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Simultaneous non-vanishing for Dirichlet L-functions

SIMULTANEOUS NON-VANISHING FOR DIRICHLET L-FUNCTIONS

by Raphaël ZACHARIAS

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ABSTRACT. — We extend the work of Fouvry, Kowalski and Michel on correlation between Hecke eigenvalues of modular forms and algebraic trace functions in order to establish an asymptotic formula for a generalized cubic moment of modular L-functions at the central point $s = \frac{1}{2}$. As an application, we exploit our recent result on the mollification of the fourth moment of Dirichlet L-functions to derive that for any pair (ω_1, ω_2) of multiplicative characters modulo a prime q, there is a positive proportion of χ (mod q) such that the central values $L(\chi, \frac{1}{2}), L(\chi\omega_1, \frac{1}{2})$ and $L(\chi\omega_2, \frac{1}{2})$ are simultaneously not too small.

RÉSUMÉ. — Nous généralisons le travail de Fouvry, Kowalski et Michel sur la corrélation entre les valeurs propres de Hecke de formes modulaires et les fonctions traces dans le but d'établir une formule asymptotique pour un moment cubique généralisé de fonctions L au point central $s = \frac{1}{2}$. Comme application, nous exploitons notre résultat récent sur la mollification du quatrième moment des fonctions L de Dirichlet et déduisons que pour ω_1, ω_2 deux charactères multiplicatifs modulo un nombre premier q, il existe une proportion positive de χ (mod q) telle que les valeurs centrales $L(\chi, \frac{1}{2}), L(\chi\omega_1, \frac{1}{2})$ et $L(\chi\omega_2, \frac{1}{2})$ soient simultanément pas trop petites.

1. Introduction and statement of results

The zeros of automorphic L-functions on the critical line have received considerable attention these last years [4, 9, 21, 28, 32]. In particular, at the central point $s = \frac{1}{2}$, an L-function is expected to vanish only for either a good reason or a trivial reason. For example, if E is an elliptic curve defined over $\mathbf{Q}$ and $L(E, s)$ is its associated L-function, then according to the Birch and Swinnerton–Dyer conjecture, $L(E, \frac{1}{2}) = 0$ if and only if the group of $\mathbf{Q}$-points $E(\mathbf{Q})$ has positive rank. A trivial reason is for instance

when the sign of the functional equation is -1, which is the case if the L-function is attached to an odd Hecke–Maass form.

A typical approach in the study of non- vanishing problems is to consider a family of L-functions $\{L(\pi, \frac{1}{2})\}$ for π varying in some finite set $\mathcal{A}$ and try to give a lower bound for the proportion of $\pi \in \mathcal{A}$ such that $L(\pi, \frac{1}{2}) \neq 0$ as $|\mathcal{A}| \to \infty$. In [20], H. Iwaniec and P. Sarnak examined $L(\chi, s)$ at $s = \frac{1}{2}$ as χ ranges over all primitive Dirichlet characters modulo q. They proved that at least $\frac{1}{3}$ of the central values $L(\chi, \frac{1}{2})$ are not zero. This proportion has been slightly improved to 0.3411 by H. M. Bui [3] and finally to $\frac{3}{8}$ by R. Khan and H. T. Ngo [25] with the restriction to prime moduli q.

In [36], P. Michel and J. Vanderkam considered simultaneous non-vanishing problems : given three distinct Dirichlet characters $\omega_1, \omega_2, \omega_3$ of fixed modulus D_1, D_2, D_3 (satisfying some technical conditions), they proved that a positive proportion of Holomorphic primitive Hecke cusp form f of weight 2, prime level q and trivial nebentypus are such that the product $L(f \otimes \omega_1, \frac{1}{2})L(f \otimes \omega_2, \frac{1}{2})L(f \otimes \omega_3, \frac{1}{2})$ is not zero for sufficiently large q (in terms of D_1, D_2, D_3).

In this paper, we let $q > 2$ be a prime number, ω_1, ω_2 be arbitrary Dirichlet characters of modulus q [(1)], $\mathcal{D}(q)$ (resp. $\mathcal{D}_{\overline{\omega}_1, \overline{\omega}_2}(q)$) the set of primitive characters modulo q (resp. different from $\overline{\omega}_1, \overline{\omega}_2$) and f a cuspidal Hecke eigenform for $\mathrm{SL}_2(\mathbf{Z})$ (holomorphic or Maass). We are interested in the distribution of the values of the two families

$$\left\{ L\left(\chi, \tfrac{1}{2}\right) L\left(\chi\omega_1, \tfrac{1}{2}\right) L\left(\chi\omega_2, \tfrac{1}{2}\right) \ : \ \chi \in \mathcal{D}_{\overline{\omega}_1, \overline{\omega}_2}(q) \right\},$$
$$\left\{ L\left(f \otimes \chi, \tfrac{1}{2}\right) L\left(\chi, \tfrac{1}{2}\right) \ : \ \chi \in \mathcal{D}(q) \right\},$$

as $q \to \infty$. Using mollification method, a technique that has made the success of many of the papers cited above, we show that for both families, a positive proportion of these central values of character twists is not zero. We give in fact a more precise statement, saying that they are simultaneously not too small. These simultaneous non-vanish results require the evaluation of the two cubic moments

$$(1.1) \qquad \mathscr{T}^3(\omega_1, \omega_2, \ell; q) := \frac{1}{|\mathcal{D}_{\overline{\omega}_1, \overline{\omega}_2}(q)|} \sum_{\substack{\chi \pmod{q} \\ \chi \neq 1, \overline{\omega}_1, \overline{\omega}_2}} \chi(\ell) \prod_{i=0}^{2} L\left(\chi\omega_i, \tfrac{1}{2}\right),$$

[(1)] We point out that there is no additional difficulty by considering fixed ω_1, ω_2 of conductors $D_1, D_2 < q$.

ω_0 being in this paper the trivial character modulo q, and

$$(1.2) \qquad \mathscr{T}^3(f, \ell; q) := \frac{1}{|\mathcal{D}(q)|} \sum_{\substack{\chi \pmod q \\ \chi \neq 1}} L\left(f \otimes \chi, \tfrac{1}{2}\right) L\left(\chi, \tfrac{1}{2}\right) \chi(\ell),$$

for $\ell \geqslant 1$ an integer coprime with q.

We now state the two main results of this paper :

THEOREM 1.1. — Let $\varepsilon > 0$ be a real number. Then there exists an explicit absolute constant $c_1 > 0$ and $Q = Q(\varepsilon) > 2$ such that for any prime $q \geqslant Q$ and every Dirichlet characters ω_1, ω_2 modulo q,

$$\left| \left\{ \chi \pmod q \ : \ \left| L\left(\chi\omega_i, \tfrac{1}{2}\right) \right| \geqslant \frac{1}{\log q}, \ i = 0, 1, 2 \right\} \right| \geqslant (c_1 - \varepsilon)(q - 1).$$

THEOREM 1.2. — Let f be a Hecke cusp of level 1 and spectral parameter t_f satisfying the Ramanujan–Petersson conjecture and let $\varepsilon > 0$. Then there exists an explicit absolute constant $c_2 > 0$ and $Q = Q(\varepsilon, t_f) > 2$ such that for any prime $q \geqslant Q$,

$$\left| \left\{ \chi \pmod q \ : \ \left| L\left(f \otimes \chi, \tfrac{1}{2}\right) \right| \geqslant \frac{1}{\log(q)^2}, \left| L\left(\chi, \tfrac{1}{2}\right) \right| \geqslant \frac{1}{\log q} \right\} \right|$$

is at least $(c_2 - \varepsilon)(q - 1)$.

The new main ingredient in the proof of Theorems 1.1 and 1.2 is the following result which establishes an asymptotic formula for the moments (1.1) and (1.2).

THEOREM 1.3. — Let $q > 2$ be a prime number, ω_1, ω_2 be Dirichlet characters of modulus q, f a primitive Hecke cusp form of level 1 or q and trivial nebentypus. Assume that f satisfies the Ramanujan–Petersson conjecture, then for any $\varepsilon > 0$, we have

$$(1.3) \qquad \mathscr{T}^3(\omega_1, \omega_2, \ell; q) = \delta_{\ell=1} + O\left(q^{-\frac{1}{64}+\varepsilon}\right),$$

$$(1.4) \qquad \mathscr{T}^3(f, \ell; q) = \delta_{\ell=1} + O\left(q^{-\frac{1}{52}+\varepsilon}\right),$$

where the implied constant only depends on $\varepsilon > 0$ and polynomially on the Archimedean parameters of f (the weight or the Laplace eigenvalue) in (1.4).

Remark 1.4. — We have not seen the details, but the case where f is of level one and $\ell = 1$ has been announced by S. Das and S. Ganguly and it seems that their method is similar to our.

When f is of level q with non trivial central character, the proof of Theorem 1.3 is similar but requires a mild extension of [29, Theorem 1.3]

for Kloosterman sums twisted by characters. We will return to this question in a coming paper.

Remark 1.5. — The asymptotic formula (1.4) is similar to the mixted cubic moment evaluated by S. Das and R. Khan in [5]

$$\frac{1}{q-1} \sum_{\substack{\chi \pmod{q} \\ \chi \neq 1}} L\left(f \otimes \chi, \tfrac{1}{2}\right) \overline{L\left(\chi, \tfrac{1}{2}\right)}.$$

As the authors explained, the complex conjugation above $L(\chi, 1/2)$ was introduced to avoid difficulties connected to the oscillations of Gauss sums. What we show here is that this difficulties are resolved using variants of the methods of [14] [29].

Following a suggestion of the referee, we point out that there is an advantage in considering the two moments without the complex conjugation. In this case we obtain a main term only when $\ell = 1$ and this main term is 1 (in particular independant of q, ω_i or f), which greatly facilitates the average over ℓ in the mollification method (see Section 7.1).

1.1. Sketch of the proof of Theorem 1.3

After an application of the approximate functional equation to (1.2) and (1.1), which expresses the central value of automorphic L-functions as a convergent series, and an average over the characters, we isolate a main term which appears only if $\ell = 1$ (cf. Section 6.1.1–6.1.3).

