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A quantum splitting principle and an application

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A QUANTUM SPLITTING PRINCIPLE AND AN APPLICATION

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ABSTRACT. — We propose an analogy of splitting principle in genus-0 Gromov–Witten theory. More precisely, we show how the Gromov–Witten theory of a variety X can be embedded into the theory of the projectivization of a vector bundle over X. An application is also given.

RÉSUMÉ. — Nous proposons un analogue du principe de décomposition en théorie de Gromov–Witten de genre zéro. Plus précisément, nous montrons comment réaliser la théorie de Gromov–Witten d'une variété X dans la théorie de la projectivisation d'un fibré vectoriel sur X. Nous donnons également une application.

1. Introduction

In geometry and topology, it is common to apply base-change in order to reduce a problem. For example, splitting principle in algebraic topology reduces a vector bundle to a split one by applying a base-change. More specifically, let X be a smooth manifold and V a vector bundle over X. The pull-back of V along the projection $\pi : \mathbb{P}_X(V) \to X$ splits off a line bundle. Applying it multiple times eventually reduces V to a split bundle. But to make it work, facts like the following are crucial.

The pull-back $\pi^* : H^*(X; \mathbb{Q}) \to H^*(\mathbb{P}_X(V); \mathbb{Q})$ is injective.

By using this, a lot of cohomological computations that is done on the pull-back of V can be related back to the ones on V.

In this paper, our focus will be on Gromov–Witten theory. But the above technique cannot be applied easily because Gromov–Witten theory lacks a nice "pull-back" operation. Let $f : Y \to X$ be a map between smooth projective varieties. The complexity already appears when f is an embedding. In this case, one might naïvely hope for some statement in the spirit of the following.

$$\text{(a part of) GWT}(Y) \text{ is a sub-theory of GWT}(X),$$

where GWT stands for Gromov–Witten theory. But this is not quite true. The closest theorem might be the quantum Lefschetz theorem ([3, 8]), which only works in genus 0 and has a restriction on the embedding.

Another common situation is when $f : Y \to X$ is a smooth morphism. When the fibration is nice (e.g. projectivization of vector bundles in splitting principle), one might wonder whether something like the following could happen.

$$\text{GWT}(X) \text{ is a sub-theory of GWT}(Y).$$

If a theorem with the above idea can be established, we might be able to imitate splitting principle to simplify some problems via base-changes. But again, the relation between GWT(X) and GWT(Y) seems complicated, and very few work has been done along these lines.

We will work on the case when Y is the projectivization of a vector bundle V over X. In this paper, Corollary 1.3 realizes genus-0 GWT(X) as a sub-theory of genus-0 GWT$(\mathbb{P}(V))$ in a reasonably simple statement. To show an application, we generalize the main theorem in [4] to $\mathbb{P}^n$-fibrations in genus 0.

It's worth noting that splitting principle has already inspired some of the techniques in [11]. Roughly speaking, they apply base-changes along blow-ups to turn a vector bundle into a split one, and relate back via degeneration formula. But the relate-back process is very complicated, mainly due to the complexity of degeneration formula.

1.1. Statements of results

Let X be a smooth projective variety, V be a vector bundle over X and $\beta \in NE(X)$ where $NE(X)$ stands for the group of numerical curve classes. We first prove the following theorem.

THEOREM 1.1. — *If V is globally generated, then the following holds.*

$$\langle \pi^*\sigma_1, \ldots, \pi^*\sigma_n \rangle^{\mathbb{P}_X(V), \mathcal{O}(-1)}_{0,n,\widetilde{\beta}} = \langle \sigma_1, \ldots, \sigma_n \rangle^{X,V}_{0,n,\beta},$$

where $\widetilde{\beta} \in NE(\mathbb{P}(V))$ is an effective class such that $\pi_*\widetilde{\beta} = \beta$ and $(\widetilde{\beta}, c_1(\mathcal{O}(1))) = 0$.

Here both sides of the equation are twisted Gromov–Witten invariants defined in Section 3.1 equation (3.1).

Remark 1.2. — If we put descendants into the invariants and match them up like in Theorem 1.1, the equality might *not* hold. But it holds if we use the pull-back of ψ_i in $\overline{\mathcal{M}}_{0,n}(X, \beta)$ on the left-hand side. The difference between ψ-classes and the pull-back of ψ-classes can be made explicit, but it's irrelevant in our paper.

Let λ be the equivariant parameter in the twisted theory. We also use the notation $[\ldots]_{\lambda^{-N}}$ for taking the λ^{-N} coefficient. We have the following corollary.

COROLLARY 1.3. — *If V is globally generated, Gromov–Witten invariants of X can be computed from that of Y according to the following.*

$$\langle \sigma_1, \ldots, \sigma_n \rangle^X_{0,n,\beta} = \left[\langle \pi^*\sigma_1, \ldots, \pi^*\sigma_n \rangle^{\mathbb{P}_X(V), \mathcal{O}(-1)}_{0,n,\widetilde{\beta}} \right]_{\lambda^{-N}},$$

where $N = \mathrm{rank}(V) + (\widetilde{\beta}, \det(T_\pi))$ with T_π the relative tangent bundle,

As an application, we prove the following generalization of [4] in genus 0.

THEOREM 1.4. — *Let $\pi_1 : P_1 \to X$, $\pi_2 : P_2 \to X$ be two projective smooth morphisms between smooth projective varieties whose fibers are isomorphic to $\mathbb{P}^n$. Suppose their Brauer classes are equal in $H^2_{\text{ét}}(X, \mathbb{G}_m)$. Furthermore, suppose there exists a ring isomorphism*

$$\mathscr{F} : H^*(P_1; \mathbb{Q}) \to H^*(P_2; \mathbb{Q}),$$

such that

(1) $\mathscr{F}$ *sends the subgroup* $H^*(P_1; \mathbb{Z})/\mathrm{tor}(H^*(P_1; \mathbb{Z}))$ *into* $H^*(P_2; \mathbb{Z})/\mathrm{tor}(H^*(P_2; \mathbb{Z}))$. *Here* $\mathrm{tor}(\cdot)$ *denotes the torsion subgroup.*

(2) $\mathscr{F}$ *restricts to identity on* $H^*(X; \mathbb{Q})$. *Here we identify* $H^*(X; \mathbb{Q})$ *as subrings under pull-backs of* π_1, π_2.

(3) $\mathscr{F}(c_1(\omega_{\pi_1})) = c_1(\omega_{\pi_2})$, *where* ω_{π_i} *is the corresponding relative canonical sheaf.*

Then we have

$$\langle \psi^{k_1}\sigma_1, \ldots, \psi^{k_n}\sigma_n \rangle_{0,n,\beta}^{P_1} = \langle \psi^{k_1}\mathscr{F}\sigma_1, \ldots, \psi^{k_n}\mathscr{F}\sigma_n \rangle_{0,n,\Psi(\beta)}^{P_2},$$

where $\Psi : N_1(P_1) \to N_1(P_2)$ *is the isomorphism induced by* $\mathscr{F}$ *under Poincaré duality.*

Remark 1.5. — When one of P_i is the projectivization of vector bundle, all these conditions are equivalent to the existence of vector bundles V_1, V_2 such that $c(V_1) = c(V_2)$ and $P_1 = \mathbb{P}(V_1), P_2 = \mathbb{P}(V_2)$. This will be explained in Lemma 4.3.

Remark 1.6. — The reduction technique in [11] by blow-ups does not seem to work in our situation, as the Brauer group is a birational invariant.

1.2. Some further questions

Our result is based on Theorem 2.2. Note that this method seems to fail in high genus. Therefore a further question is the following.

QUESTION 1.7. — *Are there any similar results as Theorem 1.1 or Corollary 1.3 in high genus?*

Although the proof of Theorem 1.4 is to mainly demonstrate the use of "splitting principle" type of idea in Gromov–Witten theory, we are still interested in the following question.

QUESTION 1.8. — *Is it essential to put the requirement that* P_1, P_2 *have the same Brauer class?*

In the meantime, we are wondering to what extent the topology can determine the GWT of a smooth fibration. In particular, one way to phrase our questions is the following.

QUESTION 1.9. — *Let* $P_1 \to X, P_2 \to X$ *be two smooth projective morphisms between smooth projective varieties. Suppose all of their fibers are isomorphic to a certain variety* F. *Can we further impose a suitable isomorphism between* $H^*(P_1, \mathbb{Q})$ *and* $H^*(P_2, \mathbb{Q})$, *and conclude* $\mathrm{GWT}(P_1) = \mathrm{GWT}(P_2)$?

