neq j \neq k} 2\lambda_{ij} \hat{F}_{i_0 j_0} \hat{F}_{ijk} \, i_{Y_{j_0}} (\beta_{i_0 j_0}) \wedge dF_i \wedge dF_j \wedge dF_k$$
$$+ \sum_{i \neq j \neq k} 2\lambda_{j_0 i_0} \hat{F}_{i_0 j_0} \hat{F}_{ijk} dF_{i_0} \wedge dF_k \wedge \beta_{ij} + 4\lambda_{j_0 i_0} \hat{F}_{i_0 j_0} \hat{F}_{i_0 j_0} dF_{i_0} \wedge d\beta_{i_0 j_0} = 0.$$

Afterward, we can remove the factor $\hat{F}_{i_0 j_0}$, select another index k_0, restrict the equation to $X_{i_0 j_0 k_0}$ and obtain

$$(3.8) \quad (\lambda_{i_0 j_0} - \lambda_{i_0 k_0} + \lambda_{j_0 k_0}) \hat{F}_{i_0 j_0 k_0} i_{Y_{j_0}} (\beta_{i_0 j_0}) dF_{i_0} \wedge dF_{j_0} \wedge dF_{k_0}$$
$$+ 2\lambda_{j_0 i_0} \hat{F}_{i_0 j_0 k_0} dF_{i_0} \wedge dF_{k_0} \wedge \beta_{i_0 j_0} + 2\lambda_{j_0 i_0} \hat{F}_{i_0 j_0 k_0} dF_{i_0} \wedge dF_{j_0} \wedge \beta_{i_0 k_0} = 0.$$

If we take the wedge product by $dF_{j_0}|_{X_{i_0 j_0 k_0}}$, then we deduce

$$(\beta_{i_0 j_0} \wedge dF_{i_0} \wedge dF_{j_0} \wedge dF_{k_0})|_{X_{i_0 j_0 k_0}} = 0.$$

Also, the same conclusion holds for $\beta_{i_0 k_0}$ and $\beta_{j_0 k_0}$. From Lemma 3.5 (division lemma) we have

$$(\beta_{i_0 j_0})|_{X_{i_0 j_0 k_0}} = \mu_{k_0}^{i_0 j_0 k_0} \wedge dF_{k_0} + \mu_{j_0}^{i_0 j_0 k_0} \wedge dF_{j_0} + \mu_{i_0}^{i_0 j_0 k_0} \wedge dF_{i_0}$$

for some homogeneous affine forms $\mu_l^{i_0 j_0 k_0}$ of degree $d_{i_0} + d_{j_0} - d_l$.

Now, we want to prove that these new forms are also divisible by dF_{i_0}, dF_{j_0} and dF_{k_0}. Select a new rational vector field $Z_{j_0} := Y_{j_0}^{i_0 j_0 k_0}$ with the corresponding δ-property, and replace the above decomposition for $\beta_{i_0 j_0}$ into (3.8) to obtain

$$((\lambda_{i_0 j_0} - \lambda_{j_0 k_0} + \lambda_{i_0 k_0}) \mu_{j_0}^{i_0 j_0 k_0} - 2\lambda_{i_0 j_0} \mu_{k_0}^{i_0 k_0 j_0}) \wedge dF_{i_0} \wedge dF_{j_0} \wedge dF_{k_0} = 0.$$

For each $l \in \{i_0, j_0, k_0\}$ define $\gamma_l^{i_0 j_0 k_0} := \mu_l^{i_0 j_0 k_0} \wedge dF_{i_0} \wedge dF_{j_0} \wedge dF_{k_0}$ (similarly with $\gamma_l^{i_0 k_0 j_0}$ and $\gamma_l^{j_0 k_0 i_0}$). Next, our last equation may be written as

$$Eq(I_{i_0 j_0 k_0}) : \quad (\lambda_{i_0 j_0} - \lambda_{j_0 k_0} + \lambda_{i_0 k_0}) \gamma_{j_0}^{i_0 j_0 k_0} - 2\lambda_{i_0 j_0} \gamma_{k_0}^{i_0 k_0 j_0} = 0,$$

where the tag $Eq(I_{i_0 j_0 k_0})$ refers to the order in which indexes were selected. Permuting these indexes, we can construct a linear system of equations in order to deduce our claim. In particular, from $Eq(I_{i_0 j_0 k_0}) + Eq(I_{k_0 j_0 i_0})$ we deduce $\gamma_{k_0}^{i_0 k_0 j_0} = \gamma_{j_0}^{i_0 j_0 k_0}$. Then, using again $Eq(I_{i_0 j_0 k_0})$ we get $\gamma_{j_0}^{i_0 j_0 k_0} = 0$ on $X_{i_0 j_0 k_0}$. With a similar argument we can also prove the other vanishing conditions for the forms $\gamma_l^{i_0 j_0 k_0}$.

