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THE PLATEAU PROBLEM FOR CONVEX CURVATURE FUNCTIONS

by Graham SMITH

Abstract. — We present a novel and comprehensive approach to the study of the parametric Plateau problem for locally strictly convex (LSC) hypersurfaces of prescribed curvature for general convex curvature functions inside general Riemannian manifolds. We prove existence of solutions to the Plateau problem with outer barrier for LSC hypersurfaces of constant or prescribed curvature for general curvature functions inside general Hadamard manifolds modulo a single scalar condition. In particular, convex curvature functions of bounded type are fully treated.

Résumé. — Nous étudions le problème de Plateau paramétrique dans des variétés riemanniennes générales pour des hypersurfaces localement strictement convexes (LSC) et à courbure prescrite pour une classe générale de fonctions de courbure convexes. Nous établissons une condition scalaire pour l'existence de solutions dans le cas où il existe une barrière externe et la variété ambiante est une variété d'Hadamard

1. Introduction

1.1. Non-linear curvature functions

In the classical theory of hypersurfaces one usually studies functions of the principal curvatures, such as their mean or product which yield respectively the mean curvature and extrinsic curvature of the hypersurface. However, many other such "curvature functions" have also been formulated and studied. In order to correctly appreciate the results of this paper and their historical context, it is worthwhile spending some time recalling the general framework of curvature functions as laid down elegantly by Caffarelli–Nirenberg–Spruck in [3] and [4]

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Let $\Lambda_+ := \Lambda^n_+ \subseteq \mathbb{R}^n$ be the open cone of vectors all of whose components are positive. Let $\Lambda := \Lambda^n \subseteq \mathbb{R}^n$ be another open cone such that:

(1) for all $x := (x_1, \ldots, x_n) \in \Lambda$ and for every permutation σ, $x_\sigma := (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ is also an element of Λ;
(2) for all $x \in \Lambda$ and for all $t > 0$, tx is also an element of Λ (in particular, the vertex of Λ lies on the origin);
(3) for all $x \in \Lambda$ and for all $y \in \Lambda_+$, $x + y$ is also an element of Λ; and
(4) Λ is convex.

A *non-linear curvature function* is a non-negative function

$$K \in C^\infty(\Lambda) \cap C^0(\overline{\Lambda})$$

such that:

(A) K is *invariant* in the sense that for all $x \in \Lambda$ and for every permutation σ, $K(x_\sigma) = K(x)$;
(B) K is *homogeneous* of order 1;
(C) K is *normalised* in the sense that $K(1, \ldots, 1) = 1$; and
(D) K is *compatible* with Λ in the sense that it is strictly positive over Λ and vanishes over $\partial\Lambda$.

Scalar notions of curvature for suitable classes of hypersurfaces are defined using non-linear curvature functions. Indeed, let $M := M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold and let $\Sigma := (i, (S, \partial S))$ be a smooth *immersed hypersurface* in M. We recall that this means that $(S, \partial S)$ is an n-dimensional manifold (possibly with boundary) and $i : S \to M$ is a smooth immersion. Let $\kappa := (\kappa_1, \ldots, \kappa_n)$ be its vector of principal curvatures. We say that Σ is *strictly K-convex* whenever $\kappa(p)$ is an element of Λ for all p in S. When Σ is strictly K-convex, we define the *K-curvature* of Σ, $K(\Sigma) : S \to]0, \infty[$, by:

$$K(\Sigma) = K(\kappa_1, \ldots, \kappa_n).$$

We now see that invariance ensures that both strict K-convexity and K-curvature are well-defined; homogeneity ensures that K-curvature transforms in a familiar manner under rescaling of the metric over M; and the normalisation condition ensures that the K-curvature of a unit sphere in Euclidean space is equal to 1. Finally, compatibility ensures that no sequence $(x_n)_{n\in\mathbb{N}}$ of points in Λ with $(K(x_n))_{n\in\mathbb{N}}$ uniformly bounded below has a limit point on $\partial\Lambda$. In geometric terms, this means that in order for a smooth limit of strictly K-convex hypersurfaces to be also strictly K-convex, it is sufficient to ensure that the K-curvatures of all hypersurfaces in the sequence are uniformly bounded below. In other words, strict K-convexity is a closed property, modulo a scalar condition that is easy to verify.

For PDE reasons, it is usual to require that a curvature function K satisfy, in addition, the following two conditions:

(E) K is *strictly elliptic* in the sense that for all $x \in \Lambda$ and for all $1 \leqslant i \leqslant n$, $\partial_i K(x) > 0$; and

(F) K is a *concave* function over Λ.

We recall that the Jacobi operator of K-curvature over any strictly K-convex hypersurface measures the infinitesimal variation of the K-curvature arising from an infinitesimal normal perturbation of the immersion.[(1)] The Jacobi operator is always a second order partial differential operator. Ellipticity of K ensures that the Jacobi operator is elliptic for every smooth strictly K-convex immersed hypersurface. Finally, the condition of concavity is natural from the perspective of non-linear elliptic PDEs, and unavoidable at the current level of technology (cf. [1, 3]).

Examples. — Setting $\Lambda = \{x \mid x_1 + \cdots + x_n > 0\}$ and $K(x) = (x_1 + \cdots + x_n)/n$, we recover the notions of local strict mean convexity and mean curvature. Setting $\Lambda = \Lambda_+ = \{x \mid x_i > 0 \ \ \forall\, i\}$ and $K(x) = (x_1 \cdot \ldots \cdot x_n)^{1/n}$, we recover the notions of local strict convexity and extrinsic curvature. More generally, for $1 \leqslant k < n$, define $f_k : \mathbb{R}^n \to \mathbb{R}$ by:

$$f_k(x) = \frac{1}{n!} \sum_{\text{permutations } \sigma} x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(k)},$$

define $\Lambda_k := \{x \mid f_1(x), \ldots, f_k(x) > 0\}$ and define $\sigma_k : \Lambda_k \to]0, \infty[$ by $\sigma_k(x) := f_k(x)^{1/k}$. With this definition of (Λ_k, σ_k), we recover the notions of local strict k-convexity and σ_k-curvature. In this notation, σ_1-curvature is mean curvature, σ_n-curvature is extrinsic curvature and, when the ambient space is $\mathbb{R}^n$, σ_2-curvature coincides with the square root of the scalar curvature of the immersed hypersurface.

K is said to be a *convex curvature function* whenever $\Lambda = \Lambda_+$. We say that a smooth immersed hypersurface is *locally strictly convex* (LSC) whenever it is strictly Λ_+-convex. Convex curvature functions are of particular

(1) More precisely, let $\Sigma := (i, S)$ be an immersed hypersurface in M. Let N be the upward-pointing unit normal vector field over Σ. For $\phi \in C_0^\infty(S)$, we define the family $(i_t)_{t \in]-\epsilon,\epsilon[}$ of immersions from S into M by $i_t(p) = \mathrm{Exp}(t\phi(p)N(p))$, where Exp is the exponential map of M. For all t, let $K_t : S \to \mathbb{R}$ be such that $K_t(p)$ is the K-curvature of i_t at p. The Jacobi operator J of K-curvature over Σ is defined such that for all such ϕ, $J\phi = \partial_t K_t|_{t=0}$.

interest as they are considerably more tractable to mathematical analysis than more general non-linear curvature functions. Indeed, whilst very general Plateau problems can be solved for convex curvature functions (in the literature discussed below, as well as in the current paper), comparatively little is known about Plateau problems for non-convex non-linear curvature functions when the desired hypersurface is anything other than a graph. In this paper, therefore, *we will only treat smooth LSC hypersurfaces and convex curvature functions* and we henceforth write Λ instead of Λ_+.

Examples. — Extrinsic curvature $K := \sigma_n$ is a convex curvature function. For all $1 \leqslant k < n$, the curvature quotient $K := \sigma_{n,k} := \sigma_n/\sigma_k$ is a convex curvature function (cf. Proposition 2.2, below). Special Lagrangian curvature $K := \rho_{(n-1)\pi/2}$ is a convex curvature function (cf. [19] and [14]). If K_1 and K_2 are both convex curvature functions, then any *convex sum* or *convex product* of K_1 and K_2 given respectively by $\alpha K_1 + (1-\alpha)K_2$ and $K_1^\alpha K_2^{1-\alpha}$ for $\alpha \in [0,1]$ is also a convex curvature function. Any non-trivial convex product of a convex curvature function with any other curvature function is also a convex curvature function.

1.2. The non-linear Plateau problem

Let $M := M^{n+1}$ be a complete $(n+1)$-dimensional Riemannian manifold. Let K be a convex curvature function. Let $\Gamma := (i, G)$ be a smooth compact codimension-2 immersed submanifold in M. The (convex) *non-linear Plateau problem* asks for the existence of a smooth compact LSC immersed hypersurface $\Sigma := (i, (S, \partial S))$ in M with boundary equal to Γ and with K-curvature, $K(\Sigma)(p)$ equal to some positive constant κ, say, for all $p \in S$. More generally, for a given smooth function $\kappa : M \to]0,\infty[$, we ask for the existence of a smooth compact LSC immersed hypersurface Σ such that $\partial\Sigma = \Gamma$ and $K(\Sigma)(p) = (\kappa \circ i)(p)$ for all $p \in S$. When this latter condition is satisfied, we say that the K-curvature of Σ is *prescribed* by κ, and we write $K(\Sigma) = \kappa$.

The non-linear Plateau problem differs from the linear case in that Γ is usually required to be the boundary of a K-convex outer barrier, $\hat{\Sigma}$, with K-curvature at all points greater than k (this is the geometric analogue of the PDE concept of supersolution). Furthermore, in treating the non-linear Plateau problem, it becomes necessary to separate the family of convex curvature functions into two classes as follows. Following Trudinger (cf. [23]), for $K : \Lambda^n \to]0,\infty[$ a convex curvature function, we define

$K_\infty : \Lambda^{n-1} \to]0,\infty]$ by:

$$K_\infty(x_1,\ldots,x_{n-1}) = \lim_{t\to\infty} K(x_1,\ldots,x_{n-1},t).$$

Since K is concave, K_∞ is either everywhere infinite or everywhere finite and concave (cf. Proposition 2.1, below). We say that K is of *unbounded type* in the former case and of *bounded type* in the latter. Convex curvature functions of bounded type present an extra level of analytic complexity and are thus much less studied in the literature.

Examples. — Extrinsic curvature $K := \sigma_n$ is of unbounded type. The curvature quotients $K := \sigma_{n,k}$ discussed in the preceeding section are of bounded type (cf. Proposition 2.2, below). Special Lagrangrian curvature $K := \rho_{(n-1)\pi/2}$ is of bounded type (cf. [19] and [14]). Any convex sum or product of two convex curvature functions of bounded or unbounded type is also respectively of bounded or unbounded type. Any non-trivial convex sum or product of a convex curvature function of bounded type with a convex curvature function of unbounded type is of unbounded type.

An extensive literature has grown around the non-linear Plateau problem with outer barrier over the past three decades. The interested reader may consult our text [21] for an in-depth study of the state of the art in the case of extrinsic curvature. The subject was essentially opened to modern mathematical analysis in the 1980's with the application by Caffarelli–Nirenberg–Spruck in [4] of the celebrated barrier technique first introduced to the study of non-linear boundary value problems by the same authors in [2] (cf. also [10]). This allowed them to prove existence of solutions to the non-linear Plateau problems with outer barrier in $\mathbb{R}^{n+1}$ for (not necessarily convex) non-linear curvature functions of unbounded type. Importantly, however, they only treat the case where Γ is the boundary of a strictly convex subset of a hyperplane and where the solution is a graph over this hyperplane. This is referred to as the *non-parametric* case, as a specific parametrisation of the solution is provided by the geometry of the problem. Much subsequent work aimed to remove this restrictive geometric condition and thus treat the so-called *parametric* case. The most notable result of this period was the existence theorem [6] of Guan–Spruck for solutions to the non-linear Plateau problem with outer barrier in $\mathbb{R}^{n+1}$ for LSC graphs over hyperspheres. Spruck then conjectured in his presentation to the 1994 ICM (cf. [22]) the existence of solutions to the parametric non-linear Plateau problem with outer barrier for hypersurfaces of constant extrinsic curvature in $\mathbb{R}^{n+1}$. This conjecture was solved simultaneously using identical techniques by Guan–Spruck in [7] and Trudinger–Wang in [24].

It was then extended by Guan–Spruck in [8] to include all convex curvature functions of unbounded type, and by Sheng–Urbas–Wang in [13] to include the curvature quotients $K := \sigma_{n,k}$ discussed in the preceeding section. The case of general convex curvature functions has remained to date unstudied.

The recent results of Guan–Spruck, Trudinger–Wang and subsequent authors are obtained by supplementing Caffarelli–Nirenberg–Spruck's barrier technique with a Perron-type argument. This beautiful approach suffers nonetheless from two significant limitations. First, the Perron method yields no information concerning the uniqueness of the solutions obtained, and second, more significantly, by requiring in a fundamental manner the existence of large families of complete, totally geodesic hypersurfaces, it cannot be applied when the ambient manifold is anything other than a space-form, with the exception of a handful of cases where extra properties of the curvature function used compensate for this limitation (such as the case of special Lagrangian curvature, cf. [19] and [18]). Nonetheless, in a different direction, using completely different techniques, Labourie showed in [11] the existence of unique solutions to the non-linear Plateau problem with outer barrier for surfaces of constant extrinsic curvature immersed inside 3-dimensional Hadamard manifolds of sectional curvature bounded above by -1. However, these techniques, involving an ingenious application of Gromov's theory of pseudo-holomorphic curves, do not extend to higher-dimensional ambient spaces.

In summary, the existing results to date leave open the following three problems, in order of significance: firstly, to extend the result of Guan–Spruck, Trudinger–Wang and Labourie to general ambient spaces of arbitrary dimension; secondly, to extend these results to general convex curvatures functions, including the case of curvature functions of bounded type; and, thirdly, to prove the uniqueness or otherwise of the solutions obtained. In [17] and [20], with the aim of addressing these open problems, we initiated a programme of developing a fully geometric approach to the original barrier technique of Caffarelli–Nirenberg–Spruck as presented in [3] and [4]. By developing furthermore a parametric Smale-type degree theory in the spirit of the work [25] and [26] of White, we extend the results of Guan–Spruck, Trudinger–Wang and Labourie to solve the parametric Plateau problem with outer barrer for hypersurfaces of prescribed extrinsic curvature inside general Hadamard manifolds of arbitrary dimension. Furthermore, although we do not, strictly speaking, prove uniqueness, the differential topological degree that we obtain nonetheless guarantees that under generic conditions, the number of solutions *counted with algebraic sign* is equal to 1.

The current paper concerns the final stage of this programme which involves extending the existence result of [20] to the parametric non-linear Plateau problem with outer barrier for general convex curvature functions inside general manifolds. To this end, we prove a-priori estimates for the norms of the shape operators of hypersurfaces of prescribed K-curvature. The challenge here is two-fold, but in each case involves a highly non-trivial reworking of Caffarelli–Nirenberg–Spruck's barrier arguments. The first challenge lies in the fact that the barrier functions used to date, being non-geometric in nature, become unworkably complex in general ambient spaces. In the case of extrinsic curvature, the multiplicative nature of the determinant function allows us to remove most of the problematic terms. However, for general convex curvature functions, we no longer have this luxury. We found that the best approach involves a complete reconstruction of Caffarelli–Nirenberg–Spruck's original barrier functions in terms of natural geometric concepts. We find the resulting barrier arguments very satisfying in their generality as well as their relative simplicity. The second challenge concerns convex curvature functions of bounded type which have not hitherto been treated in full generality. However, by once again reconstructing in terms of natural geometric objects Caffarelli–Nirenberg–Spruck's original barrier argument for the non-linear *Dirichlet* problem (cf. [3]), we successfully eliminate the extra complexities that arise in this case, thus solving, in a relatively straightforward manner, the non-linear Plateau problem for all convex curvature functions inside general Hadamard manifolds, modulo a single scalar condition on the curvature function in question. It is an interesting open question to know to what extent this condition is necessary.

Finally, the techniques used here do not restrict to Hadamard manifolds. In forthcoming work, we aim to review the straightforward geometric conditions required to extend these results to Plateau problems inside suitable open subsets of general Riemannian manifolds.

1.3. Main results

Given a convex curvature function, K, we define $\mu_\infty(K)$ by:

$$\mu_\infty(K) = \liminf_{\|x\|=1, x\to\partial\Gamma} DK_x(1,\ldots,1). \tag{1.1}$$

For all K, $\mu_\infty(K) > 1$ (cf. Proposition 6.3, below). Furthermore, when K is of unbounded type, $\mu_\infty(K) = \infty$ (cf. Proposition 6.4, below).

Given a smooth compact codimension-2 immersed submanifold $\Gamma := (i, G)$ in M, we say that Γ is *generic* whenever $T_p\Gamma \neq T_q\Gamma$ for all distinct pairs of points p and q in G. Observe that this condition is weaker than transversality. Our main result is:

THEOREM 1.1. — *Let K be a convex curvature function; let M be an $(n+1)$-dimensional Hadamard manifold; let $\kappa : M \to]0, \infty[$ be a smooth function; and let $\hat{\Sigma}$ be a smooth compact LSC immersed hypersurface in M. Suppose that:*

(1) *$K(\hat{\Sigma}) > \kappa$; and*
(2) *$\partial\hat{\Sigma}$ is generic.*

Suppose, furthermore, that there exists a point $p \in M$ and $R > 0$ such that:

(3) *$\hat{\Sigma} \subseteq B_R(p)$; and*
(4) *$\kappa(q) < \frac{1}{R}\mu_\infty(K)$ for all $q \in B_R(p)$.*

There exists a smooth compact LSC immersed hypersurface Σ in M such that:

(a) *$\partial\Sigma = \partial\hat{\Sigma}$;*
(b) *$\Sigma < \hat{\Sigma}$; and*
(c) *$K(\Sigma) = \kappa$.*

Remark. — In particular, when K is of unbounded type, Conditions (3) and (4) are trivially satisfied for all p and for all sufficiently large R.

Remark. — The notation $\Sigma < \hat{\Sigma}$ is explained in detail in Section 3.1 and illustrated in Figures 3.1 and 3.2, below. In particular, if $\Sigma < \hat{\Sigma}$ then $\partial\Sigma = \partial\hat{\Sigma}$ and Σ lies on the "inside" of $\hat{\Sigma}$.

Remark. — The fact that $\mu_\infty(K) > 1$ has the following pleasing consequence. Let $B_r(0)$ be the ball of radius r about the origin in $\mathbb{R}^{n+1}$. Let $\hat{\Sigma}$ be an open subset in $\partial B_r(0)$ with smooth boundary. For all $\kappa \in]0, 1/r]$ let $\mathcal{S}_\kappa$ be the family of smooth compact LSC immersed hypersurfaces Σ in $B_r(0)$ such that $K(\Sigma) = \kappa$ and $\Sigma < \hat{\Sigma}$. By Proposition 2.4, below, $\mu_\infty(K) \geqslant 1$, and it follows by ellipticity that $\mathcal{S}_\kappa$ varies continuously with $\kappa \in]0, 1/r[$. Since $\mu_\infty(K)$ is in fact greater than 1, our compactness results show that this family in fact various continuously even at $\kappa = 1/r$. Furthermore, using standard barrier techniques, we show that $\hat{\Sigma}$ is the only element of $\mathcal{S}_{1/r}$, so that $(\mathcal{S}_\kappa)_{\kappa \in]0,1/r[}$ converges to $\{\hat{\Sigma}\}$ as κ tends to $1/r$. In other words, the solutions approach the data as κ tends to $1/r$. This would not necessarily hold if we did not have strict inequality in the above relation.

Using a slightly different approach, we obtain the following complementary result:

THEOREM 1.2. — *Let K be a convex curvature function; let M be an $(n+1)$-dimensional Hadamard manifold of sectional curvature bounded above by -1; let $\kappa : M \to]0,1[$ be a smooth function; and let $\hat{\Sigma}$ be a smooth compact LSC immersed hypersurface in M. Suppose that:*

(1) $K(\hat{\Sigma}) > \kappa$*; and*
(2) $\partial\hat{\Sigma}$ *is generic.*

There exists a smooth compact LSC immersed hypersurface Σ in M such that:

(a) $\partial\Sigma = \partial\hat{\Sigma}$*;*
(b) $\Sigma < \hat{\Sigma}$*; and*
(c) $K(\Sigma) = \kappa$*.*

Concerning uniqueness, as indicated above, the degree theory of [12] should readily adapt to the current framework of compact hypersurfaces with boundary. It would then follow that the number of solutions (counted algebraically) would be equal to 1 for generic $\hat{\Sigma}$. In any case, in certain situations uniqueness can be obtained, essentially because the contribution of every solution to the degree is positive, although this perspective is not necessary to obtain the result:

THEOREM 1.3. — *Let K be a convex curvature function. Suppose that, for all $x := (x_1, \ldots, x_n) \in \Lambda$ such that $K(x_1, \ldots, x_n) \leqslant 1$ the derivative of K satisfies:*

$$DK_x \cdot (1, \ldots, 1) \geqslant DK_x \cdot (x_1^2, \ldots, x_2^2).$$

Then the solution obtained in Theorem 1.2 is unique.

Remark. — This property is satisfied by the curvature quotients $K := \sigma_{n,n-1}$ and $K := \sigma_{n,n-2}$ (cf. [9]), by special Lagrangian curvature (cf. [16, Lemma 7.4]) and by Gaussian curvature when the hypersurface is 2-dimensional (cf. [11, Proposition 3.2.1]).

1.4. Conventions and acknowledgements

All immersed submanifolds will be oriented. For an immersed hypersurface Σ, we say that a unit normal vector field N over Σ points *upward* whenever it is compatible with the orientation of Σ. We say it points *downward* otherwise. If Σ is LSC and lies on the boundary of a convex set, then the orientation of Σ is chosen such that the convex set lies below Σ. If Γ is the boundary of Σ, then the *right* hand side of Γ is the side on which Σ lies, and the *left* hand side is the other side.

The first draft of this paper was written in response to a question asked by Joel Spruck at the conference, "Algebraic, Geometric and Analytical Aspects of Surface Theory", held in Buzios, Brazil in April 2010. The main results of this paper first appeared in their current form (with slightly stronger hypotheses) in the second draft which has been available on arXiv since October 2010. The author is grateful to the IMPA, Rio de Janeiro, Brazil, for providing the excellent conditions required to write the second draft. Finally, Sylvestre Gallot, Lucio Rodriguez and Harold Rosenberg are warmly thanked for helpful comments made to this and earlier drafts, as well as for their continual encouragement.

2. Convex Curvature Functions

2.1. Convex curvature functions

Proposition 2.1. — *If K is a convex curvature function, then K_∞ is either everywhere infinite, or everywhere finite. Moreover, if K_∞ is everywhere finite, then:*

(1) *K_∞ is invariant;*
(2) *K_∞ is homogeneous of order 1;*
(3) *$K_\infty(1,\ldots,1) > 1$;*
(4) *K_∞ is (possibly non-strictly) elliptic; and*
(5) *K_∞ is concave;*

Remark. — Importantly, K_∞ is not necessarily continuous over $\partial\Lambda^{n-1}$.

Proof. — The limit of an increasing family of concave functions over an open set is either everywhere infinite, or everywhere finite and concave. This proves the main assertion and (5). (1), (2) and (4) follow trivially. Finally $K_\infty(1,\ldots,1) > K(1,\ldots,1) = 1$. (3) follows, and this completes the proof. □

For $1 \leqslant k < n$, let $\sigma_{n,k}$ be the curvature quotient as defined in the introduction.