Finally, we want to remark that Theorem 2.3 yields a number of new identities besides Thereom 1.1. They are summarized in section 3.2. They are interesting to us for a few reasons. For example they are relatively simple, especially because they are closed formulas and there is no generating

series or mirror map involved (compare [2]). This seems to show us that the Gromov–Witten theory of $\mathbb{P}(V)$ (maybe twisted by $\mathcal{O}(-1)$) and the Gromov–Witten theory of X twisted by V are closely related. Note that another evidence that already exists in literature is the striking similarity between their I-functions when V splits. We suspect that these identities are a tip of the iceberg, and there might be a general relationship underneath. Therefore we would like to ask the following (vague) question.

QUESTION 1.10. — *Is there a reasonably nice relationship between* $\mathrm{GWT}(\mathbb{P}(V))$ *and the* $\mathrm{GWT}(X)$ *twisted by V, that includes our identities as special cases?*

1.3. Structure of the paper

In order to prove Theorem 1.1, we need to take a detour to the naturality problem of Gromov–Witten invariants under blow-up proposed by Y. Ruan in [15]. A variation of the main theorem in [10] under the equivariant setting (Theorem 2.3) is stated in section 2. Its weaker version (Theorem 2.2) is finally proven in section 2.3, and this already implies Theorem 1.1 by comparing localization residues (section 3.2). For completeness, we finish the proof of Theorem 2.3 in section 3.3. Finally as an application, we prove Theorem 1.4 in section 4.

2. The naturality of virtual class under certain blow-up

In this section, we take a detour to study the naturality of virtual class under blow-up. Keep in mind that our ultimate goal is still Theorem 1.1, and it will be proven in section 3.2 when the tools are ready.

Recall a vector bundle V over a variety X is *convex* if for any morphism $f : \mathbb{P}^1 \to X$, $H^1(\mathbb{P}^1, f^*V) = 0$. A smooth projective variety is called *convex* if its tangent bundle is a convex vector bundle. Note that a globally generated vector bundle is always convex.

Let X be a smooth projective variety. Let V be a vector bundle of rank $\geqslant 2$ over X. We want to understand the Gromov–Witten theory of $\mathbb{P}(V)$. For our purpose, we will assume the following.

CONVENTION 2.1. — *Assume V is a globally generated vector bundle.*

This assumption can be realized by tensoring with a sufficiently ample line bundle. Define $Y = \mathbb{P}(V \oplus \mathcal{O})$. This projective bundle can be viewed as a compactification of the total space of V. The zero section induces an embedding $X \hookrightarrow Y$. We denote its image by Y_0. There is also an embedding $\mathbb{P}(V) \hookrightarrow Y$ whose image is denoted by Y_∞ (the divisor at infinity).

$\mathbb{C}^*$ can act on V by fixing the base and scaling the fibers (direct sum of weight 1 representations). This will induce a $\mathbb{C}^*$ action on Y. Consider $\widetilde{Y} = Bl_{Y_0} Y$. We denote the contraction by

$$\pi : \widetilde{Y} \to Y.$$

The $\mathbb{C}^*$ action uniquely lifts to $\widetilde{Y}$. On the other hand, it is not hard to see

$$\widetilde{Y} \cong \mathbb{P}_{\mathbb{P}(V)}(\mathcal{O}(-1) \oplus \mathcal{O}).$$

Again, it can be seen as the compactification of the line bundle $\mathcal{O}(-1)$ over $\mathbb{P}(V)$. Denote the zero section by $\widetilde{Y}_0$ and the infinity section by $\widetilde{Y}_\infty$. One can check that

$$\pi(\widetilde{Y}_0) = Y_0 \,, \quad \pi(\widetilde{Y}_\infty) = Y_\infty.$$

This $\mathbb{C}^*$ action induces actions on moduli of stable maps. There is a stabilization morphism $\tau : \overline{\mathcal{M}}_{0,n}(\widetilde{Y}, \pi^! \beta) \to \overline{\mathcal{M}}_{0,n}(Y, \beta)$. Because V is globally generated, by [10] we have (nonequivariantly)

$$\tau_* [\overline{\mathcal{M}}_{0,n}(\widetilde{Y}, \pi^! \beta)]^{vir} = [\overline{\mathcal{M}}_{0,n}(Y, \beta)]^{vir}.$$

In this note, we would like to partially extend the above identity to equivariant setting under the given $\mathbb{C}^*$ action. Let $Z \subset \overline{\mathcal{M}}_{0,n}(Y, \beta)$ be the locus consisting of the stable maps at least one of whose components map nontrivially to the closed subvariety Y_∞. Z is a closed substack. Denote $\mathscr{U} \subset \overline{\mathcal{M}}_{0,n}(Y, \beta) \backslash Z$ the complement. Let $[\mathscr{U}]^{vir,eq}$ denote the equivariant virtual class induced by the perfect obstruction theory coming from the moduli of stable map. Also denote $\tau^{-1}(\mathscr{U})$ by $\widetilde{\mathscr{U}}$. We will show the following.

THEOREM 2.2. — *The following holds for equivariant virtual classes.*

$$\tau_* [\widetilde{\mathscr{U}}]^{vir,eq} = [\mathscr{U}]^{vir,eq}.$$

Once this is established, we can compare the localization residues, and eventually prove the following by virtual localization.

THEOREM 2.3. — *The following holds for equivariant virtual classes.*

$$\tau_* [\overline{\mathcal{M}}_{0,n}(\widetilde{Y}, \pi^! \beta)]^{vir,eq} = [\overline{\mathcal{M}}_{0,n}(Y, \beta)]^{vir,eq}.$$

We prove Theorem 2.3 for the sake of completeness. But Theorem 2.2 is already strong enough to deduce Theorem 1.1.

2.1. Towards the proof of Theorem 2.2

We start with the following lemma.

LEMMA 2.4. — *Under the setting of Section 2, there exists a convex projective variety* $\mathbb{G}$, *an embedding* $i : X \to \mathbb{G}$ *and a globally generated vector bundle* $V_{\mathbb{G}}$ *such that* $i^*V_{\mathbb{G}} = V$.

Proof. — First of all because V is globally generated, there is a map $f : X \to G(k, n)$ to some Grassmannian such that $f^*Q = V$ where Q is the universal quotient bundle over $G(k, n)$. Notice Q is globally generated. The only thing different in this situation is that f may not be an embedding. However, we can pick an embedding $f' : X \to \mathbb{P}^N$ for some N. Now they induce an embedding $f \times f' : X \to G(k, n) \times \mathbb{P}^N$. It is not hard to see that $(f \times f')^*(Q \boxtimes \mathcal{O}_{\mathbb{P}^N}) = V$. $Q \boxtimes \mathcal{O}_{\mathbb{P}^N}$ is still globally generated since it is a quotient of direct sums of trivial bundles. Let $\mathbb{G} = G(k, n) \times \mathbb{P}^N$ and $V_{\mathbb{G}} = Q \boxtimes \mathcal{O}_{\mathbb{P}^N}$. $\square$

Denote $Y_{\mathbb{G}} = \mathbb{P}_{\mathbb{G}}(V_{\mathbb{G}} \oplus \mathcal{O})$ and $\widetilde{Y}_{\mathbb{G}} = \mathbb{P}_{\mathbb{P}_{\mathbb{G}}(V_{\mathbb{G}} \oplus \mathcal{O})}(\mathcal{O}(-1) \oplus \mathcal{O})$. Similar as before, $\widetilde{Y}_{\mathbb{G}}$ is the blow-up of $Y_{\mathbb{G}}$ at zero section. Let $Z_{\mathbb{G}} \subset \overline{\mathcal{M}}_{0,n}(Y_{\mathbb{G}}, \beta)$ be the closed substack containing stable maps with at least one component mapping nontrivially onto $(Y_{\mathbb{G}})_\infty$, the image of the embedding $\mathbb{P}_{\mathbb{G}}(V_{\mathbb{G}}) \subset \mathbb{P}_{\mathbb{G}}(V_{\mathbb{G}} \oplus \mathcal{O})$. Similarly, we denote $\tau_{\mathbb{G}} : \overline{\mathcal{M}}_{0,n}(\widetilde{Y}_{\mathbb{G}}, \pi^!\beta) \to \overline{\mathcal{M}}_{0,n}(Y_{\mathbb{G}}, \beta)$, $\mathscr{U}_{\mathbb{G}} = \overline{\mathcal{M}}_{0,n}(Y_{\mathbb{G}}, \beta) \backslash Z_{\mathbb{G}}$ and $\widetilde{\mathscr{U}}_{\mathbb{G}} = \tau_{\mathbb{G}}^{-1}(\mathscr{U}_{\mathbb{G}})$. We have a similar $\mathbb{C}^*$ action on $Y_{\mathbb{G}}$, $\widetilde{Y}_{\mathbb{G}}$ and their associated moduli of stable maps like before.