Now, we are able to apply again the division lemma to the forms $\mu_l^{i_0 j_0 k_0}$, and obtain the following decomposition for the original form $\beta_{i_0 j_0}$ on $X_{i_0 j_0 k_0}$:

$$(3.9) \quad \beta_{i_0 j_0} = A_{i_0 j_0 k_0}^{i_0 j_0} \mathrm{d}F_{i_0} \wedge \mathrm{d}F_{j_0} + A_{i_0 k_0 j_0}^{i_0 j_0} \mathrm{d}F_{i_0} \wedge \mathrm{d}F_{k_0} + A_{j_0 k_0 i_0}^{i_0 j_0} \mathrm{d}F_{j_0} \wedge \mathrm{d}F_{k_0}.$$

If we fix another index l_0, then all the possible decompositions for $\beta_{i_0 j_0}$ must coincide in the intersection $X_{i_0 j_0 k_0 l_0} = X_{i_0 j_0 k_0} \cap X_{i_0 j_0 l_0}$. In particular, we have

$$A_{i_0 j_0 k_0}^{i_0 j_0} \mathrm{d}F_{i_0} \wedge \mathrm{d}F_{j_0} + A_{i_0 k_0 j_0}^{i_0 j_0} \mathrm{d}F_{i_0} \wedge \mathrm{d}F_{k_0} + A_{j_0 k_0 i_0}^{i_0 j_0} \mathrm{d}F_{j_0} \wedge \mathrm{d}F_{k_0}$$

$$= A_{i_0 j_0 l_0}^{i_0 j_0} \mathrm{d}F_{i_0} \wedge \mathrm{d}F_{j_0} + A_{i_0 l_0 j_0}^{i_0 j_0} \mathrm{d}F_{i_0} \wedge \mathrm{d}F_{l_0} + A_{j_0 l_0 i_0}^{i_0 j_0} \mathrm{d}F_{j_0} \wedge \mathrm{d}F_{l_0}.$$

From the normal crossings hypothesis we deduce

$$A_{i_0 j_0 k_0}^{i_0 j_0} = A_{i_0 j_0 l_0}^{i_0 j_0} \quad \text{and} \quad A_{i_0 k_0 j_0}^{i_0 j_0} = A_{j_0 k_0 l_0}^{i_0 j_0} = 0 \quad \text{on } X_{i_0 j_0 k_0 l_0}.$$

The first condition implies that $A_{i_0 j_0 k_0}^{i_0 j_0}$ does not depend on k_0. Notice that its degree equals to zero, and so we write $\lambda'_{i_0 j_0} := A_{i_0 j_0 k_0}^{i_0 j_0}$. On the other hand, from the second condition we have

$$A_{i_0 k_0 j_0}^{i_0 j_0} = \hat{F}_{i_0 j_0 k_0} B_{i_0 k_0}^{i_0 j_0} \quad \text{and} \quad A_{j_0 k_0 i_0}^{i_0 j_0} = \hat{F}_{i_0 j_0 k_0} B_{j_0 k_0}^{i_0 j_0}$$

on $X_{i_0 j_0 k_0}$, for some homogeneous polynomials $B_{i_0 k_0}^{i_0 j_0}$ and $B_{j_0 k_0}^{i_0 j_0}$ of degrees $\deg(B_{i_0 k_0}^{i_0 j_0}) = 2d_{j_0} + d_{i_0} - d$ and $\deg(B_{j_0 k_0}^{i_0 j_0}) = 2d_{i_0} + d_{j_0} - d$. Finally, our claim follows from (3.9). $\square$

3.2.3. The balanced assumption and end of the proof. Steps 5 to 7

We need the following definition in order to restrict the possible degrees of the polynomials $B_{i_0 k_0}^{i_0 j_0}$ and $B_{j_0 k_0}^{i_0 j_0}$ introduced in the previous step.

