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# UNIFORM APPROXIMATION OF HARMONIC FUNCTIONS

**by G. F. VINCENT-SMITH**

## Introduction.

Let $\omega$ be a bounded open set in Euclidian $n$-space $(n > 1)$, with closure $\overline{\omega}$ and frontier $\omega^*$. Corollary 1 below gives a necessary and sufficient condition that each continuous real-valued function on $\overline{\omega}$ harmonic in $\omega$, may be uniformly approximated on $\overline{\omega}$ by functions harmonic in a neighbourhood of $\overline{\omega}$. The purpose of this paper is to extend corollary 1 to axiomatic potential theory.

Suppose $a_p$ is a sequence of points chosen one from each domain in $\complement\overline{\omega}$. Let $\Phi_n^{a_p}$ be the elementary harmonic functions relative to $a_p$ [10, § 1]. Then $\Phi_n^{a_p}$ is a potential of support $a_p$, $n = 1, 2, \ldots$ If $C(\overline{\omega})$ denotes the space of continuous real-valued functions on $\overline{\omega}$, then following Deny [9], [10, § 4] and de La Pradelle [16], we consider the following linear function spaces :

$M = \{f \in C(\overline{\omega}) : f$ is harmonic in $\omega\}$;

$L = \{f \in C(\overline{\omega}) : f$ extends to a function harmonic in a neighbourhood $U_f$ of $\overline{\omega}\}$;

$K = \{f \in C(\overline{\omega}) : f$ extends to the difference of two potentials with compact support contained in $\complement\overline{\omega}\}$;

$J = f \in C(\overline{\omega}) : f$ extends to a function in the linear span of the elementary harmonic functions $\Phi_n^{a_p}\}$.

Then $J \subset K \subset L \subset M$, and Deny [10, th. 5] proves the following approximation theorem.

THEOREM 1. — $J$ *is uniformly dense in* $M$ *if and only if the sets* $\complement \omega$ *and* $\complement \overline{\omega}$ *are effilé (thin) at the same points.*

The points at which $\complement \omega$ is not thin [7, ch. VII, § 1] are precisely the regular points of $\omega^*$ for the Dirichlet problem [7, ch. VIII, § 6], while the points where $\complement \overline{\omega}$ is not thin are precisely the stable points of $\omega^*$ for the Dirichlet problem.

Suppose now that $\omega$ is a relatively compact open subset of a harmonic space $\Omega$ which satisfies Brelot's axioms 1, 2 and 3, and on which there exists a strictly positive potential. Suppose also that the topology of $\Omega$ has a countable base of completely determining open sets, that potentials with the same one point support are proportional, and that adjoint potentials with one point support are proportional. De La Pradelle [16, th. 5] proves the following generalisation of theorem 1.

THEOREM 1'. — $K$ *is uniformly dense in* $M$ *if and only if the sets* $\complement \omega$ *and* $\complement \overline{\omega}$ *are thin at the same points.*

Deny's proof of theorem 1 consists of showing that the same measures on annilhilate $J$ and $M$, and the same method is used to prove theorem 1'. In this paper the conditions on $\Omega$ are relaxed, and the following corollary to theorem 1 is generalised.

COROLLARY 1. — $L$ *is uniformly dense in* $M$ *if and only if every regular point of* $\omega^*$ *is stable.*

The proof of corollary 1, using elementary harmonic functions, does not adapt to axiomatic potential theory. In example 2 we give a proof which does generalise. This proof is rather satisfying, since it uses Bauer's characterisation of regular points, and the following generalisation of the Stone-Weierstrass theorem [13, th. 5].

THEOREM 2. — *Suppose that* $X$ *is a compact Hausdorff space, that* $L$ *is a linear subspace of* $C(X)$ *which contains the* *constant functions, separates the points of* X, *and has the weak Riesz separation property, and that* L *is contained in the linear subspace* M *of* C(X). *Then* L *is uniformly dense in* M *if and only if* $\partial_L(X) = \partial_M(X)$.

L is said to have the weak Riesz separation property (R.s.p.) if whenever $\{f_1, f_2, g_1, g_2\} \subset L$ with $f_1 \vee f_2 < g_1 \wedge g_2$, there exists $h \in L$ with $f_1 \vee f_2 \leqslant h \leqslant g_1 \wedge g_2$. The Choquet boundary of M is denoted $\partial_M(X)$ [15] and Bauer [1, th. 6] shows that in the classical case $\partial_M(\overline{\omega})$ is precisely the set of regular points of $\omega^*$. Brelot [7, ch. VIII, § 1] remarks that this remains true when $\omega$ is a relatively compact open subset of a harmonic space satisfying Brelot's axioms 1, 2 and 3′, and that in this case $\partial_L(\overline{\omega})$ is precisely the set of stable points of $\omega^*$. Using Bauer's results, corollary 1 is an immediate consequence of Theorem 2, both in the classical case, and when $\omega$ is a relatively compact open subset of a harmonic space satisfying Brelot's axioms 1, 2 and 3′.