The treatment of the error term passes by the analysis of sums of the shape

$$(1.5) \quad \mathcal{S}(\omega_1, \omega_2; q)$$

$$= \frac{1}{(qN_0N_1N_2)^{1/2}} \sum_{\substack{n_i \sim N_i \\ i=0,1,2}} \overline{\omega}_1(n_1)\overline{\omega}_2(n_2)\mathrm{Kl}_3(n_0n_1n_2, \omega_1, \omega_2, 1; q),$$

$$(1.6) \qquad \mathcal{C}(f; q) = \frac{1}{(qMN)^{1/2}} \sum_{n \sim N, m \sim M} \lambda_f(n)\mathrm{Kl}_3(nm; q),$$

where Kl_3 is the classical 2-dimensional normalized hyper-Kloosterman sum, $\mathrm{Kl}_3(\omega_1, \omega_2, 1; q)$ is the twisted version as defined in (3.9), $\{\lambda_f(n)\}_{n \geqslant 1}$ are the Hecke eigenvalues of f and N_0, N_1, N_2, N, M are parameters satisfying

$$1 \leqslant N_i, N, M, \quad N_0N_1N_2 \leqslant q^{3/2+\varepsilon} \text{ and } MN \leqslant q^{3/2+\varepsilon}.$$

The ultimate goal is to obtain a bound of the form

$$\mathcal{S}(\omega_1, \omega_2; q), \mathcal{C}(f; q) = O\left(q^{-\delta}\right),$$

for some absolute constant $\delta > 0$. Using Poisson summation in the three variables in (1.5), or Voronoi formula in the n-variable in (1.6) (followed by Poisson on m) allows us to get rid of the cases where the product of the variables is larger than q; namely in Section 6.1.6 and 6.2.4, we prove

$$\mathcal{S}(\omega_1, \omega_2; q) \ll q^\varepsilon \left(\frac{q}{N_0 N_1 N_2}\right)^{1/2} \quad \text{and} \quad \mathcal{C}(f; q) \ll q^\varepsilon \left(\frac{q}{NM}\right)^{1/2}.$$

Combining these two estimates with the trivial bounds

$$\mathcal{S}(\omega_1, \omega_2; q) \ll \left(\frac{N_0 N_1 N_2}{q}\right)^{1/2} \quad \text{and} \quad \mathcal{C}(f; q) \ll q^\varepsilon \left(\frac{NM}{q}\right)^{1/2},$$

we can assume for the rest of this outline that

$$N_0 N_1 N_2 = NM = q.$$

We treat these sums differently according to the relative size of the various parameters. If $N_1 \sim 1$ (say) and $M \sim 1$, we exploit the n_0, n_2-sum (resp. the n-sum) in (1.5) (resp. in (1.6)) and average trivially over the others. Grouping $n_0 n_2$ into a long variable n and we need to analyze roughly

$$\sum_{n \sim q} \lambda_{\overline{\omega}_2}(n, 0) \mathrm{Kl}_3(n n_1, \omega_1, \omega_2; q) \quad \text{and} \quad \sum_{n \sim q} \lambda_f(n) \mathrm{Kl}_3(nm; q),$$

where for any $t \in \mathbf{R}$,

$$\lambda_{\overline{\omega}_2}(n, it) = \sum_{n_0 n_2 = n} \overline{\omega}_2(n_2) \left(\frac{n_2}{n_0}\right)^{it}.$$

In [13] and [14], Fouvry, Kowalski and Michel studied these sums when f is a fixed cusp form of level 1, $\lambda_{\overline{\omega}_2}(n, it)$ is replaced by the generalized divisor function $d_{it}(n) = \sum_{ab=n} a^{it} b^{-it}$ and a general Frobenius trace function modulo q instead of Kl_3. Our second main result is an extension of their theorems (see [14, Theorem 1.2] and [13, Theorem 1.15]) for Hecke eigenforms (cuspidal or not) of level q with arbitrary central ω. More precisely, for V a smooth and compactly supported function on $\mathbf{R}_+^*$, we consider the sums

$$(1.7) \qquad \mathcal{S}_V(f, K; q) := \sum_{n \geqslant 1} \lambda_f(n) K(n) V\left(\frac{n}{q}\right),$$

$$(1.8) \qquad \mathcal{S}_V(\omega, it, K; q) := \sum_{n \geqslant 1} \lambda_\omega(n, it) K(n) V\left(\frac{n}{q}\right).$$

We prove in Section 5 :

THEOREM 1.6. — *Let $q > 2$ be a prime number, ω a Dirichlet character of modulus q, f a primitive Hecke cusp form of level q, nebentypus ω and spectral parameter t_f. Let K be an isotypic trace function modulo q of conductor $\mathrm{cond}(K)$ such that its Fourier transform is not ω-exceptional, as defined in (3.3), (3.5) and (3.16). Let V be a function satisfying $V(C, P, Q)$ (see Definition (3.1)). Then there exists constants $s \geqslant 1$ and $A \geqslant 1$ such that*

$$\mathcal{S}_V(f, K; q) \ll_{C,\delta} (1 + |t_f|)^A \mathrm{cond}(K)^s q^{1-\delta} (PQ)^{1/2} (P + Q)^{1/2},$$

$$\mathcal{S}_V(\omega, it, K; q) \ll_{C,\delta} (1 + |t|)^A \mathrm{cond}(K)^s q^{1-\delta} (PQ)^{1/2} (P + Q)^{1/2},$$

for any $\delta < 1/16$, where A depends on ε and s is absolute.

Therefore, Theorem 1.6 provides the desired power saving (set $P = Q = 1$) in the special case where one of the variable is very small in (1.5) and $M \sim 1$ in (1.6).

Assume now that $N_0, N_1, N_2 \geqslant q^\eta$ and $M \geqslant q^\eta$ for some small real number $\eta > 0$. From now, we need to take care of the different nature of expressions (1.5) and (1.6). Indeed, for (1.5), the fact of having three free variables allows us to factorize two of them (say $n_0 n_2$) in such a way that $N_0 N_2 \geqslant q^{1/2+\eta}$. In this case, we can form a bilinear sum and use a general version of Polyá–Vinogradov (see Theorem 4.2) to obtain a power saving in the error term. The same method also works for (1.6), as long as $M \leqslant q^{1/2-\eta}$, because in this case $N \geqslant q^{1/2+\eta}$, or $M \geqslant q^{1/2+\eta}$. Hence, the critical range for the second sum, i.e. when Polyá–Vinogradov is useless, appears when $M \sim q^{1/2}$ and $N \sim q^{1/2}$ and here we apply the general result of Kowalski, Michel and Sawin concerning bilinear forms involving classical Kloosterman sums [29, Theorem 1.3].

2. Background on automorphic forms

In this section, we briefly compile the main results from the theory of automorphic forms which we shall need in Section 5. Among these are Hecke eigenbases, multiplicative properties of Hecke eigenvalues, the Kuznetsov trace formula and the spectral large sieve inequality.

2.1. Hecke eigenbases

Let $N \geqslant 1$ be an integer, ω a Dirichlet character of modulus N, $\kappa = \frac{1-\omega(-1)}{2} \in \{0,1\}$ and $k \geqslant 2$ satisfying $k \equiv \kappa$ (mod 2). We denote by $\mathcal{B}_k(N, \chi)$ (resp. $\mathcal{B}(N, \chi)$) a Hecke basis of the Hilbert space of holomorphic cusp forms of weight k (resp. of Maass cusp forms of weight κ) with respect to the Hecke congruence group $\Gamma_0(N)$ and with nebentypus ω. The continuous spectrum is continuously spanned by the Eisenstein series $E_{\mathfrak{a}}(\,\cdot\,, 1/2 + it)$ where $\mathfrak{a}$ runs over the singular cusps of $\Gamma_0(N)$.

2.1.1. The Eisenstein series in the special case $N = 2q$

Let $q > 2$ be a prime number. For some technical reasons, it is convenient for the proof of Theorem 1.6 to see the form f of level q as a form of level $2q$ (see the beginning of Section 4.1 and Section 5.5 in [14]). For arbitrary level N, the Eisenstein series $E_{\mathfrak{a}}(\,\cdot\,, 1/2 + it)$ are usually not eigenfunctions of the Hecke operators. In the special case where $N = 2q$, there are exactly four inequivalent cups for $\Gamma_0(2q)$ which are

$$\mathfrak{a} = 1 \ , \ \frac{1}{2} \ , \ \frac{1}{q} \ , \ \frac{1}{2q},$$

see for example [17, Proposition 2.6] and all are singular. The main advantage in this situation is that these Eisentein series are eigenforms of the Hecke operators T_n for $(n, 2q) = 1$. More precisely, if $\mathfrak{a} = 1/v$ with $v \in \{1, 2, q, 2q\}$, then we have for $(n, 2q) = 1$,

$$T_n E_{\mathfrak{a}}(\,\cdot\,, 1/2 + it) = \lambda_{\mathfrak{a}}(n, it) E_{\mathfrak{a}}(\,\cdot\,, 1/2 + it),$$

with explicitly

$$(2.1) \qquad \lambda_{\mathfrak{a}}(n, it) = \begin{cases} \sum_{ab=n} \omega(a) \left(\frac{a}{b}\right)^{it} & \text{if } v = q, 2q \\ \sum_{ab=n} \omega(b) \left(\frac{a}{b}\right)^{it} & \text{if } v = 1, 2, \end{cases}$$

see [11, (6.16)–(6.17)].

Remark 2.1. — In the case $N = q$, there are exactly two inequivalent cusps $\mathfrak{a} = 1, 1/q$ and the two Eisenstein series are eigenfunctions of the Hecke operators T_n for $(n, q) = 1$ with eigenvalues given by (2.1). Moreover, they are also Eisenstein series of level $2q$ after the normalization by $1/\sqrt{3}$.

