Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Rafik Imekraz

Multidimensional Paley–Zygmund theorems and sharp L^p estimates for some elliptic operators

Tome 69, nº 6 (2019), p. 2723-2809.

<http://aif.centre-mersenne.org/item/AIF_2019__69_6_2723_0>

Ann. Inst. Fourier, Grenoble
69, 6 (2019) 2723-2809

MULTIDIMENSIONAL PALEY–ZYGMUND THEOREMS AND SHARP L^p ESTIMATES FOR SOME ELLIPTIC OPERATORS

by Rafik IMEKRAZ (*)

ABSTRACT. — The goal of the paper is twofold. Firstly we study sufficient conditions of convergence for random series of eigenfunctions in L^∞. The eigenfunctions are considered with respect to a reference elliptic operator like the Laplace–Beltrami operator or a Schrödinger operator with a growing potential on the Euclidean space. That is a generalization of an old result due to Paley and Zygmund. Secondly, we obtain a few optimal L^p bounds of eigenfunctions including a generalization of the Bernstein inequality. We show that the previous two themes are intimately linked.

RÉSUMÉ. — Le but de cet article est double. Premièrement, nous étudions des conditions suffisantes de convergence pour des séries aléatoires de fonctions propres dans L^∞. Les fonctions propres sont considérées par rapport à un opérateur elliptique de référence tel que l'opérateur de Laplace–Beltrami ou un opérateur de Schrödinger avec un potentiel confinant de l'espace euclidien. Cela constitue une généralisation d'un vieux résultat de Paley et Zygmund. Dans un deuxième temps, nous obtenons quelques estimées L^p optimales de fonctions propres incluant une généralisation de l'inégalité de Bernstein. Nous montrons que ces deux thèmes sont intimement liés.

1. Introduction

This paper studies the close relation between the following two themes

(1) obtaining sufficient conditions for the almost sure convergence in L^∞ of random series of eigenfunctions of some elliptic operators (the Laplace–Beltrami operator on a boundaryless compact Riemannian manifold and similarly the quantum superquadratic oscillator

Keywords: Paley–Zygmund theorems, elliptic operators, wave equation, Sobolev embeddings.
2010 *Mathematics Subject Classification:* 60G50, 15B52, 46B09.
(*) This project is supported by the ANR UNIRANDOM.

$-\Delta + |x|^{2\alpha}$ on $\mathbb{R}^d$, with $(\alpha, d) \in \mathbb{N}^\star \times \mathbb{N}^\star$). Such results extend the classical Paley–Zygmund theorem (see below Theorem 1.1). Moreover, our results are universal with respect to the randomization.

(2) obtaining sharp L^p estimates, with $p \in [1, +\infty]$, of eigenfunctions of the above mentioned elliptic operators (or more precisely functions that are spectrally localized). In particular, we will be interested in Bernstein inequalities involving gradients of eigenfunctions.

It will appear that several elements of the proof of one of those themes are used to study the other one. As a by-product, we complete and simplify some results proven in [11, 14, 24, 43, 47] in a unified framework. Although our new results are essentially concerned with random series in L^∞, we also make a comparison with the L^p theory (with finite p) and, maybe more striking, with the almost sure convergence in *one point* (see Theorem 2.3, Theorem 4.1 and Theorem 4.5). Almost all proofs of our results are technically possible thanks to our previous paper [23].

About the L^p estimates, we shall get new and optimal estimates dealing with $-\Delta + |x|^{2\alpha}$. We thus complete some results proved in [24, 29, 41]. Those results are shortly summarized at the end of this introduction but the interested reader may directly go to Section 5.

Let us begin by recalling the known literature. For any set I, we denote by $(\varepsilon_n)_{n\in I}$ a sequence of i.i.d. Rademacher random variables, in other words $\mathbf{P}(\varepsilon_n = 1) = \mathbf{P}(\varepsilon_n = -1) = \frac{1}{2}$. Denoting by $\mathbb{T}$ the unidimensional torus, the classical Paley–Zygmund theorem for random Fourier series in $L^\infty(\mathbb{T})$ is the following (see [37, p. 347] or [51, p. 219]).

Theorem 1.1. — *For any complex sequence $(a_n)_{n\in\mathbb{Z}}$ satisfying*

$$\exists\, \gamma > 1 \qquad \sum_{\substack{n\in\mathbb{Z}\\ |n|\geqslant 2}} |a_n|^2 \ln^\gamma(|n|) < +\infty, \tag{1.1}$$

the random Fourier series $\sum_{n\in\mathbb{Z}} \varepsilon_n a_n e^{inx}$ almost surely converges in the Banach space $L^\infty_x(\mathbb{T})$.

The assumption (1.1) is fulfilled if the function $\sum_{n\in\mathbb{Z}} a_n e^{inx}$ belongs to $H^s_x(\mathbb{T})$ for some small $s > 0$. Hence, the Paley–Zygmund theorem ensures that a function belongs, in probability, in $L^\infty(\mathbb{T})$ with a weaker condition than the one given by the Sobolev embedding $H^s(\mathbb{T}) \subset L^\infty(\mathbb{T})$ (that needs the inequality $s > \frac{1}{2}$).

In modern probability theory, a natural question is the universality of such a phenomenon. Let us recall that an asymptotic probability result involving a sequence of *independent* random variables is called *universal* if its conclusion remains unchanged whenever each of the random variables is replaced with another one (with some necessary but quite general normalizations). See for instance [2, 44, 45] for several modern examples. In our context, one may reformulate this problem as follows: “is it possible to replace the Rademacher law of the random variables ε_n in $\sum \varepsilon_n a_n e^{inx}$ with other laws?”. This question was completely solved by Marcus and Pisier (see [34] or [33, p. 527, théorèmes III.5 et III.6]). Their result claims that for any sequence of independent *centered* real random variables $(X_n)_{n\in\mathbb{Z}}$ satisfying

$$0 < \inf_{n\in\mathbb{Z}} \mathbf{E}[|X_n|] \quad \text{and} \quad \sup_{n\in\mathbb{Z}} \mathbf{E}[|X_n|^2] < +\infty. \tag{1.2}$$

the following equivalence holds true

$$\begin{aligned} \sum \varepsilon_n a_n e^{inx} \quad &\text{a.s. converges in } L^\infty_x(\mathbb{T}) \\ \Leftrightarrow \quad \sum X_n a_n e^{inx} \quad &\text{a.s. converges in } L^\infty_x(\mathbb{T}). \end{aligned} \tag{1.3}$$

We stress that the previous equivalence holds true even if the random variables X_n have not the same law! The implication $\Leftarrow$ is quite classical in the theory of Banach spaces (see [25, Corollary 5.2] or [24, Theorem 5.2]). Let us now give a deep reason that convinces the reader that the weak conditions (1.2) and the converse implication $\Rightarrow$ in (1.3) are remarkable. Let us denote by g_n a sequence of i.i.d. $\mathcal{N}_\mathbb{C}(0,1)$ Gaussian random variables. The Maurey–Pisier theorem [35, corollaire 1.3] states that there is a geometric property, called the *finite cotype*, that characterizes the Banach spaces B for which the following equivalence holds true for any sequence $(f_n)_{n\in\mathbb{N}}$ of B

$$\sum \varepsilon_n f_n \ \text{ a.s. converges in } B \ \Leftrightarrow \ \sum g_n f_n \ \text{ a.s. converges in } B. \tag{1.4}$$

It is not necessary to give a precise definition of the cotype since we will not need it. For our purpose, we only need to know that $L^q(\mathbb{T}^d)$ has finite cotype if and only if q is finite. Hence, $L^\infty(\mathbb{T}^d)$ is not expected to satisfy (1.4). The equivalence (1.3) however implies, if X_n are Gaussian, that (1.4) remains true for the trigonometric functions in $B = L^\infty(\mathbb{T})$. Such a fact is astonishing because $L^\infty(\mathbb{T}^d)$ is the typical Banach space that has not finite cotype. The key of this observation is that the last statement merely means that *there is a* sequence $(f_n)_{n\in\mathbb{N}}$ in $B = L^\infty(\mathbb{T})$ for which (1.4) is false. In other words, the trigonometric functions e^{inx} have somehow a special behavior in the Banach space $L^\infty(\mathbb{T})$. The following implications summarize the previous results

$$\begin{tikzcd}
\boxed{\displaystyle\sum_{n\in\mathbb{Z}} \varepsilon_n a_n e^{inx} \in L^\infty_x(\mathbb{T}) \text{ a.s.}} \arrow[rr, leftrightarrow, "(1.3)"] & & \boxed{\displaystyle\sum_{n\in\mathbb{Z}} X_n a_n e^{inx} \in L^\infty_x(\mathbb{T}) \text{ a.s.}} \\
& \boxed{\exists\, \gamma > 1 \quad \displaystyle\sum_{\substack{n\in\mathbb{Z}\\ |n|\geqslant 2}} |a_n|^2 \ln^\gamma(n) < +\infty} \arrow[ul, "\textbf{(PZ)}"] \arrow[ur, "\textbf{(UPZ)}"'] &
\end{tikzcd}$$

Our main sequence of results extend the *universal Paley–Zygmund* implication (UPZ), with a moment condition $\sup \mathbf{E}[|X_n|^2] < +\infty$, if the trigonometric functions on $\mathbb{T}^d$ are replaced with eigenfunctions in the following two multidimensional settings:

- for the Laplace–Beltrami operator $\mathbf{\Delta}$ on a compact Riemannian manifold X without boundary,
- for superquadratic oscillators $-\Delta + |x|^{2\alpha}$ on $\mathbb{R}^d$, with $(\alpha, d) \in \mathbb{N}^\star \times \mathbb{N}^\star$.

We briefly recall that, for the first setting, the Laplace–Beltrami operator $\mathbf{\Delta}$ acts from the space $\mathcal{C}^\infty(X)$ to itself (see [5] or the introduction of [47]) and there is a natural measure on X (the Riemannian volume) that is invariant under isometries of X and allows to consider the Banach spaces $L^p(X)$ for any $p \in [1, +\infty]$. Moreover, there is a Hilbert basis $(\phi_n)_{n\in\mathbb{N}}$ of $L^2(X)$ that diagonalizes $\mathbf{\Delta}$:

$$\mathbf{\Delta}\phi_n = -\lambda_n^2 \phi_n, \qquad \lambda_0 = 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to +\infty.$$

Although the sequence $(\phi_n)_{n\in\mathbb{N}}$ may not be unique (if there are multiple eigenvalues), the sequence of eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ of $\sqrt{-\mathbf{\Delta}}$ is completely determined by the Riemannian structure of X. In the specific case $X = \mathbb{T}$, one recovers $\lambda_n = k$ associated to the eigenfunctions $x \mapsto e^{\pm ikx}$. It is thus natural to try to extend Theorem 1.1 for random linear combinations of eigenfunctions and thus to improve the usual Sobolev embeddings. Such results appear in [47]. More precisely, [47, Corollary 6] states the following:

for any complex sequence $(c_n)_{n\in\mathbb{N}}$ satisfying

$$\exists\, \gamma > 1 \qquad \sum_{n\geqslant 1} |c_n|^2 \lambda_n^{\dim(X)-1} \ln^{\gamma}(\lambda_n) < +\infty, \tag{1.5}$$

the random series $\sum \varepsilon_n c_n \phi_n$ converges almost surely in the Banach space $L^\infty(X)$. Moreover, if one looks at the universality given by the proof of [47, Theorem 5], one sees that one may replace the Rademacher random variables ε_n with any nonzero sequence of i.i.d. subgaussian random variables (we recall that a centered random variable Y is subgaussian if it satisfies the large deviations estimate $\mathbf{P}(|Y| \geqslant t) \lesssim e^{-Ct^2}$ as $t \to +\infty$, another equivalent definition is the bound $\mathbf{E}[|Y|^p]^{1/p} \lesssim \sqrt{p}$ as $p \to +\infty$). Furthermore, the choice of randomization in [47] highly depends on the sequence of the chosen Hilbert basis $(\phi_n)_n$ (that is not unique). Other choices of randomization have been studied in the following two settings:

(i) the paper [11] makes use of a randomization around the unit balls of a sequence of subspaces of $L^2(X)$ (with increasing dimensions) if X is a sphere or more generally a Riemannian compact manifold. The proofs however involve subgaussian random variables (or more generally random variables that satisfy Gaussian deviation estimates, see [11, Appendice C.1]).

(ii) in [24, 38, 39], another intrinsic randomization is used by selecting again a sequence of subspaces of $L^2(\mathbb{R}^d)$ (with increasing dimensions). The models studied were the quantum harmonic oscillator $-\Delta + |x|^2$ on $\mathbb{R}^d$ or more generally $-\Delta + |x|^{2\alpha}$ on $\mathbb{R}^d$ in [41]. Unfortunately, the proofs needed a technical condition called a *squeezing* condition and specific distributional spaces (denoted by $\mathcal{Z}^s_\phi(\mathbb{R}^d)$). We refer to [24, lines (1.5) and (1.9) and part 1.1.2] or [41, lines (1.8) and (1.9)]). It is not necessary to recall the precise above definitions but the reason why such technicalities were useful is the following: they forced *unidimensional* random series of eigenfunctions to behave as if they were *multidimensional*. Hence, estimates of eigenfunctions are replaced with estimates of spectral functions. Again, as in [11], the $L^\infty(\mathbb{R}^d)$ results involve probability measures that satisfy the Gaussian concentration of measure property (see [41]).

In both previous frameworks, the independent random variables have finite moments of all order and it is not clear if one can reach a universal moment condition of order two as explained above. The paper [23] introduced other random series (involving random matrices without any *squeezing* conditions) we think are the most fitting for multidimensional settings (see below (2.2)). The idea was to take advantage of the multidimensional Kahane–Khintchine inequalities (proven by Marcus and Pisier in [34], see (7.1)). Such inequalities allow to overcome the use of large deviations estimates or the famous phenomenon of concentration of the measure (that both need subgaussian random variables). All results in [23] were concerned by the Banach space L^p with finite p. The present work aims to study the L^∞ case with those new random series. It is now worthwhile to recall why the L^∞ case is much more difficult than L^p with finite p. The Maurey theorem states that for any $p \in [1, +\infty)$, any $N \in \mathbb{N}^\star$ and any $(f_1, \dots, f_N) \in L^p(\mathbb{R})^N$, the following holds true

$$\mathbf{E}\left[\left\|\sum_{n=1}^{N} \varepsilon_n f_n\right\|_{L^p(\mathbb{R})}\right] \simeq \left\|\sqrt{\sum_{n=1}^{N} |f_n|^2}\right\|_{L^p(\mathbb{R})}.$$

The previous equivalence obviously transfers the study of *unidimensional* random series (at least with Rademacher laws) to a deterministic question. The *multidimensional* case is done in [23] and is summarized in Theorem 12.1. To our knowledge, such a result does not exist for $p = +\infty$.

The plan of the paper is the following:

In Section 2, we define more precisely our choice of randomization and we state three multidimensional Paley–Zygmund results on Riemannian compact manifolds without boundary. In particular, we will see the role of the BMO space on compact manifolds (thus extending known results on the torus, see [42]) and a somehow strange property of *spatial universality* (see (2.6)). Those results are proved in Sections 8, 9, 10, 11 and 12.

In Section 3, we motivate our new random series (seen as random initial data) by giving a modest application to the cubic wave equation $\partial_t^2 w - \mathbf{\Delta} w + w^3 = 0$ on a three-dimensional compact manifold. The proof was initially discovered by Burq and Tzvetkov for the torus $\mathbb{T}^3$ in [14] and modified by de Suzzonni for the three-dimensional sphere $\mathbb{S}^3$ in [43]). We explain in Section 15 and 16 how our version of the Paley–Zygmund theorem allows to modify this proof to reach any boundaryless compact manifold (without any geometric restriction). Our approach avoids to search specific eigenbasis (by comparison with [14, 43]) but has to be connected with the study of the critical semi-linear wave equation in [11].

In Section 4, we state Paley–Zygmund theorems for $-\Delta + |x|^{2\alpha}$ on $\mathbb{R}^d$ and explain the differences with Section 2. Those results are proved in Sections 13, 14, 18, 19, 20 and 21.

In Section 5, we state several sharp L^p bounds for the superquadratic oscillators $-\Delta + |x|^{2\alpha}$. Those bounds complete those of [24, 29, 41]. At first sight, such bounds have a marginal relevance with our study of random linear combinations. But the connection is the following: the optimality of such bounds will be a consequence of a specific use of the multidimensional Kahane–Khintchine inequalities. In the other side, our Paley–Zygmund theorems need $L^2 \to L^\infty$ bounds and a weak Bernstein inequality $L^\infty \to \nabla L^\infty$. We shall also prove a Bernstein inequality, see below (5.4), in a stronger form that has it own interest. For instance, the strong Bernstein inequality will allow for an enlightenment of the same critical exponent $\frac{2d}{d-1}$ appearing in several models peaking at a point (without any computation involving orthogonal polynomials). The proofs are developed in Sections 22, 23. Among those inequalities, the strong Bernstein inequality is the most intricate but follows a more or less classical way. For that reason, we have decided to postpone the proof to Appendix D. The main steps of the proof are:

(a) a parametrix for the Weyl–Hörmander pseudo-differential calculus (see Appendix A),
(b) an asymptotic development via the Helffer–Sjöstrand formula (see Appendix B),
(c) several $L^r \to L^r$ homogeneous estimates of pseudo-differential operators (see Appendix C).

In Sections 6 and 7, we state and prove an abstract, universal and multidimensional Paley–Zygmund theorem, that is Theorem 6.1, that will be used for compact manifolds and for $-\Delta + |x|^{2\alpha}$.

In Section 17, we show an exponential decay property of the spectral function of $-\Delta + |x|^{2\alpha}$. Such a proprety plays a role in the proof of the Paley–Zygmund theorems and for every $L^2 \to L^p$ bounds of Section 5.

Finally, Appendices E and F contain simple lemmas about random vectors.

From now, $(X_n)_{n\in\mathbb{N}}$ will everywhere denote a sequence of real independent random variables on a probability space Ω endowed with a probability measure $\mathbf{P}$. For a given sequence of positive integers $(d_n)_{n\in\mathbb{N}}$, we denote by $(\mathcal{E}_n)_{n\in\mathbb{N}}$ a sequence of random matrices such that the law of $\mathcal{E}_n : \Omega \to U_{d_n}(\mathbb{C})$ is the normalized Haar measure of the unitary group $U_{d_n}(\mathbb{C})$. The same analysis holds true for orthogonal groups $O_{d_n}(\mathbb{R})$. Finally, all the variables and random matrices that appear in a same statement will be assumed to be mutually independent and defined on the same probability space Ω (it will be however useful to replace Ω with Ω^2 in some proofs).

Acknowledgments. The author would like to thank the anonymous referee for a careful reading of the first version of the paper.

2. Paley–Zygmund theorems on Riemannian compact manifolds

In this section X is a boundaryless Riemannian compact manifold of dimension d, and we denote by $\mathbf{\Delta}$ its non positive Laplace–Beltrami operator. Without any precision, the Hilbert space $L^2(X)$ is considered with respect to the Riemannian measure of X. For any function $g \in L^2(X)$ we define its spectrum $\mathrm{sp}(g)$ by

$$\mathrm{sp}(g) := \left\{ \lambda \in [0,+\infty), \ \ \exists\, \phi \in L^2(X) \ \ \int_X g\overline{\phi}\,\mathrm{d}x \neq 0 \ \text{ and } \ \mathbf{\Delta}\phi = -\lambda^2\phi \right\}.$$

For any $\kappa > 0$ and $n \in \mathbb{N}$ we also define the subspaces

$$\begin{aligned} & E_0 := \{g \in L^2(X), \ \ \mathrm{sp}(g) \subset [0,\kappa]\}. \\ \forall\, n \in \mathbb{N}^* \quad & E_n := \{g \in L^2(X), \ \ \mathrm{sp}(g) \subset (\kappa n, \kappa n + \kappa]\}. \end{aligned} \tag{2.1}$$

The subspaces E_n are orthogonal. It is well known that if κ is large enough then $d_n := \dim(E_n)$ is positive and behaves like n^{d-1} as $n \to +\infty$ (this is a consequence of the Weyl law with remainder, see below Lemma 8.1). We thus may consider a Hilbert basis $(\phi_{n,1}, \dots, \phi_{n,d_n})$ of E_n where $d_n \simeq n^{d-1}$.

Now consider a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in E_n$ and assume that $f = \sum_{n\in\mathbb{N}} f_n$ is a distribution on X (which practically means that the sequence $(\|f_n\|_{L^2(X)})_{n\geqslant 0}$ has a polynomial growth). We choose to randomize the distribution f by considering the random series

$$\sum_{n\in\mathbb{N}} X_n(\omega) f_n^\omega \quad \text{with} \quad f_n^\omega := \sum_{i=1}^{d_n}\sum_{j=1}^{d_n} \mathcal{E}_{n,i,j}(\omega)\langle f_n, \phi_{n,j}\rangle \phi_{n,i}. \tag{2.2}$$

In [23, Théorème 1.8], more general random matrices than $X_n\mathcal{E}_n$ are studied in $L^p(X)$, with finite p. But we believe that the notation $\sum_{n\in\mathbb{N}} X_n(\omega) f_n^\omega$ is much more explicit and simple. We stress that the random part f_n^ω is much more relevant than $X_n(\omega)$ (and we may even choose $X_n = 1$ in our paper). Also note that the law of $\omega \mapsto f_n^\omega$ merely depends on $\|f_n\|_{L^2(X)}$.

Before stating any result, let us point out that Gaussian measures on E_n are often studied, in other contexts, because explicit computations are possible (see for instance [1, 50]). It turns out that Gaussian measures are indeed particular cases of (2.2). To see that point, consider a sequence $(g_{n,k})_{1\leqslant k\leqslant d_n}$ of i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$ Gaussian random variables that are mutually independent with the random matrix $\mathcal{E}_n$. The main property of the random vector $(g_{n,1},\dots,g_{n,d_n})$ is its unitary invariance. Standard computations or more generally Lemma E.1 allow to check the following law equivalence:

$$\sum_{k=1}^{d_n} g_{n,k}(\omega)\phi_{n,k} \sim \sqrt{|g_{n,1}(\omega)|^2+\cdots+|g_{n,d_n}(\omega)|^2}(\phi_{n,1})^{\omega}.$$

Let us now state our generalization of Theorem 1.1 on $L^\infty(X)$.

Theorem 2.1. — *Assume that the random variables $(X_n)_{n\in\mathbb{N}}$ satisfy $\sup_{n\in\mathbb{N}}\mathbf{E}[|X_n|^2]<+\infty$. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ satisfying $f_n\in E_n$ for any $n\in\mathbb{N}$ and*

$$\exists\,\gamma>1 \qquad \sum_{n\geqslant 2}\|f_n\|^2_{L^2(X)}\ln^\gamma(n)<+\infty. \tag{2.3}$$

Then, for almost every $\omega\in\Omega$, the random series $\sum X_n(\omega)f_n^\omega$ converges in $L^\infty(X)$.

The previous result improves [47, Corollary 6] in two ways: firstly, one has a probabilistic universality, in particular we do not need to force any assumption of large deviation estimates on each random variable X_n. Secondly, the assumption (2.3) is much less demanding (see (1.5)). Let us add that our proof is spectral whereas the proof of Tzvetkov is spatial.

Just after stating [47, Corollary 6], Tzvetkov asked if the condition $\gamma>1$, that appears in (2.3), is sharp in a Paley–Zygmund theorem on a Riemannian compact manifold of dimension $d\geqslant 2$ (e.g. a sphere). For $X=\mathbb{T}$, the answer is positive and comes from the existence of Sidon sets [37, p. 350]. Unfortunately, the author is not aware of such a notion on a general compact Riemannian manifold. For $\gamma<1$, we shall overcome this difficulty by using the fact that the dimensions $d_n=\dim(E_n)$ tend to $+\infty$ (a property that is specific to the case $d\geqslant 2$). Our result needs to introduce the space of functions of bounded mean oscillation on the boundaryless compact manifold X. The original BMO space had been introduced for the Euclidean space in [26] and, though slightly larger than L^∞, is usually considered as a good substitute. A function $f\in L^1(X)$ belongs to $\mathrm{BMO}(X)$ if and only if the following semi-norm (vanishing for constants) is finite

$$\|f\|_{\mathrm{BMO}(X)} := \sup_{\substack{0<\varepsilon<r_0(X)\\ x\in X}} \frac{1}{\mathrm{Vol}(B(x,\varepsilon))}\int_{B(x,\varepsilon)}|f(y)-f_{B(x,\varepsilon)}|\,\mathrm{d}y, \tag{2.4}$$

where $r_0(X)$ is a positive number called the injectivity radius of X and $f_{B(x,\varepsilon)}$ is the average of f on the geodesic ball $B(x,\varepsilon)$ (that definition is studied in [9]). It turns out that $\mathrm{BMO}(X)$ allows for a satisfactory enlightenment of the almost sharpness of Theorem 2.1.

Theorem 2.2. — *Assume that the random variables $(X_n)_{n\in\mathbb{N}}$ satisfy $\sup_{n\in\mathbb{N}}\mathbf{E}[|X_n|^2]<+\infty$ and consider a sequence $(f_n)_{n\in\mathbb{N}}$, with $f_n\in E_n$ for any $n\in\mathbb{N}$, satisfying*

$$\sum_{n\geqslant 2}\|f_n\|^2_{L^2(X)}\ln(n)<+\infty. \tag{2.5}$$

Then, for almost every $\omega\in\Omega$, the random series $\sum X_n(\omega)f_n^\omega$ converges in $\mathrm{BMO}(X)$.

Conversely, in the case $\dim(X)\geqslant 2$, there is a sequence (f_n) satisfying

$$\forall\,\gamma\in[0,1)\qquad \sum_{n\geqslant 2}\|f_n\|^2_{L^2(X)}\ln^\gamma(n)<+\infty$$

and such that the random series $\sum f_n^\omega$ almost surely diverges in $\mathrm{BMO}(X)$ (and also in $L^\infty(X)$ because of the inequality $\|\cdot\|_{\mathrm{BMO}(X)}\leqslant 2\|\cdot\|_{L^\infty(X)}$).

The first statement of Theorem 2.2 is known on the torus $X=\mathbb{T}$ for analytic versions of $\mathrm{BMO}(\mathbb{T})$ (see [42]). Roughly speaking, Theorem 2.2 shows that $\gamma=1$ is the critical logarithmic exponent for the almost sure convergence in $\mathrm{BMO}(X)$ or $L^\infty(X)$. Our proof needs to understand the Littlewood–Paley theory of $\mathrm{BMO}(X)$. This is done by adapting a strategy used by Burq–Gérard–Tzvetkov in [10] that transfers a Littlewood–Paley theory on the Euclidean space to a boundaryless compact manifold.

As explained in the introduction, the L^∞ case and the L^p case, with finite p, have a completely different approach. It is therefore interesting to compare Theorem 2.1 and the following optimal result.

Theorem 2.3. — *Let us consider*

- *a sequence $(f_n)_{n\in\mathbb{N}}$, with $f_n\in E_n$ for each $n\in\mathbb{N}$,*
- *a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ satisfying for some $p\in[1,+\infty)$ the following conditions*

$$0<\inf_{n\in\mathbb{N}}\mathbf{E}[|X_n|]\quad\textit{and}\quad\sup_{n\in\mathbb{N}}\mathbf{E}[|X_n|^{\max(2,p)}]<+\infty,$$

- *a Borel probability measure ν on X.*

Then the following assertions are equivalent:

1. *the series $\sum\|f_n\|^2_{L^2(X)}$ is convergent, namely $\sum_{n\in\mathbb{N}}f_n$ belongs to $L^2(X)$,*
2. *the random series $\sum X_n(\omega)f_n^\omega$ almost surely converges in $L^p(X,\nu)$,*
3. *there is $x\in X$ such that, the random series $\sum X_n(\omega)f_n^\omega(x)$ almost surely converges in $\mathbb{C}$.*

One may also add the following (apparently stronger) statement

(4) *for every $x \in X$, for almost every $\omega \in \Omega$, the random series $\sum X_n(\omega) f_n^\omega(x)$ converges in $\mathbb{C}$.*

Let us make a few remarks about Theorem 2.3:

- Assuming p belonging to $[2, +\infty)$, ν being the normalized Riemannian volume of X and the random variables X_n satisfying uniform large deviations estimates, the author considers that the implication (1) $\Rightarrow$ (2) essentially appears in [11] in a different form.
- Whereas we were primarily interested in the universality with respect to the random variables X_n, we see that Point (2) of Theorem 2.3 also shows a spatial universality: for any two Borel probability measures ν_1 and ν_2 with *disjoint* supports on X, the following equivalence holds true:

$$\sum X_n(\omega) f_n^\omega \quad \text{a.s. converges in } L^1(X, \nu_1) \qquad \Leftrightarrow \quad \sum X_n(\omega) f_n^\omega \quad \text{a.s. converges in } L^1(X, \nu_2). \tag{2.6}$$

  That fact is not surprising if ν_2 is an isometric pushforward of ν_1 but seems to be new if ν_1 and ν_2 are *singular* and if X is a general Riemannian compact manifold.
- We stress that the condition (3) is very weak because the (almost sure) convergence holds true on *one point* $x \in X$ (one can see it as Point (2) with a Dirac measure $\nu = \delta_x$). Note now that a simple use of the Fubini–Tonelli theorem on the product space $X \times \Omega$ shows that (4) and (3) are equivalent to

  "*for almost every $\omega \in \Omega$, for* almost every *$x \in X$, the random series $\sum X_n(\omega) f_n^\omega(x)$ converges in $\mathbb{C}$.*"

  Hence, Theorem 2.3 seems to be a satisfactory generalization of the setting of random Fourier series [27, p. 46] to the Riemannian setting. Similarly to Theorem 2.2, Theorem 2.3 also motivates the study of the random series looking like $\sum X_n(\omega) f_n^\omega$.
- The proof of Theorem 2.3 will be a short consequence of a sharp bound of Hörmander on the spectral function (see Section 12) and a result previously obtained in [23] (namely Theorem 12.1 whose proof relies on the Kahane–Khintchine–Marcus–Pisier inequalities).

As explained in the beginning of the introduction, a Paley–Zygmund theorem should be considered as a probabilistic improvement of a Sobolev embedding. Remember that a spectral definition of $H^s(X)$, for any $s \in \mathbb{R}$, is given by

$$(2.7)\quad \forall\,(f_n)\in\prod_{n\in\mathbb{N}}E_n\quad \sum_{n\in\mathbb{N}}f_n\in H^s(X)\quad\Leftrightarrow\quad\sum_{n\geqslant 1}\|f_n\|^2_{L^2(X)}\,n^{2s}<+\infty.$$

Note that (2.3) and (2.5) are satisfied if $\sum_{n\in\mathbb{N}}f_n$ merely belongs to $H^s(X)$ for some $s>0$. Hence, Theorem 2.1 and Theorem 2.2 allows a probabilistic gain of almost $\frac{d}{2}$ derivatives by comparison with the Sobolev embeddings

$$\begin{aligned} &H^s(X)\subset L^\infty(X), &&\forall\, s>\frac{d}{2},\\ &H^{d/2}(X)\subset \mathrm{BMO}(X) &&(\text{see [9, p. 210]}). \end{aligned}$$

3. The cubic wave equation with random initial data

In the series of papers [12, 13, 14], Burq and Tzvetkov introduced new ideas to solve nonlinear wave equations with rough initial data (namely that belong to $H^s(X)$, see (2.7), for small values of s). They cleverly use the probabilistic gain of integrability to solve equations that are deterministically ill-posed. We go on as in the last section. The cubic wave equation

$$(\partial_t^2-\boldsymbol{\Delta})v+v^3=0,\quad (t,x)\in\mathbb{R}\times X$$

is $\frac{1}{2}$-critical in dimension 3 so it is interesting to construct solutions in $H^s(X)\times H^{s-1}(X)$ for $s\in(0,\frac{1}{2})$ (see the introductions of [7, 14]). The previous equation was studied for the three-dimensional torus $X=\mathbb{T}^3$ in [14] and for the three-dimensional sphere $X=\mathbb{S}^3$ in [43]. The previous two cases use that the Laplace–Beltrami opertators on $\mathbb{T}^3$ and on $\mathbb{S}^3$ admit a Hilbert basis of eigenfunctions that are uniformly bounded on any $L^p(X)$ space (for $2\leqslant p<+\infty$). That fact is true on $\mathbb{T}^d$ thanks to the trigonometric functions but is not obvious at all on $\mathbb{S}^d$ (see a probabilistic proof in [11, 43]). To the knowledge of the author, such a property is not known to be true on any compact manifold X. We shall overcome this issue thanks to a quantitative version of our Paley–Zygmund result (Theorem 6.1) that holds true without any geometric assumption on X. Although our approach is different from [11], we use a “spectral trick” (Lemma 8.1) that we learn from the latter paper. The first thing to do is to define the random initial data. For any $n\in\mathbb{N}$, we consider $\mathcal{E}_n:\Omega\to O_{d_n}(\mathbb{R})$ and $\mathcal{E}'_n:\Omega\to O_{d_n}(\mathbb{R})$ two random matrices whose laws are the normalized Haar measures. We also consider two sequences of real random variables (X_n) and (X'_n). As above, all the considered random variables and matrices are assumed to be mutually independent. For any $s \in \mathbb{R}$, any $(v_0, v_1) \in H^s(X) \times H^{s-1}(X)$, we define the following two initial data for any $\omega \in \Omega$:

$$\begin{aligned} v_0^\omega &:= \sum_{n\in\mathbb{N}} X_n(\omega)\left(\sum_{i=1}^{d_n}\sum_{j=1}^{d_n} \mathcal{E}_{n,i,j}(\omega)\langle v_0, \phi_{n,j}\rangle \phi_{n,i}\right), \\ v_1^\omega &:= \sum_{n\in\mathbb{N}} X'_n(\omega)\left(\sum_{i=1}^{d_n}\sum_{j=1}^{d_n} \mathcal{E}'_{n,i,j}(\omega)\langle v_1, \phi_{n,j}\rangle \phi_{n,i}\right). \end{aligned} \tag{3.1}$$

In the previous series, each eigenfunction is assumed to be real-valued.

We claim that v_0^ω and v_1^ω almost surely admit the same regularity as v_0 and v_1.

Proposition 3.1. — *We keep the above notations and assume*

$$0 < \inf_{n\in\mathbb{N}} \mathbf{E}[|X_n|] \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^2] < +\infty. \tag{3.2}$$

then the following two assertions are equivalent:

(1) *v_0 belongs to $H^s(X)$,*
(2) *for almost every $\omega \in \Omega$, the random series defining v_0^ω converges in $H^s(X)$.*

We now can use the analysis of [14] (more precisely Theorem 16.1 below) to get the following result.