DEFINITION 3.18. — *We say that an m-tuple of degrees $\mathbf{d} = (d_1, \ldots, d_m)$ is k-balanced if for each $I \subset \{1, \cdots, m\}$ of size $|I| = k$, the following inequality holds:*

$$(3.10) \qquad \sum_{i \in I} d_i = d_{i_1} + \cdots + d_{i_k} < \sum_{l \notin I} d_l.$$

Example 3.19. — If all the possible degrees are equal to 1, i.e., $\mathbf{d} = (1, \ldots, 1)$, then $\mathbf{d}$ is k-balanced if and only if $2k < m$.

Remark 3.20. — When $k = 1$, condition (3.10) is the same that appears in [9, Corollaries 5.10 and 5.11] and [5, Definition 8.16].

Remark 3.21. — If a vector $\mathbf{d}$ is k-balanced then it is k'-balanced for all k' lower than k. Although, the converse of this fact is trivially not true (not even for large m). For example:

$$\mathbf{d} = \left(1, 2, \ldots, m-2, \frac{(m-2)(m-1)}{2}, \frac{(m-2)(m-1)}{2}\right)$$

is 1-balanced but not 2-balanced for every $m > 3$.

Now, we are ready to continue with the following step.

PROPOSITION 3.22 (Step 5). — *Suppose $\mathbf{d}$ is 2-balanced. Then, for any $\beta \in \mathcal{T}_\omega\mathcal{F}_2(d, \mathbb{P}^n)$ that vanishes on $X^3_{\mathcal{D}_F}$ there exist constants $\{\lambda'_{ij}\}_{i \neq j}$ and a homogeneous affine form $\gamma \in H^0(\mathbb{C}^{n+1}, \Omega^2_{\mathbb{C}^{n+1}})$ of total degree d, with $\gamma|_{X^2_{\mathcal{D}_F}} = 0$, such that*

$$\beta = \sum_{i \neq j} \lambda'_{ij} \hat{F}_{ij} \mathrm{d}F_i \wedge \mathrm{d}F_j + \gamma.$$

Proof. — From Proposition 3.17 (step 4) we know that $\beta = \sum_{i \neq j} \hat{F}_{ij}\beta_{ij}$, where

$$\beta_{ij} = \lambda'_{ij}\mathrm{d}F_i \wedge \mathrm{d}F_j + \hat{F}_{ijk}(B^{ij}_{ik}\mathrm{d}F_i \wedge \mathrm{d}F_k + B^{ij}_{jk}\mathrm{d}F_j \wedge \mathrm{d}F_k) \quad \text{on } X_{ijk}.$$

Since $\mathbf{d}$ is 2-balanced, computing degrees we deduce that the polynomials B^{ij}_{ik} and B^{ij}_{jk} must be equal to zero. Then, for every i, j and k we have

$$(3.11) \qquad\qquad \beta_{ij} = \lambda'_{ij}\mathrm{d}F_i \wedge \mathrm{d}F_j \quad \text{on } X_{ijk}.$$

Observe that the form $(\beta_{ij} - \lambda'_{ij}\mathrm{d}F_i \wedge \mathrm{d}F_j)|_{X_{ij}}$ vanishes on the divisor $(\hat{F}_{ij}|_{X_{ij}} = 0)$. Hence, with a slight modification of the vanishing lemma (Lemma 3.8), we get $\beta_{ij} - \lambda'_{ij}\mathrm{d}F_i \wedge \mathrm{d}F_j = \hat{F}_{ij}\mu_{ij}$ on X_{ij}, for some $\mu_{ij} \in H^0(\mathbb{C}^{n+1}, \Omega^2_{\mathbb{C}^{n+1}})$ homogeneous form of degree $d_i + d_j - \sum_{k \neq i,j} d_k$. Using again our hypothesis among $\mathbf{d}$, we deduce that equality (3.11) holds in X_{ij}.

Finally, observe that β and $\sum_{i \neq j} \lambda_{ij}\hat{F}_{ij}\mathrm{d}F_i \wedge \mathrm{d}F_j$ have the same restriction to $X^2_{\mathcal{D}_F}$, and hence their difference vanishes on this stratum. $\qquad\square$

Remark 3.23. — With the notations above, notice that the forms β and $\sum_{i \neq j} \lambda'_{ij}\hat{F}_{ij}\mathrm{d}F_i \wedge \mathrm{d}F_j$ satisfy the integrability perturbation equation (2.10), and therefore the same holds for γ. But, a priori, we can not assume that γ satisfies the other perturbation equation (2.9).

