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UNIVERSAL REAL LOCALLY CONVEX LINEAR TOPOLOGICAL SPACES

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It is known that every separable Fr. Riesz-Banach space can be isometrically and isomorphically embedded into the (C) — space of all continuous functions in $\langle 0, 1 \rangle$ with norm $\|f\| = \max f(x)$, [15]. Recently E. Silverman [12] has embedded the same spaces into the space (m), i. e. the space of all bounded infinite sequences (1) with the norm $\|a\| = \sup_n |a_n|$ where $a = (a_1, a_2, \ldots)$. The underlying paper shows that the method used by Silverman can be generalized to fit the construction of universal spaces which embed the general locally convex real linear topological spaces. The obtained result discloses, at the same time, that essentially, vectors of such a space can be conceived as some real valued functions, and its topology as generated by a generalized uniform convergence. Thus the uniform convergence shows itself as a more general notion than it could be surmised.

Notations. — The elements of a linear space will be termed **vectors**, and sometimes, for the sake of clearness provided with arrows. We also call them **points**, since we may admit the Grassmann's approach to the vector calculus.

The operations on sets will be denoted by the Bourbaki symbols [4], $\cap$, $\cup$, $\bigcap$, the inclusion of sets by $\subseteqq$. The empty set will be written $\emptyset$, the set composed of the single vector $\vec{x}$ by $(\vec{x})$.

Addition of vectors, and the multiplication of a vector by a real number λ will be denoted by $\vec{x} + \vec{y}$, $\lambda\vec{x}$ respectively.

(1) This space is not separable.

Given the sets E, F of vectors, $E \dot{+} F$ will denote the set of all vectors $\vec{x} + \vec{y}$ where $\vec{x} \epsilon E$, $\vec{y} \epsilon F$. Similarly $x \dot{+} E$ will denote the set of all vectors $\vec{x} + \vec{y}$ where $\vec{y} \epsilon E$. The symbol $\lambda . E$ will denote the set of all vectors $\lambda . \vec{x}$ where $\vec{x} \epsilon E$.

$\langle x, y \rangle$ will mean the closed segment and (x, y) the open one. The domain and range of a relation **R** will be denoted by ᗡ**R**, D**R** respectively. If ᗡ**R** $\cup$ D**R** is meaningful, this set will be termed field of **R** and denoted by ⦶**R**. If a topological space is denoted by (L), the set of all its vectors will be written L.

§ 1. — Basic concepts.

1. A. Kolmogoroff [1] has introduced the notion of a general linear real topological space. This is 1) a linear space L (i. e. an abelian group with real multipliers), 2) provided with an open — set topology [4] (which is equivalent to the neighborhood topology and also to the Kuratowski's topology [2], [3]), 3) satisfying the weakest separation axiom (i. e. if $x \neq y$, then either there exists a neighborhood of x not containing y, or there exits a neighborhood of y not containing x), and 4) in which $x + y$, λx are continuous functions of the couple of both variables (not only with respect to each variable separately).

It bas been proved that the linear topological space must satisfy the Hausdorff separation axiom (i. e. if $x \neq y$, then there exist disjoint neighborhoods of x and y), and even it must be a regular topology [3].

J. v. Neumann [5] has given an equivalent definition of a linear real topological space, by axiomatizing a class of sets U, V, W, ... of vectors in L in the following way:

1° $\bigcap_{U \epsilon \mathfrak{U}} U = (\vec{o})$ (2),

2° for every U, $V \epsilon \mathfrak{U}$ there exists $W \epsilon \mathfrak{U}$ with $W \subseteq U \cap V$,

3° for every U there exists V such that

$$V \dot{+} V \subseteq U,$$

(2) This axiom was admitted by D. H. Hyers [6] instead of the original v. Neumann's axiom.

4° for every U there exists V such that

$$\alpha . V \subseteq U \quad \text{for all } \alpha \text{ where } \quad |\alpha| \leqslant 1,$$

5° for every $\vec{x}$ and U there exists λ such that

$$\vec{x} \in \lambda . U.$$

Those sets U, V, ...will be termed v. Neumann's neighborhoods (N. nbhds).

Two systems $\mathfrak{U}'$, $\mathfrak{U}''$ of N. nbhds are said to be equivalent if for every $U' \in \mathfrak{U}'$ there exists $U'' \in \mathfrak{U}''$ with $U'' \subseteq U'$ and for every $U_1'' \in \mathfrak{U}''$ there exists $U_1' \in \mathfrak{U}$ with $U_1' \subseteq U_1''$.

Let $E \subseteq L$. A point $\vec{x}$ is said to be an $\mathfrak{U}$ — interior point of E, [5], if there exists $U \in \mathfrak{U}$ such that $\vec{x} \dotplus U \subseteq E$. A set E such that, if $\vec{x} \in E$, then $\vec{x}$ is an $\mathfrak{U}$ — interior point of E is called $\mathfrak{U}$ — open. If $\mathfrak{U}$ is equivalent to $\mathfrak{U}'$, then a $\mathfrak{U}$ — interior point of E is also a $\mathfrak{U}'$ — interior point of E.

If $\mathfrak{U}$ is a system of N. nbhds, and if we replace every $U \in \mathfrak{U}$ by U^0 i. e. the set of all $\mathfrak{U}$ — interior points of U, then the system $\{U^0\}$ will be an equivalent system of N. nbhds. U^0 is never empty. If given $\mathfrak{U}$, we take all translations of the sets of $\{U^0\}$, we obtain a neighborhood-topology in L. (By a neighborhood of $\vec{x}_0$ we shall understand the $\vec{x}_0$ — translation of any U^0.) The space L provided with this topology is a Kolmogoroff linear topological space, [5]. The topologies, thus generated by two equivalent systems of N. nbhds, are equivalent.

Conversely, given a Kolmogoroff linear topological space, there exists a topologically equivalent system of neighborhoods (s) such that the neighborhoods $\mathfrak{U}$ of $\vec{o}$ satisfy v. Neumann's axioms and (s) is composed of all translations of all sets of $\mathfrak{U}$, [18].

It can be easily shown, that if we add, to the above five axioms, the axiom

6° If $U \in \mathfrak{U}$, and $|\alpha| \leqslant 1$, then

$$\alpha . U \subseteq U,$$

no restriction to the topology will be introduced [6]. Hence we may admit 6° too. N. nbhds satisfying 6° will be termed V. Neumann's star neighborhoods (N. st. nbhds).