If $\omega$ is a relatively compact open subset of one of the harmonic spaces of Boboc and Cornea [4], which are more general than those of Brelot, then the set of regular points of $\omega^*$ corresponds not to $\partial_M(\overline{\omega})$ but to $\omega^* \cap \partial_W(\overline{\omega})$, where $W \subset C(\overline{\omega})$ is the min-stable wedge of continuous functions on $\overline{\omega}$ superharmonic in $\omega$. In this case we need a strengthened form of theorem 2, which, together with this characterisation of regular points, has corollary 1 as a direct consequence. This we supply in theorem 4.

In order to strengthen theorem 2 we consider min-stable wedges $\mathcal{G} \subset W$ in C(X), and a geometric simplex $(X, \mathcal{G}, L)$. In theorem 4 we give a sufficient condition that L be uniformly dense in the space M of continuous W-affine functions on X. This condition is given in terms of the Choquet boundaries $\partial_W(X)$ and $\partial_{\mathcal{G}}(X)$. In lemma 5 a pair of conditions equivalent to this is given. These are of a more analytic nature. Theorem 4 is deduced from proposition 1, which is a characterisation of geometric simplexes. This is proved by repeated use of filtering arguments together with the folowing form of Dini's theorem.

THEOREM 3. — *If* $\{f_i : i \in I\}$ *is an upward filtering family in* C(X) *and* $g$ *is an upper bounded upper semicontinuous* *function such that* $g < \sup\{f_i : i \in I\}$, *then* $g < f_{i_0}$ *for some* $i_0 \in I$.

$f > 0$ ($\geqslant 0$) will mean that $f(x) > 0$ ($\geqslant 0$) for all $x \in X$.

## A characterisation of geometric simplexes.

Let X be a compact Hausdorff space, and let $\mathcal{S} \subset W$ be min-stable wedges in C(X). If $f \wedge g \in W$ whenever $f, g \in W$ then W is said to be *min-stable. We shall assume that* $\mathcal{S}$ *contains a function* $p \geqslant 1$ *and a function* $q < -1$. The Choquet theory for min-stable wedges has been developed in [11] [5] where proofs of the following results may be found.

The wedge W induces a partial order $\prec_W$ on the positive regular Borel measures on X given by the formula

$$\mu \prec_W \lambda, \qquad \lambda(f) \leqslant \mu(f) \text{ whenever } f \in W.$$

A measure which is maximal for $\prec_W$ is said to be W-*extremal.* A meausre $\mu$ is W-extremal if and only if

$$(1) \qquad \mu(g) = \inf\{\mu(f) : g < f \in W\}$$

whenever $g \in -W$ [5, Th. 1.2]. An extended real-valued function $g$ on X is $\omega$-*concave* if the upper integral $\overline{\int} g\, d\mu \leqslant g(x)$ whenever $\varepsilon_x \prec_W \mu$. The function $g$ is W-*affine* if both $g$ and $-g$ are W-concave. The min-stable wedge of lower bounded extended real-valued lower semicontinuous $\omega$-concave functions on X will be denoted $\hat{W}$.

Lemma 1. — [11, Th. 1] [5, Cor. 1.4 *d*)]. *Each* $f \in \hat{W}$ *is the pointwise supremum of an upward filtering family in* W.

A closed subset A of X is a W-*face* (W-absorbent set [5, § 2], W-extreme set [11, § 2]) if for each $x \in A$

$$\mu(X \setminus A) = 0 \quad \text{whenever } \varepsilon_x \prec_W \mu.$$

If A is a W-face and $f \in \hat{W}$ then the function $f_A^\infty$, equal to $f$ on A and to $+\infty$ on $X \setminus A$, belongs to $\hat{W}$ [11, § 2]. The W-faces are ordered by inclusion, and each W-face contains a minimal W-face. The measure $\varepsilon_x$ is W-extremal if and only if $x$ belongs to a minimal W-face. The Choquet boundary of W is the union of all minimal W-faces of X, and is denoted $\partial_W(X)$ [5, § 2]. Each $\mathcal{G}$-face is a W-face, so that each minimal $\mathcal{G}$-face contains at least one minimal W-face.

LEMMA 2. — [2, Satz 2] [5, Cor. 2.1] *A function* $f \in \hat{W}$ *is positive if and only if it is positive on* $\partial_W(X)$.

We say that W *distinguishes* the points $x, y \in X$ if there exists $f, g \in W$ such that

$$f(x)g(y) \neq f(y)g(x).$$

If W contains the constant functions, then W distinguishes $x$ and $y$ if and only if W separates $x$ and $y$. The subspace $(W - W)/p = \{(f - g)/p : f, g \in W\}$ is a sublattice of C(X) containing the constant functions. $(W - W)/p$ separates points of X if and only if W distinguishes points of X. By Stone's theorem, W − W is uniformly dense in C(X) if and only if W distinguishes points of X. The following lemma is an immediate consequence of [5, Th. 2.1 *c*)].

LEMMA 3. — W *distinguishes* $x, y \in \partial_W(X)$ *if and only if* $x$ *and* $y$ *belong to different minimal* W*-faces of* X.