Theorem 3.2. — *Assume* $\dim(X) = 3$. *Consider a real number $s \in (0, \frac{1}{2})$ and two sequences of independent random variables X_n and X'_n such that*

$$0 < \inf_{n\in\mathbb{N}} \mathbf{E}[|X_n|], \quad 0 < \inf_{n\in\mathbb{N}} \mathbf{E}[|X'_n|], \quad \sup_{n\in\mathbb{N}} \mathbf{E}\left[|X_n|^3 + |X'_n|^3\right] < +\infty. \tag{3.3}$$

For any couple of real-valued functions $(v_0, v_1) \in H^s(X) \times H^{s-1}(X)$, for almost every $\omega \in \Omega$, the random function (v_0^ω, v_1^ω) belongs to $H^s(X) \times H^{s-1}(X)$ and the cubic wave equation

$$(\partial_t^2 - \mathbf{\Delta})v + v^3 = 0, \qquad v(0,\cdot) = v_0^\omega, \qquad \dot{v}(0,\cdot) = v_1^\omega,$$

admits a unique global solution v that satisfies

$$v(t) - \cos(t\sqrt{-\mathbf{\Delta}})v_0^\omega - \frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}} v_1^\omega \in \mathcal{C}_t^0(\mathbb{R}, H^1(X)) \cap \mathcal{C}_t^1(\mathbb{R}, L^2(X)).$$

Proposition 3.1 has to be compared with [12, Lemma B.1] that uses the apparently weaker assumption

$$\exists\, C > 0 \qquad \inf_{n\in\mathbb{N}} \mathbf{P}(|X_n| \geqslant C) > 0. \tag{3.4}$$

But we believe that (3.2) is more natural in the framework of random series (see [34] or [33, Théorème III.4, p. 125]). Let us explain why there is essentially no mathematical loss to assume (3.2) or (3.4) for our purpose. Note indeed that the assumption $\sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^2] < +\infty$ is always satisfied (for instance implied by (3.3) in our paper or by [12, line (1.4)]). Then the Paley–Zygmund inequality (see [27, Page 8]) easily shows the following equivalence:

$$0 < \inf_{n\in\mathbb{N}} \mathbf{E}[|X_n|] \quad \text{and} \quad \sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^2] < +\infty \\ \Leftrightarrow \quad (3.4) \quad \text{and} \quad \sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^2] < +\infty.$$

Our proof of Proposition 3.1 is however different from that of [12] since we do not use the same random series for initial data.

4. Paley–Zygmund theorems for the superquadratic oscillator

Before entering into details, we present two results that give a real contrast with the setting of compact manifolds. In the latter setting, the almost sure convergence in *one point* is equivalent to the probabilistic L^p convergence for any finite p (see Theorem 2.3), and also to the deterministic L^2 convergence. The next two results show that the lack of compactness of $\mathbb{R}^d$ forbids a similar property for the harmonic oscillator $-\Delta + |x|^2$. We moreover emphasize the need to avoid $x = 0$ in the next statement.

Theorem 4.1. — *Assume $d \geqslant 2$ and denote by E_n the eigenspace of $-\Delta + |x|^2$ in $L^2(\mathbb{R}^d)$ associated to the eigenvalue $d + 2n$. Now consider*

- *a sequence $(f_n)_{n\in\mathbb{N}}$, with $f_n \in E_n$,*
- *a real number $p \in [1, +\infty)$ and a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ satisfying*

$$0 < \inf_{n\in\mathbb{N}} \mathbf{E}[|X_n|] \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^{\max(2,p)}] < +\infty,$$

- *a Borel probability measure ν on $\mathbb{R}^d\backslash\{0\}$.*

Then the following assertions are equivalent

(1) *the series $\sum \|f_n\|^2_{L^2(\mathbb{R}^d)} n^{-\frac{d}{2}}$ converges (which means that $\sum_{n\in\mathbb{N}} f_n$ belongs to $\mathcal{H}_1^{-\frac{d}{2}}(\mathbb{R}^d)$, see below (4.4))*

(2) *for almost every $\omega \in \Omega$, the random series $\sum X_n(\omega) f_n^\omega$ (as in (2.2)) converges in $L^p(\mathbb{R}^d, \nu)$,*

(3) *there is* $x \in \mathbb{R}^d \backslash \{0\}$ *such that, for almost every* $\omega \in \Omega$*, the random series* $\sum X_n(\omega) f_n^\omega(x)$ *is convergent in* $\mathbb{C}$*.*

As above, we may add the (apparently stronger) assertion:

(4) *for every* $x \in \mathbb{R}^d$*, for almost every* $\omega \in \Omega$*, the random series* $\sum X_n(\omega) f_n^\omega(x)$ *converges in* $\mathbb{C}$*.*

Proposition 4.2. — *There exists a sequence of functions* $(f_n)_{n \in \mathbb{N}}$*, with* $f_n \in E_n$ *for any* $n \in \mathbb{N}$*, and that satisfies* (1)–(4) *but the random series* $\sum f_n^\omega$ *almost surely diverges in* $L^p(\mathbb{R}^d)$ *for any* $p \in [1, +\infty)$*.*

Let us give a few comments about Theorem 4.1:

- The complete solution of the almost sure convergence in $L^p(\mathbb{R}^d)$, for finite p, is given in [23, Théorème 1.4]. But the latter result is useless to prove Proposition 4.2 because we need to deal with an uncountable set of p. We shall overcome this issue in Section 14 by introducing a simple Banach space that embeds in any $L^p(\mathbb{R}^d)$ for any $p \in [1, +\infty)$.
- Remembering that any function in E_{2n+1} is odd, we obviously have $f_{2n+1}^\omega(0) = 0$. That fact explains why we cannot choose $x = 0$ in (3).

We now present similar results for the operators $-\Delta + |x|^{2\alpha}$ on $L^2(\mathbb{R}^d)$ where $d \geqslant 2$ and $\alpha \geqslant 1$ are two integers (such operators are sometimes referred as quantum superquadratic oscillators [40, 48, 49]). Due to the fact that the spectral analysis of such operators is not explicit, our results are slightly weaker than those of the case $\alpha = 1$ and we merely concentrate our efforts in obtaining sufficient conditions of almost sure convergence. From now, we denote by $V_{2\alpha}$ a positive 2α-homogeneous polynomial, where "positive" means that $V_{2\alpha}$ is positive on $\mathbb{R}^d \backslash \{0\}$. We recall that $L^2(\mathbb{R}^d)$ admits a Hilbert basis $(\varphi_j)_{j \geqslant 0}$ made of eigenfunctions of $-\Delta + V_{2\alpha}$ and let $(\lambda_j)_{j \geqslant 0}$ be the associated sequence of eigenvalues which is non-decreasing, positive and tends to $+\infty$. Thanks to the paper [19], it is well understood that $-\Delta + V_{2\alpha}$ should not be considered as a differential operator of order 2 but as a pseudo-differential operator of order $\frac{2\alpha}{\alpha+1}$. For $\alpha = 1$, this consideration means that the quantum harmonic oscillator $-\Delta + |x|^2$ should be seen as an operator of order 1! Such unintuitive statements will be clarified thanks to the Weyl–Hörmander symbolic calculus (see the computation (A.5) of the λ-function associated to a natural metric for $-\Delta + V_{2\alpha}$). Therefore, a good choice of clusters for $-\Delta + V_{2\alpha}$ is given by the sequence of intervals

$$I_n := (\kappa n^{\frac{2\alpha}{\alpha+1}}, \kappa (n+1)^{\frac{2\alpha}{\alpha+1}}] \tag{4.1}$$

and $I_0 := [0, \kappa]$ for some fixed constant $\kappa > 0$. This is perfectly coherent with the harmonic oscillator $-\Delta + |x|^2$ whose spectrum is $d + 2\mathbb{N}$. We then introduce

$$\forall\, n \in \mathbb{N} \qquad E_n := \left\{ f \in L^2(\mathbb{R}^d), \quad f = \sum_{\lambda_j \in I_n} \langle f, \varphi_j \rangle \varphi_j \right\}.$$

We clearly have the Hilbert decomposition $L^2(\mathbb{R}^d) = \bigoplus_{n\in\mathbb{N}} E_n$. As above, it will be more convenient to denote by $(\phi_{n,1}, \dots, \phi_{n,d_n})$ a Hilbert basis of E_n with $d_n := \dim(E_n)$. By comparison with the two settings of Riemannian compact manifolds and the harmonic oscillator, the following gives another reason to consider the clusters I_n:

$$\forall\, \kappa \gg 1 \quad \exists\, C \geqslant 1 \quad \forall\, n \geqslant 1 \quad \frac{1}{C} n^{d-1} \leqslant \dim(E_n) \leqslant C n^{d-1}. \tag{4.2}$$

The proof of (4.2) is detailed in Section 18 as a consequence of a Weyl formula with remainder obtained by Helffer–Robert [19, Théorème 6-4, p. 840].

As for the setting of compact manifolds, we aim to study random series of functions of E_n. The case $\alpha = 1$, namely random linear combinations of Hermite functions, has been already studied in [24, Theorem 2.6] but the proofs involve *squeezing* conditions, specific distributional spaces and subgaussian random variables. The case $\alpha \geqslant 2$ does not appear in the known literature. We can now state a Paley–Zygmund theorem for $-\Delta + V_{2\alpha}$ without any *squeezing* condition or specific distributional spaces (and we thus improve [24, Theorem 2.6] for $\alpha = 1$). Moreover, the following statement merely needs a moment condition of order 2.

Theorem 4.3. — *Assume $d \geqslant 2$ and consider a sequence $(f_n)_{n\in\mathbb{N}}$, with $f_n \in E_n$ for each $n \in \mathbb{N}$. We moreover assume that the supremum $\sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^2]$ is finite and that the following holds true*

$$\exists\, \gamma > 1 \qquad \sum_{n\geqslant 2} \|f_n\|^2_{L^2(\mathbb{R}^d)}\, n^{-\frac{d}{\alpha+1}} \ln^\gamma(n) < +\infty. \tag{4.3}$$

Then, for almost every $\omega \in \Omega$, the random series $\sum X_n(\omega) f_n^\omega$ converges in $L^\infty(\mathbb{R}^d)$ (where f_n^ω is defined as in (2.2)).

As a direct consequence, we obtain the following "Paley–Zygmund" phenomenon that does not occur for compact manifolds (see the second assertion of Theorem 2.2).

Corollary 4.4. — *Assume $d \geqslant 2$ and consider a function $\sum_{n\in\mathbb{N}} f_n$ belonging to $L^2(\mathbb{R}^d)$ with $f_n \in E_n$ for each $n \in \mathbb{N}$. Then the random series $\sum f_n^\omega$ almost surely converges in $L^p(\mathbb{R}^d)$ for any $p \in [2, +\infty]$.*

Proof. — With probability 1, the series $\sum f_n^\omega$ converges in $L^2(\mathbb{R}^d)$ because the functions f_n^ω are orthogonal and fulfill $\|f_n^\omega\|_{L^2(\mathbb{R}^d)} = \|f_n\|_{L^2(\mathbb{R}^d)}$. The convergence in $L^\infty(\mathbb{R}^d)$ comes from (4.3). We conclude by interpolation. □

Although Theorem 2.1 looks like Theorem 4.3, the proof of the latter is much more difficult because $\mathbb{R}^d$ is not compact. We shall overcome this issue by proving a precise decay property (see Proposition 17.1) of the associated spectral function. As for compact manifolds, it is worth comparing with the L^p theory, for finite p. By using estimates proven in the papers [23, 41] (see details in Sections 20 and 21), one may prove the next two results. Remark that one almost recovers (4.3) as p tends to $+\infty$ in Theorem 4.6.

THEOREM 4.5. — *Assume $d \geqslant 2$ and consider a sequence $(f_n)_{n\in\mathbb{N}}$, with $f_n \in E_n$ for each $n \in \mathbb{N}$. We moreover assume*

$$\exists\, p \in [2, +\infty) \quad \sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^p] < +\infty \quad \textit{and} \quad \sum_{n\geqslant 1} \|f_n\|^2_{L^2(\mathbb{R}^d)}\, n^{-\frac{d}{\alpha+1}} < +\infty,$$

then

1. (1) *for every Borel probability measure ν on $\mathbb{R}^d$, the random series $\sum X_n(\omega) f_n^\omega$ almost surely converges in $L^p(\mathbb{R}^d, \nu)$,*
2. (2) *the random series $\sum X_n(\omega) f_n^\omega$ almost surely converges in $L^p_{loc}(\mathbb{R}^d)$,*
3. (3) *for every $x \in \mathbb{R}^d$, the random series $\sum X_n(\omega) f_n^\omega(x)$ almost surely converges in $\mathbb{C}$,*
4. (4) *the random series $\sum X_n(\omega) f_n^\omega(x)$ almost surely converges in $\mathbb{C}$ for almost every $x \in \mathbb{R}^d$.*

THEOREM 4.6. — *Assume $d \geqslant 2$ and consider a sequence $(f_n)_{n\in\mathbb{N}}$, with $f_n \in E_n$ for each $n \in \mathbb{N}$. We moreover assume that there is $p \in [2, +\infty)$ for which we have*

$$\sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^p] < +\infty \quad \textit{and} \quad \sum_{n\geqslant 1} \|f_n\|^2_{L^2(\mathbb{R}^d)}\, n^{-\frac{d}{\alpha+1}(1-\frac{2}{p})} < +\infty.$$

Then the random series $\sum X_n(\omega) f_n^\omega$ almost surely converges in $L^p(\mathbb{R}^d)$.

Let us now make a comparison with deterministic results. We need to recall the Sobolev spaces associated to $-\Delta + V_{2\alpha}$: for any $s \in \mathbb{N}$ one defines the subspace $\mathcal{H}^s_\alpha(\mathbb{R}^d)$ of the functions $f \in L^2(\mathbb{R}^d)$ satisfying $(-\Delta + V_{2\alpha})^{s/2} f \in L^2(\mathbb{R}^d)$. From [49, Lemma 2.4 with $p = 2$]), we have

$$\left\|(-\Delta + V_{2\alpha})^{s/2} f\right\|_{L^2(\mathbb{R}^d)} \simeq \|f\|_{H^s(\mathbb{R}^d)} + \sqrt{\int_{\mathbb{R}^d} (1 + |x|)^{2\alpha s} |f(x)|^2 \mathrm{d}x}. \tag{4.4}$$

The definition of the clusters (4.1) leads to the equivalent spectral definition

$$f \in \mathcal{H}^s_\alpha(\mathbb{R}^d) \quad \Leftrightarrow \quad \sum_{n \geqslant 1} n^{\frac{2\alpha s}{\alpha+1}} \, \|f_n\|^2_{L^2(\mathbb{R}^d)} < +\infty.$$

Forgetting the logarithmic term in (4.3), Theorem 4.3 roughly means that if f belongs to $\mathcal{H}_\alpha^{\frac{-d}{2\alpha}}(\mathbb{R}^d)$ then the randomized series $\sum X_n(\omega) f_n^\omega$ associated to f almost surely belongs to $L^\infty(\mathbb{R}^d)$. This "probabilistic Sobolev embedding" should be compared to the inclusion $\mathcal{H}^s_\alpha(\mathbb{R}^d) \subset H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ that holds true for any $s > \frac{d}{2}$. In other words, the probability theory allows a gain of almost $\frac{d}{2}\left(1 + \frac{1}{\alpha}\right)$ derivatives. A similar discussion is possible for Theorem 4.5 and Theorem 4.6.

5. Three optimal $L^2 \to L^p$ bounds and a Bernstein inequality

We keep the notations of the last part. Our first result proves the sharpness of $L^2 \to L^\infty$ bounds of Robert–Thomann [41, Proposition 2.4, $\delta = 1$, $\theta = 0$, $r = \infty$] and Koch–Tataru [29, Corollary 3.2, $p = \infty$] for the harmonic oscillator.

Proposition 5.1. — *Let us assume $d \geqslant 2$ and fix $\kappa \gg 1$ large enough (in the sense of (4.2)), then the following estimate holds true*

$$\forall\, n \gg 1 \qquad \sup_{\substack{f \in E_n \\ f \neq 0}} \frac{\|f\|_{L^\infty(\mathbb{R}^d)}}{\|f\|_{L^2(\mathbb{R}^d)}} \simeq n^{\frac{1}{2}\left(\frac{\alpha d}{\alpha+1} - 1\right)}, \tag{5.1}$$

where E_n is the subspace of functions of $L^2(\mathbb{R}^d)$ that are spectrally localized in $(\kappa n^{\frac{2\alpha}{\alpha+1}}, \kappa(n+1)^{\frac{2\alpha}{\alpha+1}}]$ with respect to $-\Delta + V_{2\alpha}$.

It is interesting to note that Proposition 5.1 interpolates the optimal results for the harmonic oscillator $\alpha = 1$ and for Riemannian compact manifolds $\alpha = +\infty$. Although Proposition 5.1 has its own interest, its optimality and the equivalence $d_n \simeq n^{d-1}$ show that our abstract Paley–Zygmund result (see below Corollary 6.2) cannot provide a better assumption in (4.3).

Our contribution in Proposition 5.1 is the lower bound in (5.1). The upper bound is proved in [41]. It turns out that Robert and Thomann also proved lower bounds but for larger spectral windows. Let us explain this point. Let h be the usual semiclassical parameter that tends to 0^+ and set $[\frac{a_h}{h}, \frac{b_h}{h}]$ a spectral window associated to $(-\Delta + V_{2\alpha})^{\frac{\alpha+1}{2\alpha}}$. We assume for some $\delta \in (0, 1]$

$$\lim_{h \to 0^+} a_h \leqslant \lim_{h \to 0^+} b_h \qquad \text{and} \qquad b_h - a_h \gtrsim h^\delta.$$

Lemma 3.7 of [41] proves an optimal two-side estimate of the L^p norm of the spectral function of $(-\Delta + V_{2\alpha})^{\frac{\alpha+1}{2\alpha}}$ on $[\frac{a_h}{h}, \frac{b_h}{h}]$ but the lower bound needs the assumption $\delta < \frac{2}{3}$. For our purpose, one has $[\frac{a_h}{h}, \frac{b_h}{h}] = [\kappa^{\frac{\alpha+1}{2\alpha}} n, \kappa^{\frac{\alpha+1}{2\alpha}}(n+1)]$ with $h = \frac{1}{n}$, $a_h = \kappa^{\frac{\alpha+1}{2\alpha}}$ and $b_h = \kappa^{\frac{\alpha+1}{2\alpha}}(1+h)$. Therefore, we are in the case $\delta = 1$ which is not covered by [41]. We overcome this pseudo-differential issue by proving a decay property of the so-called spectral function of $-\Delta + V_{2\alpha}$ (see Proposition 17.1 for a precise statement).

Let us present another application of this decay property combined with a probabilistic argument. It is known that large deviations estimates or the concentration measure phenomenon both allow to obtain sharp L^p estimates (see [11, 39]) but the case $p \in [1, 2]$ is not studied and we give here an alternative point of view based on the multidimensional Kahane–Khintchine inequalities.

Proposition 5.2. — *Assume $d \geqslant 2$ and also fix $\kappa \gg 1$ large enough (in the sense of (4.2)), then the following estimates hold true*

$$\forall\, p \in [2, +\infty) \qquad \forall\, n \gg 1 \qquad \inf_{\substack{f \in E_n \\ f \neq 0}} \frac{\|f\|_{L^p(\mathbb{R}^d)}}{\|f\|_{L^2(\mathbb{R}^d)}} \simeq n^{-\frac{d}{\alpha+1}\left(\frac{1}{2}-\frac{1}{p}\right)}, \tag{5.2}$$

$$\forall\, p \in [1, 2] \qquad \forall\, n \gg 1 \qquad \sup_{\substack{f \in E_n \\ f \neq 0}} \frac{\|f\|_{L^p(\mathbb{R}^d)}}{\|f\|_{L^2(\mathbb{R}^d)}} \simeq n^{\frac{d}{\alpha+1}\left(\frac{1}{p}-\frac{1}{2}\right)}. \tag{5.3}$$

where E_n is the subspace of functions of $L^2(\mathbb{R}^d)$ that are spectrally localized in $(\kappa n^{\frac{2\alpha}{\alpha+1}}, \kappa(n+1)^{\frac{2\alpha}{\alpha+1}}]$ with respect to $-\Delta + V_{2\alpha}$.

Our final result is a Bernstein inequality that we will use (at least in a weaker form) to prove our L^∞ Paley–Zygmund theorem for $-\Delta + V_{2\alpha}$ (Theorem 4.3).

Theorem 5.3. — *For any dimension $d \geqslant 1$, there is $C > 0$ (that depends on d and $V_{2\alpha}$) such that for any real number $\rho \geqslant 1$ and any function $f \in L^2(\mathbb{R}^d)$ that is spectrally localized in $[0, \rho]$ with respect to $-\Delta + V_{2\alpha}$ (in particular f is smooth), the following Bernstein inequality holds true*

$$\forall\, r \in [1, +\infty] \quad \|\nabla f\|_{L^r(\mathbb{R}^d)} \leqslant C\sqrt{\rho}\, \|f\|_{L^r(\mathbb{R}^d)}\,. \tag{5.4}$$

To the knowledge of the author, the paper [16] gives a general strategy, via gradient estimates of heat kernels, to prove Bernstein inequalities like (5.4). Although there is a wide literature on heat kernels and their gradient estimates, we have not found suitable bounds for $-\Delta + |x|^{2\alpha}$. Our proof of Theorem 5.3 is of pseudo-differential nature and is explained in several appendices.

We stress that the proofs of Proposition 5.1 and Proposition 5.2 do not give any explicit example of $f \in E_n$ that optimizes the considered bounds. Let us now recall that a concentration phenomenon of eigenfunctions is usually expected to have an impact on the L^p estimates. For instance, the lower bound (5.2) will come from the exponential decay outside a ball $B(0, \frac{c}{n^{1/(\alpha+1)}})$ (see Section 22). Let us explain another concentration phenomenon that is known for explicit examples but is enlightened by the Bernstein inequality. Consider a sequence of functions $f_n \in E_n$ that optimize the $L^2 \to L^\infty$ bounds (5.1). For any sequence $(x_n)_{n \geqslant 0}$ of $\mathbb{R}^d$, one may use (5.4) with $r = +\infty$ to get

$$\forall\, x \in \mathbb{R}^d \quad |f_n(x)| + |x - x_n| n^{\frac{\alpha}{\alpha+1}} \left\| f_n \right\|_{L^\infty(\mathbb{R}^d)} \gtrsim |f_n(x_n)|.$$

Now choose $x_n \in \mathbb{R}^d$ satisfying $|f_n(x_n)| \geqslant \frac{1}{2} \left\| f_n \right\|_{L^\infty(\mathbb{R}^d)}$. Hence, the function f_n must concentrate in a small ball centered in x_n:

$$|x - x_n| \lesssim n^{-\frac{\alpha}{\alpha+1}} \quad \Rightarrow \quad |f_n(x)| \gtrsim \left\| f_n \right\|_{L^\infty(\mathbb{R}^d)} \gtrsim n^{\frac{1}{2}\left(\frac{\alpha d}{\alpha+1} - 1\right)} \left\| f_n \right\|_{L^2(\mathbb{R}^d)}.$$

Such a concentration around a point implies the following bound from below

$$\begin{aligned} \forall\, p \geqslant 2 \quad \left\| f_n \right\|_{L^p(\mathbb{R}^d)} &\gtrsim n^{\frac{1}{2}\left(\frac{\alpha d}{\alpha+1} - 1\right) - \frac{d}{p}\frac{\alpha}{\alpha+1}} \left\| f_n \right\|_{L^2(\mathbb{R}^d)} \\ &= n^{\frac{d\alpha}{\alpha+1}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}} \left\| f_n \right\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

By comparison with (5.2), the last bound is of interest only if

$$\frac{d\alpha}{\alpha+1}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2} > -\frac{d}{\alpha+1}\left(\frac{1}{2} - \frac{1}{p}\right),$$

that is $p > \frac{2d}{d-1}$. In other words, the exponent $p := \frac{2d}{d-1}$ is the smallest exponent above which the peaking concentration is relevant for the L^p bounds. This interpretation is well-known in the following settings:

- The sphere $\mathbb{S}^d$ can be seen as a limit model corresponding to the case $\alpha = +\infty$. Let $(Z_n)_{n \geqslant 1}$ be a sequence of zonal harmonics with $\left\| Z_n \right\|_{L^2(\mathbb{S}^d)} = 1$ and write $\Delta Z_n = -\rho_n Z_n$ with $\rho_n = n(n+d-1)$. The following estimates are known (see for instance [23, line (6)]):

$$\begin{aligned} 1 \leqslant p < \frac{2d}{d-1} \quad &\Rightarrow \quad \left\| Z_n \right\|_{L^p(\mathbb{S}^d)} \simeq 1, \\ \frac{2d}{d-1} < p \leqslant \infty \quad &\Rightarrow \quad \left\| Z_n \right\|_{L^p(\mathbb{S}^d)} \simeq n^{\frac{d-1}{2} - \frac{d}{p}}. \end{aligned}$$

  It is moreover known that each Z_n concentrates on a ball $B(\wp, \frac{c}{\sqrt{\rho_n}})$ for some $c > 0$ and where $\wp$ is a pole of $\mathbb{S}^d$. It also optimizes the $L^2 \to L^\infty$ bound (5.1), the $L^2 \to L^p$ bounds (5.2) and (5.3) for

$p \in [1, \frac{2d}{d-1})$. Note that a more reasonable limit model would be the unit ball of $\mathbb{R}^d$ with Dirichlet boundary conditions but similar observations are true (see [3, Lemma 2.5]).

- For the harmonic oscillator $-\Delta + |x|^2$ on $L^2(\mathbb{R}^d)$, it has been recently remarked in [24, Proposition 2.4] that the optimal $L^2 \to L^\infty$ bounds (5.1) are reached for the radial eigenfunctions we denote by ψ_n. We still write $(-\Delta + |x|^2)\psi_n = \rho_n \psi_n$ with $\rho_n = 4n + d$. Similarly, it turns out that ψ_n concentrates on a ball $B(0, \frac{c}{\sqrt{\rho_n}}) \subset \mathbb{R}^d$. Combined with [24, Proposition 2.4 (iii)], it also appears that the functions ψ_n have optimal $L^2 \to L^p$ bounds (5.2) for $p \in [2, \frac{2d}{d-1})$.

An intriguing remark is that the same critical exponent $\frac{2d}{d-1}$ appears in those limit models whereas it is usually computed with intricate properties of Jacobi polynomials, Bessel functions or Laguerre polynomials. Although the boundary case needs further investigations, the Bernstein inequality shows that this is not an accident for the boundaryless case.

6. An abstract and multidimensional Paley–Zygmund theorem

Here we want to state an abstract Paley–Zygmund theorem that will encompass the last settings. So we choose to denote by X a Riemannian manifold (not necessarily compact). We also consider a sequence $(E_n)_{n\in\mathbb{N}}$ of nonzero finite dimensional subspaces of $L^2(X) \cap L^\infty(X) \cap \mathcal{C}^0(X)$. We write $d_n := \dim(E_n)$. A fundamental object in our study is the so-called spectral function of E_n:

$$\forall\, x \in X \qquad e_n(x) := |\phi_{n,1}(x)|^2 + \cdots + |\phi_{n,d_n}(x)|^2, \tag{6.1}$$

where $\phi_{n,1}, \ldots, \phi_{n,d_n}$ is a Hilbert basis of E_n. An easy but crucial property of the function $x \mapsto e_n(x)$ is that it does not depend on the specific choice of the latter Hilbert basis. In other words, the function e_n merely depends on the subspace E_n. We now consider the following assumptions that will be satisfied in our examples:

(A1) there is a constant $S \in \mathbb{R}$ such that

$$\exists\, C > 0 \quad \forall\, n \gg 1 \quad \frac{\|e_n\|_{L^\infty(X)}}{\dim(E_n)} \leqslant C n^S.$$

We now recall that $\|e_n\|_{L^\infty(X)}$ admits the following expression

$$\begin{aligned}\|e_n\|_{L^\infty(X)} &= \sup_{x\in X}\sup_{(a_1,\dots,a_{d_n})\in\mathbb{C}^{d_n}\setminus\{0\}} \frac{|a_1\phi_1(x)+\cdots+a_{d_n}\phi_{d_n}(x)|^2}{|a_1|^2+\cdots+|a_{d_n}|^2}\\ &= \sup_{u_n\in E_n\setminus\{0\}} \frac{\|u_n\|^2_{L^\infty(X)}}{\|u_n\|^2_{L^2(X)}}.\end{aligned} \tag{6.2}$$

Hence, (A1) is equivalent to

(A1′) there is a constant $S\in\mathbb{R}$ such that

$$\exists\, C>0 \quad \forall\, n\gg 1 \quad \forall\, u\in E_n\setminus\{0\} \quad \frac{\|u\|^2_{L^\infty(X)}}{\|u\|^2_{L^2(X)}} \leqslant Cn^S\dim(E_n).$$

We also need a *finite subset concentration* assumption:

(A2) for any $N\in\mathbb{N}^\star$, there are a constant $C>1$ and a finite set $\mathcal{X}_N\subset X$ of cardinal $\operatorname{Card}(\mathcal{X}_N)\leqslant CN^C$ such that

$$\forall\, u\in E_0+\cdots+E_N \qquad \|u\|_{L^\infty(X)}\leqslant 2\max_{x\in\mathcal{X}_N}|u(x)|.$$

The assumption (A2) roughly means that a function belonging to $E_0+\cdots+E_N$ is essentially concentrated in a finite subset of X (independent of the function). The following theorem is a multidimensional and quantitative Paley–Zygmund theorem for L^∞.

Theorem 6.1. — *We assume that* (A2) *holds true and that the sequence of random variables* $(X_n)_{n\in\mathbb{N}}$ *satisfies*

$$\exists\, p\in[2,+\infty) \qquad \sup_{n\in\mathbb{N}}\mathbf{E}[|X_n|^p]<+\infty.$$

Consider now a sequence of matrices $(b_n)_{n\in\mathbb{N}}$, *with* $b_n\in\mathcal{M}_{d_n}(E_n)$, *satisfying*

$$\exists\,\gamma>1 \qquad \sum_{n\geqslant 2}\ln^\gamma(n)\sup_{x\in X}\|b_n(x)\|^2_{HS}<+\infty,$$

where $\|\cdot\|_{HS}$ *stands for the Hilbert–Schmidt norm:*

$$\|b_n(x)\|^2_{HS}=\sum_{i=1}^{d_n}\sum_{j=1}^{d_n}|b_{n,i,j}(x)|^2.$$

Then the following statements hold true

(1) *For any integer $N \geqslant 2$, one has*

$$\mathbf{E}_\omega \left[\left\| \sum_{n=2}^{N} X_n(\omega) \sqrt{d_n} \mathbf{tr}(\mathcal{E}_n(\omega) b_n) \right\|_{L^\infty(X)}^p \right] \leqslant C(X,p,\gamma) \left(\sup_{n \geqslant 2} \mathbf{E}\left[|X_n|^p\right] \right) \left(\sum_{n \geqslant 2} \ln^\gamma(n) \sup_{x \in X} \|b_n(x)\|_{HS}^2 \right)^{p/2},$$

where all random variables are assumed to be mutually independent (we recall that the law of the random matrix $\mathcal{E}_n : \Omega \to U_{d_n}(\mathbb{C})$ is the normalized Haar measure of the unitary group $U_{d_n}(\mathbb{C})$ but the same conclusion would hold true for the orthogonal group $O_{d_n}(\mathbb{R})$).

(2) *The random series $\sum_{n\in\mathbb{N}} X_n(\omega)\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n(\omega) b_n(x))$ converges in the Banach space $L^p_\omega(\Omega, L^\infty_x(X))$ and almost surely converges in $L^\infty_x(X)$.*

At a first glance, only Point (2) seems to be of interest. But we will see that Point (1) will be used in the proof of Theorem 3.2 dealing with the cubic wave equation $\partial_t^2 w - \mathbf{\Delta} w + w^3 = 0$.

Before proving Theorem 6.1, we state its main corollary.

Corollary 6.2. — *Assume* (A1) *(with a real number S) and* (A2). *Consider a sequence $(f_n)_{n\in\mathbb{N}}$ of $L^2(X)$, with $f_n \in E_n$ for each $n \in \mathbb{N}$, and we also assume the following condition*

$$\exists\, \gamma > 1 \qquad \sum_{n \geqslant 2} \|f_n\|_{L^2(X)}^2 \, n^S \ln^\gamma(n) < +\infty.$$

We define f_n^ω as in (2.2) *and we moreover consider a sequence of independent random variables $(X_n)_{n\in\mathbb{N}}$ satisfying $\sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^2] < +\infty$. Then the random series $\sum_{n\in\mathbb{N}} X_n(\omega) f_n^\omega$ almost surely converges in $L^\infty(X)$.*

Proof. — We choose $b_n(x) = \frac{1}{\sqrt{d_n}} \langle f_n, \phi_{n,j}\rangle \phi_{n,i}(x)$. Hence, we have

$$\|b_n(x)\|_{HS}^2 = \frac{e_n(x)}{d_n} \|f_n\|_{L^2(X)}^2 \quad \text{and} \quad X_n(\omega) f_n^\omega = X_n(\omega)\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n(\omega) b_n).$$

Theorem 6.1, with $p = 2$, directly gives the conclusion. □

7. Proof of Theorem 6.1

We first recall the multidimensional Kahane–Khintchine inequality proven by Marcus and Pisier (see [34, p. 81, line (2.1) and p. 91, Corollary 2.12]).

Proposition 7.1. — *For any real numbers $p \geqslant q \geqslant 1$, there is a constant $K_{p,q} \geqslant 1$ such that, for any complex Banach space B, for any integer $N \in \mathbb{N}$ and for any sequence of matrices $b_n \in \mathcal{M}_{d_n}(B)$, one has*

$$\mathbf{E}\left[\left\|\sum_{n=0}^{N} \mathbf{tr}(\mathcal{E}_n b_n)\right\|_B^p\right]^{\frac{1}{p}} \leqslant K_{p,q}\mathbf{E}\left[\left\|\sum_{n=0}^{N} \mathbf{tr}(\mathcal{E}_n b_n)\right\|_B^q\right]^{\frac{1}{q}}. \tag{7.1}$$

Moreover, there is an absolute constant $K \geqslant 1$ such that $K_{p,q} \leqslant K_{p,1} \leqslant K\sqrt{p}$.