2. D. H. Hyers [6] has introduced the pseudo-normed linear spaces which are identical with the topological linear spaces. His approach is this :

Let $\mathbf{R}$ be an E. H. Moore- H. L. Smith stream ordering, i. e. a partial ordering such that
if $d_1, d_2 \in \Phi\mathbf{R}$, then there exists $d_3 \in \Phi\mathbf{R}$ with

$$d_1 \mathbf{R} d_3, \qquad d_2 \mathbf{R} d_3 \text{ (}^3\text{)}.$$

Let us attach to every $\vec{x} \in L$ and every $d \in \Phi\mathbf{R}$ a number $H(\vec{x}, d)$, called $\mathbf{R}$-pseudonorm of $\vec{x}$, and satisfying the following conditions:

1° $$H(\vec{x}, d) \geqslant 0,$$

2° if $H(\vec{x}, d) = 0$ for all $d \in \Phi\mathbf{R}$, then $\vec{x} = \vec{0}$,

3° $H(\lambda\vec{x}, d) = |\lambda| . H(\vec{x}, d)$ for all real numbers λ,

4° for every $\eta > 0$ and $e \in \Phi\mathbf{R}$ there exist $\delta > 0$ and $d \in \Phi\mathbf{R}$ such that for every $\vec{x}, \vec{y} \in L$ we have :

$$\text{if } H(\vec{x}, d) < \delta, \ H(\vec{y}, d) < \delta, \ \text{then } H(\vec{x} + \vec{y}, e) < \eta,$$

5° if $d_1 \mathbf{R} d_2$, then $H(\vec{x}, d_1) \leqslant H(\vec{x}, d_2)$ for every $\vec{x}$.

A linear space with pseudo-norm is termed pseudo-normed linear space.

Given such a space, if we define

$$U(d, \alpha) \underset{\text{df}}{=} \hat{\vec{x}} \{ H(\vec{x}, d) < \alpha \} \text{ for all } \alpha > 0,$$

the class $\{U(d, \alpha)\}$, where $d \in \Phi\mathbf{R}$, $\alpha > 0$ will satisfy J. V. Neumann's axioms, and the condition 6° too, so the $U(d, \alpha)$ are N. st. nbhds.

(3) By a partial ordering we understand a not empty relation [9] $\mathbf{R}$ such that
1) $a\mathbf{R}a$ whenever $a \in \Phi\mathbf{R}$,
2) if $a\mathbf{R}b$, $b\mathbf{R}c$, then $a\mathbf{R}c$.
3) for $a, b \in \Phi\mathbf{R}$ the following are equivalent

$$\text{I)} \quad a\mathbf{R}b, \ b\mathbf{R}a, \qquad \text{II)} \quad a = b. \ [10].$$

If, given a stream ordering $\mathbf{R}$, we attach to every element a of the field $\Phi\mathbf{R}$ of $\mathbf{R}$ an element taken from a not empty set E, we get a function $f(a)$ which will be termed $\mathbf{R}$- stream sequence of elements of E.

The term, commonly used, for a stream-sequence is « directed set », though this is clearly no set at all, but may be rather understood as the ordered couple $(\mathbf{R}, f)$. The stream-sequence is a generalisation of the ordinary sequence.

Conversely given a system $\mathfrak{U}$ of N. st. nbhds we can define a Hyers' pseudo-norm in the following natural way by putting

1) $$H(\vec{x}, U) \underset{\text{df}}{=} \inf_{>0} \widehat{\alpha} \{\vec{x} \in \alpha . U\},$$

2) $$U_1 \mathbf{R} U_2 . \underset{\text{df}}{=} . U_2 \subseteq U_1 \text{ for } U_1, U_2 \in \mathfrak{U}.$$

The second definition organizes $\mathfrak{U}$ into a stream ordering, and one can prove that $H(\vec{x}, U)$ thus defined satisfies the Hyers' conditions 1°-5°.

Thus, given a topological linear space (L), there is at least one stream ordering attached to it.

The Hyers' pseudo-norm-approach to linear spaces has been put into a simpler form by J. P. La Salle [7].

3. The linear (real) topological space is said to be convex (locally convex) if there exists a system of N. nbhds $\mathfrak{U}$ such that every $V \in \mathfrak{U}$ is a convex set (4), which condition is equivalent to

$$V + V \subseteq 2 . V.$$

The linear space is convex if and only if there exists a pseudo-norm $H(\vec{x}, d)$ such that for every $d, \vec{x}, \vec{y}$

$$H(\vec{x} + \vec{y}, d) \leqslant H(\vec{x}, d) + H(\vec{y}, d).$$

We shall deal only with real linear topological convex spaces (L).

We fix a system $\mathfrak{U}$ of convex N. st. nbhds and take the corresponding Hyers' pseudonorm which we shall denote by

$$||x||_U \underset{\text{df}}{=} H(\vec{x}, U),$$

and where the corresponding stream ordering is defined by

$$U_1 \mathbf{R} U_2 . \underset{\text{df}}{=} . U_2 \subseteq U_1, \quad (U_1, U_2 \in \mathfrak{U})$$

(4) Given a linear space L, a subset E of L is said to be convex if the following condition is satisfied:

$$\text{if } \vec{x}_1, \vec{x}_2 \in E, \text{ then } \lambda_1 \vec{x}_1 + \lambda_2 \vec{x}_2 \in E \text{ for every } \lambda_1, \lambda_2 \text{ with } \lambda_1 \geqslant 0,\ \lambda_2 \geqslant 0,\ \lambda_1 + \lambda_2 = 1.$$

Theorem. Every U is a convex body in (L), having $\vec{o}$ as its linearly inner point(5).

It may be proved that for U the linearly inner point coincide with the topologically inner points, but we do not need it.

Proof. Let $\vec{x} \neq \vec{o}$ and $U \epsilon \mathfrak{U}$. By V. Neumann's axiom 5 there exists β such that $\vec{x} \epsilon \beta . U$, $\beta > o$.

Hence $\frac{1}{\beta} \cdot \vec{x} \in U$.

and then, for all γ with $|\gamma| < \frac{1}{\beta}$, we have $\gamma . \vec{x} \epsilon U$.

Hence on the line l passing through $\vec{o}$ and $\vec{x}$ there exists an open segment containing o and included in U.

Theorem. — If we replace every U by its linear closure $\overline{U}$, we obtain an equivalent system of convex N. st. nbhds(6).

Proof. Let $U \epsilon \mathfrak{U}$. There exists U_1 with $U_1 \dotplus U_1 \subseteq U$, hence $2U_1 \subseteq U$. Let $\vec{A}$ be a boundary point of U_1. We have $\vec{A} \neq \vec{o}$, because $\vec{o}$ is a linearly inner point of U_1. Fut $\vec{B} = 2 . \vec{A}$. We have $(\vec{o}, \vec{B}) \subseteq U$ and then $\vec{A} \epsilon U$.

Since every boundary point of U_1 belongs to U, we have $\overline{U_1} \subseteq U$. On the other hand we have $U \subseteq \overline{U}$, which completes the proof.