*Example* 1. — Let $X = [0, 1] \times [0, 1]$, and let $\mathcal{G} = \{f \in C(X) : y \rightsquigarrow f(x, y)$ is convex for each $x$, and $x \rightsquigarrow f(x, y)$ is affine with $f(1, y) = 2f(0, y)$ for each $y\}$. Then the sets $A = \{(x, 0) : x \in [0, 1]\}$ and $B = \{(x, 1) : x \in [0, 1]\}$ are minimal $\mathcal{G}$-faces. $\mathcal{G}$ separates, yet does not distinguish the points of A. The Choquet boundary

$$\partial_{\mathcal{G}}(X) = A \cup B.$$

The $\mathcal{G}$-affine functions are the $f \in \mathcal{G}$ which are affine in $y$ for each $x$.

LEMMA 4. — *If* $\mathcal{G} \subset W$ *are min-stable wedges in* C(X), *and if* $\mathcal{G}$ *contains a positive function* $p$ *and a negative function* $q$, *then the following conditions are equivalent*:

(i) *For each pair of (disjoint) minimal* $\omega$*-faces* $A_1, A_2$, *there exists a pair of (disjoint)* $\mathcal{G}$*-faces* $B_1, B_2$, *such that* $A_1 \subset B_1$ *and* $A_2 \subset B_2$;

(ii) *Same statement as* (i) *but with* $B_1, B_2$ *minimal* $\mathcal{G}$*-faces*;

(iii) $\partial_W(X) \subset \partial_{\mathcal{S}}(X)$ *and* $\mathcal{S}$ *distinguishes points of* $\partial_W(X)$ *which are distinguished by* W.

*Proof.* — (i) $\Longrightarrow$ (ii). Let A be a minimal W-face, and put $G = \bigcap \{F : F$ is an $\mathcal{S}$-face and $A \subset F\}$. Then G is an $\mathcal{S}$-face, and contains a minimal $\mathcal{S}$-face H. Now H is a W-face and contains a minimal W-face A′. If $A \cap A' = \emptyset$, then there exist disjoint $\mathcal{S}$-faces B, B′ such that $A \in B$ and $A' \in B'$. Then $B \cap G$ is an $\mathcal{S}$-face properly contained in G, which contradicts the definition of G. Therefore $A = A'$, so that $G \subset H$ and G is a minimal $\mathcal{S}$-face. It follows immediately that if $A_1$, $A_2$ are disjoint minimal W-faces, then $A_1 \subset G_1$ and $A_2 \subset G_2$, where $G_1$ and $G_2$ are disjoint minimal $\mathcal{S}$-faces.

(ii) $\Longrightarrow$ (iii). $\partial_W(X) = \bigcup \{A : A$ is a minimal W-face$\}$ $\subset \bigcup \{B : B$ is a minimal $\mathcal{S}$-face$\} = \partial_{\mathcal{S}}(X)$. Suppose W distinguishes $x_1$ and $x_2 \in \partial_\omega(X)$, then by lemma 3 there are disjoint minimal W-faces $A_1$ and $A_2$ with $x_1 \in A_1$ and $x_2 \in A_2$. Therefore there are disjoint minimal $\mathcal{S}$-faces $B_1$, $B_2$ with $x_1 \in A_1 \subset B_1$ and $x_2 \in A_2 \subset B_2$, and by lemma 3 $\mathcal{S}$ distinguishes $x_1$ and $x_2$.

(iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i). If $A_1$ and $A_2$ are disjoint minimal W-faces, then the points $x_1 \in A_1$ and $x_2 \in A_2$ are distinguished by W. Therefore $x_1$ and $x_2$ are distinguished by $\mathcal{S}$. Since $x_1$, $x_2 \in \partial_W(X) \subset \partial_{\mathcal{S}}(X)$ there are disjoint minimal $\mathcal{S}$-faces $B_1$, $B_2$ with $x_1 \in B_1$ and $x_2 \in B_2$. Since $A_1$ is minimal $A_1 \subset A_1 \cap B_1$, so that $A_1 \subset B_1$. Similarly $A_2 \subset B_2$.

If L and M are linear subspaces of C(X), then we will put

$$\mathcal{L} = \{f_1 \wedge \cdots \wedge f_r : f_i \in L, \quad i = 1 \ldots r\}$$

and

$$\mathcal{M} = \{f_1 \wedge \cdots \wedge f_r : f_i \in M, \quad i = 1 \ldots r\}.$$

Then $\mathcal{L}$ and $\mathcal{M}$ are min-stable wedges in C(X) and if the functions in L are $\mathcal{S}$-affine then $\mathcal{L} \subset \hat{\mathcal{S}}$.

Suppose L is a linear subspace of continuous $\mathcal{S}$-affine functions on X. The triple $(X, \mathcal{S}, L)$ is a *geometric simplex* if given $f \in -\mathcal{S}$ and $g \in \mathcal{S}$ with $f < g$, then there exists

$h \in L$ with $f \leqslant h \leqslant g$ [5, § 4]. We have assumed that $p, q \in \mathcal{S}$ with $p > 0$ and $q < 0$, so that $\alpha p < q$ for some $\alpha < 0$. If $(X, \mathcal{S}, L)$ is a geometric simplex it follows that L contains an element $l > 0$.