2.2. Multiplicative and boundedness properties of Hecke eigenvalues

Let f be any Hecke eigenform of level N and nebentypus ω and let $\lambda_f(n)$ be the corresponding Hecke eigenvalues for T_n. Then for $(nm, N) = 1$, we have

$$(2.2) \qquad \lambda_f(m)\lambda_f(n) = \sum_{d|(n,m)} \omega(d)\lambda_f\left(\frac{nm}{d^2}\right),$$

$$(2.3) \qquad \overline{\lambda_f(n)} = \overline{\omega}(n)\lambda_f(n).$$

Note that if f is holomorphic, then by the work of Deligne and Serre, we have the Ramanujan–Petersson conjecture, namely

$$(2.4) \qquad |\lambda_f(n)| \leqslant \tau(n).$$

The same bound is of course trivial in the special case $N = 2q$ or q and the Eisenstein series are eigenfunctions with eigenvalues given by (2.1). In the case of a Maass cusp form f, the best result is due to Kim and Sarnak [26] and it is given by

$$(2.5) \qquad |\lambda_f(n)| \leqslant \tau(n)n^\theta \ , \quad \theta = \frac{7}{64},$$

with an analogous bound for the spectral parameter

$$(2.6) \qquad |\Im m(t_f)| \leqslant \theta,$$

where $1/4 + t_f^2$ is the Laplace eigenvalue of f. However, the conjecture is true on average, in the sense that for every $X \geqslant 1$,

$$(2.7) \qquad \sum_{n \leqslant X} |\lambda_f(n)|^2 \ll (N(1 + |t_f|))^\varepsilon X,$$

with an implied constant depending only on ε [11, Proposition 19.6]. We will also need later similar bound for the fourth-power on average and it is enough for our purpose to restrict to prime numbers p not dividing the level N

$$(2.8) \qquad \sum_{\substack{p \leqslant X \\ (p,N)=1}} |\lambda_f(p)|^4 \ll (XN(1 + |t_f|))^\varepsilon X,$$

for any $\varepsilon > 0$ and the constant only depends on ε. This bound is a consequence of the automorphy of the symmetric square $\mathrm{Sym}^2 f$ and Rankin–Selberg theory (see for example [35]).

2.3. Hecke eigenvalues and Fourier coefficients

Let f be a modular form. For $z = x + iy \in \mathbf{H}$, we write the Fourier expansion as

$$f(z) = \sum_{n=1}^{\infty} \rho_f(n) n^{\frac{k-1}{2}} e(nz) \ \text{ for } f \in \mathcal{B}_k(N, \omega),$$

$$f(z) = \sum_{n \neq 0} \rho_f(n) |n|^{-1/2} W_{\frac{|n|}{n} \frac{\kappa}{2}, it_f}(4\pi |n| y) e(nx), \ \text{ for } f \in \mathcal{B}(N, \omega),$$

where $W_{\alpha, \beta}$ is a Whittaker function, as defined in [11, Section 4]. For an Eisenstein series $E_{\mathfrak{a}}(z, 1/2 + it)$, we write

$$E_{\mathfrak{a}}(z, 1/2 + it) = c_{1,\mathfrak{a}}(t) y^{1/2+it} + c_{2,\mathfrak{a}}(t) y^{1/2-it}$$

$$+ \sum_{n \neq 0} \rho_{\mathfrak{a}}(n, it) |n|^{-1/2} W_{\frac{|n|}{n} \frac{\kappa}{2}, it}(4\pi |n| y) e(nx).$$

When f is a Hecke eigenform, there is a closed relation between the Fourier coefficients and the Hecke eigenvalues $\lambda_f(n)$; for $(m, N) = 1$ and $n \geqslant 1$, one has

$$(2.9) \qquad \lambda_f(n) \rho_f(n) = \sum_{d | (m,n)} \omega(d) \rho_f\left(\frac{mn}{d^2}\right).$$

In particular, for all $(m, q) = 1$,

$$(2.10) \qquad \lambda_f(m) \rho_f(1) = \rho_f(m).$$

If f is primitive, the relations (2.9) and (2.10) are valid for every $m \geqslant 1$. We will also need lower bounds for the first coefficient $\rho_f(1)$; we have for any $\varepsilon > 0$

$$(2.11) \qquad |\rho_f(1)|^2 \gg_{\varepsilon} \begin{cases} \frac{\cosh(\pi t_f)}{N(1+|t_f|)^{\kappa}(N+|t_f|)^{\varepsilon}} & \text{if } f \in \mathcal{B}(N, \omega) \\ \frac{(4\pi)^{k-1}}{(k-1)! N^{1+\varepsilon} k^{\varepsilon}} & \text{if } f \in \mathcal{B}_k(N, \omega), \end{cases}$$

see [11, (6.22), (7.16)] and [34, Lemma 2.2 and (2.23)]. For an Eisenstein series $E_{\mathfrak{a}}(\,\cdot\,, 1/2 + it)$, we have

$$(2.12) \qquad |\rho_{\mathfrak{a}}(1, it)|^2 \gg \frac{\cosh(\pi t)}{N(1 + |t|)^{\kappa}(\log(N + |t|))^2},$$

see [11, (6.23), (7.15)].

2.4. Summation formula, trace formula and the spectral large sieve inequality

2.4.1. Voronoi summation formula

We state a version of Voronoi summation formula for the cuspidal case. For our purpose, it is enough to restrict to modular form of prime level q.

PROPOSITION 2.2. — *Let $q > 2$ be a prime number, ω a Dirichlet character of modulus q and f a primitive Hecke cusp form of level q and nebentypus ω with associated Hecke eigenvalues $(\lambda_f(n))_{n \geqslant 1}$. Given an integer a coprime with q and $g : \mathbf{R}_+^* \to \mathbf{C}$ a smooth and compactly supported function, we set*

$$\mathcal{V}(f, a; q) := \sum_{n \geqslant 1} \lambda_f(n) e\left(\frac{an}{q}\right) g(n).$$

Then

$$(2.13) \qquad \mathcal{V}(f, a; q) = \frac{\overline{\omega}(a)}{q} \sum_{\pm} \sum_{n \geqslant 1} \lambda_f(n) e\left(\mp \frac{\overline{a}n}{q}\right) g_{\pm}\left(\frac{n}{q^2}\right),$$

where

$$g_{\pm}(y) = \int_0^\infty g(x) \mathcal{J}_{\pm}(4\pi\sqrt{xy}) \, \mathrm{d}x,$$

with

$$\mathcal{J}_+(x) = 2\pi i^k J_{k-1}(x) \ , \quad \mathcal{J}_-(x) = 0$$

if f is holomorphic of weight k and

$$\mathcal{J}_+(x) = -\frac{\pi}{\sin(\pi i t_f)} \left(J_{2it_f}(x) - J_{-2it_f}(x)\right),$$

$$\mathcal{J}_-(x) = \varepsilon_f 4\cos(\pi i t_f) K_{2it_f}(x)$$

if f is a Maass form of parity $\varepsilon_f \in \{-1, +1\}$ and spectral parameter t_f.

Proof. — The proof is contained in [30, Appendix A.3–A.4] and also [35, Lemma 2.3.1] for the holomorphic case. □

Finally, we consider the decay properties of the Bessel transforms $g_{\pm}$ (see [1, Lemma 2.4]).

LEMMA 2.3. — *Let $g : \mathbf{R}_+^* \to \mathbf{C}$ be a smooth and compactly supported function satisfying*

$$(2.14) \qquad x^i g^{(i)}(x) \ll_{i,\varepsilon} q^{\varepsilon i}$$

for any $\varepsilon > 0$ and $j \geqslant 0$. In the non-holomorphic case set $\vartheta = \Re(it_f)$, otherwise set $\vartheta = 0$. Then for any $\varepsilon > 0$, for any $i, j \geqslant 0$ and all $y > 0$, we have

$$y^j g_{\pm}^{(j)}(y) \ll_{i,j,\varepsilon} \frac{(1+y)^{j/2}}{\left(1 + (yq^{-\varepsilon})^{1/2}\right)^i} \left(1 + y^{-2\vartheta - \varepsilon}\right).$$

2.4.2. The Petersson formula

For $k \geqslant 2$ an integer such that $k \equiv \kappa \pmod 2$, the Petersson trace formula expresses an average of product of Fourier coefficients over $\mathcal{B}_k(N, \omega)$ in terms of sums of Kloosterman sums (see [18, Theorem 9.6] and [19, Proposition 14.5]) : for any integers $n, m > 0$, we have

$$(2.15) \qquad \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{g \in \mathcal{B}_k(N,\omega)} \rho_g(n)\overline{\rho_g(m)} = \delta(m,n) + \Delta_{N,k}(m,n),$$

where

$$\Delta_{N,k}(m,n) := 2\pi i^{-k} \sum_{N|c} \frac{1}{c} S_\omega(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

and the Kloosterman sum $S_\omega(m, n; c)$ is defined by

$$S_\omega(m,n;c) = \sum_{\substack{d \pmod c \\ (d,c)=1}} \overline{\omega}(d) e\left(\frac{m\overline{d} + nd}{c}\right).$$

2.4.3. The Kuznetsov formula

Let $\phi : \mathbf{R}_+ \to \mathbf{C}$ be a smooth function satisfying $\phi(0) = \phi'(0) = 0$ and $\phi^{(j)}(x) \ll_\varepsilon (1+x)^{-2-\varepsilon}$ for $0 \leqslant j \leqslant 3$ and every $\varepsilon > 0$. For $\kappa \in \{0, 1\}$, we define the two Bessel transforms

$$\dot{\phi}(k) = i^k \int_0^\infty J_{k-1}(x)\phi(x)\frac{\mathrm{d}x}{x},$$

$$\widetilde{\phi}(t) = \frac{it^\kappa}{2\sinh(\pi t)} \int_0^\infty \left(J_{2it}(x) - (-1)^\kappa J_{-2it}(x)\right)\phi(x)\frac{\mathrm{d}x}{x}.$$

Then for every integers $m, n > 0$, we have the following spectral decomposition of the Kloosterman sums,

$$(2.16) \quad \sum_{\substack{k \equiv \kappa \bmod 2 \\ k > \kappa \\ g \in \mathcal{B}_k(N, \omega)}} \dot{\phi}(k) \frac{(k-1)!}{\pi(4\pi)^{k-1}} \overline{\rho_g(m)} \rho_g(n)$$

$$+ \sum_{g \in \mathcal{B}(N, \omega)} \widetilde{\phi}(t_g) \frac{4\pi}{\cosh(\pi t_g)} \overline{\rho_g(m)} \rho_g(n)$$

$$+ \sum_{\mathfrak{a} \text{ singular}} \int_0^\infty \widetilde{\phi}(t) \frac{1}{\cosh(\pi t)} \overline{\rho_\mathfrak{a}(m, it)} \rho_\mathfrak{a}(n, it) \, \mathrm{d}t = \Delta_{N, \phi}(m, n),$$

where

$$\Delta_{N, \phi}(m, n) = \sum_{N | c} \frac{1}{c} S_\omega(m, n; c) \phi\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

see [18, Theorem 9.4 and 9.8].