LEMMA 2.5. — *Theorem 2.2 is true when* $X = \mathbb{G}$ *and* $V = V_{\mathbb{G}}$.

Its proof is given in Section 2.2. In Section 2.3 we show the above lemma implies Theorem 2.2.

2.2. The case when X is convex

The argument here is basically parallel to the proof of [10, Theorem 1.4]. The situation here is even simpler, since the "Assumption $*$" in [10] is automatically satisfied because of the following.

LEMMA 2.6. — *The open substack* $\mathscr{U}_{\mathbb{G}} \subset \overline{\mathcal{M}}_{0,n}(Y_{\mathbb{G}}, \beta)$ *is unobstructed, thus smooth.*

Proof. — It suffices to prove that for any stable map $(C, x_1, \ldots, x_n, f) \in \mathscr{U}_{\mathbb{G}}$, $H^1(C, f^*TY_{\mathbb{G}}) = 0$. By passing to normalization, one may assume $C = \mathbb{P}^1$. We have the following Euler sequence of $Y_{\mathbb{G}}$.

$$0 \to \mathcal{O} \to (p_{\mathbb{G}})^*(V \oplus \mathcal{O}) \oplus \mathcal{O}_{\mathbb{P}_{\mathbb{G}}(V \oplus \mathcal{O})}(1) \to T_{p_{\mathbb{G}}} \to 0,$$

where $T_{p_{\mathbb{G}}}$ is the relative tangent bundle of $p_{\mathbb{G}} : Y_{\mathbb{G}} \to \mathbb{G}$ the projection. We also have the short exact sequence

$$0 \to T_{p_{\mathbb{G}}} \to T_{Y_{\mathbb{G}}} \to (p_{\mathbb{G}})^* T_{\mathbb{G}} \to 0.$$

Since they are vector bundles, this short exact sequence pulls back to

$$0 \to f^* T_{p_{\mathbb{G}}} \to f^* T_{Y_{\mathbb{G}}} \to f^* (p_{\mathbb{G}})^* T_{\mathbb{G}} \to 0$$

over $\mathbb{P}^1$. Now $f^*(p_{\mathbb{G}})^* T_{\mathbb{G}}$ is already convex. If we can prove $f^* T_{p_{\mathbb{G}}}$ is also convex, then $f^* T_{Y_{\mathbb{G}}}$ will be convex as well. We will show that there is a point $x \in \mathbb{P}^1$ such that the global sections of $T_{p_{\mathbb{G}}}$ generates its fiber at $f(x)$. One point is enough because if a vector bundle over $\mathbb{P}^1$ has a negative factor, the global sections would not generate any fiber over $\mathbb{P}^1$.

We choose the point x to be one that $f(x) \notin (Y_{\mathbb{G}})_\infty$ (by definition of $\mathscr{U}_{\mathbb{G}}$, such an x exists). $f(x)$ could a priori be any point in $Y_{\mathbb{G}} \backslash (Y_{\mathbb{G}})_\infty$. In the meantime, $T_{p_{\mathbb{G}}}$ is a quotient of $(p_{\mathbb{G}})^* (V \oplus \mathcal{O}) \oplus \mathcal{O}_{\mathbb{P}_{\mathbb{G}}(V \oplus \mathcal{O})}(1)$. Therefore it suffices to show that global sections of $(p_{\mathbb{G}})^* (V \oplus \mathcal{O}) \oplus \mathcal{O}_{\mathbb{P}_{\mathbb{G}}(V \oplus \mathcal{O})}(1)$ generates the fiber over any point in $Y_{\mathbb{G}} \backslash (Y_{\mathbb{G}})_\infty$. This can be easily seen because

(1) $(p_{\mathbb{G}})^* (V \oplus \mathcal{O})$ is globally generated;
(2) $\mathcal{O}_{\mathbb{P}_{\mathbb{G}}(V \oplus \mathcal{O})}(1)$ has a section s whose vanishing locus is exactly $(Y_{\mathbb{G}})_\infty$.

One can multiply suitable sections with the section s to generate the fiber of any given point in $Y_{\mathbb{G}} \backslash (Y_{\mathbb{G}})_\infty$. $\qquad\square$

Therefore we have $[\mathscr{U}]^{vir} = [\mathscr{U}]$. This is also why we choose this open substack.

For the rest of this subsection, we assume $X = \mathbb{G}$ is a convex variety. Let $U = Y \backslash Y_0$. Since U doesn't intersect with the blow-up center, it is also an open subset of $\widetilde{Y}$.

LEMMA 2.7. — *The open substack $\mathscr{U} \cap \overline{\mathcal{M}}_{0,n}(U, \beta) \subset \mathscr{U}$ is dense.*

Proof. — Recall V is globally generated. Since the rank of V is larger than 1, a general section is disjoint from the zero section. Choose any section $s_V : X \to V$ that is disjoint from the zero section. Also note that $N_{Y_0/Y} = V$. We claim that this section s_V of $N_{Y_0/Y}$ lifts to the tangent space TY. It's easy to check that $N_{Y_0/Y} = T_p|_{Y_0}$ the restriction of relative tangent. By the Euler sequence of the relative tangent, we can check that

$$(s_V \oplus 1) \otimes s \in \Gamma(p^*(V \oplus \mathcal{O}) \otimes \mathcal{O}_{\mathbb{P}_X(V \oplus \mathcal{O})}(1))$$

is a lifting of s_V, where s is the section of $\mathcal{O}_{\mathbb{P}_X(V \oplus \mathcal{O})}(1)$ that vanishes along Y_∞.

Now that $\mathscr{U}$ is unobstructed, for any point $(C, x_1, \ldots, x_n, f) \in \mathscr{U}$, the pull-back of this tangent vector field along f is integrable and there is a one-parameter family whose generic member is in $\mathscr{U} \cap \overline{\mathcal{M}}_{0,n}(U, \beta)$. □

Let $d = \dim \mathscr{U} = \dim(\overline{\mathcal{M}}_{0,n}(Y, \beta))$ and $\mathcal{Z} = \mathscr{U} \backslash \mathscr{U} \cap \overline{\mathcal{M}}_{0,n}(U, \beta)$. The previous lemma implies

$$\dim(\mathcal{Z}) < d.$$

We have the exact sequence

$$A_d^{\mathbb{C}^*}(\mathcal{Z}) \to A_d^{\mathbb{C}^*}(\mathscr{U}) \to A_d^{\mathbb{C}^*}(\mathscr{U} \cap \overline{\mathcal{M}}_{0,n}(U, \beta)) \to 0.$$

Recall that $A_*^{\mathbb{C}^*}(\cdot)$ is bounded above by the dimension, even if it is an equivariant Chow group. Therefore the first term $A_d^{\mathbb{C}^*}(\mathcal{Z}) = 0$ for dimensional reason.