Proposition 1. — $(X, \mathcal{S}, L)$ *is a geometric simplex if and only if* L *has the weak* R.s.p., $\partial_{\mathcal{S}}(X) \subset \partial_{\mathcal{L}}(X)$ *and* $\mathcal{L}$ *distinguishes points of* $\partial_{\mathcal{S}}(X)$ *which are distinguished by* $\mathcal{S}$.

*Proof.* — Let $(X, \mathcal{S}, L)$ be a geometric simplex and suppose that $\{f_1, f_2, g_1, g_2\} \subset L$ with $f_1 \vee f_2 < g_1 \wedge g_2$. Since $g_1 \wedge g_2 \in \hat{\mathcal{S}}$ there exists a family $\Lambda = \{h_i \in \mathcal{S} : h_i < g_1 \wedge g_2, i \in I\}$ filtering up to $g_1 \wedge g_2$. By Dini's theorem there exists $h_{i_0} \in \Lambda$ such that $f_1 \vee f_2 < h_{i_0} < g_1 \wedge g_2$. Similarly, there exists $h_{j_0} \in -\mathcal{S}$ such that $f_1 \vee f_2 < h_{j_0} < h_{i_0} < g_1 \wedge g_2$. Since $(X, \mathcal{S}, L)$ is a geometric simplex there exists $h \in L$ such that

$$f_1 \vee f_2 \leqslant h_{j_0} \leqslant h \leqslant h_{i_0} \leqslant g_1 \wedge g_2$$

and L has the weak R.s.p.

Suppose $x_i \in \partial_{\mathcal{S}}(X)$, $i = 1, 2$, and $f_j \in -\mathcal{L}$, $j = 1, 2$. Then $f_j \in -\overline{\mathcal{S}}$ and by (1)

$$\begin{aligned} f_j(x_i) &= \inf \{h(x_i) : \ f_j < h \in \mathcal{S}\}, \\ &= \inf \{g(x_i) : \ g \in L, f_j < g < h \in \mathcal{S}\}, \end{aligned} \tag{2}$$

since $(X, \mathcal{S}, L)$ is a geometric simplex. Therefore $x_i \in \partial_{\mathcal{L}}(X)$, and $\partial_{\mathcal{S}}(X) \subset \partial_{\mathcal{L}}(X)$. If $\varepsilon > 0$ then by (2) there exists $g_1$, $g_2 \in L$ such hat

$$|g_j(x_i) - f_j(x_i)| < \varepsilon, \quad i, j = 1, 2.$$

If $f_1$ and $f_2$ distinguish $x_1$ and $x_2$, and $\varepsilon$ is small enough, then $g_1$ and $g_2$ distinguish $x_1$ and $x_2$, and the conditions of the proposition are necessary.

Suppose that $(X, \mathcal{S}, L)$ satisfies the given conditions, and that $f \in -\mathcal{S}$, $g \in \mathcal{S}$ with $f < g$. If A is a minimal $\mathcal{S}$-face, then by lemma 4 A is contained in a minimal $\mathcal{L}$-face B. If $\alpha$ is the smallest real number such that $\alpha l \geqslant f$ on B, then

$$D = \{x \in B : \ (\alpha l - f)(x) = 0\} = \{x \in X : \ (\alpha l - f)\ {}^{\infty}_{B}\ (x) = 0\}$$

is a $\mathcal{S}$-face [5, prop. 2.2]. D contains a minimal $\mathcal{S}$-face A′, and by lemma 4, A = A′. Similarly

$$A \subset \{x \in B : \quad (g - \beta l)(x) = 0\},$$

where $\beta$ is the greatest real number such that $\beta l \leqslant g$ on B. Since $l$ is strictly positive, $\alpha < \beta$, and if $\alpha < \gamma < \beta$, then $f < \gamma l < g$ on B. By lemma 1, the function $(\gamma l)_B^\infty$ is the supremum of an increasing filtering family $\{f_i \in \mathcal{L} : i \in I\}$. Since $f < (\gamma l)_B^\infty$, it follows from Dini's theorem that $f < f_{i_0}$ $(= h_1 \wedge \cdots \wedge h_n : h_r \in L,\ r = 1, \ldots, n)$ for some $i_0 \in I$. Therefore there exists $h \in L$ with $f < h$ on X and $h < g$ on B.

Suppose that $f < h_1 \wedge h_2$ with $h_1, h_2 \in L$. Since L has the weak R.s.p. and contains a positive function, the family $\{k \in L : k < h_1 \wedge h_2\}$ filters up. Therefore

$$\bar{k} = \sup \{k' \in \mathcal{L} : k' < h_1 \wedge h_2\} = \sup \{k \in L : k < h_1 \wedge h_2\}.$$

Thus $\bar{k}$ is the supremum of a filtering family of continuous $\mathcal{L}$-affine functions and is therefore $\mathcal{L}$-affine and lower semicontinuous. Therefore $\bar{k} \in \hat{\mathcal{S}}$. It follows from (1) that $\bar{k} = h_1 \wedge h_2$ on $\partial_{\mathcal{L}}(x)$. Since $\partial_{\mathcal{S}}(X) \subset \partial_{\mathcal{L}}(X)$, the function $\bar{k} - f$ is strictly positive on $\partial_{\mathcal{S}}(X)$. By lemma 2, $\bar{k} > f$. By Dini's theorem there exists $h \in L$ such that $f < h < h_1 \wedge h_2$, and the family $\mathcal{F} = \{h \in L : f < h$ is filtering down.