PROPOSITION 3.24 (Step 6). — *Assume $\mathbf{d}$ is 2-balanced. If γ is a homogeneous affine 2-form of degree d satisfying the integrability perturbation equation (2.10) and $\gamma|_{X^2_{\mathcal{D}_F}} = 0$, then $\gamma = 0$.*

Proof. — Using the vanishing lemma, we get $\gamma = \sum_l \hat{F}_l \gamma_l$, for some homogeneous forms $\gamma_l \in H^0(\mathbb{C}^{n+1}, \Omega^2_{\mathbb{C}^{n+1}})$ of total degree d_l.

With the same idea as in other steps, we fix i_0 and j_0 distinct, and choose a rational local vector field $Y_{j_0} = Y^{j_0}_{i_0 j_0}$ with the corresponding δ-property. In this case, the restriction of the integrability perturbation equation to $X_{i_0 j_0}$ reduces to

$$\lambda_{j_0 i_0}(\hat{F}_{i_0 j_0})^2 \mathrm{d}F_{i_0} \wedge \mathrm{d}F_{j_0} \wedge \gamma_{i_0} = 0.$$

Then it follows from the division lemma that

$$\gamma_{i_0} = \mu_{i_0 j_0} \wedge \mathrm{d}F_{i_0} + \nu_{i_0 j_0} \wedge \mathrm{d}F_{j_0} \quad \text{on} \quad X_{i_0 j_0},$$

for certain homogeneous affine forms $\mu_{i_0 j_0}$ and $\nu_{i_0 j_0}$. But since $\mathrm{d}F_{i_0}$ and γ_{i_0} have the same degree, we deduce that $\mu_{i_0 j_0} = 0$. Now, select a new index k_0, and notice that $\gamma_{i_0} = \nu_{i_0 j_0} \wedge \mathrm{d}F_{j_0} = \nu_{i_0 k_0} \wedge \mathrm{d}F_{k_0}$ on $X_{i_0 j_0 k_0}$. If we take the wedge product by $\mathrm{d}F_{k_0}$, then we get $\nu_{i_0 j_0} \wedge \mathrm{d}F_{j_0} \wedge \mathrm{d}F_{k_0} = 0$. Again by the division lemma, we can select $A_{i_0 j_0 k_0}$ such that

$$\gamma_{i_0} = A_{i_0 j_0 k_0} \mathrm{d}F_{j_0} \wedge \mathrm{d}F_{k_0} \quad \text{on} \quad X_{i_0 j_0 k_0}.$$

Furthermore, $A_{i_0 j_0 k_0}$ is divisible by $\hat{F}_{i_0 j_0 k_0}$. This is just a consequence of the equality

$$A_{i_0 j_0 k_0} \mathrm{d}F_{j_0} \wedge \mathrm{d}F_{k_0} = A_{i_0 j_0 l} \mathrm{d}F_{j_0} \wedge \mathrm{d}F_l \quad \forall \, l \neq i_0, j_0, k_0$$

and the normal crossings hypothesis. Then, we obtain

$$\gamma_{i_0} = \hat{F}_{i_0 j_0 k_0} B_{i_0 j_0 k_0} \mathrm{d}F_{j_0} \wedge \mathrm{d}F_{k_0}$$

on $X_{i_0 j_0 k_0}$, for some new homogeneous polynomials $B_{i_0 j_0 k_0}$ of degree $d_{i_0} - \sum_{l \neq i_0} d_l$. But notice that this degree is negative because $\mathbf{d}$ is also 1-balanced (see Remark 3.21). As a consequence, $\gamma_i|_{X_{ijk}} = 0$ for all $i \neq j \neq k$.

Finally, for each index i, we deduce that γ_i vanishes on the subvariety of $X^3_{\mathcal{D}_F}$ defined by $\bigcup_{j,k \neq i} X_{ijk}$. It is also clear that its corresponding homogeneous saturated ideal is generated by $\{\hat{F}_{ij}\}_{j \neq i}$. With a slight modification of the vanishing lemma (Lemma 3.8), we have $\gamma_i = \sum_{j \neq i} \hat{F}_{ij} \gamma_{ij}$, for some homogeneous affine forms γ_{ij}. Since $\mathbf{d}$ is 2-balanced, each form γ_i must be equal to zero, and hence the claim. $\qquad\square$

The next proposition summarizes the last step of the proof. We finally show that each Zariski tangent vector at ω vanishing on $X^3_{\mathcal{D}_F}$ lies in the image of $\mathrm{d}\rho(\lambda, \underline{F})$.