Thanks to the Hölder inequality, note that the converse inequality is obvious in the last statement:

$$\mathbf{E}\left[\left\|\sum_{n=0}^{N} \mathbf{tr}(\mathcal{E}_n b_n)\right\|_B^q\right]^{\frac{1}{q}} \leqslant \mathbf{E}\left[\left\|\sum_{n=0}^{N} \mathbf{tr}(\mathcal{E}_n b_n)\right\|_B^p\right]^{\frac{1}{p}}.$$

We stress that the inequalities are stated in [34] with the random series $\sum d_n \mathbf{tr}(\mathcal{E}_n b_n)$ but one can consider each integer d_n as a part of the matrix b_n (so we are reduced to (7.1)). In the specific case $B = \mathbb{C}$ (or more generally if B is a Hilbert space), one can compute the moment of order two thanks to the Hilbert–Schmidt norms.

Proposition 7.2. — *For any integer $N \in \mathbb{N}$ and for any sequence of matrices $b_n \in \mathcal{M}_{d_n}(\mathbb{C})$, the following equality holds true:*

$$\mathbf{E}\left[\left|\sum_{n=0}^{N} \sqrt{d_n}\mathbf{tr}(\mathcal{E}_n b_n)\right|^2\right] = \sum_{n=0}^{N} \|b_n\|_{HS}^2.$$

Proof. — See [23, Lemme 2.9]. □

We now write a simple “$\sqrt{\ln}$ lemma” that allows to estimate the expectation of a supremum of random variables.

Lemma 7.3. — *Let us consider two integers $\alpha \geqslant 2$ and $N \geqslant 0$. For any integers $\beta \in [1, \alpha]$ and $n \in [0, N]$ one also considers a matrix $b_{(n,\beta)} \in \mathcal{M}_{d_n}(\mathbb{C})$. Then the following estimate holds true*

$$\mathbf{E}_\omega\left[\sup_{1 \leqslant \beta \leqslant \alpha}\left|\sum_{n=0}^{N} \sqrt{d_n}\mathbf{tr}(\mathcal{E}_n(\omega) b_{(n,\beta)})\right|\right] \leqslant C\sqrt{\ln(\alpha)\sup_{1 \leqslant \beta \leqslant \alpha}\left(\sum_{n=0}^{N} \left\|b_{(n,\beta)}\right\|_{HS}^2\right)}.$$

Proof. — For any real number $p \in [1, +\infty)$, one makes use of the Hölder inequality, the multidimensional Kahane–Khintchine inequalities (7.1) and

Proposition 7.2:

$$\begin{aligned}
\mathbf{E}\left[\sup_{1\leqslant\beta\leqslant\alpha}\left|\sum_{n=0}^{N}\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n b_{(n,\beta)})\right|\right]
&\leqslant \mathbf{E}\left[\left(\sum_{\beta=1}^{\alpha}\left|\sum_{n=0}^{N}\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n b_{(n,\beta)})\right|^p\right)^{\frac{1}{p}}\right]\\
&\leqslant \mathbf{E}\left[\sum_{\beta=1}^{\alpha}\left|\sum_{n=0}^{N}\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n b_{(n,\beta)})\right|^p\right]^{\frac{1}{p}}\\
&\leqslant \left(\sum_{\beta=1}^{\alpha}\mathbf{E}\left[\left|\sum_{n=0}^{N}\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n b_{(n,\beta)})\right|^p\right]\right)^{\frac{1}{p}}\\
&\leqslant \alpha^{\frac{1}{p}}\sup_{1\leqslant\beta\leqslant\alpha}\mathbf{E}\left[\left|\sum_{n=0}^{N}\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n b_{(n,\beta)})\right|^p\right]^{\frac{1}{p}}\\
&\leqslant K\alpha^{\frac{1}{p}}\sqrt{p}\sup_{1\leqslant\beta\leqslant\alpha}\mathbf{E}\left[\left|\sum_{n=0}^{N}\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n b_{(n,\beta)})\right|^2\right]^{\frac{1}{2}}\\
&\leqslant K\alpha^{\frac{1}{p}}\sqrt{p}\sup_{1\leqslant\beta\leqslant\alpha}\sqrt{\sum_{n=0}^{N}\left\|b_{(n,\beta)}\right\|_{HS}^2}.
\end{aligned}$$

In the case $\alpha \geqslant 3$, one may choose $p = \ln(\alpha) \geqslant 1$. For the remaining case $\alpha = 2$, the choice $p = 1$ is convenient. □

We shall need a generalization of the Salem–Zygmund inequality that holds true on a quite general framework. This type of inequality is usually presented as an estimate of a probability ([27, Chapter 6], [24, Theorem 2.5] or [11, Théorème 2]). But it will be much more efficient to estimate expectations because we want to reach a sharp moment condition.

Proposition 7.4. — *Assume* (A2) *and let us consider* $N+1$ *matrices* $(b_n)_{0\leqslant n\leqslant N}$ *with* $b_n \in \mathcal{M}_{d_n}(E_n)$. *Then the following Salem–Zygmund inequality holds true:*

$$\mathbf{E}\left[\left\|\sum_{n=0}^{N}\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n b_n(x))\right\|_{L^\infty_x(X)}\right] \leqslant C\sqrt{\ln(N)}\left\|\sqrt{\sum_{n=0}^{N}\|b_n(x)\|_{HS}^2}\right\|_{L^\infty_x(X)}. \tag{7.2}$$

Proof. — This is a straightforward consequence of the *finite subset concentration* property (A2) and the previous lemma:

$$\mathbf{E}\left[\left\|\sum_{n=0}^{N}\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n b_n(x))\right\|_{L^\infty_x(X)}\right] \leqslant 2\mathbf{E}\left[\sup_{x\in\mathcal{X}_N}\left|\sum_{n=0}^{N}\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n b_n(x))\right|\right] \leqslant C\sqrt{\ln(\mathrm{Card}(\mathcal{X}_N))}\sup_{x\in\mathcal{X}_N}\sqrt{\sum_{n=0}^{N}\|b_n(x)\|_{HS}^2}. \qquad \square$$

We now have all the ingredients to prove Theorem 6.1.

Proof of (1) $\Rightarrow$ (2). — By considering Cauchy sequences, it is clear that Point (1) implies that the random series $\sum X_n(\omega)\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n(\omega)b_n)$ converges in the Banach space $L^p(\Omega, L^\infty(X))$ to a random variable $U : \Omega \to L^\infty(X)$. The Markov inequality thus implies that the random series $\sum X_n(\omega)\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n(\omega)b_n)$ converges in probability to U. The following classical result of Paul Lévy ([33, Théorème II.3, p. 119] or [31, Theorem 2.4]) ensures that the random series is almost surely convergent in $L^\infty(X)$.

Theorem 7.5 (Paul Lévy). — *Consider a sequence (U_n) of independent random variables that take value in a Banach space B. Then the following statements are equivalent:*

(1) *there is a random variable $U : \Omega \to B$ such that the series $\sum U_n$ converges in probability to U:*

$$\forall\, \varepsilon > 0 \qquad \lim_{n\to+\infty}\mathbf{P}\left(\|U_1+\cdots+U_n-U\|_B > \varepsilon\right) = 0.$$

(2) *for almost every $\omega \in \Omega$, the series $\sum U_n(\omega)$ converges in B.*

Proof of Point (1). — We now make use of a well-known trick in the theory of random series. One can change the probability space Ω by Ω^2 without any effect on our expectations. Using that all the involved random variables and random matrices are mutually independent (see Appendix F), we may prove the following formula

$$\mathbf{E}_\omega\left[\left\|\sum_{n=2}^{N}X_n(\omega)\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n(\omega)b_n)\right\|_{L^\infty(X)}^p\right] = \mathbf{E}_{\omega'}\mathbf{E}_\omega\left[\left\|\sum_{n=2}^{N}X_n(\omega')\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n(\omega)b_n)\right\|_{L^\infty(X)}^p\right]. \tag{7.3}$$

By invoking the multidimensional Kahane–Khintchine inequalities (7.1) in the Banach space $L^\infty(X)$, we get

$$\begin{aligned}(7.4)\quad \mathbf{E}_\omega\left[\left\|\sum_{n=2}^N X_n(\omega)\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n(\omega)b_n(x))\right\|^p_{L^\infty_x(X)}\right] \\ \leqslant C\mathbf{E}_{\omega'}\left[\mathbf{E}_\omega\left[\left\|\sum_{n=2}^N X_n(\omega')\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n(\omega)b_n(x))\right\|_{L^\infty_x(X)}\right]^p\right],\end{aligned}$$

where C may depend on p. We now freeze the variable ω' and work on the last expectation $\mathbf{E}_\omega$. Choose N of the form 2^{2^k} with $k\in\mathbb{N}$ and apply the Salem–Zygmund inequality (7.2):

$$\begin{aligned}\mathbf{E}_\omega&\left[\left\|\sum_{2^{2^k}\leqslant n<2^{2^{k+1}}} X_n(\omega')\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n(\omega)b_n(x))\right\|_{L^\infty_x(X)}\right] \\ &\leqslant C\sqrt{\ln(2^{2^{k+1}})}\left\|\sqrt{\sum_{2^{2^k}\leqslant n<2^{2^{k+1}}}\|X_n(\omega')b_n(x)\|^2_{HS}}\right\|_{L^\infty_x(X)} \\ &\leqslant C2^{\frac{k}{2}}\sqrt{\sum_{2^{2^k}\leqslant n<2^{2^{k+1}}}|X_n(\omega')|^2\sup_{x\in X}\|b_n(x)\|^2_{HS}} \\ &\leqslant \frac{C}{2^{(\gamma-1)k/2}}\sqrt{\sum_{2^{2^k}\leqslant n<2^{2^{k+1}}}|X_n(\omega')|^2\ln^\gamma(n)\sup_{x\in X}\|b_n(x)\|^2_{HS}},\end{aligned}$$

where the constant C is independent of ω'. Using the assumption $\gamma>1$ and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\sum_{k\in\mathbb{N}}\mathbf{E}_\omega&\left[\left\|\sum_{2^{2^k}\leqslant n<2^{2^{k+1}}} X_n(\omega')\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n(\omega)b_n(x))\right\|_{L^\infty_x(X)}\right] \\ &\leqslant C\sqrt{\sum_{n\geqslant 2}|X_n(\omega')|^2\ln^\gamma(n)\sup_{x\in X}\|b_n(x)\|^2_{HS}}.\end{aligned}$$

Let us now explain why the following inequality holds true for any integer $N\geqslant 2$ and any $\omega'\in\Omega$:

$$\begin{aligned}(7.5)\quad \mathbf{E}_\omega\left[\left\|\sum_{n=2}^N X_n(\omega')\sqrt{d_n}\mathbf{tr}(\mathcal{E}_n(\omega)b_n)\right\|_{L^\infty(X)}\right] \\ \leqslant C\sqrt{\sum_{n\geqslant 2}|X_n(\omega')|^2\ln^\gamma(n)\sup_{x\in X}\|b_n(x)\|^2_{HS}}.\end{aligned}$$

From the previous analysis, if N is of the form $2^{2^k}-1$, then (7.5) is just a consequence of the triangular inequality. To see that (7.5) still holds true for any $N \geqslant 2$, we just have to check the monotonicity in N:

$$\mathbf{E}_\omega\left[\left\|\sum_{n=2}^{N} X_n(\omega')\sqrt{d_n}\,\mathbf{tr}\left(\mathcal{E}_n(\omega)b_n\right)\right\|_{L^\infty(X)}\right] \leqslant \mathbf{E}_\omega\left[\left\|\sum_{n=2}^{N+1} X_n(\omega')\sqrt{d_n}\,\mathbf{tr}\left(\mathcal{E}_n(\omega)b_n\right)\right\|_{L^\infty(X)}\right].$$

The last inequality is a consequence of the independence of the random matrices $\mathcal{E}_0,\dots,\mathcal{E}_{N+1}$ and of the Jensen inequality (by integration with respect to the last d_{N+1}^2 variables of the random matrices $\mathcal{E}_{N+1}$ that are all centered: $\mathbf{E}[\mathcal{E}_{N+1,i,j}]=0$). Hence, (7.5) is proved.

We are close to conclude. We recall that the constant C in (7.5) is independent of the variable ω'. Combining (7.4) and (7.5) gives us

$$\mathbf{E}_\omega\left[\left\|\sum_{n=2}^{N} X_n(\omega)\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n(\omega)b_n)\right\|_{L^\infty(X)}^p\right] \leqslant C\mathbf{E}_{\omega'}\left[\left(\sum_{n=2}^{+\infty}|X_n(\omega')|^2\ln^\gamma(n)\sup_{x\in X}\|b_n(x)\|_{HS}^2\right)^{\frac{p}{2}}\right]. \tag{7.6}$$

As we assumed the inequality $p\geqslant 2$ in the assumption of Theorem 6.1, one may apply the triangular inequality in $L^{p/2}_{\omega'}(\Omega)$:

$$\mathbf{E}_\omega\left[\left\|\sum_{n=2}^{N} X_n(\omega)\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n(\omega)b_n)\right\|_{L^\infty(X)}^p\right] \leqslant C\left(\sum_{n=2}^{+\infty}\mathbf{E}_{\omega'}[|X_n(\omega')|^p]^{\frac{2}{p}}\ln^\gamma(n)\sup_{x\in X}\|b_n(x)\|_{HS}^2\right)^{\frac{p}{2}}. \tag{7.7}$$

The last inequality clearly shows Point (1) of Theorem 6.1. □

8. Proof of Theorem 2.1

It suffices to prove (A1), (A2) and to apply Corollary 6.2. Here E_n is defined by (2.1) and e_n is its spectral function (see (6.1)). We shall need the following result that proves (A1) with $S=0$.

Lemma 8.1. — *There is a constant $\kappa(X) > 0$ such that for any $\kappa \geqslant \kappa(X)$ and $(n,x) \in \mathbb{N} \times X$, one has*

$$\begin{aligned}\frac{1}{C(X,\kappa)}(1+n)^{d-1} \leqslant &\quad e_n(x) \quad \leqslant C(X,\kappa)(1+n)^{d-1},\\ \frac{1}{C(X,\kappa)}(1+n)^{d-1} \leqslant &\dim(E_n) \leqslant C(X,\kappa)(1+n)^{d-1}.\end{aligned}$$

Proof. — We use the same idea as that of [11, p. 923] with $b_h = \kappa(n+1)h$, $a_h = \kappa n h$ and $h \simeq \frac{1}{n}$. Remark that the $L^2(X)$-normalized function $\frac{1}{\sqrt{\operatorname{Vol}(X)}}$ belongs to E_0. Consequently, one has $e_0(x) \geqslant \frac{1}{\operatorname{Vol}(X)}$ and $\dim(E_0) \geqslant 1$. The continuity of e_0 and the compactness of X make obvious the case $n = 0$.

We handle the case $n \geqslant 1$ with an accurate estimate of the spectral function (see [21]):

$$e_0(x) + \cdots + e_n(x) = (2\pi)^{-d} \operatorname{Vol}(\mathbb{B}_d(0,1))(\kappa n + \kappa)^d + (\kappa n + \kappa)^{d-1}\mathcal{O}(1)$$

where the remainder $\mathcal{O}(1)$ is uniform with respect to x and $\kappa > 0$. By integraton on X, the Weyl formula reads

$$\sum_{k=0}^{n} \dim(E_k) = (2\pi)^{-d} \operatorname{Vol}(\mathbb{B}_d(0,1)) \operatorname{Vol}(X)(\kappa n + \kappa)^d + (\kappa n + \kappa)^{d-1}\mathcal{O}(1).$$

Setting $c(d) = (2\pi)^{-d} \operatorname{Vol}(\mathbb{B}_d(0,1))$, we thus get

$$\begin{aligned}e_n(x) &= c(d)\kappa^d \left[(n+1)^d - n^d\right] + \kappa^{d-1}n^{d-1}\mathcal{O}(1),\\ \dim(E_n) &= c(d) \operatorname{Vol}(X)\kappa^d \left[(n+1)^d - n^d\right] + \kappa^{d-1}n^{d-1}\mathcal{O}(1).\end{aligned}$$

Choosing $\kappa = \frac{\kappa^d}{\kappa^{d-1}}$ large enough, we get the conclusion. □

It remains to check (A2). Let us consider $u \in E_0 + \cdots + E_N$ and write

$$u := \sum_{n=0}^{N}\sum_{i=1}^{d_n} \langle u, \phi_{n,i}\rangle \phi_{n,i}.$$

Hence, the following inequality holds true

$$\begin{aligned}|\langle u, \phi_{n,i}\rangle| &\leqslant \|u\|_{L^\infty(X)} \|\phi_{n,i}\|_{L^1(X)}\\ &\leqslant \sqrt{\operatorname{Vol}(X)}\, \|u\|_{L^\infty(X)} \|\phi_{n,i}\|_{L^2(X)}\\ &\leqslant \sqrt{\operatorname{Vol}(X)}\, \|u\|_{L^\infty(X)},\end{aligned}$$

which implies for every $(x,y) \in X^2$:

$$|u(x) - u(y)| \leqslant \sqrt{\operatorname{Vol}(X)}\, \|u\|_{L^\infty(X)} \sum_{n=0}^{N}\sum_{i=1}^{d_n} |\phi_{n,i}(x) - \phi_{n,i}(y)|.$$

Remember now the Sobolev embedding $H^\tau(X) \subset W^{1,\infty}(X)$ that holds true for any $\tau > 1 + \frac{d}{2}$. We consequently control the Lipschitz constant of each $\phi_{n,i}$ by a polynomial bound in N. The asymptotics $d_n \simeq n^{d-1}$ (see Lemma 8.1) gives a weak Bernstein inequality

$$(8.1) \quad \exists\, c_1(X) > 0 \quad \exists\, c_2(X) > 0 \quad \forall\, (x,y) \in X^2$$
$$|u(x) - u(y)| \leqslant c_1(X) N^{c_2(X)} \|u\|_{L^\infty(X)} \operatorname{dist}(x,y).$$

The compactness of X ensures that there is a maximal finite subset $\mathcal{X}_N := \{x_1, \dots, x_\alpha\} \subset X$ for the following property: the open balls centered in $x_1, \dots, x_\alpha$ and of radius $r = \frac{1}{4c_1(X)N^{c_2(X)}}$ are disjoint. Each of those balls has a volume greater or equal to $Cr^{\dim(X)}$. Remarking that the volume of the disjoint union of those balls is bounded by $\mathrm{Vol}(X)$, we infer

$$\alpha \leqslant C(X) N^{c_2(X)\dim(X)}.$$

The weak Bernstein inequality (8.1) and the maximal property of $\{x_1, \dots, x_\alpha\}$ ensure that, for any $x \in X$, one may find x_β such that $d(x, x_\beta) \leqslant \frac{1}{2c_1(X)N^{c_2(X)}}$ holds and hence

$$|u(x) - u(x_\beta)| \leqslant c_1(X) N^{c_2(X)} \|u\|_{L^\infty(X)} \operatorname{dist}(x, x_\beta) \leqslant \frac{1}{2} \|u\|_{L^\infty(X)}.$$

By choosing $x \in X$ satisfying $|u(x)| = \|u\|_{L^\infty(X)}$, we get $\|u\|_{L^\infty(X)} \leqslant \frac{1}{2}\|u\|_{L^\infty(X)} + |u(x_\beta)|$. The assertion (A2) is then proved.

9. Littlewood–Paley theory for $\mathrm{BMO}(X)$

Once and for all, we consider a dyadic partition of the unity $\widetilde{\theta} \in \mathcal{C}_c^\infty(\mathbb{R}^+, \mathbb{R})$ and $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^+ \backslash \{0\}, \mathbb{R})$, in the sense of Littlewood–Paley:

$$(9.1) \qquad \forall\, \lambda \geqslant 0 \quad \widetilde{\theta}(\lambda) + \sum_{j \geqslant 1} \theta(2^{-2j}\lambda) = 1.$$

In view to make a functional calculus with second order elliptic operators, we prefer writing $\theta(2^{-2j}\lambda)$ in (9.1) instead of the usual scaling $\theta(2^{-j}\lambda)$. The goal of this part is to prove the following result that roughly says that $\mathrm{BMO}(X)$ is between the two Besov spaces $B^0_{\infty,\infty}(X)$ and $B^0_{\infty,2}(X)$.

Theorem 9.1. — *Consider a smooth function* $\sigma \in \mathcal{C}_c^\infty(\mathbb{R}^+\backslash\{0\},\mathbb{R})$. *Then for any distribution f of the Riemannian boundaryless compact manifold X, the following inequalities hold true*

$$\sup_{0<h\leqslant 1} \left\|\sigma(-h^2\boldsymbol{\Delta})f\right\|_{L^\infty(X)} \lesssim \|f\|_{\mathrm{BMO}(X)}\,, \tag{9.2}$$

$$\|f\|^2_{\mathrm{BMO}(X)} \lesssim \left\|\widetilde{\theta}(-\boldsymbol{\Delta})f\right\|^2_{L^\infty(X)} + \sum_{j\geqslant 1} \left\|\theta(-2^{-2j}\boldsymbol{\Delta})f\right\|^2_{L^\infty(X)}\,, \tag{9.3}$$

where $\boldsymbol{\Delta}$ *stands for the non positive Laplace–Beltrami operator of* X.

An essential ingredient of the proof of Theorem 9.1 is the Littlewood–Paley theory of the space $\mathrm{bmo}(\mathbb{R}^d)$ (see [46, p. 93 (Theorem 2) and p. 50]).

Proposition 9.2. — *Let* $\mathrm{bmo}(\mathbb{R}^n)$ *be the space of the locally integrable functions* $f:\mathbb{R}^d\to\mathbb{C}$ *for which the following norm is finite:*

$$\|f\|_{\mathrm{bmo}} := \sup_{\substack{Q\subset\mathbb{R}^d\\ \mathrm{Vol}(Q)\leqslant 1}} \frac{1}{\mathrm{Vol}(Q)}\int_Q |f-f_Q|\mathrm{d}x + \sup_{\substack{Q\subset\mathbb{R}^d\\ \mathrm{Vol}(Q)\geqslant 1}} \frac{1}{\mathrm{Vol}(Q)}\int_Q |f(x)|\mathrm{d}x, \tag{9.4}$$

where we denote by Q a cube of $\mathbb{R}^d$ and by f_Q the average of f on Q. Then $\mathrm{bmo}(\mathbb{R}^n)$ *is exactly the subspace of the tempered distributions* $f\in\mathcal{S}'(\mathbb{R}^d)$ *that can be written*

$$f = \widetilde{\theta}(-\Delta)f_0 + \sum_{j\geqslant 1}\theta(-2^{-2j}\Delta)f_j, \qquad \forall\, j\in\mathbb{N}\quad f_j\in L^\infty(\mathbb{R}^d), \tag{9.5}$$

with $\sup_{x\in\mathbb{R}^d}\sqrt{\sum_{j\in\mathbb{N}}|f_j(x)|^2}<+\infty$. *Finally, we have*

$$\|f\|_{\mathrm{bmo}} \simeq \inf_{(f_j)}\sup_{x\in\mathbb{R}^d}\sqrt{\sum_{j\in\mathbb{N}}|f_j(x)|^2}$$

where (f_j) runs over all admissible representations (9.5).

We now need the following flat but pseudo-differential version of Theorem 9.1.

Theorem 9.3. — *Consider two real numbers $\beta>\alpha>0$ and a smooth symbol $\sigma:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}$ with support included in the ring $\alpha\leqslant|\xi|\leqslant\beta$, then the following inequality holds true for any distribution f on $\mathbb{R}^d$:*

$$\sup_{0<h\leqslant 1}\|\sigma(x,hD)f\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{\mathrm{bmo}(\mathbb{R}^d)}\,,$$

where we set

$$\sigma(x,hD)f(x)=\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\sigma(x,h\xi)\widehat{f}(\xi)\frac{\mathrm{d}\xi}{(2\pi)^d}.$$

Moreover we have

$$\|f\|^2_{\mathrm{bmo}(\mathbb{R}^d)} \lesssim \left\|\widetilde{\theta}(-\Delta)f\right\|^2_{L^\infty(\mathbb{R}^d)} + \sum_{j\geqslant 1}\left\|\theta(-2^{-2j}\Delta)f\right\|^2_{L^\infty(\mathbb{R}^d)}.$$

Proof. — Let $W^{2s,1}_\xi(\mathbb{R}^d)$ be the usual Sobolev space for some integer $s>\frac{d}{2}$. It is easy to check an inequality of the form

$$\sup_{0<h\leqslant 1}\|\sigma(x,hD)f\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\sigma\|_{L^\infty_x(\mathbb{R}^d,W^{2s,1}_\xi(\mathbb{R}^d))}\|f\|_{L^\infty(\mathbb{R}^d)}. \tag{9.6}$$

More precisely, it is sufficient to get an adequate bound of the kernel $K_h(x,y)$ of $\sigma(x,hD)$:

$$K_h(x,y)=\int_{\mathbb{R}^d} e^{i\langle x-y,\xi\rangle}\sigma(x,h\xi)\frac{\mathrm{d}\xi}{(2\pi)^d}$$

$$\int_{\mathbb{R}^d} K_h(x,y)f(y)\mathrm{d}y=\sigma(x,hD)f(x).$$

A change of variable and an integration by parts give

$$\int_{\mathbb{R}^d}|K_h(x,y)|\mathrm{d}y=\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\frac{e^{i\langle\frac{x-y}{h},\xi\rangle}}{(1+h^{-2}|x-y|^2)^s}(1-\Delta_\xi)^s\sigma(x,\xi)\frac{\mathrm{d}\xi}{(2\pi h)^d}\right|\mathrm{d}y,$$

which is less than

$$\begin{aligned}\int_{\mathbb{R}^d}|(1-\Delta_\xi)^s\sigma(x,\xi)|\left(\int_{\mathbb{R}^d}\frac{1}{(1+h^{-2}|x-y|^2)^s}\frac{\mathrm{d}y}{(2\pi h)^d}\right)\mathrm{d}\xi&\\ \leqslant C(d,s)\int_{\mathbb{R}^d}|(1-\Delta_\xi)^s\sigma(x,\xi)|\mathrm{d}\xi&\\ \leqslant C(d,s)\|\sigma\|_{L^\infty_x(\mathbb{R}^d,W^{2s,1}_\xi(\mathbb{R}^d))}&\end{aligned}$$

which clearly gives (9.6).

To prove the first inequality of the statement, we assume that f belongs to $\mathrm{bmo}(\mathbb{R}^d)$ and we consider a decomposition of f as in (9.5). We then write for any integer $j\geqslant 1$

$$\sigma(x,hD)\theta(-2^{-2j}\Delta)f_j(x)=\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\sigma(x,h\xi)\theta(2^{-2j}|\xi|^2)\widehat{f_j}(\xi)\frac{\mathrm{d}\xi}{(2\pi)^d}. \tag{9.7}$$

For convenience, we assume that the support of θ is included in $[a^2,b^2]$ with $b>a>0$. Let $\Xi(h)$ be the set of $j\in\mathbb{N}^\star$ satisfying $\frac{\alpha}{b}\leqslant h2^j\leqslant\frac{\beta}{a}$. Passing to the logarithm shows that the cardinal of $\Xi(h)$ is bounded with respect to $h\in(0,1]$ (this is the crucial property that comes from the assumption on the support of σ). For any $j\notin\Xi(h)$ the two intervals $[\frac{\alpha}{h},\frac{\beta}{h}]$ and $[a2^j,b2^j]$ are disjoint and the symbol $(x,\xi)\mapsto \sigma(x,h\xi)\theta(2^{-2j}|\xi|^2)$ identically vanishes. Coming back to (9.5) and (9.7), we get

$$\sigma(x,hD)f=\sigma(x,hD)\widetilde{\theta}(-\Delta)f_0+\sum_{j\in\Xi(h)}\sigma(x,hD)\theta(-2^{-2j}\Delta)f_j. \tag{9.8}$$

Note that the inequality (9.6) shows that the operators $\widetilde{\theta}(-\Delta)$ and $\theta(-h^2\Delta)$ are also uniformly bounded from $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ with respect to $h\in(0,1]$. Hence we get

$$\begin{aligned}\|\sigma(x,hD)f\|_{L^\infty(\mathbb{R}^d)}&\lesssim\|f_0\|_{L^\infty(\mathbb{R}^d)}+\sum_{j\in\Xi(h)}\|f_j\|_{L^\infty(\mathbb{R}^d)}\\&\lesssim(1+\mathrm{Card}(\Xi(h)))\sup_{j\in\mathbb{N}}\|f_j\|_{L^\infty(\mathbb{R}^d)}\\&\lesssim\left\|\sqrt{\sum_{j\in\mathbb{N}}|f_j(x)|^2}\right\|_{L^\infty(\mathbb{R}^d)}.\end{aligned}$$

Since this last estimate is independent of the sequence (f_j), Proposition 9.2 finally proves the inequality

$$\sup_{0<h\leqslant1}\|\sigma(x,hD)f\|_{L^\infty(\mathbb{R}^d)}\lesssim\|f\|_{\mathrm{bmo}(\mathbb{R}^d)}.$$

It remains to prove the second inequality of the statement

$$\|f\|^2_{\mathrm{bmo}(\mathbb{R}^d)}\lesssim\left\|\widetilde{\theta}(-\Delta)f\right\|^2_{L^\infty(\mathbb{R}^d)}+\sum_{j\geqslant1}\left\|\theta(-2^{-2j}\Delta)f\right\|^2_{L^\infty(\mathbb{R}^d)}.$$

Let $\widetilde{\Theta}\in\mathcal{C}^\infty_c(\mathbb{R}^+,\mathbb{R})$ (respectively $\Theta\in\mathcal{C}^\infty_c(\mathbb{R}^+\backslash\{0\},\mathbb{R})$) be a function that identically coincides with 1 on the support of $\widetilde{\theta}$ (respectively θ). Hence, we have $\widetilde{\theta}\widetilde{\Theta}=\widetilde{\theta}$ and $\theta\Theta=\theta$. So the Littlewood–Paley decomposition of f allows to set a natural sequence of (f_j) in (9.5):

$$\begin{aligned}f&=\widetilde{\theta}(-\Delta)f+\sum_{j\geqslant1}\theta(-2^{-2j}\Delta)f\\&=\widetilde{\theta}(-\Delta)\underbrace{\widetilde{\Theta}(-\Delta)(f)}_{=f_0}+\sum_{j\geqslant1}\theta(-2^{-2j}\Delta)\underbrace{\Theta(-2^{-2j}\Delta)f}_{=f_j}.\end{aligned}\tag{9.9}$$

Thanks to Proposition 9.2, we obtain

$$\|f\|^2_{\mathrm{bmo}(\mathbb{R}^d)}\lesssim\left\|\widetilde{\Theta}(-\Delta)f\right\|^2_{L^\infty(\mathbb{R}^d)}+\sum_{j\geqslant1}\left\|\Theta(-2^{-2j}\Delta)f\right\|^2_{L^\infty(\mathbb{R}^d)}.$$

By a standard argument (indeed almost the same as that of (9.8)), if one plugs (9.9) in each f of the right-hand side and if we invoke that the operators $\widetilde{\Theta}(-\Delta)$ and $\Theta(-h^2\Delta)$ are uniformly bounded from $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$, we easily get the conclusion. □

Similarly to (9.4), we recall the definition of the BMO semi-norm on $\mathbb{R}^d$:

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^d)} := \sup_{Q\subset\mathbb{R}^d} \frac{1}{\mathrm{Vol}(Q)} \int_Q |f - f_Q|\mathrm{d}x. \tag{9.10}$$

And we have the simple inequalities

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^d)} \leqslant 2\,\|f\|_{\mathrm{bmo}(\mathbb{R}^d)} \leqslant 2\,\|f\|_{\mathrm{BMO}(\mathbb{R}^d)} + 2\,\|f\|_{L^1(\mathbb{R}^d)}\,. \tag{9.11}$$

As we will work on compact manifolds, we need a few tools to transfer local estimates. The first one is the following result proved by Brezis and Niremberg.

Proposition 9.4. — *For any $p \in [1,+\infty)$ and any $\chi \in \mathcal{C}^1(X)$, one may find $C > 0$ such that for any $g \in \mathrm{BMO}(X)$ the following inequality holds true*

$$\|g\|_{L^p(X)} + \|\chi g\|_{\mathrm{BMO}(X)} \leqslant C\left(\|g\|_{\mathrm{BMO}(X)} + \left|\int_X g(x)\mathrm{d}x\right|\right). \tag{9.12}$$

Proof. — This inequality may be deduced from Lemma A.1, Lemma B.3 and Lemma B.8 of [9]. □

With a small modification of Lemma A.10 of [9], we also have the following intuitive result.

Proposition 9.5. — *Let U be a bounded open subset of $\mathbb{R}^d$ and V be an open subset of the boundaryless compact manifold X. We assume that there is a diffeomorphism $\varrho : U \to V$ that extends as a diffeomorphism from a neighborhood of $\overline{U}$ to a neighborhood of $\overline{V}$. Then there is a constant $C \geqslant 1$ such that, for any function $g \in L^1(X)$ with support in V, the following holds true*

$$\frac{1}{C}\,\|g\|_{\mathrm{BMO}(X)} \leqslant \|g\circ\varrho\|_{\mathrm{BMO}(\mathbb{R}^d)} \leqslant C\,\|g\|_{\mathrm{BMO}(X)}\,. \tag{9.13}$$

where $g\circ\varrho$ is extended by 0 outside U.

Proof. — We shall explain the strategy for the second bound in (9.13) but the first one is similar. All the ideas are in [9] but we have to carefully increase U and choose small balls. In view to bound $\|g\circ\varrho\|_{\mathrm{BMO}(\mathbb{R}^d)}$, we recall that we obtain an equivalent definition of $\|g\circ\varrho\|_{\mathrm{BMO}(\mathbb{R}^d)}$ by replacing cubes in (9.10) with Euclidean balls. So we have to prove the following inequality (uniformly with respect to $x \in \mathbb{R}^d$ and $r > 0$):

$$\int_{B(x,r)} |g\circ\varrho(y) - (g\circ\varrho)_{B(x,r)}| \frac{\mathrm{d}y}{Vol_{\mathbb{R}^d}(B(x,r))} \lesssim \|g\|_{\mathrm{BMO}(X)}\,, \tag{9.14}$$

where we recall that $(g \circ \varrho)_{B(x,r)}$ is the average of $g \circ \rho$ on the ball $B(x, r)$ (see (9.4)). Let us choose $R > 0$ small enough such that ϱ extends as a diffeomorphism from the following bounded open subset

$$U_R := \{x \in \mathbb{R}^d, \ \operatorname{dist}(x, U) < R\}$$

to an open subset of X. We notice that there is a constant $K > 0$ such that the following holds true

$$\forall\, x \in U_{R/2} \quad \forall\, r \in (0, R/4) \quad \varrho(B(x, r)) \subset B(\varrho(x), Kr) \subset X.$$

Since the definition (2.4) of the BMO semi-norm on X involves the injectivity radius $r_0(X)$, it will be convenient to assume that R is also small enough so that $\frac{KR}{4} < r_0(X)$ holds true.