On the other hand, because of the open immersion $U \subset \widetilde{Y}$, $\overline{\mathcal{M}}_{0,n}(U, \beta)$ has an open immersion into $\overline{\mathcal{M}}_{0,n}(\widetilde{Y}, \pi^! \beta)$ (may not be dense). We have a similar exact sequence for $A_d^{\mathbb{C}^*}(\widetilde{\mathscr{U}})$, except that the first term may not vanish (we don't know the dimension of $\tau^{-1}\mathcal{Z}$). Because the blow-up π restricts to isomorphism on the open subset $U \subset Y$, we have

$$\widetilde{\mathscr{U}} \cap \overline{\mathcal{M}}_{0,n}(U, \beta) \cong \mathscr{U} \cap \overline{\mathcal{M}}_{0,n}(U, \beta),$$

where we slightly abuse notations by regarding U as a subset of $\widetilde{Y}$ on the left hand side and regarding U as a subset of Y on the right hand side. To put the two exact sequences together, we have

$$
\begin{array}{ccc}
A_d^{\mathbb{C}^*}(\widetilde{\mathscr{U}}) & \longrightarrow A_d^{\mathbb{C}^*}(\widetilde{\mathscr{U}} \cap \overline{\mathcal{M}}_{0,n}(U, \beta)) \longrightarrow 0 \\
\downarrow {\scriptstyle \tau_*} & \downarrow {\scriptstyle \cong} \\
0 \longrightarrow A_d^{\mathbb{C}^*}(\mathscr{U}) \xrightarrow{\ \cong\ } A_d^{\mathbb{C}^*}(\mathscr{U} \cap \overline{\mathcal{M}}_{0,n}(U, \beta)) \longrightarrow 0.
\end{array}
$$

Recall we want to show that $\tau_*[\widetilde{\mathscr{U}}]^{vir,eq} = [\mathscr{U}]^{vir,eq}$ (via the left column). If we let the class $[\widetilde{\mathscr{U}}]^{vir,eq}$ go through the arrow to the right, we notice that $[\widetilde{\mathscr{U}}]^{vir,eq}$ is sent to $[\widetilde{\mathscr{U}} \cap \overline{\mathcal{M}}_{0,n}(U, \beta)]^{vir,eq}$ because virtual class is pulled back to virtual class via open immersions. Then we proceed by going through the right vertical arrow and the inverse of the bottom horizontal arrow. Since they are isomorphisms, we easily conclude that Lemma 2.5 is true.

2.3. Virtual pull-back

We are under the setting of Lemma 2.4. There is the following diagram.

(2.1)
$$
\begin{array}{ccc}
\overline{\mathcal{M}}_{0,n}(\widetilde{Y}, \pi^! \beta) & \xrightarrow{\widetilde{\iota}} & \overline{\mathcal{M}}_{0,n}(\widetilde{Y}_{\mathbb{G}}, \pi^! \beta) \\
\downarrow{\tau} & & \downarrow{\tau_{\mathbb{G}}} \\
\overline{\mathcal{M}}_{0,n}(Y, \beta) & \xrightarrow{\iota} & \overline{\mathcal{M}}_{0,n}(Y_{\mathbb{G}}, \beta).
\end{array}
$$

It is Cartesian by [13, Remark 5.7]. By base-change to an open substack, one has the following Cartesian diagram

(2.2)
$$
\begin{array}{ccc}
\widetilde{\mathcal{U}} & \xrightarrow{\widetilde{\iota}} & \widetilde{\mathcal{U}}_{\mathbb{G}} \\
\downarrow{\tau} & & \downarrow{\tau_{\mathbb{G}}} \\
\mathcal{U} & \xrightarrow{\iota} & \mathcal{U}_{\mathbb{G}},
\end{array}
$$

where we slightly abuse notations by using the same $\iota, \widetilde{\iota}, \tau, \tau_{\mathbb{G}}$ on open substacks.

In [13], virtual pull-backs are constructed (in our case, [13, Construction 3.13] applies). Note that for a torus-equivariant DM-type morphism, if there is an equivariant perfect relative obstruction theory, then the construction in [13] can be done in the equivariant Chow in parallel. In our case, we denote the following virtual pull-backs under equivariant context.

$$
\iota^! : A_*^{\mathbb{C}^*}(\mathcal{U}_{\mathbb{G}}) \to A_{*-\delta}^{\mathbb{C}^*}(\mathcal{U}),
$$

$$
\widetilde{\iota}^! : A_*^{\mathbb{C}^*}(\widetilde{\mathcal{U}}_{\mathbb{G}}) \to A_{*-\delta'}^{\mathbb{C}^*}(\widetilde{\mathcal{U}}),
$$

where δ, δ' are the differences of the corresponding virtual dimensions. Note that they are defined because the relative obstruction theory of ι is perfect over $\mathcal{U}$, which further follows from Lemma 2.6 under a similar argument to [13, Remark 3.15]. By [13, Corollary 4.9], these morphisms send virtual classes to virtual classes.

LEMMA 2.8. — *Lemma 2.5 implies Theorem 2.2.*

It follows from a completely parallel argument to [13, Proposition 5.14]. Briefly speaking, one first check that the diagram (2.1) is Cartesian. And then,

(2.3) $\quad \tau_*[\widetilde{\mathcal{U}}]^{vir,eq} = \tau_* \widetilde{\iota}^! [\widetilde{\mathcal{U}}_{\mathbb{G}}]^{vir,eq} = \iota^! (\tau_{\mathbb{G}})_* [\widetilde{\mathcal{U}}_{\mathbb{G}}]^{vir,eq} = \iota^! [\mathcal{U}_{\mathbb{G}}] = [\mathcal{U}]^{vir,eq}.$

For the reason of each equality, one is also referred to the proof of [13, Proposition 5.14].

3. Consequences via localization

We would like to briefly recall the virtual localization in [6] and make the set-up in the next subsection.

Let $\mathscr{X}$ be a smooth projective variety admitting an action by a torus $T = (\mathbb{C}^*)^m$. It induces an action of T on $\overline{\mathcal{M}}_{g,n}(\mathscr{X}, \beta)$. Let $\overline{\mathcal{M}}_\alpha$ be the connected components of the fixed locus $\overline{\mathcal{M}}_{g,n}(\mathscr{X}, \beta)^T$ labeled by α with the inclusion $i_\alpha : \overline{\mathcal{M}}_\alpha \to \overline{\mathcal{M}}_{g,n}(\mathscr{X}, \beta)$. The virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(\mathscr{X}, \beta)]^{vir}$ can be written as

$$[\overline{\mathcal{M}}_{g,n}(\mathscr{X}, \beta)]^{vir} = \sum_\alpha (i_\alpha)_* \frac{[\overline{\mathcal{M}}_\alpha]^{vir}}{e_T(N_\alpha^{vir})},$$

where $[\overline{\mathcal{M}}_\alpha]^{vir}$ is constructed from the fixed part of the restriction of the perfect obstruction theory of $\overline{\mathcal{M}}_{g,n}(\mathscr{X}, \beta)$, the virtual normal bundle N_α^{vir} is the moving part of the two term complex in the perfect obstruction theory of $\overline{\mathcal{M}}_{g,n}(\mathscr{X}, \beta)$, and e_T stands for the equivariant Euler class. Sometimes we call $\frac{[\overline{\mathcal{M}}_\alpha]^{vir}}{e_T(N_\alpha^{vir})}$ the localization residue of α. In Gromov–Witten theory, one ingredient of these localization residues is the twisted theory.

3.1. Twisted Gromov–Witten theory

Let X be a smooth projective variety, E be a vector bundle over X. Let

$$ft_{n+1} : \overline{\mathcal{M}}_{g,n+1}(X, \beta) \to \overline{\mathcal{M}}_{g,n}(X, \beta)$$

be the map forgetting the last marked point. Under this map $\overline{\mathcal{M}}_{g,n+1}(X, \beta)$ can be viewed as the universal family over $\overline{\mathcal{M}}_{g,n}(X, \beta)$. Furthermore, the evaluation map of the last marked point

$$ev_{n+1} : \overline{\mathcal{M}}_{g,n+1}(X, \beta) \to X$$

serves as the universal stable map from the universal family over $\overline{\mathcal{M}}_{g,n}(X, \beta)$.

Let $\mathbb{C}^*$ act on X trivially and on E by scaling (weight 1 on every 1-dimensional linear subspace of a fiber). Denote λ the corresponding equivariant parameter. Let $E_{g,n,\beta} = [R(ft_{n+1})_* ev_{n+1}^* E] \in K^0_{\mathbb{C}^*}(\overline{\mathcal{M}}_{g,n}(X, \beta))$. This class can be represented by the difference of two vector bundles $E^0_{g,n,\beta} - E^1_{g,n,\beta}$ such that $\mathbb{C}^*$ acts on each of them by scaling as well.