Therefore the function $\underline{h} = \inf \{h \in L : f < h\}$ is upper semicontinuous $\mathcal{L}$-affine and $\mathcal{S}$-affine. If A is a minimal $\mathcal{S}$-face, then there exists $h \in \mathcal{F}$ with $h < g$ on A. Therefore $\underline{h} < g$ on $\partial_{\mathcal{S}}(X)$, and by lemma 2, $\underline{h} < g$. By Dini's theorem there exists $h \in L$ such that $f < h < g$. Therefore $(X, \mathcal{S}, L)$ is a geometric simplex.

We may now extend the density theorem in [13].

THEOREM 4. — *Suppose that* $\mathcal{S} \subset W$ *are min-stable wedges in* C(X), *and that* $\mathcal{S}$ *contains a positive function* $p$ *and a negative function* $q$. *Let* $M = \{f \in C(X) : f$ *is* W-*affine*$\}$ *and let* $L \subset C(X)$ *be a linear subspace of* $\mathcal{S}$-*affine functions. If* $(X, \mathcal{S}, L)$ *is a geometric simplex and if* $\partial_W(X) \subset \partial_{\mathcal{S}}(X)$ *and if* $\mathcal{S}$ *distinguishes points of* $\partial_W(X)$ *which are distinguished by* W, *then* L *is uniformly dense in* M.

*Proof.* — It follows from proposition 1 that $\partial_W(X) \subset \partial_{\mathcal{L}}(X)$ and that $\mathcal{L}$ distinguishes points of $\partial_W(X)$ distinguished by W. Therefore (X, W, L) is a geometric simplex. If $f \in M$ and $\varepsilon > 0$, then by lemma 1 and by Dini's theorem there exist $h \in -W$, $k \in W$ such that

$$f + \varepsilon q < h < k < f + \varepsilon p.$$

Since $(X, \mathcal{W}, L)$ is a geometric simplex, there exists $g \in L$ such that $f + \varepsilon q < h \leqslant g \leqslant k < f + \varepsilon p$, and L is uniformly dense in M.

Suppose that $L \subset M$ are linear subspaces of C(X) containing the constant functions, and that L has the weak R.s.p. Then $\mathcal{L}$ and $\mathcal{M}$ are min-stable wedges, $\partial_{\mathcal{L}}(X) = \partial_L(X)$ the Choquet boundary of L, and $\partial_{\mathcal{M}}(X) = \partial_M(X)$, the Choquet boundary of M [15], and $(X, \mathcal{L}, L)$ is a geometric simplex. Since L contains the constant functions, points are distinguished by $\mathcal{L}$ (resp. $\mathcal{M}$) if and only if they are separated by L (resp. M). We have therefore the following corollary to theorem 4.

COROLLARY 1. — [13, cor. to th. 5]. *If* $\partial_L(X) = \partial_M(X)$ *and* L *separates the points of* $\partial_M(X)$ *which are separated by* M, *then* L *is uniformly dense in* M.

We may replace the conditions in proposition 1 and theorem by a pair of conditions very similar to those used by D. A. Edwards [12].

Suppose we are given wedges $W_0$ and $\mathcal{G}_0$ such that the min-stable wedges $\{f_1 \wedge \cdots \wedge f_r : f_i \in \mathcal{W}_0,\ i = 1, \ldots, r\}$ and $\{f_1 \wedge \cdots \wedge f_r : f_i \in \mathcal{G}_0,\ i = 1, \ldots, r\}$ are uniformly dense in W and $\mathcal{G}$ respectively. For example, in corollary 1 we could take $M = W_0$ and $L = \mathcal{G}_0$. Since $\mathcal{G}$ contains a positive element it follows that $\mathcal{G}_0$ contains a positive element which we may take as $p$. We consider the following conditions:

(*a*) If $x \in \partial_W(X)$, $\varepsilon > 0$ and $f_1, f_2 \in \mathcal{G}_0$, then there exists $g \in -\mathcal{G}$ such that $g < f_1 \wedge f_2$ and $f_1 \wedge f_2(x) < g(x) + \varepsilon$.

(*a'*) Same as (*a*), but with $g \in -\mathcal{G}_0$.

(*b*) If $x_1$ and $x_2 \in \partial_W(X)$, $\varepsilon > 0$ and $0 < f \in W_0$, then there exists $g \in \mathcal{G}_0$ such that $|f(x_i) - g(x_i)| < \varepsilon$, $i = 1, 2$.

Suppose that $\mathcal{G}_0$ satisfies condition (*a*). Then there exists $\{h_1, \ldots, h_n\} \subset -\mathcal{G}_0$ such that $g \leqslant h_1 \vee \cdots \vee h_n < f_1 \wedge f_2$.