Remark. It may happen that the whole space L belongs to $\mathfrak{U}$, but, if we drop it, we obtain an equivalent system of N. nbhds, unless the topology is trivial with L as the only neighborhood.

(5) Given a linear real space (L), (even without any topology considered therein), a convex body in V is a convex set E containing at least one linearly inner point, i. e. a point x_0 such that on every straight line, in L, passing through x_0 there exists an open segment (x', x'') containing x_0 and belonging to E. [11].

(6) If (L) is a linear space, $E \subseteq L$, then E is said to be linearly-closed, if for every straight line l in L the set $E \cap l$ is a closed set in the natural topology on the straight line l.

By the linear closure $\overline{E}$ of a set $E \subseteq L$ we understand the smallest linearly closed set containing E. If B is a convex body and, $\vec{x}_0$ its linearly inner point, then $\overline{B}$ is also a convex body in which $\vec{x}_0$ is a linearly inner point, and vice versa.

The points of $\overline{B}$ which are no linearly inner points of B (hence of $\overline{B}$) are termed boundary points of B. (and of $\overline{B}$) and its collection is termed the linear boundary of B. Points of L which do not belong to $\overline{B}$ are termed linearly outer points for B (and for $\overline{B}$) [11].

It may be proved that the linear closure of U coïncides with its topological closure, but we do not need it.

In the sequel we shall suppose that (L) is not trivial, and that $\mathfrak{U}$ does not contain L. Besides we may admit, without loss of generality, that every U is a linearly closed convex body.

§ 2. — Some linear functionals.

4. We shall need an important theorem by J. V. Wehausen ([18], p. 162); it will be states at 6. 2. It has been proved, by relying on the known theorem of Hahn and Banach, but we shall derive it in a geometrical way, by relying on a theorem on convex bodies. This will give the Wehausen's theorem the geometrical evidence, (see also [13]).

Let us have a fixed system $\mathfrak{U}$ of N. st. nbhds where $L\bar{\in}\mathfrak{U}$, where every $U\in\mathfrak{U}$ is a linearly closed convex body with $\vec{o}$ as its linearly inner point. Take the corresponding pseudo-norm $||x||_U$. We have $||x||_U > 1$, < 1, or $= 1$, if $\vec{x}$ is linearly exterior to U, interior of U or a boundary point of U respectively.

Consider the cartesian product $(L^*) \underset{\mathrm{df}}{=} (L) \times (-\infty, +\infty)$ of the given linear topological space (L) and the topological space $(-\infty, +\infty)$. Its vectors are ordered couples $(\vec{x}, \lambda)$ where $\vec{x}\in L$, and λ is a real number. We define addition and multiplication by

$$(\vec{x}, \lambda) + (\vec{y}, \mu) \underset{\mathrm{df}}{=} (\vec{x} + \vec{y}, \lambda + \mu),$$

$$\alpha . (\vec{x}, \lambda) \underset{\mathrm{df}}{=} (\alpha \vec{x}, \alpha\lambda).$$

Define U^* as the set of all $(\vec{x}, \lambda)$ where $x\in U$, $\lambda\in\langle -\varepsilon, +\varepsilon\rangle$, $(\varepsilon > 0)$. The class $\{U^*\}$ satisfies V. Neumann's axioms for neigborhoods. Hence (L^*) is a real linear topological space.

All U^* are linearly closed convex bodies in L^* with $(\vec{o}, 0)$ as a linearly inner point.

Having this, take $U\in\mathfrak{U}$. Define E_U^* as the set of all couples $(\vec{x}, \lambda)$ where $\lambda \geqslant ||x||_U$.

4. 1. The set E_U^* is convex in (L^*).

4. 2. The point $(\vec{o}, 1)$ is a linearly inner point of E_U^*[in (L^*)]. E_U^* is a convex body in (L^*).

Proof. Choose $\vec{x}_0 \neq \vec{o}$; take the points $(\vec{x}_0, 0)$, $(\vec{o}, 1)$ and the straight line l^* passing through them. The point

$$P^*_\alpha \underset{\text{df}}{=} \left(\alpha \vec{x}_0,\ 1 - \alpha\right)$$

is lying on l^*.

If $\alpha = 0$, we have $P^*_\alpha = \left(\vec{0},\ 1\right)$. We want to find $\alpha_1 > 0$ such that if $|\alpha| < \alpha_1$, then $P^*_\alpha \in E^*_U$. A simple geometric construction will do this, and its arithmetical equivalent is as follows.

Put

$$\alpha_0 \underset{\text{df}}{=} \frac{1}{1 + ||x_0||_U}; \qquad \text{we have} \quad 0 < \alpha_0 \leqslant 1, \qquad ||x_0||_U = \frac{1 - \alpha_0}{\alpha_0}.$$

Let $|\alpha| \leqslant \alpha_0$.

It follows

$$1 - \alpha \geqslant |\alpha| . \frac{1 - \alpha_0}{\alpha_0} = |\alpha| . ||\vec{x}_0||_U = ||\alpha \vec{x}_0||_U.$$

This gives:

$$||\alpha x_0||_U \leqslant 1 - \alpha, \qquad \text{and then} \qquad P^*_\alpha \in E^*_U.$$

Hence, if a straight line l^* passing through $\left(\vec{0},\ 1\right)$ is not parallel to L, there exists an open segment on l^* containing $\left(\vec{0},\ 1\right)$ and contained in E^*_U.

Now let l^* be parallel to L. Its points are $\left(\alpha \vec{x}_0, 1\right)$ where $\vec{x}_0 \neq \vec{0}$. If $||\vec{x}_0||_U = 0$, we have $||\alpha \vec{x}_0||_U \leqslant 1$, and hence $l^* \subseteqq E^*_U$. If $||\vec{x}_0||_U > 0$, we have for $|\alpha| \leqslant \dfrac{1}{||\vec{x}_0||_U}$ the inequality

$$||\alpha \vec{x}_0||_U = |\alpha| . ||\vec{x}_0||_U \leqslant 1,$$

and then the open interval $\left(\left(-\varepsilon . \vec{x}_0,\ 1\right),\ \left(\varepsilon . \vec{x}_0,\ 1\right)\right)$, where $\varepsilon \underset{\text{df}}{=} \dfrac{1}{||\vec{x}_0||_U}$, belongs to E^*_U.

Thus we have proved that $\left(\vec{0},\ 1\right)$ is a linearly inner point of E^*_U. Hence E^*_U is a convex body in (L^*).

4. 3. If $\left(\vec{x},\ \lambda\right) \in E^*_U$, $\vec{x} \neq \vec{0}$, $\lambda > 0$, then $\left(\alpha \vec{x},\ \alpha\lambda\right) \in E^*_U$ for every $\alpha \geqslant 0$.