Define the twisted Gromov–Witten invariants to be

$$(3.1) \quad \langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n \rangle_{g,n,\beta}^{X,E}$$

$$= \int_{[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{vir}} \frac{1}{e_{\mathbb{C}^*}(E_{g,n,\beta})} \cup \prod_{i=1}^n \psi_i^{k_i} ev_i^*\alpha_i \in \mathbb{C}[\lambda, \lambda^{-1}],$$

where $\frac{1}{e_{\mathbb{C}^*}(E_{g,n,\beta})} = \frac{e_{\mathbb{C}^*}(E_{g,n,\beta}^1)}{e_{\mathbb{C}^*}(E_{g,n,\beta}^0)}$.

Remark 3.1. — For a more detailed discussion about a more general set-up, one can see for example [3].

3.2. Theorem 1.1 and other consequences

Back to our case, we can apply virtual localizations to $\widetilde{Y}$ and Y respectively. In each case we will index fixed locus by suitable bipartite graphs. For the assignment of decorated graphs to invariant stable maps, one can refer to for example [5, 12], among others. Given $\widetilde{\Gamma}$ a decorated graph for an invariant stable map $(C, x_1, \ldots, x_n, f)$ to $\widetilde{Y}$, one can assign a decorated graph to the image of $(C, x_1, \ldots, x_n, f)$ under τ (still invariant because τ is equivariant). By a slight abuse of notation, we denote the resulting graph by $\tau(\widetilde{\Gamma})$.

One thing to be careful about is that we have chosen to work on some open substacks. Since $\tau : \overline{\mathcal{M}}_{0,n}(\widetilde{Y}, \pi^!\beta) \to \overline{\mathcal{M}}_{0,n}(Y, \beta)$ is proper, its base-change $\tau : \widetilde{\mathscr{U}} \to \mathscr{U}$ is also proper. Therefore its push-forward on Chow groups are defined. Localization formula alone may not produce meaningful computational result, because $\widetilde{\mathscr{U}}$ and $\mathscr{U}$ are not proper. We need the following "correspondence of residues".

PROPOSITION 3.2. — Fixing a connected component $\overline{\mathcal{M}}_\Gamma$ of the fixed locus in $\mathscr{U}$, Theorem 2.2 implies the following equality in $A_*^{\mathbb{C}^*}(\overline{\mathcal{M}}_\Gamma)$.

$$\tau_* \left(\sum_{\tau(\widetilde{\Gamma})=\Gamma} \frac{[\overline{\mathcal{M}}_{\widetilde{\Gamma}}]^{vir}}{e_{\mathbb{C}^*}(N_{\widetilde{\Gamma}}^{vir})} \right) = \frac{[\overline{\mathcal{M}}_\Gamma]^{vir}}{e_{\mathbb{C}^*}(N_\Gamma^{vir})},$$

where we index fixed components of $\overline{\mathcal{M}}_{0,n}(\widetilde{Y}, \beta)$ and $\overline{\mathcal{M}}_{0,n}(Y, \beta)$ by $\overline{\mathcal{M}}_{\widetilde{\Gamma}}$ and $\overline{\mathcal{M}}_\Gamma$ respectively (with $\widetilde{\Gamma}, \Gamma$ corresponding decorated graphs). $N_{\widetilde{\Gamma}}^{vir}$ and N_Γ^{vir} are the corresponding virtual normal bundles as well.

Note that this proposition yields relations between numerical invariants. It is because in our case, $\overline{\mathcal{M}}_\Gamma$ and $\overline{\mathcal{M}}_{\widetilde{\Gamma}}$ are all proper. A quick proof of the "correspondence of residues" is included in Appendix A.

Note that different choices of Γ yield different equalities according to the above corollary. We will show that suitable choices of Γ give rise to neat relations between Gromov–Witten invariants of $\mathbb{P}_X(V)$ and (twisted) Gromov–Witten invariants of X. In the rest of this section, we will only show what choices of Γ lead to what relations. Some details of the computations of $e_{\mathbb{C}^*}(N_\Gamma^{vir})$ can be found in Appendix B

Theorem 1.1 can be deduced by the following choice of graph. Let Γ be the graph consisting of a single vertex over Y_0 of class β with n markings without any edge. Then we have

$$(3.2) \qquad \langle \pi^* \sigma_1, \ldots, \pi^* \sigma_n \rangle_{0,n,\pi^! \beta}^{\mathbb{P}_X(V),\mathcal{O}(-1)} = \langle \sigma_1, \ldots, \sigma_n \rangle_{0,n,\beta}^{X,V}.$$

This is exactly Theorem 1.1.

One can choose other graphs to obtain other types of relations. We will list a few examples. Let $f \in N_1(\mathbb{P}(V))$ be the class of a line in a fiber. Let Γ be the graph consisting of a vertex over Y_0 of class β, a vertex over Y_∞ of degree 0 and an edge of class f connecting them. Put n markings on the vertex over Y_0 and 1 on the one over Y_∞. Then we have

COROLLARY 3.3.

$$(3.3) \quad \left\langle \frac{h^e \pi^* \alpha}{(h - \lambda)(\lambda - h - \psi)}, \pi^* \sigma_1, \ldots, \pi^* \sigma_n \right\rangle_{0,n+1,\pi^! \beta + f}^{\mathbb{P}_X(V),\mathcal{O}(-1)}$$

$$= \left\langle \pi_* \left(\frac{h^e \pi^* \alpha}{(h - \lambda)(\lambda - h - \psi)} \right), \sigma_1, \ldots, \sigma_n \right\rangle_{0,n+1,\beta}^{X,V},$$

where $h = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$.

One can similarly consider the case when Γ consists of a vertex over Y_0 and k edges, each of class f, coming out of this vertex. For each edge one attach a vertex at the other end over Y_∞ of degree 0 with 1 marking. One gets

COROLLARY 3.4.

$$(3.4) \quad \left\langle \frac{h^e \pi^* \alpha_1}{(h-\lambda)(\lambda - h - \psi)}, \ldots, \frac{h^e \pi^* \alpha_m}{(h-\lambda)(\lambda - h - \psi)}, \right.$$

$$\left. \pi^* \sigma_1, \ldots, \pi^* \sigma_n \right\rangle^{\mathbb{P}_X(V), \mathcal{O}(-1)}_{0, n+m, \pi^! \beta + kf}$$

$$= \left\langle \pi_* \left(\frac{h^e \pi^* \alpha_1}{(h-\lambda)(\lambda - h - \psi)} \right), \ldots, \pi_* \left(\frac{h^e \pi^* \alpha_m}{(h-\lambda)(\lambda - h - \psi)} \right), \right.$$

$$\left. \sigma_1, \ldots, \sigma_n \right\rangle^{X, V}_{0, n+m, \beta}.$$

One can also consider the Γ consisting of a vertex over Y_0, an edge of class kf, and a vertex of degree 0 at Y_∞ end with 1 marking. In this case, we get

COROLLARY 3.5.

$$(3.5) \quad \left\langle \frac{h^e \pi^* \alpha}{\left(\frac{(\lambda - h)}{k} - \psi \right) \prod_{m=1}^{k} \left(\frac{m}{k}(h - \lambda) \right) \prod_{m=1}^{k-1} \left(\frac{m}{k}(\lambda - h) \right)}, \right.$$

$$\left. \pi^* \sigma_1, \ldots, \pi^* \sigma_n \right\rangle^{\mathbb{P}_X(V), \mathcal{O}(-1)}_{0, n+1, \pi^! \beta + kf}$$

$$= \left\langle \pi_* \left(\frac{h^e \pi^* \alpha}{\left(\frac{(\lambda - h)}{k} - \psi \right) \prod_{m=1}^{k} \left(\frac{m}{k}(h - \lambda) \right) \prod_{m=1}^{k-1} \left(c_V(h + \frac{m}{k}(\lambda - h)) \right)} \right), \right.$$

$$\left. \sigma_1, \ldots, \sigma_n \right\rangle^{X, V}_{0, n+1, \beta}.$$

3.3. Completing the proof of Theorem 2.3

Theorem 2.3 is true if and only if the "correspondence of residues" in Proposition 3.2 holds for all $\overline{\mathcal{M}}_\Gamma$ no matter whether it is inside $\mathscr{U}$ or not (see Remark A.2). Now that Theorem 2.2 is proven, we will show that the correspondence of residues over $\mathscr{U}$ is enough to establish the correspondence of all residues.

For each vertex v in a decorated graph, we denote the valence of v by $\mathrm{val}(v)$ and the number of markings on v by $n(v)$.