Proposition 2.2. — *For $1 \leqslant k < n$, $\sigma_{n,k}$ possesses Properties* (A) *to* (F) *listed in Section 1.1 and is of bounded type.*

Proof. — Denote $K := \sigma_{n,k}$. (A) follows from the definition. (B) and (C) are trivial. Define $\Phi :]0,\infty[^n \to]0,\infty[^n$ by:

$$\Phi(x_1,\ldots,x_n) = (x^{-1},\ldots,x^{-n}).$$

Then:

$$K\circ\Phi = \sigma_l^{-1/l},$$

where $l = n-k$. Trivially, if any component of $x \in]0,\infty]^n$ is infinite, then so is $\sigma_l(x)$ and so K extends to a continuous function over $\overline{\Lambda}$ which vanishes over $\partial\Lambda$. This proves (D). For $x \in]0,\infty[^n$, if we denote $y = \Phi(x)$, then, for each i, by the chain rule:

$$(\partial_i K)(y) = c_1 y_i^2 \sigma_l^{-(l+1)/l}\sigma_{l-1}(x_1,\ldots,\hat{x}_i,\ldots,x_n),$$

for some positive constant c_1. This is trivially positive, and (E) follows. (F) is proven in [3]. Choose $(y_1,\ldots,y_n) \in]0,\infty[^{n-1}$. Trivially:

$$\lim_{t\to 0}\sigma_l(y_1,\ldots,y_{n-1},t) = c_2\sigma_l(y_1,\ldots,y_{n-1}),$$

for some positive constant c_2. It follows that $K_\infty = c_2\sigma_{n-1,k-1}$. In particular, K is of bounded type. This completes the proof. □

We relate convex curvature functions to $O(n)$-invariant functions over the space of symmetric matrices. Let $\mathrm{Symm}(n)$ be the space of real valued symmetric n-dimensional matrices and let $\Lambda := \Lambda^n \subseteq \mathrm{Symm}(n)$ now be the open convex cone of positive definite matrices. Let f be a convex curvature function. We define $F \in C^\infty(\Lambda)\cap C^0(\overline{\Lambda})$ by:

$$F(A) = f(\lambda_1,\ldots,\lambda_n),$$

where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of A. Interpreting F in this manner, yields the following important characterisation of F_∞:

Proposition 2.3. — *For all $A \in \Lambda^{n-1}$:*

$$F_\infty(A) = \sup\left\{F(B) \mid B \in \Lambda^n,\ B|_{\mathbb{R}^{n-1}} = A\right\}.$$

Furthermore, this supremum is not attained by any $B \in \Lambda^n$.

Proof. — We denote the supremum by F. Trivially, $F_\infty(A) \leqslant F$. Conversely, let $B \in \Lambda^n$ be such that its restriction to $\mathbb{R}^{n-1}$ concides with A. Let $\lambda_1 \leqslant \cdots \leqslant \lambda_{n-1}$ be the eigenvalues of A. Let $\mu_1 \leqslant \cdots \leqslant \mu_n$ be the eigenvalues of B. By the classical minimax principle for eigenvalues of a real valued symmetric matrix (cf., for example, [16, Lemma 10.2]), for all $1 \leqslant \alpha \leqslant n-1$, $\mu_\alpha \leqslant \lambda_\alpha$. Thus, bearing in mind ellipticity of f

$$F(B) = f(\mu_1,\ldots,\mu_n) \leqslant f(\lambda_1,\ldots,\lambda_{n-1},\mu_n) < f_\infty(\lambda_1,\ldots,\lambda_{n-1}) = F_\infty(A).$$

Taking the supremum over all B, it follows that $F_\infty(A) \geqslant F$ and that $F_\infty(A)$ is not attained by any element of Λ^n, as desired. □

We finally list the properties of F:

Proposition 2.4. — *Let f be a convex curvature function. For all $A \in \Lambda \subseteq \mathrm{Symm}(n)$, there exists a unique matrix $B \in \mathrm{Symm}(n)$ such that, for all $M \in \mathrm{Symm}(n)$:*

$$DF_A(M) = \mathrm{Tr}(BM).$$

Moreover:

(1) *B is positive definite; and*
(2) *A and B are simultanously diagonalisable.*

In addition, if $e_1, \ldots, e_n$ is a system of shared eigenvectors for A and B and if $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ are the corresponding eigenvalues of A and B respectively, then:

(3) *for all $i \neq j$:*

$$\lambda_i \geqslant \lambda_j \Leftrightarrow \mu_i \leqslant \mu_j;$$

(4) *$DF_A(A) = \sum_{i=1}^n \lambda_i \mu_i = F(A)$;*
(5) *for all $A' \in A$, $DF_A(A') \geqslant F(A')$;*
(6) *$\sum_{i=1}^n \mu_i = DF_A(\mathrm{Id}) \geqslant 1$; and*
(7) *for all $M \in \mathrm{Symm}(n)$:*

$$-(D^2 f)_A(M, M) \geqslant \sum_{i \neq j}^n \frac{\mu_j - \mu_i}{\lambda_i - \lambda_j} M_{ij}^2 \geqslant 0.$$

In particular, F is concave.

Proof. — $DF_A : \mathrm{Symm}(n) \to \mathrm{Symm}(n)$ is linear. There therefore exists a unique matrix $B \in \mathrm{Symm}(n)$ such that, for all $M \in \mathrm{Symm}(n)$:

$$DF_A(M) = \mathrm{Tr}(BM).$$

By invariance of f, F is $O(n)$ invariant. Thus, for all $A \in \Lambda$ and $M \in O(n)$:

$$F(M^t A M) = F(A).$$

Differentiating yields, for all antisymmetric M:

$$DF_A(MA - AM) = 0 \quad \Rightarrow \quad \mathrm{Tr}([AB]M) = 0.$$

However, since A and B are both symmetric, $[AB]$ is antisymmetric, and so, since M is arbitrary:

$$[AB] = 0.$$

(2) now follows. Let $e_1,\ldots,e_n$ be a system of shared eigenvectors of A and B and let $\lambda_1,\ldots,\lambda_n$ and $\mu_1,\ldots,\mu_n$ be the respective corresponding eigenvalues. By strict ellipticity of f, for all i:

$$\mu_i = DF_A(e_i\otimes e_i) > 0.$$

(1) now follows. (3) follows by concavity of f and $O(n)$-invariance and (4) follows by homogeneity. Likewise, by concavity, for all $A'\in\Lambda$:

$$DF_A(A'-A) \geqslant F(A') - F(A).$$

Thus, by (4), $DF_A(A')\geqslant F(A')$, and (5) follows. (6) is a special case of (5). Finally, (7) follows by concavity as in Lemma 2.3 of [13]. This completes the proof. □

3. Convex Cobordisms and Embedding Radii

3.1. Definition of convex cobordisms

Convex cobordisms provide a concise means of expressing the outer barrier condition. They are defined as follows. Let $M := M^{n+1}$ be an $(n+1)$-dimensional Hadamard manifold. For $m\in\{1,2\}$, let $\Sigma_m := (i_m,(S_m,\partial S_m))$ be a smooth compact LSC immersed hypersurface in M. We say that Σ_2 *bounds* Σ_1, and we denote $\Sigma_1 < \Sigma_2$ whenever there exists a smooth compact $(n{+}1)$-dimensional manifold $(N,\partial N)$ with piecewise smooth boundary and a smooth immersion $I : N\to M$ such that ∂N consists of 2 smooth components $\partial_1 N$ and $\partial_2 N$ and:

(1) the boundaries of $\partial_1 N$ and $\partial_2 N$ coincide;
(2) $\partial_1 N$ and $\partial_2 N$ meet transversally along their shared boundary;
(3) the immersed hypersurface $(\partial_1 N, I)$ coincides with Σ_1 and furthermore N lies above this LSC hypersurface;
(4) the immersed hypersurface $(\partial_2 N, I)$ coincides with Σ_2, and furthermore N lies below this LSC hypersurface; and
(5) furnishing N with the unique metric that makes I into a local isometry, N is foliated by the geodesic segments normal to $\partial_1 N$.

We refer to (N,I) as the *convex cobordism* from Σ_1 to Σ_2. When (N,I) only satisfies conditions (1) to (4), we call it a *convex precobordism* from Σ_1 to Σ_2. This weaker concept is useful in proving technical results.

The concept of bounding generalises the concept of graph for LSC immersions. Indeed, we recall that Σ_2 is said to be a graph over Σ_1 whenever there exists a smooth non-negative function $f : S_1\to [0,\infty[$ and a smooth

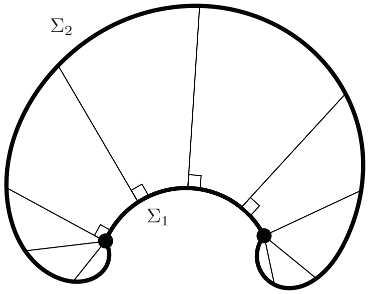

Figure 3.1. A convex cobordism - This convex cobordism from Σ_1 to Σ_2 is not a graph.

diffeomorphism $\alpha : S_1 \to S_2$ such that f vanishes along ∂S_1 and for all $p \in S_1$, $(i_2 \circ \alpha)(p) = \mathrm{Exp}(f(p)N_1(p))$, where N_1 is the upward-pointing unit normal vector field over Σ_1. We define the manifold $N \subseteq S_1 \times \mathbb{R}$ and the smooth immersion $I : N \to M$ by:

$$N = \{(p,t) \mid 0 \leqslant t \leqslant f(p)\}, \qquad I(p,t) = \mathrm{Exp}(tN_1(p)).$$

(N, I) defines a convex cobordism from Σ_1 to Σ_2. We conclude that Σ_2 bounds Σ_1 whenever Σ_2 is a graph over Σ_1. Conversely, one may show that if Σ_2 bounds Σ_1 and makes an angle of less than $\pi/2$ with Σ_1 at every point of their common boundary, then Σ_2 is a graph over Σ_1, although we will have no need for this result.

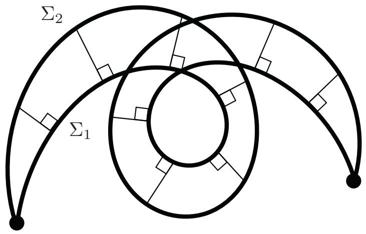

Figure 3.2. Another convex cobordism - This convex cobordism from Σ_1 to Σ_2 is a graph.

In order for boundedness to be used as part of a Smale-type degree theory, we require that it constitute an open and closed condition modulo suitable hypotheses. Sections 3.2 to 3.5 are devoted to establishing these properties.

3.2. Geometry of convex precobordisms

Let Σ_1 and Σ_2 be as before. Let $N := (N, I)$ be a convex precobordism from Σ_1 to Σ_2. The geometry of N is encoded in the following:

Proposition 3.1. — *Choose $p \in N \setminus S_1$, let $q \in S_1$ be a point minimising distance to p and let $\gamma : [0,1] \to N$ be a length minimising rectifiable curve such that $\gamma(0) = q$ and $\gamma(1) = p$. Then:*

(1) *$\gamma(]0,1[)$ lies in the interior of N;*
(2) *$\gamma(]0,1[)$ is a smooth geodesic;*
(3) *if $p \in S_2 \setminus \partial S_2$, then γ is transverse to S_2 at p;*
(4) *if $q \in S_1 \setminus \partial S_1$, then γ makes a right angle with S_1 at q; and*
(5) *if $q \in \partial S_1$, then γ is normal to ∂S_1 at q and:*

$$\langle \partial_t \gamma(0), N_{\partial S_1} \rangle \geqslant 0,$$

where $N_{\partial S_1}$ is the outward-pointing normal vector to ∂S_1 along S_1.

Proof. — Define $t_0 \in [0,1]$ by:

$$t_0 = \inf \{ t \in]0,1] \mid \gamma(t) \in S_2 \}.$$

If $q \in S_1 \setminus \partial S_1$, then trivially $t_0 > 0$. If $q \in \partial S_1$, then likewise, by local strict convexity of S_2, $t_0 > 0$ unless γ is trivial. Since $\gamma([0, t_0])$ lies inside S_2, by local strict convexity of S_2, it cannot be tangent to S_2 at t_0. It follows that γ does not minimise length unless $t_0 = 1$ or $t_0 = +\infty$ (the latter case occuring when γ never intersects S_2) and this proves (1). (2) is trivial. Likewise, taking $t_0 = 1$, we see that γ is transverse to S_2 at $\gamma(1) = p$, and this proves (3). (4) and (5) are trivial, since γ is length minimising. This completes the proof. □

We characterise convex cobordisms amongst convex precobordisms:

Proposition 3.2. — *If there exists no geodesic arc $\gamma : [0,1] \to N$ such that $\gamma(0) \in S_1$; $\gamma(1) \in S_1$; and γ is normal to S_1 at $\gamma(0)$, then (N, I) is a convex cobordism from Σ_1 to Σ_2.*

Proof. — For $p \in N$, let $Q(p)$ be the set of points $q \in S_1$ such that there exists a geodesic $\gamma_q : [0,1] \to N$ such that $\gamma_q(0) = q$; γ_q is normal to S_1 at q; and $\gamma_q(1) = p$. Since S_1 is LSC, and since N is non-positively curved, $Q(p)$ is discrete. Since S_1 is compact, $Q(p)$ is therefore finite. Denote $D(p) = |Q(p)|$. It suffices to show that $D(p) = 1$ for all p. Choose $q \in Q(p)$. As in Proposition 3.1 (1), γ_q may only intersect S_2 at p. Furthermore, if $p \in S_2$, then γ_q is transverse to S_2 at this point. Finally, by hypothesis, γ_q only intersects S_1 at q. It follows that for all $q \in Q(p)$, and for all p' sufficiently close to p, $\gamma(q)$ may be perturbed to another geodesic segment normal to S_1 and terminating at p'. $D(p)$ is therefore locally constant. However, by hypothesis, $D(p)$ is equal to 1 along S_1. It follows by connectedness that $D(p) = 1$ for all p, as desired. □

3.3. Geometry of convex cobordisms

Let Σ_1 and Σ_2 continue to be as before. Let $N := (N, I)$ be a convex cobordism from Σ_1 to Σ_2.

Proposition 3.3 (Uniqueness). — *If (N', I') is another convex cobordism from Σ_1 to Σ_2, then there exists a diffeomorphism $\alpha : N \to N'$ such that $I = I' \circ \alpha$.*

Proof. — Let $\mathcal{F}$ and $\mathcal{F}'$ be the foliations of N and N' respectively by geodesic segments normal to S_1. Observe that $\mathcal{F}$ and $\mathcal{F}'$ are canonically homeomorphic. We thus identify $\mathcal{F}$ and $\mathcal{F}'$. Observe that, as $\mathcal{F}$ is also homeomorphic to Σ_2, it has the structure of a topological manifold with boundary. For $L \in \mathcal{F}$, let $h(L)$ and $h'(L)$ be the length of the leaf L in N and N' respectively. It suffices to show that $h = h'$ throughout $\mathcal{F}$. Let $\Omega \subseteq \mathcal{F}$ be the set of all points where $h = h'$. Observe that $\partial\mathcal{F} \subseteq \Omega$. Let Ω_0 be the connected component of Ω containing $\partial\mathcal{F}$. Ω_0 is open and, by continuity, it is also closed. It follows by connectedness that $\Omega_0 \subseteq \mathcal{F}$. The result follows. □

Let $d : N \to \mathbb{R}$ be the distance in N to S_1. The main consequence of the foliation condition is the following result:

Proposition 3.4. — *For all $p \in N$, there exists a unique point $q \in S_1$ minimising distance to p.*

Proof. — Let $q \in S_1$ minimise distance to p. Let $\gamma : [0, 1] \to N$ be a length minimising curve from p to q. By Proposition 3.1 (2), (4) and (5), γ is a geodesic in N which is normal to S_1 at q. However, by hypothesis, there is only one such geodesic passing through p. The result follows. □

Proposition 3.5. — *The function d is strictly convex.*

Proof. — Choose $p \in N \setminus S_1$. By Proposition 3.4, there exists a unique point $q \in S_1$ minimising distance to p. Since S_1 is LSC and since N is non-positively curved, the result now follows. □

Furthermore, Proposition 3.4 yields a well-defined closest-point projection $\pi : N \to S_1$. Since S_1 is LSC and since N is non-positively curved, π is distance decreasing. Considering the restriction of π to S_2, we therefore obtain:

Proposition 3.6. — *For each i, let $\mathrm{Diam}(\Sigma_i)$ and $\mathrm{Vol}(\Sigma_i)$ denote the diameter and volume of Σ_i respectively. Then:*

$$\mathrm{Diam}(\Sigma_2) \geqslant \mathrm{Diam}(\Sigma_1), \qquad \mathrm{Vol}(\Sigma_2) \geqslant \mathrm{Vol}(\Sigma_1).$$

3.4. Glueing and excision

For $m \in \{1,2,3\}$, let $\Sigma_m := (i_m, (S_m, \partial S_m))$ be smooth compact LSC immersed hypersurfaces in M. Suppose that $\Sigma_1 < \Sigma_2$ and $\Sigma_2 < \Sigma_3$. Let (N_{12}, I_{12}) and (N_{23}, I_{23}) be convex cobordisms from Σ_1 to Σ_2 and from Σ_2 to Σ_3 respectively. We define $N_{12} \cup N_{23}$ by joining N_{12} to N_{23} along S_2. We define $I_{12} \cup I_{23} : N_{12} \cup N_{23} \to M$ by:

$$(I_{12} \cup I_{23})(x) = \begin{cases} I_{12}(x) & \text{if } x \in N_{12}; \\ I_{23}(x) & \text{if } x \in N_{23}. \end{cases}$$

$N_{12} \cup N_{23}$ is trivially a convex precobordism from Σ_1 to Σ_3.

Proposition 3.7. — *There exists no non-trivial geodesic arc $\gamma : [0,1] \to N_{12} \cup N_{23}$ such that $\gamma(0), \gamma(1) \in S_1$.*

Proof. — Suppose the contrary. Let $\gamma : [0,1] \to N_{12} \cup N_{23}$ be a geodesic arc such that $\gamma(0), \gamma(1) \in S_1$. By Proposition 3.5, γ is not contained in N_{12}. There therefore exists $t_0 \in \,]0,1[$ such that $\gamma(t_0)$ lies in the interior of N_{23}, and there exist $t_1 < t_0 < t_2$ such that $\gamma([t_1,t_2])$ is contained in N_{23} and $\gamma(t_1)$ and $\gamma(t_2)$ both lie in S_2. However, this is also absurd by Proposition 3.5, and this completes the proof. □

We thus obtain a glueing operation for convex cobordisms:

Proposition 3.8 (Glueing). — *$(N_{12} \cup N_{23}, I_{12} \cup I_{23})$ defines a convex cobordism from Σ_1 to Σ_3.*

Proof. — This follows from Propositions 3.2 and 3.7. □

Now, for $m \in \{1,2\}$, let $\Sigma_m := (i_m, (S_m, \partial S_m))$ be a smooth compact LSC immersed hypersurface in M. Suppose that $\Sigma_1 < \Sigma_2$. Let (N, I) be the convex cobordism from Σ_1 to Σ_2. Denote by $\mathcal{F}$ the foliation of N by geodesic segments normal to S_1. Let $(S_3, \partial S_3)$ be a smooth compact LSC embedded submanifold of N such that $\partial S_3 = \partial S_1 = \partial S_2$; S_3 is transverse to every leaf of $\mathcal{F}$; and S_3 is not tangent to S_1 or to S_2 at any point.

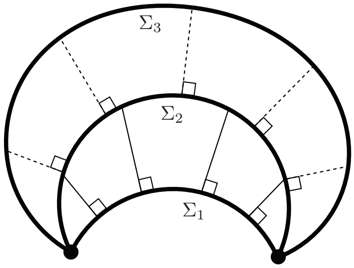

Figure 3.3. The glueing operation - The convex cobordism from Σ_1 to Σ_2 glues to the convex cobordism from Σ_2 to Σ_3 to yield a convex cobordism from Σ_1 to Σ_3.

PROPOSITION 3.9. — *S_3 meets every leaf of $\mathcal{F}$ at exactly 1 point.*

Proof. — For L in $\mathcal{F}$, since S_3 is transverse to L, $L \cap S_3$ is discrete. Since S_3 is compact, $L \cap S_3$ is therefore finite. Denote $D(L) = |L \cap S_3|$. By transversality and finiteness, $D(L)$ is locally constant. However, close to $\partial S_1 = \partial S_2$, $D(L) = 1$. It follows by connectedness that $D(L) = 1$ for all L, as desired. □

PROPOSITION 3.10. — *S_3 divides N into two connected components.*

Proof. — This follows from Proposition 3.9 since each leaf divides into two subintervals lying below and above S_3 respectively. □

Let i_3 be the restriction of I to S_3. Denote $\Sigma_3 := (i_3, (S_3, \partial S_3))$. Denote by N_- and N_+ the closures of the connected components of $N \setminus S_3$ lying below and above S_3 respectively.

PROPOSITION 3.11. — *(N_-, I) and (N_+, I) define convex precobordisms from Σ_1 to Σ_3 and from Σ_3 to Σ_2 respectively.*

Proof. — It suffices to show that S_1 and S_2 lie below and above S_3 respectively. However, since this holds along the boundary, the result follows. □

This yields the first excision operation for convex cobordisms:

PROPOSITION 3.12 (Excision I). — *(N_-, I) defines a convex cobordism from Σ_1 to Σ_3.*

Proof. — Since S_3 meets every leaf of $\mathcal{F}$ at exactly 1 point, the geodesic segments normal to S_1 also foliate N_-, and the result follows. □

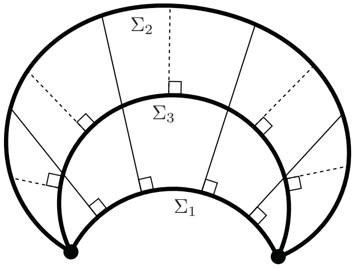

Figure 3.4. The excision operation - The convex cobordism from Σ_1 to Σ_2 yields convex cobordisms from Σ_1 to Σ_3 and from Σ_3 to Σ_2.

Proposition 3.13. — *There exists no geodesic segment $\gamma : [0,1] \to N_+$ such that $\gamma(0) \in S_3$; $\gamma(1) \in S_3$; and γ is normal to S_3 at $\gamma(0)$.*

Proof. — Suppose the contrary. Let γ be such a geodesic. Let $d : N \to [0,\infty[$ be the distance to S_1 along $\mathcal{F}$. Since γ points upwards from S_2, $\langle\dot\gamma(0),(\nabla d\circ\gamma)(0)\rangle > 0$. However, by Proposition 3.5, d is convex. Thus, since γ is a geodesic:

$$\partial_t\langle\dot\gamma(t),(\nabla d\circ\gamma)(t)\rangle = \mathrm{Hess}(d)(\dot\gamma(t),\dot\gamma(t)) \geqslant 0.$$

In particular, $\langle\dot\gamma(1),(\nabla d\circ\gamma)(1)\rangle > 0$. That is, γ meets S_2 from below at $\gamma(1)$. In other words, for t sufficiently close to 1, $\gamma(1) \in N_- \setminus S_3$. This is absurd, and the result follows. □

This yields the second excision operation for convex cobordisms:

Proposition 3.14 (Excision II). — *(N_+, I) defines a convex cobordism from Σ_3 to Σ_2.*

Proof. — This follows from Propositions 3.2 and 3.13. □

3.5. Openness and closedness

Let Σ_1^-,Σ_1,Σ_2 and Σ_2^+ be smooth compact LSC immersed hypersurfaces in M such that $\Sigma_1^- < \Sigma_1 < \Sigma_2 < \Sigma_2^+$. By Proposition 3.8, $\Sigma_1^- < \Sigma_2^+$. Let N be the convex cobordism from Σ_1^- to Σ_2^+. By Proposition 3.3, N is unique. Observe that Σ_1 and Σ_2 identify with smooth compact LSC embedded hypersurfaces in N which we denote by S_1 and S_2 respectively.