The proof is obvious.

4. 4. Every point $(\vec{o}, \lambda)$, $(\lambda > o)$ is a linearly inner point of E^*_U.

Proof. Let l^*_λ be a straight line passing through $(\vec{o}, \lambda)$. Denote by l^* the straight line passing through $(\vec{o}, 1)$ and parallel to l^*_λ. Since $(\vec{o}, 1)$ is a linearly inner point of E^*_U, there exists an open segment (x^*_1, x^*_2) with $(\vec{o}, 1) \in (x^*_1, x^*_2)$, $(x^*_1, x^*_2) \subseteq l^*$. If $x^* \in (x^*_1, x^*_2)$, we have $\lambda x^* \in (\lambda x^*_1, \lambda x^*_2) \subseteq l^*_\lambda$. Since, by the preceding theorem, $\lambda . x^* \in E^*_U$, it follows that

$$(\vec{o}, \lambda) \in (\lambda x^*_1, \lambda x^*_2) \subseteq l^*_\lambda \cap E^*_U,$$

and then, that $(\vec{o}, \lambda)$ is a linearly inner point of E^*_U.

4. 5. Every point $(\vec{x}, \lambda)$, where $\|\vec{x}\|_U < \lambda$, $(\lambda > o)$ is a linearly inner point of E^*_U.

Proof. Consider the line l^*, $(\beta\vec{x}, \lambda)$ where β varies in $(-\infty, +\infty)$. l^* passes through $(\vec{o}, \lambda)$.

There exists $\varepsilon > o$ such that

$$\|\vec{x}\|_U . (1 + \varepsilon) < \lambda.$$

It follows that if $-\varepsilon < \alpha < 1 + \varepsilon$, we have

$$\|\alpha . \vec{x}\|_U < \lambda.$$

Hence the segment

$$\left((-\varepsilon\vec{x}, \lambda), ((1 + \varepsilon)\vec{x}, \lambda)\right) \qquad \text{belongs to } E^*_U.$$

The point $(\vec{o}, \lambda)$ is an inner point of this segment.

Now we can apply the following theorem on convex bodies [11]:

If G is a convex body, y_0 its linearly inner point, and (y_1, y_2) an open segment containing y_0 and included in G, then all points of (y_1, y_2) are linearly inner points of G.

It follows that $(\vec{x}, \lambda)$ is a linearly inner point of E^*_U.

4. 6. Every point $(\vec{x}, \lambda)$, where $\|\vec{x}\|_U = \lambda$, is a boundary point of E^*_U; if $\|\vec{x}\|_U > \lambda$, it is a linearly exterior point of E^*_U.

Proof. Take the set of all points $(\vec{x}, \mu)$ in L^* for which $\mu > \lambda$. They are all linearly inner points of E^*_U and are lying on the straight

line l^* composed of the points $(\vec{x}, \alpha)$ where $\alpha \in (-\infty, +\infty)$. If $\alpha < \lambda$, the point $(\vec{x}, \alpha)$ does not belong to E_U^*.
Choose a number $\lambda_0 > \lambda$. The point $(\vec{x}, \lambda_0)$ is a linearly inner point of E_U^*. If we choose on l^* the direction in which $(\vec{x}, \lambda_0)$ precedes $(\vec{x}, \lambda)$, we see that the supremum of points of E_U^* lying on l^* is $(\vec{x}, \lambda)$. Hence, by a theorem on convex bodies [11]. $(\vec{x}, \lambda)$ must be a boundary point of E_U^*, and $(\vec{x}, \lambda')$ for every $(\lambda' < \lambda)$ an exterior point. The theorem is proved.

4. 7. It follows that $(\vec{x}, \lambda)$ is a linearly interior, exterior or boundary point of E_U^* according to whether $\|\vec{x}\|_U < \lambda$, $> \lambda$ or $= \lambda$ respectively. It also follows, by a theorem on convex bodies [11], that E_U^* is a linearly closed convex body in L^*.

4. 8. We see that, if $(\vec{x}, \lambda)$ is a boundary point of E_U^*, then for every $\alpha \geqslant 0$, $(\alpha\vec{x}, \lambda\alpha)$ is also a boundary point.

5. Take a vector $\vec{x}$ and a neighborhood U with $\|\vec{x}\|_U > 0$. The point

$$P^* \underset{\mathrm{df}}{=} \left(\vec{x}, \|\vec{x}\|_U\right)$$

is a boundary point of E_U^*. Since E_U^* is a linearly closed convex body, there exists at P^* a hyperflat F^* of support of E_U^* in L^* (7).

5. 1. Let F^* be a hyperflat of support of E_U^* at P^*, and let M^* be the halfspace with boundary F^* and such that

$$E_U^* \subseteq M^*.$$

We have $P^* \neq (\vec{0}, 0)$, because $\vec{x} \neq \vec{0}$; hence the ray R^* issuing

(7) If L is a linear space, then by a linear variety in L we understand a not empty subset E of L such that, if x_1, $x_2 \in E$, then $\lambda_1 x_1 + \lambda_2 x_2 \in E$ for every real λ_1 and λ_2.

By a flat in L we understand a translation of a linear variety. By hyperflat in L we understand a flat $F \neq L$ for which there exists a vector $\vec{x}$ such that the smallest flat containing F and $\vec{x}$ coincides with L.

A hyperflat F determines two halfspaces M_1, M_2, such that $M_1 \cup M_2 = L$, $M_1 \cap M_2 = F$. They are linearly closed convex bodies with F as common linear boundary.

J. Dieudonné [13] has proved that, if G is a linearly closed convex body, x its boundary point, then there exists at least one hyperflat F such that $x \in F$, and that G, is contained in one of the two halfspaces determined by F. Such a hyperflat is termed hyperflat of support for G at x. Dieudonné's proof is algebraic. A geometrical proof is given in [11].

from $(\vec{o}, o)$ and passing through P^* is well determined. It must belong to F^*. Indeed $R^* \subseteq E_U^*$ (on account of 4. 7).

Suppose $R^* \subseteq F^*$. Since $P^* \epsilon R^* \cap F^*$, there would exist on R^* a segment (P^*, P_1^*) composed of linearly inner points of one halfspace, and another segment (P^*, P_2^*) on R^* composed of linearly inner points of the other halfspace. Since $\langle P_1^*, P_2^* \rangle$ belongs to the boundary of E_U^*, we could deduce that E_U^* possesses inner points in both halfspaces (8) which is impossible. Thus $R^* \subseteq F^*$, and then $(\vec{o}, o) \epsilon F^*$.