Choose any $\overline{\mathcal{M}}_\Gamma$. The decorated graph Γ might involve vertices over Y_∞ with nontrivial degrees. Suppose $v_1, \ldots, v_l$ are those vertices with degree $\beta_1, \ldots, \beta_l$ correspondingly. Let $\Gamma(v_i)$ be the graph consisting of a single vertex v_i of degree β_i with $\mathrm{val}(v_i) + n(v_i)$ markings. In the meantime, we can break the graph Γ into pieces $\Gamma_1, \ldots, \Gamma_m$ along $v_1, \ldots, v_l$, where the original vertices of $v_1, \ldots, v_l$ in each Γ_i are replaced by unstable vertices each of degree 0 with 1 marking. To sum it up, we just break the decorated graph Γ into $\Gamma(v_1), \ldots, \Gamma(v_l)$ and $\Gamma_1, \ldots, \Gamma_m$. By gluing the corresponding markings of $\Gamma(v_1), \ldots, \Gamma(v_l), \Gamma_1, \ldots, \Gamma_m$, we can recover Γ.

By definition we have $\overline{\mathcal{M}}_{\Gamma_i} \in \mathscr{U}$. In terms of fixed loci of moduli space, we have

$$\overline{\mathcal{M}}_\Gamma = \left(\prod_{i=1}^m \overline{\mathcal{M}}_{\Gamma_i} \right) \times_\Gamma \left(\prod_{j=1}^l \overline{\mathcal{M}}_{0, \mathrm{val}(v_j) + n(v_j)}(Y_\infty, \beta_j) \right),$$

where $\times_\Gamma$ is the fiber product gluing the corresponding markings according to the splitting of the graph Γ. In $\overline{\mathcal{M}}_{0,n}(\widetilde{Y}, \beta)$, the fixed locus $\overline{\mathcal{M}}_{\widetilde{\Gamma}}$ such that $\tau(\widetilde{\Gamma}) = \Gamma$ can be described by a similar gluing process. We have

$$\overline{\mathcal{M}}_{\widetilde{\Gamma}} = \left(\prod_{i=1}^m \overline{\mathcal{M}}_{\widetilde{\Gamma}_i} \right) \times_{\widetilde{\Gamma}} \left(\prod_{j=1}^l \overline{\mathcal{M}}_{0, \mathrm{val}(v_j) + n(v_j)}(\widetilde{Y}_\infty, \beta_j) \right),$$

where $\widetilde{\Gamma}_i$ is the graph such that $\tau(\widetilde{\Gamma}_i) = \Gamma_i$.

Now $e_{\mathbb{C}^*}(N_\Gamma^{vir})$ can be computed using $e_{\mathbb{C}^*}(N_{\Gamma_i}^{vir})$, ψ-classes in $\overline{\mathcal{M}}_{0, \mathrm{val}(v_i) + n(v_i)}(Y_\infty, \beta_i))$ (smoothing the nodes glued by "$\times_\Gamma$"), $TY|_{Y_\infty}$, and $(\mathcal{O}(-1))_{0, \mathrm{val}(v_i) + n(v_i), \beta_i}$ in $K^0(\overline{\mathcal{M}}_{0, \mathrm{val}(v_i) + n(v_i)}(Y_\infty, \beta_i)))$. If we use $\mathrm{Res}(\Gamma)$ to denote the localization residue of the graph Γ, the computation can be schematically written as follows.

$$\mathrm{Res}(\Gamma) = \int_{\overline{\mathcal{M}}_\Gamma} \prod_{i=1}^m \mathrm{Res}(\overline{\mathcal{M}}_{\Gamma_i}) \prod_{j=1}^l \prod_{e \in E(v_j)} \frac{1}{(\lambda + h)/d_e - \psi_{(e, v_j)}} \prod_{k=1}^l \mathrm{Res}(\Gamma(v_k)),$$

where $E(v_j)$ is the set of edges incident to v_j, d_e is the degree of the edge e, $\psi_{(e, v_j)}$ is the psi-class in $\overline{\mathcal{M}}_{\Gamma(v_j)}$ corresponding to the marking that was supposed to glue with edge e in the original graph Γ. Note that if v_j is an unstable vertex, the corresponding part of the formula needs a slight modification. This is standard and we do not spell it out to introduce unnecessary notations.

Similar thing happens to the localization residue of $\widetilde{\Gamma}$ as well.

$$\mathrm{Res}(\widetilde{\Gamma}) = \int_{\overline{\mathcal{M}}_{\widetilde{\Gamma}}} \prod_{i=1}^m \mathrm{Res}(\overline{\mathcal{M}}_{\widetilde{\Gamma}_i}) \prod_{j=1}^l \prod_{e \in E(v_j)} \frac{1}{(\lambda + h)/d_e - \psi_{(e, v_j)}} \prod_{k=1}^l \mathrm{Res}(\Gamma(v_k)).$$

Here we use the fact that $Y_\infty \cong \widetilde{Y}_\infty$, and identify the residues of vertices over Y_∞ with the ones over $\widetilde{Y}_\infty$. By using the correspondence of residues between $\overline{\mathcal{M}}_{\Gamma_i}$ and $\overline{\mathcal{M}}_{\widetilde{\Gamma}_i}$ and apply projection formula to the rest of the factors, one can see that the localization residues for $\overline{\mathcal{M}}_\Gamma$ and $\overline{\mathcal{M}}_{\widetilde{\Gamma}}$ match.

4. $\mathbb{P}^n$-fibrations

Let $\pi : P \to X$ be a smooth morphism whose fibers are isomorphic to projective spaces. Then P is a *Brauer–Severi* scheme over X. We would like to recall a few standard facts about Brauer–Severi schemes. According to [7, I, Théorème 8.2], it is étale-locally a product with projective space. It also induces a class in $H^2_{\text{ét}}(X, \mathbb{G}_m)$ called *Brauer class*. Let's denote it by $\alpha(P)$. $\alpha(P) = 0$ if and only if the fibration $\pi : P \to X$ comes from projectivization of a vector bundle. We also have the following standard fact.

LEMMA 4.1. — $\widetilde{\pi} : P \times_X P \to P$ *is the projectivization of a vector bundle over P.*

This can be seen by, for example, observing $\widetilde{\pi}$ has a section, and then apply [1, Lemma 2.4]. As a result, we also have the following.

LEMMA 4.2. — *The induced pull-back $\pi^* : H^2_{\text{ét}}(X; \mathbb{G}_m) \to H^2_{\text{ét}}(P; \mathbb{G}_m)$ sends $\alpha(P)$ to 0.*

Now we want to prove Theorem 1.4. Let's recall the set-up. Let X be a smooth projective variety, $\pi_1 : P_1 \to X$, $\pi_2 : P_2 \to X$ be projective smooth morphisms whose fibers are isomorphic to $\mathbb{P}^n$. Suppose

$$\alpha(P_1) = \alpha(P_2) \in H^2_{\text{ét}}(X; \mathbb{G}_m).$$

We also assume the existence of a ring isomorphism

$$\mathscr{F} : H^*(P_1; \mathbb{Q}) \to H^*(P_2; \mathbb{Q}),$$

such that

(1) $\mathscr{F}$ sends the subgroup $H^*(P_1; \mathbb{Z})/\text{tor}(H^*(P_1; \mathbb{Z}))$ into $H^*(P_2; \mathbb{Z})/\text{tor}(H^*(P_2; \mathbb{Z}))$. Here $\text{tor}(\,\cdot\,)$ denotes the torsion subgroup.

(2) $\mathscr{F}$ restricts to identity on $H^*(X; \mathbb{Q})$. Here we identify $H^*(X; \mathbb{Q})$ as subrings under pull-backs of π_1, π_2.

(3) $\mathscr{F}(c_1(\omega_{\pi_1})) = c_1(\omega_{\pi_2})$, where ω_{π_i} is the corresponding relative canonical sheaf.

LEMMA 4.3. — If $\alpha(P_1) = \alpha(P_2) = 0$, the existence of such $\mathscr{F}$ is equivalent to the existence of vector bundles V_1, V_2 over X such that $c(V_1) = c(V_2)$ and $P_1 \cong \mathbb{P}(V_1), P_2 \cong \mathbb{P}(V_2)$.