Proposition 3.15. — *S_1 lies below S_2 and both S_1 and S_2 are transverse to the foliation of N by geodesic segments normal to S_1^-.*

Proof. — The convex cobordism N may be constructed by glueing the convex cobordism from Σ_1 to Σ_2 to the convex cobordism from Σ_1^- to Σ_1 along S_1 and then glueing the convex cobordism form Σ_2 to Σ_2^+ to this convex cobordism along S_2. The result follows. □

We now obtain openness:

PROPOSITION 3.16 (Openness). — *Let $(\Sigma_n)_{n\in\mathbb{N}}$ and $(\hat{\Sigma}_n)_{n\in\mathbb{N}}$ be sequences of smooth compact LSC immersed hypersurfaces in M converging to the smooth compact LSC immersed hypersurfaces Σ_∞ and $\hat{\Sigma}_\infty$ respectively. If $\Sigma_\infty < \hat{\Sigma}_\infty$, then for all sufficiently large n, $\Sigma_n < \hat{\Sigma}_n$.*

Proof. — Upon perturbing Σ_∞ and $\hat{\Sigma}_\infty$ we obtain smooth compact LSC immersed hypersurfaces Σ'_∞ and $\hat{\Sigma}'_\infty$ respectively such that $\Sigma'_\infty < \Sigma_\infty$ and $\hat{\Sigma}_\infty < \hat{\Sigma}'_\infty$. By Proposition 3.8, $\Sigma'_\infty < \hat{\Sigma}'_\infty$. Let N be the convex cobordism from Σ'_∞ to $\hat{\Sigma}'_\infty$. Denote by S'_∞ and $\hat{S}'_\infty$ the lower and upper boundary components of N respectively. Denote by $\mathcal{F}$ the foliation of N by geodesic segments normal to S'_∞. Σ_∞ and $\hat{\Sigma}_\infty$ identify with smooth compact LSC embedded hypersurfaces in N which we denote by S_∞ and $\hat{S}_\infty$ respectively. By Proposition 3.15, S_∞ lies below $\hat{S}_\infty$ and both S_∞ and $\hat{S}_\infty$ are transverse to $\mathcal{F}$. For sufficiently large n, Σ_n and $\hat{\Sigma}_n$ identify with smooth compact LSC embedded hypersurfaces in N which we denote by S_n and $\hat{S}_n$ respectively. Upon increasing n further if necessary, we may suppose that S_n lies below $\hat{S}_n$ and that both S_n and $\hat{S}_n$ are transverse to $\mathcal{F}$. By Proposition 3.12, $\Sigma'_\infty < \hat{\Sigma}_n$, and by Proposition 3.14, $\Sigma_n < \hat{\Sigma}_n$, as desired. □

We obtain closedness modulo a condition on the curvature:

PROPOSITION 3.17 (Closedness). — *Let $(\Sigma_n)_{n\in\mathbb{N}}$ and $(\hat{\Sigma}_n)_{n\in\mathbb{N}}$ be sequences of smooth compact LSC immersed hypersurfaces in M converging to the smooth compact LSC immersed hypersurfaces Σ_∞ and $\hat{\Sigma}_\infty$ respectively. If $\Sigma_n < \hat{\Sigma}_n$ for all n, and if $K(\Sigma_\infty) < K(\hat{\Sigma}_\infty)$, then $\Sigma_\infty < \hat{\Sigma}_\infty$.*

Proof. — Upon perturbing Σ_∞ and $\hat{\Sigma}_\infty$ we obtain smooth compact LSC immersed hypersurfaces Σ'_∞ and $\hat{\Sigma}'_\infty$ respectively such that $\Sigma'_\infty < \Sigma_\infty$ and $\hat{\Sigma}_\infty < \hat{\Sigma}'_\infty$. By Proposition 3.16, there exists n_0 such that for all $n \geqslant n_0$, $\Sigma'_\infty < \Sigma_n$ and $\hat{\Sigma}_n < \hat{\Sigma}'_\infty$. Thus, by Proposition 3.8, $\Sigma'_\infty < \hat{\Sigma}'_\infty$. Let N be the convex cobordism from Σ'_∞ to $\hat{\Sigma}'_\infty$. Denote by S'_∞ and $\hat{S}'_\infty$ the lower and upper boundary components of N respectively. Denote by $\mathcal{F}$ the foliation of N by geodesic segments normal to S'_∞. By uniqueness of N, for $n \geqslant n_0$, Σ_n and $\hat{\Sigma}_n$ identify with smooth compact LSC embedded hypersurfaces in N which we denote by S_n and $\hat{S}_n$ respectively. By Proposition 3.15, for $n \geqslant n_0$, S_n lies below $\hat{S}_n$ and both S_n and $\hat{S}_n$ are transverse to $\mathcal{F}$. Upon taking limits, it follows that Σ_∞ and $\hat{\Sigma}_\infty$ identify with smooth compact LSC embedded hypersurfaces in N which we denote by S_∞ and $\hat{S}_\infty$ respectively. We claim that S_∞ is transverse to $\mathcal{F}$. Indeed, otherwise, taking limits, there exists a geodesic segment normal to S'_∞ which is an interior tangent to S_∞ at some point. This is absurd, and it follows that S_∞ is transverse to $\mathcal{F}$ as asserted. Likewise, $\hat{S}_\infty$ is also transverse to $\mathcal{F}$. Furthermore, S_∞ lies below $\hat{S}_\infty$, and since $K(S_\infty) < K(\hat{S}_\infty)$, it follows from the geometric maximum principle that S_∞ is not tangent to $\hat{S}_\infty$ at any point. By Proposition 3.12, $\Sigma'_\infty < \hat{\Sigma}_\infty$, and by Proposition 3.14, $\Sigma_\infty < \hat{\Sigma}_\infty$, as desired. □

3.6. Boundedness and embedding radii

In Proposition 4.1.1 of [20], we obtain a-priori lower bounds of the radius about any boundary point over which a smooth compact LSC immersed hypersurface is embedded. We introduce the terminology required to use this result within the current framework. *The definitions and results of this section, though highly technical, are of central importance to the sequel.* We recommend the reader study them carefully.

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Hadamard manifold. Let $\Sigma := (i,(S,\partial S))$ be a smooth immersed submanifold in M. Let p be a point in S which we identify with its image in M. Let U be a neighbourhood of p in M. Let S' be the connected component of $i^{-1}(U)$ containing p. We denote $\Sigma \cap_p U := (i,(S',\partial S'))$. We refer to $\Sigma \cap_p U$ as the *connected component* of $\Sigma \cap U$ containing p.

Let $\Gamma := (i, G)$ be a smooth oriented compact codimension-2 submanifold of M. Let $\mathrm{N}\Gamma$ be the circle bundle of unit normal vectors over Γ. For $N \in \mathrm{N}\Gamma$, we define $A_\Gamma(N)$ to be the *second fundamental form* of Γ in the direction of N. In other words, if X and Y are vector fields tangent to Γ, then:

$$A_\Gamma(N)(X,Y) = -\langle \nabla_X Y, N\rangle. \tag{3.1}$$

For all $p \in \Gamma$, we define $\mathrm{CN}_p\Gamma$ by:

$$\mathrm{CN}_p\Gamma = \{N \in \mathrm{N}_p\Gamma \mid A_\Gamma(N) > 0\},$$

where, for any matrix A, we write $A > 0$ whenever A is positive definite. We refer to $\mathrm{CN}\Gamma$ as the bundle of *convex normal vectors* over Γ at p. We say that Γ is *locally strictly convex* (LSC) whenever $\mathrm{CN}\Gamma$ has non-trivial fibre above every point. In this case, by Proposition 4.1.2 of [20], for all $p \in \Gamma$, $\mathrm{CN}_p\Gamma$ is an open subinterval of $\mathrm{N}_p\Gamma$ of length at most π. Furthermore, by

Proposition 4.1.3 of [20], $\mathrm{CN}_p\Gamma$ varies continuously with p in the Hausdorff topology.

Let $\mathcal{G}$ be a family of smooth oriented compact codimension-2 LSC immersed submanifolds in M. Suppose that $\mathcal{G}$ is compact in the C^∞-sense. For all $\Gamma = (i, G)$ in $\mathcal{G}$, for all points p in G, and for all $N \in \mathrm{CN}_p\Gamma$, there exists a (non-complete) smooth LSC embedded hypersurface H in M such that p lies in H; N is the upward-pointing normal to H at p; and $\Gamma \cap_p B_r(p)$ is contained in $H \cap_p B_r(p)$ for some small r. Although H is, strictly speaking, not canonical, we assume throughout the sequel that H depends continuously on $\Gamma \in \mathcal{G}$ and $N \in \mathrm{CN}\Gamma$. We then refer to H as the *LSC extension* of Γ at p with normal N. We denote $\Gamma \cap_p B_r(p)$ and $H \cap_p B_r(p)$ by $\Gamma_{p,r}$ and $H_r(\Gamma, N)$ respectively. Observe that, upon reducing r if necessary, $\Gamma_{p,r}$ divides $H_r(\Gamma, N)$ into two connected components. We denote the closure of the component lying to the left (resp. right) of $\Gamma_{p,r}$ by $H_r^-(\Gamma, N)$ (resp. $H_r^+(\Gamma, N)$).

A subset X of M is said to be *convex* whenever the shortest geodesic in M joining any two points of X is also contained in X. Let $\mathcal{G}$ be as above. We make a uniform choice of the radius r as follows. For $\theta > 0$, there exists $r > 0$ with the following property: for all $\Gamma = (i, G)$ in $\mathcal{F}$; for all points p in G; and for all $N \in \mathrm{CN}_p\Gamma$ such that N makes an angle of at least θ with each end-point of $\mathrm{CN}_p\Gamma$, if $H := H_r(\Gamma, N)$ is the LSC extension of Γ at p with normal N, then H is closed in $B_r(p)$; H meets $\partial B_r(p)$ transversally; and H divides $B_r(p)$ into two connected components, one of which is convex. We denote the closure of the convex component by $X_r(\Gamma, N)$.

This construction yields a canonical order on $\mathrm{CN}_p\Gamma$. Indeed, choose Γ in $\mathcal{G}$. Let p be a point in Γ. Let N and N' be distinct vectors in $\mathrm{CN}_p\Gamma$. Let $\theta > 0$ be such that both N and N' make an angle of at least θ with each end-point of $\mathrm{CN}_p\Gamma$. Let r be defined as above. Since $H_r(\Gamma, N)$ and $H_r(\Gamma, N')$ meet transversally at p, upon reducing r if necessary, we may assume that $H_r^+(\Gamma, N')$ is wholly contained in one of the connected components of the complement of $H_r(\Gamma, N)$ in $B_r(p)$. We say that N lies *above* N' whenever $H_r^+(\Gamma, N')$ is wholly contained in the convex component $X_r(\Gamma, N)$. We define $\partial^+\mathrm{CN}_p\Gamma$ (resp. $\partial^-\mathrm{CN}_p\Gamma$) to be the upper (resp. lower) end-point of $\mathrm{CN}_p\Gamma$.

Proposition 4.1.1 of [20] yields a priori lower bounds of the radius over which a smooth compact LSC hypersurface with generic boundary is embedded. In the present context, this is expressed as follows:

Theorem 3.18. — *Let $\mathcal{G}$ be a family of smooth oriented compact generic codimension-2 LSC immersed submanifolds in M. Suppose that*

$\mathcal{G}$ is compact in the C^∞-sense. For all $\theta > 0$, there exists $r > s > 0$ with the following property. If $\Sigma = (i, (S, \partial S))$ is a smooth compact LSC immersed hypersurface with boundary $\Gamma := \partial\Sigma$ in $\mathcal{G}$; if p is a boundary point of Σ; if N is the upward-pointing unit normal to Σ at p; if $N' \in \mathrm{CN}_p\Gamma$ is a convex normal vector to Γ lying above N; and if the angles between $\partial^+\mathrm{CN}_p\Gamma$ and N' and N' and N are no less than θ, then $\Sigma \cap_p B_r(p)$ is embedded; meets $\partial B_r(p)$ transversally; is contained within $X_r(\Gamma, N')$; and divides $X_r(\Gamma, N')$ into two connected components, one of which is convex and contains a ball of radius s. Furthermore $\Gamma_{p,r}$ is the only component of Γ in $\Sigma \cap_p B_r(p)$.

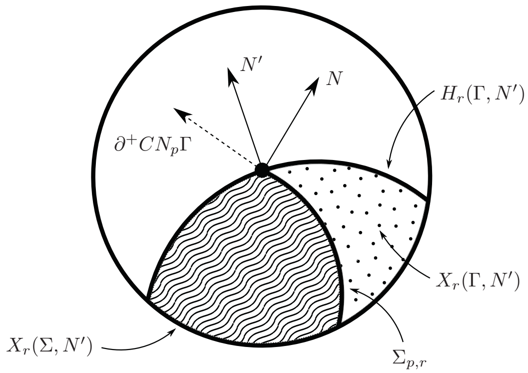

Figure 3.5. The embedding radius - The convex set $X_r(\Sigma, N')$ is uniformly bounded below in the sense that it contains a ball of radius s.

We denote $\Sigma_{p,r} := \Sigma \cap_p B_{r,p}$. We denote the closure of the convex component of the complement of $\Sigma_{p,r}$ in $X_r(\Gamma, N')$ by $X_r(\Sigma, N')$. Observe that the boundary of $X_r(\Sigma, N')$ is the union of $H_r^-(\Gamma, N')$, $\Sigma_{p,r}$ and some open subset of $\partial B_r(p)$.

Let $\mathcal{F}$ be a family of smooth compact LSC immersed hypersurfaces with generic boundaries in M. Suppose that $\mathcal{F}$ is compact in the C^∞ topology. Let $\mathcal{G}$ be the family of all $\partial\hat{\Sigma}$ where $\hat{\Sigma} \in \mathcal{F}$. Choose $\hat{\Sigma} \in \mathcal{F}$. Denote $\Gamma = \partial\hat{\Sigma}$. Let $\hat{N}$ be the upward-pointing unit normal vector field over $\hat{\Sigma}$. Let p be a boundary point of $\hat{\Sigma}$. Observe that $\hat{N}(p) \in \mathrm{CN}_p\Gamma$. We denote by $\mathrm{CN}_p^+\hat{\Sigma}$ (resp. $\mathrm{CN}_p^-\hat{\Sigma}$) the open subinterval of $\mathrm{CN}_p\Gamma$ lying above (resp. below) $\hat{N}(p)$. We extend $\hat{\Sigma}$ smoothly across its boundary to obtain a larger smooth compact LSC immersed hypersurface $\hat{\Sigma}'$. Although $\hat{\Sigma}'$ is, strictly speaking, not canonical, we assume throughout the sequel that it depends continuously on $\hat{\Sigma}\in\mathcal{F}$. Furthermore, we may suppose that $H_r(\Gamma,\hat{N}(p))=\hat{\Sigma}'\cap_p B_r(p)$, and we denote $H_r(\Gamma,\hat{N})$ and $X_r(\Gamma,\hat{N})$ by $\hat{\Sigma}_{p,r}$ and $X_{p,r}(\hat{\Sigma})$ respectively. Likewise, if Σ is another smooth compact LSC immersed hypersurface in M such that $\Sigma<\hat{\Sigma}$ and Σ makes an angle of at least θ with $\hat{\Sigma}$ along their common boundary, for suitable values of $r>0$, we denote $X_{p,r}(\Sigma,\hat{\Sigma})$ instead of $X_r(\Sigma,\hat{N}(p))$.

3.7. Supporting normal vectors

We recall the definition of supporting normal vectors to convex sets. Let X be any subset of M. Let p be a point in X and let $N\in T_pM$ be a unit vector to M at p. Let $P\subseteq T_pM$ be the hyperplane normal to N. We identify P with its image under the exponential map in M. We orient P such that N points upwards. Observe that P divides M into two connected components. We say that N is a *supporting normal* to X at p whenever X lies in the closure of the connected component lying below P. We recall that the set of supporting normals to X at p is a closed convex subset of the sphere of unit vectors in T_pM (cf. [21]). Furthermore, when the set X is convex, the set of supporting normals to X is non-trivial at all of its boundary points (cf. [21]).

We recall that the set of supporting normal vectors to a convex set varies upper-semicontinously in the Hausdorff sense. This permits us to obtain uniform moduli of continuity as follows. Let $(\hat{\Sigma}_n)_{n\in\mathbb{N}}$, $\hat{\Sigma}$ be smooth compact LSC immersed hypersurfaces with generic boundaries in M such that $(\hat{\Sigma}_n)_{n\in\mathbb{N}}$ converges to $\hat{\Sigma}$ in the C^∞ topology. For all n, let p_n be a boundary point of $\hat{\Sigma}_n$. Suppose that $(p_n)_{n\in\mathbb{N}}$ converges to the boundary point p of $\hat{\Sigma}$. Choose $\theta>0$ and for all n, let Σ_n be a smooth compact LSC immersed hypersurface in M such that $\Sigma_n<\hat{\Sigma}_n$ and Σ_n makes an angle of at least θ with $\hat{\Sigma}_n$ along their common boundary. For all n, let N_n be the upward-pointing unit normal vector field over Σ_n. Let $r>s>0$ be as in Theorem 3.18. For all n, denote $X_n:=X_{p_n,r}(\Sigma_n,\hat{\Sigma}_n)$. For all n, by definition, X_n is contained in $B_r(p_n)$. Furthermore, by Theorem 3.18, for all n, X_n contains a ball of radius s. Thus, by compactness of the family of convex sets, we may assume that there exists a convex set X with non-trivial interior towards which $(X_n)_{n\in\mathbb{N}}$ converges in the Hausdorff sense. Observe that p is a boundary point of X. Let I be the set of supporting normals to X at p:

Proposition 3.19. — *I is contained in the closure of* $\mathrm{CN}_p^-\hat{\Sigma}$.

Proof. — For all n, let Γ_n be the boundary of $\hat{\Sigma}_n$. Let Γ be the boundary of $\hat{\Sigma}$. For all n, $\Gamma_{n,p_n,r}$ is contained in X_n. Thus, taking limits, $\Gamma_{p,r}$ is contained in X. It follows that I is contained in the closure of $\mathrm{CN}_p\Gamma$. For all n, $\hat{\Sigma}^-_{n,p_n,r}$ is contained in X_n. Thus, taking limits, $\hat{\Sigma}^-_{p,r}$ is contained in X. It follows that I is contained in the closure of $\mathrm{CN}^-_p\hat{\Sigma}$, as desired. □

Proposition 3.20. — *Upon extracting a subsequence, there exist subsets $(I_n)_{n\in\mathbb{N}}$ of UM and a continuous function $m : [0,\infty[\to [0,\infty[$ such that:*

1. *for all n, I_n is contained in the closure of $\mathrm{CN}^-_{p_n}\hat{\Sigma}_n$;*
2. *$(I_n)_{n\in\mathbb{N}}$ converges to I in the Hausdorff topology;*
3. *$m(0) = 0$; and*
4. *for all n, and for all $q \in \Sigma_{n,p_n,r}$:*

$$D(I_n, N_n(q)) \leqslant m(d(p,q)),$$

where D and d are the distances in the total space of UM and in M respectively.

Remark. — Observe that, if $X, Y \in UM$ are two unit vectors in the same fibre, then $D(X,Y)$ is the angle between X and Y.

Proof. — Let l be the length of I. For all n, we define I_n to be the shortest interval of the closure of $\mathrm{CN}^-_{p_n}\hat{\Sigma}_n$ which contains both $\hat{N}_n(p_n)$ and $N_n(p_n)$ and has length at least l. $(I_n)_{n\in\mathbb{N}}$ trivially satisfies Condition (1). By upper-semicontinuity of the sets of supporting normals to convex sets (cf. [21]), upon extracting a subsequence, we may suppose that $(N_n(p_n))_{n\in\mathbb{N}}$ converges to a vector in I. Condition (2) follows. Let $(n_k, q_k)_{k\in\mathbb{N}}$ be such that for all k, $n_k \in \mathbb{N}$ and $q_k \in \Sigma_{n_k,p_{n_k},r}$. Denote $p_\infty := p$ and $I_\infty := I$. Suppose that $(n_k, q_k)_{n\in\mathbb{N}}$ converges to (n_∞, p_{n_∞}) where $n_\infty \in \mathbb{N}\cup\{\infty\}$. By upper-semincontinuity of the sets of supporting normals to convex sets, every limit point of the sequence $(N_{n_k}(q_k))_{k\in\mathbb{N}}$ is contained in I_{n_∞}. The result follows. □

4. First Order Lower Estimates

4.1. Main results

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. Let K be a convex curvature function. Let $\mathcal{F}$ be a family of pairs $(\hat{\Sigma}, \kappa)$ where $\hat{\Sigma}$ is a smooth compact LSC immersed hypersurface with generic boundary in

M; κ is a smooth positive function over M; and $K(\hat{\Sigma}) > \kappa$. We furnish $\mathcal{F}$ with the product topology of the C^∞ topology in the first component and the C^∞_{loc} topology in the second. We obtain lower estimates for the normals of LSC hypersurfaces of prescribed K-curvature.

Proposition 4.1. — *If $\mathcal{F}$ is compact, then there exists $\theta > 0$ such that for all smooth compact LSC immersed hypersurfaces Σ in M such that $K(\Sigma) = \kappa$ and $\Sigma < \hat{\Sigma}$ for some element $(\hat{\Sigma}, \kappa) \in \mathcal{F}$, if N is the upward pointing unit normal vector field of Σ, then $N(p)$ makes an angle of at least θ with each of the end-points of $\text{CN}_p\partial\Sigma$.*

When K is of bounded type, we require the following finer estimate:

Proposition 4.2. — *Suppose that K is of bounded type. If $\mathcal{F}$ is compact, then there exists $\delta > 0$ such that for all smooth compact LSC immersed hypersurfaces Σ in M such that $K(\Sigma) = \kappa$ and $\Sigma < \hat{\Sigma}$ for some element $(\hat{\Sigma}, \kappa) \in \mathcal{F}$, and for all $p \in \partial\Sigma$:*

$$K_\infty(A_\Gamma(N(p)) \geqslant \kappa(p) + \delta,$$

where N is the outward pointing unit normal vector field over Σ and A_Γ is the second fundamental form of $\Gamma := \partial\Sigma$, as defined in Section 3.6.

Logically, Proposition 4.1 precedes Proposition 4.2. However, the proofs are similar, and we prove Proposition 4.2 first as it is more involved. We then review the modifications required to prove Proposition 4.1.

The following technical result will be used repeatedly throughout the sequel:

Lemma 4.3. — *Let $f : M \to \mathbb{R}$ be a smooth function. Let $\text{Hess}(f)$ and $\text{Hess}^\Sigma(f)$ denote the Hessians of f and the restriction of f to Σ respectively. Then:*

$$\text{Hess}^\Sigma(f) = \text{Hess}(f)|_{T\Sigma} - \langle \nabla f, N \rangle II,$$

where N is the upward-pointing unit normal vector field over Σ and II is the corresponding second fundamental form.