2.4.4. The spectral large sieve inequality

LEMMA 2.4. — *Let $N \geqslant 1$ be an integer and ω a Dirichlet character of modulus N whose conductor is odd and squarefree. Then for any $m, n \in \mathbf{Z}$ and $N | c$, we have the Weil bound*

$$|S_\omega(m, n; c)| \leqslant \tau(c)(m, n, c)^{1/2} c^{1/2}.$$

Proof. — The proof is a consequence of the twisted multiplicativity of Kloosterman sums and [27, Propositions 9.7 and 9.8]. □

PROPOSITION 2.5. — *Let $N \geqslant 1$ be an integer, ω a Dirichlet character of modulus N with squarefree and odd conductor and $\kappa \in \{0, 1\}$ such that $\omega(-1) = (-1)^\kappa$. Let $T \geqslant 1$, $M \geqslant 1/2$ and $(a_m)_{M < m \leqslant 2M}$ a sequence of complex numbers. Then for any $\varepsilon > 0$, each of the following three quantities*

$$\sum_{\substack{\kappa < k \leqslant T \\ k \equiv \kappa \pmod 2}} \Gamma(k) \sum_{g \in \mathcal{B}_k(N, \omega)} \left| \sum_{M < m \leqslant 2M} a_m \rho_g(m) \right|^2,$$

$$\sum_{\substack{g \in \mathcal{B}(N, \omega) \\ |t_g| \leqslant T}} \frac{(1 + |t_g|)^\kappa}{\cosh(\pi t_g)} \left| \sum_{M < m \leqslant 2M} a_m \rho_g(m) \right|^2,$$

$$\sum_{\mathfrak{a} \text{ singular}} \int_{-T}^{T} \frac{(1 + |t|)^\kappa}{\cosh(\pi t)} \left| \sum_{M < m \leqslant 2M} a_m \rho_\mathfrak{a}(m, t) \right|^2,$$

is bounded, up to a constant depending only on ε, by

$$(2.17) \qquad \left(T^2 + \frac{M^{1+\varepsilon}}{N} \right) \sum_{M < m \leqslant 2M} |a_m|^2.$$

Proof. — This result has been proved in [8, Proposition 4.7], except that the bound (2.17) is replaced by $\left(T^2 + c(\omega)^{1/2} M^{1+\varepsilon} N^{-1} \right) \|a_m\|_2^2$, where $c(\omega)$ is the conductor of ω. The extra factor $c(\omega)$ in the proof of this Proposition appears in [8, Lemma 4.6, (4.20)] and is a consequence of the general estimation [27, Theorem 9.3]

$$S_\omega(m, n; c) \ll \tau(c)^{O(1)} (m, n, c)^{1/2} (c(\omega)c)^{1/2}.$$

By the hypothesis on the conductor of ω (that it is squarefree), we can apply Lemma 2.4 whose consequence is the cancellation of the factor $c(\omega)^{1/2}$ in [8, Lemma 4.6, (4.20)] and the rest of the proof is completely similar. $\qquad \square$

2.5. L-functions and functional equations

2.5.1. Dirichlet L-functions

Let χ be a non-principal Dirichlet character of modulus $q > 2$ with q prime, $\kappa \in \{0, 1\}$ satisfying $\chi(-1) = (-1)^\kappa$ and define the complete L-function

$$\Lambda(\chi, s) := q^{s/2} L_\infty(\chi, s) L(\chi, s),$$

where

$$(2.18) \qquad L_\infty(\chi, s) := \pi^{-s/2} \Gamma\left(\frac{s + \kappa}{2} \right).$$

It is well known that $\Lambda(\chi, s)$ admits an analytic continuation to the whole complex plane and satisfies the functional equation (cf. [19, Theorem 4.15])

$$(2.19) \qquad \Lambda(\chi, s) = i^\kappa \varepsilon(\chi) \Lambda(\overline{\chi}, 1 - s),$$

where $\varepsilon(\chi)$ is the normalized Gauss sum defined by

$$(2.20) \qquad \varepsilon(\chi) := \frac{1}{q^{1/2}} \sum_{x \in \mathbf{F}_q^\times} \chi(x) e\left(\frac{x}{q} \right).$$

Using (2.19), we can express the central value of a Dirichlet L-function as a convergent series (see [19, Theorem 5.3]) and thus, extend in an easy way the proof to a product of three L-functions.

LEMMA 2.6. — *Let* χ, ω_1, ω_2 *be Dirichlet characters such that* $\chi \neq \omega_0, \overline{\omega}_1, \overline{\omega}_2$. *If* κ *(resp* κ_1, κ_2*)* $\in \{0,1\}$ *are such that* $\chi(-1) = (-1)^\kappa$ *(resp* $\omega_1(-1) = (-1)^{\kappa_1}, \omega_2(-1) = (-1)^{\kappa_2}$*), then we have*

$$\prod_{i=0}^{2} L\left(\chi\omega_i, \tfrac{1}{2}\right) = \sum\sum_{n_0,n_1,n_2 \geqslant 1}\sum \frac{\chi(n_0 n_1 n_2)\chi_1(n_1)\chi_2(n_2)}{(n_0 n_1 n_2)^{1/2}} \mathbf{V}_\chi\left(\frac{n_0 n_1 n_2}{q^{3/2}}\right)$$

$$+ \chi(-1)i^{\kappa + \kappa_1 + \kappa_2}\varepsilon(\chi)\varepsilon(\chi\omega_1)\varepsilon(\chi\omega_2)$$

$$\times \sum\sum_{n_0,n_1,n_2 \geqslant 1}\sum \frac{\overline{\chi}(n_0 n_1 n_2)\overline{\chi}_1(n_1)\overline{\chi}_2(n_2)}{(n_0 n_1 n_2)^{1/2}} \mathbf{V}_\chi\left(\frac{n_0 n_1 n_2}{q^{3/2}}\right),$$

where

$$(2.21) \qquad \mathbf{V}_\chi(x) := \frac{1}{2\pi i}\int_{(2)} \prod_{i=0}^{2} \frac{L_\infty\left(\chi\omega_i, \tfrac{1}{2}+s\right)}{L_\infty\left(\chi\omega_i, \tfrac{1}{2}\right)} x^{-s}Q(s)\frac{\mathrm{d}s}{s},$$

for some entire and even function $Q(s)$ *with exponential decay in vertical strips and satisfying* $Q(0) = 1$.

Remark 2.7. — *For given* ω_1, ω_2, *the function* $\mathbf{V}_\chi$ *depends on* χ *only through its parity. Shifting the* s-*contour to the right in* (2.21) *and we see that for* $x \geqslant 1$ *and any* $A \geqslant 0$, *we have the estimation*

$$\mathbf{V}_\chi(x) \ll_A x^{-A}.$$

Now moving the s-line to $\Re e(s) = -\frac{1}{2} + \varepsilon$, we pass a simple pole at $s = 0$ of residu 1 and thus, we obtain for $x \leqslant 1$

$$\mathbf{V}_\chi(x) = 1 + O_\varepsilon\left(x^{1/2-\varepsilon}\right).$$

2.5.2. Twisted *L*-functions

Let $q > 2$ be a prime number, ω a Dirichlet character of modulus q and f a primitive Hecke cusp of level q and nebentypus ω. For χ a non trivial character modulo q, we can construct the twisted modular form $f \otimes \chi$ whose n-th Fourier coefficient is given by $\rho_f(n)\chi(n)$. This form is a Hecke eigenform of level q^2 with nebentypus $\omega\chi^2$ and eigenvalues $\lambda_f(n)\chi(n)$ for $(n, q) = 1$ (see [17, Chapter 7]). The following proposition says when $f \otimes \chi$ stills a primitive form and derive the functional equation of the associated *L*-function (see for example [15] and [16]).

PROPOSITION 2.8. — *Let* $q > 2$ *be a prime number,* f, ω *and* χ *as above and assume further that* $\chi \neq \omega_0, \overline{\omega}$. *Then the twisted modular form* $f \otimes \chi$ *is*

a primitive Hecke cusp form of level q^2 and nebentypus $\chi^2\omega$ with associated Hecke eigenvalues $\chi(n)\lambda_f(n)$ for every $n \geqslant 1$. If

$$(2.22) \qquad L(f \otimes \chi, s) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s}, \ \Re e(s) > 1$$

is its associated L-function, then there exists a factor $L_\infty(f \otimes \chi, s)$ such that the product

$$\Lambda(f \otimes \chi, s) := q^s L_\infty(f \otimes \chi, s) L(f \otimes \chi, s)$$

extends holomorphically to $\mathbf{C}$ and satisfies the functional equation

$$(2.23) \qquad \Lambda(f \otimes \chi, s) = \varepsilon_\infty(f, \chi)\varepsilon(f \otimes \chi)\Lambda(\overline{f} \otimes \overline{\chi}, 1 - s),$$

where $\varepsilon(f \otimes \chi) = \varepsilon(\chi)\varepsilon(\omega\chi)$, $\varepsilon(\chi), \varepsilon(\omega\chi)$ are defined by (2.20) and the infinite factors $\varepsilon_\infty(f, \chi)$ and $L_\infty(f \otimes \chi, s)$ satisfy $|\varepsilon_\infty(f, \chi)| = 1$ and both depends on χ only trough its parity.

Remark 2.9. — The infinite factor presents as a product of Gamma functions

$$L_\infty(f \otimes \chi, s) = \pi^{-s}\Gamma\left(\frac{s - \mu_{1,f\otimes\chi}}{2}\right)\Gamma\left(\frac{s - \mu_{2,f\otimes\chi}}{2}\right),$$

where $\mu_{i,f\otimes\chi}$ are the local parameters at the infinite place (encodes the weight in the holomorphic setting or the Laplace eigenvalue if f is a Maass form) and we recall that they depend on χ at most trough its parity. In any case, a consequence of the work of Kim and Sarnak [26] toward the Ramanujan–Petersson conjecture is that

$$(2.24) \qquad \Re e(\mu_{i,f\otimes\chi}) \leqslant \frac{7}{64}.$$

We finally state the analogous of Lemma 2.6 for the product $L(f \otimes \chi, s)L(\chi, s)$ on the critical point $s = 1/2$ (see [35, Section 1.3.2] for the general result).