5.2. If $(\vec{y}, \lambda_1) \epsilon F^*$ and $(\vec{y}, \lambda_2) \epsilon F^*$, then $\lambda_1 = \lambda_2$.

Proof. Suppose $\lambda_1 < \lambda_2$. The straight line passing through $(\vec{y}, \lambda_1)$, $(\vec{y}, \lambda_2)$ belongs to F^*. Hence $(\vec{y}, o) \epsilon F^*$. If $\vec{y} = \vec{o}$, we would have $(\vec{o}, 1) \epsilon F^*$ which is impossible, because $(o, \vec{1})$ is a linearly interior point of E_U^*. Hence $\vec{y} \neq \vec{o}$. Since $(\vec{o}, o) \epsilon F^*$ and $(\vec{y}, o) \epsilon F^*$, the straight line l joining these two points is contained in F^*. Hence the plane passing through l and $(\vec{y}, 1)$ belongs to F^*, and hence $(\vec{o}, 1) \epsilon F^*$ which is impossible, since $(\vec{o}, 1)$ is a linearly inner point of E_U^*.

5.3. For every $\vec{x}$ there exists λ such that

$$(\vec{x}, \lambda) \epsilon F^*.$$

Proof. Suppose that for a given $\vec{x}_0$ we have

$$(\vec{x}_0, \lambda) \bar{\epsilon} F^* \text{ for all } \lambda.$$

Consider the straight line l^* composed of all points $(\vec{x}_0, \lambda)$ where $\lambda \epsilon (-\infty, +\infty)$. Take the hyperflat E^* parallel to F^* and passing through $(\vec{o}, o)$. Since $(\vec{o}, o) \epsilon F^*$, we have $E^* = F^*$. The vector

$$\vec{\xi} \underset{df}{=} (\vec{x}_0, +1) - (\vec{x}_0, -1) \epsilon E^*,$$

because if not, $\vec{\xi}$ would be independent of E^* and hence l^* would intersect F^*.

Hence the line $(\vec{o}, \lambda)$ with varying λ would belong to F^* and then $(\vec{o}, o)$ would be no inner point of E_U^*.

(8) There is the following theorem. If G is a convex body, x_0 its linearly inner point and a its boundary point then the open segment (a, x_0) is composed of linearly inner points of G, [11].

5.4. Let us remark that given a U there exists $\vec{x}$ such that $\|\vec{x}\|_U > 0$.

Since U differs from the whole space, there exists at least one boundary point $\vec{x}$ of U.

For this point we have $\|\vec{x}\|_U = 1$ as was already proved.

5.5. Given an $\vec{x} \neq \vec{0}$, there exists U such that

$$\|\vec{x}\|_U > 0.$$

Ifnot, we would have $\|\vec{x}\|_U = 0$ for all U, and then by Hyers' first axiom, $\vec{x} = \vec{0}$.

The set E_U^* is lying in one of the two linearly closed halfspaces determined by F^*. One of them M_1 contains $(\vec{0}, -1)$ the other M_2 contains $(\vec{0}, 1)$. Clearly $E_U^* \subseteq M_2$.

5.6. If $\vec{x} \in L$, and $(\vec{x}, \lambda) \in F^*$, then $\lambda \leqslant \|x\|_U$.

Indeed if we had $\lambda > \|x\|_U$, the point (x, λ) would be a linearly interior point of E_U^*, and then $(\vec{x}, \lambda) \bar{\in} F^*$.

6. Take a neighborhood $U \in \mathfrak{U}$. Since there exists $\vec{x}_0$ with $\|\vec{x}_0\|_U > 0$, the corresponding set E_U^* is not empty, and then the flat F^*, as defined before, exists. Choose F^*. Let us define the function $f(\vec{x})$ by putting:

$$f(\vec{x}) = \lambda \qquad \text{where} \qquad (\vec{x}, \lambda) \in F^*.$$

Such a number λ exists and is unique.

6.1. We easily see that $f(\vec{x})$ is a linear functional in L. This will mean that

$$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y}) \qquad \text{for all} \qquad \vec{x}, \vec{y} \in \mathrm{L},$$

and $f(\alpha\vec{x}) = \alpha f(\vec{x})$ for all $\vec{x} \in L$ and all real numbers α. Of course,

$f(\vec{x})$ depends not only on U but also on the choice of the support-hyperflat F^* for E_U^* (9).

6.2. Theorem. (Wehausen) For every $\vec{x}_1$ and U there exists a linear continuous functional $f_{x_1, U}(x)$ in L such that

$$\sup_{x \in U} |f_{x_1, U}(x)| = 1, \qquad f_{x_1, U}(x_1) = \|\vec{x}_1\|_U.$$

Proof.

Suppose first that

$$\|\vec{x}_1\|_U > 0.$$

Consider the set E_U^*. We know that the point $P^* \underset{\mathrm{df}}{=} (\vec{x}_1, \|x_1\|_U)$ is lying on the boundary of E_U^*, and that this point differs from $(\vec{0}, 0)$.

Take a support-hyperflat F^* at P^* to E_U^* in L^*, and consider the corresponding linear continuous functional $f(x)$.

Put

$$\vec{y} \underset{\mathrm{df}}{=} \frac{\vec{x}}{\|\vec{x}_1\|_U}.$$

The point $(\vec{y}, 1)$ lies on the boundary of E_U^*, and especially on the line joining $(\vec{0}, 0)$ with P^*.

Hence $(\vec{y}, 1) \in F^*$. Hence $f(\vec{y}) = 1$, and then

$$f(x_1) = f(\|\vec{x}_1\|_U \cdot \vec{y}) = \|\vec{x}_1\|_U \quad \ldots\ldots \quad (1)$$

Now we have for $x \in U$,

$$f(x) \leqslant \|x\|_U \leqslant 1.$$

(9) Let $\vec{x} \in U$. We have $\|\vec{x}\|_U \leqslant 1$. Since $(\vec{x}, f(\vec{x})) \in F^*$, we have by what has been proved,

$$f(\vec{x}) \leqslant \|\vec{x}\|_U \leqslant 1.$$

Let U^0 be the set of all points of U which are $\mathfrak{U}$-interior points. Hence U^0 is an open set in the linear topological space. We have for $x \in U^0$ also the inequality

$$f(\vec{x}) \leqslant 1.$$

Thus we see that $f(x)$ is bounded from above on a non empty topologically open set U^0.

The following theorem is true [15].

If, in a linear space, provided with a topology T such that the following conditions are satisfied:

1° if E is a T-open set, then every translation of E is so,

2° if E is a T-open set then $\alpha . E$ for $\alpha \neq 0$ is also a T-open set, there exists a T-open set, on which the linear function $f(x)$ is bounded from above, then $f(x)$ is continuous in this topology.