Then $h_i < f_1 \wedge f_2$ and $f_1 \wedge f_2(x) < h_i(x) + \varepsilon$ for some $i$ with $1 \leqslant i \leqslant n$. Therefore $(a)$ implies $(a')$ and since $(a')$ implies $(a)$, the two conditions are equivalent.

Lemma 5. — $\partial_W(X) \subset \partial_{\mathcal{G}}(X)$ *if and only if* $\mathcal{G}_0$ *satisfies condition* $(a)$.

*Proof.* — It follows from (1) that $x \in \partial_{\mathcal{G}}(X)$ if and only if whenever $f \in \mathcal{G}$ there exists $g \in -\mathcal{G}$ with $g < f$ and $f(x) < g(x) + \varepsilon$. Therefore the condition is necessary.

If $\mathcal{G}_0$ satisfies condition $(a)$ then it satisfies $(a')$. Consider $x \in \partial_W(X)$, $\varepsilon > 0$ and $f \in \mathcal{G}$. If $\delta > 0$ choose $\{f_i, \ldots, f_n\} \subset \mathcal{G}$. such that $|f - f_1 \wedge \cdots \wedge f_n| < \delta$. Let

$$c = \min \{f_i(x) : \quad i = 1, \ldots, n\}.$$

By condition $(a')$ there exists $k \in \mathcal{G}_0$ such that $k(x) = -c$ and $\{g_1, \ldots, g_n\} \subset -\mathcal{G}_0$ such that

$$g_i < (f_i + k) \wedge 0, \qquad g_i(x) > -\varepsilon/n, \quad i = 1, \ldots, n.$$

Then

$$g_0 = \Sigma\{g_i : i = 1, \ldots, n\} < (f_1 + k) \wedge \cdots \wedge (f_n + k) = f_1 \wedge \cdots \wedge f_n + k,$$

and $g_0(x) > -\varepsilon$. Therefore $g_0 - k = h \in -\mathcal{G}_0$ and $h < f_1 \wedge \cdots \wedge f_n < f + \delta$ with $h(x) > c - \varepsilon > f(x) - \delta - \varepsilon$. Choosing $\delta$ such that $\delta(1 + p(x)) < \varepsilon$ and then putting $g = h - \delta p$ it follows that $g < f$ and $g(x) > f(x) - 2\varepsilon$. It follows from (1) that $x \in \partial_{\mathcal{G}}(X)$ and that $\partial_W(X) \subset \partial_{\mathcal{G}}(X)$.

Lemma 6. — $\partial_W(X) \subset \partial_{\mathcal{G}}(X)$ *and* $\mathcal{G}$ *distinguishes points of* $\partial_W(X)$ *which are distinguished by* W *if and only if* $\mathcal{G}_0$ *and* $W_0$ *satisfy conditions* $(a)$ *and* $(b)$.

*Proof.* — If W distinguishes the points $x_1$ and $x_2$ of $\partial_W(X)$, then there exists $f \in W$ such that

$$f(x_1)p(x_2) \neq f(x_2)p(x_1).$$

Since $p \in W$, we may assume that $f > 0$. If $\mathcal{G}_0$ satisfies condition $(b)$ and $\varepsilon < 0$, then there exists $g \in \mathcal{G}_0$ such that $|g(x_i) - f(x_i)| < \varepsilon$, $i = 1, 2$. If $\varepsilon$ is small enough, then $g(x_1)p(x_2) \neq g(x_2)p(x_1)$, and $\mathcal{G}$ distinguishes $x_1$ and $x_2$.

If $\mathcal{S}_0$ also satisfies condition $(a)$ then $\partial_W(X) \subset \partial_{\mathcal{S}}(X)$, by lemma 4.

Conversely, suppose that $x_1$, $x_2 \subset \partial_W(X)$, $\varepsilon > 0$ and $0 < f \in W_0$. We consider the following cases:

(i) $f(x_1)p(x_2) = f(x_2)p(x_1)$. Choose real $c$ such that $cp(x_1) = f(x_1)$ and $cp(x_2) = f(x_2)$. Then $cp = g \in \mathcal{S}_0$ and $|f(x_i) - g(x_i)| = 0 < \varepsilon$, $i = 1, 2$.

(ii) $f(x_1)p(x_2) < f(x_2)p(x_1)$. If $\partial_W(X) \subset \partial_{\mathcal{S}}(X)$ and $\mathcal{S}$ distinguishes points of $\partial_W(X)$ distinguished by W, then $\mathcal{S}$ distinguishes $x_1$ and $x_2$, and $x_1$ belongs to a minimal $\mathcal{S}$-face A. Then the function $0_A^\infty \in \hat{\mathcal{S}}$. It follows from lemma 1 that there exists $k \in \mathcal{S}$ such that $k(x_1) < 0$ and $k(x_2) > 0$. Since $\mathcal{S}_0$ is a wedge containing $p$, there exists $h \in \mathcal{S}_0$ such that $h(x_1) = 0$ and $h(x_2) > 0$. Define $g \in \mathcal{S}_0$ by the formula

$$g = \frac{f(x_1)}{p(x_1)}\, p + \frac{f(x_2)p(x_1) - f(x_1)p(x_2)}{f(x_1)h(x_2)}\, h$$

Then $|f(x_i) - g(x_i)| = 0 < \varepsilon$, $i = 1, 2$, and $W_0$ and $\mathcal{S}_0$ satisfy the conditions $(a)$ and $(b)$.