Proposition 2.10. — Let $q > 2$ be a prime number, ω a Dirichlet character of modulus q and f a primitive Hecke cusp form of level q and nebentypus ω with associated Hecke eigenvalues $\lambda_f(n)$ for all $n \geqslant 1$. Then for every character χ modulo q such that $\chi \neq \omega_0, \overline{\omega}$, $\chi(-1) = (-1)^\kappa$ with

$\kappa \in \{0,1\}$, we have

$$(2.25) \quad L\left(f \otimes \chi, \tfrac{1}{2}\right) L\left(\chi, \tfrac{1}{2}\right)$$

$$= \sum_{n,m \geqslant 1} \frac{\lambda_f(n)\chi(n)\chi(m)}{(nm)^{1/2}} \mathbf{V}_{f,\chi}\left(\frac{nm}{q^{3/2}}\right)$$

$$+ i^\kappa \varepsilon_\infty(f,\chi)\varepsilon(f \otimes \chi)\varepsilon(\chi) \sum_{n,m \geqslant 1} \frac{\overline{\lambda_f(n)}\,\overline{\chi}(n)\overline{\chi}(m)}{(nm)^{1/2}} \mathbf{V}_{f,\chi}\left(\frac{nm}{q^{3/2}}\right),$$

where

$$(2.26) \quad \mathbf{V}_{f,\chi}(x) := \frac{1}{2\pi i} \int_{(2)} \frac{L_\infty\left(f \otimes \chi, s + \tfrac{1}{2}\right) L_\infty\left(\chi, s + \tfrac{1}{2}\right)}{L_\infty\left(f \otimes \chi, \tfrac{1}{2}\right) L_\infty\left(\chi, \tfrac{1}{2}\right)} x^{-s} P(s) \frac{ds}{s},$$

for some suitable entire even function $P(s)$ with exponential decay in vertical strips and satisfying $P(0) = 1$.

Remark 2.11. — Shifting the s-contour on the right in (2.26) and we obtain that for every $x \geqslant 1$ and any $A > 0$,

$$\mathbf{V}_{f,\chi}(x) \ll_A x^{-A}.$$

By (2.24), moving the s-line to $\Re e(s) = -1/4$, we catch a simple pole at $s = 0$ of residue 1 and thus

$$\mathbf{V}_{f,\chi}(x) = 1 + O(x^{1/4}) \text{ for } 0 < x \leqslant 1.$$

3. ℓ-adic twists of modular forms

In this section, we fix $q > 2$ a prime number, ω a Dirichlet character modulo q, f a primitive Hecke cusp form of level q and nebentypus ω and we denote by $\{\lambda_f(n)\}_{n \geqslant 1}$ the Hecke eigenvalues of f. For any $t \in \mathbf{R}$, we also define the twisted divisor function $\lambda_\omega(n, it)$ by

$$(3.1) \qquad \lambda_\omega(n, it) := \sum_{ab=n} \omega(a) \left(\frac{a}{b}\right)^{it},$$

which, for $(n, q) = 1$, appears as Hecke eigenvalues of Eisenstein series $E_{\mathfrak{a}}(\cdot, 1/2 + it)$ of level q and nebentypus ω for a suitable choice of cusp $\mathfrak{a}$ (cf. Section 2.1.1 and (2.1)).

As announced in Section 1.1, for $K : \mathbf{Z} \longrightarrow \mathbf{C}$ a q-periodic function, a crucial step in the proof of Theorem 1.3 requires non trivial estimates for sums of the shape

$$(3.2) \qquad \mathcal{S}_V(f, K; q) = \sum_{n \geqslant 1} \lambda_f(n) K(n) V\left(\frac{n}{q}\right),$$

$$(3.3) \qquad \mathcal{S}_V(\omega, it, K; q) = \sum_{n \geqslant 1} \lambda_\omega(n, it) K(n) V\left(\frac{n}{q}\right),$$

where V is a smooth a compactly supported function on $\mathbf{R}_+^*$. Assuming that $|K(n)| \leqslant M$ for every $n \in \mathbf{Z}$, we obtain by Cauchy–Schwarz inequality and (2.7),

$$(3.4) \qquad \mathcal{S}_V(\omega, it, K; q), \mathcal{S}_V(f, K; q) \ll M q^{1+\varepsilon},$$

with an implied constant depending only on V and the spectral parameter t_f and this bound can be seen as the trivial one. Theorem 1.6 improves on (3.4) with a power saving in the q-aspect, namely

$$\mathcal{S}_V(\omega, it, K; q), \mathcal{S}_V(f, K; q) \ll q^{1 - \frac{1}{16} + \varepsilon},$$

for any $\varepsilon > 0$ and with an implied constant depending on ε, V, t_f (or t) and controlled by some invariant of K, called the conductor (see Definition (3.5)). As in [14, Definition 1.1], we make the following definition about the test function V.

DEFINITION 3.1 (Condition $V(C, P, Q)$). — *Let $P > 0$ and $Q \geqslant 1$ be real numbers and let $C = (C_\nu)_{\nu \geqslant 0}$ be a sequence of non-negative real numbers. A smooth compactly supported function V on $\mathbf{R}$ satisfies Condition $(V(C, P, Q))$ if*

(1) *The support of V is contained in the dyadic interval $[P, 2P]$;*
(2) *For all $x > 0$ and all integer $\nu \geqslant 0$, we have the inequality*

$$\left| x^\nu V^{(\nu)}(x) \right| \leqslant C_\nu Q^\nu.$$

In practice, the test function V is not compactly supported, but rather in the Schwartz class. We give here a simple Corollary of Theorem 1.6.

COROLLARY 3.2. — *Let $q > 2$ be a prime number, f, ω and K as in Theorem 1.6. Let $Q \geqslant 1$, $C = (C_\nu)_\nu$ a sequence of non-negative real numbers and V a smooth function on $\mathbf{R}$ with the property that for any $A > 0$,*

$$(3.5) \qquad V(x) \ll_A \frac{1}{(1 + |x|)^A} \quad \text{and} \quad \left| x^\nu V^{(\nu)}(x) \right| \leqslant C_\nu Q^\nu, \quad \nu \geqslant 0.$$

Then for every $X > 0$ and every $\varepsilon > 0$, we have

$$\sum_{n \geqslant 1} \lambda_f(n) K(n) V\left(\frac{n}{X}\right)$$

$$\ll_{C, \varepsilon} (qX)^\varepsilon (1 + |t_f|)^A \mathrm{cond}(K)^s X Q^2 \left(1 + \frac{q}{X}\right)^{1/2} q^{-1/16},$$

$$\sum_{n\geqslant 1} \lambda_\omega(n, it)K(n)V\left(\frac{n}{X}\right)$$

$$\ll_{C,\varepsilon} (qX)^\varepsilon(1+|t|)^A \mathrm{cond}(K)^s XQ^2\left(1+\frac{q}{X}\right)^{1/2} q^{-1/16},$$

Proof. — We consider the cuspidal case since the other is completely similar. Applying a partition of unity to $[1,\infty)$ leads to the decomposition

$$\sum_{n\geqslant 1} \lambda_f(n)K(n)V\left(\frac{n}{X}\right) = \sum_N\sum_{n\geqslant 1} \lambda_f(n)K(n)V\left(\frac{n}{X}\right)W\left(\frac{n}{N}\right),$$

where W is a smooth and compactly supported function on $(1/2, 2)$ satisfying $|x^j W^{(j)}(x)| \leqslant \widetilde{c}_j$ for some sequence $\widetilde{c} = (\widetilde{c}_j)$ of non-negative real numbers and N runs over real numbers of the form 2^i with $i \geqslant 0$. Since V has fast decay at infinity, we can focus on the contribution of $1 \leqslant N \leqslant q^\varepsilon X$ at the cost of an error of size $O(q^{-10})$, (say). Hence, we just need to bound $O(\log(qX))$ sums of the form

$$\sum_{n\geqslant 1} \lambda_f(n)K(n)V\left(\frac{n}{X}\right)W\left(\frac{n}{N}\right).$$

By Mellin inversion formula, we have for any $\varepsilon > 0$

$$\sum_{n\geqslant 1} \lambda_f(n)K(n)V\left(\frac{n}{X}\right)W\left(\frac{n}{N}\right)$$

$$= \frac{1}{2\pi i}\int_{(\varepsilon)}\left(\frac{X}{N}\right)^s \widetilde{V}(s)\left(\sum_{n\geqslant 1}\lambda_f(n)K(n)W_s\left(\frac{n}{N}\right)\right)\mathrm{d}s,$$

where the function $W_s(x) := x^{-s}W(x)$ satisfies

(3.6) $$x^j W_s^{(j)}(x) \ll_{\widetilde{c},j} (1+|s|)^j.$$

For fixed s with $\Re e(s) = \varepsilon$, we apply Theorem 1.6 to the inner sum with the function $V(x) = W_s(xq/N)$ which satisfies condition $V(\widetilde{C}, N/q, 1+|s|)$ for some $\widetilde{C}$ depending on $\widetilde{c}$, obtaining the bound

$$(1+|t_f|)^A\mathrm{cond}(K)^s q^{\frac{1}{2}-\frac{1}{16}+\varepsilon}\left(\frac{X}{N}\right)^\varepsilon$$

$$\times \int_{(\varepsilon)}|\widetilde{V}(s)|(N(1+|s|))^{1/2}\left(\frac{N}{q}+1+|s|\right)^{1/2}\mathrm{d}s.$$

Using the fact that the Mellin transform $\widetilde{V}(s)$ satisfies

$$\widetilde{V}(s) \ll \left(\frac{Q}{1+|s|}\right)^B,$$

for every $B > 0$ with an implied constant depending on B and $\Re e(s)$, we see that we can restrict the integral to $|s| \leqslant q^\varepsilon Q$. Hence replacing $1 + |s|$ by its maximal value, maximizing over $N \leqslant q^\varepsilon X$ and average trivially over $|s| \leqslant q^\varepsilon Q$ in the integral yields the desire result. $\qquad\square$

3.1. Trace functions of ℓ-adic sheaves

The functions to which we will apply Theorem 1.6 are called *trace functions* modulo q, which we now define formally.