Case 1: $r < \frac{R}{4}$ and $x \in U_{R/2}$. — We then use the second argument given in [9, p. 243]. More precisely, by writing $(g \circ \varrho)_{B(x,r)}$ as an integral, we see that the left-hand side of (9.14) is bounded by

$$\int_{B(x,r)} \int_{B(x,r)} |g \circ \varrho(y) - g \circ \varrho(y')| \frac{\mathrm{d}y \mathrm{d}y'}{\operatorname{Vol}_{\mathbb{R}^d}(B(x, r))^2}.$$

Although we have agreed to extend $g \circ \varrho$ by 0 outside U in our statement, it is clear the formula $\varrho(U) = V$ shows that the equalities $g \circ \varrho(x) = 0 = g(\varrho(x))$ still hold true for any $x \in U_R \backslash U$. So we can make a change of variables to change the last double integral as

$$\int_{\varrho(B(x,r))} \int_{\varrho(B(x,r))} |g(z) - g(z')| \frac{J(z)J(z')\mathrm{d}z\mathrm{d}z'}{\operatorname{Vol}_{\mathbb{R}^d}(B(x, r))^2}, \tag{9.15}$$

where J is a Jacobian function. Since we have $B(x, r) \subset U_{\frac{3R}{4}}$ and $U_{3R/4}$ is a relatively compact subset of U_R on which ϱ is a diffeomorphism, the function J appears to be bounded on $\varrho(U_{3R/4})$. We finally notice the obvious bounds

$$\operatorname{Vol}_{\mathbb{R}^d}(B(x, r)) \gtrsim r^d \gtrsim \operatorname{Vol}_X(B(\varrho(x), Kr)).$$

The combination of the last facts allows us to control (9.15) by

$$\int_{B(\varrho(x),Kr)} \int_{B(\varrho(x),Kr)} |g(z) - g(z')| \frac{\mathrm{d}z\mathrm{d}z'}{\operatorname{Vol}_X(B(\varrho(x), Kr))^2}.$$

Since Kr is less that the injectivity radius $r_0(X)$, it turns out that the last term is bounded by $\|g\|_{\mathrm{BMO}(X)}$ (see [9, p. 202]).

Case 2: $r < \frac{R}{4}$ and $x \notin U_{R/2}$. — The function $g \circ \varrho$ has support in U so vanishes over the ball $B(x, r)$. Consequently, the left-hand side of (9.14) vanishes.

Case 3: $r \geqslant \frac{R}{4}$. — We merely bound the left-hand side of (9.14) by

$$\frac{2}{\mathrm{Vol}(B(x,r))}\int_{B(x,r)} |g\circ\varrho(y)|\mathrm{d}y \leqslant \frac{C}{\mathrm{Vol}(B(0,R/4))}\int_X |g(z)|\mathrm{d}z$$

which is controlled by $\|g\|_{\mathrm{BMO}(X)} + |\int_X g(z)\mathrm{d}z|$ thanks to (9.12). Remembering that BMO semi-norms vanish on constant functions, we could have added from the beginning the assumption $\int_X g(z)\mathrm{d}z = 0$ so that we get the wanted conclusion. ☐

We now have all the tools to prove Theorem 9.1. Let us adapt the technique of the semi-classical functional calculus developed in [10, part 2.1]. Due to the compactness of the manifold X, we may assume that X admits coordinate patches similar of those of Proposition 9.5. Let $\varrho : U \subset \mathbb{R}^d \to V \subset X$ be one of them and consider $\chi_1, \chi_2 \in \mathcal{C}_0^\infty(V)$ such that $\chi_2 = 1$ near the support of χ_1. From [10, Proposition 2.1], if one fixes a symbol $\sigma \in \mathcal{C}_c^\infty(\mathbb{R}^+\backslash\{0\}, \mathbb{R})$ and an integer $N \in \mathbb{N}^\star$ large enough such that $H^N(\mathbb{R}^d)$ embeds in $L^\infty(\mathbb{R}^d)$ then one can find several smooth symbols $\sigma_0, \dots, \sigma_N$ with compact support in $\mathbb{R}^d \times \mathbb{R}^d$ such that the following holds true for any $h \in (0,1]$ and any $f \in L^2(X)$:

$$\varrho^\star(\chi_1\sigma(-h^2\boldsymbol{\Delta})f) = \sum_{j=0}^{N} h^j \sigma_j(x, hD)\varrho^\star(\chi_2 f) + R_{N,h}(f), \tag{9.16}$$

where the remainder satisfies $\|R_{N,h}(f)\|_{H^N(\mathbb{R}^d)} \lesssim h\,\|f\|_{L^2(X)}$ uniformly in $h \in (0,1]$. Our assumption on N even gives

$$\|R_{N,h}(f)\|_{L^\infty(\mathbb{R}^d)} \lesssim h\,\|f\|_{L^2(X)}\,. \tag{9.17}$$

Furthermore, we have $\sigma_0(x,\xi) = \chi_1(\varrho(x))\sigma(\wp(x,\xi))$ where $\wp$ is the principal symbol of $-\boldsymbol{\Delta}$ in the coordinate patch $\rho : U \to X$ (that is the last statement of [10, Proposition 2.1]). As in [10, p. 578], the symbol $\wp(x,\xi)$ behaves like $|\xi|^2$:

$$\exists\, C \geqslant 1 \quad \forall\, x \in U \quad \forall\, \xi \in \mathbb{R}^d \quad \frac{1}{C}|\xi|^2 \leqslant \wp(x,\xi) \leqslant C|\xi|^2.$$

We then note the important following fact that comes from the assumption $\sigma \in \mathcal{C}_c^\infty(\mathbb{R}^+\backslash\{0\}, \mathbb{R})$: the symbol $\sigma_0(x,\xi)$ is supported in a ring with respect to ξ. Moreover, the same property is shared by $\sigma_1, \dots, \sigma_N$ thanks to more sophisticated expressions involving the Cauchy formula and the Helffer–Sjöstrand formula (see [10, p. 577]).

Proof of (9.2). — Noting the equality $\sigma(-h^2\boldsymbol{\Delta})1 = 0$, we see that (9.2) is consistent with the fact that the semi-norm $\|\cdot\|_{\mathrm{BMO}(X)}$ vanishes for constant functions. So we can assume that $\int_X f(x)\mathrm{d}x$ equals 0. It is now sufficient to prove the inequality

$$\left\|\varrho^\star(\chi_1\sigma(-h^2\boldsymbol{\Delta})f)\right\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{\mathrm{BMO}(X)}\,. \tag{9.18}$$

In (9.16), the contribution of $R_{N,h}(f)$ is directly controlled with (9.12) and (9.17). We now turn to the pseudo-differential operators $\sigma_j(x,hD)$. Since N is fixed, one can use Theorem 9.3 and the three inequalities (9.11), (9.12) and (9.13) to get

$$\begin{aligned}\left\|\sum_{j=0}^N h^j\sigma_j(x,hD)\varrho^\star(\chi_2 f)\right\|_{L^\infty(\mathbb{R}^d)}&\\ &\lesssim \|\varrho^\star(\chi_2 f)\|_{\mathrm{bmo}(\mathbb{R}^d)} = \|(\chi_2 f)\circ\varrho\|_{\mathrm{bmo}(\mathbb{R}^d)}\\ &\lesssim \|(\chi_2 f)\circ\varrho\|_{\mathrm{BMO}(\mathbb{R}^d)} + \|(\chi_2 f)\circ\varrho\|_{L^1(\mathbb{R}^d)}\\ &\lesssim \|\chi_2 f\|_{\mathrm{BMO}(X)} + \|\chi_2 f\|_{L^1(X)}\\ &\lesssim \|f\|_{\mathrm{BMO}(X)}\,.\end{aligned}$$

The inequality (9.18) is proved. □

Proof of (9.3). — All the ideas are in [10, Corollary 2.3] but we must play here with $\mathrm{BMO}(X)$, $L^\infty(X)$, $\mathrm{BMO}(\mathbb{R}^d)$ and $\mathrm{bmo}(\mathbb{R}^d)$ instead of the mere two spaces $L^p(X)$ and $L^p(\mathbb{R}^d)$. Writing f as a sum of functions $\chi_1 f$ (with a partition of unity localized in charts of X), we see that it is sufficient to prove the inequality

$$\|\chi_1 f\|^2_{\mathrm{BMO}(X)} \lesssim \left\|\widetilde{\theta}(-\boldsymbol{\Delta})f\right\|^2_{L^\infty(X)} + \sum_{k\geqslant 1}\left\|\theta(-2^{-2k}\boldsymbol{\Delta})f\right\|^2_{L^\infty(X)}\,. \tag{9.19}$$

Following (9.1), we write a Littlewood–Paley decomposition of the distribution f of X:

$$\varrho^\star(\chi_1 f) = \varrho^\star(\chi_1\widetilde{\theta}(-\boldsymbol{\Delta})f) + \sum_{k\geqslant 1}\varrho^\star(\chi_1\theta(-2^{-2k}\boldsymbol{\Delta})f).$$

Thanks to the semi-classical functional calculus of $\theta(-h^2\boldsymbol{\Delta})$ (as in (9.16) but replacing σ with θ), we infer

$$\begin{aligned}\varrho^\star(\chi_1 f) &= \varrho^\star(\chi_1\widetilde{\theta}(-\boldsymbol{\Delta})f)\\ &\quad+\sum_{k\geqslant 1}\left(\sum_{j=0}^N 2^{-kj}\theta_j(x,2^{-k}D)\varrho^\star(\chi_2 f) + R^\theta_{N,2^{-k}}(f)\right).\end{aligned} \tag{9.20}$$

We finish in several steps.

Step 1. — The inequality $\|\cdot\|_{\mathrm{bmo}(\mathbb{R}^d)} \leqslant 3\,\|\cdot\|_{L^\infty(\mathbb{R}^d)}$ is obvious thanks to (9.4). So we have

$$\begin{aligned}\left\|\varrho^\star(\chi_1\widetilde{\theta}(-\boldsymbol{\Delta})f)\right\|_{\mathrm{bmo}(\mathbb{R}^d)} &\leqslant 3\left\|\varrho^\star(\chi_1\widetilde{\theta}(-\boldsymbol{\Delta})f)\right\|_{L^\infty(\mathbb{R}^d)}\\ &\leqslant 3\left\|\chi_1\widetilde{\theta}(-\boldsymbol{\Delta})f\right\|_{L^\infty(X)}\\ &\lesssim \left\|\widetilde{\theta}(-\boldsymbol{\Delta})f\right\|_{L^\infty(X)}.\end{aligned}$$

Step 2. — Using (9.17), one easily get rid of the remainders:

$$\sum_{k\geqslant 1}\left\|R^\theta_{N,2^{-k}}(f)\right\|_{\mathrm{bmo}(\mathbb{R}^d)} \leqslant 3\sum_{k\geqslant 1}\left\|R^\theta_{N,2^{-k}}(f)\right\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L^2(X)}.$$

Step 3. — In this step, we fix j once for all and we remark each $\theta_j(x,2^{-k}\xi)$ has support in a ring $\frac{1}{C}2^k \leqslant |\xi| \leqslant C2^k$. As a consequence, $\theta_j(x,2^{-k}\xi)$ is bounded by 2^{-k} for any semi-norm in the Hörmander pseudo-differential class $S^1_{1,0}$: for any $(\alpha,\beta)\in\mathbb{N}^d\times\mathbb{N}^d$ and $k\in\mathbb{N}^\star$ we have

$$\sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d}(1+|\xi|)^{-1+|\beta|}|\partial_x^\alpha\partial_\xi^\beta\{\theta_j(x,2^{-k}\xi)\}| \lesssim 2^{-k}. \tag{9.21}$$

Similarly, the symbol $\theta(2^{-2\ell}|\xi|^2)$ has support in a ring $\frac{1}{C'}2^\ell \leqslant |\xi| \leqslant C'2^\ell$ and we can write

$$\forall\,(\alpha,\beta)\in\mathbb{N}^d\times\mathbb{N}^d \quad \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d}(1+|\xi|)^{-1+|\beta|}|\partial_x^\alpha\partial_\xi^\beta\{\theta(2^{-2\ell}|\xi|^2)\}| \lesssim 2^{-\ell}. \tag{9.22}$$

Now we choose ν large enough such that if $|\ell-k|\geqslant\nu$ holds true then $\theta(2^{-2\ell}|\xi|^2)$ and $\theta_j(x,2^{-k}\xi)$ have disjoint supports. Consequently, their symbolic calculus ensures that the operator $\theta(-2^{-2\ell}\Delta)\theta_j(x,2^{-k}D)$ is smoothing (see [32, Theorem 1.1.20]). In particular, if we stop that symbolic calculus at the order $N+2$, it appears that the symbol of $\theta(-2^{-2\ell}\Delta)\theta_j(x,2^{-k}D)$ belongs to $S^{-N}_{1,0}$. From the proof of Theorem 1.1.20 of [32] and from the Calderon–Vaillancourt Theorem, we may bound, up to a multiplicative constant independent of ℓ and k, the norm

$$\left\|\theta(-2^{-2\ell}\Delta)\theta_j(x,2^{-k}D)\right\|_{L^2(\mathbb{R}^d)\to H^N(\mathbb{R}^d)}$$

by a product of two semi-norms of $\theta(-2^{-2\ell}|\xi|^2)$ and $\theta_j(x,2^{-k}\xi)$ in $S^1_{1,0}$. Then (9.21) and (9.22) give us

$$\left\|\theta(-2^{-2\ell}\Delta)\sum_{\substack{k\geqslant 1\\|k-\ell|>\nu}} 2^{-kj}\theta_j(x,2^{-k}D)\varrho^\star(\chi_2 f)\right\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{-\ell}\left\|\varrho^\star(\chi_2 f)\right\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-\ell}\|f\|_{L^2(X)}. \tag{9.23}$$

Similarly, upon increasing $\nu\in\mathbb{N}$, we have

$$\left\|\widetilde{\theta}(-\Delta)\sum_{k>\nu} 2^{-kj}\theta_j(x,2^{-k}D)\varrho^\star(\chi_2 f)\right\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L^2(X)}. \tag{9.24}$$

We may also assume that the same ν is chosen for each $j\in\{0,\dots,N\}$.

Step 4. — Thanks to (9.11) and (9.13), we first connect the three BMO spaces:

$$\|\chi_1 f\|^2_{\mathrm{BMO}(X)} \lesssim \|\varrho^\star(\chi_1 f)\|^2_{\mathrm{BMO}(\mathbb{R}^d)} \leqslant 2\,\|\varrho^\star(\chi_1 f)\|^2_{\mathrm{bmo}(\mathbb{R}^d)}.$$

Looking at (9.20) and using Step 1 and Step 2 allow us to bound $\|\chi_1 f\|^2_{\mathrm{BMO}(X)}$ by

$$\|f\|^2_{L^2(X)} + \left\|\widetilde{\theta}(-\boldsymbol{\Delta})f\right\|^2_{L^\infty(X)} + \left\|\sum_{k\geqslant 1}\left(\sum_{j=0}^{N} 2^{-kj}\theta_j(x,2^{-k}D)\varrho^\star(\chi_2 f)\right)\right\|^2_{\mathrm{bmo}(\mathbb{R}^d)}.$$

We now use Theorem 9.3 and the two inequalities (9.23) and (9.24) to get the bound from above

$$\begin{aligned}\|f\|^2_{L^2(X)} &+ \left\|\widetilde{\theta}(-\boldsymbol{\Delta})f\right\|^2_{L^\infty(X)}\\ &+ \left\|\widetilde{\theta}(-\Delta)\sum_{0\leqslant k\leqslant\nu}\sum_{j=0}^{N} 2^{-kj}\theta_j(x,2^{-k}D)\varrho^\star(\chi_2 f)\right\|^2_{L^\infty(\mathbb{R}^d)}\\ &+\sum_{\ell\geqslant 1}\left\|\theta(-2^{-2\ell}\Delta)\sum_{\substack{k\geqslant 1\\|k-\ell|\leqslant\nu}}\sum_{j=0}^{N} 2^{-kj}\theta_j(x,2^{-k}D)\varrho^\star(\chi_2 f)\right\|^2_{L^\infty(\mathbb{R}^d)}.\end{aligned}$$

As seen in the proof of Theorem 9.3, the Fourier multipliers $\widetilde{\theta}(-\Delta)$ and $\theta(-2^{-2\ell}\Delta)$ are uniformly bounded from $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. And if we plug the remainders of (9.16), we can bound $\|\chi_1 f\|^2_{\mathrm{BMO}(X)}$ by

$$\|f\|^2_{L^2(X)} + \left\|\widetilde{\theta}(-\mathbf{\Delta})f\right\|^2_{L^\infty(X)} + \sum_{k\geqslant 1} \left\|\varrho^\star \chi_1 \theta(-2^{-k}\mathbf{\Delta})f - R^\theta_{N,2^{-k}}(f)\right\|^2_{L^\infty(\mathbb{R}^d)}.$$

We again make use of (9.17) in order to write

$$\|\chi_1 f\|^2_{\mathrm{BMO}(X)} \lesssim \|f\|^2_{L^2(X)} + \left\|\widetilde{\theta}(-\mathbf{\Delta})f\right\|^2_{L^\infty(X)} + \sum_{k\geqslant 1} \left\|\theta(-2^{-k}\mathbf{\Delta})f\right\|^2_{L^\infty(X)}.$$

Since $L^2(X)$ is a Hilbert space, the following Littlewood–Paley inequality holds true

$$\begin{aligned}\|f\|^2_{L^2(X)} &\simeq \left\|\widetilde{\theta}(-\mathbf{\Delta})f\right\|^2_{L^2(X)} + \sum_{k\geqslant 1} \left\|\theta(-2^{-k}\mathbf{\Delta})f\right\|^2_{L^2(X)} \\ &\lesssim \left\|\widetilde{\theta}(-\mathbf{\Delta})f\right\|^2_{L^\infty(X)} + \sum_{k\geqslant 1} \left\|\theta(-2^{-k}\mathbf{\Delta})f\right\|^2_{L^\infty(X)}.\end{aligned}$$

Finally, (9.19) is proved. □

10. Proof of Theorem 2.2, part 1

We need the following consequence of Theorem 9.1.

Proposition 10.1. — *Consider a sequence $(f_n)_{n\geqslant 1}$ of $L^2(X)$, such that $f_n \in E_n$ for any $n \geqslant 1$ (see (2.1)). Then there is a constant $C \geqslant 1$ such that the following inequality holds true:*

$$\left\|\sum_{n\geqslant 1} f_n\right\|^2_{\mathrm{BMO}(X)} \leqslant C \sum_{k\in\mathbb{N}} \left\|\sum_{2^k\leqslant n<2^{k+1}} f_n\right\|^2_{L^\infty(X)}. \tag{10.1}$$

Proof. — Let us assume that the right-hand side of (10.1) is finite. The inclusion $L^\infty(X) \subset L^2(X)$ ensures that the series $\sum_{n\geqslant 1} f_n$ is well-defined as an element of $L^2(X)$. Now we want to apply Theorem 9.1 to $f = \sum_{n\geqslant 1} f_n$. Let $b > a > 0$ be two real numbers such that $\mathrm{Supp}(\theta) \subset (a^2, b^2)$. By using that each f_n is spectrally localized in $[\kappa n, \kappa(n+1)]$ with respect to $\sqrt{-\mathbf{\Delta}}$, we see that there is $\nu \in \mathbb{N}$ such that the following holds true for any positive integers j and k:

$$\begin{aligned}|j-k| > \nu \ &\Rightarrow\ [\kappa 2^k, \kappa(2^{k+1}+1)] \cap [a2^j, b2^j] = \emptyset, \\ &\Rightarrow\ \theta(-2^{-2j}\mathbf{\Delta}) \sum_{2^k\leqslant n<2^{k+1}} f_n = 0.\end{aligned}$$

We now invoke the fact that the operators $\theta(-h^2\boldsymbol{\Delta})$ are uniformly bounded from $L^\infty(X)$ to $L^\infty(X)$ with respect to $h \in (0,1]$ (see [10, Corollary 2.2]). Hence, we can write

$$\begin{aligned}\left\|\theta(-2^{-2j}\boldsymbol{\Delta})f\right\|_{L^\infty(X)} &= \Bigg\|\theta(-2^{-2j}\boldsymbol{\Delta}) \sum_{\substack{k\in\mathbb{N}\\ |j-k|\leqslant\nu}} \sum_{2^k\leqslant n<2^{k+1}} f_n\Bigg\|_{L^\infty(X)}\\ &\leqslant C \sum_{\substack{k\in\mathbb{N}\\ |j-k|\leqslant\nu}} \Bigg\|\sum_{2^k\leqslant n<2^{k+1}} f_n\Bigg\|_{L^\infty(X)}\\ \left\|\theta(-2^{-2j}\boldsymbol{\Delta})f\right\|^2_{L^\infty(X)} &\leqslant C^2(2\nu+1) \sum_{\substack{k\in\mathbb{N}\\ |j-k|\leqslant\nu}} \Bigg\|\sum_{2^k\leqslant n<2^{k+1}} f_n\Bigg\|^2_{L^\infty(X)}.\end{aligned}$$

By using that $\widetilde{\theta}$ has support in $[0,\widetilde{b}^2]$ for some $\widetilde{b}>0$, we can choose $\widetilde{\nu}\in\mathbb{N}$ satisfying $\kappa 2^{\widetilde{\nu}} > \widetilde{b}$ and we similarly find

$$\begin{aligned}\left\|\widetilde{\theta}(-\boldsymbol{\Delta})f\right\|_{L^\infty(X)} &\leqslant \left\|\widetilde{\theta}(-\boldsymbol{\Delta})\right\|_{L^\infty\to L^\infty} \sum_{1\leqslant k\leqslant\widetilde{\nu}} \Bigg\|\sum_{2^k\leqslant n<2^{k+1}} f_n\Bigg\|_{L^\infty(X)}\\ \left\|\widetilde{\theta}(-\boldsymbol{\Delta})f\right\|^2_{L^\infty(X)} &\leqslant \widetilde{\nu}\left\|\widetilde{\theta}(-\boldsymbol{\Delta})\right\|^2_{L^\infty\to L^\infty} \sum_{1\leqslant k\leqslant\widetilde{\nu}} \Bigg\|\sum_{2^k\leqslant n<2^{k+1}} f_n\Bigg\|^2_{L^\infty(X)}.\end{aligned}$$

Straightforward computations lead to

$$\begin{aligned}&\left\|\widetilde{\theta}(-\boldsymbol{\Delta})f\right\|^2_{L^\infty(X)} + \sum_{j\geqslant 1} \left\|\theta(-2^{-2j}\boldsymbol{\Delta})f\right\|^2_{L^\infty(X)}\\ &\qquad\qquad \lesssim \sum_{k\in\mathbb{N}} \Bigg\|\sum_{2^k\leqslant n<2^{k+1}} f_n\Bigg\|^2_{L^\infty(X)}. \qquad \square\end{aligned}$$

We can begin the proof of the first statement of Theorem 2.2. Remembering that $\mathrm{BMO}(X)$ is a Banach space (once we identify functions differing by a constant), we have to prove that the random series $\sum X_n f_n^\omega$ satisfies the Cauchy convergence test for almost every $\omega\in\Omega$. Thanks to Proposition 10.1, it is enough to prove that the following finiteness almost surely holds:

$$\sum_{k\in\mathbb{N}} \Bigg\|\sum_{2^k\leqslant n<2^{k+1}} X_n(\omega) f_n^\omega\Bigg\|^2_{L^\infty(X)} < +\infty. \tag{10.2}$$

We follow the same strategy as that of Theorem 6.1. We introduce a new variable ω' that we momentarily freeze, then we invoke the Salem–Zygmund inequality (7.2) and the same computations we made in the proof of Corollary 6.2:

$$\forall\, k \geqslant 1 \quad \mathbf{E}_\omega\left[\left\|\sum_{2^k \leqslant n < 2^{k+1}} X_n(\omega') f_n^\omega\right\|_{L^\infty(X)}\right] \lesssim \sqrt{k}\left\|\sqrt{\sum_{2^k \leqslant n < 2^{k+1}} |X_n(\omega')|^2 \left\|f_n\right\|_{L^2(X)}^2 \frac{e_n(x)}{\dim(E_n)}}\right\|_{L^\infty(X)}.$$

But we have already seen that, in the compact manifold setting, the function $\frac{e_n}{\dim(E_n)}$ is essentially constant (see Lemma 8.1). The Kahane–Khintchine–Marcus–Pisier (7.1) inequalities then give

$$\mathbf{E}_\omega\left[\left\|\sum_{2^k \leqslant n < 2^{k+1}} X_n(\omega') f_n^\omega\right\|_{L^\infty(X)}\right] \lesssim \sqrt{\sum_{2^k \leqslant n < 2^{k+1}} \ln(n) |X_n(\omega')|^2 \left\|f_n\right\|_{L^2(X)}^2}$$

$$\mathbf{E}_\omega\left[\left\|\sum_{2^k \leqslant n < 2^{k+1}} X_n(\omega') f_n^\omega\right\|_{L^\infty(X)}^2\right] \lesssim \sum_{2^k \leqslant n < 2^{k+1}} \ln(n) |X_n(\omega')|^2 \left\|f_n\right\|_{L^2(X)}^2.$$

To get rid off ω', we invoke (7.3) (see Appendix F for more details)

$$\begin{aligned}\mathbf{E}_\omega\left[\left\|\sum_{2^k \leqslant n < 2^{k+1}} X_n(\omega) f_n^\omega\right\|_{L^\infty(X)}^2\right] &= \mathbf{E}_{\omega'}\left[\mathbf{E}_\omega\left[\left\|\sum_{2^k \leqslant n < 2^{k+1}} X_n(\omega') f_n^\omega\right\|_{L^\infty(X)}^2\right]\right] \\ &\lesssim \left(\sup_{n \geqslant 2} \mathbf{E}[|X_n|^2]\right) \sum_{2^k \leqslant n < 2^{k+1}} \ln(n) \left\|f_n\right\|_{L^2(X)}^2.\end{aligned}$$

The assumptions of Theorem 2.2 finally give

$$\sum_{k \in \mathbb{N}} \mathbf{E}_\omega\left[\left\|\sum_{2^k \leqslant n < 2^{k+1}} X_n(\omega) f_n^\omega\right\|_{L^\infty(X)}^2\right] < +\infty,$$

which in turn gives (10.2) for almost every $\omega \in \Omega$.

11. Proof of Theorem 2.2, part 2

We need the following result giving a necessary condition of almost sure convergence.

Lemma 11.1. — *Assume $d \geqslant 2$ and consider a sequence $(f_n)_{n\in\mathbb{N}}$ of $L^2(X)$ such that f_n belongs to E_n for each $n \in \mathbb{N}$. If the random series $\sum f_n^\omega$ almost surely converges in $\mathrm{BMO}(X)$, then the following limit holds true*

$$\lim_{n\to+\infty} \sqrt{\ln(n)}\, \|f_n\|_{L^2(X)} = 0.$$

Proof. — Thanks to the formula (2.2) and [34, p. 92, Theorem 2.14, i) $\Rightarrow$ iv)] (see also [23, Théorème 2.8]), we know that the almost sure convergence of the random series $\sum f_n^\omega$ in $\mathrm{BMO}(X)$ is equivalent to that in $L^1(\Omega, \mathrm{BMO}(X))$. As a consequence, we get $\lim_{n\to+\infty} \mathbf{E}[\|f_n^\omega\|_{\mathrm{BMO}(X)}] = 0$. Remember that each random function f_n^ω is spectrally localized in $[\kappa n, \kappa(n+1)] \subset [\frac{1}{h}, \frac{\sqrt{2}}{h}]$ with $h = \frac{1}{\kappa n}$ provided that $n \gg 1$. Now choose a function $\sigma \in \mathcal{C}_c^\infty(\mathbb{R}^+\backslash\{0\}, \mathbb{R})$ that identically coincides with 1 over $[1,2]$, so we have $\sigma(-h^2\mathbf{\Delta})f_n^\omega = f_n^\omega$. The inequality (9.2) then gives $\lim_{n\to+\infty} \mathbf{E}[\|f_n^\omega\|_{L^\infty(X)}] = 0$. But [11, Théorème 5, p. 930] ensures that the previous sequence is almost equivalent to $\sqrt{|\ln(h)|}\, \|f_n\|_{L^2(X)} \simeq \sqrt{\ln(n)}\, \|f_n\|_{L^2(X)}$. □

We can finish the proof of Theorem 2.2. Let $(n_k)_{k\in\mathbb{N}}$ be an increasing sequence of positive integers such that the series $\sum \frac{1}{\ln\ln(n_k)}$ is convergent. Consider now a sequence of functions (f_n), with $f_n \in E_n$, such that $\|f_n\|_{L^2(X)} = \frac{1}{\sqrt{\ln(n_k)}}$ holds true if $n = n_k$ for some k, and 0 either. Lemma 11.1 ensures that the random series $\sum f_n^\omega$ does not almost surely converge in $\mathrm{BMO}(X)$ (and in fact almost surely diverges due to the Kolmogorov's zero-one law). But the series $\sum_{n\in\mathbb{N}} \|f_n\|^2_{L^2(X)} \ln^\gamma(n) = \sum_{k\in\mathbb{N}} \frac{1}{\ln^{1-\gamma}(n_k)}$ is convergent for any $\gamma < 1$.

12. Proof of Theorem 2.3

We need a result proved in [23] (that may be compared to Theorem 6.1).

Theorem 12.1. — *Let (X,ν) be a σ-finite measure space, $(d_n)_{n\in\mathbb{N}}$ be a sequence of positive integers, $(b_n)_{n\in\mathbb{N}}$ be a sequence of matrices with*

$b_n \in \mathcal{M}_{d_n}(L^p(X,\nu))$ for some $p \in [1,+\infty)$, and a sequence $(X_n)_{n\geqslant 0}$ of independent random variables satisfying

$$0 < \inf_{n\in\mathbb{N}} \mathbf{E}[|X_n|] \qquad \text{and} \qquad \sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^{\max(2,p)}] < +\infty.$$

Then the following assertions are equivalent:

(a) *the function $x \mapsto \sum_{n\geqslant 0}\sum_{i=1}^{d_n}\sum_{j=1}^{d_n} |b_{n,i,j}(x)|^2$ belongs to $L^{\frac{p}{2}}(X,\nu)$,*

(b) *for almost every $\omega \in \Omega$, the random series $\sum X_n(\omega)\sqrt{d_n}\,\mathbf{tr}\,(\mathcal{E}_n(\omega)b_n)$ converges in the Banach space $L^p(X,\nu)$.*

Finally, if one merely assumes $\sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^{\max(2,p)}] < +\infty$, then the implication (a) $\Rightarrow$ (b) holds true.

Proof. — Apply [23, Théorème 2.1] with $M_n = X_n\mathcal{E}_n$. Finally, the last statement comes from [23, Corollaire 2.14]. □

With the above notations and those of Theorem 2.3, set $b_{n,i,j}(x) = \frac{1}{\sqrt{d_n}}\langle f_n, \phi_{n,j}\rangle \phi_{n,i}(x)$ for any $x \in X$ (as in the proof of Corollary 6.2). Let us now begin the proof of Theorem 2.3.

(1) $\Leftrightarrow$ (2). — This equivalence is indeed already contained in [23, Théorème 2.2]. We write it because it involves a computation that we will use in the sequel of the proof. Note that (2) is nothing else than the condition (b) of Theorem 12.1. It remains to check that (1) is the condition (a) of Theorem 12.1:

$$\sum_{i=1}^{d_n}\sum_{j=1}^{d_n} |b_{n,i,j}(x)|^2 = \frac{\|f_n\|^2_{L^2(X)}}{d_n}\sum_{i=1}^{d_n}|\phi_{n,i}(x)|^2 = \frac{\|f_n\|^2_{L^2(X)}\,e_n(x)}{d_n}, \tag{12.1}$$

which is equivalent to $\|f_n\|^2_{L^2(X)}$, uniformly in x, thanks to the accurate estimate of the spectral function on a boundaryless Riemannian compact manifold (see Lemma 8.1). Since ν is assumed to be a probability measure in Theorem 2.3, the condition (a) means that the series $\sum \|f_n\|^2_{L^2(X)}$ converges.

(3) $\Rightarrow$ (1) *and* (1) $\Rightarrow$ (4). — Look at Theorem 12.1 in the very particular case $X = \{x\}$ and $p = 2$. Hence, $L^2(X)$ is just the unidimensional space of functions that send x to a constant. We thus can identify $L^1(X)$ and $L^2(X)$ with $\mathbb{C}$. Again, (12.1) gives the conclusion.

(4) $\Rightarrow$ (3). — Obvious. □

13. Proof of Theorem 4.1

For pedagogical reasons, we prove here Theorem 4.1. We follow the same strategy as that of Theorem 2.3. We merely explain the slight modifications with the needed inequalities of the spectral function (proved in [23, Proposition 4.1]). We first recall that it is well known that $d_n := \dim(E_n)$ is a polynomial of degree $d-1$ with respect to n.

(1) ⇒ (4). — We need to know the following bound $\|e_n\|_{L^\infty(\mathbb{R}^d)} \lesssim n^{\frac{d}{2}-1}$. Therefore, one may use

$$\forall\, x \in \mathbb{R}^d \quad \frac{\|f_n\|^2_{L^2(\mathbb{R}^d)}\, e_n(x)}{d_n} \lesssim \|f_n\|^2_{L^2(\mathbb{R}^d)}\, n^{-\frac{d}{2}}.$$

(4) ⇒ (3). — Obvious.

(3) ⇒ (1). — We need to use that there is a universal constant $\sigma \in (0,1)$ and a constant $C(d) > 0$ such that for any $n \gg 1$ and any $x \in \mathbb{R}^d \backslash \{0\}$ one has

$$\frac{C(d)}{\sqrt{n}} \leqslant |x| \leqslant \sigma\sqrt{2n+1} \quad \Rightarrow \quad e_n(x) \simeq n^{\frac{d}{2}-1}. \tag{13.1}$$

We finish as in the proof of Theorem 2.3 by taking account the following

$$\forall\, x \in \mathbb{R}^d\backslash\{0\} \quad \exists\, n_x \in \mathbb{N} \quad \forall\, n \geqslant n_x \quad \frac{\|f_n\|^2_{L^2(\mathbb{R}^d)}\, e_n(x)}{d_n} \simeq \|f_n\|^2_{L^2(\mathbb{R}^d)}\, n^{-\frac{d}{2}}.$$

(1) ⇔ (2). — From Theorem 12.1 and (12.1), we know that (2) is equivalent to the condition

$$\sqrt{\sum_{n\geqslant 1} \|f_n\|^2_{L^2(\mathbb{R}^d)}\, \frac{e_n(x)}{d_n}} \in L^p_x(\mathbb{R}^d, \nu).$$

By using the inequality $e_n(x) \lesssim n^{\frac{d}{2}-1}$ and the fact that ν is a Borel probability measure on $\mathbb{R}^d\backslash\{0\}$, one immediately gets the implication (1) ⇒ (2). For the other side, we remark that (2) implies that the series $\sum \|f_n\|^2_{L^2(\mathbb{R}^d)} \frac{e_n(x)}{d_n}$ converges at least at one point $x \in \mathbb{R}^d\backslash\{0\}$. Then (13.1) gives (1). □

Remark 13.1. — The proof of (13.1) relies on very specific properties of Hermite functions. It is maybe possible to get a more interesting proof of (13.1) by looking at the details of [17, Lemma 10]. Note however that the Bernstein inequality (5.4) and the equality $e_{2n+1}(0) = 0$ forbids to replace $\frac{C(d)}{\sqrt{n}}$ in (13.1) with a better bound like $\frac{C(d)}{n^\theta}$ and $\theta > \frac{1}{2}$.