Proof. — Choose $p \in \Sigma$. Let X and Y be tangent vector fields over Σ which are parallel at p. Let ∇^M denote the Levi-Civita covariant derivative of M. Since Y is parallel at p as a vector field over Σ, $\nabla^M_X Y = -II(X,Y)N$. Thus:

$$\begin{aligned}\text{Hess}(f)(X,Y)(p) &= (X(Yf))(p) - df(\nabla^M_X Y)(p)\\ &= \text{Hess}^\Sigma(f)(X,Y)(p) + \langle \nabla f, N \rangle II(X,Y)(p).\end{aligned}$$

The result follows. □

4.2. Analytic properties of the barrier function

In this and the following section, we assume that K is of bounded type. Choose $(\hat{\Sigma},\kappa)\in\mathcal{F}$. Denote $\Gamma:=\partial\hat{\Sigma}$. Let p be a point in Γ. Let d_p be the distance to p in M. Let $\hat{N}$ be the upward pointing unit normal vector to $\hat{\Sigma}$ at p. Let $\mathrm{CN}_p\Gamma$ and $\mathrm{CN}^{\pm}_p\hat{\Sigma}$ be defined as in Section 3.6. Choose $N\in\mathrm{CN}^{-}_p\hat{\Sigma}$ such that $K_\infty(A_\Gamma(N))=\kappa(p)$. We aim to show that it is not possible for a sequence of smooth compact LSC hypersurfaces satisfying the hypotheses of Proposition 4.2 to have as a limit a hypersurface whose normal at p is equal to N.

Let Φ_0 be a smooth function defined in a neighbourhood of p such that $\nabla\Phi_0(p)=N$ and Φ_0 vanishes along Γ. The required barrier function is constructed by perturbing Φ_0 as follows. Choose $V\in\mathrm{CN}^{+}_p\hat{\Sigma}$. We denote $H_0:=H_{p,r}(\Gamma,V)$ for sufficiently small r. For V sufficiently close to $\hat{N}$, we may suppose that $K(H_0)(p)>\kappa(p)$. Upon perturbing H_0 slightly and reversing the orientation, we obtain a smooth locally strictly *concave* embedded hypersurface H in M such that H passes through p; the upward pointing normal to H at p is equal to $-V$; $K(-H)(p)>\kappa(p)$, where $-H$ is H furnished with the reverse orientation; and H_0 lies above the graph of ϵd_p^2 over H for some $\epsilon>0$. Observe in particular that $\Gamma_{p,r}$ also lies above the graph of ϵd_p^2 over H. Furthermore, since V lies above $\hat{N}$, $\hat{\Sigma}^{+}_{p,r}$ also lies above the graph of ϵd_p^2 over H. We define d_H near p to be the signed distance to H in M with sign chosen so that it is positive above H. For appropriate functions x and h defined near p, we define Φ_1 by:

$$\Phi_1=\Phi_0+x(d_H-h). \tag{4.1}$$

Given h, the function x is determined as follows. For any two functions f and g with non-colinear derivatives at p, we define the $(n-2)$-dimensional distribution $E(f,g)$ near p by:

$$E(f,g)=\langle\nabla f,\nabla g\rangle^{\perp},$$

where $\langle X,Y\rangle$ here denotes the space spanned by the vectors X and Y. Let $e_1,\dots,e_{n-1}$ be an orthonormal basis for $T_p\Gamma$ with respect to which $A_\Gamma(N)$ is diagonal. Observe that $\nabla d_H=-V$ and $\nabla\Phi_0=N$ are non-colinear at p. We thus extend $e_1,\dots,e_{n-1}$ to a local frame in TM such that, at p, for all X and for all $1\leqslant\alpha\leqslant n-1$:

$$\begin{aligned}\langle\nabla_X e_\alpha,\nabla d_H\rangle&=-\operatorname{Hess}(d_H)(e_\alpha,X),\\ \langle\nabla_X e_\alpha,\nabla\Phi_0\rangle&=-\operatorname{Hess}(\Phi_0)(e_\alpha,X).\end{aligned} \tag{4.2}$$

We define the distribution E near p to be the linear span of $e_1,\ldots,e_{n-1}$. Bearing in mind Lemma 4.3, for a smooth function f, we define $K_{\infty,E}(f)$ by:

$$K_{\infty,E}(f)=\frac{1}{\|\nabla f\|}K_\infty(\mathrm{Hess}(f)|_E),$$

where $\mathrm{Hess}(f)|_E$ is the restriction to E of the Hessian of f. Given h, the function x is now determined by the following result:

Proposition 4.4. — *If N is not an end-point of $\mathrm{CN}_p\Gamma$, then, for V sufficiently close to $\hat{N}$ and for all h defined near p such that:*

(1) $h(p)=0$;
(2) $\nabla h(p)=0$;
(3) *the Hessian of the restriction of d_H-h to Γ vanishes at p; and*
(4) *the restriction of* $\mathrm{Hess}(h)$ *to H is positive definite,*

there exists a function x such that $x(p)=0$, $\mathrm{Hess}(x)(p)=0$ and:

$$K_{\infty,E}(\Phi_0+x(d_H-h))\leqslant\kappa+O(d_p^2).$$

Proof. — By Lemma 4.3, the restriction of $\|\nabla\Phi_0\|^{-1}\,\mathrm{Hess}(\Phi_0)$ to E_p coincides with $A_\Gamma(N)$. Thus, by definition of N, at p, $K_{\infty,E}(\Phi_0)=\kappa(p)$. The gradient of $x(d_H-h)$ vanishes at p. The Hessian of xh vanishes at p. The Hessian of xd_H vanishes on $(\nabla d_H)^\perp$ at p and thus so too does its restriction to E. It follows that for all x, at p, $K_{\infty,E}(\Phi_1)=\kappa(p)$.

Let $\lambda_1,\ldots,\lambda_{n-1}$ be the eigenvalues of $A_\Gamma(N)$. Since N is not an end-point of $\mathrm{CN}_p\Gamma$, $\lambda_i>0$ for all i. Thus, by concavity, K_∞ has a finite supporting tangent $(\mu_1,\ldots,\mu_{n-1})$ at $(\lambda_1,\ldots,\lambda_{n-1})$. Suppose first that all the λ_i are distinct. Define $\tilde{K}_\infty$ such that, for all $\lambda_1',\ldots,\lambda_{n-1}'$:

$$\tilde{K}_\infty(\lambda_1',\ldots,\lambda_{n-1}'):=K_\infty(\lambda_1,\ldots,\lambda_{n-1})+\sum_{i=1}^n\mu_i(\lambda_i'-\lambda_i).$$

By concavity, for all $\lambda_1',\ldots,\lambda_{n-1}'$:

$$K_\infty(\lambda_1',\ldots,\lambda_{n-1}')\leqslant\tilde{K}_\infty(\lambda_1',\ldots,\lambda_{n-1}').$$

For a smooth function f, we define:

$$\tilde{K}_{\infty,E}(f)=\frac{1}{\|\nabla f\|}\tilde{K}_\infty(\mathrm{Hess}(f)|_E).$$

Denote $P=x(d_H-h)$. At p:

$$\mathrm{Hess}(P)=\nabla x\otimes\nabla d_H+\nabla d_H\otimes\nabla x.$$

At p, for all $1 \leqslant \alpha \leqslant n-1$, by definition, $\langle e_\alpha, \nabla d_H \rangle = 0$. Thus, by (4.2), for all X, and for all $1 \leqslant \alpha, \beta \leqslant n-1$:

$$\begin{aligned} &X \operatorname{Hess}(P)(e_\alpha, e_\beta) \\ &\quad = (\nabla_X \operatorname{Hess}(P))(e_\alpha, e_\beta) + \operatorname{Hess}(P)(\nabla_X e_\alpha, e_\beta) + \operatorname{Hess}(P)(e_\alpha, \nabla_X e_\beta) \\ &\quad = (\nabla_X \operatorname{Hess}(P))(e_\alpha, e_\beta) - \operatorname{Hess}(d_H)(X, e_\alpha)x_{;\beta} - \operatorname{Hess}(d_H)(X, e_\beta)x_{;\alpha}. \end{aligned}$$

We extend $e_1, \ldots, e_{n-1}$ to a basis $e_0, \ldots, e_n$ for T_pM. Observe, in particular, that the plane spanned by e_0 and e_n coincides with the plane spanned by V and N. Since all the λ_i are distinct, they are smooth in a neighbourhood of p and thus, with respect to this basis, for all $1 \leqslant \alpha \leqslant n-1$ and $0 \leqslant k \leqslant n$, bearing in mind those terms which vanish at p, we obtain:

$$\begin{aligned} \partial_k \lambda_\alpha &= \partial_k \operatorname{Hess}(\Phi_0)(e_\alpha, e_\alpha) + \partial_k \operatorname{Hess}(P)(e_\alpha, e_\alpha) \\ &= \partial_k \operatorname{Hess}(\Phi_0)(e_\alpha, e_\alpha) - 2x_{;\alpha}h_{;\alpha k} - x_{;k}h_{;\alpha\alpha} + x_{;k}d_{H;\alpha\alpha}. \end{aligned}$$

The Hessian of the restriction of $d_H - h$ to Γ vanishes at P. Furthermore $\nabla(d_H - h)(p) = \nabla d_H(p) = -V$. Thus, by Lemma 4.3, at p, for all α:

$$d_{H;\alpha\alpha} - h_{;\alpha\alpha} = A_\Gamma(\nabla d_H)_{\alpha\alpha} = -A_\Gamma(V)_{\alpha\alpha}.$$

Thus, for all $1 \leqslant \alpha \leqslant n-1$ and for all $0 \leqslant k \leqslant n$ at p:

$$\partial_k \lambda_\alpha = \partial_k \operatorname{Hess}(\Phi_0)(e_\alpha, e_\alpha) - 2x_{;\alpha}h_{;\alpha k} - x_{;k}A_\Gamma(V)_{\alpha\alpha}. \tag{4.3}$$

Thus, for all k, at p bearing in mind that $\tilde{K}_{\infty,E}(\Phi_1) = \kappa$ at p:

$$\partial_k \tilde{K}_{\infty,E}(\Phi_1) = \partial_k \tilde{K}_{\infty,E}(\Phi_0) + (M\nabla x)_k,$$

where, for any vector U:

$$(MU)_k = \kappa\langle N, V\rangle U_k + \kappa\langle N, U\rangle V_k - 2\sum_{\alpha=1}^{n-1} \mu_\alpha U_\alpha h_{;\alpha k} - \sum_{\alpha=1}^{n-1} \mu_\alpha A_\Gamma(V)_{\alpha\alpha} U_k.$$

We claim that for V sufficiently close to $\hat{N}$, M is invertible. Indeed, suppose that $MU = 0$ for some non-trivial U. Taking the inner product with $(0, \mu_1 U_1, \ldots, \mu_{n-1}U_{n-1}, 0)$ yields:

$$2\sum_{\alpha,\beta=1}^{n-1} (\mu_\alpha U_\alpha)(\mu_\beta U_\beta)h_{;\alpha\beta} + \left(\sum_{\alpha=1}^{n-1} \mu_\alpha A_\Gamma(V)_{\alpha\alpha} - \kappa\langle N, V\rangle\right)\sum_{\beta=1}^{n-1} \mu_\beta U_\beta^2 = 0$$

However, by construction, and bearing in mind Proposition 2.3:

$$K_\infty(A_\Gamma(V)) > K(-H)(p) > \kappa(p).$$

Thus, bearing in mind concavity and Proposition 2.4 (4) applied to K_∞:

$$\begin{aligned}\sum_{\alpha=1}^{n-1}\mu_\alpha A_\Gamma(V)_{\alpha\alpha} &= K_\infty(A_\Gamma(N)) + \sum_{\alpha=1}^{n-1}\mu_\alpha(A_\Gamma(V)_{\alpha\alpha}-\lambda_\alpha)\\ &\geqslant K_\infty(A_\Gamma(V))\\ &> \kappa(p).\end{aligned}$$

Hence:

$$\sum_{\alpha=1}^{n-1}\mu_\alpha A_\Gamma(V)_{\alpha\alpha} - \kappa\langle N,V\rangle > 0. \tag{4.4}$$

In particular:

$$\sum_{\alpha,\beta=1}^{n-1}(\mu_\alpha U_\alpha)(\mu_\beta U_\beta)h_{;\alpha\beta} = 0.$$

Since the restriction of Hess(h) to H is positive definite, it follows that $\mu_\alpha U_\alpha = 0$ for all $1 \leqslant \alpha \leqslant n-1$. Taking the inner product of MU with $(0, U_1, \ldots, U_{n-1}, 0)$ now yields:

$$\left(\sum_{\alpha=1}^{n-1}\mu_\alpha A_\Gamma(V)_{\alpha\alpha} - \kappa\langle N,V\rangle\right)\sum_{\beta=1}^{n-1}U_\beta^2 = 0.$$

Thus by (4.4), $U_\alpha = 0$ for all $1 \leqslant \alpha \leqslant n-1$. Now observe that M preserves the plane generated by e_0 and e_n. Observe, furthermore, that this plane coincides with the plane generated by N and V. With respect to the basis (N, V), the matrix of the restriction of M to this plane is given by:

$$M|_{\langle N,V\rangle} = \begin{pmatrix}\kappa\langle N,V\rangle - \lambda(V) & 0\\ \kappa & 2\kappa\langle N,V\rangle - \lambda(V)\end{pmatrix},$$

where, for all W:

$$\lambda(W) = \sum_{\alpha=1}^{n-1}\mu_\alpha A_\Gamma(W)_{\alpha\alpha}.$$

By linearity, for any constant, c, the set of all unit vectors W in this plane such that $c\langle N,W\rangle - \lambda(W) = 0$ is either empty, consists of two points, or is the entire circle. However, on the one hand, by (4.4):

$$\kappa\langle N,V\rangle - \lambda(V) \neq 0.$$

On the other hand, by Proposition 2.4 (4):

$$\lambda(N) = \sum_{\alpha=1}^{n-1}\mu_\alpha A_\Gamma(N)_{\alpha\alpha} = K_\infty(A_\Gamma(N)) = \kappa(p).$$

In particular, $2\kappa\langle N,N\rangle - \lambda(N) \neq 0$. The restriction of M to the plane generated by e_0 and e_n is therefore non-invertible for at most 4 distinct values of V. In particular, for V sufficiently close to $\hat{N}$, it is invertible. It follows that, $U_0 = U_n = 0$, and so $U = 0$. M is therefore invertible, as asserted.

There therefore exists x such that, at p, for all k:

$$(M\nabla x)_{;k} = -\partial_k \tilde{K}_{\infty,E}(\Phi_0) + \kappa_{;k}.$$

Consequently:

$$\tilde{K}_{\infty,E}(\Phi) = \kappa + O(d_p^2) \quad \Rightarrow \quad K_{\infty,E}(\Phi) \leqslant \kappa + O(d_p^2).$$

Finally, if $\lambda_i = \lambda_j$ for some $i \neq j$, then, by convexity, we may choose μ such that $\mu_i = \mu_j$, and we proceed as before. This completes the proof. □

For $M > 0$, we define Φ by:

$$\Phi = \Phi_0 + x(d_H - h) + M d_H^2. \tag{4.5}$$

Proposition 4.5. — *If D represents the Grassmannian distance between two $(n-2)$-dimensional subspaces then:*

$$D(E, E(\Phi, d_H)) = O(d_p^2) + O(d_H).$$

Proof. — Bearing in mind, (4.2), for all i, at p:

$$X\langle e_i, \nabla d_H\rangle = \langle \nabla_X e_i, \nabla d_H\rangle + \langle e_i, \nabla_X \nabla d_H\rangle = 0.$$

Likewise, $X\langle e_i, \nabla\Phi_0\rangle = 0$. It follows that $D(E, E(\Phi_0, d_H)) = O(d_p^2)$. Moreover, since xh is of order 3, near p:

$$\nabla\Phi = \nabla\Phi_0 + (x + 2Md_H)\nabla d_H + O(d_p^2) + O(d_H).$$

Thus:

$$\langle \nabla\Phi, \nabla d_H\rangle = \langle \nabla\Phi_0 + O(d_p^2) + O(d_H), \nabla d_H\rangle.$$

It follows that $D(E(\Phi_0, d_H), E(\Phi, d_H)) = O(d_p^2) + O(d_H)$. The result now follows by the triangle inequality. □

Proposition 4.6. — *For $d_p \leqslant 1$, and for $Md_H \leqslant 1$:*

$$1/\|\nabla\Phi\| \leqslant 1 + O(d_p) + O(Md_H),$$
$$\|\nabla\Phi_1\|/\|\nabla\Phi\| \leqslant 1 + 2Md_H + O(Md_H d_p) + O(M^2 d_H^2).$$

Proof. — Indeed:

$$\nabla\Phi = \nabla\Phi_1 + 2Md_H\nabla d_H.$$

Thus:

$$\|\nabla\Phi\|^2 \geqslant \|\nabla\Phi_1\|^2 + 4Md_H\langle \nabla d_H, \nabla\Phi_1\rangle.$$

Observe that:

$$\|\nabla\Phi_1\| \leqslant 1 + O(d_p). \tag{4.6}$$

Furthermore $\|\nabla d_H\| = 1$. It follows by the Cauchy–Schwarz inequality that:

$$\|\nabla\Phi\|^2 \geqslant \|\nabla\Phi_1\|^2 - 4Md_H + O(Md_Hd_p).$$

Since $Md_H \leqslant 1$, by it follows by Taylor's Theorem that:

$$1/\|\nabla\Phi\| \leqslant 1/\|\nabla\Phi_1\| + 2Md_H/\|\nabla\Phi_1\|^3 + O(Md_Hd_p) + O(M^2d_H^2).$$

The first inequality follows by (4.6). Furthermore, by (4.6) again:

$$\|\nabla\Phi_1\|/\|\nabla\Phi\| \leqslant 1 + 2Md_H + O(Md_Hd_p) + O(M^2d_H^2),$$

as desired. □

Proposition 4.7. — *If N is not an end-point of $\mathrm{CN}_p\Gamma$, then there exists $\delta > 0$ such that for $M \geqslant 1$, for $d_p \leqslant 1$ and for $Md_H \leqslant 1$:*

$$K_{\infty,E}(\Phi) \leqslant \kappa - \delta Md_H + O(d_p^2) + O(d_H) + O(Md_Hd_p) + O(M^2d_H^2).$$

Proof. — By Proposition 4.4:

$$K_{\infty,E}(\Phi_1) \leqslant \kappa + O(d_p^2).$$

Since $\mathrm{Hess}(\Phi_1) = O(1)$, by Proposition 4.5:

$$K_{\infty,E(\Phi,d_H)}(\Phi_1) \leqslant \kappa + O(d_p^2) + O(d_H).$$

Thus, by Proposition 4.6:

$$\frac{\|\nabla\Phi_1\|}{\|\nabla\Phi\|}K_{\infty,E(\Phi,d_H)}(\Phi_1) \leqslant (1 + 2Md_H)\kappa + O(d_p^2) + O(d_H) + O(Md_Hd_p) + O(M^2d_H^2).$$

However, differentiating Md_H^2 yields:

$$\mathrm{Hess}(Md_H^2) = 2M\nabla d_H \otimes \nabla d_H + 2Md_H\,\mathrm{Hess}(d_H).$$

Observe that $\nabla d_H \otimes \nabla d_H$ vanishes along $(\nabla d_H)^\perp$. Furthermore, since H is locally strictly concave, the restriction of $\mathrm{Hess}(d_H)$ to $E(\Phi, d_H)$ is negative definite. Thus, if the restriction of $\mathrm{Hess}(\Phi_1)$ to $E(\Phi, d_H)$ is not positive definite, then the restriction of $\mathrm{Hess}(\Phi)$ to $E(\Phi, d_H)$ is also not positive definite, and we are done. Otherwise, let A and B be the restrictions of $\|\Phi\|^{-1}\,\mathrm{Hess}(\Phi_1)$ and $2Md_H\|\Phi\|^{-1}\,\mathrm{Hess}(d_H)$ to $E(\Phi, d_H)$ respectively. By concavity of K_∞:

$$\|\nabla\Phi\|^{-1}K_\infty(\mathrm{Hess}(\Phi)|_{E(\Phi,d_H)}) = K_\infty(A + B) \leqslant K_\infty(A) + DK_{\infty,A}(B).$$

However by construction, $K(-H)(p) > \kappa(p)$, where $-H$ is H furnished with the reverse orientation. Thus, bearing in mind Proposition 2.3, Proposition 2.4 (5) and Lemma 4.3, at p:

$$\begin{aligned} DK_{\infty,A}(-\operatorname{Hess}(d_H)|_{E(\Phi,d_H)}) &\geqslant K_\infty(-\operatorname{Hess}(d_H|_{E(\Phi,d_H)})) \\ &> K(-\operatorname{Hess}(d_H)|_{(\nabla d_H)^\perp}) \\ &= K(-H)(p) \\ &> \kappa(p). \end{aligned}$$

Thus, bearing in mind Proposition 4.6, there exists $\delta > 0$ such that:

$$DK_{\infty,A}(B) \leqslant -Md_H(2\kappa+\delta) + O(Md_Hd_p) + O(M^2d_H^2).$$

Combining these relations yields:

$$K_{\infty,E}(\Phi) \leqslant \kappa - \delta Md_H + O(d_p^2) + O(d_H) + O(Md_Hd_p) + O(M^2d_H^2),$$

as desired. □

4.3. Geometric properties of the barrier

We now suppose the contrary in Proposition 4.2. Let $(\hat{\Sigma}_n, \kappa_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ converging to $(\hat{\Sigma}, \kappa)$, say. For convenience, we assume that $(\hat{\Sigma}_n, \kappa_n) = (\hat{\Sigma}, \kappa)$ for all n. Let Γ be the boundary of $\hat{\Sigma}$. Let $\hat{N}$ be the upward-pointing unit normal vector to $\hat{\Sigma}$ at p. For all n, let Σ_n be a smooth compact LSC immersed hypersurface such that $K(\Sigma_n) = \kappa_n$ and $\Sigma_n < \hat{\Sigma}_n$. For all n, let N_n be the upward-pointing unit normal vector field over Σ_n. For all n, let p_n be a point in Γ. Suppose that $(p_n)_{n\in\mathbb{N}}$ converges to the point p in Γ and that $(N_n(p_n))_{n\in\mathbb{N}}$ converges to a limit N, say. For convenience, we assume that $p_n = p$ for all n. By Proposition 4.1, we may suppose that there exists $\theta > 0$ such that for all n, $N_n(p)$ makes an angle of at least θ with the lower end-point of $\mathrm{CN}_p(\Gamma)$. In particular, N is an element of $\mathrm{CN}_p\Gamma$. $A_\Gamma(N)$ is therefore positive-definite.

We assume that $K_\infty(A_\Gamma(N)) \leqslant \kappa(p)$ and obtain a contradiction. Let V, H, d_H, Φ_0 and Φ_1 be defined as in the preceeding section. Define the family $\mathcal{G}$ by $\mathcal{G} := \{\partial\hat{\Sigma} \mid (\hat{\Sigma}, \kappa) \in \mathcal{F}\}$. Let $\theta > 0$ be such that both $\partial^+\mathrm{CN}_p\Gamma$ and V and V and $\hat{N}$ make angles of at least θ. Suppose furthermore that $\hat{N}$ and N_n make angles of at least θ for all n. Let $r > s > 0$ be as in Theorem 3.18 for the family $\mathcal{G}$ and the angle θ. For all n, we henceforth identify Σ_n, $\hat{\Sigma}$ and Γ with $\Sigma_{n,p,r}$, $\hat{\Sigma}_{p,r}$ and $\Gamma_{p,r}$. By Theorem 3.18, for all n, Σ_n is embedded; meets $\partial B_r(p_n)$ transversally; and is contained within both $X_{p,r}(\hat{\Sigma})$ and $X_r(\Gamma, V)$. We denote $X_n := X_{p,r}(\Sigma_n, \hat{\Sigma})$. By Theorem 3.18, for all n, X_n

contains a ball of radius s. Thus, by compactness of the family of convex sets, we may suppose that there exists a convex subset X with non-trivial interior towards which $(X_n)_{n\in\mathbb{N}}$ converges in the Hausdorff sense. Observe that p is a boundary point of X. Let I be the set of supporting normals to X at p. By Proposition 3.19, I is contained in the closure of $\mathrm{CN}_p^-\hat{\Sigma}$.