Let q be a prime number and $\ell \neq q$ be an auxiliary prime. To any constructible $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ on $\mathbf{A}^1_{\mathbf{F}_q}$ and any point $x \in \mathbf{A}^1_{\mathbf{F}_q}$, we have a linear action of a geometric Frobenius F_x acting on a finite dimensional $\overline{\mathbf{Q}}_\ell$-vector space $\mathcal{F}_{\overline{x}}$. We can thus consider the trace $\mathrm{Tr}(\mathrm{F}_x | \mathcal{F}_{\overline{x}})$. Because this trace takes value in $\overline{\mathbf{Q}}_\ell$, we also fix a field isomorphism

$$\iota : \overline{\mathbf{Q}}_\ell \xrightarrow{\simeq} \mathbf{C},$$

and we consider functions of the shape

$$(3.7) \qquad K(x) = \iota\left(\mathrm{Tr}(\mathrm{F}_x | \mathcal{F}_{\overline{x}})\right),$$

as defined in [24, (7.3.7)].

DEFINITION 3.3 (Trace sheaves).

(1) *A constructible $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ on $\mathbf{A}^1_{\mathbf{F}_q}$ is called a trace sheaf if it is a middle extension sheaf (in the sens of [12, Section 1]) whose restriction on any non empty open subset $U \subset \mathbf{A}^1_{\mathbf{F}_q}$ where $\mathcal{F}$ is lisse is pointwise pure of weight zero.*

(2) *A trace sheaf is called a Fouriertrace sheaf if in addition, it is a Fourier sheaf in the sense of [23, Definition 8.2.2].*

(3) *We say that $\mathcal{F}$ is an isotypic trace sheaf if it is a Fourier trace sheaf and if for every non empty open subset U as in (1), the associated ℓ-adic representation*

$$\pi_1(U \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q, \overline{\eta}) \longrightarrow \mathrm{GL}(\mathcal{F}_{\overline{\eta}}),$$

of the geometric etale fondamental group is an isotypic representation [23, Chapter 2]. We define similarly an irreducible trace sheaf.

DEFINITION 3.4. — *Let q be a prime number. A function $K : \mathbf{F}_q \longrightarrow \mathbf{C}$ is called a trace function (resp. Fourier trace function, isotypic trace function) if there exists a trace sheaf (resp Fourier trace sheaf, isotypic trace sheaf) $\mathcal{F}$ such that K is given by (3.7).*

There is an important invariant which measure the complexity of a trace function which we define now.

DEFINITION 3.5 (Conductor). — *Let $\mathcal{F}$ be a constructible $\overline{\mathbf{Q}}_\ell$-sheaf on $\mathbf{A}^1_{\mathbf{F}_q}$ with $n(\mathcal{F})$ singularities in $\mathbf{P}^1_{\mathbf{F}_q}$. The conductor of $\mathcal{F}$ is the integer defined by*

$$\operatorname{cond}(\mathcal{F}) := \operatorname{rank}(\mathcal{F}) + n(\mathcal{F}) + \sum_{x \in \mathbf{P}^1_{\mathbf{F}_q}} \operatorname{Swan}_x(\mathcal{F}),$$

where $\operatorname{Swan}_x(\mathcal{F}) = 0$ if $\mathcal{F}$ is lisse at x (see for example [22, Section 4.6]). If K is a trace function, the conductor $\operatorname{cond}(K)$ of K is the smallest conductor of a trace sheaf $\mathcal{F}$ with trace function K.

Let $\mathcal{F}$ be a trace sheaf with associated trace function $K : \mathbf{F}_q \to \mathbf{C}$. The normalized Fourier transform of K, denoted by $\widehat{K}$ or $\operatorname{FT}(K)$, is the function on $\mathbf{F}_q$ defined by

$$\widehat{K}(y) := \frac{1}{q^{1/2}} \sum_{x \in \mathbf{F}_q} K(x) e\left(\frac{xy}{q}\right).$$

When $\mathcal{F}$ is a Fourier sheaf, there is a deep interpretation of the Fourier transform at the level of sheaves who was discovered by Deligne and developed by Laumon, specially in [31]. To be precise, there exists a Fourier sheaf $\mathcal{G}$ whose conductor satisfies

$$(3.8) \qquad\qquad \operatorname{cond}(\mathcal{G}) \leqslant 10 \operatorname{cond}(\mathcal{F})^2,$$

and with the property that

$$\iota\left(\operatorname{Tr}\left(\mathbf{F}_x | \mathcal{G}_{\overline{x}}\right)\right) = -\widehat{K}(x).$$

Moreover, the sheaf $\mathcal{G}$ is geometrically isotypic (resp. geometrically irreducible) if and only if $\mathcal{F}$ has this property [14, Lemma 8.1].

3.1.1. Kloosterman sheaves

Let $k \geqslant 2$ be an integer, $\omega_1, \ldots, \omega_k$ be multiplicative characters on $\mathbf{F}_q^\times$. The twisted rank k Kloosterman sum $\operatorname{Kl}_k(\omega_1, \ldots, \omega_k; q)$ is the function on $\mathbf{F}_q^\times$ defined by

$$(3.9) \quad \operatorname{Kl}_k(a, \omega_1, \ldots, \omega_k; q)$$

$$:= \frac{1}{q^{\frac{k-1}{2}}} \sum_{\substack{x_1, \ldots, x_k \in \mathbf{F}_q^\times \\ x_1 \cdots x_k = a}} \omega_1(x_1) \cdots \omega_k(x_k) e\left(\frac{x_1 + \cdots + x_k}{q}\right),$$

for every $a \in \mathbf{F}_q^{\times}$. If $\omega_1 = \cdots = \omega_k = 1$, we write $\mathrm{Kl}_k(a; q)$ instead of $\mathrm{Kl}_k(a, 1, \ldots, 1 : q)$. The main result is the existence of Kloosterman sheaves and it is due to Deligne [23, Theorem 4.1.1].

THEOREM 3.6 (Kloosterman sheaves). — For every prime $\ell \neq q$, there exists a constructible $\overline{\mathbf{Q}}_{\ell}$-sheaf on $\mathbf{A}^1_{\mathbf{F}_q}$, denoted by $\mathcal{K}\ell_k(\omega_1, \ldots, \omega_k; q)$ (or simply $\mathcal{K}\ell$), of rank k satisfying the following properties :

(1) For every $a \in \mathbf{F}_q^{\times}$, we have the equality
$$\iota \left(\mathrm{Tr} \left(\mathrm{F}_a | (\mathcal{K}\ell)_{\overline{a}} \right) \right) = (-1)^{k-1} \mathrm{Kl}_k(a, \omega_1, \ldots, \omega_k; q);$$

(2) $\mathcal{K}\ell$ is geometrically irreducible, lisse on $\mathbf{G}_{m,\mathbf{F}_q}$ and pointwise pure of weight zero;

(3) $\mathcal{K}\ell$ has wild ramification at ∞ with $\mathrm{Swan}_{\infty}(\mathcal{K}\ell) = 1$, tamely ramified at 0 and has conductor $k + 3$.

In other words, since the rank is $\geqslant 2$, $\mathcal{K}\ell$ is an irreducible trace sheaf in the sense of Definition 3.3 (3).

COROLLARY 3.7. — For every $a \in \mathbf{F}_q^{\times}$, we have the sharp bound

(3.10) $|\mathrm{Kl}_k(a, \omega_1, \ldots, \omega_k; q)| \leqslant k.$

It will be convenient in Section 6, specially because of the Poisson summation and Fourier inversion formula, to present Kl as a Fourier transform of a function defined in $\mathbf{F}_q$. For this, we let

$$\mathrm{Kl}_1(a, \omega_1; q) := \omega_1(a) e\left(\frac{a}{q} \right),$$

and we see that for any $k \geqslant 2$ and $a \in \mathbf{F}_q^{\times}$, $\mathrm{Kl}_k(a, \omega_1, \ldots, \omega_k; q)$ is given by the formula

(3.11) $\mathrm{Kl}_k(a, \omega_1, \ldots, \omega_k; q)$
$$= \omega_k(a) \mathrm{FT}\left(\mathbf{F}_q \ni x \mapsto \omega_k(x) \mathbf{K}_{k-1}(x, \omega_1, \ldots, \omega_{k-1}; q) \right)(a),$$

where the function $\mathbf{K}_{k-1}$ is defined by

(3.12) $\mathbf{K}_{k-1}(x, \omega_1, \ldots, \omega_{k-1}; q) := \begin{cases} \mathrm{Kl}_{k-1}(\overline{x}, \omega_1, \ldots, \omega_{k-1}; q) & \text{if } x \in \mathbf{F}_q^{\times}, \\ 0 & \text{if } x = 0. \end{cases}$

Remark 3.8. — There are several ways to extend the function Kl_k to $a = 0$. One can choose for example the extension by zero. We choose here the middle extension, i.e. that $\mathrm{Kl}_k(0, \omega_1, \ldots, \omega_k; q)$ coincides with the trace of the Frobenius at $x = 0$. It is a deep result of Deligne that the estimate (3.10) remains valid for $a = 0$ (see [6, (1.8.9)]).

3.2. Twisted Correlation Sums and the ω-Möbius Group

The strategy in the proof of Theorem 1.6 is to estimate an amplified second moment of $\mathcal{S}_V(g, K; q)$ for g varying in a basis of cusp forms of level q and nebentypus ω. After completing the spectral sum, applying Kuznetsov–Petersson and Poisson formula, we have to confront some sums that we call *twisted correlation sums*, which we now define.

We let $\mathrm{PGL}_2(\mathbf{F}_q)$ acts on the projective line $\mathbf{P}^1(\mathbf{F}_q)$ by fractional linear transformations

$$\gamma z = \frac{az + b}{cz + d}, \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{F}_q).$$

DEFINITION 3.9 (Twisted correlation sum). — *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbf{F}_q)$. *For* ω *a multiplicative character modulo* q *and* $K : \mathbf{F}_q \to \mathbf{C}$, *we define the twisted correlation sum* $\mathcal{C}(K, \omega; \gamma)$ *by*

$$\mathcal{C}(K, \omega; \gamma) := \sum_{\substack{z \in \mathbf{F}_q \\ z \neq -d/c}} \overline{\omega}(cz + d) K(\gamma z) \overline{K(z)}.$$

Remark 3.10. — Note that for $\gamma \in \mathrm{PGL}_2(\mathbf{F}_q)$, $\mathcal{C}(K, \omega; \gamma)$ is well defined up to multiplication by $\omega(-1) \in \{-1, +1\}$. This is in fact not a problem since only the complex modulus $|\mathcal{C}(K, \omega; \gamma)|$ will be considered later. We also mention that unlike the definition of correlation sum that the authors made for the original Theorem (see [14, (1.10)]), we have the presence here of a twist by the nebentypus of the modular form f. This is because the Kloosterman sums that we obtain after the application of Kuznetsov trace formula are also twsited by ω.