This theorem allows to conclude that $f(x)$ is a continuous linear functional.

If $x \in U$, then $-x \in U$, because U is a star-neighborhood. Hence

$$-f(x) = f(-x) \leqslant \|-x\|_U = \|x\|_U \leqslant 1.$$

It follows that for every $x \in U$ we have

$$|f(x)| \leqslant 1.$$

Since $\|\vec{y}\|_U = 1$, we have $\vec{y} \in U$, for U is linearly closed.

Hence

$$\sup_{x \in U} |f(x)| = 1 \quad \ldots\ldots \qquad (2)$$

Thus the functional $f(x)$ satisfies the conditions stated in the theorem.

Now suppose that $\|x_1\|_U = 0$.

There exists x_2 such that $\|\vec{x}_2\|_U > 0$; find a linear functional $f(x)$, as above, for $\vec{x}_2$.

We have

$$\sup_{x \in U} |f(x)| = 1.$$

Besides

$$f(x_1) \leqslant \|\vec{x}_1\|_U = 0.$$

Since $x_1 \in U$, we have $-x_1 \in U$,
and then

$$-f(x_1) = f(-x_1) \leqslant \|-\vec{x}_1\|_U = 0.$$

Thus

$$|f(x_1)| \leqslant 0, \text{ hence } f(x_1) = 0,$$

and then

$$f(x_1) = \|\vec{x}_1\|_U.$$

The theorem is proved.

6. 3. The functional $f_{x,U}(\vec{y})$ of the preceding theorem has the property:

$$(1) \quad \ldots\ldots \quad |f_{x_1,U}(\vec{y})| \leqslant \|\vec{y}\|_U \quad \text{for every } \vec{y}.$$

Proof. Let $\|\vec{y}\|_U = 0$. We have for $n = 1, 2, \ldots$

$$\|n\vec{y}\|_U = n \cdot \|\vec{y}\|_U = 0.$$

Since $\|\overrightarrow{ny}\|_U \leqslant 1$, we have $\overrightarrow{ny} \in U$,
and then

$$\left|f_{x_1, U}(\overrightarrow{ny})\right| \leqslant 1.$$

It follows

$$\left|f_{x_1, U}(\vec{y})\right| \leqslant \frac{1}{n} \qquad \text{for} \quad n = 1, 2, \ldots, \quad \text{hence}$$

$f_{x_1, U}(\vec{y}) = 0$, and then in this case the inequality (1) is proved. Let $\|\vec{y}\|_U > 0$. Put $\vec{z} \underset{\mathrm{df}}{=} \dfrac{\vec{y}}{\|y\|_U}$.

We have $\|\vec{z}\|_U = 1$, hence $z \in U$, and then

$$\left|f_{x_1, U}(\vec{z})\right| \leqslant 1, \text{ i. e.}$$
$$\left|f_{x_1, U}(\vec{y})\right| \leqslant \|\vec{y}\|_U.$$

The theorem is proved.

§ 3. — The matrix space.

7. Let $\aleph$ be a cardinal $\geqslant \aleph_0$ and **R** a stream ordering. Take a set J of any elements α, β, ... whose power is $\aleph$; these elements, as well as the elements $d, e, f \ldots$ of $\oplus\mathbf{R}$ will be used as indices. By a matrix we shall understand a function $A = \{A_{\alpha, d}\}$ defined for all $\alpha \in J$ and all $d \in \oplus\mathbf{R}$ and whose values are real numbers (hence $\{A_{\alpha, d}\}$ is a function of two variables α, d).

Let $\mathfrak{M}$ be the class of all matrices $A = \{A_{\alpha, d}\}$ such that for every α and every $d_0 \in \oplus\mathbf{R}$

$$\sup_{\substack{d\mathbf{R}d_0, \\ \alpha \in J}} |A_{\alpha, d}| < \infty. \quad \ldots\ldots\ldots \quad (0)$$

7. 1. We organize $\mathfrak{M}$ into a linear space $(\mathfrak{M})$ by defining the addition of matrices and the multiplication of matrix by a real number :

$$\{A_{\alpha, d}\} + \{B_{\alpha, d}\} \underset{\mathrm{df}}{=} \{A_{\alpha, d} + B_{\alpha, d}\},$$

$$\lambda . \{A_{\alpha, d}\} \underset{\mathrm{df}}{=} \{\lambda . A_{\alpha, d}\}.$$

The matrix with all elements $= 0$ is the null-vector of this space.

Put

$$\|A\|_{d_0} \underset{\mathrm{df}}{=} \sup_{\substack{d\mathbf{R}d_0 \\ \alpha \in J}} |A_{\alpha, d}|. \quad \ldots\ldots\ldots \quad (1)$$

7.2. We easily check that this function of A and d_0 satisfies all the Hyers' conditions for the **R**-convex pseudonorm.

Hence $\mathfrak{M}$ is organized into a pseudo-normed space with the pseudo-norm (1).

§ 4. — Embedding of (L) into a matrix space.

8. Let (L) be a real linear convex topological space; by its separability-cardinal we shall understand the smallest cardinal $\aleph$ such that there exists a subset Q of L with power $\aleph$ and everywhere dense in (L), i. e. for every vector $\vec{x}_0 \in L$ and every topologically open set G containing $\vec{x}_0$, there exists $\vec{x} \in Q \cap G$.

Let us choose a system $\mathfrak{U}$ of N. star. nbhds, such that $L\bar{\in}\mathfrak{U}$ and such that, if $U \in \mathfrak{U}$ then U is linearly closed. Construct the corresponding Hyers' pseudonorm $||x||_U$ with the corresponding stream ordering **R**, defined by $U_1 \mathbf{R} U_2 \cdot \underset{\mathrm{df}}{=} \cdot U_2 \subseteq U_1, \quad (U_1, U_2 \in \mathfrak{U})$.

Let $\aleph$ be the separability-cardinal of (L) according to the topology induced by $\mathfrak{U}$, and choose an everywhere dense set Q of power $\aleph$ in L. We suppose that $\aleph \geqslant \aleph_0$.

According to what was made in § 2, attach to every $\xi \in Q$ and $U_0 \in \mathfrak{U}$ a linear continuous functional $f(x) = f_{\vec{\xi}, U_0}(x)$ with the properties:

$$\sup_{x \in U_0} |f(x)| = 1, \quad f(\vec{\xi}) = ||\vec{\xi}||_{U_0} \quad \ldots\ldots \quad (1)$$

Put

$$A(\vec{x}) = \{A_{\xi, U_0}(\vec{x})\} \underset{\mathrm{df}}{=} \{f_{\xi, U_0}(\vec{x})\} \text{ for all } \vec{x} \in L,$$

$\xi \in Q$ and $U_0 \in \ominus \mathbf{R} = \mathfrak{U}$.