14. Proof of Proposition 4.2

Let us explain a little probabilistic issue in the statement of Proposition 4.2 by considering the following two assertions:

(a) for any fixed $p \in [1, +\infty)$, the random series $\sum f_n^\omega$ almost surely diverges in $L^p(\mathbb{R}^d)$,

(b) the random series $\sum f_n^\omega$ almost surely diverges in $L^p(\mathbb{R}^d)$ for any $p \in [1, +\infty)$.

The assertion (a) means that for any $p \in [1, +\infty)$, one may find an event $\Omega_p \subset \Omega$ of probability 1 such that for any $\omega \in \Omega_p$ the concerned random series diverges in $L^p(\mathbb{R}^d)$. It is not clear at all that one may deduce (b) from (a) since the uncountable intersection of the events Ω_p has no reason to be of probability 1 and, even worse, to be an event. We shall overcome this difficulty be considering a weighted Lebesgue space that continuously embeds in any $L^p(\mathbb{R}^d)$ for $p \in [1, +\infty)$. Before introducing this Banach space, we begin by the following result that will also play a fundamental role in the proof of Proposition 5.2.

Proposition 14.1. — *Let X be a σ-finite measure space. For any $p \in [1, +\infty)$ and any positive measurable function $W : X \to (0, \infty)$, we denote by $L^p(X, W(x)dx)$ the weighted Lebesgue space defined by the norm*

$$\forall\, f \in L^p(X, W(x)\mathrm{d}x) \qquad \|f\|_{L^p(X,W(x)dx)} := \left(\int_X |f(x)|^p W(x)\mathrm{d}x\right)^{1/p}.$$

Consider now a nonzero finite-dimensional subspace E of the vector space $L^2(X) \cap L^p(X, W(x)dx)$ and let $(\phi_1, \ldots, \phi_\delta)$ be a Hilbert basis of E. For any $f \in E$ satisfying $\|f\|_{L^2(X)} = 1$, the following holds true

$$\mathbf{E}\left[\left\|\sum_{i=1}^{\delta}\sum_{j=1}^{\delta} \mathcal{E}_{n,i,j}(\omega)\langle f, \phi_j\rangle \phi_i(x)\right\|_{L^p_x(X,W(x)\mathrm{d}x)}\right] \simeq \left\|\frac{\sqrt{|\phi_1(x)|^2 + \cdots + |\phi_\delta(x)|^2}}{\sqrt{\delta}}\right\|_{L^p_x(X,W(x)\mathrm{d}x)}$$

where $\simeq$ means that the quotient belongs to $[\frac{1}{C(p)}, C(p)]$ for some $C(p) \geqslant 1$.

Proof. — Thanks to (7.1), it is sufficient to compute the following expectation

$$\mathbf{E}_\omega\left[\left\|\sum_{i=1}^{\delta}\sum_{j=1}^{\delta} \mathcal{E}_{n,i,j}(\omega)\langle f, \phi_j\rangle \phi_i(x)\right\|^p_{L^p_x(X,W(x)\mathrm{d}x)}\right].$$

Applying the Fubini theorem, the previous estimate is nothing else than

$$\int_X \mathbf{E}\left[\left|\sum_{i=1}^{\delta}\sum_{j=1}^{\delta}\mathcal{E}_{n,i,j}(\omega)\langle f,\phi_j\rangle\phi_i(x)\right|^p\right]W(x)\mathrm{d}x.$$

Again, (7.1) shows that the previous is equivalent, up to a multiplicative loss merely depending on p, to the following

$$\int_X \mathbf{E}\left[\left|\sum_{i=1}^{\delta}\sum_{j=1}^{\delta}\mathcal{E}_{n,i,j}(\omega)\langle f,\phi_j\rangle\phi_i(x)\right|^2\right]^{p/2}W(x)\mathrm{d}x.$$

Using Proposition 7.2 and the equality $\|f\|_{L^2(X)}=1$, we easily conclude. □

We now need the following variant of Lemma 11.1.

Lemma 14.2. — *With the same notations as in Theorem 4.1. If the random series $\sum f_n^\omega$ almost surely converges in the weighted Lebesgue space $L^1_x\left(\mathbb{R}^d,\frac{\mathrm{d}x}{1+|x|^d}\right)$, then*

$$\lim_{n\to+\infty} n^{-\frac{d}{4}}\ln(n)\,\|f_n\|_{L^2(\mathbb{R}^d)}=0.$$

Proof. — Remembering the proof of Lemma 11.1, we know that the expectation $\mathbf{E}\Big[\|f_n^\omega\|_{L^1(\mathbb{R}^d,\frac{\mathrm{d}x}{1+|x|^d})}\Big]$ tends to 0. Thanks to Proposition 14.1 and (13.1), we may write for $n\gg 1$

$$\begin{aligned}
&\mathbf{E}\left[\|f_n^\omega\|_{L^1(\mathbb{R}^d,\frac{\mathrm{d}x}{1+|x|^d})}\right]\\
&\qquad\simeq \|f_n\|_{L^2(\mathbb{R}^d)}\left\|\frac{\sqrt{|\phi_{n,1}(x)|^2+\cdots+|\phi_{n,d_n}(x)|^2}}{\sqrt{d_n}}\right\|_{L^1_x(\mathbb{R}^d,\frac{\mathrm{d}x}{1+|x|^d})}\\
&\qquad\simeq \|f_n\|_{L^2(\mathbb{R}^d)}\frac{1}{n^{\frac{d-1}{2}}}\int_{\mathbb{R}^d}\frac{\sqrt{e_n(x)}}{1+|x|^d}\mathrm{d}x\\
&\qquad\gtrsim \|f_n\|_{L^2(\mathbb{R}^d)}\frac{n^{\frac{d}{4}-\frac{1}{2}}}{n^{\frac{d-1}{2}}}\int_1^{\sigma\sqrt{2n+1}}\frac{r^{d-1}\mathrm{d}r}{1+r^d}\\
&\qquad\gtrsim \|f_n\|_{L^2(\mathbb{R}^d)}\,n^{-\frac{d}{4}}\ln(n).
\end{aligned}$$

□

The end of the proof of Proposition 4.2 is now easy. Consider an increasing sequence of integers $(n_k)_{k\geqslant 0}$, with $n_k\geqslant 2$, such that the series $\sum\frac{1}{\ln^2(n_k)}$ converges. Choose now a sequence $(f_n)_{n\in\mathbb{N}}$, with $f_n\in E_n$, such that

$$n \notin \{n_0, n_1, \ldots\} \Rightarrow f_n = 0,$$
$$\exists\, k \in \mathbb{N} \quad n = n_k \Rightarrow \|f_n\|_{L^2(\mathbb{R}^d)}\, n^{-\frac{d}{4}} = \frac{1}{\ln(n)}.$$

Hence, the sequence $\sum \|f_n\|^2_{L^2(\mathbb{R}^d)}\, n^{-\frac{d}{2}}$ is convergent. Due to the previous lemma, there is an event $\Omega_0 \subset \Omega$, of positive probability, such that for any $\omega \in \Omega_0$ the series $\sum f_n^\omega$ diverges in $L^1\big(\mathbb{R}^d, \frac{1}{1+|x|^d}\big)$. By independence of the random functions f_n^ω, the Kolmogorov's zero-one law ensures that $\mathbf{P}(\Omega_0)$ indeed equals 1. But the Hölder inequality gives:

$$\forall\, p \in [1, +\infty) \quad \forall\, \varphi \in L^p(\mathbb{R}^d) \quad \int_{\mathbb{R}^d} \frac{|\varphi(x)|}{1+|x|^d} \mathrm{d}x \lesssim \left(\int_{\mathbb{R}^d} |\varphi(x)|^p \mathrm{d}x \right)^{1/p}.$$

As a consequence, for any $\omega \in \Omega_0$ and for any $p \in [1, +\infty)$ the series $\sum f_n^\omega$ diverges in $L^p(\mathbb{R}^d)$. □

15. Proof of Proposition 3.1

We need the following corollary of Theorem 12.1.

Proposition 15.1. — *Let H be a Hilbert space, $(d_n)_{n\in\mathbb{N}}$ be a sequence of positive integers, $(b_n)_{n\in\mathbb{N}}$ be a sequence of matrices with $b_n \in \mathcal{M}_{d_n}(H)$. For any sequence $(X_n)_{n\in\mathbb{N}}$ of independent random variables satisfying*

$$0 < \inf_{n\in\mathbb{N}} \mathbf{E}[|X_n|] \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^2] < +\infty,$$

the following assertions are equivalent:

(1) *the series $\sum_{n\geqslant 0} \sum_{i=1}^{d_n} \sum_{j=1}^{d_n} \|b_{n,i,j}\|^2_H$ converges,*
(2) *for almost every $\omega \in \Omega$, the random series $\sum_{n\geqslant 0} X_n(\omega)\sqrt{d_n}\, \mathbf{tr}(\mathcal{E}_n(\omega) b_n)$ converges in H.*

If one merely assumes $\sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^2] < +\infty$ then (1) *implies* (2).

Proof. — By considering a closed subspace that contains the elements $b_{n,i,j}$, one may assume that H is separable and hence equals $L^2(\nu)$ for a σ-finite measure ν. Theorem 12.1 therefore ensures that (2) is equivalent to

$$\sum_{n\in\mathbb{N}} \sum_{i=1}^{d_n} \sum_{j=1}^{d_n} |b_{n,i,j}|^2 \in L^1(\nu).$$

By integrating over ν, one recovers (1). □

The proof of Proposition 3.1 is now straightforward. We first decompose $v_0 = \sum_{n\in\mathbb{N}} v_{0,n}$ following the Hilbert sum $L^2(X) = \bigoplus E_n$. Thanks to Proposition 15.1, the series (3.1) defining the random initial data v_0^ω almost surely converges in the Hilbert space $H^s(X)$ if and only if the following holds

$$\sum_{n\in\mathbb{N}}\sum_{i=1}^{d_n}\sum_{j=1}^{d_n}\frac{1}{d_n}|\langle v_0,\phi_{n,i}\rangle|^2\,\|\phi_{n,j}\|^2_{H^s(X)} < +\infty \tag{15.1}$$
$$\sum_{n\in\mathbb{N}}\frac{\|v_{0,n}\|^2_{L^2(X)}}{d_n}\sum_{j=1}^{d_n}\|\phi_{n,j}\|^2_{H^s(X)} < +\infty.$$

Remember now that $\phi_{n,j}$ is spectrally localized, with respect to $\sqrt{-\boldsymbol{\Delta}}$, in $[\kappa n, \kappa n + n]$ (see (2.1)). So $\|\phi_{n,j}\|_{H^s(X)}$ behaves like n^s as n tend to $+\infty$ and it turns out that (15.1) means that v_0 belongs to $H^s(X)$ (see (2.7)). Thus, the proof of Proposition 3.1 is complete. □

16. Proof of Theorem 3.2

We essentially combine our Paley–Zygmund theorem with the analysis of [14].

THEOREM 16.1 (Burq–Tzvetkov). — *Consider $s > 0$ a real number and X a boundaryless Riemannian compact manifold of dimension 3. Also consider a function*

$$\begin{aligned}\Omega &\to H^s(X)\times H^{s-1}(X)\\ \omega &\mapsto (v_0^\omega, v_1^\omega)\end{aligned} \tag{16.1}$$

that satisfies the following property for almost every $\omega\in\Omega$:

$$\cos(t\sqrt{-\boldsymbol{\Delta}})v_0^\omega(x) + \frac{\sin(t\sqrt{-\boldsymbol{\Delta}})}{\sqrt{-\boldsymbol{\Delta}}}v_1^\omega(x) \in L^3_{t,\mathrm{loc}}(\mathbb{R}, L^6_x(X)) \cap L^1_{t,\mathrm{loc}}(\mathbb{R}, L^\infty_x(X)). \tag{16.2}$$

Then, for almost every $\omega\in\Omega$, the cubic wave equation on X with initial data (v_0^ω, v_1^ω)

$$(\partial_t^2 - \boldsymbol{\Delta})v + v^3 = 0, \quad v(0,\cdot) = v_0^\omega, \quad \dot v(0,\cdot) = v_1^\omega,$$

admits a unique global solution satisfying

$$v(t) - \cos(t\sqrt{-\boldsymbol{\Delta}})v_0^\omega - \frac{\sin(t\sqrt{-\boldsymbol{\Delta}})}{\sqrt{-\boldsymbol{\Delta}}}v_1^\omega \in \mathcal{C}^0_t(\mathbb{R}, H^1(X)) \cap \mathcal{C}^1_t(\mathbb{R}, L^2(X)).$$

Proof. — This is the meaning of the proofs of Proposition 2.1 and Proposition 2.2 of [14]. □

Let us comment on the previous result. The fact that X is of dimension 3 is used for the local existence [14, Proposition 2.1] thanks to the Sobolev embedding $H^1(X) \subset L^6(X)$. The proof of Theorem 16.1 is *deterministic* and the main interest of the probability theory is that it provides a random process (16.1) that almost surely fulfills (16.2). This property is proved for $X = \mathbb{T}^3$ in [14, Appendix A] and for the sphere $X = \mathbb{S}^3$ in [43]. In both cases, the crucial fact is that the torus and the sphere admit a Hilbert basis of eigenfunctions that are uniformly bounded in all $L^p(X)$ spaces, for any $p \in [1, +\infty)$.

Let us now comment on the part $L^1_{t,\mathrm{loc}}(\mathbb{R}, L^\infty_x(X))$ of (16.2). It is usually proved thanks to the Sobolev embedding $W^{s,p}(X) \subset L^\infty(X)$ (that holds true for any $p > \frac{3}{s}$). From a probability point of view, this embedding forces to consider random variables that have a p-th moment (so $p > \frac{3}{s} \gg 1$ if s is near 0^+). Our quantitative Paley–Zygmund theorem (see Theorem 6.1) allows to consider random variables that merely have a third moment (this is probably sharp since this moment is directly linked to the cubic nonlinearity).

Proposition 16.2. — *Consider a real number $s > 0$, a couple of real-valued functions $(v_0, v_1) \in H^s(X) \times H^{s-1}(X)$ and a sequence of independent random variables $X_n : \Omega \to \mathbb{R}$ satisfying $\sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^3] < +\infty$. Then for almost every $\omega \in \Omega$, the following two functions belong to $L^3_{t,\mathrm{loc}}(\mathbb{R}, L^\infty_x(X))$:*

$$\begin{aligned}
&\cos(t\sqrt{-\mathbf{\Delta}}) \sum_{n\in\mathbb{N}} X_n(\omega) \left(\sum_{i=1}^{d_n} \sum_{j=1}^{d_n} \mathcal{E}_{n,i,j}(\omega) \langle v_0, \phi_{n,j} \rangle \phi_{n,i}(x) \right), \\
&\frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}} \sum_{n\in\mathbb{N}} X_n(\omega) \left(\sum_{i=1}^{d_n} \sum_{j=1}^{d_n} \mathcal{E}_{n,i,j}(\omega) \langle v_1, \phi_{n,j} \rangle \phi_{n,i}(x) \right).
\end{aligned} \tag{16.3}$$

Consequently, the latter two random series almost surely belong to the space $L^3_{t,\mathrm{loc}}(\mathbb{R}, L^6_x(X)) \cap L^1_{t,\mathrm{loc}}(\mathbb{R}, L^\infty_x(X))$.

Proof. — We merely consider (16.3) but the first random series is easier. For any integer $N \geqslant 2$, we introduce the partial sum

$$F_N(\omega, t, x) = \frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}} \sum_{n=2}^{N} X_n(\omega) \left(\sum_{i=1}^{d_n} \sum_{j=1}^{d_n} \mathcal{E}_{n,i,j}(\omega) \langle v_1, \phi_{n,j} \rangle \phi_{n,i}(x) \right)$$

$$= \sum_{n=2}^{N} X_n(\omega)\left(\sum_{i=1}^{d_n}\sum_{j=1}^{d_n}\mathcal{E}_{n,i,j}(\omega)\langle v_1, \phi_{n,j}\rangle \frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}}\phi_{n,i}(x)\right).$$

For any integer $T \geqslant 1$, the Fubini–Tonelli theorem gives us

$$\mathbf{E}_\omega\left[\|F_N(\omega,t,x)\|^3_{L^3_t([-T,T],L^\infty_x(X))}\right] = \int_{-T}^{T}\mathbf{E}_\omega\left[\|F_N(\omega,t,x)\|^3_{L^\infty_x(X)}\right]\mathrm{d}t \tag{16.4}$$

We denote by $v_{1,n}$ the orthogonal projection of v_1 on E_n. By invoking Point (1) of Theorem 6.1 for any choice of $\gamma > 1$ and

$$b_{n,i,j}(x) = \frac{1}{\sqrt{d_n}}\langle v_1, \phi_{n,j}\rangle \frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}}\phi_{n,i}(x),$$

we see that $\mathbf{E}_\omega\left[\|F_N(\omega,t,x)\|^3_{L^\infty_x(X)}\right]$ is controlled by

$$\left(\sup_{n\geqslant 2}\mathbf{E}[|X_n|^3]\right)\left(\sum_{n\geqslant 2}\ln^\gamma(n)\frac{\|v_{1,n}\|^2_{L^2(X)}}{d_n}\sup_{x\in X}\sum_{i=1}^{d_n}\left|\frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}}\phi_{n,i}(x)\right|^2\right)^{3/2}. \tag{16.5}$$

Using a similar argument as that of (6.2), we get

$$\sup_{x\in X}\sum_{i=1}^{d_n}\left|\frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}}\phi_{n,i}(x)\right|^2 = \sup_{\substack{u_n\in E_n\\ \|u_n\|_{L^2(X)}=1}}\left\|\frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}}u_n\right\|^2_{L^\infty(X)}.$$

We now use the Sogge $L^2 \to L^\infty$ bound:

$$\begin{aligned}\sup_{x\in X}\sum_{i=1}^{d_n}\left|\frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}}\phi_{n,i}(x)\right|^2 &\lesssim n^{d-1}\sup_{\substack{u_n\in E_n\\ \|u_n\|_{L^2(X)}=1}}\left\|\frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}}u_n\right\|^2_{L^2(X)}\\ &\lesssim \frac{n^{d-1}}{n^2}.\end{aligned}$$

Note that the last bound is uniform with respect to $t \in [-T,T]$. Remembering the asymptotic $d_n \simeq n^{d-1}$ and the assumption $s > 0$, we can bound (16.5) by

$$\left(\sup_{n\geqslant 2}\mathbf{E}[|X_n|^3]\right)\left(\sum_{n\geqslant 2}\frac{\ln^\gamma(n)}{n^2}\,\|v_{1,n}\|^2_{L^2(X)}\right)^{3/2} \lesssim \left(\sup_{n\geqslant 2}\mathbf{E}[|X_n|^3]\right)\|v_1\|^3_{H^{s-1}(X)}\,.$$

Hence, (16.4) is controlled by $T\left(\sup_{n\geqslant 2}\mathbf{E}[|X_n|^3]\right)\|v_1\|^3_{H^{s-1}(X)}$. By considering Cauchy sequences, such a bound easily shows that the random series in (16.3) converges in the Banach space $L^3_\omega(\Omega, L^3_t([-T,T], L^\infty_x(X)))$, so converges in probability in the Banach space $L^3_t([-T,T], L^\infty_x(X))$. Then Theorem 7.5 ensures that (16.3) almost surely converges in $L^3_t([-T,T], L^\infty_x(X))$. Since T runs over the countable set $\mathbb{N}^\star$, one can conclude that (16.3) almost surely converges in $L^3_{t,\mathrm{loc}}(\mathbb{R}, L^\infty_x(X))$. □

17. Exponential decay of the spectral function

The goal of this part is to prove Proposition 17.1 that will play a role in the proofs of Theorem 4.3, Proposition 5.1 and Proposition 5.2. It is well-known that if $V\colon \mathbb{R}^d \to [0,+\infty[$ is a smooth potential satisfying $\lim_{|x|\to+\infty} V(x) = +\infty$ then any eigenfunction $\psi \in L^2(\mathbb{R}^d)$ of $-\Delta + V$ tends to 0 at infinity (see [4, Corollary 3.1, p. 169]). In the specific case $V(x) = |x|^{2\alpha}$ with $\alpha \in \mathbb{N}^\star$, note that the latter property can be seen as a consequence of the description (4.4) of the Sobolev spaces of $-\Delta + |x|^{2\alpha}$ and of a classical Sobolev embedding. We explain here how to modify the proofs of [4] to obtain a precise asymptotic behavior for spectral functions.

Proposition 17.1. — *Consider a real number $\rho > 1$ and a smooth potential $V : \mathbb{R}^d \to \mathbb{R}$ satisfying $V(x) \geqslant m|x|^{2\alpha}$ for any $x \in \mathbb{R}^d$ and for some $m > 0$. There is a constant $C \geqslant 1$ that merely depends on (d, α, ρ, m) such that the following holds true. For any $T \in \mathbb{N}^\star$ and any eigenfunctions $\psi_1, \dots, \psi_T$ of $-\Delta + V(x)$, whose largest eigenvalue is μ, then the following holds true*

$$(17.1)\quad |x| \geqslant \rho m^{-\frac{1}{2\alpha}}\mu^{\frac{1}{2\alpha}} \Rightarrow \sum_{k=1}^T |\psi_k(x)|^2 \leqslant C\left\|\sum_{k=1}^T |\psi_k|^2\right\|_{L^\infty(\mathbb{R}^d)} \exp\left(-\frac{1}{C}|x|^{\alpha+1}\right).$$

We stress that the eigenfunctions $\psi_1, \dots, \psi_T$ in (17.1) are neither assumed to be $L^2(\mathbb{R}^d)$-normalized or associated to successive eigenvalues. The previous result gives a precise statement of the following well-known fact in mathematical physics: an eigenfunction ψ of $-\Delta + V$ associated to an eigenvalue μ is essentially concentrated in the allowed region $\{x \in \mathbb{R}^d, V(x) \lesssim \mu\}$. Note that the particular case $\alpha = 1$ explains the Gaussian tail of the Hermite functions (see [36, line (2.3)]). More generally, we refer to [39, Lemma 3.2], [41] and [28, Theorem 4]. The proofs of [28, 39] seem to be quite specific to the harmonic oscillator since they use for instance a Mehler formula. The proof of [41] is based on the work [30] and involves estimates of the Schwartz kernel of the spectral function. By comparison with the previous works, our assumption on the potential is merely $V(x) \gtrsim |x|^{2\alpha}$ and our sub-exponential remainder in (17.1) has not exactly the same form (it is independent of the largest frequency). Our proof is quite elementary and is inspired from the maximum principle technique used in [4, Theorem 3.3, p. 173]. The main idea is that [4] contains estimates on eigenfunctions ψ whose proofs are convex with respect to ψ^2. For that reason, we succeed to consider in (17.1) several eigenfunctions. Our proof needs two preliminary results.

Lemma 17.2. — *Consider a smooth function $\Theta : (0, +\infty) \to \mathbb{R}$ satisfying $\lim_{t \to +\infty} \Theta(t) = 1$. Then there is a solution $h \in \mathcal{C}^\infty((0, +\infty), \mathbb{R})$ of the differential equation*

$$-h'' + \Theta h = 0 \tag{17.2}$$

such that for any $\sigma > 1$ one may find $C > 1$ such that

$$\forall\, t \gg 1 \qquad \frac{1}{C} e^{-\sigma t} \leqslant h(t) \leqslant C e^{\frac{-t}{\sigma}}.$$

Proof. — From [4, Theorem 3.2 and Theorem 3.3, p. 59–61], we know that (17.2) admits a unique solution h which is positive in a neighborhood of $t = +\infty$ and that satisfies $\lim_{t \to +\infty} h(t) = 0$. To get the end of the statement, we slightly shorten the proof of [4, Theorem 3.4, p. 62]. Consider $t_0 > 0$ such that, for any $t \geqslant t_0$, the inequalities $0 < \Theta(t) \leqslant \sigma^2$ hold true. So we have the following differential inequality

$$\forall\, t \geqslant t_0 \qquad -\frac{\mathrm{d}^2}{\mathrm{d}t^2}(e^{-\sigma t}) + \Theta(t) e^{-\sigma t} \leqslant 0.$$

Fix now $C > 1$ large enough such that $\frac{1}{C} e^{-\sigma t_0} - h(t_0) \leqslant 0$. From (17.2), we get

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{C} e^{-\sigma t} - h(t)\right) + \Theta(t)\left(\frac{1}{C} e^{-\sigma t} - h(t)\right) \leqslant 0.$$

Remembering that the second derivative of a smooth function on a maximum is non-positive, one sees that the function $t \mapsto \frac{1}{C}e^{-\sigma t} - h(t)$ cannot have a *positive* local maximum on $[t_0, +\infty)$. Combining that fact with the limit $\lim_{t\to+\infty} \frac{1}{C}e^{-\sigma t} - h(t) = 0$, one necessarily gets

$$\forall\, t \geqslant t_0 \quad \frac{1}{C}e^{-\sigma t} - h(t) \leqslant 0.$$

We similarly prove the inequality $h(t) \leqslant Ce^{-\frac{t}{\sigma}}$ by assuming $\frac{1}{\sigma^2} \leqslant \Theta(t)$ for $t \gg 1$. □

The previous lemma allows us to prove the following result (in which "radial" means "even" for $d = 1$).

Proposition 17.3. — *For any $\alpha \in \mathbb{N}^\star$ and any $b > 0$, there exists a radial solution $E_{d,\alpha,b} \in \mathcal{C}^\infty(\mathbb{R}^d \backslash \{0\})$ of the equation*

$$(-\Delta + b|x|^{2\alpha})E_{d,\alpha,b} = 0 \tag{17.3}$$

such that for any $\sigma > 1$ there are $R = R(d,\alpha,b,\sigma) > 0$ and $C = C(d,\alpha,b,\sigma) > 1$ such that the following inequality holds true for any $x \in \mathbb{R}^d$ satisfying $|x| \geqslant R$:

$$\frac{1}{C}\exp\left(\frac{-\sqrt{b}}{\alpha+1}\sigma|x|^{\alpha+1}\right) \leqslant |x|^{\frac{\alpha+d-1}{2}} E_{d,\alpha,b}(x) \leqslant C\exp\left(\frac{-\sqrt{b}}{(\alpha+1)}\frac{|x|^{\alpha+1}}{\sigma}\right).$$

Proof. — We shall make several changes of variables and functions f, g and h summarized by the following equalities

$$E_{d,\alpha,b}(x) = f(r) = g(t) = t^{-\gamma/2}h(t),$$
$$r = |x|, \qquad t = \frac{\sqrt{b}}{\alpha+1}r^{\alpha+1}, \qquad \gamma := \frac{\alpha+d-1}{\alpha+1}.$$

Reducing the radial equation (17.3) of $E_{d,\alpha,b}$, we get the following equation on f:

$$\forall\, r > 0 \qquad -f''(r) - \frac{d-1}{r}f'(r) + br^{2\alpha}f(r) = 0.$$

We now compute the derivatives, with respect to r, of $f(r) = g(t)$:

$$f'(r) = g'(t)\sqrt{b}r^\alpha,$$
$$f''(r) = g''(t)br^{2\alpha} + g'(t)\alpha\sqrt{b}r^{\alpha-1}.$$

Hence, the equation in g is

$$\forall\, t > 0 \qquad -g''(t) - \frac{\gamma}{t}g'(t) + g(t) = 0.$$

Let us compute the derivatives, with respect to t, of $g(t) = t^{-\gamma/2}h(t)$ for any $t > 0$. We get

$$g'(t) = -\frac{\gamma}{2}t^{-\frac{\gamma}{2}-1}h(t) + t^{\frac{-\gamma}{2}}h'(t),$$

$$g''(t) = \frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)t^{-\frac{\gamma}{2}-2}h(t) - \gamma t^{-\frac{\gamma}{2}-1}h'(t) + t^{-\frac{\gamma}{2}}h''(t).$$

and so h satisfies an equation without the first order term:

$$\forall\, t > 0 \qquad -h''(t) + h(t)\left[1 + \frac{1}{t^2}\frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right)\right] = 0.$$

We then apply Lemma 17.2 and may conclude by coming back to $E_{d,\alpha,b}$:

$$E_{d,\alpha,b}(x) = \left(\frac{\sqrt{b}}{\alpha+1}\right)^{-\frac{\gamma}{2}} |x|^{\frac{-\gamma(\alpha+1)}{2}} h\left(\frac{\sqrt{b}}{\alpha+1}|x|^{\alpha+1}\right). \qquad \square$$

Proof of Proposition 17.1. — For simplicity, we assume $m = 1$. The reader will easily check that there is no loss of generality in the proof. Consider a real-valued eigenfunction ψ of $-\Delta + V$ associated to the eigenvalue μ. Then one easily writes

$$\Delta(\psi^2) = 2\psi\Delta\psi + 2|\nabla\psi|^2,$$
$$-\Delta(\psi^2) = -2(V-\mu)\psi^2 - 2|\nabla\psi|^2.$$

For any $\rho > 1$, we consider the real number $b \in (0,2)$ such that $\rho^{-\alpha} = 1 - \frac{b}{2}$ holds true. We can write

$$(-\Delta + b|x|^{2\alpha})(\psi^2)(x) = -2\left[V(x) - \frac{b}{2}|x|^{2\alpha} - \mu\right]\psi(x)^2 - 2|\nabla_x\psi|^2.$$

As we assumed the inequality $V(x) \geqslant |x|^{2\alpha}$, we get

$$V(x) - \frac{b}{2}|x|^{2\alpha} - \mu \geqslant \rho^{-\alpha}|x|^{2\alpha} - \mu.$$

So the choice of b ensures the following implication

$$|x| \geqslant \sqrt{\rho}\mu^{\frac{1}{2\alpha}} \quad \Rightarrow \quad (-\Delta + b|x|^{2\alpha})(\psi^2)(x) \leqslant 0.$$

Note that the previous implication is still true if one fixes the eigenfunction ψ but changes the eigenvalue μ by any greater number. Consider now $\psi_1, \dots, \psi_T$ as in the statement of Proposition 17.1. Using the fact the real and imaginary parts of each ψ_k are real-valued eigenfunctions of $-\Delta + V$ and the relation $|\psi_k|^2 = (\mathrm{Re}(\psi_k))^2 + (\mathrm{Im}(\psi_k))^2$, we see that it is sufficient to prove (17.1) if each ψ_k is real-valued. Assume now that μ is the largest eigenvalue among those of $\psi_1, \dots, \psi_T$. By summing several eigenfunctions and going beyond the largest eigenvalue μ, one clearly has

$$(17.4) \qquad |x| \geqslant \sqrt{\rho}\mu^{\frac{1}{2\alpha}} \quad \Rightarrow \quad (-\Delta + b|x|^{2\alpha}) \left(\psi_1^2 + \cdots + \psi_T^2\right)(x) \leqslant 0.$$

We now fix a real number σ satisfying the condition

$$(17.5) \qquad 1 < \sigma < \rho^{\frac{\alpha+1}{4}}.$$

The relevance of this condition will appear at the end of the proof. Given b and σ as above, we can introduce the function $E_{d,\alpha,b}$, the constants R and C of Proposition 17.3. In the sequel, we shall need the following number

$$M := \frac{1}{E_{d,\alpha,b}(\max(\sqrt{\rho}\mu^{\frac{1}{2\alpha}}, R))} \left\| \sum_{k=1}^{T} \psi_k^2 \right\|_{L^\infty(\mathbb{R}^d)}.$$

Note that (17.4) is equivalent to

$$(17.6) \ |x| \geqslant \sqrt{\rho}\mu^{\frac{1}{2\alpha}} \ \Rightarrow \ (-\Delta + b|x|^{2\alpha})\big(\psi_1^2 + \cdots + \psi_T^2 - ME_{d,\alpha,b}\big)(x) \leqslant 0.$$

Moreover, the definition of M and the fact that $E_{d,\alpha,b}$ is radial give us

$$|x| = \max(\sqrt{\rho}\mu^{\frac{1}{2\alpha}}, R) \quad \Rightarrow \quad \psi_1(x)^2 + \cdots + \psi_T(x)^2 - ME_{d,\alpha,b}(x) \leqslant 0.$$

Remember now that the function $E_{d,\alpha,b}$ and each eigenfunction ψ_k tend to 0 if $|x|$ tends to $+\infty$ (see Proposition 17.3 and the discussion above the statement of Proposition 17.1), so we get the following limit

$$\lim_{|x| \to +\infty} \psi_1(x)^2 + \cdots + \psi_T(x)^2 - ME_{d,\alpha,b}(x) = 0.$$

We shall make the same reasoning as that of the end of Lemma 17.2. We claim that the continuous function $\psi_1^2 + \cdots + \psi_T^2 - ME_{d,\alpha,b}$ cannot have a positive maximum on the domain $\{|x| \geqslant \max(\sqrt{\rho}\mu^{\frac{1}{2\alpha}}, R)\}$. Assume the contrary and call x_0 such a point on which the maximum is attained. One necessarily has $|x_0| > \max(\sqrt{\rho}\mu^{\frac{1}{2\alpha}}, R)$ and hence we can show the following inequality by a local analysis around x_0 in all directions:

$$\Delta(\psi_1^2 + \cdots + \psi_T^2 - ME_{d,\alpha,b})(x_0) \leqslant 0$$

and we thus get

$$0 < (-\Delta + b|x|^{2\alpha})(\psi_1^2 + \cdots + \psi_T^2 - ME_{d,\alpha,b})(x_0).$$

That is a contradiction with (17.6). In other words, we have just proved the implication

$$|x| \geqslant \max(\sqrt{\rho}\mu^{\frac{1}{2\alpha}}, R) \quad \Rightarrow \quad \psi_1(x)^2 + \cdots + \psi_T(x)^2 - ME_{d,\alpha,b}(x) \leqslant 0.$$

The definition of M leads us to bound $\frac{E_{d,\alpha,b}(x)}{E_{d,\alpha,b}(\max(\sqrt{\rho}\mu^{\frac{1}{2\alpha}},R))}$. We now want to use Proposition 17.3. First note the inequalities $|x| \geqslant \max(\sqrt{\rho}\mu^{\frac{1}{2\alpha}}, R)$ and $\alpha + d - 1 \geqslant 0$ imply the following one

$$\left(\frac{\max(\sqrt{\rho}\mu^{\frac{1}{2\alpha}}, R)}{|x|}\right)^{\frac{\alpha+d-1}{2}} \leqslant 1.$$

As a consequence, one may bound $\frac{E_{d,\alpha,b}(x)}{E_{d,\alpha,b}(\max(\sqrt{\rho}\mu^{\frac{1}{2\alpha}},R))}$ thanks to Proposition 17.3 and we get that $\sum_{k=1}^{T} \psi_k(x)^2$ is bounded from above by

$$C^2 \left\| \sum_{k=1}^{T} \psi_k^2 \right\|_{L^\infty(\mathbb{R}^d)} \exp\left[-\frac{\sqrt{b}}{\alpha+1} \left(\frac{|x|^{\alpha+1}}{\sigma} - \sigma \max(\sqrt{\rho}\mu^{\frac{1}{2\alpha}}, R)^{\alpha+1} \right) \right].$$

We indeed can simplify the previous bound thanks to our choice of σ. The condition (17.5) indeed implies the existence of a constant $K(\alpha,\rho,\sigma) \in (0,1)$, more precisely $K(\alpha,\rho,\sigma) := 1 - \frac{\sigma^2}{\rho^{(\alpha+1)/2}}$, such that the following implication holds true for any $y \in \mathbb{R}^d$:

$$|y| \geqslant \sqrt{\rho} \quad \Rightarrow \quad |y|^{\alpha+1} - \sigma^2 \geqslant K(\alpha,\rho,\sigma)|y|^{\alpha+1}.$$

We now choose $y = \frac{|x|}{\max(\sqrt{\rho}\mu^{1/2\alpha},R)}$ and thus force

$$|x| \geqslant \sqrt{\rho} \max\left(\sqrt{\rho}\mu^{\frac{1}{2\alpha}}, R\right).$$

Under the last restriction on x, we see that there is a constant $\xi > 0$ that merely depends on (d,α,ρ,σ) such that

$$\sum_{k=1}^{T} \psi_k(x)^2 \leqslant C^2 \left\| \sum_{k=1}^{T} \psi_k^2 \right\|_{L^\infty(\mathbb{R}^d)} \exp\left(-\xi|x|^{\alpha+1}\right), \tag{17.7}$$

To get the conclusion, we consider three cases:

Case $R \leqslant \sqrt{\rho}\mu^{1/2\alpha}$. — The condition $|x| \geqslant \sqrt{\rho}\max(\sqrt{\rho}\mu^{\frac{1}{2\alpha}}, R)$ is reduced to $|x| \geqslant \rho\mu^{\frac{1}{2\alpha}}$. So we get the wanted conclusion (17.1).