Proposition 4.8. — *If $N'\in I$ lies strictly below N, then*

$$K_\infty(A_\Gamma(N')) < K_\infty(A_\Gamma(N)) \leqslant \kappa(p).$$

Proof. — Observe that $\hat{N}(p)$ lies strictly above N. Furthermore, by Proposition 2.3, $K_\infty(A_\Gamma(\hat{N}(p))) > \kappa(p)$. Thus, if $K_\infty(A_\Gamma(N')) \geqslant \kappa(p)$, then, by linearity of A_Γ and concavity of K_∞, $K_\infty(A_\Gamma(N)) > \kappa(p)$. This is absurd, and the result follows. □

Let N_0 be the lower end-point of I. We shall see presently that when $K_\infty(A_\Gamma(N_0)) < \kappa(p)$ it is relatively straightforward to construct comparison hypersurfaces of K-curvature strictly less than κ. The case of equality is more subtle. We therefore assume that $N_0 = N$ and that $K_\infty(A_\Gamma(N)) = \kappa(p)$.

Let Σ be the closure of the intersection of ∂X with the interior of $X_{p,r}(\hat{\Sigma})$. For $\rho,\epsilon>0$, we define $U_{\rho,\epsilon}$ by:

$$U_{\rho,\epsilon} = \left\{q\in M \mid d_p(q)<\epsilon,\ d_H(q)<\rho\epsilon^2\right\}.$$

Proposition 4.9. — *There exists $\rho>0$ such that for all sufficiently small ϵ, and for all $p\in\Sigma\cap\partial U_{\rho,\epsilon}$, $d_p(q)<\epsilon$ and $d_H(q)=\rho\epsilon^2$.*

Proof. — Indeed, by definition of H, there exists $\rho>0$ such that for sufficiently small ϵ and for all $p\in X_r(\Gamma,V)\cap\partial U_{\rho,\epsilon}$, $d_p(q)<\epsilon/2$ and $d_H(q)=\rho\epsilon^2$. Since Σ is contained in $X_r(\Gamma,V)$, the result follows. □

We choose ρ as in Proposition 4.9, and henceforth keep it fixed. Since Γ is LSC and lies strictly above H, we may define the function h near p such that $h(p)=0$; $\nabla(h)(p)=0$; the restriction of $\mathrm{Hess}(h)(p)$ to T_pH is positive definite; and $d_H - h = O(d_p^3)$ along Γ. We choose x as in Proposition 4.4. We recall that for $M>0$, the function Φ is given by:

$$\Phi = \Phi_0 + x(d_H - h) + Md_H^2.$$

Observe that the only parameters that remain to be determined at this stage are ϵ and M. M is chosen as a function of ϵ using the following result:

Proposition 4.10. — *There exists a continuous function $B:[0,\infty[\,\to[0,\infty[$ such that $B(0)=0$ and for all sufficiently small ϵ, if $M\geqslant B(\epsilon)\epsilon^{-2}$, then $\Phi\geqslant 0$ along $\partial(\Sigma\cap U_{\rho,\epsilon})$.*

Proof. — Let $m : [0,\infty[\rightarrow [0,\infty[$ be as in Proposition 3.20. Recall that, by definition, $-\nabla d_H(p) = V$ lies strictly above $\hat{N}(p)$. This vector therefore also lies strictly above every element of I. By Proposition 3.20, there therefore exists $c > 0$ such that, upon reducing r if necessary, for all sufficiently large n and for all $q \in \Sigma_n$:

$$\|\pi_{n,q}(\nabla d_H)\| \geqslant c,$$

where $\pi_{n,q}$ is the orthogonal projection of T_pM onto $T_q\Sigma_n$. Furthermore, by definition $\nabla\Phi_0(p) = N_0$. Thus, upon modifying m if necessary, we may suppose that, for all sufficiently large n, and for all $q \in \Sigma_n$:

$$\langle \pi_{n,q}(\nabla\Phi_0), \pi_{n,q}(\nabla d_H)\rangle \geqslant -m(d_p(q)).$$

Now consider $q \in \Sigma_n \cap \partial U_{\rho,\epsilon}$. By Proposition 4.9, for sufficiently large n, $d_p(q) < \epsilon$ and $d_H(q) = \rho\epsilon^2$. Let γ be an integral curve of $-\pi_{n,q}(\nabla d_H)$ in Σ_n starting at q. Since Σ_n is compact and since $-\pi_{n,q}(\nabla d_H)$ never vanishes, γ meets another boundary point q' of Σ_n after finite time. Furthermore, since d_H is strictly decreasing along γ, q' is not an element of $\partial U_{\rho,\epsilon}$, and is therefore an element of Γ. Thus, since $d_H(q) = \rho\epsilon^2$ and $d_H = 0$ along Γ, $\mathrm{length}(\gamma) \leqslant c^{-1}\rho\epsilon^2$. Since Φ_0 vanishes over Γ, integration over γ yields:

$$\Phi_0(q) \geqslant -m(\epsilon)c^{-2}\rho\epsilon^2.$$

Now choose $q \in \Sigma \cap \partial U_{\rho,\epsilon}$. Taking limits, we obtain:

$$\Phi_0(q) \geqslant -m(\epsilon)c^{-2}\rho\epsilon^2.$$

Thus, since $x(q) = O(d_p) = O(\epsilon)$, $d_H(q) = \rho\epsilon^2$ and $h(q) = O(d_p^2) = O(\epsilon^2)$, upon modifying m if necessary:

$$\Phi(q) = (\Phi_0 + x(d_H - h) + Md_H^2)(q) \geqslant Md_H^2(q) - m(\epsilon)O(\epsilon^2). \tag{4.7}$$

However, $(\partial\Sigma) \cap U_{\rho,\epsilon}$ is contained in Γ. Thus, since Φ_0 vanishes along Γ, by definition of h, along $(\partial\Sigma) \cap U_{\rho,\epsilon}$:

$$\Phi(q) \geqslant Md_H^2 - O(d_p^4). \tag{4.8}$$

However, by definition of H, along Γ, $d_H \geqslant O(d_p^2)$, and along $\Sigma \cap (\partial U_{\rho,\epsilon})$, $d_H = \rho\epsilon^2$. Thus, by (4.7) and (4.8), there exists a continuous function $B : [0,\infty[\rightarrow [0,\infty[$ such that $B(0) = 0$ and, for sufficiently small $\epsilon > 0$, if $M \geqslant B(\epsilon)\epsilon^{-2}$, then $\Phi \geqslant 0$ along $\partial(\Sigma \cap U_{\rho,\epsilon})$, as desired. □

For all ϵ small, we henceforth choose $M := M(\epsilon) = B(\epsilon)\epsilon^{-2}$.

Proposition 4.11. — *For sufficiently small $\epsilon > 0$, $\nabla\Phi$ is non-vanishing over $U_{\rho,\epsilon}$.*

Proof. — As in the proof of Proposition 4.5:

$$\begin{aligned}\nabla\Phi &= \nabla\Phi_0 + (d_H - h)\nabla x + x\nabla(d_H - h) + 2Md_H\nabla d_H\\ &= \nabla\Phi_0 + 2Md_H\nabla d_H + O(d_p) + O(d_H).\end{aligned}$$

However, by definition, $\nabla\Phi_0$ and ∇d_H are non-colinear, and the result follows. □

In particular, the level sets of Φ are smooth hypersurfaces of $U_{\rho,\epsilon}$. We orient these hypersurfaces such that $\nabla\Phi/\|\nabla\Phi\|$ is the upward-pointing unit normal vector.

Proposition 4.12. — *For all sufficiently small $\epsilon > 0$, there exists $\delta > 0$ such that if L is a level subset of Φ in $U_{\rho,\epsilon}\cap X_{p,r}(\hat{\Sigma})$, if q is a point of L, and if A is the shape operator of L at q, then, either A is not positive-definite, or $K(A) \leqslant \kappa(q) - \delta$.*

Proof. — Let q be a point in the closure of $U_{\rho,\epsilon} \cap X_{p,r}(\hat{\Sigma})$. Let A be the shape operator of the level hypersurface of Φ passing through q. By compactness, we may suppose that q maximises $K(A)$ over the closure of $U_{\rho,\epsilon}$. It suffices to show that $K(A) < \kappa(q)$ when A is positive definite. However, by Lemma 4.3, A is the restriction of $\|\nabla\Phi\|^{-1}\,\mathrm{Hess}(\Phi)$ to $\nabla\Phi^{\perp}$. Thus, if A is positive definite, then bearing in mind Proposition 2.3:

$$\begin{aligned}K(A) &= K(\|\nabla\Phi\|^{-1}\,\mathrm{Hess}(\Phi)|_{(\nabla\Phi)^{\perp}})\\ &< K_{\infty}(\|\nabla\Phi\|^{-1}\,\mathrm{Hess}(\Phi)|_{E(\Phi,d_H)})\\ &= K_{\infty,E}(\Phi).\end{aligned}$$

However, $d_H \geqslant O(d_p^2)$ over $U_{\rho,\epsilon} \cap X_{p,r}(\hat{\Sigma})$. Thus, by Proposition 4.7, for sufficiently small $\epsilon > 0$, $K_{\infty,E}(\Phi) \leqslant \kappa$. In particular, $K(A) < \kappa(q)$, as desired. □

We now complete the proof of Proposition 4.2:

Proof of Proposition 4.2. — Recall that N_0 is the lower end-point of I. We first suppose that $K_{\infty}(A_{\Gamma}(N_0)) = \kappa(p)$. Choose $\epsilon > 0$ sufficiently small so that Propositions 4.10, 4.11 and 4.12 hold. In particular, $\Phi \geqslant 0$ along $\partial(\Sigma \cap U_{\rho,\epsilon})$ and there exists $\delta > 0$ such that for all $q \in U_{\rho,\epsilon} \cap X_{p,r}(\hat{\Sigma})$, if A is the shape operator of the level hypersurface of Φ passing through q, then either A is not positive-definite, or $K(A) \leqslant \kappa(q) - \delta$. Let $X \in \mathrm{N}_p\Gamma$ be the unit vector normal to N_0 pointing into Σ. Since $\nabla\Phi(p) = 0$, we may perturb Φ to a function Φ' such $\Phi' \geqslant 0$ over $\partial(\Sigma \cap U_{\rho,\epsilon})$; for all $q \in U_{\rho,\epsilon} \cap X_{p,r}(\hat{\Sigma})$, if A is the shape operator of the level hypersurface of Φ passing through q, then either A is not positive-definite, or $K(A) < \kappa(q)$; and $\nabla\Phi'(p) = \lambda X$ for some $\lambda < 0$. In particular, the restriction of Φ to

$\Sigma\cap U_{\rho,\epsilon}$ achieves its minimum value at some interior point $q\in\Sigma\cap U_{\rho,\epsilon}$. If L is the level hypersurface of Φ passing through q, then L is an interior tangent to Σ at q. However, since Σ is a limit of smooth hypersurfaces of K-curvature prescribed by κ, the K-curvature of Σ is prescribed by κ in the viscosity sense. That is, the K-curvature of L at p is no less than $\kappa(p)$. This yields a contradiction in the case where $K_\infty(A_\Gamma(N_0))=\kappa(p)$. When $K_\infty(A_\Gamma(N_0))<\kappa(p)$, choosing $x=0$ in the above construction yields a function Φ with the required properties, and we thus also obtain a contradiction in this case. This completes the proof. □

4.4. The unbounded case

We now consider general K. We outline the modifications required in the proof of Proposition 4.2 to obtain Proposition 4.1. Choose $(\hat{\Sigma},\kappa)\in\mathcal{F}$. Denote $\Gamma=\partial\hat{\Sigma}$. Choose $p\in\Gamma$. We define V and H as before. Let N be the lower end-point of $\mathrm{CN}_p\Gamma$. Using this N, we define Φ_0 as before. For appropriate functions x and h defined near p, we define Φ_1 as before. Define E as before. For a smooth function, f, we define $\lambda_{1,E}(f)$ to be the least eigenvalue of the restriction of $\mathrm{Hess}(f)$ to E. Proposition 4.4 becomes:

Proposition 4.13. — *For all h defined near p satisfying the hypotheses of Proposition 4.4, there exists a function x, defined near p such that $x(p)=0$, $\mathrm{Hess}(x)(p)=0$, and:*

$$\lambda_{1,E}(\Phi_1)\leqslant O(d_p^2).$$

Remark. — Importantly, in contrast to Proposition 4.4, it is not necessary to assume that N is not an end-point of $\mathrm{CN}_p\Gamma$. This is because the function T used in the proof of Proposition 4.13 is differentiable at the point $(0,\ldots,0,\lambda_{m+1},\ldots,\lambda_{n-1})$, which is not necessarily the case for general K_∞.

Proof. — As in the proof of Proposition 4.4, the restrictions of $\mathrm{Hess}(\Phi_0)$ and $\mathrm{Hess}(\Phi_1)$ to E are both equal to $A_\Gamma(N)$. Let $\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_{n-1}$ be the eigenvalues of this restriction. By definition of N, $\lambda_1=0$. Let $1\leqslant m\leqslant n-1$ be such that $\lambda_m=0$ but $\lambda_{m+1}>0$. For any function f, we define $T(f)$ to be the sum of the first m eigenvalues of the restriction of $\mathrm{Hess}(f)$ to E. In particular $T(\Phi_1)(p)=T(\Phi_0)(p)=0$. By (4.3), for all k, at p:

$$\partial_kT(\Phi_1)=\partial_kT(\Phi_0)-(M\nabla x)_k,$$

where, for any vector U:

$$(MU)_k = -\sum_{\alpha=1}^{m} \left(2U_\alpha h_{;\alpha k} + U_k A_\Gamma(V)_{\alpha\alpha}\right).$$

We claim that M is invertible. Indeed, suppose that $MU = 0$ for some non-trivial U. Taking the inner product with $(0, U_1, \ldots, U_m, 0, \ldots, 0)$ yields:

$$\sum_{\alpha,\beta=1}^{m} U_\alpha U_\beta 2h_{;\alpha\beta} + \sum_{\alpha,\beta=1}^{m} U_\alpha^2 A_\Gamma(V)_{\beta\beta} = 0.$$

Since the restriction of $\mathrm{Hess}(h)$ to E is positive definite, and since $A_\Gamma(V)$ is positive definite, it follows that $U_1 = \cdots = U_m = 0$. Thus, for all k:

$$U_k = -(MU)_k / \sum_{\alpha=1}^{m} A_\Gamma(V)_{\alpha\alpha} = 0.$$

It follows that the kernel of M is trivial. M is therefore invertible, as asserted. There therefore exists x such that, at p, for all k, $\partial_k T(\Phi_1) = 0$. For such a choice of x, $T(\Phi_1) = O(d_p^2)$. Since $\lambda_{1,E}(\Phi_1) \leqslant T(\Phi_1)/m$, the result follows. □

For $M > 0$, we define Φ as before. Proposition 4.7 now becomes:

Proposition 4.14. — *There exists $\delta > 0$ such that for all M:*

$$\lambda_{1,E(\Phi,d_H)} \leqslant -M\delta d_H + O(d_p^2) + O(d_H).$$

Proof. — By Proposition 4.13:

$$\lambda_{1,E}(\Phi_1) \leqslant O(d_p^2).$$

Since $\mathrm{Hess}(\Phi_1) = O(1)$, by Proposition 4.5:

$$\lambda_{1,E(\Phi,d_H)}(\Phi_1) \leqslant O(d_p^2) + O(d_H).$$

Differentiating Md_H^2 yields:

$$\mathrm{Hess}(Md_H^2) = 2M\nabla d_H \otimes \nabla d_H + 2Md_H \,\mathrm{Hess}(d_H).$$

The first term vanishes along $(\nabla d_H)^\perp$. On the other hand, for sufficiently small ϵ, there exists $\delta_1 > 0$ such that for all $X \in (\nabla d_H)^\perp$, $\mathrm{Hess}(d_H)(X, X) \leqslant -\delta\|X\|^2/2$. Thus:

$$\lambda_{1,E(\Phi,d_H)}(\Phi) \leqslant -M\delta d_H + O(d_p^2) + O(d_H),$$

as desired. □

We repeat the geometric construction of Section 4.3. Observe that in this case, it is not necessary to assume that N is an interior point of $\mathrm{CN}_p\Gamma$ (cf. the remark following Proposition 4.13). Proposition 4.12 becomes:

Proposition 4.15. — *For all sufficiently small $\epsilon > 0$, if L is a level subset of Φ in $U_{\rho,\epsilon}\cap X_{p,r}(\hat{\Sigma})$, then L has at least one non-positive principal curvature at every point.*

Proof. — Indeed, by Proposition 4.14, for sufficiently small ϵ, $\lambda_{1,E(\Phi,d_H)}(\Phi) \leqslant 0$ throughout $U_{\rho,\epsilon}\cap X_{p,r}(\hat{\Sigma})$. In other words, at every point $p\in U_{\rho,\epsilon}\cap X_{p,r}(\hat{\Sigma})$, the restriction of $\mathrm{Hess}(\Phi)$ to $E(\Phi,d_H)$ is not positive definite. In particular, for all $p\in U_{\rho,\epsilon}\cap X_{p,r}(\hat{\Sigma})$, the restriction of $\mathrm{Hess}(\Phi)$ to $(\nabla\Phi)^{\perp}$ is also not positive definite. The result now follows by Lemma 4.3. □

We now prove of Proposition 4.1:

Proof of Proposition 4.1. — In order to obtain a contradiction, we suppose that N_0 is the lower end-point of $\mathrm{CN}_p\Gamma$. Choose $\epsilon > 0$ sufficiently small so that Propositions 4.10, 4.11 and 4.15 hold. In particular, $\Phi \geqslant 0$ over $\partial(\Sigma\cap U_{\rho,\epsilon})$. By Proposition 4.15, for all $q\in\Sigma\cap U_{\rho,\epsilon}$, the level hypersurface of Φ passing through q has at least one non-positive principal curvature at q. Let $X\in \mathrm{N}_p\Gamma$ be the unit vector normal to N_0 pointing into Σ. Since $\nabla\Phi(p)=0$, we may perturb Φ to a function Φ' such that $\Phi'\geqslant 0$ over $\partial(\Sigma\cap U_{p,\epsilon})$; for all $q\in\Sigma\cap U_{\rho,\epsilon}$, if A is the shape operator of the level hypersurface of Φ passing through q, then either A is not positive-definite or $K(A) < \kappa(q)$; and $\nabla\Phi'(p) = \lambda X$ for some $\lambda < 0$. In particular, the restriction of Φ to $\Sigma\cap U_{\rho,\epsilon}$ achieves its minimum value at some interior point $q\in\Sigma\cap U_{\rho,\epsilon}$. If L is the level hypersurface of Φ passing through q, then L is an interior tangent to Σ at q. However, since Σ is a limit of smooth hypersurfaces of K-curvature prescribed by κ, the K-curvature of Σ is prescribed by κ in the viscosity sense. That is, the K-curvature of L at p is no less than $\kappa(p)$. This is absurd, and the result follows. □

5. Second Order Boundary Estimates

5.1. Main result

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. Let K be a convex curvature function. Let $\mathcal{F}$ be a family of pairs $(\hat{\Sigma},\kappa)$ where $\hat{\Sigma}$ is a smooth compact LSC immersed hypersurface with generic boundary in M; κ is a smooth positive function over M; and $K(\hat{\Sigma}) > \kappa$. We furnish $\mathcal{F}$ with the product topology of the C^∞ topology in the first component and the C^∞_{loc} topology in the second. We obtain a-priori estimates along the boundary for the shape operators of LSC hypersurfaces of prescribed K-curvature.

PROPOSITION 5.1. — *If $\mathcal{F}$ is compact, then, for $\theta > 0$ small, there exists $B > 0$ such that for all smooth compact LSC immersed hypersurfaces Σ in M such that $K(\Sigma) = \kappa$; $\Sigma < \hat{\Sigma}$; and Σ makes an angle of at least θ with $\hat{\Sigma}$ along their common boundary, and for all $p \in \partial\Sigma$, if A is the shape operator of Σ at p, then $\|A\| \leqslant B$.*

5.2. Preliminary Results

The required barrier will be constructed in the following sections. Upon reducing θ if necessary, we may suppose that for all $(\hat{\Sigma}, \kappa) \in \mathcal{F}$ and for every boundary point p of Σ, the upward-pointing unit normal vector to $\hat{\Sigma}$ at p makes an angle of at least θ with the upper end-point of $\mathrm{CN}_p\partial\hat{\Sigma}$. Denote $\mathcal{G} = \{\partial\hat{\Sigma} \,|\, (\hat{\Sigma}, \kappa) \in \mathcal{F}\}$. Let $r > s > 0$ be as in Theorem 3.18 for the family $\mathcal{G}$ and the angle θ.

Choose $(\hat{\Sigma}, \kappa) \in \mathcal{F}$. Let Γ be the boundary of $\hat{\Sigma}$. Let p be a point of Γ. We henceforth identify Γ and $\hat{\Sigma}$ with $\Gamma_{p,r}$ and $\hat{\Sigma}_{p,r}$ respectively. For brevity, in the sequel, we denote by $\mathcal{B}$ the family of all quantities which only depend upon $\hat{\Sigma}$ and the restrictions to $B_r(p)$ of κ and the metric of M. For any supplementary object, X, we denote by $\mathcal{B}(X)$ those terms that also depend on the restriction of X to $B_r(p)$, if X is a submanifold or function or so on, or just on X otherwise.

Let I be a closed subinterval of the closure of $\mathrm{CN}_p^-\hat{\Sigma}$. Let $m : [0,\infty[\to [0,\infty[$ be a continuous function such that $m(0) = 0$. Both I and m will be determined presently. Let N_0 be the lower end-point if I. Let Σ be a smooth compact LSC immersed hypersurface in M such that $K(\Sigma) = \kappa$; $\Sigma < \hat{\Sigma}$; and Σ makes an angle of at least θ with $\hat{\Sigma}$ along their common boundary. By Theorem 3.18, $\Sigma_{p,r}$ is embedded; meets $\partial B_r(p)$ transversally; is contained within $X_r(\hat{\Sigma})$; and divides $X_r(\hat{\Sigma})$ into two connected components, one of which is convex and which we denote by $X_r(\Sigma, \hat{\Sigma})$. We identify Σ with $\Sigma_{p,r}$. Let N and A be the upward-pointing unit normal vector field and shape operator of Σ respectively. Bearing in mind Proposition 3.20, we suppose that for all $q \in \Sigma$:

$$D(I, N(q)) \leqslant m(d(p, q)), \tag{5.1}$$

where D and d are the distance functions in the total space of UM and in M respectively.