## Application to axiomatic potential theory.

Let $\omega$ be an open relatively compact MP subset [4, § 2] of a harmonic space which satisfies one of the axiomatic systems [4, $H_0$, ..., $H_4$] [3, $A_1$, ..., $A_3$]. Let

$W = \{f \in C(\overline{\omega}) : f$ is superharmonic in $\omega\}$,

$\mathcal{S} = \{f \in C(\overline{\omega}) : f$ extends to a function superharmonic in an open neighbourhood $U_f$ of $\overline{\omega}\}$,

and define L and M as in the introduction. Then $\mathcal{S} \subset W$ are min-stable wedges in $C(\overline{\omega})$, M is the space of continuous W-affine functions, and L is the space of conitnuous $\mathcal{S}$-affine functions on $\overline{\omega}$. We suppose that $\mathcal{S}$ *contains a positive function* $p$ *and a negative function* $q$, *and distinguishes points of* $\omega^*$.

LEMMA 7. — *If* A *is a minimal* W-*face of* $\overline{\omega}$, *then* $A \cap \omega^* \neq \emptyset$.

*Proof.* — The function $O_A$ belongs to $\hat{W}$ and is therefore hyperharmonic [4, § 1]. Suppose $A \cap \omega^* = \emptyset$, then $0_A^\infty - p$

is non-negative on $\omega\setminus A$, and for any point $x_0 \in \omega^*$, $\liminf \{(0_A^\infty - p)(x) : x \to x_0\} = \infty$. Since $\omega$ is an MP set, $0_A^\infty - p > 0$ and therefore $A = \emptyset$. Therefore $A \cap \omega^* \neq \emptyset$.

We now recall the definitions and some properties of regular and stable points of $\omega^*$. If $f \in C(\omega^*)$ put $\Phi_f^\omega = \{v : v$ is hyperharmonic in $\omega$ and

$$\liminf \{v(x) : x \in \omega,\ x \to x_0\} \geqslant f(x_0),\ x \in \omega^*\},$$

put $\overline{H}_f^\omega = \inf \{v : v \in \Phi_f^\omega\}$, and put $\underline{H}_f^\omega = -\overline{H}_{(-f)}^\omega$. Since $(\mathcal{S} - \mathcal{S})|_{\omega^*}$ is uniformly dense in $C(\omega^*)$ it may be shown as in [7, ch. VIII, § 3] [14] [3, Satz 24], that $\underline{H}_f^\omega = \overline{H}_f^\omega = H_f^\omega$ whenever $f \in C(\omega^*)$. Moreover $f \rightsquigarrow H_f$ is a linear map from $C(\omega^*)$ to the bounded continuous functions on $\omega$, which is continuous for the supremum norms. A point $x_0 \in \omega^*$ is *regular* if $\lim \{H_f(x) : x \in \omega,\ x \to x_0\} = f(x_0)$ whenever $f \in C(\omega^*)$. Since $(\mathcal{S} - \mathcal{S})|_{\omega^*}$ is dense in $C(\omega^*)$ and the map $f \rightsquigarrow H_f^\omega$ is continuous, $x_0$ is regular if and only if $\lim \{H_f^\omega(x) : x \in \omega,\ x \to x_0\} = f(x_0)$ whenever $f \in -\mathcal{S}|_{\omega^*}$.

If $f \in C(\omega^*)$ then put $\Psi_f^\omega = \{v : v$ is hyperharmonic in a neighbourhood of $\overline{\omega}$ and

$$\liminf \{v(x) : x \in \complement\overline{\omega},\ x \to x_0\} \geqslant f(x_0)\},$$

put $\overline{K}_f^\omega = \inf \{v : v \in \Psi_f^\omega\}$ and put $\underline{K}_f^\omega = -\overline{K}_{(-f)}^\omega$. As in [6, § 2] it may be shown that $\underline{K}_f^\omega = \overline{K}_f^\omega = K_f^\omega$, a continuous function on $\overline{\omega}$, harmonic in $\omega$, whenever $f \in C(\omega^*)$. The map $f \rightsquigarrow K_f^\omega$ is a linear map from $C(\omega^*)$ to $C(\overline{\omega})$ continuous for the supremum norms. If $f(x) = K_f^\omega(x)$ whenever $f \in C(\omega^*)$ then $x$ is a *stable* point of $\omega^*$. As with regular points, $x$ is stable if and only if $f(x) = K_f^\omega(x)$ whenever $f \in -\mathcal{S}|_{\omega^*}$.