Case $\sqrt{\rho}\mu^{1/2\alpha} \leqslant R$ and $\sqrt{\rho}R \leqslant |x|$. — Again, (17.7) gives the conclusion.

Case $\sqrt{\rho}\mu^{1/2\alpha} \leqslant R$ and $\rho\mu^{1/2\alpha} \leqslant |x| \leqslant \sqrt{\rho}R$. — This is the only case which is not covered by (17.7). We can however easily conclude since everything takes place in a fixed compact. Let us consider a constant $C' > 1$ such that

$$1 \leqslant \inf_{|x| \leqslant \sqrt{\rho}R} C' \exp\left(-\frac{1}{C'}|x|^{\alpha+1}\right).$$

Hence, we have

$$\sum_{k=1}^{T} \psi_k(x)^2 \leqslant C' \left\| \sum_{k=1}^{T} \psi_k^2 \right\|_{L^\infty(\mathbb{R}^d)} \exp\left(-\frac{1}{C'}|x|^{\alpha+1}\right).$$

Hence, (17.1) is proved in all the cases and the proof of Proposition 17.1 is complete. □

18. Proof of (4.2)

From [19, Théorème 6-4, p. 840] we know that the spectral counting function of $(-\Delta + V_{2\alpha})^{\frac{\alpha+1}{2\alpha}}$ behaves as that of the operator $\sqrt{-\Delta}$ on a compact Riemannian manifold. Namely, for a suitable constant $C > 0$, one has

$$\forall\, \mu \geqslant 1 \quad \mathrm{Card}\left\{\ell \in \mathbb{N}, \quad \lambda_\ell^{\frac{\alpha+1}{2\alpha}} \leqslant \mu\right\} = C\mu^d + \mathcal{O}(\mu^{d-1}).$$

For any $K > 0$ (to be chosen below) and any $n \gg 1$, the number

$$\mathrm{Card}\left\{\ell \in \mathbb{N}, \quad \lambda_\ell^{\frac{\alpha+1}{2\alpha}} \in (Kn, Kn + K]\right\}$$

has the asymptotic

$$K^{d-1}\left[CK[(n+1)^d - n^d] + \mathcal{O}(n^{d-1})\right].$$

Using the inequalities $dn^{d-1} \leqslant (n+1)^d - n^d \leqslant (2^d - 1)n^{d-1}$, one sees that K may be chosen large enough such that

$$\mathrm{Card}\left\{\ell \in \mathbb{N}, \quad \lambda_\ell^{\frac{\alpha+1}{2\alpha}} \in (Kn, Kn + K]\right\} \simeq n^{d-1}. \qquad \square$$

19. Proof of Theorem 4.3

We have to fulfill the assumptions of Corollary 6.2. Thanks to (4.2) and Proposition 5.1, we see that (A1) holds true with

$$S = \frac{\alpha d}{\alpha + 1} - 1 - (d - 1) = \frac{-d}{\alpha + 1}.$$

Note that we merely need the bound from above of Proposition 5.1 that has been proved in [41]. We now use the exponential decay of the spectral function to prove (A2) in two steps.

Step 1. — We will show that if N is large enough then, for a suitable constant $K>0$, we have

$$\forall\, u\in E_0\oplus\cdots\oplus E_N \qquad \|u\|_{L^\infty(\mathbb{R}^d)}=\|u\|_{L^\infty(B(0,KN^{1/(\alpha+1)}))}\,. \tag{19.1}$$

Consider $\psi_1,\ldots,\psi_T$ a Hilbert basis of $E_0\oplus\cdots\oplus E_N$ made of real-valued eigenfunctions of $-\Delta+V_{2\alpha}$ and we still denote by $e_0,\ldots,e_N$ the respective spectral functions of $E_0,\ldots,E_N$. The Cauchy–Schwarz inequality allows us to write for any $u\in E_0\oplus\cdots\oplus E_N$:

$$u(x)=\sum_{k=1}^{T}\left(\int_{\mathbb{R}^d}\psi_k(y)u(y)\mathrm{d}y\right)\psi_k(x)=\int_{\mathbb{R}^d}\left(\sum_{k=1}^{T}\psi_k(y)\psi_k(x)\right)u(y)\mathrm{d}y$$

$$\begin{aligned}|u(x)|&\leqslant\sqrt{\psi_1(x)^2+\cdots+\psi_T(x)^2}\int_{\mathbb{R}^d}\sqrt{\psi_1(y)^2+\cdots+\psi_T(y)^2}|u(y)|\mathrm{d}y\\&\leqslant\sqrt{e_0(x)+\cdots+e_N(x)}\left\|\sqrt{e_0+\cdots+e_N}\right\|_{L^1(\mathbb{R}^d)}\|u\|_{L^\infty(\mathbb{R}^d)}\,.\end{aligned}$$

We now apply (17.1). For a suitable constant $K>0$ and for any $x\in\mathbb{R}^d$, the number $\sqrt{\frac{e_0(x)+\cdots+e_N(x)}{\|e_0+\cdots+e_N\|_{L^\infty(\mathbb{R}^d)}}}$ is less than

$$\mathbf{1}_{\{|x|<KN^{\frac{1}{\alpha+1}}\}}+\sqrt{C}\exp\left(-\frac{1}{2C}|x|^{\alpha+1}\right)\mathbf{1}_{\{|x|\geqslant KN^{\frac{1}{\alpha+1}}\}}$$

From this inequality, we get for any $N\gg 1$

$$\left\|\sqrt{e_0(x)+\cdots+e_N(x)}\right\|_{L^1(\mathbb{R}^d)}\lesssim N^{\frac{d}{\alpha+1}}\|e_0+\cdots+e_N\|_{L^\infty(\mathbb{R}^d)}^{\frac12}\,.$$

Combining the last three inequalities, we obtain for any $N\gg1$ and any x satisfying $|x|\geqslant KN^{\frac{1}{\alpha+1}}$:

$$|u(x)|\lesssim\|u\|_{L^\infty(\mathbb{R}^d)}N^{\frac{d}{\alpha+1}}\|e_0+\cdots+e_N\|_{L^\infty(\mathbb{R}^d)}\exp\left(-\frac{K^{\alpha+1}}{2C}N\right).$$

Using the triangular inequality $\|e_0+\cdots+e_N\|_{L^\infty(\mathbb{R}^d)}\leqslant\|e_0\|_{L^\infty(\mathbb{R}^d)}+\cdots+\|e_N\|_{L^\infty(\mathbb{R}^d)}$ and a polynomial bound of $\|e_N\|_{L^\infty(\mathbb{R}^d)}$ (this is nothing else than (A1)), one may exploit the decaying exponential to get the following implication provided that $N\gg1$

$$|x|\geqslant KN^{\frac{1}{\alpha+1}}\quad\Rightarrow\quad|u(x)|\leqslant\frac12\|u\|_{L^\infty(\mathbb{R}^d)}\,.$$

In other words, (19.1) is proved.

Step 2. — Thanks to (19.1) and a Bernstein inequality for $-\Delta+V_{2\alpha}$ (see Theorem 5.3 that we can admit for the moment), we can prove (A2) by the same "mesh strategy" on the closed ball $B(0,KN^{1/(\alpha+1)})$ as in the setting of Riemannian compact manifolds (see the end of Section 8). □

20. Proof of Theorem 4.5

We follow exactly the same ideas as those of the proofs of Theorem 2.3 and Theorem 4.1 (see Sections 12 and 13). More precisely, Theorem 12.1 and the inequality $\|e_n\|_{L^\infty(\mathbb{R}^d)} \lesssim n^{\frac{\alpha d}{d+1}-1}$ (see below (21.2)) easily imply the conclusion. The connexion with (4) needs the Fubini–Tonelli theorem as follows.

(3) $\Rightarrow$ (4). — Let Υ the measurable subset of $\Omega \times \mathbb{R}^d$ of pairs (ω, x) such that the series $\sum X_n(\omega) f_n^\omega(x)$ diverges in $\mathbb{C}$. The assertion (3) and the Fubini–Tonelli theorem give

$$\int_\Omega \left(\int_{\mathbb{R}^d} \mathbf{1}_\Upsilon(\omega, x) \mathrm{d}x \right) \mathrm{d}\mathbf{P}(\omega) = \int_{\mathbb{R}^d} \left(\int_\Omega \mathbf{1}_\Upsilon(\omega, x) \mathrm{d}\mathbf{P}(\omega) \right) \mathrm{d}x = \int_{\mathbb{R}^d} 0 \,\mathrm{d}x = 0.$$

As a consequence, with probability 1, the integral $\int_{\mathbb{R}^d} \mathbf{1}_\Upsilon(\omega, x)\mathrm{d}x$ vanishes. In other words, with probability 1, for almost every $x \in \mathbb{R}^d$, the series $\sum X_n(\omega) f_n^\omega(x)$ converges. □

21. Proof of Theorem 4.6

Thanks to the last statement of Theorem 12.1, we know that the almost sure convergence in $L^p(\mathbb{R}^d)$ of the random series $\sum_{n\geqslant 0} X_n(\omega) f_n^\omega$ is a consequence of the deterministic condition

$$\sum_{n\geqslant 0} \frac{e_n(x)}{d_n} \|f_n\|^2_{L^2(\mathbb{R}^d)} \in L^{p/2}(\mathbb{R}^d). \tag{21.1}$$

The conclusion of Theorem 4.6 will come from the implication

$$\sum_{n\geqslant 1} n^{-\frac{d}{\alpha+1}(1-\frac{2}{p})} \|f_n\|^2_{L^2(\mathbb{R}^d)} < +\infty \quad \Rightarrow \; (21.1).$$

At this point, we need the following accurate estimate of the $L^{p/2}$ norm of the spectral function of $-\Delta + V_{2\alpha}$

$$\forall\, p \in [2, +\infty] \quad \|e_n\|_{L^{p/2}(\mathbb{R}^d)} \lesssim n^{\frac{d}{\alpha+1}(\alpha+\frac{2}{p})-1}. \tag{21.2}$$

Such inequalities are proved in [41, Proposition 2.4, $\delta = 1, \theta = 0$] by interpolating the trivial case $p = 2$ and the much more difficult case $p = +\infty$. The triangular inequality in $L^{\frac{p}{2}}(\mathbb{R}^d)$ finally gives:

$$\begin{aligned} \left\| \sum_{n\geqslant 1} \frac{e_n(x)}{d_n} \|f_n\|^2_{L^2(\mathbb{R}^d)} \right\|_{L_x^{\frac{p}{2}}(\mathbb{R}^d)} &\lesssim \sum_{n\geqslant 1} \frac{\|e_n\|_{L^{p/2}(\mathbb{R}^d)}}{n^{d-1}} \|f_n\|^2_{L^2(\mathbb{R}^d)} \\ &\lesssim \sum_{n\geqslant 1} n^{-\frac{d}{\alpha+1}(1-\frac{2}{p})} \|f_n\|^2_{L^2(\mathbb{R}^d)} . \end{aligned}$$ □

22. Proof of Proposition 5.1

Thanks to (6.2) and (21.2), it is sufficient to prove the inequality

$$\|e_n\|_{L^\infty(\mathbb{R}^d)} \gtrsim n^{\frac{\alpha d}{\alpha+1}-1}.$$

As explained below the statement of Proposition 5.1, the proof of the lower bounds in [41] does not cover the cases we are concerned with. The exponential decay of the spectral function and an elementary interpolation argument allow to reverse (21.2) and complete it below 2.

PROPOSITION 22.1. — *With the above notations, for any $p \in [1, +\infty]$, we have*

$$\forall\, n \gg 1 \qquad \|\sqrt{e_n}\|_{L^p(\mathbb{R}^d)} \simeq n^{\frac{d}{2(\alpha+1)}\left(\alpha+\frac{2}{p}\right)-\frac{1}{2}}. \tag{22.1}$$

Proof. — Let us introduce the exponent $\Theta(p) := \frac{d}{2(\alpha+1)}\left(\alpha+\frac{2}{p}\right)-\frac{1}{2}$ and assume for a moment that there is $M \geqslant 1$ such that for any $n \gg 1$ the following three inequalities hold true

$$\|\sqrt{e_n}\|_{L^\infty(\mathbb{R}^d)} \leqslant M n^{\Theta(\infty)}, \quad \|\sqrt{e_n}\|_{L^2(\mathbb{R}^d)} \geqslant \frac{n^{\Theta(2)}}{M}, \quad \|\sqrt{e_n}\|_{L^1(\mathbb{R}^d)} \leqslant M n^{\Theta(1)}. \tag{22.2}$$

Since $\Theta(p)$ is affine in $\frac{1}{p}$, one may prove by interpolation that the last inequalities would imply the stronger ones:

$$\forall\, p \in [1, +\infty] \qquad \frac{n^{\Theta(p)}}{M^3} \leqslant \|\sqrt{e_n}\|_{L^p(\mathbb{R}^d)} \leqslant M n^{\Theta(p)}.$$

As recalled in (21.2), Robert and Thomann have already obtained the upper bound of $\left\|\sqrt{e_n}\right\|_{L^\infty(\mathbb{R}^d)}$. The middle bound in (22.2) is obvious because $\left\|\sqrt{e_n}\right\|_{L^2(\mathbb{R}^d)}$ equals $\sqrt{d_n} \simeq n^{\frac{d-1}{2}}$. The last bound in (22.2) now needs the exponential decay of the spectral function. From (17.1) with $\mu \simeq n^{\frac{2\alpha}{\alpha+1}}$, there is $c > 0$ and $C \geqslant 1$ such that

$$\forall\, x \in \mathbb{R}^d \quad e_n(x) \leqslant n^{\frac{\alpha d}{\alpha+1}-1}\left(\mathbf{1}_{B(0,cn^{1/(\alpha+1)})}(x) + C\exp\left(-\frac{1}{C}|x|^{\alpha+1}\right)\right). \tag{22.3}$$

By integration, we get

$$\int_{\mathbb{R}^d} \sqrt{e_n(x)}\mathrm{d}x \lesssim n^{\frac{\alpha d}{2(\alpha+1)}-\frac{1}{2}} n^{\frac{d}{\alpha+1}} = n^{\Theta(1)}. \qquad \square$$

23. Proof of Proposition 5.2

For any $f \in E_n \backslash \{0\}$, one may decompose f on a Hilbert basis $\phi_{n,1}, \dots, \phi_{n,d_n}$ of E_n made of eigenfunctions, and thus use the Cauchy–Schwarz inequality to get

$$\forall\, x \in \mathbb{R}^d \quad |f(x)|^2 \leqslant \|f\|^2_{L^2(\mathbb{R}^d)} \sum_{k=1}^{d_n} |\phi_{n,k}(x)|^2 = \|f\|^2_{L^2(\mathbb{R}^d)}\, e_n(x).$$

We again use the exponential decay (22.3) with $K = \frac{\alpha d}{\alpha+1} - 1$ to get for any $n \gg 1$ and any $x \in \mathbb{R}^d$ satisfying $|x| \geqslant cn^{\frac{1}{\alpha+1}}$:

$$|f(x)|^2 \leqslant Cn^K \|f\|^2_{L^2(\mathbb{R}^d)} \exp\left(\frac{-1}{C}|x|^{\alpha+1}\right). \tag{23.1}$$

Case $p \in [2, +\infty)$. — Use now the Hölder inequality:

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x)|^2 \mathrm{d}x &= \int_{|x|<cn^{\frac{1}{\alpha+1}}} |f(x)|^2 \mathrm{d}x + \int_{|x|\geqslant cn^{\frac{1}{\alpha+1}}} |f(x)|^2 \mathrm{d}x \\ &\leqslant \mathrm{Vol}(B(0, cn^{\frac{1}{\alpha+1}}))^{1-\frac{2}{p}} \|f\|^2_{L^p(\mathbb{R}^d)} \\ &\qquad + Cn^K \|f\|^2_{L^2(\mathbb{R}^d)} \int_{|x|\geqslant cn^{\frac{1}{\alpha+1}}} \exp\left(-\frac{1}{C}|x|^{\alpha+1}\right) \mathrm{d}x.\end{aligned}$$

The remainder can be controlled as follows for $n \gg 1$:

$$Cn^K \int_{|x|\geqslant cn^{1/(\alpha+1)}} \exp\left(-\frac{1}{C}|x|^{\alpha+1}\right) \mathrm{d}x \leqslant \frac{1}{2}.$$

So we have

$$\frac{1}{2}\|f\|^2_{L^2(\mathbb{R}^d)} \leqslant \mathrm{Vol}(B(0, cn^{\frac{1}{\alpha+1}}))^{1-\frac{2}{p}} \|f\|^2_{L^p(\mathbb{R}^d)}$$

which implies

$$n^{-\frac{d}{\alpha+1}\left(\frac{1}{2}-\frac{1}{p}\right)} \lesssim \inf_{\substack{f\in E_n \\ f\neq 0}} \frac{\|f\|_{L^p(\mathbb{R}^d)}}{\|f\|_{L^2(\mathbb{R}^d)}}. \tag{23.2}$$

Case $p \in [1, 2]$. — We write similarly thanks to (23.1) and the Hölder inequality:

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x)|^p \mathrm{d}x &= \int_{|x|<cn^{\frac{1}{\alpha+1}}} |f(x)|^p \mathrm{d}x + \int_{|x|\geqslant cn^{\frac{1}{\alpha+1}}} |f(x)|^p \mathrm{d}x \\ &\leqslant \mathrm{Vol}(B(0, cn^{\frac{1}{\alpha+1}}))^{1-\frac{p}{2}} \|f\|^p_{L^2\mathbb{R}^d)} \\ &\qquad + C^{\frac{p}{2}} n^{\frac{Kp}{2}} \|f\|^p_{L^2(\mathbb{R}^d)} \int_{|x|\geqslant cn^{\frac{1}{\alpha+1}}} e^{-\frac{p}{2C}|x|^{\alpha+1}} \mathrm{d}x.\end{aligned}$$

And by the same idea, the last term is smaller than the first one for $n \gg 1$. So we get

$$\sup_{\substack{f \in E_n \\ f \neq 0}} \frac{\|f\|_{L^p(\mathbb{R}^d)}}{\|f\|_{L^2(\mathbb{R}^d)}} \lesssim n^{\frac{d}{\alpha+1}\left(\frac{1}{p}-\frac{1}{2}\right)}. \tag{23.3}$$

The probabilistic argument for any $p \in [1, +\infty)$. — We now have to reverse (23.2) and (23.3). We make use of a standard idea of randomization [11, 38, 39, 41] but our approach is slightly different. Instead of large deviations estimates or the principle of the concentration of the measure, we will use Proposition 14.1 with $W = 1$ (whose proof relies on the multidimensional Kahane–Khintchine inequalities). Fix any element $f \in E_n$ with $\|f\|_{L^2(\mathbb{R}^d)} = 1$. As the random matrix $\mathcal{E}_n$ is unitary, one has

$$\forall\, \omega \in \Omega \qquad \left\| \sum_{i=1}^{d_n} \sum_{j=1}^{d_n} \mathcal{E}_{n,i,j}(\omega) \langle f, \phi_{n,j} \rangle \phi_{n,i} \right\|_{L^2(\mathbb{R}^d)} = 1.$$

As a consequence of the two-side bound given by Proposition 14.1, one sees that there are a constant $C(p) \geqslant 1$ and two functions $f_1 \in E_n$ and $f_2 \in E_n$, with $\|f_1\|_{L^2(\mathbb{R}^d)} = \|f_2\|_{L^2(\mathbb{R}^d)} = 1$, such that

$$\frac{\|f_1\|_{L^p(\mathbb{R}^d)}}{C(p)} \leqslant \left\| \frac{\sqrt{|\phi_{n,1}|^2 + \cdots + |\phi_{n,d_n}|^2}}{\sqrt{d_n}} \right\|_{L^p(\mathbb{R}^d)} \leqslant C(p) \, \|f_2\|_{L^p(\mathbb{R}^d)} \,.$$

The middle bound can easily be controlled thanks to the asymptotic $d_n \simeq n^{d-1}$ and (22.1):

$$\begin{aligned} \left\| \frac{\sqrt{|\phi_{n,1}|^2 + \cdots + |\phi_{n,d_n}|^2}}{\sqrt{d_n}} \right\|_{L^p(\mathbb{R}^d)} &= \left\| \frac{\sqrt{e_n}}{\sqrt{d_n}} \right\|_{L^p(\mathbb{R}^d)} \\ &\simeq \frac{n^{\frac{d}{2(\alpha+1)}\left(\alpha+\frac{2}{p}\right)-\frac{1}{2}}}{n^{\frac{d-1}{2}}} = n^{\frac{d}{\alpha+1}\left(\frac{1}{p}-\frac{1}{2}\right)}. \end{aligned}$$

Finally, the function f_1 allows to reverse (23.2) whereas f_2 reverses (23.3). The proof of Proposition 5.2 is finished. □

Appendix A. Parametrix for the Weyl–Hörmander pseudo-differential calculus

The goal of this part is to extend the classical construction of an arbitrary precise parametrix to the Weyl–Hörmander pseudo-differential calculus (see for instance the strategy in [18, Proposition 2.5] or [15, Part 8]).

Let us begin by setting the notations of the Weyl–Hörmander pseudo-differential calculus (see [22, Parts 18.4, 18.5, 18.6] or [32]). For any Schwartz functions $\sigma \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ and $u \in \mathcal{S}(\mathbb{R}^d)$, one can give a sense to the following Weyl quantization

$$\forall\, x \in \mathbb{R}^d \quad \mathrm{Op}^w(\sigma)u(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\langle x-y,\xi\rangle}\sigma\left(\frac{x+y}{2},\xi\right)u(y)\frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi)^d}. \tag{A.1}$$

One can still give a sense to the previous expression for any tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ as an operator $Op^w(\sigma) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. If σ_1 and σ_2 are two symbols, for instance belonging to $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, then there is a unique symbol $\sigma_1 \# \sigma_2$ such that $\mathrm{Op}^w(\sigma_1\#\sigma_2) = \mathrm{Op}^w(\sigma_1) \circ \mathrm{Op}^w(\sigma_2)$. The Weyl–Hörmander pseudo-differential calculus now needs

- a Riemannian metric g on the phase space $\mathbb{R}^d \times \mathbb{R}^d$ (more precisely, a measurable map from $\mathbb{R}^d \times \mathbb{R}^d$ into the cone of positive-definite quadratic forms on $\mathbb{R}^d \times \mathbb{R}^d$),
- a function $M : \mathbb{R}^d \times \mathbb{R}^d \to (0,+\infty)$ called a weight.

The class of symbols $S(M,g)$ is then defined as the linear subspace of symbols $\sigma \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$ such that for any $k \in \mathbb{N}$ there is $C_k > 0$ for which the following estimates hold true:

$$\forall\, (x,\xi) \in \mathbb{R}^{2d} \quad \forall\, (T_1,\dots,T_k) \in (\mathbb{R}^{2d})^k \quad |\partial_{T_1}\dots\partial_{T_k}\sigma(x,\xi)| \leqslant C_k M(x,\xi)\prod_{j=1}^{k}\sqrt{g_{x,\xi}(T_j)}, \tag{A.2}$$

where $\partial_{T_1}\dots\partial_{T_k}\sigma$ equals the differential expression $(d^k\sigma)(T_1,\dots,T_k)$. Note that the symmetry of the multilinear map $d^k\sigma$ allows to replace (A.2) with

$$\forall\, T \in \mathbb{R}^{2d} \quad |\partial_T^k\sigma(x,\xi)| \leqslant C_k M(x,\xi) g_{x,\xi}(T)^{k/2}. \tag{A.3}$$

If C_k is the best constant in (A.2), then one defines

$$\|\sigma\|_{S(M,g)}^{(k)} = \max(C_0,\dots,C_k).$$

There is a special function, denoted $\lambda : \mathbb{R}^d \times \mathbb{R}^d \to (0,+\infty)$, that is associated to the metric g (see [32, Part 2.2.3]). To define λ, we recall that the classical symplectic classification of metrics (see also [20, Part 1.7]) states that there is a symplectic basis $\mathbf{B}_{x,\xi}$ of $\mathbb{R}^d \times \mathbb{R}^d$ in which the metric g can be written

$$\sum_{j=1}^{d}\frac{\mathrm{d}x_j^2 + \mathrm{d}\xi_j^2}{\lambda_j(x,\xi)}, \qquad 0 < \lambda_1(x,\xi) \leqslant \dots \leqslant \lambda_d(x,\xi), \tag{A.4}$$

where the numbers $\lambda_1(x,\xi),\dots,\lambda_d(x,\xi)$ are uniquely determined by $g_{x,\xi}$ and the canonic symplectic structure of $\mathbb{R}^d\times\mathbb{R}^d$. We then define

$$\lambda(x,\xi)=\lambda_1(x,\xi).$$

To get a symbolic calculus, the theory needs to assume that g is an admissible metric, namely that is slowly varying, is temperate and satisfies the uncertainty principle. We also assume that M is admissible with respect to g. It is not necessary for us to recall the precise above definitions. It is however important to give the main example we will use in Appendix B. For any $\alpha\in\mathbb{N}^\star$, we introduce the following admissible metric and admissible weight

$$g_{x,\xi}=\sum_{j=1}^{d}\frac{\mathrm{d}x_j^2}{M(x,\xi)^{1/\alpha}}+\frac{\mathrm{d}\xi_j^2}{M(x,\xi)},\qquad M(x,\xi):=1+|\xi|^2+|x|^{2\alpha}.$$

Then the smooth symbol $|\xi|^2+|x|^{2\alpha}$ belongs to $S(M,g)$ and is associated, in the Weyl–Hörmander pseudo-differential calculus, to the superquadratic oscillator $-\Delta+|x|^{2\alpha}$. The symplectic map $(x,\xi)\mapsto(M^{-\frac14(1-\frac1\alpha)}x,M^{\frac14(1-\frac1\alpha)}\xi)$ of $\mathbb{R}^{2d}$ puts the latter metric to the form (A.4) so that the corresponding function $\lambda=\lambda_1=\cdots=\lambda_d$ reads

$$\lambda(x,\xi)=(1+|\xi|^2+|x|^{2\alpha})^{\frac{\alpha+1}{2\alpha}}. \tag{A.5}$$

Consequently, the Weyl–Hörmander symbolic calculus (see below (A.6)) gives an enlightenment of the reason why $(-\Delta+|x|^{2\alpha})^{\frac{\alpha+1}{2\alpha}}$ is usually considered as an operator of order 1 (see [19]).

We can now state the fundamental symbolic calculus [22, Theorem 18.5.4] or [32, Theorem 2.3.7]: for any admissible weights M and M' with respect to an admissible metric g, for any symbols $a\in S(M,g)$ and $b\in S(M',g)$, for any integer $N\in\mathbb{N}$, one has

$$a\#b-\sum_{n=0}^{N}T_n(a,b)\in S(MM'\lambda^{-N-1},g), \tag{A.6}$$

where $T_n(a,b)$ is the bilinear differential operator defined by

$$\begin{aligned}\frac{i^n}{n!2^n}&(\langle D_{\xi_1},D_{x_2}\rangle-\langle D_{x_1},D_{\xi_2}\rangle)^n\,a(x_1,\xi_1)b(x_2,\xi_2)|_{(x_1,\xi_1)=(x_2,\xi_2)=(x,\xi)}\\&=\frac{1}{(2i)^n}\sum_{\substack{(s,t)\in\mathbb{N}^d\times\mathbb{N}^d\\|s+t|=n}}\frac{(-1)^{|s|}}{s!t!}(\partial_x^s\partial_\xi^t a(x,\xi))(\partial_\xi^s\partial_x^t b(x,\xi)),\end{aligned} \tag{A.7}$$

where $\langle D_{\xi_1},D_{x_2}\rangle-\langle D_{x_1},D_{\xi_2}\rangle$ is the differential operator of $(\mathbb{R}^d\times\mathbb{R}^d)^2$ that can be written as $\sum_{j=1}^d\frac{\partial^2}{\partial_{\xi_{2,j}}\partial_{x_{1,j}}}-\frac{\partial^2}{\partial_{\xi_{1,j}}\partial_{x_{2,j}}}$ in any symplectic coordinates

(x,ξ) of $\mathbb{R}^d \times \mathbb{R}^d$. For instance, the first two terms are $T_0(a,b) = ab$ and $T_1(a,b) = \frac{1}{2i}\{a,b\}$ where $\{\,\cdot\,,\cdot\,\}$ is the Poisson bracket. Moreover, for any $N \in \mathbb{N}$ and $\ell \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that

$$\left\| a\#b - \sum_{n=0}^{N} T_n(a,b) \right\|^{(\ell)}_{S(MM'\lambda^{-N-1},g)} \lesssim \|a\|^{(m)}_{S(M,g)} \, \|b\|^{(m)}_{S(M',g)} \,. \tag{A.8}$$

One can check that λ is an admissible weight for g. The previous symbolic calculus gives a motivation to state the following heuristic rule:

> "*the class $S(\lambda,g)$ can be considered as the class of symbols of order* 1."

It will be convenient to set the following convention: for any $a \in S(1,g)$ and any sequence $(a_j)_{j\geqslant 0}$ of symbols satisfying $a_j \in S(\lambda^{-j},g)$, we write a symbolic expansion as follows

$$a \sim \sum_{j\geqslant 0} a_j \qquad \Leftrightarrow \qquad \forall\, k \in \mathbb{N} \quad a - \sum_{j=0}^{k} a_j \in S(\lambda^{-k},g). \tag{A.9}$$

From this point, Γ will denote the open subset $\{z \in \mathbb{C},\ 0 < |\mathrm{Im}(z)| < 1\}$. The following result, whose scheme of proof is classical, gives the parametrix of the resolvent of an "elliptic" symbol.

Proposition A.1. — *Consider an admissible weight $M \gtrsim 1$ for an admissible metric g. Consider also an integer $N \geqslant 2$ and a real symbol $p \in S(M,g)$ satisfying $1 + p \gtrsim M$. Then there are*

- *a symbol $r_N(\,\cdot\,,\cdot\,,z) \in S(\lambda^{-N},g)$ that depends of $z \in \Gamma$,*
- *an integer $N' \geqslant 2$ and symbols $q_2,\dots,q_{N'}$ where q_k belongs to the set $S(M^k\lambda^{-2},g)$ for each k,*

such that the following equality holds

$$\forall\, z \in \Gamma \quad (z-p)\#\left(\frac{1}{z-p} + \frac{q_2}{(z-p)^3} + \cdots + \frac{q_{N'}}{(z-p)^{N'+1}}\right) = 1 + r_N$$

where the remainder r_N moreover satisfies the following estimates:

$$\forall\, \ell \in \mathbb{N} \quad \exists\, m \in \mathbb{N}^\star \quad \forall\, z \in \Gamma \qquad \|r_N(\,\cdot\,,\cdot\,,z)\|^{(\ell)}_{S(\lambda^{-N},g)} \leqslant C(p,N,\ell)\frac{(1+|z|)^m}{|\mathrm{Im}(z)|^m}. \tag{A.10}$$

For $N = 2$, one may choose $N' = 2$ and $q_2 = 0$.

In the next lemma, we define for any integer $N \in \mathbb{N}$ a subfamily Υ_{-N} of (meromorphic) symbols of $S(\lambda^{-N},g)$ that appear in the construction of a parametrix.

Lemma A.2. — *Consider an admissible weight $M \gtrsim 1$ for an admissible metric g and a real symbol $p \in S(M,g)$ satisfying $1+p \gtrsim M$. For any integer $N \in \mathbb{N}$, we define Υ_{-N} the linear space of symbols $\sigma : \mathbb{R}^d \times \mathbb{R}^d \times \Gamma \to \mathbb{C}$ of the form*

$$\sigma(x,\xi,z) = \sum_{j=2}^{N'} \frac{\sigma_j(x,\xi)}{(z-p(x,\xi))^j}, \quad N' \geqslant 2, \quad \sigma_j \in S(M^j\lambda^{-N}, g). \tag{A.11}$$

Then for any integer $n \geqslant N$, one has

$$\forall\, \sigma \in \Upsilon_{-N} \qquad T_{n-N}\left(z-p, \frac{\sigma}{z-p}\right) \in \Upsilon_{-n},$$

where T_{n-N} is defined in (A.7).