For $q \in \Sigma$, let $e_1, \dots, e_n$ be an orthonormal basis diagonalising DK_A and let $\mu_1, \dots, \mu_n$ be its corresponding eigenvalues. We denote:

$$\mu := \sum_{i=1}^{n} \mu_i = DK_A(\mathrm{Id}) \geqslant 1. \tag{5.2}$$

We define the operator Δ^K on functions over Σ by:

$$\Delta^K f = \sum_{i=1}^{n} \mu_i \operatorname{Hess}^{\Sigma}(f)_{ii}, \tag{5.3}$$

where $\operatorname{Hess}^{\Sigma}$ is the Hessian of the Levi-Civita covariant derivative of Σ. Throughout the sequel, we aim to construct barrier functions that are superharmonic with respect to this generalised Laplacian. The following result plays a fundamental role. It yields a general construction of barrier functions in the non-linear setting.

Lemma 5.2. — *Let $\phi : M \to \mathbb{R}$ be a smooth function such that:*

(1) $\|\nabla\phi\| = 1$*; and*

(2) *the level sets of ϕ are LSC with K-curvature greater than κ.*

Then the restriction of ϕ to Σ satisfies:

$$\Delta^K \phi \geqslant -\|\operatorname{Hess}(\phi)\| \sum_{i=1}^{n} \mu_i \phi_{;i}\phi_{;i}.$$

Proof. — Choose $q \in \Sigma$. We construct two orthonormal bases for T_qM. Let L_q be the level set of ϕ passing through q. Suppose first that $\nabla\phi$ and N are not colinear at q. Then L_q and Σ meet transversally at this point. Let $f_1, \dots, f_{n-1}$ be an orthonormal basis of $T_qL_q \cap T_q\Sigma$ and complete this to an orthonormal basis $f_1, \dots, f_n$ of $T_q\Sigma$. For $1 \leqslant i \leqslant n-1$, denote $f'_i = f_i$, and complete $f'_1, \dots, f'_{n-1}$ to an orthonormal basis $f'_1, \dots, f'_{n+1}$ of T_qM such that f'_n is tangent to L_q; f'_n makes an angle of at most $\pi/2$ with f_n; and f'_{n+1} is normal to L_q.

Let $\theta \in \,]0, \pi/2]$ be the angle between f_n and f'_n. Then:

$$f_n = \cos(\theta) f'_n \pm \sin(\theta) f'_{n+1}.$$

Let m_{ij} and m'_{ij} be the matrices of the restrictions of $\operatorname{Hess}(\phi)$ to $T_q\Sigma$ and T_qL_q respectively with respect to these bases. Since $\|\nabla\phi\| = 1$ and $f'_{n+1} = \pm\nabla\phi$:

$$\operatorname{Hess}(\phi)(f'_{n+1}, \cdot\,) = 0.$$

Consequently:

$$(m_{ij}) = \begin{pmatrix} (m'_{ij}) & \cos(\theta)(m'_{in}) \\ \cos(\theta)(m'_{ni}) & \cos^2(\theta) m'_{nn} \end{pmatrix}.$$

That is:

$$(m_{ij}) = \cos(\theta)(m'_{ij}) + (1-\cos(\theta))\begin{pmatrix}(m'_{ij}) & 0\\ 0 & m'_{nn}\end{pmatrix} - \sin^2(\theta)\begin{pmatrix}0 & 0\\ 0 & m'_{nn}\end{pmatrix}.$$

Thus, since (m'_{ij}) is positive definite:

$$(m_{ij}) \geqslant \cos(\theta)(m'_{ij}) - \sin^2(\theta)\begin{pmatrix}0 & 0\\ 0 & m'_{nn}\end{pmatrix}.$$

Let B^{ij} be the matrix of DK_A with respect to $f_1,\ldots,f_n$. Then, since B^{ij} is positive definite:

$$\sum_{i,j=1}^{n} B^{ij}m_{ij} \geqslant \cos(\theta)\sum_{i,j=1}^{n} B^{ij}m'_{ij} - \sin^2(\theta)B^{nn}m'_{nn}.$$

By concavity of K, bearing in mind Proposition 2.4(4):

$$\sum_{i,j=1}^{n} B^{ij}(m'_{ij} - A_{ij}) \geqslant K(m'_{ij}) - K(A_{ij}) \Rightarrow \sum_{i,j=1}^{n} B^{ij}m'_{ij} \geqslant K(m'_{ij}).$$

However, since $K(L_q) > \kappa(q)$, by Lemma 4.3:

$$K(m'_{ij}) > \kappa(q).$$

Thus:

$$\sum_{i,j=1}^{n} B^{ij}m_{ij} \geqslant \cos(\theta)\kappa(q) - \sin^2(\theta)B^{nn}m'_{nn}.$$

However, by Lemma 4.3 again:

$$\mathrm{Hess}^{\Sigma}(\phi) = \mathrm{Hess}(\phi)|_{T\Sigma} - \langle\nabla\phi, N\rangle A.$$

Thus, bearing in mind Proposition 2.4(4) again:

$$\Delta^K\phi \geqslant \cos(\theta)\kappa(q) - \cos(\theta)\sum_{i=1}^{n}\mu_i\lambda_i - \sin^2(\theta)B^{nn}m'_{nn} = -\sin^2(\theta)B^{nn}m'_{nn}.$$

Finally, $\sin(\theta)f_n$ is the orthogonal projection onto $T\Sigma$ of $\pm\nabla\phi$. Thus:

$$\left|m'_{nn}B^{nn}\sin^2(\theta)\right| \leqslant \|\mathrm{Hess}(\phi)\|\sum_{i=1}^{n}\mu_i\phi_{;i}\phi_{;i}.$$

The result follows in the case where $\nabla\phi$ and N are non-colinear. The case where $\nabla\phi$ and N are colinear follows directly from the concavity of K, and this completes the proof. □

We require the following modification of this result. For $\phi \in C^\infty(M)$, we define the operator $\mathcal{D}_\phi$ over Σ by:

$$\mathcal{D}_\phi f = \sum_{i=1}^{n} \mu_i \phi_{;i} f_{;i}.$$

Corollary 5.3. — *Let ϕ be as in Lemma 5.2. There exists $\delta, C > 0$ in $\mathcal{B}(\phi)$ such that:*

$$(\Delta^K + C\mathcal{D}_\phi)\phi \geqslant \delta \sum_{i=1}^{n} \mu_i \operatorname{Hess}(\phi)_{ii}.$$

Proof. — In the proof of Lemma 5.2, there exists $\delta > 0$ in $\mathcal{B}$ such that:

$$\sum_{i,j=1}^{n} B^{ij} m'_{ij} > \frac{1}{1-\delta}\kappa(q).$$

Thus, since $\sin^2(\theta) B^{nn} m'_{nn} \geqslant 0$:

$$\sum_{i,j=1}^{n} B^{ij} m_{ij} \geqslant \frac{1}{1-\delta}(\cos(\theta)\kappa(q) - \sin^2(\theta) B^{nn} m'_{nn})$$

$$\Rightarrow \sum_{i,j=1}^{n} B^{ij} m_{ij} \geqslant \delta \sum_{i,j=1}^{n} B^{ij} m_{ij} + \cos(\theta)\kappa(q) - \sin^2(\theta) B^{nn} m'_{nn}.$$

Thus:

$$\Delta^K \phi \geqslant \delta \sum_{i,j=1}^{n} B^{ij} \operatorname{Hess}(\phi)_{ij} - \|\operatorname{Hess}(\phi)\| \sum_{i=1}^{n} \mu_i \phi_{;i} \phi_{;i}.$$

The result now follows for $C \geqslant \|\operatorname{Hess}(\phi)\|$. □

We also require the following straightforward relations. Let R^M be the Riemann curvature tensor of M. Let R^Σ be the Riemann curvature tensor of Σ. Let the subscript ; denote covariant differentiation with respect to the Levi-Civita covariant derivative of Σ. We recall the commutation rules of covariant differentiation in a Riemannian manifold:

Proposition 5.4.

(1) *For all i, j, k:*

$$A_{ij;k} = A_{kj;i} + R^M_{ki\nu j},$$

where ν represents the direction normal to Σ; and

(2) *for all i, j, k, l:*

$$A_{ij;kl} = A_{ij;lk} + R^{\Sigma}_{kli}{}^{p} A_{pj} + R^{\Sigma}_{klj}{}^{p} A_{pi}.$$

Proof. — This is an elementary calculation (cf. [17, Lemma 6.3]). □

Proposition 5.5.

(1) *For all p:*

$$\sum_{i=1}^{n} \mu_i A_{ii;p} = \kappa_{;p}.$$

(2) *For all p, q:*

$$\sum_{i=1}^{n} \mu_i A_{ii;pq} = -(D^2K)^{ij,mn} A_{ij;p} A_{mn;q} + \kappa_{;pq}.$$

Proof. — This follows by differentiating the equation $K(A) = \kappa$. □

Let the subscript : denote covariant differentiation with respect to the Levi-Civita covariant derivative of M:

Proposition 5.6. — *Let f be the signed distance function to Σ, let $\nu := (n+1)$ denote the upward pointing normal direction to Σ.*

(1) *Along Σ, for all $1 \leqslant i, j \leqslant n$:*

$$f_{:ij} = A_{ij}, \qquad f_{:i\nu} = f_{:\nu i} = f_{:\nu\nu} = 0,$$

where A is the shape operator of Σ; and

(2) *along Σ, for all $1 \leqslant i, j, k \leqslant n$:*

$$f_{:ijk} = (\nabla^\Sigma A)_{ijk}, \qquad f_{:\nu ij} = -A^2_{ij},$$

where ∇^Σ is Levi-Civita covariant derivative of Σ.

Proof. — This is an elementary calculation (cf. [14, Lemma 3.16]). □

5.3. Constructing the barrier - Part I

The barrier function required to prove Proposition 5.1 consists of three components. We construct the first component as follows. Let X be a vector field defined near p. Let f be the signed distance to Σ in M with sign chosen so that it is positive above Σ. We define the function ϕ_X near p by:

$$\phi_X = \langle X, \nabla f \rangle. \tag{5.4}$$

Proposition 5.7. — *The restriction of ϕ_X to Σ satisfies:*

$$\Delta^K \phi_X = O(1)(1+\mu) - \phi_X \sum_{i=1}^{n} \mu_i \lambda_i^2,$$

where $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ and $\mu_1 \leqslant \cdots \leqslant \mu_n$ are the eigenvalues of A and DK_A respectively, and $O(1)$ represents terms controlled by B, for some $B \in \mathcal{B}(X)$.

Proof. — Choose $q \in \Sigma$ and let $e_1, \ldots, e_n$ be an orthonormal basis of $T\Sigma$ at q with respect to which A and DF_A are diagonalised. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ and $\mu_1 \leqslant \cdots \leqslant \mu_n$ be the corresponding eigenvalues of A and DF_A respectively. We extend $e_1, \ldots, e_n$ to an orthonormal basis of M at q by defining $e_{n+1} = N$. In the sequel, $\nu := (n+1)$ denotes the upward-pointing normal direction.

We recall that : denotes covariant differentiation with respect to the Levi-Civita covariant derivative of M. By Propositions 5.4 (1), and 5.6 (2), for all $1 \leqslant j \leqslant n$:

$$\sum_{i=1}^{n} \mu_i f_{:jii} = \sum_{i=1}^{n} \mu_i (f_{:iij} + R^M_{ij\nu i}) = \kappa_{:j} + O(\mu) = O(1) + O(\mu).$$

By Proposition 5.6 (2), again:

$$\sum_{i=1}^{n} \mu_i f_{:\nu ii} = -\sum_{i=1}^{n} \mu_i \lambda_i^2.$$

By Proposition 2.4 (4):

$$\sum_{i=1}^{n} \mu_i \lambda_i = \kappa.$$

In particular, for all i, since $\mu_i \lambda_i \geqslant 0$, $\lambda_i \mu_i = O(1)$. By Proposition 5.6 (1), $f_{:ij} = \delta_{ij} \lambda_i$. Thus, recalling that $\|\nabla f\| = 1$:

$$\begin{aligned}\sum_{i=1}^{n} \mu_i \operatorname{Hess}^M(\phi_X)_{ii} &= \sum_{i=1}^{n} \mu_i ({X^j}_{:ii} f_{:j} + 2{X^j}_{:i} f_{:ji} + X^j f_{:jii}) \\ &= O(1)(1+\mu) - X^\nu \sum_{i=1}^{n} \mu_i \lambda_i^2 \\ &= O(1)(1+\mu) - \phi_X \sum_{i=1}^{n} \mu_i \lambda_i^2.\end{aligned}$$

Finally, by Lemma 4.3, for any function h:

$$\operatorname{Hess}^\Sigma(h) = \operatorname{Hess}(h)|_{T\Sigma} - \langle \mathsf{N}, \nabla h \rangle A.$$

Moreover:

$$\langle \mathsf{N}, \nabla \phi_X \rangle = {X^k}_{:\nu} f_{:k} + X^k f_{:k\nu}.$$

By Proposition 5.6 (1), the second term on the right hand side vanishes along Σ, and so, using Proposition 2.4 (4) again:

$$\sum_{i=1}^{n} \langle \mathsf{N}, \nabla \phi_X \rangle \mu_i \lambda_i = O(1).$$

Thus:

$$\Delta^K\phi_X = \sum_{i=1}^n \mu_i \operatorname{Hess}^\Sigma(\phi_X)_{;ii} = O(1)(1+\mu) - \phi_X\sum_{i=1}^n\mu_i\lambda_i^2,$$

as desired. □

The final term on the right-hand side on Proposition 5.7 presents an obstacle to the direct use of ϕ_X in the construction of the barrier. We remove it by modifying ϕ_X as follows. Let A_Γ be the shape operator of Γ, as defined in Section 3.6. Let N_1 be the unit vector in $\mathrm{N}_p\Gamma$ normal to N_0 chosen such that $\langle\hat{N},N_1\rangle > 0$. Define the interval $J\subseteq \mathrm{N}_p\Gamma$ by:

$$J = \left\{U\in I\ \middle|\ \langle\hat{N},U\rangle > 0, \langle N_0,U\rangle > 0, \langle N_1,U\rangle > (\kappa(p)/\hat{\kappa})\langle\hat{N},N_1\rangle\right\},$$

where $\hat{\kappa}$ is the K-curvature of $\hat{\Sigma}$ at p. Observe that since $\kappa(p) < \hat{\kappa}$ and since I has length strictly less than π, J is non-empty. Let V_p be the mid-point of J. The reason for such a meticulous choice of V_p will become clear in Propositions 5.8, 5.10 and 5.11 below. Let V be a vector field defined near p such that $V(p) = V_p$ and $(\nabla V)(p) = 0$.

Proposition 5.8. — *There exist $\rho, c > 0$ in $B(V,I,m)$ such that, throughout $\Sigma\cap B_\rho(p)$, $\phi_V > c$.*

Proof. — By construction, there exists $\eta > 0$ such that every $N'\in I$ makes an angle of at most $\pi/2-2\eta$ with V_p. Let $\rho>0$ be such that $m(\rho) < \eta/2$. Upon reducing ρ if necessary, we may assume that $D(V(p),V(q)) < \eta/2$ for all $q\in B_\rho(p)$. Then, by definition of m and the triangle inequality, for all $q\in B_\rho(p)$, $D(V(q),N(q)) > \pi/2-\eta$, and the result now follows with $c_0 := \cos(\pi/2-\eta)$. □

We define the first order operator $\mathcal{D}_1$ on functions over Σ by:

$$\mathcal{D}_1 h = \frac{2}{\phi_V}\sum_{i=1}^n \mu_i\phi_{V;i}h_{;i},$$

and we define the operator $\mathcal{L}_1$ by:

$$\mathcal{L}_1 = \Delta^K + \mathcal{D}_1. \tag{5.5}$$

Proposition 5.9. — *Using the notation of Proposition 5.7, the restriction of $\phi_X\phi_V^{-1}$ to $\Sigma\cap B_\rho(p)$ satisfies:*

$$\mathcal{L}_1(\phi_X\phi_V^{-1}) = O(1)(1+\mu),$$

where $O(1)$ represents terms controlled by B, for some $B\in\mathcal{B}(X,V,I,m)$.

Proof. — We use the notation of the proof of Proposition 5.7. By the product rule and the chain rule:

$$\begin{aligned}\Delta^K(\phi_X\phi_V^{-1}) &= \sum_{i=1}^n \mu_i(\phi_V^{-1}\,\mathrm{Hess}^\Sigma(\phi_X)_{ii} + 2(\nabla^\Sigma\phi_X)_i(\nabla^\Sigma\phi_V^{-1})_i \\ &\qquad + \phi_X\,\mathrm{Hess}^\Sigma(\phi_V^{-1})_{ii}) \\ &= \sum_{i=1}^n \mu_i(\phi_V^{-1}\,\mathrm{Hess}^\Sigma(\phi_X)_{ii} - 2\phi_V^{-2}(\nabla^\Sigma\phi_X)_i(\nabla^\Sigma\phi_V)_i \\ &\qquad - \phi_X\phi_V^{-2}\,\mathrm{Hess}^\Sigma(\phi_V)_{ii} + 2\phi_X\phi_V^{-3}(\nabla^\Sigma\phi_V)_i(\nabla^\Sigma\phi_V)_i) \\ &= \phi_V^{-1}(\Delta^K\phi_X) - 2\phi_V^{-2}\sum_{i=1}^n\mu_i(\phi_X)_{;i}(\phi_V)_{;i} \\ &\qquad + 2\phi_X\phi_V^{-3}\sum_{i=1}^n\mu_i(\phi_V)_{;i}(\phi_V)_{;i} - \phi_X\phi_V^{-2}(\Delta^K\phi_V).\end{aligned}$$

Thus, by Proposition 5.7, bearing in mind that $|\phi_X|, |\phi_V|, |\phi_V^{-1}| = O(1)$ over $\Sigma\cap B_\rho(p)$:

$$\begin{aligned}\Delta^K(\phi_X\phi_V^{-1}) &= O(1)(1+\mu) - 2\phi_V^{-1}\sum_{i=1}^n\mu_i\ln(\phi_V)_{;i}(\phi_X)_{;i} \\ &\qquad - 2\phi_X\sum_{i=1}^n\mu_i\ln(\phi_V)_{;i}(\phi_V^{-1})_{;i} \\ &= O(1)(1+\mu) - 2\sum_{i=1}^n\mu_i\ln(\phi_V)_{;i}(\phi_X\phi_V^{-1})_{;i} \\ &= O(1)(1+\mu) - \mathcal{D}_1(\phi_X\phi_V^{-1}).\end{aligned}$$

This completes the proof. □

5.4. Constructing the Barrier - Part II

The second component of the barrier function is constructed by taking the signed distance function to a carefully chosen locally strictly concave embedded hypersurface. This hypersurface is constructed as follows. Let $P\subseteq T_pM$ be the hyperplane normal to N_0. We identify P with its image under the exponential map in M. Observe that P is totally geodesic at p. We orient P such that N_0 points upward. Observe that P is transverse to $\hat{\Sigma}$ at P. Thus, upon reducing r if necessary, we may assume that $P\cap_p B_r(p)$

meets $\hat{\Sigma}$ along a smooth codimension-2 submanifold, Γ', say of M. Let $A_{\Gamma'}$ be the shape operator of Γ', as defined in Section 3.6.

Proposition 5.10. — *$K_\infty(A_{\Gamma'}(V)) > \kappa(p)$.*

Proof. — Recall the construction of V_p in the preceeding section. Let N_1 be the unit vector in $\mathrm{N}_p\Gamma$ normal to N_0 such that $\langle \hat{N}, N_1\rangle > 0$. By linearity:

$$A_{\Gamma'}(\hat{N}) = \langle \hat{N}, N_1\rangle A_{\Gamma'}(N_1) + \langle \hat{N}, N_0\rangle A_{\Gamma'}(N_0).$$

Since P is totally geodesic at p, $A_{\Gamma'}(N_0) = 0$. Thus,

$$A_{\Gamma'}(\hat{N}) = \langle \hat{N}, N_1\rangle A_{\Gamma'}(N_1).$$

Likewise, $A_{\Gamma'}(V) = \langle V, N_1\rangle A_{\Gamma'}(N_1)$. Combining these relations yields:

$$A_{\Gamma'}(V) = \frac{\langle V, N_1\rangle}{\langle \hat{N}, N_1\rangle} A_{\Gamma'}(\hat{N}).$$

Thus, bearing in mind Proposition 2.3:

$$K_\infty(A_{\Gamma'}(V)) = \frac{\langle V, N_1\rangle}{\langle \hat{N}, N_1\rangle} K_\infty(A_{\Gamma'}(\hat{N})) > (\kappa(p)/\hat{\kappa})\hat{\kappa} = \kappa(p),$$

as desired. □

By Proposition 5.10, we may extend Γ' to a smooth LSC embedded hypersurface H_0 passing through p such that the upward pointing normal to H_0 at p is $V(p)$ and $K(H_0)(p) > \kappa(p)$. Upon perturbing H_0 slightly, and reversing the orientation, we obtain a smooth locally strictly *concave* embedded hypersurface H passing through p such that the upward pointing normal to H at p is $-V(p)$; if A is the shape operator of H at p, then $K(-A) > \kappa(p)$; and $H_0 \cap_p B_r(p)$ lies above the graph of ϵd_p^2 over H, for some $\epsilon > 0$.

We will show that H is the desired hypersurface. Let d_H be the signed distance to H in M with sign chosen so that d_H is positive above H. In this section, we prove superharmonicity of d_H. Later we prove non-negativity of the restriction of d_H to $\partial\Sigma$. This will depend upon a suitable choice of N_0. For $C > 0$, define the first order operator $\mathcal{D}_2$ on functions over Σ such that:

$$\mathcal{D}_2 h = -C\sum_{i=1}^n \mu_i d_{H;i} h_{;i}.$$

We define $\mathcal{L}_2$ by:

$$\mathcal{L}_2 = \Delta^K + \mathcal{D}_1 + \mathcal{D}_2.$$

C is determined by the following result:

Proposition 5.11. — *Using the notation of Proposition 5.7, there exists $\rho, C \in \mathcal{B}(V, I, m, H)$ such that the restriction of d_H to $\Sigma \cap B_\rho(p)$ satisfies:*

$$\mathcal{L}_2 d_H \leqslant 0.$$

Proof. — Choose $\rho > 0$ such that the K-curvature of every level set of d_H in $B_\rho(p)$ is strictly greater than κ. Bearing in mind that $-d_H$ is convex, by Corollary 5.3, there exists $C, \delta_a > 0$ in $\mathcal{B}(H)$ such that:

$$(\Delta^K + \mathcal{D}_2) d_H \leqslant \delta_a \sum_{i=1}^n \mu_i \operatorname{Hess}(d_H)_{ii}.$$

However, since $\nabla d_H(p) = -V(p)$, as in Proposition 5.8, there exists $\delta_b > 0$ in $\mathcal{B}(I, m, H)$ such that, upon reducing ρ if necessary, $\langle \nabla d_H, N \rangle < -\delta_b$ throughout $\Sigma \cap B_\rho(p)$. Since H is strictly convex, there therefore exists $\delta_c > 0$ in $\mathcal{B}(I, m, H)$ such that throughout $\Sigma \cap B_\rho(p)$ and for all $1 \leqslant i \leqslant n$:

$$\operatorname{Hess}(d_H)_{ii} \leqslant -\delta_b.$$

Thus, denoting $\delta_d = \delta_a \delta_c$:

$$(\Delta^K + \mathcal{D}_2) d_H \leqslant -\delta_d \mu. \tag{5.6}$$

It remains to consider the contribution from $\mathcal{D}_1$. As in the proof of Proposition 5.7:

$$(\phi_V)_{;i} = V^k{}_{:i} f_k + V^k f_{:ki} = V^\nu{}_{:i} + \lambda_i V^i.$$

Since $\nabla V(p) = 0$, bearing in mind Proposition 5.8, upon reducing ρ if necessary, we may assume that:

$$\left| 2\phi_V^{-1} \sum_{i=1}^n \mu_i V^\nu{}_{:i} d_{H;i} \right| \leqslant \frac{\delta_d \mu}{2}. \tag{5.7}$$

Moreover, since $(\nabla d_H + V)(p) = 0$ at p, after reducing ρ further if necessary, we may assume that:

$$\|\nabla d_H + V\| \leqslant \frac{\delta_d \phi_V \mu}{4\kappa \|V\|}.$$

Thus, bearing in mind Proposition 2.4 (4), and the fact that $\mu_i \lambda_i \geqslant 0$ for all i:

$$\begin{aligned} 2\phi_V^{-1} &\sum_{i=1}^n \mu_i \lambda_i V^i d_{H;i} \\ &= -2\phi_V^{-1} \sum_{i=1}^n \mu_i \lambda_i V^i V^i + 2\phi_V^{-1} \sum_{i=1}^n \mu_i \lambda_i V^i (d_{H;i} + V^i) \leqslant \frac{\delta_d \mu}{2}. \end{aligned} \tag{5.8}$$

Combining these relations yields:

$$\mathcal{L}_2 d_H \leqslant 0,$$

as desired. □

We now verify that the addition of the term $\mathcal{D}_2$ does not affect the conclusion of Proposition 5.9. Indeed:

PROPOSITION 5.12. — *Using the notation of Proposition 5.9, the restriction of $\phi_X\phi_V^{-1}$ to $\Sigma\cap B_\rho(p)$ satisfies:*

$$\mathcal{L}_2(\phi_X\phi_V^{-1}) = O(1)(1+\mu),$$

where $O(1)$ represents terms controlled by B for some B in $\mathcal{B}(X,V,I,m,H)$.