Proof. — The existence of such V_1, V_2 clearly induces such an $\mathscr{F}$. It suffices to prove the other direction. Since $\alpha(P_1) = 0$, there exists a V_1 such that $P_1 = \mathbb{P}(V_1)$. We have $\omega_{\pi_1} = -\mathcal{O}_{\mathbb{P}(V_1)}(n+1) - \pi_1^* \det(V_1)$. The same happens for P_2 and we have $P_2 = \mathbb{P}(V_2)$ and $\omega_{\pi_2} = -\mathcal{O}_{\mathbb{P}(V_2)}(n+1) - \pi_1^* \det(V_2)$. $\mathscr{F}$ preserving relative canonical sheaf implies that

$$\mathscr{F}(c_1(\mathcal{O}_{\mathbb{P}(V_1)}(1))) = c_1(\mathcal{O}_{\mathbb{P}(V_2)}(1)) - \pi_2^* \left(\frac{c_1(\det(V_1) - \det(V_2))}{n+1} \right).$$

But since everything else is in $H^2(P_2; \mathbb{Z}) / \operatorname{tor}(H^2(P_2; \mathbb{Z}))$, we also have that

$$\pi_2^* \left(\frac{c_1(\det(V_1) - \det(V_2))}{n+1} \right) \in H^2(P_2; \mathbb{Z}) / \operatorname{tor}(H^2(P_2; \mathbb{Z})).$$

Since π_2^* is an embedding on $H^2(X; \mathbb{Z}) / \operatorname{tor}(H^2(X; \mathbb{Z}))$, we have

$$\left(\frac{c_1(\det(V_1) - \det(V_2))}{n+1} \right) \in H^2(X; \mathbb{Z}) / \operatorname{tor}(H^2(X; \mathbb{Z})).$$

On the other hand, this class lies in $H^{1,1}(X)$. By Lefschetz theorem, there exists a divisor $L \in \operatorname{Pic}(X)$ such that $c_1(L) = \frac{c_1(\det(V_1) - \det(V_2))}{n+1}$. If one changes V_2 into $V_2 \otimes L$, one can verify that its Chern class will match the one with V_1 because $\mathscr{F}$ is a ring homomorphism. $\square$

In order to reduce the problem to the projectivization of vector bundles, we do a base-change along $\pi_1 : P_1 \to X$ (π_2 works equally well, but we need to fix one here). Consider the following diagram for both $i = 1, 2$

$$
\begin{array}{ccc}
P_1 \times_X P_i & \xrightarrow{pr_{2,i}} & P_i \\
\downarrow{\scriptstyle pr_{1,i}} & & \downarrow{\scriptstyle \pi_i} \\
P_1 & \longrightarrow & X.
\end{array}
$$

Because $\alpha(P_1) = \alpha(P_2)$ and Lemma 4.2, this class will be mapped to 0 via pull-back along either π_1 or π_2. As a result, the four maps $pr_{1,i}$ and $pr_{2,i}$ for $i = 1, 2$ all come from the projectivization of vector bundles.

Because of the degeneration of Leray spectral sequence at E_2, one can check that $H^*(P_1 \times_X P_i; \mathbb{Q}) = H^*(P_i; \mathbb{Q}) \otimes_{H^*(X; \mathbb{Q})} H^*(P_1; \mathbb{Q})$. The ring isomorphism $\mathscr{F}$ passes to $P_1 \times_X P_i$ over P_1 and all conditions are preserved (by slight abuse of notation, we still use $\mathscr{F}$ for it). By Lemma 4.3 and the

main theorem in [4], the Gromov–Witten theory of $P_1 \times_X P_1$ and $P_1 \times_X P_2$ are identified via $\mathscr{F}$. To be precise,

$$\langle \psi^{k_1}\sigma_1, \ldots, \psi^{k_n}\sigma_n \rangle^{P_1 \times_X P_1}_{0,n,\beta} = \langle \psi^{k_1}\mathscr{F}\sigma_1, \ldots, \psi^{k_n}\mathscr{F}\sigma_n \rangle^{P_1 \times_X P_2}_{0,n,\Psi(\beta)}$$

Finally, we want to relate back via $pr_{2,i}$ to prove such identification for P_1 and P_2. Suppose $P_1 \times_X P_i = \mathbb{P}_{P_i}(V_i)$ for $i = 1, 2$. Because we can tensor each of them with $\det(T_{\pi_i}) \otimes \pi_i^* L$ where L is a sufficiently ample line bundle, we can assume both V_1 and V_2 are globally generated. Now we are at a place to apply our Corollary 1.3. Since twisted Gromov–Witten invariant can be computed by descendant Gromov–Witten invariants, we also have

$$\langle \psi^{k_1}\sigma_1, \ldots, \psi^{k_n}\sigma_n \rangle^{\mathbb{P}_{P_1}(V_1),\mathcal{O}(-1)}_{0,n,\beta} = \langle \psi^{k_1}\mathscr{F}\sigma_1, \ldots, \psi^{k_n}\mathscr{F}\sigma_n \rangle^{\mathbb{P}_{P_2}(V_2),\mathcal{O}(-1)}_{0,n,\Psi(\beta)}.$$

Immediately applying Corollary 1.3, we have

$$\langle \sigma_1, \ldots, \sigma_n \rangle^{P_1}_{0,n,\beta} = \langle \mathscr{F}\sigma_1, \ldots, \mathscr{F}\sigma_n \rangle^{P_2}_{0,n,\Psi(\beta)}.$$

This is almost the form of Theorem 1.4 except we don't have descendants here. However, since we only work on genus-0 Gromov–Witten theory, descendants invariants can be computed via the topological recursion relation from the absolute invariants. Since the cohomology rings of $H^*(P_1; \mathbb{Q})$ and $H^*(P_2; \mathbb{Q})$ are identified, each step in the computation can be matched. This concludes our proof for Theorem 1.4.

Appendix A. Correspondence of residues

Let's put everything into a general setting. Let $f : X \to Y$ be a proper map between Deligne–Mumford stacks. Suppose $T = \mathbb{C}^*$ acts on both X and Y and f is equivariant. Let the connected components of the fixed loci of X and Y be $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$, respectively. Denote $i_{X_j} : X_j \hookrightarrow X, i_{Y_i} : Y_i \hookrightarrow Y$ as the corresponding embeddings. Suppose that there are equivariant perfect obstruction theories of X and Y. They induce natural perfect obstruction theories on components X_j, Y_i ([6]). We want to prove the following theorem.

PROPOSITION A.1. — *Assume X and Y admit equivariant embeddings into smooth Deligne–Mumford stacks with $\mathbb{C}^*$ actions. Suppose f_* preserves equivariant virtual classes, i.e., $f_*[X]^{vir,eq} = [Y]^{vir,eq}$. Fixing a connected component Y_i of the fixed locus in Y. We have the following equality in*

$A^{\mathbb{C}^*}_*(Y_i)$.

$$f_*\left(\sum_{f(X_j)\subset Y_i}\frac{[X_j]^{vir}}{e_{\mathbb{C}^*}(N^{vir}_{X_j/X})}\right)=\frac{[Y_i]^{vir}}{e_{\mathbb{C}^*}(N^{vir}_{Y_i/Y})},$$

where $N^{vir}_{X_j/X}$ and $N^{vir}_{Y_i/Y}$ are the corresponding virtual normal bundles, and we slightly abuse the notation by writing $f|_{f^{-1}(Y_i)}$ simply as f.

Proof. — This is basically a direct consequence of the decomposition of equivariant Chow groups in [9, Theorem 5.3.5] and the explicit localization formula in [6]. More precisely, by [9, Theorem 5.3.5], the equivariant Chow groups of X and Y admit the following decompositions.

$$A^{\mathbb{C}^*}_*(X)\otimes_{\mathbb{Q}}\mathbb{Q}[\lambda,\lambda^{-1}]=\bigoplus_{j=1}^{n}A^{\mathbb{C}^*}_*(X_j)\otimes_{\mathbb{Q}[\lambda]}\mathbb{Q}[\lambda,\lambda^{-1}],$$

$$A^{\mathbb{C}^*}_*(Y)\otimes_{\mathbb{Q}}\mathbb{Q}[\lambda,\lambda^{-1}]=\bigoplus_{i=1}^{m}A^{\mathbb{C}^*}_*(Y_i)\otimes_{\mathbb{Q}[\lambda]}\mathbb{Q}[\lambda,\lambda^{-1}].$$

Since f maps fixed locus into fixed locus, f_* respects these decompostions. Now [6] tells us that under the above decompositions, the X_j component of $[X]^{vir,eq}$ can be explicitly written as $\frac{[X_j]^{vir}}{e_{\mathbb{C}^*}(N^{vir}_{X_j/X})}$. Since f_* respects the decomposition, the Y_i component of $f_*[X]^{vir,eq}$ is $f_*\left(\sum_{f(X_j)\subset Y_i}\frac{[X_j]^{vir}}{e_{\mathbb{C}^*}(N^{vir}_{X_j/X})}\right)$. On the other hand, the Y_i component of $[Y]^{vir,eq}$ is $\frac{[Y_i]^{vir}}{e_{\mathbb{C}^*}(N^{vir}_{Y_i/Y})}$. Since $f_*[X]^{vir,eq}=[Y]^{vir,eq}$, the result follows. □

Remark A.2. — Conversely, if all localization residues match, one can conclude that $f_*[X]^{vir,eq}=[Y]^{vir,eq}$ by similar argument. Again, the decomposition [9, Theorem 5.3.5] is the key.