Proof. — It is sufficient to check the following for any integer $j \geqslant 2$:

$$\forall\, \sigma_j \in S(M^j\lambda^{-N}, g) \qquad T_{n-N}\left(z-p, \frac{\sigma_j}{(z-p)^{j+1}}\right) \in \Upsilon_{-n}.$$

The case $n = N$ is obvious so we may assume that $n \geqslant N+1$ holds. Using (A.7), one sees that the number

$$T_{n-N}\left(z-p, \frac{\sigma_j}{(z-p)^{j+1}}\right) = T_{n-N}\left(-p, \frac{\sigma_j}{(z-p)^{j+1}}\right)$$

becomes

$$\sum_{|s+t|=n-N} C_{s,t}\left(\partial_x^s \partial_\xi^t p\right)\left(\partial_\xi^s \partial_x^t \left\{\frac{\sigma_j}{(z-p)^{j+1}}\right\}\right), \tag{A.12}$$

for suitable complex constants $C_{s,t}$. Applying the Leibniz rule rises to a formula like

$$T_{n-N}\left(z-p, \frac{\sigma_j}{(z-p)^{j+1}}\right) = \sum_{J=j+1}^{\widetilde{N}} \frac{\sigma_{j,J}}{(z-p)^J}.$$

where $\widetilde{N}$ is an integer and each symbol $\sigma_{j,J}$ is a product of p, σ_j and their derivatives. We now have to explain why $\sigma_{j,J}$ indeed belongs to $S(M^J\lambda^{-n}, g)$. To see this point, it is much interesting to fix (x,ξ) and to compute the differential expression $\langle D_{\xi_1}, D_{x_2}\rangle - \langle D_{x_1}, D_{\xi_2}\rangle$ in the symplectic basis $\mathbf{B}_{x,\xi}$ (see (A.4)) that diagonalizes the metric $g_{x,\xi}$. One still has a formula like (A.12) but the main advantage is that each derivative at (x,ξ) with respect to any direction of an element of $\mathbf{B}_{x,\xi}$ gives $\frac{1}{\sqrt{\lambda(x,\xi)}}$ as a multiplicative gain. Hence (A.12) gives $\frac{1}{\lambda(x,\xi)^{n-N}}$ as a global multiplicative gain. Moreover, each symbol $\frac{\sigma_{j,J}}{(z-p)^J}$ comes from the Leibniz rule in (A.12), so belongs to $S(\lambda^{-n},g)$. By fixing z, one sees that $\sigma_{j,J}$ belongs to $S(M^J\lambda^{-n},g)$ as announced above. $\square$

We can now prove Proposition A.1.

For any $N\geqslant 2$, we denote by $H(N)$ the following induction hypothesis: there are $\sigma\in\Upsilon_{-2}$ (see Lemma A.2) and a sequence $(A_{-j})_{j\geqslant N}$ of symbols satisfying $A_{-j}\in\Upsilon_{-j}$ for any integer $j\geqslant N$ such that the following symbolic expansion, in the sense of (A.9), holds true

$$(z-p)\#\left(\frac{1}{z-p}+\frac{\sigma}{z-p}\right)\sim 1+\sum_{j\geqslant N}A_{-j}. \tag{A.13}$$

We moreover assume that for any integers $K\geqslant N$ and $\ell\geqslant 0$, one may find an integer $m>0$ such that the following estimates hold true for any $z\in\Gamma$

$$\left\|(z-p)\#\left(\frac{1}{z-p}+\frac{\sigma}{z-p}\right)-1-\sum_{j=N}^{K-1}A_{-j}\right\|^{(\ell)}_{S(\lambda^{-K},g)}\leqslant C\frac{(1+|z|)^m}{|\mathrm{Im}(z)|^m}, \tag{A.14}$$

where C depends on (p,N,K,ℓ). Note that the conclusion of Proposition A.1 is merely the case $K=N$ (for which $\sum_{j=N}^{K-1}A_{-j}$ vanishes) but our proof needs to consider any integer $K\geqslant N$ in the induction hypothesis.

Proof of $H(2)$. — We choose $\sigma=0$. Remember that for any functions $u\in\mathcal{C}^1(\mathbb{R}^{2d},\mathbb{C}^\star)$ and $F\in\mathcal{C}^1(\mathbb{C}^\star,\mathbb{C})$, the Poisson bracket $\{u,F(u)\}=0$ vanishes. Notice now that the elliptic assumption on $1+p\gtrsim M$ ensures that $(z-p)^{-1}$ belongs to $S(M^{-1},g)$. Hence, the beginning of the symbolic calculus of $z-p\in S(M,g)$ and $(z-p)^{-1}\in S(M^{-1},g)$ gives

$$\begin{aligned}(z-p)\#\frac{1}{z-p}\sim 1+\frac{1}{2i}\left\{z-p,\frac{1}{z-p}\right\}+\sum_{n\geqslant 2}T_n\left(z-p,\frac{1}{z-p}\right)\\ \in 1+0+S(\lambda^{-2},g).\end{aligned}$$

By using (A.12), one gets a formula like

$$\forall\, n\geqslant 2\qquad T_n\left(z-p,\frac{1}{z-p}\right)=\sum_{j=2}^{\widetilde{N}}\frac{\sigma_j(x,\xi)}{(z-p(x,\xi))^j}.$$

for a suitable integer $\widetilde{N}\geqslant 2$ and symbols $\sigma_j\in S(M^j\lambda^{-n},g)$. In other words, $T_n(z-p,\frac{1}{z-p})$ belongs to Υ_{-n}.

The adequate estimates on the semi-norms of the remainders will be a consequence of the following two observations:

- the obvious relation $T_n(z-p,\frac{1}{z-p})=T_n(-p,\frac{1}{z-p})$ kills the constant symbol z,

- one has $z\#\frac{1}{z-p} = \frac{z}{z-p}$ (that is a consequence of the definition of the operation $\#$).

Hence, for any integer $K \geqslant 2$, one has

$$\begin{aligned}(z-p)\#\left(\frac{1}{z-p}\right) - 1 - \sum_{n=2}^{K-1} T_n\left(z-p, \frac{1}{z-p}\right) \\ = -p\#\left(\frac{1}{z-p}\right) + \frac{p}{z-p} - \sum_{n=2}^{K-1} T_n\left(-p, \frac{1}{z-p}\right).\end{aligned}$$

Using (A.8), for any $\ell \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that

$$\begin{aligned}&\left\|(z-p)\#\left(\frac{1}{z-p}\right) - 1 - \sum_{n=2}^{K-1} T_n(z-p, \frac{1}{z-p})\right\|^{(\ell)}_{S(\lambda^{-K},g)} \\ &\quad = \left\|-p\#\left(\frac{1}{z-p}\right) + \frac{p}{z-p} - \sum_{n=2}^{K-1} T_n(-p, \frac{1}{z-p})\right\|^{(\ell)}_{S(\lambda^{-K},g)} \\ &\quad \leqslant C(p,K,\ell)\left\|\frac{1}{z-p}\right\|^{(m)}_{S(M^{-1},g)}.\end{aligned}$$

We now adapt the argument of [32, Lemma 2.2.22, p. 80 with $f(t) = 1/t$] by using the Faà di Bruno formula. For any $k \in \mathbb{N}$ and $T \in \mathbb{R}^{2d}$, we can prove that $\partial_T^k\big(\frac{1}{z-p}\big)$ equals

$$\begin{aligned}&k! \sum_{\substack{k_1+\cdots+k_r=k \\ 1\leqslant r\leqslant k \\ k_j\geqslant 1}} \frac{1}{r!}\left.\frac{\mathrm{d}^r(1/t)}{\mathrm{d}t^r}\right|_{t=z-p} \frac{1}{k_1!\dots k_r!}\partial_T^{k_1}(z-p)\dots\partial_T^{k_r}(z-p) \\ &\quad = k!(-1)^k \sum_{\substack{k_1+\cdots+k_r=k \\ 1\leqslant r\leqslant k, k_j\geqslant 1}} \frac{(-1)^r}{(z-p)^{r+1}}\frac{1}{k_1!\dots k_r!}\partial_T^{k_1}(p)\dots\partial_T^{k_r}(p).\end{aligned}$$

Hence, we get

$$\left|\partial_T^k\left(\frac{1}{z-p}\right)\right| \lesssim \left(\sum_{r=1}^{k} |z-p(x,\xi)|^{-(r+1)} M(x,\xi)^{r+1}\right)\frac{g_{x,\xi}(T)^{k/2}}{M(x,\xi)}.$$

Thanks to (A.3), we obtain

$$\left\|\frac{1}{z-p}\right\|^{(m)}_{S(M^{-1},g)} \lesssim \sup_{\substack{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d \\ 1\leqslant k\leqslant m+1}} \frac{M(x,\xi)^k}{|z-p(x,\xi)|^k}.$$

Then we clearly have

$$\frac{M(x,\xi)}{|z-p(x,\xi)|} \lesssim \frac{1+p(x,\xi)}{|\mathrm{Im}(z)|+|\mathrm{Re(z)}-p(x,\xi)|}$$

$$\lesssim \frac{1+|\mathrm{Re}(z)-p(x,\xi)|+|\mathrm{Re(z)}|}{|\mathrm{Im}(z)|+|\mathrm{Re(z)}-p(x,\xi)|}.$$

We finally get

$$\frac{M(x,\xi)}{|z-p(x,\xi)|} \lesssim \max\left(1, \frac{1}{|\mathrm{Im}(z)|}\right) + \frac{|\mathrm{Re}(z)|}{|\mathrm{Im}(z)|}$$

$$\lesssim \frac{1+|z|}{|\mathrm{Im}(z)|}, \tag{A.15}$$

$$\left\|\frac{1}{z-p}\right\|^{(m)}_{S(M^{-1},g)} \lesssim \frac{(1+|z|)^{m+1}}{|\mathrm{Im}(z)|^{m+1}}.$$

Proof of $H(N) \Rightarrow H(N+1)$. — We consider A_{-N} that appears in (A.13). The classical idea to recursively construct the parametrix is to consider the symbolic expansion given by the symbolic calculus of $z-p$ and $\frac{-A_{-N}}{z-p}$

$$(z-p)\#\left(\frac{-A_{-N}}{z-p}\right) \sim -A_{-N} + \sum_{n\geqslant N+1} B_{-n},$$

$$B_{-n} := T_{n-N}\left(z-p, \frac{-A_{-N}}{z-p}\right) \in S(\lambda^{-n}, g).$$

Lemma A.2 ensures that B_{-n} indeed belongs to Υ_{-n}. By summing (A.13) and the previous symbolic expansion, we infer the following one

$$(z-p)\#\left(\frac{1}{z-p} + \frac{\sigma - A_{-N}}{z-p}\right) \sim 1 + \sum_{n\geqslant N+1} A_{-n} + B_{-n}.$$

Remember that one has $\sigma \in \Upsilon_{-2}$ and $A_{-N} \in \Upsilon_{-N}$. So $\sigma - A_{-N}$ belongs to Υ_{-2}. The general form predicted by the induction is checked.

To get the good estimates on the semi-norms, one may use the trick, used in the proof of $H(2)$, to eliminate the constant z in the first factor $z-p$. We indeed remark the equality $T_{n-N}(z-p, \frac{A_{-N}}{z-p}) = T_{n-N}(-p, \frac{A_{-N}}{z-p})$ and the obvious exact symbolic calculus

$$z\#\frac{A_{-N}}{z-p} = \frac{zA_{-N}}{z-p}.$$

Then we write as above for any integer $K \geqslant N$

$$(z-p)\#\left(\frac{A_{-N}}{z-p}\right) - A_{-N} + \sum_{n=N+1}^{K} B_{-n} = -p\#\left(\frac{A_{-N}}{z-p}\right) + \frac{pA_{-N}}{z-p} + \sum_{n=N+1}^{K} B_{-n}.$$

This relation and the remainder estimate (A.8) give us for any $\ell \in \mathbb{N}$ and some integer $\overline{m} \in \mathbb{N}$ an inequality of the form

$$\left\| (z-p)\#\left(\frac{A_{-N}}{z-p}\right) - A_{-N} + \sum_{n=N+1}^{K-1} B_{-n} \right\|_{S(\lambda^{-K},g)}^{(\ell)} \lesssim \left\| \frac{A_{-N}}{z-p} \right\|_{S(M^{-1}\lambda^{-N},g)}^{(\overline{m})}.$$

To bound the last term, we write A_{-N} like the right-hand side of (A.11). For suitable symbols $\sigma_j \in S(M^j\lambda^{-N})$, we have

$$\left\| \frac{A_{-N}}{z-p} \right\|_{S(M^{-1}\lambda^{-N},g)}^{(\overline{m})} \leqslant \sum_{j=2}^{N'} \left\| \frac{\sigma_j}{(z-p)^{j+1}} \right\|_{S(M^{-1}\lambda^{-N},g)}^{(\overline{m})}.$$

Using the Leibniz rule with (A.15), we get

$$\left\| (z-p)\#\left(\frac{A_{-N}}{z-p}\right) - A_{-N} + \sum_{n=N+1}^{K-1} B_{-n} \right\|_{S(\lambda^{-K},g)}^{(\ell)} \lesssim \left(\frac{(1+|z|)^{\overline{m}+1}}{|\mathrm{Im}(z)|^{\overline{m}+1}} \right)^{N'+1}. \tag{A.16}$$

Note that $1+|z|$ is larger than $|\mathrm{Im}(z)|$. Hence, the bounds from above still remain true if one increases m and $\overline{m}$ in both (A.14) and (A.16) such that $m = (\overline{m}+1)(N'+1)$. A trivial triangular inequality with (A.14) and (A.16) finally gives the expected estimates of the remainders. The assumption $H(N+1)$ is proven. □

Appendix B. An asymptotic development via the Helffer–Sjöstrand formula

We shall use the classical idea of functional calculus of a pseudo-differential operator based on the Helffer–Sjöstrand formula (see [10, Proposition 2.1], [15, Theorem 8.7], [8, Theorem 1.5] or [6, Theorem 4]). Let us fix an integer $\alpha \geqslant 1$ and a positive 2α-homogeneous polynomial $V_{2\alpha}$ on $\mathbb{R}^d$. We want to apply Proposition A.1 with

$$\begin{gathered} p(x,\xi) = |\xi|^2 + V_{2\alpha}(x), \quad M(x,\xi) = 1 + |\xi|^2 + |x|^{2\alpha}, \\ g = \sum_{j=1}^{d} \frac{\mathrm{d}x_j^2}{M^{1/\alpha}} + \frac{\mathrm{d}\xi_j^2}{M}. \end{gathered} \tag{B.1}$$

We have checked in (A.5) that the λ-function associated to the metric g is given by $\lambda = M^{\frac{\alpha+1}{2\alpha}}$. We now recall the following easy lemma.

Lemma B.1. — *For any $s > \frac{d}{2}$ and any $r \in [1, +\infty]$, one has the continuous embeddings*

$$\mathcal{H}_\alpha^s(\mathbb{R}^d) \subset L^r(\mathbb{R}^d) \subset \mathcal{H}_\alpha^{-s}(\mathbb{R}^d) \quad \textit{and} \quad \mathcal{H}_\alpha^{s+1}(\mathbb{R}^d) \subset W^{1,r}(\mathbb{R}^d),$$

where $W^{1,r}(\mathbb{R}^d)$ stands for the usual Sobolev space.

Proof. — Looking at (4.4), we have already remarked that the Sobolev embedding $H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ implies the inclusion $\mathcal{H}_\alpha^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$. Now remark for any $s' > \frac{d}{2\alpha}$ that the inclusion $\mathcal{H}_\alpha^{s'}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ holds true: for any $u \in L^1(\mathbb{R}^d)$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x)| \mathrm{d}x &\leqslant \sqrt{\int_{\mathbb{R}^d} (1+|x|)^{-2\alpha s'} \mathrm{d}x} \sqrt{\int_{\mathbb{R}^d} (1+|x|)^{2\alpha s'} |u(x)|^2 \,\mathrm{d}x} \\ &\lesssim \|u\|_{\mathcal{H}_\alpha^{s'}(\mathbb{R}^d)} . \end{aligned}$$

As a consequence, we obtain

$$\mathcal{H}_\alpha^s(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subset L^r(\mathbb{R}^d). \tag{B.2}$$

The other side is obtained by duality. To prove the inclusion involving $W^{1,r}(\mathbb{R}^d)$, we first remark that for any $j \in \{1, \dots, d\}$, the pseudo-differential operator $\frac{\partial}{\partial x_j} \circ (1 - \Delta + |x|^{2\alpha})^{-1/2}$ admits a symbol belonging to

$$S(\sqrt{M} M^{-1/2}, g) = S(1, g).$$

Hence, $\frac{\partial}{\partial x_j} \circ (1 - \Delta + |x|^{2\alpha})^{-1/2}$ is bounded on any Sobolev space and we infer the following

$$\begin{aligned} \frac{\partial}{\partial x_j} \mathcal{H}_\alpha^{s+1}(\mathbb{R}^d) &= \frac{\partial}{\partial x_j} \circ (1 - \Delta + |x|^{2\alpha})^{-1/2} \circ (1 - \Delta + |x|^{2\alpha})^{1/2} \mathcal{H}_\alpha^{s+1}(\mathbb{R}^d) \\ &\subset \mathcal{H}_\alpha^s(\mathbb{R}^d). \end{aligned}$$

Using twice (B.2), we get the inclusion $\mathcal{H}_\alpha^{s+1}(\mathbb{R}^d) \subset W^{1,r}(\mathbb{R}^d)$. □

We are ready for the Helffer–Sjöstrand formula.

Proposition B.2. — *Let us consider a smooth function* $\Psi : \mathbb{R} \to \mathbb{R}$, *with compact support. There are an integer* $N' \geqslant 2$ *and symbols* $q_2, \ldots, q_{N'}$, *where* q_k *belongs to* $S(M^k\lambda^{-2}, g)$ *for any* k, *such that for any* $r \in [1, +\infty]$ *and any* $u \in L^r(\mathbb{R}^d)$, *the following function from* $(0,1]$ *to the Sobolev space* $W^{1,r}(\mathbb{R}^d)$

$$h \mapsto \Psi(-h^2\Delta + h^2V_{2\alpha})u - Op^w\left(\Psi(h^2p)\right)u - \sum_{k=2}^{N'} h^{2k} Op^w\left(\Psi^{(k)}(h^2p)q_k\right)u$$

is uniformly bounded by $C\left\|u\right\|_{L^r(\mathbb{R}^d)}$.

Before proving Proposition B.2, we point out two things:

- Proposition B.2 is *not semi-classical* because the parameter h^2 is also in front of $V_{2\alpha}$. This is the reason why the symbols appearing are not of the form $q_k(x, h\xi)$,
- it is not necessary to show that $\Psi(-h^2\Delta + h^2V_{2\alpha})$ is a pseudo-differential operator. However, the proof needs that some powers of $-\Delta + V_{2\alpha}$ are pseudo-differential operators (as in the proof of Lemma B.1). The proof of Proposition B.2 is divided in the following steps.

Step 1. — We need the notion of "almost analytic extension" for which we refer for instance to [15, Chapter 8]. We recall that the function $\Psi : \mathbb{R} \to \mathbb{R}$ of the statement of Proposition B.2 admits an almost analytic extension $\Psi : \mathbb{C} \to \mathbb{C}$ in the following sense:

$$\text{(B.3)} \qquad \forall\, m \in \mathbb{N} \qquad \exists\, C_m > 0 \qquad \forall\, z \in \mathbb{C} \qquad \left|\frac{\partial \Psi}{\partial \overline{z}}\right| \leqslant C_m |\mathrm{Im}(z)|^m.$$

Moreover, such an extension $\Psi : \mathbb{C} \to \mathbb{C}$ can be constructed such that its support is as close as wanted to the one of $\Psi : \mathbb{R} \to \mathbb{R}$. As a consequence, we can assume that the support of $\Psi : \mathbb{C} \to \mathbb{C}$ is included in the strip $\Gamma = \{z \in \mathbb{C}, |\mathrm{Im}(z)| < 1\}$. The Helffer–Sjöstrand formula then reads

$$\text{(B.4)} \quad \Psi(-h^2\Delta + h^2V_{2\alpha}) = \frac{-1}{\pi}\int_{\mathbb{C}} \frac{\partial}{\partial \overline{z}}\left\{\Psi(h^2z)\right\}(z + \Delta - V_{2\alpha})^{-1}\mathrm{d}L(z),$$

where $\mathrm{d}L(z)$ stands for the Lebesgue measure on $\mathbb{C}$. We now need a parametrix of the resolvent of $-\Delta + V_{2\alpha}$ as in Proposition A.1. For any integer $N \geqslant 2$ (to be chosen below at the end of Step 2), we get the following formula for a suitable integer $N' \geqslant 2$ and suitable symbols $q_2, \dots, q_{N'}$:

(B.5) $\forall\, z \in \Gamma$

$$(z - \Delta + V_{2\alpha})^{-1} = \mathrm{Op}^w\left(\frac{1}{z-p} + \sum_{k=2}^{N'} \frac{q_k}{(z-p)^{k+1}}\right) - R_N(z),$$

with

$$R_N(z) := (z - \Delta + V_{2\alpha})^{-1} \circ \mathrm{Op}^w(r_N(z))$$

where the symbol $r_N(z)$ belongs to $S(\lambda^{-N}, g)$ for any $z \in \Gamma$ and satisfies (A.10).

Step 2. — We now explain how to bound the norm

$$\|R_N(z)\|_{L^\infty(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)}$$

with respect to z. We recall that $(1-\Delta+V_{2\alpha})^{\frac{N(\alpha+1)}{2\alpha}}$ is a pseudo-differential operator (see for instance [19] or [6, Theorem 4]) and

$$(1 - \Delta + V_{2\alpha})^{\frac{N(\alpha+1)}{2\alpha}} \in \mathrm{Op}^w S(M^{\frac{N(\alpha+1)}{2\alpha}}, g)$$
$$(1 - \Delta + V_{2\alpha})^{\frac{N(\alpha+1)}{2\alpha}} \in \mathrm{Op}^w S(\lambda^N, g).$$

As a consequence, we get

$$(1 - \Delta + V_{2\alpha})^{\frac{N(\alpha+1)}{2\alpha}} \circ \mathrm{Op}^w(r_N(z)) \in \mathrm{Op}^w S(1, g).$$

And thus, the last operator is bounded on any Sobolev space based on $-\Delta + V_{2\alpha}$. Let us fix a real number $s' > \frac{d}{2}$ and use the classical estimates of the norm operator of a pseudo-differential operator with (A.10), we get for a suitable integer m_0 the following estimate

$$\left\|(1 - \Delta + V_{2\alpha})^{\frac{N(\alpha+1)}{2\alpha}} \circ \mathrm{Op}^w(r_N(z))\right\|_{\mathcal{H}_\alpha^{-s'}(\mathbb{R}^d)\to\mathcal{H}_\alpha^{-s'}(\mathbb{R}^d)} \lesssim \frac{(1+|z|)^{m_0}}{|\mathrm{Im}(z)|^{m_0}}.$$

The behavior of the last bound with respect to s' (both in the multiplicative loss and the exponent m_0) is not relevant and we are merely interested in its dependence with respect to z. Writing

$$\mathrm{Op}^w(r_N(z)) = (1 - \Delta + V_{2\alpha})^{-\frac{N(\alpha+1)}{2\alpha}} \circ (1 - \Delta + V_{2\alpha})^{\frac{N(\alpha+1)}{2\alpha}} \circ \mathrm{Op}^w(r_N(z)),$$

we also obtain

$$\|\mathrm{Op}^w(r_N(z))\|_{\mathcal{H}_\alpha^{-s'}(\mathbb{R}^d)\to\mathcal{H}_\alpha^{\frac{N(\alpha+1)}{\alpha}-s'}(\mathbb{R}^d)} \lesssim \frac{(1+|z|)^{m_0}}{|\mathrm{Im}(z)|^{m_0}}.$$

If one considers $-\Delta + V_{2\alpha}$ as an operator of the Hilbert space $\mathcal{H}_\alpha^{\frac{N(\alpha+1)}{\alpha}-s'}(\mathbb{R}^d)$ with domain $\mathcal{H}_\alpha^{\frac{N(\alpha+1)}{\alpha}-s'+2}(\mathbb{R}^d)$, then one clearly gets a self-adjoint operator. One consequently obtains

$$\forall\, z \in \mathbb{C}\backslash\mathbb{R} \quad \left\|(z-\Delta+V_{2\alpha})^{-1}\right\|_{\mathcal{H}_\alpha^{\frac{N(\alpha+1)}{\alpha}-s'}(\mathbb{R}^d)\to\mathcal{H}_\alpha^{\frac{N(\alpha+1)}{\alpha}-s'}(\mathbb{R}^d)} \leqslant \frac{1}{|\mathrm{Im}(z)|}.$$

Combining the previous inequalities, we have proved

$$\|R_N(z)\|_{\mathcal{H}_\alpha^{-s'}(\mathbb{R}^d)\to\mathcal{H}_\alpha^{\frac{N(\alpha+1)}{\alpha}-s'}(\mathbb{R}^d)} \lesssim \frac{(1+|z|)^{m_0}}{|\mathrm{Im}(z)|^{m_0+1}}.$$

Thanks to Lemma B.1, one may choose N large enough so that the inclusion $\mathcal{H}_\alpha^{\frac{N(\alpha+1)}{\alpha}-s'}(\mathbb{R}^d) \subset W^{1,r}(\mathbb{R}^d)$ holds true. As we have chosen $s' > \frac{d}{2}$, we finally get

$$\|R_N(z)\|_{L^r(\mathbb{R}^d)\to W^{1,r}(\mathbb{R}^d)} \lesssim \frac{(1+|z|)^{m_0}}{|\mathrm{Im}(z)|^{m_0+1}}.$$

Step 3. — We come back to the almost analytic extension $\Psi : \mathbb{C} \to \mathbb{C}$ and we assume that Ψ has support in $\{z \in \mathbb{C}, |z| \leqslant \rho\}$. We now choose $m = m_0 + 1$ in (B.3) to get, uniformly in $h \in (0,1]$, the following bounds

$$\begin{aligned}
&\left\|\int_{\mathbb{C}} \frac{\partial}{\partial \overline{z}}\left\{\Psi(h^2 z)\right\} R_N(z)\mathrm{d}L(z)\right\|_{L^r(\mathbb{R}^d)\to W^{1,r}(\mathbb{R}^d)} \\
&\qquad\lesssim \int_{|h^2 z|\leqslant\rho} h^2 \left|\frac{\partial\{\Psi\}}{\partial\overline{z}}(h^2 z)\right| \frac{(1+|z|)^{m_0}}{|\mathrm{Im}(z)|^{m_0+1}}\mathrm{d}L(z) \\
&\qquad\lesssim \frac{\rho^2}{h^4} \sup_{|z|\leqslant h^{-2}\rho} h^2|\mathrm{Im}(h^2 z)|^{m_0+1} \frac{(1+|z|)^{m_0}}{|\mathrm{Im}(z)|^{1+m_0}} \\
&\qquad\lesssim \rho^2 \sup_{|z|\leqslant h^{-2}\rho} h^{2m_0}(1+|z|^{m_0}) \\
&\qquad\lesssim 1.
\end{aligned}$$

A classical computation, as used for instance in [15, p. 103] or [10, p. 577], based on the Cauchy formula and the fact that $\frac{1}{\pi z}$ is a fundamental solution of the Cauchy–Riemann operator $\frac{\partial}{\partial\overline{z}}$, shows the following equality

$$\forall\, k \in \mathbb{N} \quad \frac{-1}{\pi}\int_{\mathbb{C}} \frac{1}{(z-p)^{k+1}} \frac{\partial}{\partial\overline{z}}\left\{\Psi(h^2 z)\right\}\mathrm{d}L(z) = \frac{h^{2k}}{k!}\Psi^{(k)}(h^2 p).$$

Combining the Helffer–Sjöstrand formula (B.4) and (B.5), we can finish the proof of Proposition B.2. $\square$

Appendix C. About $L^r \to L^r$ estimates of pseudo-differential operators

A pseudo-differential of order 0 on $\mathbb{R}^d$, in a suitable setting, is expected to be bounded on $L^2(\mathbb{R}^d)$. The case $L^p(\mathbb{R}^d)$, with finite p, has also been studied in the literature. We have not found known estimates of the case $L^\infty(\mathbb{R}^d)$ that can be appropriate for our needs. For this reason, we prove elementary results that covers this extremal case. Before stating our results, we make a remark that will motivate the sequel. Consider $r \in [1, +\infty]$ and a symbol $\sigma \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$. We expect to bound $\|\mathrm{Op}^w(\sigma)\|_{L^r(\mathbb{R}^d)\to L^r(\mathbb{R}^d)}$ by a suitable norm on σ (and certainly its derivatives). For any $\lambda > 0$, consider now the following linear isometry of $L^r(\mathbb{R}^d)$:

$$\begin{aligned} U_{\lambda,r} : L^r(\mathbb{R}^d) &\to L^r(\mathbb{R}^d) \\ f &\mapsto (x \mapsto \lambda^{\frac{d}{r}} f(\lambda x)). \end{aligned}$$

Due to the definition (A.1) of the Weyl quantization, one easily checks the formula

$$U_{\lambda,r}\mathrm{Op}^w(\sigma)U_{\lambda,r}^{-1} = \mathrm{Op}^w\left(\sigma\left(\lambda x, \frac{\xi}{\lambda}\right)\right).$$

This remark implies that any reasonable bound from above of the norm $\|\mathrm{Op}^w(\sigma)\|_{L^r(\mathbb{R}^d)\to L^r(\mathbb{R}^d)}$ should be scale-invariant by replacing the symbol σ with $\sigma\big(\lambda x, \frac{\xi}{\lambda}\big)$. This is the purpose of the following result relying on the Schur test.

Proposition C.1. — *For any integer $s > \frac{d}{2}$, there is $C(d,s) > 0$ such that for any symbol $\sigma \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$ with compact support, for any $r \in [1, +\infty]$, the norm $\|Op^w(\sigma)\|_{L^r(\mathbb{R}^d)\to L^r(\mathbb{R}^d)}$ is bounded by*

$$C(d,s)\left(\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\sigma(x,\xi)|\mathrm{d}\xi\right)^{1-\frac{d}{2s}}\left(\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|(-\Delta_\xi)^s\sigma(x,\xi)|\mathrm{d}\xi\right)^{\frac{d}{2s}}.$$

Proof. — By interpolating, one merely has to prove the cases $r = 1$ and $r = +\infty$. From (A.1), the Schwartz kernel of $\mathrm{Op}^w(\sigma)$ is clearly given by

$$K(x,y) = \int_{\mathbb{R}^d} e^{i\langle x-y,\xi\rangle}\sigma\left(\frac{x+y}{2},\xi\right)\frac{\mathrm{d}\xi}{(2\pi)^d}.$$

Hence, the bounds $L^\infty \to L^\infty$ and $L^1 \to L^1$ will respectively come from the bounds of

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|K(x,y)|\mathrm{d}y \quad \text{and} \quad \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}|K(x,y)|\mathrm{d}x. \tag{C.1}$$

Since σ has a compact support, we may integrate by parts with the differential operator $I+(-\Delta_\xi)^s$ to get

$$|K(x,y)| = \left|\int_{\mathbb{R}^d} \frac{e^{i\langle x-y,\xi\rangle}}{1+|x-y|^{2s}}(I+(-\Delta_\xi)^s)\sigma\left(\frac{x+y}{2},\xi\right)\frac{\mathrm{d}\xi}{(2\pi)^d}\right|$$

which is less than

$$\begin{aligned}&\frac{1}{1+|x-y|^{2s}}\sup_{X\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\sigma(X,\xi)|+|(-\Delta)^s\sigma(X,\xi)|\frac{\mathrm{d}\xi}{(2\pi)^d}\\ &\quad\leqslant \frac{(2\pi)^{-d}}{1+|x-y|^{2s}}\left(\sup_{X\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\sigma(X,\xi)|\mathrm{d}\xi+\sup_{X\in\mathbb{R}^d}\int_{\mathbb{R}^d}|(-\Delta)^s\sigma(X,\xi)|\mathrm{d}\xi\right).\end{aligned}$$

As a result, the two suprema in (C.1) are bounded by

$$C(d,s)\left(\sup_{X\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\sigma(X,\xi)|\mathrm{d}\xi+\sup_{X\in\mathbb{R}^d}\int_{\mathbb{R}^d}|(-\Delta)^s\sigma(X,\xi)|\mathrm{d}\xi\right).$$

From the remark made in the beginning of this part, we may improve the last bound by

$$C(d,s)\inf_{\lambda>0}\left(\sup_{X\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\sigma_\lambda(X,\xi)|\mathrm{d}\xi+\sup_{X\in\mathbb{R}^d}\int_{\mathbb{R}^d}|(-\Delta)^s\sigma_\lambda(X,\xi)|\mathrm{d}\xi\right)$$

where we denote by σ_λ the symbol $\sigma(\lambda x,\xi/\lambda)$. This reduces to

$$C(d,s)\inf_{\lambda>0}\left[\lambda^d\sup_{X\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\sigma(X,\xi)|\mathrm{d}\xi+\lambda^{-(2s-d)}\sup_{X\in\mathbb{R}^d}\int_{\mathbb{R}^d}|(-\Delta_\xi)^s\sigma(X,\xi)|\mathrm{d}\xi\right].$$

The conclusion comes from the easy fact

$$\forall\, A,B\geqslant 0\qquad \inf_{\lambda>0}\lambda^dA+\lambda^{-(2s-d)}B\simeq A^{1-\frac{d}{2s}}B^{\frac{d}{2s}}.\qquad\square$$

The previous result allows to prove two corollaries needed to deal with the pseudo-differential operators appearing in Proposition B.2.