Proof. — Indeed, for any vector field, Y:

$$\phi_{Y;i} = Y^\nu{}_{;i} + Y_i\lambda_i.$$

It follows by Proposition 2.4 (4), and the fact that $\lambda_i\mu_i \geqslant 0$ for all i that:

$$\mathcal{D}_2(\phi_X\phi_V^{-1}) = O(1)(1+\mu).$$

The result now follows by Proposition 5.9. □

5.5. Constructing the Barrier - Part III

The third component of the barrier function is simply the squared distance to p in M:

PROPOSITION 5.13. — *There exists $\epsilon > 0$ in $\mathcal{B}(V,I,m,H)$ such that, after reducing ρ if necessary, the restriction of d_p^2 to $\Sigma\cap B_\rho(p)$ satisfies:*

$$\mathcal{L}_2 d_p^2 \geqslant \frac{1}{2}(1+\mu).$$

Proof. — We continue to use the notation of the proof of Proposition 5.7. Since $\mu \geqslant 1$, by Proposition 2.4 (4):

$$\Delta^K d_p^2 \geqslant (1+\mu) - 2d_p\langle\nabla d_p, N\rangle\kappa.$$

For $\rho < 1/8$, throughout $\Sigma\cap B_\rho(p)$:

$$2d_p\langle\nabla d_p, \mathsf{N}\rangle\kappa < 1/4.$$

Thus, throughout $B_\rho(p)$:

$$\Delta^K(d_p^2) \geqslant \frac{3}{4}(1+\mu).$$

Reducing ρ further if necessary, by Proposition 5.8:

$$\left|4d_p\phi_V^{-1}\sum_{i=1}^n \mu_i V^\nu{}_{;i}d_{p;i}\right| \leqslant \frac{1}{16}\mu.$$

Moreover, upon reducing ρ further if necessary, for all i, by Proposition 2.4 (4), bearing in mind that $\mu_i\lambda_i \geqslant 0$ for all i:

$$\left|4d_p\phi_V^{-1}\mu_i\lambda_i V^i d_{p;i}\right| \leqslant \frac{\mu}{16n}$$

Combining these relations yields:

$$\left|\mathcal{D}_1 d_p^2\right| \leqslant \frac{1}{8}(1+\mu).$$

In like manner, after reducing ρ yet further if necessary:

$$\left|\mathcal{D}_2 d_p^2\right| \leqslant \frac{1}{8}(1+\mu).$$

Thus:

$$\mathcal{L}_2(d_p^2) \geqslant \frac{1}{2}(1+\mu),$$

as desired. □

We now prove Proposition 5.1:

Proof of Proposition 5.1. — We assume the contrary. Let $(\hat{\Sigma}_n, \kappa_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ converging to $(\hat{\Sigma}, \kappa)$. For convenience, we suppose that $\hat{\Sigma}_n = \hat{\Sigma}$ and $\kappa_n = \kappa$ for all n. Let Γ be the boundary of $\hat{\Sigma}$ and let $\hat{N}$ be its upward-pointing unit normal vector field. Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of smooth compact LSC immersed hypersurfaces such that for all n, $K(\Sigma_n) = \kappa$; $\Sigma_n < \hat{\Sigma}$; and Σ makes an angle of at least θ with $\hat{\Sigma}$ along their common boundary. For all n, let N_n be the upward-pointing unit normal vector field over Σ_n and let A_n be its shape operator. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of points of Γ converging to p and suppose that $(\|A_n(p_n)\|)_{n\in\mathbb{N}}$ converges to $+\infty$. For convenience, we suppose that $p_n = p$ for all n.

We henceforth identify Σ_n, $\hat{\Sigma}$ and Γ with $\Sigma_{n,p,r}$, $\hat{\Sigma}_{p,r}$ and $\Gamma_{p,r}$ respectively. For all n, we denote $X_n = X_{p,r}(\Sigma_n, \hat{\Sigma})$. Observe that, for all n, X_n is contained in a ball of radius r. Furthermore, by Theorem 3.18 there exists $s > 0$ such that for all n, X_n contains a ball of radius s. Thus, upon extracting a subsequence, there exists a convex subset X with non-trivial interior towards which $(X_n)_{n\in\mathbb{N}}$ converges in the Hausdorff sense.

Observe that p is a boundary point of X. Let I be the set of supporting normal vectors to X at p. By Proposition 3.19, I is a subset of the closure of $\mathrm{CN}_p^-\hat{\Sigma}$. By Proposition 3.20, upon extracting a subsequence, we may suppose that there exists a sequence $(I_n)_{n\in\mathbb{N}}$ of closed subintervals of $\mathrm{CN}_p^-\hat{\Sigma}$ and a continuous function $m : [0,\infty[\, \to [0,\infty[$ such that $(I_n)_{n\in\mathbb{N}}$ converges to I in the Hausdorff sense; $m(0) = 0$; and for all n and for all $q \in \Sigma_n$, $D(N_n(q), I_n) \leqslant m(d(q,p))$. In particular, for all n, $N_n(p)$ is an element of I_n. For all n, let $N_{n,0}$ be the lower end-point of I_n, and with this choice of I_n, define V, P, H and d_H as above.

We claim that, upon reducing r if necessary, there exists $\delta_a > 0$ such that, for all sufficiently large n, $d_H \geqslant \delta_a d_p^2$ along $(\partial\Sigma_n) \cap B_\rho(p) = \Gamma \cap B_\rho(p)$ and $d_H \geqslant \delta_a$ along $\Sigma_n \cap (\partial B_\rho(p))$. Indeed, let Y be the convex component of the complement of $H_0 \cap_p B_\rho(p)$ in $B_\rho(p)$. Observe that $P \cap \hat{\Sigma}$ divides $\hat{\Sigma}$ into two connected components. We denote by $\hat{\Sigma}^-$ the component lying to the left of this hypersurface. Since V lies below $\hat{N}_p$, $\hat{\Sigma}^-$ is contained in Y. Likewise, $P \cap \hat{\Sigma}$ divides P into two connected components. We denote by P^+ the component lying to the right of this hypersurface. Since V lies above N_0, P^+ is also contained in Y. We denote by Y_1 the closure of the convex component of the complement of $\hat{\Sigma}$ in $B_\rho(p)$, and by Y_2 the closure of the connected component of the complement of P in $B_\rho(p)$ lying below P. Since the boundary of $Y_1 \cap Y_2$ coincides with the union of $\hat{\Sigma}^-$ with P^+, $Y_1 \cap Y_2 \subseteq Y$. However, by construction, $X \subseteq Y_1$. Furthermore, since N_0 is a supporting normal to X at p, by convexity $X \subseteq Y_2$. It follows that X lies in Y. In particular, Γ is contained in Y, and so, by definition of H, for sufficiently small $\delta_a > 0$, $d_H \geqslant \delta_a d_p^2$ along $\Gamma \cap B_\rho(p)$ as asserted. Likewise, upon reducing δ further if necessary, $d_H(q) \geqslant 2\delta_a$ for all $q \in X \cap (\partial B_\rho(p))$. Thus, for sufficiently large n, $d_H(q) \geqslant \delta_a$ for all $q \in \Sigma_n \cap (\partial B_\rho(p)) \subseteq X_n \cap (\partial B_\rho(p))$, as asserted.

Now let X be any vector field over $B_r(p)$ which is tangent along Γ. Denote $\phi = \phi_X \phi_V^{-1}$. Observe that ϕ vanishes along Γ. By Propositions 5.12 and 5.13, there exists $A_- > 0$ in $\mathcal{B}(X, V, I, m, H)$ such that, throughout $\Sigma_n \cap B_\rho(p)$:

$$\mathcal{L}_2(\phi - A_- d_p^2) < 0.$$

Bearing in mind Proposition 5.11 and the preceeding paragraph, upon reducing ρ if necessary, there therefore exists $B_- > 0$ in $\mathcal{B}(X, V, I, m, H)$ such that:

(1) $\mathcal{L}_2(\phi + B_- d_H - A_- d_p^2) < 0$ throughout $\Sigma_n \cap B_\rho(p)$; and
(2) $\phi + B_- d_H - A_- d_p^2 \geqslant 0$ along $\partial(\Sigma_n \cap B_\rho(p))$.

It thus follows by the maximum principle that, throughout $\Sigma_n \cap B_\rho(p)$:

$$\phi \geqslant A_- d_p^2 - B_- d_H.$$

Likewise, reducing ρ further if necessary, there exists A_+ and B_+ in $\mathcal{B}(X, V, I, m, H)$ such that, throughout $\Sigma_n \cap B_\rho(p)$:

$$\phi \leqslant -A_+ d_p^2 + B_+ d_H.$$

We thus obtain a-priori bounds on $d\phi$ at p. Let f be the signed distance function to Σ_n. For all Y, since $\phi_X(p) = 0$:

$$\mathrm{Hess}(f)(X, Y) = \langle \nabla\phi, Y\rangle \phi_V(p) - \langle \nabla_Y X, \mathsf{N}\rangle.$$

Thus, since X is arbitrary, we obtain a-priori bounds on $\mathrm{Hess}(f)(X, Y)$ for all pairs of vectors $X, Y \in T_p\Sigma_n$ where at least one of X or Y is tangent to $\partial\Sigma_n$. By Lemma 4.3, the second fundamental form of Σ_n is the restriction to $T\Sigma_n$ of the hessian of f, we deduce that there exists B in $\mathcal{B}(X, V, I, m, H)$ such that:

$$\|A_n(X, Y)\|(p) \leqslant B\|X\|\|Y\|,$$

for all n and for all such pairs of vectors. Since, by hypotheses, $\|A_n(p)\| \to +\infty$, it follows that $\|A_n(X_n, X_n)\| \to +\infty$ where, for all n, X_n is the unit vector normal to $\partial\Sigma_n$ in $T_p\Sigma_n$.

However, we may assume that $(X_n)_{n\in\mathbb{N}}$ converges to X_∞, say, which is normal to Γ at p. Let $\lambda'_1 \leqslant \cdots \leqslant \lambda'_{n-1}$ be the eigenvalues of $A_\Gamma(X_\infty)$. For all m, let $\lambda_{1,m} \leqslant \cdots \leqslant \lambda_{n,m}$ be the eigenvalues of A_n. By the above discussion, $(\lambda_{n,m})_{m\in\mathbb{N}} \to +\infty$. By Lemma 1.2 of [3] and the bounds already obtained, for all $1 \leqslant i \leqslant n-1$:

$$(\lambda_{i,m})_{m\in\mathbb{N}} \to \lambda'_i.$$

Suppose first that K is of bounded type. By Proposition 4.1, $\lambda'_i > 0$ for all i. By Proposition 4.2, $K_\infty(\lambda'_1, \ldots, \lambda'_{n-1}) > \kappa(p)$. However, by concavity, $K(x_1, \ldots, x_{n-1}, t)$ converges locally uniformly to $K_\infty(x_1, \ldots, x_{n-1})$ in $(x_1, \ldots, x_{n-1})$ as $t \to +\infty$. That is:

$$\lim_{m\to+\infty} K(\lambda_{1,m}, \ldots, \lambda_{n,m}) = K_\infty(\lambda'_1, \ldots, \lambda'_{n-1}) > \kappa(p_0),$$

which is absurd.

Suppose now that K is of unbounded type. By Proposition 4.1, $\lambda'_i > 0$ for all i. In the same manner, we obtain:

$$\lim_{m\to+\infty} K(\lambda_{1,m}, \ldots, \lambda_{n,m}) = +\infty > \kappa(p),$$

which is likewise absurd. There therefore exists $B_2 \geqslant 0$ such that, for all n:

$$\|A_n(p_n)\| \leqslant B_2,$$

which is absurd, and this completes the proof. □

6. Global Second Order Estimates

6.1. Main results

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. Let K be a convex curvature function. Let $\kappa : M \to]0,\infty[$ be a smooth positive function. In this section we obtain global a-priori estimates for the shape operators of smooth compact LSC immersed hypersurfaces in M of K-curvature prescribed by κ. Let X be a compact geodesically convex subset of M. As in Section 5.2, we denote by $\mathcal{B}$ the family of all quantities which only depend on the restrictions to X of κ and the metric of M.

Proposition 6.1. — *Choose $R > 0$. Suppose that $\kappa(p) < \mu_\infty(K)/R$ for all $p \in X$. There exists $B > 0$ in $\mathcal{B}(R)$ with the property that if Σ is a smooth compact LSC immersed hypersurface in M such that $K(\Sigma) = \kappa$ and $\Sigma \subseteq X \cap B_R(q)$, for some $q \in M$, then, for all $p \in \Sigma$:*

$$\|A(p)\| \leqslant B\left(1 + \sup_{p' \in \partial\Sigma} \|A(p')\|\right).$$

Using a slightly different approach, we obtain the following complementary result:

Proposition 6.2. — *Suppose that the sectional curvature of M is bounded above by -1 and that $\kappa(p) \in]0,1[$ for all $p \in X$. There exists $B > 0$ in $\mathcal{B}$ with the property that if Σ is a smooth compact LSC immersed hypersurface in M such that $K(\Sigma) = \kappa$ and $\Sigma \subseteq X$, then, for all $p \in \Sigma$:*

$$\|A(p)\| \leqslant B\left(1 + \sup_{p' \in \partial\Sigma} \|A(p')\|\right).$$

6.2. Asymptotic behaviour of curvature functions

We briefly clarify the hypotheses of Proposition 6.1. Observe that for Plateau problems in $\mathbb{R}^n$, we would want open subsets of the sphere of radius R to serve as barriers for hypersurfaces of constant curvature equal to k for all $k \in]0, 1/R]$, and in particular for $k = 1/R$. This is guaranteed by the following result.

Proposition 6.3. — $\mu_\infty(K) > 1$.

Remark. — This follows from the fact that K is C^1 over the interior of Γ.

Proof. — Denote $\mathbb{I} := (1,\ldots,1)$. Let $X \subseteq \Gamma$ be the set of all points $(x_1,\ldots,x_n)$ such that $0 \leqslant x_i \leqslant 1$ for all i, and $x_i = 1$ for at least one i. Since K is homogeneous of order 1, $DK_x(\mathbb{I})$ is homogeneous of order 0. It thus suffices to show that:

$$\liminf_{x\in X, x\to\partial\Gamma} DK_x(\mathbb{I}) > 1.$$

Suppose the contrary. By Proposition 2.4 (6), $DK_x(\mathbb{I}) \geqslant 1$ for all x. We suppose therefore that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in ∂X converging to $x_\infty \in \Gamma$ such that for all n, $DK_{x_n}(\mathbb{I}) = 1$. We claim that $DK_{\mathbb{I}}(x_n) = K(x_n)$ for all n. Indeed, by Proposition 2.4 (4), for all $x \in X$, $DK_x(x) = K(x)$. Thus, if $DK_x(\mathbb{I}) = 1$, then $DK_x(\mathbb{I}-x) = 1 - K(x)$. However, by concavity, for all $t \in [0,1]$, $DK_{t\mathbb{I}+(1-t)x}(\mathbb{I}-x) \leqslant DK_x(\mathbb{I}-x)$. Thus, for all $t \in [0,1]$, $DK_{t\mathbb{I}+(1-t)x}(\mathbb{I}-x) \leqslant 1-K(x)$. Since the integral equals $1-K(x)$, it follows that for all $t \in [0,1]$, $DK_{t\mathbb{I}+(1-t)x}(\mathbb{I}-x) = 1-K(x)$. In particular, since K is C^1, $DK_{\mathbb{I}}(\mathbb{I}-x) = 1-K(x)$. Thus, if $DK_x(\mathbb{I}) = 1$, then $DK_{\mathbb{I}}(x) = K(x)$. In particular, $DK_{\mathbb{I}}(x_n) = K(x_n)$ for all n, as asserted. By compatibility, taking limits yields $DK_{\mathbb{I}}(x_\infty) = K(x_\infty) = 0$. However, since $x_\infty \in \partial X$, $DK_{\mathbb{I}}(x_\infty) = x_{\infty,1}/n + \cdots + x_{\infty,n}/n \geqslant 1/n$. This is absurd, and the result follows. □

When K is of unbounded type, the hypotheses of Proposition 6.1 are trivially satisfied for sufficiently large R. Indeed:

Proposition 6.4. — *If K is of unbounded type, then $\mu_\infty(K) = \infty$.*

Proof. — Indeed, choose $B > 0$. Since K is of unbounded type, there exists $C > 0$ such that $K(1,\ldots,1,C) \geqslant B\sqrt{n}$. For $c \geqslant 0$, let X_c be the set of all points $(x_1,\ldots,x_n)$ such that $c \leqslant x_i \leqslant C$ for all i and $x_i = C$ for at least one i. By homogeneity, it suffices to show that there exists a neighbourhood U of ∂X_0 in X_0 such that for all $x \in U$, $DK_x(\mathbb{I}) \geqslant B$. By invariance and ellipticity of K, $K \geqslant B\sqrt{n}$ throughout X_1. However, by compatibility, K vanishes over ∂X_0. For every point $x \in \partial X_0$, let y_x be such that $x + y_x$ is the closest point to x in ∂X_1. Observe that for all $x \in \partial X_0$, $y_x \in [0,1]^n$ and $y_{x,i} = 1$ for at least one i. By the intermediate value theorem, for all $x \in \partial X_1$, there exists $t_x \in [0,1]$ such that $DK_{t_x y_x}(y_x) \geqslant B\sqrt{n}$. By concavity, for all $t \in \,]0,t_x[$, $DK_{ty_x}(y_x) \geqslant DK_{t_x y_x}(y_x) \geqslant B\sqrt{n}$. In particular, $\|DK_{ty_x}\| \geqslant B$ for all such t. By ellipticity, every component of DK_{ty_x} is positive. Thus $DK_{ty_x}(\mathbb{I}) \geqslant \|DK_{ty_x}\| \geqslant B$. Since y_x and t_x may be chosen to vary continuously with $x \in \partial X_0$, there exists a neighbourhood U of ∂X_0 in X_0 such that for all $x \in U$, $DK_x(\mathbb{I}) \geqslant B$, as desired. □

6.3. Preliminary results

Let Σ be a smooth compact LSC immersed hypersurface in M such that $K(\Sigma) = \kappa$ and $\Sigma \subseteq X$. Let N be the upward-pointing unit normal vector field over Σ. Let A be the shape operator of Σ. Observe that, since Σ is LSC, A is everywhere positive definite. Let R^M be the Riemann curvature tensor of M. Let R^Σ be the Riemann curvature tensor of Σ. Let the subscripts : and ; denote covariant differentiation with respect to the Levi-Civita covariant derivatives of M and Σ respectively. Let the subscript $\nu := n+1$ denote the upward-pointing normal direction.

Proposition 6.5. — *For all i, j, k and l:*

$$A_{ij;kl} = A_{kl;ij} + R^M_{kj\nu i;l} + R^M_{li\nu k;j} + R^{\Sigma}_{jlk}{}^{p} A_{pi} + R^{\Sigma}_{jli}{}^{p} A_{pk}.$$

Proof. — This follows immediately from Proposition 5.4 (cf. [17, Corollary 6.4]). □

Choose $p \in \Sigma$. Let $e_1, \ldots, e_n$ be an orthonormal basis of $T_p\Sigma$ diagonalising A and DK_A. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ and $\mu_1 \leqslant \cdots \leqslant \mu_n$ be the corresponding eigenvalues of A and DK_A respectively. We extend λ_1 to a continuous function near p such that for all q, $\lambda_1(q)$ is the greatest eigenvalue of $A(q)$. We define the operator Δ^K as in Section 5.2. For $\phi \in C^\infty(M)$, we define the homogeneous first order operator $\mathcal{D}_\phi$ on functions over Σ by:

$$\mathcal{D}_\phi f = \sum_{i=1}^{n} \mu_i \phi_{;i} f_{;i}.$$

We define $I, J \subseteq \{1, \ldots, n\}$ by:

$$I = \{1 \leqslant i \leqslant n \mid \mu_i \leqslant 4\mu_1\}, \qquad J = \{1 \leqslant i \leqslant n \mid \mu_i > 4\mu_1\}.$$

We recall that a function f is said to satisfy $(\Delta^K + C\mathcal{D}_\phi)f \geqslant g$ in the weak sense at p if and only if there exists a smooth function a, defined near p such that $f \geqslant a$ near p; $f(p) = a(p)$ at p; and $(\Delta^K + CD_\phi)a \geqslant g$ at p. We obtain:

Proposition 6.6. — *For all $\phi \in C^\infty(M)$, $C \geqslant 0$, there exists $K \geqslant 0$ in $\mathcal{B}(\phi, C)$ such that if $\lambda_1(p) \geqslant 1$, then, at p:*

$$(\Delta^K + C\mathcal{D}_\phi)\log(\lambda_1) \geqslant -K(1+\mu) - \sum_{i \in I} \frac{\mu_i}{\lambda_1^2} A_{11;i}^2,$$

in the weak sense.

We extend $e_1, \ldots, e_n$ to a frame near p by parallel transport along geodesics. We define the function a near p by $a = A(e_1, e_1)$. Observe that $\lambda_1 \geqslant a$ and $\lambda_1(p) = a(p)$.