Note that for K a trace function, we have the bound $\|K\|_\infty \leqslant \mathrm{cond}(K)$. Hence using Cauchy–Schwarz and Parseval identity, we get

$$(3.13) \qquad\qquad |\mathcal{C}(K, \omega; \gamma)| \leqslant \mathrm{cond}(K)^2 q.$$

In order to obtain better bounds, we introduce a geometric object associated to the correlation sum $\mathcal{C}(K, \omega; \gamma)$.

DEFINITION 3.11. — *Let* q *be prime number and* $\mathcal{F}$ *an isotypic trace sheaf on* $\mathbf{A}^1_{\mathbf{F}_q}$. *Let* ω *be a multiplicative character modulo* q *and* $\mathcal{L}_\omega$ *the associated Kummer sheaf. The* ω-*Möbius group* $\mathbf{G}_{\mathcal{F},\omega}$ *is the subgroup of* $\mathrm{PGL}_2(\mathbf{F}_q)$ *defined by*

$$\mathbf{G}_{\mathcal{F},\omega} := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{F}_q) \ \middle| \ \mathcal{F} \simeq_{\mathrm{geom}} \gamma^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(cX + d)} \right\}.$$

Remark 3.12. — Note that Definition 3.11 makes sense in the sense that if $\gamma, \gamma' \in \mathrm{GL}_2(\mathbf{F}_q)$ are equal in $\mathrm{PGL}_2(\mathbf{F}_q)$, then $\gamma = \pm \mathrm{I}_2 \gamma'$ and thus

$$\gamma^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(cX+d)} = \gamma'^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(-c'X-d')}$$
$$\simeq_{\mathrm{geom}} \gamma'^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(c'X+d')} \otimes \mathcal{L}_{\overline{\omega}(-1)}$$
$$\simeq_{\mathrm{geom}} \gamma'^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(c'X+d')}.$$

The key property here is that $\mathbf{G}_{\mathcal{F},\omega}$ is indeed a subgroup of $\mathrm{PGL}_2(\mathbf{F}_q)$.

PROPOSITION 3.13. — $\mathbf{G}_{\mathcal{F},\omega}$ is a subgroup of $\mathrm{PGL}_2(\mathbf{F}_q)$.

Proof. — Let $\mathscr{F}$ be the set of geometric isomorphism classes of trace sheaves. To show that $\mathbf{G}_{\mathcal{F},\omega}$ is a subgroup, it is enough to prove that the map $\mathscr{F} \times \mathrm{PGL}_2(\mathbf{F}_q) \longrightarrow \mathscr{F}$ given by

$$(3.14) \qquad (\mathcal{F}, \gamma) \longmapsto \mathcal{F} \cdot \gamma := \gamma^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(cX+d)},$$

defines a right group action because $\mathbf{G}_{\mathcal{F},\omega}$ will be the stabilizer of $\mathcal{F}$. For this, we will use the fact that we have geometric isomorphisms (we use the notation $\simeq$ instead of $\simeq_{\mathrm{geom}}$)

$$(3.15) \qquad \mathcal{L}_{\overline{\omega}(d)} \simeq \overline{\mathbf{Q}}_\ell \simeq \mathcal{L}_{\overline{\omega}(cX+d)} \otimes \mathcal{L}_{\omega(cX+d)},$$

where $\overline{\mathbf{Q}}_\ell$ denotes the constant sheaf. The first isomorphism implies that the identity matrix acts trivially. For the second part, note that for $\gamma_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ and $\gamma_2 = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix} \right) \in \mathrm{PGL}_2(\mathbf{F}_q)$, we have

$$\mathcal{L}_{\overline{\omega}(cX+d)} \simeq j(\gamma)^* \mathcal{L}_\chi, \quad j(\gamma) := \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix},$$

and

$$j(\gamma_1 \gamma_2) = \begin{pmatrix} 0 & 1 \\ ca' + dc' & cb' + dd' \end{pmatrix}, \quad j(\gamma_1)\gamma_2 = \begin{pmatrix} c' & d' \\ ca' + dc' & cb' + dd' \end{pmatrix}.$$

Combining the above equality with the second isomorphism in (3.15) leads to

$$j(\gamma_1 \gamma_2)^* \mathcal{L}_\omega \simeq (j(\gamma_1)\gamma_2)^* \mathcal{L}_\omega \otimes j(\gamma_2)^* \mathcal{L}_\omega.$$

Hence we obtain

$$\mathcal{F} \cdot (\gamma_1 \gamma_2) \simeq (\gamma_1 \gamma_2)^* \mathcal{F} \otimes j(\gamma_1 \gamma_2)^* \mathcal{L}_\omega \simeq \gamma_2^* \gamma_1^* \mathcal{F} \otimes \gamma_2^* j(\gamma_1)^* \mathcal{L}_\omega \otimes j(\gamma_2)^* \mathcal{L}_\omega$$
$$\simeq \gamma_2^* \left(\gamma_1^* \mathcal{F} \otimes j(\gamma_1)^* \mathcal{L}_\omega \right) \otimes j(\gamma_2)^* \mathcal{L}_\omega$$
$$\simeq (\mathcal{F} \cdot \gamma_1) \cdot \gamma_2,$$

which completes the proof of this Proposition. $\qquad \square$

We will also need the following fact about the conductor of $\mathcal{F} \cdot \gamma$.

LEMMA 3.14. — *Let $\mathcal{F}$ be a trace sheaf and $\gamma \in \mathrm{PGL}_2(\mathbf{F}_q)$. Then*

$$|\mathrm{cond}(\mathcal{F} \cdot \gamma) - \mathrm{cond}(\mathcal{F})| \leqslant 2.$$

Proof. — Since the Kummer sheaves are of rank one and tamely ramified at the singularities, we have for any $x \in \mathbf{P}^1_{\mathbf{F}_q}$ (by definition (3.14) of the action of γ on $\mathcal{F}$),

$$\mathrm{rank}(\gamma^*\mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(cX+d)}) = \mathrm{rank}(\gamma^*\mathcal{F}) = \mathrm{rank}(\mathcal{F}), \ \ \mathrm{Swan}_x(\mathcal{F}\cdot\gamma) = \mathrm{Swan}_x(\mathcal{F}),$$

see [22, 4.6 (iv)]. Moreover, if $n(\mathcal{F})$ (resp $n(\mathcal{L}_{\overline{\omega}(cX+d)})$) denotes the number of singularities of $\mathcal{F}$ (resp. of $\mathcal{L}_{\overline{\omega}(cX+d)}$), the tensor product $\gamma^*\mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(cX+d)}$ satisfies (see for example [33, Proposition 1.23])

$$\begin{aligned}
n(\mathcal{F}) - n(\mathcal{L}_{\overline{\omega}(cX+d)}) &= n(\gamma^*\mathcal{F}) - n(\mathcal{L}_{\overline{\omega}(cX+d)}) \\
&\leqslant n(\gamma^*\mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(cX+d)}) \\
&\leqslant n(\gamma^*\mathcal{F}) + n(\mathcal{L}_{\overline{\omega}(cX+d)}) = n(\mathcal{F}) + n(\mathcal{L}_{\overline{\omega}(cX+d)}),
\end{aligned}$$

which completes the proof since $n(\mathcal{L}_{\overline{\omega}(cX+d)}) = 0$ or 2 depending on whether $c = 0$ or not. $\square$

The following proposition establishes the link between the correlation sum $\mathcal{C}(K, \omega; \gamma)$ and the ω-Möbius group $\mathbf{G}_{\mathcal{F},\omega}$. The proof uses deep results on ℓ-adic cohomology for varieties over finite fields.

PROPOSITION 3.15. — *Let $q > 2$ be a prime number, $\mathcal{F}$ an isotypic trace sheaf with associated trace function K modulo q of conductor $\mathrm{cond}(K)$. Let $\omega : \mathbf{F}_q^\times \to \mathbf{C}^\times$ be a multiplicative character. Then there exists absolute constants $A, s \geqslant 1$ such that for any $\gamma \in \mathrm{PGL}_2(\mathbf{F}_q) \setminus \mathbf{G}_{\mathcal{F},\omega}$,*

$$|\mathcal{C}(K, \omega; \gamma)| \leqslant A(\mathrm{cond}(K))^s q^{1/2}.$$

Proof. — See [14, Theorem 9.1]. $\square$

The last part of this section is devoted to the structure of the group $\mathbf{G}_{\mathcal{F},\omega} \subset \mathrm{PGL}_2(\mathbf{F}_q)$. For some technical reasons due to the amplification method and the fact that we are dealing with forms of level q, we want to avoid the presence of unipotent elements in our group $\mathbf{G}_{\mathcal{F},\omega}$ because in contrary to [14, Theorem 1.2], parabolic matrices could appear in our case and their contribution seems to be big. We therefore impose an additional hypothesis on our sheaf $\mathcal{F}$ and prove that under this extra assumption, the group $\mathbf{G}_{\mathcal{F},\omega}$ does not contain any unipotent elements. Before doing this, we introduce the following notation :

- For $x \neq y$ in $\mathbf{P}^1$, the pointwise stabilizer of x and y is denoted by $\mathrm{T}^{x,y}$ (this is a maximal torus) and its stabilizer in PGL_2 (or the stabilizer of the set $\{x, y\}$) is denoted by $\mathrm{N}^{x,y}$.

DEFINITION 3.16. — Let $\mathcal{F}$ be an isotypic trace sheaf. We say that $\mathcal{F}$ is ω-exceptional if its irreducible component is of the form $\mathcal{L}_\psi \cdot \gamma = \gamma^* \mathcal{L}_\psi \otimes \mathcal{L}_{\omega(cX+d)}$ for some Artin–Schreier sheaf $\mathcal{L}_\psi$ and some $\gamma \in \mathrm{PGL}_2(\mathbf{F}_q)$.

PROPOSITION 3.17. — Let $q > 2$ be a prime number, $\mathcal{F}$ an isotypic trace sheaf on $\mathbf{A}^1_{\mathbf{F}_q}$ and $\omega : \mathbf{F}_q^\times \to \mathbf{C}^\times$. Assume that $\mathcal{F}$ is not ω-exceptional and that q is large enough compared to the conductor $\mathrm{cond}(K)$. Then $\mathbf{G}_{\mathcal{F},\omega}$ one of the following type :

(1) $|\mathbf{G}_{\mathcal{F},\omega}| \leqslant 60$ and the non trivial elements belong to at most 59 different tori.