Corollary C.2. — *Consider an integer $\alpha\geqslant 1$ and a positive 2α-homogeneous polynomial $V_{2\alpha}$ on $\mathbb{R}^d$. Set now $p(x,\xi)=|\xi|^2+V_{2\alpha}(x)$. For any smooth function $\Psi:\mathbb{R}\to\mathbb{R}$, with compact support, and any smooth function $q:\mathbb{R}\to\mathbb{C}$, the following estimate holds true*

$$\forall\, r\in[1,+\infty]\qquad \sup_{0<h\leqslant 1}\left\|Op^w(\Psi(h^2p)q(h\xi))\right\|_{L^r(\mathbb{R}^d)\to L^r(\mathbb{R}^d)}<+\infty.$$

Proof. — Let us assume that Ψ has a support in $[-R, R]$ for some real number $R > 0$. For any $x \in \mathbb{R}^d$ and $h \in (0,1]$, we can write

$$\begin{aligned}\int_{\mathbb{R}^d} &|\Psi(h^2 V_{2\alpha}(x) + h^2|\xi|^2) q(h\xi)| \mathrm{d}\xi \\ &= h^{-d} \int_{\mathbb{R}^d} |\Psi(h^2 V_{2\alpha}(x) + |\xi|^2) q(\xi)| \mathrm{d}\xi \\ &\leqslant h^{-d} \operatorname{Vol}_{\mathbb{R}^d}(B(0,\sqrt{R})) \, \|\Psi\|_{L^\infty(\mathbb{R}^d)} \, \|q\|_{L^\infty(B(0,\sqrt{R}))} = \mathcal{O}(h^{-d}).\end{aligned}$$

Fix now an integer $s > \frac{d}{2}$ and let us write

$$\begin{aligned}\int_{\mathbb{R}^d} &|(-\Delta_\xi)^{(s)} \{\Psi(h^2 V_{2\alpha}(x) + h^2|\xi|^2) q(h\xi)\}| \mathrm{d}\xi \\ &= h^{2s-d} \int_{\mathbb{R}^d} |(-\Delta_\xi)^{(s)} \{\Psi(h^2 V_{2\alpha}(x) + |\xi|^2) q(\xi)\}| \mathrm{d}\xi,\end{aligned}$$

which is uniformly bounded with respect to x, with a similar argument, by $\mathcal{O}(h^{2s-d})$. Proposition C.1 gives us the expected bound for any $r \in [1, +\infty]$:

$$\begin{aligned}\text{(C.2)} \quad &\left\|\operatorname{Op}^w(\Psi(h^2 p) q(h\xi))\right\|_{L^r(\mathbb{R}^d) \to L^r(\mathbb{R}^d)} \\ &\leqslant \mathcal{O}\left((h^{-d})^{1-\frac{d}{2s}} (h^{2s-d})^{\frac{d}{2s}}\right) = \mathcal{O}(1). \quad \square\end{aligned}$$

For the next result, we stress that we consider cut-off functions whose support is far from 0. This is a little trick that will allow to deal with the terms $h^{2k}\operatorname{Op}^w(\Psi^{(k)}(h^2 p) q_k)$, $k \geqslant 1$, of Proposition B.2 under the reasonable additional assumption that Ψ is identically constant near 0.

Corollary C.3. — *As above, we consider an integer $\alpha \geqslant 1$, a positive 2α-homogeneous polynomial $V_{2\alpha}$ on $\mathbb{R}^d$. Consider as above the symbol p, the weight M and the admissible metric g associated to $-\Delta + V_{2\alpha}$ defined in* (B.1). *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a smooth function whose support is compact and included in $(0, +\infty)$. Fix $\eta \in \mathbb{R}$ and let q be a symbol belonging to the class $S(M^\eta, g)$. Then the following holds*

$$\forall\, r \in [1, +\infty] \quad \forall\, h \in]0,1] \quad \left\|Op^w(\Phi(h^2 p) q)\right\|_{L^r(\mathbb{R}^d) \to L^r(\mathbb{R}^d)} \lesssim h^{-2\eta}.$$

Proof. — As for Corollary C.2, we aim to use Proposition C.1. For any $x \in \mathbb{R}^d$ and $h \in (0,1]$, we begin by writing

$$\begin{aligned}\int_{\mathbb{R}^d} &|\Phi(h^2 V_{2\alpha}(x) + h^2|\xi|^2) q(x,\xi)| \mathrm{d}\xi \\ &\lesssim \int_{\mathbb{R}^d} |\Phi(h^2 V_{2\alpha}(x) + h^2|\xi|^2)| (1 + |x|^{2\alpha} + |\xi|^2)^\eta \mathrm{d}\xi.\end{aligned}$$

Since the support of Φ is far from 0, one has $V_{2\alpha}(x)+|\xi|^2 \simeq h^{-2}$ in the last integral. As h belongs to $(0,1]$ and $V_{2\alpha}$ is 2α-homogeneous, we also have the equivalence $1+|x|^{2\alpha}+|\xi|^2 \simeq h^{-2}$. Whatever the sign of η is, a similar argument than that of the proof of Corollary C.2 gives us

$$\begin{aligned}\int_{\mathbb{R}^d} |\Phi(h^2V_{2\alpha}(x)+h^2|\xi|^2)q(x,\xi)|\mathrm{d}\xi &\lesssim h^{-2\eta}\int_{\mathbb{R}^d} |\Phi(h^2V_{2\alpha}(x)+h^2|\xi|^2)|\mathrm{d}\xi \\ &\lesssim h^{-2\eta-d}\int_{\mathbb{R}^d} |\Phi(h^2V_{2\alpha}(x)+|\xi|^2)|\mathrm{d}\xi. \\ &\lesssim \mathcal{O}(h^{-2\eta-d}).\end{aligned}$$

Choose now an integer $s>\frac{d}{2}$ and make use of the Leibniz rule:

$$\begin{aligned}&\int_{\mathbb{R}^d} \left|(-\Delta_\xi)^{(s)}\{\Phi(h^2|x|^{2\alpha}+h^2|\xi|^2)q(x,\xi)\}\right|\mathrm{d}\xi \\ &\quad\leqslant C(s)\sum_{\substack{(s_1,s_2)\in\mathbb{N}^d\times\mathbb{N}^d\\ |s_1|+|s_2|=2s}}\int_{\mathbb{R}^d} |D_\xi^{s_1}\{\Phi(h^2|x|^{2\alpha}+h^2|\xi|^2)\}||D_\xi^{s_2}\{q(x,\xi)\}|\mathrm{d}\xi.\end{aligned}$$

We now use the gain of derivatives in ξ of the symbol $q\in S(M^\eta,g)$ (see (B.1)). We then get

$$\begin{aligned}&\int_{\mathbb{R}^d} \left|(-\Delta_\xi)^{(s)}\{\Phi(h^2V_{2\alpha}(x)+h^2|\xi|^2)q(x,\xi)\}\right|\mathrm{d}\xi \\ &\lesssim \sum_{\substack{(s_1,s_2)\in\mathbb{N}^d\times\mathbb{N}^d\\ |s_1|+|s_2|=2s}}\int_{\mathbb{R}^d} |D_\xi^{s_1}\{\Phi(h^2V_{2\alpha}(x)+h^2|\xi|^2)\}|(1+|x|^{2\alpha}+|\xi|^2)^{\eta-\frac{|s_2|}{2}}\mathrm{d}\xi \\ &\lesssim \sum_{\substack{(s_1,s_2)\in\mathbb{N}^d\times\mathbb{N}^d\\ |s_1|+|s_2|=2s}} h^{-2\eta+|s_2|}\int_{\mathbb{R}^d} |D_\xi^{s_1}\{\Phi(h^2V_{2\alpha}(x)+h^2|\xi|^2)\}|\mathrm{d}\xi \\ &\lesssim \sum_{\substack{(s_1,s_2)\in\mathbb{N}^d\times\mathbb{N}^d\\ |s_1|+|s_2|=2s}} h^{-2\eta+|s_2|+|s_1|-d}\int_{\mathbb{R}^d} |D_\xi^{s_1}\{\Phi(h^2V_{2\alpha}(x)+|\xi|^2)\}|\mathrm{d}\xi,\end{aligned}$$

which is $\mathcal{O}(h^{-2\eta+2s-d})$. Again, Proposition C.1 gives the conclusion with the same computation made in (C.2). □

Appendix D. Proof of Theorem 5.3

We shall first prove the following result.

THEOREM D.1. — *Consider an integer $\alpha \geqslant 1$ and a 2α-homogeneous positive*[1] *polynomial $V_{2\alpha}$ on $\mathbb{R}^d$. For any smooth function $\Psi : \mathbb{R} \to \mathbb{R}$, with compact support, and that is identically constant near 0, there is $C > 0$ such that for any $r \in [1, +\infty]$ and any $f \in L^r(\mathbb{R}^d)$ the following holds true*

$$\sup_{0<h\leqslant 1} \left\| \Psi(-h^2\Delta + h^2 V_{2\alpha}) f \right\|_{L^r(\mathbb{R}^d)} \leqslant C \left\| f \right\|_{L^r(\mathbb{R}^d)}. \tag{D.1}$$

$$\sup_{0<h\leqslant 1} \left\| \nabla\{\Psi(-h^2\Delta + h^2 V_{2\alpha}) f\} \right\|_{L^r(\mathbb{R}^d)} \leqslant \frac{C}{h} \left\| f \right\|_{L^r(\mathbb{R}^d)}. \tag{D.2}$$

It is easy to check that Theorem D.1 implies Theorem 5.3. We firstly remark that any $f \in L^2(\mathbb{R}^d)$ that is spectrally localized in $[0, \rho]$ (with respect to $-\Delta + V_{2\alpha}$) automatically belongs to any Sobolev space $\mathcal{H}^s_\alpha(\mathbb{R}^d)$, with $s \gg 1$, so belongs to $L^r(\mathbb{R}^d)$ (see Lemma B.1). Choose now a function $\Psi \in \mathcal{C}^\infty_0(\mathbb{R}, \mathbb{R})$ such that $\Psi \equiv 1$ on $[0, 1]$ and set $h = \frac{1}{\sqrt{\rho}}$, we immediately obtain the equality $\psi(-h^2\Delta + h^2 V_{2\alpha}) f = f$. Hence, (D.2) implies the conclusion of Theorem 5.3. The inequality (D.1) is useless for us but its proof is easier than that of (D.2) and may be useful for further developments.

Let us now explain the proof of Theorem D.1. We use the same notations as those of Appendix B (in particular (B.1)) and we apply Proposition B.2. We shall merely invoke that each symbol q_k belongs to $S(M^k, g)$ instead of $S(M^k\lambda^{-2}, g)$.

Proof of (D.1). — Corollary C.2 (with $q = 1$) and Corollary C.3 (with $\Phi = \Psi^{(k)}$ and $q_k \in S(M^k, g)$) respectively give us

$$\sup_{h\in(0,1]} \left\| \mathrm{Op}^w(\Psi(h^2 p)) \right\|_{L^r(\mathbb{R}^d)\to L^r(\mathbb{R}^d)} < +\infty,$$

$$\forall\, k \in \{2, \ldots, N'\} \sup_{h\in(0,1]} \left\| h^{2k} \mathrm{Op}^w(\Psi^{(k)}(h^2 p) q_k) \right\|_{L^r(\mathbb{R}^d)\to L^r(\mathbb{R}^d)} < +\infty. \quad \square$$

Proof of (D.2). — It is sufficient to prove for any $j \in \{1, \ldots, d\}$:

$$\left\| \frac{\partial}{\partial x_j} \mathrm{Op}^w(\Psi(h^2 p)) \right\|_{L^r(\mathbb{R}^d)\to L^r(\mathbb{R}^d)} \leqslant \frac{C_j}{h}, \tag{D.3}$$

(D.4)

$$\forall\, k \in \{2, \ldots, N'\} \quad \left\| \frac{\partial}{\partial x_j} h^{2k} \mathrm{Op}^w\left(\Psi^{(k)}(h^2 p) q_k\right) \right\|_{L^r(\mathbb{R}^d)\to L^r(\mathbb{R}^d)} \leqslant \frac{C_{k,j}}{h}.$$

[1] we recall that "positive" means positive on $\mathbb{R}^d \backslash \{0\}$, see Section 4.

For any $u \in \mathcal{S}(\mathbb{R}^d)$, the definition (A.1) ensures that $\frac{\partial}{\partial x_j}\mathrm{Op}^w(\Psi(h^2p))u(x)$ equals

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} i\xi_j e^{i\langle x-y,\xi\rangle}\Psi\left(h^2p\left(\frac{x+y}{2},\xi\right)\right)u(y)\frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi)^d}$$
$$+\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{i\langle x-y,\xi\rangle}\frac{h^2}{2}\Psi'\left(h^2p\left(\frac{x+y}{2},\xi\right)\right)\partial_{x_j}p\left(\frac{x+y}{2},\xi\right)u(y)\frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi)^d}.$$

In other words, we have

$$\frac{\partial}{\partial x_j}\mathrm{Op}^w(\Psi(h^2p)) = \mathrm{Op}^w\left(i\xi_j\Psi(h^2p)\right) + \mathrm{Op}^w\left(\frac{h^2}{2}\Psi'(h^2p)\partial_{x_j}p\right).$$

Notice that the last formula is predicted by the exact symbolic calculus (A.6) and (A.7). For the metric $g=\sum_{\ell=1}^d \frac{\mathrm{d}x_\ell^2}{M^{1/\alpha}}+\frac{\mathrm{d}\xi_\ell^2}{M}$, the symbol $\partial_{x_j}p$ belongs to $S(M^{1-\frac{1}{2\alpha}},g)$. Corollary C.2 (with $q(h\xi)=ih\xi_1$) and Corollary C.3 allow us to get (D.3):

$$\left\|\frac{\partial}{\partial x_j}\mathrm{Op}^w(\Psi(h^2p))\right\|_{L^r(\mathbb{R}^d)\to L^r(\mathbb{R}^d)} \lesssim \frac{1}{h}+h^2h^{-2\left(1-\frac{1}{2\alpha}\right)}\lesssim\frac{1}{h}.$$

Proving the inequality (D.4) is essentially similar because of the following equality

$$\begin{aligned}\frac{\partial}{\partial x_j}&h^{2k}\mathrm{Op}^w(\Psi^{(k)}(h^2p)q_k)\\ &= h^{2k}\mathrm{Op}^w(i\xi_j\Psi^{(k)}(h^2p)q_k)+h^{2k}\mathrm{Op}^w\left(\frac{h^2}{2}\Psi^{(k+1)}(h^2p)q_k\partial_{x_j}p\right)\\ &\qquad + h^{2k}\mathrm{Op}^w\left(\Psi^{(k)}(h^2p)\frac{1}{2}\partial_{x_j}q_k\right).\end{aligned}$$

Note that $\xi_jq_k=\frac{\partial M}{2\partial\xi_j}q_k$ clearly belongs to $S(M^{k+\frac12},g)$, that $q_k\partial_{x_j}p$ belongs to $S(M^{k+1-\frac{1}{2\alpha}},g)$ and $\partial_{x_j}q_k$ belongs to $S(M^{k-\frac{1}{2\alpha}},g)$. Thanks to Corollary C.3, we obtain

$$\begin{aligned}&\left\|\frac{\partial}{\partial x_j}h^{2k}\mathrm{Op}^w(\Psi^{(k)}(h^2p)q_k)\right\|_{L^r(\mathbb{R}^d)\to L^r(\mathbb{R}^d)}\\ &\qquad\qquad \lesssim \frac{1}{h}+h^2h^{-2(1-\frac{1}{2\alpha})}+h^{2\times\frac{1}{2\alpha}}\simeq\frac{1}{h}. \qquad \square\end{aligned}$$

Appendix E. About unitarily invariant random vectors

In the sequel, we denote by $\|\cdot\|$ the Euclidean norm of $\mathbb{C}^d$.

LEMMA E.1. — *Let $\Upsilon : \Omega \to \mathbb{C}^d$ be a random vector that almost surely never vanishes and whose law is unitarily invariant, then*

(1) *$\|\Upsilon\|$ and $\frac{\Upsilon}{\|\Upsilon\|}$ (which is almost surely well defined) are independent,*
(2) *the law of $\frac{\Upsilon}{\|\Upsilon\|}$ is the normalized volume measure of the sphere $\mathbb{S}^{2d-1} \subset \mathbb{C}^d$.*

Proof. — The law of the random vector $\frac{\Upsilon}{\|\Upsilon\|}$ is a unitarily invariant Borel probability measure on the sphere $\mathbb{S}^{2d-1}$, hence (2) is proved. It remains to prove (1). Considering two measurable subsets $\mathcal{A} \subset [0, +\infty)$ and $\mathcal{B} \subset \mathbb{S}^{2d-1}$, we want to show the equality

$$\mathbf{P}\left(\|\Upsilon\| \in \mathcal{A} \quad \text{and} \quad \frac{\Upsilon}{\|\Upsilon\|} \in \mathcal{B}\right) = \mathbf{P}(\|\Upsilon\| \in \mathcal{A})\mathbf{P}\left(\frac{\Upsilon}{\|\Upsilon\|} \in \mathcal{B}\right),$$

which indeed means

$$\mathbf{P}(\Upsilon \in \mathcal{A}\mathcal{B}) = \mathbf{P}(\|\Upsilon\| \in \mathcal{A})\mathbf{P}\left(\frac{\Upsilon}{\|\Upsilon\|} \in \mathcal{B}\right). \tag{E.1}$$

Considering ν_d the normalized Haar measure of the unitary group $U_d(\mathbb{C})$, we clearly have

$$\begin{aligned} \forall\, p \in U_d(\mathbb{C}) \quad \mathbf{P}(\Upsilon \in \mathcal{A}\mathcal{B}) &= \mathbf{P}(p\Upsilon \in \mathcal{A}\mathcal{B}), \\ \mathbf{P}(\Upsilon \in \mathcal{A}\mathcal{B}) &= \int_{U_d(\mathbb{C})} \mathbf{P}(p\Upsilon \in \mathcal{A}\mathcal{B})\mathrm{d}\nu_d(p) \\ &= \mathbf{E}_\omega\left[\int_{U_d(\mathbb{C})} \mathbf{1}_{\mathcal{A}\mathcal{B}}(p\Upsilon(\omega))\mathrm{d}\nu_d(p)\right]. \end{aligned} \tag{E.2}$$

Let us now explain that we indeed have

$$\forall\, \omega \in \Omega \quad \int_{U_d(\mathbb{C})} \mathbf{1}_{\mathcal{A}\mathcal{B}}(p\Upsilon(\omega))\mathrm{d}\nu_d(p) = \mathbf{1}_{\mathcal{A}}(\|\Upsilon(\omega)\|)\mathbf{P}\left(\frac{\Upsilon}{\|\Upsilon\|} \in \mathcal{B}\right). \tag{E.3}$$

If $\|\Upsilon(\omega)\|$ does not belong to $\mathcal{A}$ then $p\Upsilon(\omega)$ does not belong to $\mathcal{A}\mathcal{B}$ and hence the two sides of (E.3) vanish. If $\|\Upsilon(\omega)\|$ belongs to $\mathcal{A}$ then we have $\mathbf{1}_{\mathcal{A}\mathcal{B}}(p\Upsilon(\omega)) = \mathbf{1}_{\mathcal{B}}\left(p\frac{\Upsilon(\omega)}{\|\Upsilon(\omega)\|}\right)$. Whatever is a fixed point $x \in \mathbb{S}^{2d-1}$, it is known that the pushforward measure of ν_d via the map $p \in U_d(\mathbb{C}) \mapsto px \in \mathbb{S}^{2d-1}$ is the normalized volume measure of $\mathbb{S}^{2d-1}$ and so also equals the distribution of $\frac{\Upsilon}{\|\Upsilon\|}$. Choosing $x = \frac{\Upsilon(\omega)}{\|\Upsilon(\omega)\|}$, it appears that (E.3) is completely proved. Finally, (E.2) and (E.3) imply (E.1). □

Appendix F. Splitting independent random vectors

We recall that the proof of Theorem 6.1 begins with the formula (7.3). Similar identities are usually used if each random variable X_n is symmetric (see [33, p. 125 and 531] or [34, p. 83]). We give here a self-contained argument showing that (7.3) essentially comes from the mutual independence. More precisely, we apply the lemma below for

- $V_n = \mathbb{C}$ and $V_{N+n} = \mathcal{M}_{d_n}(\mathbb{C})$ for any $n \in \{1, \dots, N\}$;
- X_n taking values in $\mathbb{C}$;
- $Y_n = \mathcal{E}_n$ taking values in the unitary group $U_{d_n}(\mathbb{C}) \subset \mathcal{M}_{d_n}(\mathbb{C})$;
- $F(x_1, \dots, x_N, y_1, \dots, y_N) = \left\|\sum_{n=2}^{N} x_n \sqrt{d_n}\mathbf{tr}(y_n b_n)\right\|_{L^\infty(X)}^p$ for any element $(x_1, \dots, x_N, y_1, \dots, y_N) \in \prod_{n=1}^{2N} V_n$. We also recall that b_n belongs to $\mathcal{M}_{d_n}(L^\infty(X))$ (see Theorem 6.1).

Lemma F.1. — *Consider an integer $N \in \mathbb{N}^\star$, a sequence $V_1, \dots, V_{2N}$ of finite-dimensional $\mathbb{R}$-vector spaces, a measurable function $F : \prod_{n=1}^{2N} V_n \to \mathbb{R}^+$ and a family of $2N$ random vectors $X_1, \dots, X_N, Y_1, \dots, Y_N$ satisfying*

1. *for any $n \in \{1, \dots, N\}$, the random vector X_n takes values in V_n;*
2. *for any $n \in \{1, \dots, N\}$, the random vector Y_n takes values in V_{N+n};*
3. *the $2N$ random vectors $X_1, \dots, X_N, Y_1, \dots, Y_N$ are mutually independent.*

Then the following equality holds true

$$\begin{aligned}\mathbf{E}[F(X_1, \dots, X_N, Y_1, \dots, Y_N)]\\ = \mathbf{E}_{\omega'}\mathbf{E}_{\omega}[F(X_1(\omega'), \dots, X_N(\omega'), Y_1(\omega), \dots, Y_N(\omega))],\end{aligned}$$

where the last double expectation has to be understood as

$$\int_\Omega \int_\Omega F(X_1(\omega'), \dots, X_N(\omega'), Y_1(\omega), \dots, Y_N(\omega))\mathrm{d}\mathbf{P}(\omega')\mathrm{d}\mathbf{P}(\omega). \tag{F.1}$$

Proof. — At first reading, the reader may consider the case $V_1 = \cdots = V_{2N} = \mathbb{R}$ for the sake of clarity. Let $\mathbf{P}_{X_1}, \dots, \mathbf{P}_{X_N}, \mathbf{P}_{Y_1}, \dots, \mathbf{P}_{Y_N}$ be the distributions of the $2N$ random vectors $X_1, \dots, X_N, Y_1, \dots, Y_N$. Assertion (3) means that the distribution of the random vector

$$\omega \mapsto (X_1(\omega), \dots, X_N(\omega), Y_1(\omega), \dots, Y_N(\omega))$$

is the tensorial product measure $\mathbf{P}_{X_1} \otimes \cdots \otimes \mathbf{P}_{X_N} \otimes \mathbf{P}_{Y_1} \otimes \cdots \otimes \mathbf{P}_{Y_N}$ on the product vector space $\prod_{n=1}^{2N} V_n$. Still thanks to Assertion (3), $\mathbf{P}_X := \mathbf{P}_{X_1} \otimes \cdots \otimes \mathbf{P}_{X_N}$ is the distribution of $(X_1, \dots, X_N)$. Similarly, we note $\mathbf{P}_Y := \mathbf{P}_{Y_1} \otimes \cdots \otimes \mathbf{P}_{Y_N}$ the distribution of $(Y_1, \dots, Y_N)$.

Setting $x=(x_1,\dots,x_N)\in\prod_{n=1}^{N}V_n$ and $y=(y_1,\dots,y_N)\in\prod_{n=N+1}^{2N}V_n$, we have

$$\begin{aligned}&\mathbf{E}[F(X_1,\dots,X_N,Y_1,\dots,Y_N)]\\&\qquad=\int_{\prod_{n=1}^{2N}V_n}F(x_1,\dots,x_N,y_1,\dots,y_N)\mathrm{d}\mathbf{P}_X(x)\mathrm{d}\mathbf{P}_Y(y),\end{aligned}$$

which also equals the following integral thanks to the Fubini–Tonelli theorem

$$\int_{\prod_{n=N+1}^{2N}V_n}\left[\int_{\prod_{n=1}^{N}V_n}F(x_1,\dots,x_N,y_1,\dots,y_N)\mathrm{d}\mathbf{P}_X(x)\right]\mathrm{d}\mathbf{P}_Y(y),$$

which becomes

$$\int_{\prod_{n=N+1}^{2N}V_n}\mathbf{E}[F(X_1,\dots,X_N,y_1,\dots,y_N)]\mathrm{d}\mathbf{P}_Y(y).$$

Moreover, we have

$$\begin{aligned}&\mathbf{E}[F(X_1,\dots,X_N,y_1,\dots,y_N)]\\&\qquad=\int_\Omega F(X_1(\omega'),\dots,X_N(\omega'),y_1,\dots,y_N)\mathrm{d}\mathbf{P}(\omega').\end{aligned}$$

So we have proved that $\mathbf{E}[F(X_1,\dots,X_N,Y_1,\dots,Y_N)]$ equals

$$\begin{aligned}&\int_{\prod_{n=N+1}^{2N}V_n}\left[\int_\Omega F(X_1(\omega'),\dots,X_N(\omega'),y_1,\dots,y_N)\mathrm{d}\mathbf{P}(\omega')\right]\mathrm{d}\mathbf{P}_Y(y)\\&=\int_\Omega\left[\int_{\prod_{n=N+1}^{2N}V_n}F(X_1(\omega'),\dots,X_N(\omega'),y_1,\dots,y_N)\mathrm{d}\mathbf{P}_Y(y)\right]\mathrm{d}\mathbf{P}(\omega').\end{aligned}$$

Still remembering that the N random vectors $Y_1,\dots,Y_N$ are mutually independent, we can again simplify the integral in the brackets and we get

$$\int_\Omega\left[\int_\Omega F(X_1(\omega'),\dots,X_N(\omega'),Y_1(\omega),\dots,Y_N(\omega))\mathrm{d}\mathbf{P}(\omega)\right]\mathrm{d}\mathbf{P}(\omega').$$

The Fubini–Tonelli theorem finally gives (F.1). □

BIBLIOGRAPHY

[1] N. Anantharaman, "Topologie des hypersurfaces nodales de fonctions aléatoires gaussiennes", in *Séminaire Bourbaki. Volume 2015/2016*, Astérisque, vol. 390, Société Mathématique de France, 2017.

[2] J. Angst, V.-H. Pham & G. Poly, "Universality of the nodal length of bivariate random trigonometric polynomials", *Trans. Am. Math. Soc.* **370** (2018), no. 12, p. 8331-8357.

[3] A. Ayache & N. Tzvetkov, “L^p properties for Gaussian random series”, *Trans. Am. Math. Soc.* **360** (2008), no. 8, p. 4425-4439.

[4] F. A. Berezin & M. A. Shubin, *The Schrödinger equation*, Mathematics and Its Applications. Soviet Series, vol. 66, Kluwer Academic Publishers, 1991.

[5] M. Berger, P. Gauduchon & E. Mazet, *Le spectre d’une variété riemannienne*, Lecture Notes in Mathematics, vol. 194, Springer, 1971, vii+251 pages.

[6] J.-M. Bony, “On the Characterization of Pseudodifferential Operators (Old and New)”, in *Studies in Phase Space Analysis with Applications to PDEs*, Progress in Nonlinear Differential Equations and their Applications, vol. 84, Springer, 2013, p. 21-34.

[7] A. de Bouard, “Construction de solutions pour des edp sur-critiques à données initiales aléatoires”, in *Bourbaki seminar. Volume 2013/2014*, vol. 367-368, Astérisque, no. 1074, Société Mathématique de France, 2015.

[8] J.-M. Bouclet, “Semi-classical calculus on manifolds with ends and weighted L^p estimates”, *Ann. Inst. Fourier* **61** (2011), no. 3, p. 1181-1223.

[9] H. Brézis & L. Nirenberg, “Degree theory and BMO. I: Compact manifolds without boundaries”, *Sel. Math., New Ser.* **1** (1995), no. 2, p. 197-263.

[10] N. Burq, P. Gérard & N. Tzvetkov, “Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds”, *Am. J. Math.* **126** (2004), no. 3, p. 569-605.

[11] N. Burq & G. Lebeau, “Injections de Sobolev probabilistes et applications”, *Ann. Sci. Éc. Norm. Supér.* **46** (2013), no. 6, p. 917-962.

[12] N. Burq & N. Tzvetkov, “Random data Cauchy theory for supercritical wave equations I: local theory”, *Invent. Math.* **173** (2008), no. 3, p. 449-475.

[13] ———, “Random data Cauchy theory for supercritical wave equations II: A global existence result”, *Invent. Math.* **173** (2008), no. 3, p. 477-496.

[14] ———, “Probabilistic well-posedness for the cubic wave equation”, *J. Eur. Math. Soc.* **16** (2014), no. 1, p. 1-30.

[15] M. Dimassi & J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society Lecture Note Series, vol. 268, Cambridge University Press, 1999.

[16] F.-D. Filbir & H. N. Mhaskar, “A quadrature formula for diffusion polynomials corresponding to a generalized heat kernel”, *J. Fourier Anal. Appl.* **16** (2010), no. 5, p. 629-657.

[17] B. Hanin, S. Zelditch & P. Zhou, “Nodal sets of random eigenfunctions for the isotropic harmonic oscillator”, *Int. Math. Res. Not.* **2015** (2015), no. 13, p. 4813-4839.

[18] B. Helffer & D. Robert, “Comportement semi-classique du spectre des hamiltoniens quantiques elliptiques”, *Ann. Inst. Fourier* **31** (1981), no. 3, p. 169-223.

[19] ———, “Proprietes asymptotiques du spectre d’operateurs pseudo-differentiels sur R^n”, *Commun. Partial Differ. Equations* **7** (1982), no. 7, p. 795-882.

[20] H. Hofer & E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Modern Birkhäuser Classics, Birkhäuser, 2011.

[21] L. Hörmander, “The spectral function of an elliptic operator”, *Acta Math.* **121** (1968), no. 1, p. 193-218.

[22] ———, *The analysis of linear partial differential operators. III : Pseudo-differential operators*, Grundlehren der Mathematischen Wissenschaften, vol. 274, Springer, 1985.

[23] R. Imekraz, “Concentration et randomisation universelle de sous-espaces propres”, *Anal. PDE* **11** (2018), no. 2, p. 263-350.

[24] R. Imekraz, D. Robert & L. Thomann, "On random Hermite series", *Trans. Am. Math. Soc.* **368** (2016), no. 4, p. 2763-2792.

[25] N. C. Jain & M. B. Marcus, "Integrability of infinite sums of independent vector-valued random variables", *Trans. Am. Math. Soc.* **212** (1975), p. 1-36.

[26] F. John & L. Nirenberg, "On functions of bounded mean oscillation", *Commun. Pure Appl. Math.* **14** (1961), no. 3, p. 415-426.

[27] J.-P. Kahane, *Some random series of functions*, Heath Mathematical Monographs, D.C. Heath and Company, 1968.

[28] G. E. Karadzhov, "Riesz summability of multiple Hermite series in L^p spaces", *Math. Z.* **219** (1995), no. 1, p. 107-118.

[29] H. Koch & D. Tataru, "L^p eigenfunction bounds for the Hermite operator", *Duke Math. J.* **128** (2005), no. 2, p. 369-392.

[30] H. Koch, D. Tataru & M. Zworski, "Semiclassical L^p estimates", *Ann. Henri Poincaré* **8** (2007), no. 5, p. 885-916.

[31] M. Ledoux & M. Talagrand, *Probability in Banach Spaces. Isoperimetry and processes*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 23, Springer, 1991.

[32] N. Lerner, *Metrics on the phase space and non-selfadjoint pseudo-differential operators*, Pseudo-Differential Operators. Theory and Applications, vol. 3, Birkhäuser, 2010.

[33] D. Li & H. Queffélec, *Introduction à l'étude des espaces de Banach*, Cours Spécialisés (Paris), vol. 12, Société Mathématique de France, 2004.

[34] M. B. Marcus & G. Pisier, *Random Fourier series with applications to harmonic analysis*, Annals of Mathematics Studies, vol. 101, Princeton University Press, 1981.

[35] B. Maurey & G. Pisier, "Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach", *Stud. Math.* **58** (1976), p. 45-90.

[36] B. Muckenhoupt, "Mean convergence of Hermite and Laguerre series. II", *Trans. Am. Math. Soc.* **147** (1970), no. 2, p. 433-460.

[37] R. E. A. C. Paley & A. Zygmund, "On some series of functions. I", *Proc. Camb. Philos. Soc.* **26** (1930), p. 337-357.

[38] A. Poiret, D. Robert & L. Thomann, "Probabilistic global well-posedness for the supercritical nonlinear harmonic oscillator", *Anal. PDE* **7** (2014), no. 4, p. 997-1026.

[39] ———, "Random weighted Sobolev inequalities on $\mathbb{R}^d$ and applications to Hermite functions", *Ann. Henri Poincaré* **16** (2015), no. 2, p. 651-689.

[40] L. Robbiano & C. Zuily, "Remark on the Kato smoothing effect for Schrödinger equation with superquadratic potentials", *Commun. Partial Differ. Equations* **33** (2008), no. 4, p. 718-727.

[41] D. Robert & L. Thomann, "Random weighted Sobolev inequalities and application to quantum ergodicity", *Commun. Math. Phys.* **335** (2015), no. 3, p. 1181-1209.

[42] W. T. Sledd, "Random series which are BMO or Bloch", *Mich. Math. J.* **28** (1981), no. 3, p. 259-266.

[43] A.-S. de Suzzoni, "Consequences of the choice of a particular basis of $L^2(S^3)$ for the cubic wave equation on the sphere and the Euclidian space", *Commun. Pure Appl. Anal.* **13** (2014), no. 3, p. 991-1015.

[44] T. Tao, *Topics in random matrix theory*, Graduate Studies in Mathematics, vol. 132, American Mathematical Society, 2012.

[45] T. Tao & V. Vu, "Random matrices: The distribution of the smallest singular values", *Geom. Funct. Anal.* **20** (2010), no. 1, p. 260-297.

[46] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser, 1983.

[47] N. Tzvetkov, “Riemannian analogue of a Paley–Zygmund theorem”, *Sémin. Équ. Dériv. Partielles* **2008-2009** (2010), article ID XV (14 pages).

[48] K. Yajima & G. Zhang, “Smoothing property for Schrödinger equations with potential superquadratic at infinity”, *Commun. Math. Phys.* **221** (2001), no. 3, p. 573-590.

[49] ———, “Local smoothing property and Strichartz inequality for Schrödinger operator with potentials superquadratic at infinity”, *J. Differ. Equations* **202** (2004), no. 1, p. 81-110.

[50] S. Zelditch, “Real and complex zeros of Riemannian random waves”, in *Spectral analysis in geometry and number theory*, Contemporary Mathematics, vol. 484, American Mathematical Society, 2009, p. 321-342.

[51] A. Zygmund, *Trigonometric series. II*, Cambridge University Press, 2002.

Manuscrit reçu le 24 juillet 2017,
révisé le 16 mars 2018,
accepté le 26 avril 2018.

Rafik IMEKRAZ
Institut de Mathématiques de Bordeaux, UMR 5251
Université de Bordeaux
351 cours de la Libération
F33405 Talence Cedex (France)
rafik.imekraz@math.u-bordeaux.fr