PROPOSITION 3.25 (Step 7). — *Suppose* $\mathbf{d}$ *is 2-balanced, and let* $\omega = \rho(\lambda, \underline{F}) = \sum \lambda_{ij} \hat{F}_{ij} \mathrm{d}F_i \wedge \mathrm{d}F_j$ *be a logarithmic 2-form of type* $\mathbf{d}$, *with* $\lambda \in Gr(2, \mathbb{C}_{\mathbf{d}}^m)$. *If* β *is a Zariski tangent vector of* $\mathcal{F}_2(d, \mathbb{P}^n)$ *at* ω *and also* $\beta|_{X_{\mathcal{D}_F}^3} = 0$, *then there exists* $\lambda' \in \mathcal{T}_\lambda Gr(2, \mathbb{C}_{\mathbf{d}}^m)$ *such that*

$$\beta = \sum_{i \neq j} \lambda'_{ij} \hat{F}_{ij} \mathrm{d}F_i \wedge \mathrm{d}F_j = \mathrm{d}\rho(\lambda, \underline{F})(\lambda', 0).$$

Proof. — We can use Propositions 3.22 and 3.24 (steps 5 and 6) to deduce that any Zariski tangent vector with our hypotheses may be written as $\beta = \sum_{i \neq j} \lambda'_{ij} \hat{F}_{ij} \mathrm{d}F_i \wedge \mathrm{d}F_j$. Notice that $i_R(\beta) = 0$ if and only if $i_{\mathbf{d}}(\lambda') = 0$. Hence it is easy to see that $\lambda' = (\lambda'_{ij})$ can be considered as an element of $\bigwedge^2(\mathbb{C}_{\mathbf{d}}^m)/\langle\lambda\rangle$. Also, recall from Section 2.4 that $\mathcal{T}_\lambda Gr(2, \mathbb{C}_{\mathbf{d}}^m)$ can be represented by vectors $\lambda' \in \bigwedge^2(\mathbb{C}_{\mathbf{d}}^m)/\langle\lambda\rangle$ such that $\lambda \wedge \lambda' = 0$. Now, from the decomposability perturbation equation (2.9), that is $\beta \wedge \omega = 0$, we get

$$F^2 \cdot \sum_{i \neq j \neq k \neq l} (\lambda' \wedge \lambda)_{ijkl} \frac{\mathrm{d}F_i}{F_i} \wedge \frac{\mathrm{d}F_j}{F_j} \wedge \frac{\mathrm{d}F_k}{F_k} \wedge \frac{\mathrm{d}F_l}{F_l} = 0.$$

Then from Corollary 2.8 it follows that $\lambda' \wedge \lambda = 0$, and this implies our claim. $\qquad\square$

COROLLARY 3.26. — *Combining Propositions 3.15 and 3.25 (steps 3 and 7) we conclude the whole proof of the surjectivity of* $\mathrm{d}\rho$ *(Proposition 3.2), which also implies our Theorem 3.1.*

COROLLARY 3.27. — *Assume* $m, n \in \mathbb{N}_{>4}$. *From Example 3.19 we deduce that a generic linear logarithmic 2-form is stable. In other words,* $\mathcal{L}_2(\mathbf{d}, n)$ *determines an irreducible component of* $\mathcal{F}_2(d, \mathbb{P}^n)$ *when* $\mathbf{d} = (1, \ldots, 1)$. *Furthermore, observe that any logarithmic form of type* $\mathbf{d} = (d_1, \ldots, d_m)$ *in* $\mathbb{P}^n$ *is the pullback of a linear logarithmic form by a quasi-homogeneous rational map.*

Remark 3.28. — We expect that Theorem 3.1 could be also proved for a larger codimension q with $2 \leqslant q < n - 1$ and $m > q + 1$. It seems that the argument could be performed using the same proof schema, studying the restriction of a Zariski tangent vector to each stratum $X_{\mathcal{D}_F}^k$ for $k \leqslant q + 2$. In fact, it is possible to see that each Zariski tangent vector of $\mathcal{F}_q(d, \mathbb{P}^n)$ at a logarithmic q-form of type $\mathbf{d}$ as in Definition 2.21 necessarily vanishes on $X_{\mathcal{D}_F}^{q+2}$. However, most of the steps of the proof of Proposition 3.2 are quite technical for $q > 2$ and require new combinatorial ideas, even if we assume that $\mathbf{d}$ is q-balanced. In addition, the open question stated in Remark 2.20 seems to be important for obtaining a generalized proof.

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