8.1. We shall prove that the matrix just defined satisfies the condition (o), § 3.

We have

$$|A_{\xi, U}(\vec{x})| = |f_{\xi, U}(\vec{x})| \leqslant ||\vec{x}||_U \leqslant ||x||_{U_0}$$

whenever $U \mathbf{R} U_0$. Hence if we fix $\vec{x}$ and U_0, we have

$$\sup_{\substack{\xi \in Q \\ U \mathbf{R} U_0}} |A_{\xi, U}(\vec{x})| \leqslant ||\vec{x}||_{U_0} < \infty \quad \ldots\ldots \quad (2)$$

8.2. Let us take the correspondence **S** which attaches to every $\vec{x} \in L$ the matrix $A(\vec{x})$ taken from $\mathfrak{M}$.

The correspondence **S** is an algebraic homomorphism from L into $\mathfrak{M}$.

8.3. The correspondence **S** preserves the Hyers' speudo-norm. Proof. Let $\vec{x}$ be arbitrary. Take a neighborhood U_0. We have

$$\|A(\vec{x})\|_{U_0} = \sup_{\substack{U\mathbf{R}U_0 \\ \xi \in Q}} |A_{\xi, U}(\vec{x})| = \sup_{\substack{U\mathbf{R}U_0 \\ \xi \in Q}} |f_{\xi, U}(\vec{x})|.$$

We have

$$|f_{\xi, U}(\vec{x})| \leqslant \|\vec{x}\|_U \leqslant \|\vec{x}\|_{U_0} \quad \text{if } U\mathbf{R}U_0,$$

hence

$$\|A(\vec{x})\|_{U_0} \leqslant \|\vec{x}\|_{U_0}. \quad \ldots\ldots \qquad (3)$$

Take $\varepsilon > 0$ and find a $V \in \mathfrak{U}$ such that if $\vec{y} \in V^0 \dot{+} \vec{x}$ (V^0 means the topological interior of V), we have

$$\|\vec{y} - \vec{x}\|_{U_0} \leqslant \varepsilon \quad \ldots\ldots \qquad (4)$$

For $U\mathbf{R}U_0$ we have

$$|f_{\xi, U}(\vec{y}) - f_{\xi, U}(\vec{x})| = |f_{\xi, U}(\vec{y} - \vec{x})| \leqslant \|\vec{y} - \vec{x}\|_U \leqslant \|\vec{y} - \vec{x}\|_{U_0} \leqslant \varepsilon,$$

hence

$$|f_{\xi,U}(\vec{y}) - f_{\xi,U}(\vec{x})| \leqslant \varepsilon \quad \ldots\ldots \qquad (5)$$

for every $\xi \in Q$, every $U\mathbf{R}U_0$ and $\vec{y} \in V^0$.

From (4) we have, because of the convexity condition,

$$\big|\|\vec{y}\|_U - \|\vec{x}\|_U\big| \leqslant \|\vec{y} - \vec{x}\|_U \leqslant \|\vec{y} - \vec{x}\|_{U_0} \leqslant \varepsilon$$

for every $U\mathbf{R}U_0$ and every $\vec{y} \in V^0 \dot{+} \vec{x}$.

Since Q is topologically everywhere dense in L, there exists

$$\vec{\eta} \in Q \cap [V^0 \dot{+} \vec{x}].$$

We have

$$\big|\|\vec{\eta}\|_U - \|\vec{x}\|_U\big| \leqslant \varepsilon$$

and from (5),

$$|f_{\xi,U}(\vec{\eta}) - f_{\xi,U}(\vec{x})| \leqslant \varepsilon$$

for every $\xi \in Q$ and every $U\mathbf{R}U_0$.

Hence

$$\left|f_{\eta,U}(\vec{\eta})-f_{\eta,U}(\vec{x})\right|\leqslant\varepsilon,$$

and then

$$\left|\|\vec{\eta}\|_U - A_{\eta,U}(\vec{x})\right|\leqslant\varepsilon.$$

It follows from (5):

$$\left|A_{\eta,U}(\vec{x})-\|\vec{x}\|_U\right|\leqslant 2\varepsilon$$

for every $U\mathbf{R}U_0$.

Hence

$$\|\vec{x}\|_{U_0}-2\varepsilon\leqslant A_{\eta,U_0}(\vec{x})\leqslant\left|A_{\eta,U_0}(\vec{x})\right|.$$

Consequently

$$\|\vec{x}\|_{U_0}-2\varepsilon\leqslant\sup_{\substack{U\mathbf{R}U_0\\ \xi\in Q}}\left|A_{\xi,U}(\vec{x})\right| \qquad \text{for any} \qquad \varepsilon>0.$$

Hence $$\|\vec{x}\|_{U_0}\leqslant\|A(x)\|_{U_0}. \quad \ldots\ldots \tag{6}$$

From (3) and (6) it follows

$$\|\vec{x}\|_{U_0}=\|A(x)\|_{U_0}.$$

Q. E. D.

Hence we have proved that **S** preserves the pseudo-norm.

8. 4. Finally, let us prove that **S** is a one-to-one relation. Suppose that $A(\vec{x}_1)=A(\vec{x}_2)$.

$$A(\vec{x}_1-\vec{x}_2) \qquad \text{is the o-matrix.}$$

Since $$\|\vec{x}_1-\vec{x}_2\|_U=\|A(\vec{x}_1-\vec{x}_2)\|_U=0$$

for all U, it follows that $\vec{x}_1-\vec{x}_2=\vec{0}$, and then $\vec{x}_1=\vec{x}_2$.

Thus the correspondence **S** is an isometric isomorphism from L into $\mathfrak{M}$ and then also a homeomorphism. It follows that $\mathfrak{M}$ is a universal linear convex topological space in which are isometrically and isomorphically embedded all real topological linear spaces having isomorphic stream orderings of N. str. nbhds and the same separability cardinal $\aleph$.

9. Now let Φ be a not empty set of real linear convex topological spaces (L_i) where i ranges over a not empty set I. If $\mathbf{R}_i$ is a streamordering generated by L_i, and $\aleph_i$ is its separability cardinal, we can put

$\aleph \underset{\mathrm{df}}{=} \sup\limits_{i \in I} \aleph_i$, and find a stream ordering $\mathbf{R}$ such that $\mathbf{R}_i$ is isomorphic with a sub-partial ordering of $\mathbf{R}$, and consequently have a matrix-space which embraces isomorphically and isometrically all spaces of Φ.

10. Now let $\vec{x}_U \vec{\mathbf{R}} \vec{x}$, which means that for every topologically open set G with $\vec{o} \in G$, there exists U_0 such that if $U_0 \mathbf{R} U$, then $\vec{x}_U \in \vec{x} \dotplus G$.