Suppose that $F \in -\mathcal{S}$, and let $\overline{F}$ be a continuous subharmonic function defined on an open neighbourhood $U_F$ of $\overline{\omega}$, which equals $F$ on $\overline{\omega}$. If $\overline{\omega} = \bigcap \{\omega_i : i \in I\}$ the intersection of a decreasing filtering family of open subsets of $U_F$, then (by an abuse of language) $\{H_F^{\omega_i} : i \in I\}$ is a decreasing filtering family in L, and $K_F = \inf \{H_F^{\omega_i} : i \in I\}$ [6, § 2]. If $x_0 \in \omega^*$ is stable, then

$$F(x_0) = \inf \{H_F^{\omega_i}(x_0) : i \in I\} \geqslant \inf \{h(x_0) : F < h \in \mathcal{S}\},$$

so that $x_0 \in \partial_{\mathcal{G}}(\overline{\omega})$ by (1). Conversely, if $x_0 \in \partial_{\mathcal{G}}(\overline{\omega}) \cap \omega^*$ and $F \in -\mathcal{G}$, $G \in \mathcal{G}$ with $F < G$, then $\overline{F}|_{\omega_i} < \overline{G}|_{\omega_i}$ for some $i \in I$. Therefore $F < H_F^{\omega_i} < G$ on. Therefore $(\overline{\omega}, \mathcal{G}, L)$ is a geometric simplex [11, prop. 5] [5, p. 521]. It follows that $F(x_0) = \inf g(x_0) : F < g \in \mathcal{G}\} \geqslant \inf \{H_F^{\omega_i}(x_0) : i \in I\} \geqslant F(x_0)$. Therefore $x_0$ is stable and the following lemma holds.

Lemma 7. — *The set of stable points of* $\omega^*$ *is precisely* $\partial_{\mathcal{G}}(\overline{\omega}) \cap \omega^*$.

*Example* 2. — *The classical case.* Let $\omega$ be a bounded open subset of $R^n$, $n > 1$. The affine functions on $R^n$ are harmonic, $\partial_M(\overline{\omega})$ is precisely the set of regular points of $\omega^*$, while $\partial_L(\overline{\omega})$ is precisely the set of stable points of $\omega^*$. Since L contains the constant functions, separates the points of $\overline{\omega}$, and has the weak R.s.p., the following theorem is an immediate consequence of theorem 2.

Theorem 5. — L *is uniformly dense in* M *if and only if every regular point of* X *is stable.*

We now return to the general case.

Theorem 6. — *If every regular point of* $\omega^*$ *is stable, then* L *is uniformly dense in* M.

*Proof.* — Suppose $x_i$ belongs to the minimal W-face $A_i$, $i = 1, 2$. Since $\mathcal{G}$ distinguishes points of $\omega^*$ it follows from lemma 3, that $A_i \cap \omega^*$ is a one point set $\{y_i\}$. If $F \in -\mathcal{G}$ and $f = F|_{\omega^*}$ then $\inf \{G : G \in \omega, F < G\} \geqslant H_f^{\omega} \geqslant F$ on $\omega$. Since $y_i \in \partial_W(\overline{\omega})$, $F(y_i) = \inf \{G(y_i) : G \in W, F < G\}$. Therefore $\lim \{H_f(x) : x \in \omega, x \to y_i\} = f(y_i)$, and $y_i$ is regular. Therefore $y_i$ is stable. By lemma 7 there exist minimal $\mathcal{G}$-faces $B_i$, with $y_i \in B_i$, $i = 1, 2$. Since $A_i \cap B_i \neq \emptyset$ and $A_i$ is minimal, $A_i \subset B_i$. Therefore $\partial_W(\overline{\omega}) \subset \partial_{\mathcal{G}}(\overline{\omega})$. If $\omega$ distinguishes $x_1$ and $x_2$ then by lemma 3 $\omega$ distinguishes $y_1$ and $y_2$, and $y_1 \neq y_2$. Therefore $\mathcal{G}$ distinguishes $y_1$ and $y_2$ so that $B_1 \neq B_2$, and $\mathcal{G}$ distinguishes $x_1$ and $x_2$. It follows from theorem 4 that L is uniformly dense in M.

Boboc and Cornea [5, th. 4.3], with the additional hypothesis that $\omega$ is weakly determining, show that $(\overline{\omega}, W, M)$ is a geometric simplex, and that the set of regular points of $\omega^*$ is precisely $\partial_W(\overline{\omega}) \cap \omega^*$. In this case we have a complete generalisation of theorem 5 to axiomatic potential theory.

COROLLARY 2. — *If* $\omega$ *is weakly determining, then* L *is uniformly dense in* M *if and only if every regular point of* $\omega^*$ *is stable.*

*Proof.* — If $x$ is a regular point of $\overline{\omega}$ then $x \in \partial_\omega(\overline{\omega})$ [5, th. 4.3]. $(\omega, W, M)$ is a geometric simplex so by proposition 1, $x \in \partial_M(\overline{\omega})$. If L is dense in M, then $\mathcal{L}$-faces are $\mathcal{M}$-faces, and $x$ belongs to a minimal $\mathcal{L}$-face A. Since $(\overline{\omega}, \mathcal{S}, L)$ is a geometric simplex, it follows from proposition 1 and lemma 4 that A contains a unique minimal $\mathcal{S}$-face B and a unique minimal W-face C. Therefore $x \in C \subset B$, so that $x \in \partial_{\mathcal{S}}(\overline{\omega})$ and $x$ is stable by lemma 7. The corollary is now an immediate consequence of theorem 6.

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