Appendix B. Computing the virtual normal bundles

Recall that $Y=\mathbb{P}_X(V\oplus\mathcal{O})$, $\widetilde{Y}=\mathbb{P}_{\mathbb{P}(V)}(\mathcal{O}(-1)\oplus\mathcal{O})$, and the map $\pi:\widetilde{Y}\to Y$ contracts the zero section $\widetilde{Y}_0=\mathbb{P}_{\mathbb{P}(V)}(\mathcal{O})\subset\widetilde{Y}$ to the zero section $Y_0=\mathbb{P}_X(\mathcal{O})\subset Y$. We would like to give some extra explanations to the derivation of equation (3.2)–(3.5).

B.1. A single vertex over Y_0

Now Γ consists of a single vertex over Y_0. First we want to compute $e_{\mathbb{C}^*}(N^{vir}_\Gamma)$. Recall there exists a two term complex $[E^{-1}\to E^0]$ consisting of vector bundles resolving the relative perfect obstruction theory of

$\overline{\mathcal{M}}_{0,n}(Y,\beta)$ over the moduli of prestable curves. Note that $E^\bullet$ is quasi-isomorphic to $(R^\bullet(ft_{n+1})_* ev_{n+1}^* TY)^\vee$ (Notations see Section 3.1). Since Y_0 is the fixed locus and the curve completely maps to Y_0, one can see that the moving part of $(E^\bullet)^\vee$ is a complex resolving $R^\bullet(ft_{n+1})_* ev_{n+1}^* N_{Y_0/Y}$. Comparing with the definition in Section 3.1, the localization residue is exactly the Gromov–Witten invariants twisted by $V \cong N_{Y_0/Y}$. To be precise, if we fix the insertions $\sigma_1, \ldots, \sigma_n \in H^*(X)$, the pushforward of the localization residue of Γ to a point results in the following invariant (Also see the right hand side of equation (3.2)).

$$\langle \sigma_1, \ldots, \sigma_n \rangle_{0,n,\beta}^{X,V}.$$

On the other hand, one sees that there is only one graph $\widetilde{\Gamma}$ that lifts Γ. $\widetilde{\Gamma}$ consists of only one vertex over $\widetilde{Y}_0$. By a similar argument, the corresponding localization residue gives rise to the Gromov–Witten invariant of $\mathbb{P}_X(V) \cong \widetilde{Y}_0$ twisted by $\mathcal{O}(-1)$. To be precise, it is

$$\langle \pi^*\sigma_1, \ldots, \pi^*\sigma_n \rangle_{0,n,\pi^!\beta}^{\mathbb{P}_X(V),\mathcal{O}(-1)}.$$

B.2. A vertex with edges

Now suppose the graph Γ consists of a vertex v_1 over Y_0 of class β with n markings, a vertex v_2 over Y_∞ of class 0 (i.e. degree 0) with 1 marking, and an edge of class f (the curve class of line inside a fiber) connecting these two vertices. Label the marking on v_2 as the 1-st marking, and the n markings on v_1 as $2, \ldots, n+1$-th markings. On the 1-st marking, we choose $h^e \pi^* \alpha$ as our insertion. On the $2, \ldots, n+1$-th markings, $\pi^*\sigma_1, \ldots, \pi^*\sigma_n$ are the corresponding insertions.

The lifting of the graph Γ is a graph $\widetilde{\Gamma}$ with a vertex $\widetilde{v}_1$ over $\widetilde{Y}_0$ of class $\pi^!\beta + f$, a vertex $\widetilde{v}_2$ over $\widetilde{Y}_\infty$ of class 0 and an edge between them.

The explicit localization formula for an arbitrary $\mathbb{C}^*$-action is already studied in different papers, for example, [5] and [14], among others. We have

$$\mathrm{Res}(\widetilde{\Gamma}) = \int_{\overline{\mathcal{M}}_{\widetilde{\Gamma}}} \frac{1}{e_{\mathbb{C}^*}(\mathcal{O}(-1)_{0,n+1,\pi^!\beta+f})} \frac{ev_1^*(h^e\pi^*\alpha)\prod_{i=1}^n ev_{i+1}^*\pi^*\sigma_n}{ev_1^*(h-\lambda)(ev_1^*(\lambda-h)-\psi_1)}.$$

Heuristically, $ev_1^*(h-\lambda)$ corresponds to the deformation of the 1-st marking, and $ev_1^*(\lambda-h)-\psi_i$ corresponds to the formal deformation of smoothing the node between $\widetilde{v}_1$ and the edge. We note that $\overline{\mathcal{M}}_{\widetilde{\Gamma}} \cong \overline{\mathcal{M}}_{0,n+1}(\mathbb{P}(V),\pi^!\beta+f)$. Fiber product does not affect the space because the target of ev_1 and the deformation space of the edge are both $\mathbb{P}(V)$.

Due to the similarity of the graphs, we have a similar expression for the residue of Γ.

$$\text{Res}(\Gamma) = \int_{\overline{\mathcal{M}}_\Gamma} \frac{1}{e_{\mathbb{C}^*}(V_{0,n+1,\beta+f})} \frac{ev_1^*(h^e\pi^*\alpha) \prod_{i=1}^n ev_{i+1}^* \pi^*\sigma_n}{ev_1^*(h-\lambda)(ev_1^*(\lambda-h)-\psi_1)}.$$

But note that $\overline{\mathcal{M}}_\Gamma \cong \overline{\mathcal{M}}_{0,n+1}(X,\beta) \times_{(ev_1,\pi)} \mathbb{P}_X(V)$, where $\times_{(ev_1,\pi)}$ means the fiber product of ev_1 and π. Here we abuse notation by using the same ev_1 for the evaluation map of $\overline{\mathcal{M}}_{0,n+1}(X,\beta)$. We warn the reader that this evaluation map is only used to glue with the edge. This map is completely different from the ev_1 of $\overline{\mathcal{M}}_\Gamma$ whose target is $\mathbb{P}(V)$ instead of X.

Although the above integrands are similar, in order to get a numerical result, we need to push them forward to a point. It's very straightforward for $\overline{\mathcal{M}}_{\widetilde{\Gamma}}$. But for $\overline{\mathcal{M}}_\Gamma$, there is an extra projection along the fibers of $\mathbb{P}(V)$. This is why we get

$$\left\langle \pi_* \left(\frac{h^e\pi^*\alpha}{(h-\lambda)(\lambda-h-\psi)} \right), \sigma_1, \ldots, \sigma_n \right\rangle_{0,n+1,\beta}^{X,V}$$

on the right hand side of equation (3.3).

For equations (3.4) and (3.5), the reasonings are similar. We note that in order to derive equation (3.5), the k-fold covering of the edge give rise to extra factors in the virtual normal bundles. It corresponds to a factor

$$ev_1^* \frac{1}{\prod_{m=1}^k (\frac{m}{k}(h-\lambda)) \prod_{m=1}^{k-1}(\frac{m}{k}(\lambda-h))}$$

in $\overline{\mathcal{M}}_{\widetilde{\Gamma}}$, and to a factor

$$ev_1^* \frac{1}{\prod_{m=1}^k \left(\frac{m}{k}(h-\lambda)\right) \prod_{m=1}^{k-1} \left(c_V(h+\frac{m}{k}(\lambda-h))\right)}$$

in $\overline{\mathcal{M}}_\Gamma$. The rest of the argument is similar and straightforward.

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