If we put $\vec{y}_U \underset{\mathrm{df}}{=} \vec{x}_U - \vec{x}$, the above is equivalent to the statement: for every, open G with $\vec{o} \in G$ there exists U_0 such that if $U_0 \mathbf{R} U$, then

$$\vec{y}_U \in G.$$

This is equivalent to the statement:

(1) For every Nnbhd U there exists U', such that, if $U_0 \mathbf{R} U$, then

$$\vec{y}_U \in U'.$$

We shall prove that (1) is equivalent to the following statement:

(2) For every U' and $\varepsilon > 0$ there exists U_0 such that if $U_0 \mathbf{R} U$, then

$$\|\vec{y}_U\|_{U'} \leqslant \varepsilon.$$

Proof. Suppose (1). Let p be a natural number with $\frac{1}{p} \leqslant \varepsilon$, and take the topologically open set $\left(\frac{1}{p} \cdot U'\right)^{\circ}$. There exists U_0 such that, if $U_0 \mathbf{R} U$, then

$$y_U \in \left(\frac{1}{p} \cdot U'\right)^{\circ},$$

hence

$$y_U \in \frac{1}{p} \cdot U',$$

hence $p \cdot y_U \in U'$,

hence

$$\|p \cdot y_U\|_{U'} \leqslant 1,$$

$$\|y_U\|_{U'} \leqslant \frac{1}{p} \leqslant \varepsilon, \qquad \text{hence (2).}$$

Suppose (2). Take U' and $\varepsilon = 1$. There exists U_0 such that, if $U_0 \mathbf{R} U$, then $\|\vec{y}_U\|_{U'} \leqslant 1$.

Hence

$$\vec{y}_U \in U', \qquad \text{hence (1).}$$

10. 1. Having this, take the matrix-image $A(\vec{y}_U)$ of $\vec{y}_U$.

(2) is equivalent to the statement :

(3) for every U' and $\varepsilon > 0$ there exists U_0 such that if $U_0 \mathbf{R} U$, then $\|A(\vec{y}_U)\|_{U'} \leqslant \varepsilon$,

because

$$\|A(\vec{y}_U)\|_{U'} = \|\vec{y}_U\|_{U'},$$

$$\|A(\vec{y}_U)\|_{U'} = \sup_{\substack{W' \mathbf{R} U' \\ \alpha \in Q}} |A_{\alpha, W'}(\vec{y}_U)|.$$

Hence (3) is equivalent to the statement :

(4) ...... for every U' and $\varepsilon > 0$ there exists U_0 such that, if $U_0 \mathbf{R} U$, we have

$$|A_{\alpha, W'}(\vec{y}_U)| \leqslant \varepsilon$$

for all $\alpha \in Q$ and all $W' \mathbf{R} U'$.

Now $A_{\alpha}, W'(\vec{y}_U)$ can be conceived as a function $\varphi_U(\alpha, W')$ of two variables α, W' defined for all $\alpha \in Q$ and all W' with $W' \mathbf{R} U'$.

The statement (4) says that the stream sequence $\{\varphi_U\}$ converges uniformly, i. e. For every $\varepsilon > 0$ there exists U_0, such that for all U with $U_0 \mathbf{R} U$ and all (α, W') we have

$$|\varphi_U(\alpha, W')| \leqslant \varepsilon.$$

Hence for every U' the stream sequence $\varphi_U(\alpha, W)$ of functions restricted to those (α, W) for which $W \mathbf{R} U'$, converges uniformly to the O-matrix. We can say that every matrix, restricted from above, converges uniformly.

Hence the correspondance **S** between vectors of the space L and matrices transforms every convergent **R**-stream sequence $\{\vec{x}_U\}$ of vectors into a corresponding stream sequence of functions $\varphi_U(\alpha, W)$ whose all restrictions from above (i. e. $W \mathbf{R} W_0$, where W_0 is fixed) converge uniformly.

This reminds an ordinary sequence of functions $g_n(p, q)$, $(n = 1, 2 \ldots)$ defined for $0 \leqslant q$, $0 \leqslant p \leqslant 1$, which converges uniformly in every rectangle $0 \leqslant p \leqslant 1$, $0 \leqslant q \leqslant q_0$.

REFERENCES

[1] A. Kolmogoroff. Zur Normierbarkeit eines allgemeinen topologischen linearen Raumes. *Studia Math.* 5 (1934), pp. 29-33.

[2] C. Kuratowski. *Topologie I. Monografie Matematyczne.* Warszawa-Lwów (1933).

[3] P. Alexandroff & H. Hopf. Topologie. Berlin (1935).

[4] N. Bourbaki. *Topologie générale.* Chap. 1. Structures topologiques. *Actual. Sci. et Industr.* Hermann, Paris (1940).

[5] J. v. Neumann. On complete topological spaces. *Trans. Amer. Math. Soc.* 37 (1935), pp. 1-20.

[6] D. H. Hyers. Pseudo-normed linear spaces and abelian groups. *Duke Math. J.*, vol. 5 (1939), pp. 628-634.

[7] J. P. La Salle. Pseudo-normed linear spaces. *Duke Math. J.*, vol. 8, (1941).

[8] E. H. Moore & H. L. Smith. A general theory of limits. *Amer. J. of Math.* 44 (1922), pp. 102-121.

[9] A. N. Whitehead et Bertrand Russel. *Principia Mathematica.* vol. 1. Cambridge (England) Univ. Press. 1910.

[10] H. M. Mac Neille. Partially ordered sets. *Trans. Amer. Math. Soc.* 42. (1937), pp. 416-460.

[11] W. D. Berg et O. M. Nikodým. *On convex sets in abstract linear spaces.* I, II (To appear in an american periodical).

[12] Edward Silverman. Definitions of Lebesgue area for surfaces in metric spaces. *Rivista di matematica della Università di Parma* (1951), (in print).

[13] Jean Dieudonné. Sur le théorème de Hahn-Banach. *Revue scientifique.* 79. N° 12 (1914), pp. 642-643.

[14] S. Banach. *Opérations linéaires* (Warszàwa, 1932.)

[15] O. M. Nikodým. *Criteria for continuity of a linear functional in real linear spaces.* (To be published in an american periodical.)

[16] Leon Alaoglu. Week topologies of normed linear spaces. *Ann. of Math.* 41 (1940), pp. 252-267. (Where one can find another universal space for the Banach spaces.)

[17] Garrett Birkhoff. Moore-Smith convergence in general topology. *Ann. of Math.* 38 (1937), pp. 39-56.

[18] John V. Wehausen. Transformations in linear topological spaces. *Duke Math. Journ.*, vol. 4, 1938, pp. 157-169.

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