Proposition 6.7. — *For all i, at p, $a_{;i} = A_{11;i}$ and $a_{;ii} = A_{11;ii}$.*

Proof. — This is an elementary calculation (cf. [17, Proposition 6.5]). □

Proposition 6.8. — *For all $\phi \in C^\infty(M)$, $C \geqslant 0$, there exists $K > 0$ in $\mathcal{B}(\phi, C)$ such that, if $a(p) \geqslant 1$, then, at p:*

$$(\Delta^K + C\mathcal{D}_\phi)\log(a)(p) \geqslant -K(1+\mu) - \sum_{i\in I}\frac{\mu_i}{\lambda_1^2}A_{11;i}^2.$$

Proof. — By Propositions 6.5 and 6.7:

$$a_{;ii} = A_{11;ii} = A_{ii;11} + R^M_{i1\nu1;i} + R^M_{i1\nu i;1} + R^{\Sigma}_{1ii}{}^{p}A_{p1} + R^{\Sigma}_{1i1}{}^{p}A_{pi}.$$

However, at p, by Proposition 5.5 (2):

$$\sum_{i=1}^n \frac{\mu_i}{\lambda_1}A_{ii;11} = -\frac{1}{\lambda_1}(D^2K)^{ij,mn}A_{ij;1}A_{mn;1} + \frac{1}{\lambda_1}\kappa_{;11}.$$

Thus, at p:

$$\begin{aligned}\Delta^K\log(a) = \frac{1}{\lambda_1}\kappa_{;11} - \frac{1}{\lambda_1}(D^2K)^{ij,mn}A_{ij;1}A_{mn;1} - \sum_{i=1}^n\frac{\mu_i}{\lambda_1^2}A_{11;i}A_{11;i}\\ + \sum_{i=1}^n\frac{\mu_i}{\lambda_1}(R^M_{i1\nu1;i} + R^M_{i1\nu i;1}) + \sum_{i,j=1}^n\frac{\mu_i}{\lambda_1}(R^{\Sigma}_{1ii}{}^{p}A_{p1} + R^{\Sigma}_{1i1}{}^{p}A_{pi}).\end{aligned}$$

We consider each contribution seperately. Since, for all $a, b \in \mathbb{R}$ and for all $\eta > 0$, $(a+b)^2 \leqslant (1+\eta)a^2 + (1+\eta^{-1})b^2$, by Lemma 5.4 (1), there exists K_1 in $\mathcal{B}$ such that for all $i \in J$:

$$\frac{9}{8}A_{11;i}^2 = \frac{9}{8}(A_{i1;1} + R^M_{i1\nu1})^2 \leqslant \frac{5}{4}A_{i1;1}^2 + K_1.$$

Thus, by Proposition 2.4 (7), bearing in mind the definition of J and the fact that $\lambda_1 \geqslant 1$:

$$\begin{aligned}-\frac{1}{\lambda_1}(D^2K)^{ij,mn}A_{ij;1}A_{mn;1} - \frac{9}{8}\sum_{i\in J}\frac{\mu_i}{\lambda_1^2}A_{11;i}A_{11;i}\\ \geqslant \sum_{i\in J}\Big(\frac{2(\mu_i-\mu_1)}{\lambda_1(\lambda_1-\lambda_i)} - \frac{5}{4}\frac{\mu_i}{\lambda_1^2}\Big)A_{i1;1}^2 - K_1\mu\\ \geqslant \sum_{i\in J}\frac{\mu_1}{(\lambda_1-\lambda_i)\lambda_1}A_{i1;1}^2 - K_1\mu\\ \geqslant -K_1\mu.\end{aligned}$$

Thus:

$$-\frac{1}{\lambda_1}(D^2K)^{ij,mn}A_{ij;1}A_{mn;1}-\sum_{i\in J}\frac{\mu_i}{\lambda_1^2}A_{11;i}A_{11;i} \geqslant \frac{1}{8}\sum_{i\in J}\frac{\mu_i}{\lambda_1^2}A_{11;i}A_{11;i}-K_1\mu. \tag{6.1}$$

For all ξ, X and Y:

$$\nabla^\Sigma\xi(Y;X)=\nabla^M\xi(Y;X)-A(X,Y)\xi(N);$$
$$\text{and}\qquad X\xi(N)=\nabla^M\xi(N;X)+\xi(AX).$$

Thus:

$$R^M_{i1\nu1;i}=R^M_{i1\nu1:i}+\lambda_i(1-\delta_{i1})R^M_{1\nu\nu1}+\lambda_iR^M_{i1i1},$$
$$R^M_{i1\nu i;1}=R^M_{i1\nu i:1}-\lambda_1(1-\delta_{i1})R^M_{i\nu\nu i}-\lambda_1R^M_{i1i1}.$$

Bearing in mind that $\lambda_1\geqslant 1$, by Proposition 2.4(4), there exists K_2 in $\mathcal{B}$ such that:

$$\sum_{i=1}^n\frac{\mu_i}{\lambda_1}(R^M_{i1\nu1;i}+R^M_{i1\nu i;1})\geqslant -K_2(1+\mu). \tag{6.2}$$

Next:

$$R^{\Sigma\ p}_{1ii}A_{p1}+R^{\Sigma\ p}_{1i1}A_{pi}=R^M_{1ii1}(\lambda_1-\lambda_i)+\lambda_1\lambda_i(\lambda_1-\lambda_i).$$

Thus, bearing in mind that $\lambda_1\geqslant 1$ and that $\lambda_1\geqslant\lambda_i$ for all i, there exists K_3 in $\mathcal{B}$ such that:

$$\sum_{i,j=1}^n\frac{\mu_i}{\lambda_1}(R^{\Sigma\ p}_{1ii}A_{p1}+R^{\Sigma\ p}_{1i1}A_{pi})\geqslant -K_3(1+\mu). \tag{6.3}$$

Finally, bearing in mind Lemma 4.3:

$$\begin{aligned}\kappa_{;11}&=\mathrm{Hess}^\Sigma(\kappa)(e_1,e_1)\\&=\mathrm{Hess}^M(\kappa)(e_1,e_1)-\langle\nabla\kappa,N\rangle A_{11}\\&=\mathrm{Hess}^M(\kappa)(e_1,e_1)-\lambda_1 d\kappa(N).\end{aligned}$$

Bearing in mind that $\lambda_1\geqslant 1$, there thus exists K_4 in $\mathcal{B}$ such that:

$$\frac{1}{\lambda_1}\kappa_{;11}\geqslant -K_4. \tag{6.4}$$

Combining the above relations, there exists K_5 in $\mathcal{B}$ such that:

$$\Delta^K\log(a)\geqslant -K_5(1+\mu)-\sum_{i\in I}\frac{\mu_i}{\lambda_1^2}A^2_{11;i}+\frac{1}{8}\sum_{i\in J}\frac{\mu_i}{\lambda_1^2}A^2_{11;i}.$$

Finally, bearing in mind that $\lambda_1 = a \geqslant 1$, there exists K_6 in $\mathcal{B}(\phi, C)$ such that:

$$\begin{aligned} C\mathcal{D}_\phi \log(a) &= C\sum_{i=1}^n \frac{\mu_i}{\lambda_1}\phi_{;i}A_{11;i} \\ &\geqslant -\frac{1}{8}\sum_{i=1}^n \frac{\mu_i}{\lambda_1^2}A_{11;i}^2 - K_6\mu. \end{aligned}$$

The result now follows by combining the above relations. □

We now prove Proposition 6.6:

Proof of Proposition 6.6. — This follows immediately from Proposition 6.8. □

6.4. Global second order a priori estimates - Part I

Let p be a point in M. Let d_p be the distance in M to p.

Proposition 6.9. — *Let $R > 0$ be such that $\kappa(q) < \mu_\infty(K)/R$ for all $q \in X \cap B_R(p)$. There exist $c, \epsilon > 0$ in $\mathcal{B}(R)$ such that if $\Sigma \subseteq X \cap B_R(p)$, then, over Σ:*

$$\lambda_1 \geqslant c \ \Rightarrow \Delta^K d_p^2 \geqslant \epsilon(1+\mu).$$

Proof. — Observe that, by homogeneity:

$$\liminf_{K(x)=1, x\to\infty} DK_x(\mathbb{I}) = \mu_\infty(K).$$

Since M has non-positive curvature, bearing in mind Proposition 2.4 and Lemma 4.3:

$$\begin{aligned} \mathrm{Hess}^M\left(\frac{1}{2}d_p^2\right) \geqslant \mathrm{Id} \Rightarrow \mathrm{Hess}^\Sigma\left(\frac{1}{2}d_p^2\right) &\geqslant \mathrm{Id} - d_p\langle \mathsf{N}, \nabla d_p\rangle A \\ &\Rightarrow \Delta^K \frac{1}{2}d_p^2 \geqslant \mu - \kappa d_p \geqslant \mu - \kappa R. \end{aligned}$$

Thus, by definition of R, there exists $c, \epsilon > 0$ in $\mathcal{B}(R)$ such that, for $\lambda_1 \geqslant c$:

$$\Delta^K d_p^2 \geqslant 2\epsilon\mu \geqslant \epsilon(1+\mu),$$

as desired. □

Proposition 6.10. — *Let $R > 0$ be such that $\kappa(q) < \mu_\infty(K)/R$ for all $q \in X \cap B_R(p)$. There exist $C, c > 0$ in $\mathcal{B}(R)$ and a homogeneous first order operator $\mathcal{D}$ such that if $\Sigma \subseteq X \cap B_R(p)$ and if $\lambda_1(q) \geqslant c$, then, at q:*

$$(\Delta^K + \mathcal{D})(\log(\lambda_1) + C\delta_p^2) > 0,$$

in the weak sense.

Proof. — Choose $q \in \Sigma$. Define a near q as in Section 6.3. By Propositions 6.8 and 6.9, there exists $c_1, C > 0$ in $\mathcal{B}(R)$ such that, for $\lambda_1 > c_1$, at q:

$$\Delta^K(\log(a) + Cd_p^2) \geqslant 1 - \sum_{i\in I} \frac{\mu_i}{\lambda_1^2} A_{11;i}^2.$$

Denote $\Phi := \log(a) + Cd_p^2$. For all k, at q:

$$\frac{1}{\lambda_1} A_{11;k} = \log(a)_{;k} = -2Cd_p d_{p;k} + \Phi_{;k}.$$

There therefore exist smooth functions $(D_i)_{i\in I}$ such that:

$$\sum_{i\in I} \frac{\mu_i}{\lambda_1^2} A_{11;i}^2 = 4C^2 d_p^2 \sum_{i\in I} \mu_i d_{p;k}^2 - \sum_{i\in I} D_i \Phi_{;k}.$$

By Proposition 2.4 (4), $\lambda_1\mu_1 \leqslant \kappa$. Thus $\mu_1 \leqslant \kappa/\lambda_1$, and so $\mu_i \leqslant 4\kappa/\lambda_1$ for all $i \in I$. There therefore exists $c_2 \in \mathcal{B}(R)$ such that:

$$\Delta^K \Phi + \sum_{i\in I} D_i \Phi_{;i} \geqslant 1 - c_2/\lambda_1.$$

The result follows with $c = \max(c_1, c_2)$. □

Proof of Proposition 6.1. — Let $C, c > 0$ be as in Proposition 6.10. Consider the function $\|A\| e^{Cd_p^2} = \lambda_1 e^{Cd_p^2}$. If this function acheives its maximum along $\partial\Sigma$, then the result follows since $e^{Cd_p^2}$ is uniformly bounded above and below. Otherwise, it acheives its maximum in the interior of Σ, in which case, by Proposition 6.10 and the maximum principle, at this point $\|A\| \leqslant \lambda_1 \leqslant c$. The result follows. □

6.5. Global second order a priori bounds - Part II

We require the following variant of Lemma 5.2:

Proposition 6.11. — *Let ϕ be as in Lemma 5.2. Suppose that $\phi \geqslant \delta > 0$. There exists ϵ, α, C in $\mathcal{B}(\phi, \delta)$ such that:*

$$(\Delta^K + CD_\phi)\phi^{1+\alpha} \geqslant \epsilon(1 + \mu).$$

Proof. — Let π be the orthogonal projection along $\nabla\phi$. Observe that for all $\alpha > 0$ and for all $X \in T_pM$:

$$\mathrm{Hess}(\phi^{1+\alpha})(X, X) \geqslant \mathrm{Hess}(\phi^{1+\alpha})(\pi(X), \pi(X)).$$

Defining m and m' as in the proof of Lemma 5.2, we thus obtain:

$$(m_{ij}) \geqslant \begin{pmatrix} (m'_{ij}) & \cos(\theta)(m'_{in}) \\ \cos(\theta)(m'_{ni}) & \cos^2(\theta) m'_{nn} \end{pmatrix}.$$

As in the proof of Corollary 5.3, there exists $\epsilon_1 > 0$ in $\mathcal{B}$ such that:

$$\Delta^K \phi^{1+\alpha} \geqslant \epsilon_1 \sum_{i=1}^n \mu_i \operatorname{Hess}(\phi^{1+\alpha})_{ii} - \|\operatorname{Hess}(\phi^{1+\alpha})\| \sum_{i=1}^n \mu_i \phi_{;i}\phi_{;i}.$$

However, there exists $\epsilon > 0$ in $\mathcal{B}(\phi)$ such that:

$$\epsilon_1 \sum_{i=1}^n \mu_i \operatorname{Hess}(\phi^{1+\alpha})_{ii} \geqslant \epsilon\mu.$$

Finally:

$$D_\phi \phi^{1+\alpha} = (1+\alpha)\phi^\alpha \sum_{i=1}^n \mu_i \phi_{;i}\phi_{;i}.$$

Thus, for $(1+\alpha)\delta^\alpha C \geqslant \|\operatorname{Hess}(\phi^{1+\alpha})\|$, the result follows. □

Let p be a point in M. Let d_p be the distance in M to p.

Proposition 6.12. — *If $\kappa < 1$ and if the sectional curvature of M is bounded above by -1, then there exists $\epsilon, \alpha, C > 0$ in $\mathcal{B}$ such that, over Σ and away from p:*

$$(\Delta^K + C\mathcal{D}_{d_p}) d_p^{1+\alpha} \geqslant \epsilon(1+\mu).$$

Proof. — Trivially, $\|\nabla d_p\| = 1$. Moreover, since the level sets of d_p are geodesic spheres and since M is a Hadamard manifold, they are strictly convex. Since the sectional curvature of M is bounded above by -1, by Properties (C) and (E) of K, the level sets have K-curvature greater than 1. The result now follows by Proposition 6.11. □

Proposition 6.13. — *If $\kappa < 1$ and if the section curvature of M is bounded above by -1, then there exists $C, c > 0$ in $\mathcal{B}$ and a first order homogeneous operator $\mathcal{D}$ such that over Σ:*

$$\lambda_1 \geqslant c \ \Rightarrow (\Delta^K + \mathcal{D})(\log(\lambda_1) + C\delta) > 0,$$

in the weak sense.

Proof. — By Propositions 6.12 and 6.6, there exist $C_1, C_2 > 0$ and $\alpha \in]0,1[$ in $\mathcal{B}$ such that, over Σ and away from p:

$$(\Delta + C\mathcal{D}_{d_p})(\log(a) + C d_p^{1+\alpha}) \geqslant 1 - \sum_{i\in I} \frac{\mu_i}{\lambda_1^2} A_{11;i}^2.$$

The result now follows as in the proof of Proposition 6.10. □

Proof of Proposition 6.2. — This is identical to the proof of Proposition 6.1, with Proposition 6.13 used instead of Proposition 6.10. □

7. Existence

7.1. Compactness

Let K be a convex curvature function. Bearing in mind the two complementary results of Section 6.1, we obtain the following two complementary compactness results:

Proposition 7.1. — *Let K be a convex curvature function; let M be an $(n+1)$-dimensional Hadamard manifold; let $\mathcal{F}$ be a family of triplets $(\hat{\Sigma},\kappa,R)$ where $\hat{\Sigma}$ is a smooth compact LSC immersed hypersurface with generic boundary in M; κ is a smooth positive function on M such that $\kappa < \mu_\infty(K)/R$; $\hat{\Sigma}$ is contained in $B_R(p)$ for some $p \in M$; and $K(\hat{\Sigma}) > \kappa$. Let $\mathcal{G}$ be the family of smooth compact LSC immersed hypersurfaces Σ in M such that $K(\Sigma) = \kappa$ and $\Sigma < \hat{\Sigma}$ for some triplet $(\hat{\Sigma},\kappa,R) \in \mathcal{F}$. If $\mathcal{F}$ is compact, then so too is $\mathcal{G}$.*

Proof. — By compactness, upon perturbing every element of $\mathcal{F}$ if necessary, we may assume that there exists $\theta > 0$ such that for all $\Sigma \in \mathcal{G}$, there exists $(\hat{\Sigma},\kappa,R) \in \mathcal{F}$ such that $K(\Sigma) = \kappa$; $\Sigma < \hat{\Sigma}$; and Σ makes an angle of at least θ with $\hat{\Sigma}$ along their common boundary. Let $p \in M$ be such that $\hat{\Sigma}$ is contained in $B_R(p)$. We first show that Σ is also contained in $B_R(p)$. Suppose the contrary. Let $q \in \Sigma$ be the point lying furthest from p. Let $L \subseteq M$ be the geodesic ray leaving q in the outward-pointing normal direction from Σ. Trivially L does not intersect $B_R(p)$. However, since $\hat{\Sigma}$ bounds Σ, L meets $\hat{\Sigma}$ at some point. This is absurd, and it follows that Σ is contained in $B_R(p)$ as asserted.

Proposition 5.1 yields a priori C^2 bounds for elements of $\mathcal{G}$ along the boundary. By the preceeding paragraph, Proposition 6.1 yields global a priori C^2-bounds for elements of $\mathcal{G}$. Theorem 1 of [1] yields a priori $C^{2,\alpha}$-bounds for elements of $\mathcal{G}$. The Schauder estimates (cf. [5]) yield a priori C^k-bounds for elements of $\mathcal{G}$ for all $k \in \mathbb{N}$. Proposition 3.6 yields a priori diameter bounds for elements of $\mathcal{G}$. It follows from the Arzela–Ascoli Theorem for immersed hypersurfaces (cf. [15]) that every sequence in $\mathcal{G}$ contains a convergent subsequence.

It remains to show that $\mathcal{G}$ is closed. Let Σ by a limit point of $\mathcal{G}$. Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}$ converging to Σ. Let $(\hat{\Sigma}_n,\kappa_n,R_n)$ be a sequence in $\mathcal{F}$ such that, for all n, $\Sigma_n < \hat{\Sigma}_n$ and $K(\Sigma_n) = \kappa_n$. Since $\mathcal{F}$ is compact, we may suppose that $(\hat{\Sigma}_n,\kappa_n,R_n)$ converges to $(\hat{\Sigma},\kappa,R)$, say. Taking limits, $K(\Sigma) = \kappa$. Furthermore, by Proposition 3.17, $\Sigma < \hat{\Sigma}$. It follows that Σ is an element of $\mathcal{G}$, and this completes the proof. □

Proposition 7.2. — *Let K be a convex curvature function; let M be an $(n+1)$-dimensional Hadamard manifold with sectional curvature bounded above by -1; let $\mathcal{F}$ be a family of pairs $(\hat{\Sigma}, \kappa)$ where $\hat{\Sigma}$ is a smooth compact LSC immersed hypersurface with generic boundary in M; $\kappa : M \to]0,1[$ is a smooth positive function; and $K(\hat{\Sigma}) > \kappa$. Let $\mathcal{G}$ be the family of smooth compact LSC immersed hypersurfaces Σ in M such that $\Sigma < \hat{\Sigma}$ and $K(\Sigma) = \kappa$ for some pair $(\hat{\Sigma}, \kappa) \in \mathcal{F}$. If $\mathcal{F}$ is compact, then so too is $\mathcal{G}$.*

Proof. — This is proven in the same way as Proposition 7.1, using Proposition 6.2 instead of Proposition 6.1. □

7.2. Proof of Main Results

We first show:

Proposition 7.3. — *Let M be a Hadamard manifold. Let Σ be a smooth compact LSC immersed hypersurface in M. Σ is isotopic through smooth compact LSC immersed hypersurfaces to an immersed submanifold lying on a geodesic sphere.*

Proof. — Let N_Σ be the outward pointing, unit, normal vector field over Σ and define $I : \Sigma \times [0, \infty[\to M$ by:

$$I(p,t) = \mathrm{Exp}(t\mathsf{N}_\Sigma(p)).$$

Let $d : \Sigma \times [0,\infty[\to \mathbb{R}$ be the distance in $\Sigma \times [0,\infty[$ to $\Sigma \times \{0\}$. d is a locally convex function. For all $t \in [0,\infty[$, let $\Sigma_t = d^{-1}(\{t\})$ be the level set of d at height t. Choose $p \in M$ and let d_p be the distance to p in M. d_p is also a convex function and we identify it with $d_p \circ I$.

Observe that $\partial(\Sigma \times [0,\infty[)$ consists of 2 components, being Σ and $(\partial\Sigma) \times [0,\infty[$. Choose $R \geqslant 0$ such that, for all $q \in \Sigma$:

$$d_p(q) < R.$$

Observe that, since M is non-positively curved, as $d(q)$ tends to $+\infty$, the angle between ∇d and ∇d_p at q tends to 0. Thus, increasing R if necessary, we may assume that, for $d_p(q) \geqslant R$:

$$\langle \nabla d, \nabla d_p \rangle (q) > 0.$$

For $s \in [0,1]$ define d_s and $\tilde{\Sigma}_s$ by:

$$d_s = sd_p + (1-s)d, \qquad \tilde{\Sigma}_s = d_s^{-1}(\{R\}).$$

For all s, and for all q such that $d_p(q) \geqslant R$:

$$\langle \nabla d, \nabla d_s \rangle(q) > 0.$$

Thus, for all $s \in [0,1]$, $\tilde{\Sigma}_s \cap \Sigma_0 = \emptyset$ and $\tilde{\Sigma}_s$ is transverse to $\partial\Sigma \times [0,\infty[$. Moreover, for all s, d_s is convex, and so $(\Sigma_s)_{s\in[0,1]}$ defines an isotopy through smooth compact LSC immersed hypersurfaces from $\Sigma_R = \tilde{\Sigma}_0$ to $\tilde{\Sigma}_1 \subseteq \partial B_R(p)$. Since Σ_0 is isotopic to Σ_R, the result follows. □

This allows us to prove Theorems 1.1 and 1.2:

Proof of Theorems 1.1 and 1.2. — By Proposition 3.16, boundedness is an open property and so differential topological techniques may be applied. The compactness result of Propositions 7.1 and 7.2 together with the isotopy result of Proposition 7.3 then allow us to apply the differential topological degree theory developed for the case of extrinsic curvature in [20]. Existence follows. □

We prove Theorem 1.3:

Proof of Theorem 1.3. — Let $\kappa : M \to]0,1[$ be a smooth function. Let $\Sigma = (i,(S,\partial S))$ be a smooth compact LSC immersed hypersurface $K(\Sigma) = \kappa$. Let N be the upward- pointing unit normal vector field over Σ. Let J be the Jacobi operator of K-curvature over Σ. As in Proposition 3.1.1 of [11], for all $f \in C^\infty(S)$:

$$\mathcal{L}f = (DK_A(W) - DK_A(A^2))f - DK_A(\mathrm{Hess}(f)),$$

where $W : TS \to TS$ is given by:

$$W \cdot X = R_{\mathsf{N}X}\mathsf{N}.$$

Since the sectional curvature of M is bounded above by -1, for all $X \in TS$:

$$\langle W\cdot X, X\rangle = \langle R_{\mathsf{N}X}\mathsf{N}, X\rangle \geqslant \|X\|^2 \ \Rightarrow\ W \geqslant \mathrm{Id} \ \Rightarrow\ DK_A(W) \geqslant DK_A(\mathrm{Id}).$$

It thus follows from the hypotheses on K that:

$$DK_A(W) - DK_A(A^2) > 0.$$

Thus, for all $f \in C^\infty(S)$:

$$\langle \mathcal{L}f, f\rangle \geqslant 0,$$

with equality if and only if $f = 0$.

We may conclude in one of two different ways. First, we may interpret this in terms of the degree theory of [12], in which case we see that the contribution of any solution to the degree of κ is equal to $+1$, and that there is therefore only one solution. Alternatively, we reason more directly, as in the proof of Lemma 3.0.2 of [11], to reach the same conclusion. This completes the proof. □

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