(2) $\mathbf{G}_{\mathcal{F},\omega}$ is cyclic and is contained in the normalizer $\mathrm{N}^{x,y}$ of a certain maximal torus $\mathrm{T}^{x,y}$ for $x \neq y$ in $\mathbf{P}^1$.

(3) $\mathbf{G}_{\mathcal{F},\omega}$ is dihedral of order $2r$. Its cyclic subgroup of order r is contained in a maximal torus $\mathrm{T}^{x,y}$ and any element not contained in it is in the normalizer $\mathrm{N}^{x,y}$ $(x \neq y)$.

In particular, $\mathbf{G}_{\mathcal{F},\omega}$ does not contain parabolic elements.

Proof. — If the order of $\mathbf{G}_{\mathcal{F},\omega}$ is coprime with q, the first paragraph in the proof of [14, Theorem 1.14] says that $\mathbf{G}_{\mathcal{F},\omega}$ is one of the three types of groups cited above.

We now show that the order of $\mathbf{G}_{\mathcal{F},\omega}$ cannot be divisible by q. Assume by contradiction that it is the case and fix an element $\gamma_0 \in \mathbf{G}_{\mathcal{F},\omega}$ of order q. Then γ_0 is necessarily parabolic, so it has a unique fixed point $x \in \mathbf{P}^1(\mathbf{F}_q)$. Let $\sigma \in \mathrm{PGL}_2(\mathbf{F}_q)$ be such that

$$\sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sigma^{-1} = \gamma_0,$$

and define

$$\mathcal{G} := \mathcal{F} \cdot \sigma = \sigma^* \mathcal{F} \otimes \mathcal{L}_{\overline{\omega}(cX+d)}, \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since geometrically we have $[+1]^* \mathcal{F} \simeq \mathcal{F} \cdot \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$, we see that we have a geometric isomorphism

$$[+1]^* \mathcal{G} \simeq \mathcal{G}.$$

Suppose first that $\mathcal{G}$ is ramified at some $y \in \mathbf{A}^1(\overline{\mathbf{F}}_q)$, then by the above, $\mathcal{G}$ is also ramified at $y+1, \ldots, y+p-1$ and we obtain by Lemma 3.14

$$\mathrm{cond}(\mathcal{F}) \geqslant \mathrm{cond}(\mathcal{G}) - 2 \geqslant q - 2 + \mathrm{rank}(\mathcal{G}) \geqslant q - 1,$$

which is a contradiction with the fact that $\mathrm{cond}(\mathcal{F}) < q - 1$. Assume now that $\mathcal{G}$ is lisse on $\mathbf{A}^1(\overline{\mathbf{F}}_q)$. Since $\mathcal{F}$ is geometrically isotypic, the same is true for $\mathcal{G}$ and the geometrically irreducible component $\mathcal{G}_1$ of $\mathcal{G}$ also satisfies

$[+1]^* \mathcal{G}_1 \simeq \mathcal{G}_1$. Using [12, Lemma 5.4, (2)] with $G = \mathbf{F}_q$ and $P_h = 0$, we have either

$$\mathrm{cond}(\mathcal{G}) \geqslant \mathrm{Swan}_\infty(\mathcal{G}_1) \geqslant q + \mathrm{rank}(\mathcal{G}_1)$$

and in this case we are done as before, or $\mathcal{G}_1$ is geometrically isomorphic to some Artin–Schreier sheaf $\mathcal{L}_\psi$ for some additive character ψ. It follows that $\mathcal{G}$ is geometrically isomorphic to a direct sum of copies of $\mathcal{L}_\psi$ and thus, by definition of $\mathcal{G}$, we have a geometric isomorphism

$$\mathcal{F} \simeq \left(\bigoplus \mathcal{L}_\psi \right) \cdot \sigma^{-1} = \bigoplus \mathcal{L}_\psi \cdot \sigma^{-1},$$

which contradicts the fact that $\mathcal{F}$ is not ω-exceptional. $\qquad\square$

4. Bilinear forms involving trace functions

We begin with a classical result, which is a simple consequence of the Poisson summation formula.

PROPOSITION 4.1 (Polyá–Vinogradov method). — *Let q be a prime number and $\mathcal{F}$ be a Fourier trace sheaf on $\mathbf{A}^1_{\mathbf{F}_q}$ with corresponding trace function K modulo q. Let f be a smooth and compactly supported function on $\mathbf{R}$ and $N > 0$ be a real number. Then for any $\varepsilon > 0$, we have*

$$\sum_{n \in \mathbf{Z}} K(n) f\left(\frac{n}{N} \right) \ll \min \left\{ N, \frac{N}{q^{1/2}} \left(1 + \frac{q^{1+\varepsilon}}{N} \right) \right\},$$

where the implied constant depends on ε, f and the conductor of $\mathcal{F}$.

A more elaborate treatment of the Polyá–Vinogradov method can be used to obtain bounds for bilinear sums [13, Theorem 1.17].

THEOREM 4.2. — *Let K be an isotypic trace function modulo q associated to an isotypic ℓ-adic sheaf $\mathcal{F}$ such that $\mathcal{F}$ does not contain a sheaf of the form $\mathcal{L}_\omega \otimes \mathcal{L}_\psi$ in his irreducible component. Let $M, N \geqslant 1$ be parameters and $(\alpha_m)_m, (\beta_n)_n$ two sequences of complex numbers supported on $[M/2, 2M]$ and $[N/2, 2N]$ respectively.*

(1) *We have*

$$\sum_{\substack{n,m \\ (m,q)=1}} \alpha_m \beta_n K(mn)$$

$$\ll \|\alpha\|_2 \|\beta\|_2 (NM)^{1/2} \left(\frac{1}{q^{1/4}} + \frac{1}{M^{1/2}} + \frac{q^{1/4} \log^{1/2} q}{N^{1/2}} \right),$$

with

$$\|\alpha\|_2^2 = \sum_m |\alpha_m|^2 , \ \|\beta\|_2^2 = \sum_n |\beta_n|^2.$$

(2) We have

$$\sum_{(m,q)=1} \alpha_m \sum_{n\leqslant N} K(mn) \ll \left(\sum_m |\alpha_m|\right) N \left(\frac{1}{q^{1/2}} + \frac{q^{1/2}\log q}{N}\right).$$

In both estimates, the implicit constants depend only, and at most polynomially, on the conductor of $\mathcal{F}$.

The above theorem beats the trivial bound and gives a power saving in the error term as long as $\max(N, M) \geqslant q^{1/2+\delta}$ and $\min(M, N) \geqslant q^{\delta}$ for some $\delta > 0$. In the critical case where $N \sim M \sim q^{1/2}$, we have the powerful result of Kowalski, Michel and Sawin, which still saves a small power of q, but has been proved in the special case of classical hyper-Kloosterman sums [29, Theorem 1.3] and [1, Theorem 5.1].

THEOREM 4.3. — Let q be a prime number and a an integer coprime with q. Let $M, N \geqslant 1$ be such that

$$(4.1) \qquad 1 \leqslant M \leqslant N^2, \quad N < q, \quad MN < q^{3/2}.$$

Let $(\alpha_m)_{m\leqslant M}$ be a sequence of complex numbers and $\mathcal{N} \subset [1, q-1]$ be an interval of length N. Then for any $\varepsilon > 0$, we have

$$(4.2) \quad \sum_{n\in\mathcal{N}} \sum_{1\leqslant m\leqslant M} \alpha_m \mathrm{Kl}_k(anm; q)$$

$$\ll q^\varepsilon \|\alpha\|_1^{1/2} \|\alpha\|_2^{1/2} M^{1/4} N \left(\frac{M^2 N^5}{q^3}\right)^{-1/12},$$

with

$$\|\alpha\|_1 = \sum_{1\leqslant m\leqslant M} |\alpha_m|$$

where the implied constant in (4.2) only depends on ε and k.

5. Proof of Theorem 1.6

This section is devoted to the proof of Theorem 1.6. For the cuspidal case, we will indicate the necessary changes in [14, Sections 4, 5 and 6] due to the level q and the presence of a nebentypus. Finally, we will explain how to adapt [13, Section 2] and put together with Section 5.1 to obtain the conclusion of Theorem 1.6 in the Eisenstein case.

5.1. The cuspidal case

5.1.1. The amplification method

Let $q > 2$ be a prime number, ω a Dirichlet character of modulus q and $\kappa \in \{0, 1\}$ such that $\omega(-1) = (-1)^\kappa$. Let f be a L^2-normalized primitive Hecke cusp form of weight $k_f \equiv \kappa \pmod 2$ (resp. with Laplace eigenvalue $1/4 + t_f^2$) if f is holomorphic (resp. if f is a Maass form) of level q and nebentypus ω. For some technical reasons, it is convenient to view f as a modular form of level $2q$ (see the beginning of Section 2.1.1) under the isometric embedding (with respect to the Petersson inner product)

$$f(z) \mapsto \frac{f(z)}{[\Gamma_0(q) : \Gamma_0(2q)]^{1/2}} = \frac{f(z)}{\sqrt{3}},$$

which can be embedded in a suitable orthonormal basis of modular cusp forms of level $2q$, i.e. either $\mathcal{B}_{k_f}(2q, \omega)$ or $\mathcal{B}(2q, \omega)$. The strategy is therefore to estimate an amplified second moment of the sum $\mathcal{S}_V(g, K; q)$ where g runs over a basis of $\mathcal{B}_{k_f}(2q, \omega)$ and $\mathcal{B}(2q, \omega)$.

To be precise, let $L \geqslant 1$ and (b_ℓ) a sequence of coefficients supported on $1 \leqslant \ell \leqslant 2L$. For any modular form g, we let

$$B(g) := \sum_{1 \leqslant \ell \leqslant 2L} b_\ell \lambda_g(\ell).$$

For an Eisenstein series $E_\mathfrak{a}(\,\cdot\,, 1/2 + it)$, we set

$$B(\mathfrak{a}, it) := \sum_{1 \leqslant \ell \leqslant 2L} b_\ell \lambda_\mathfrak{a}(\ell, it),$$

where for any singular cusp $\mathfrak{a}$, $\lambda_\mathfrak{a}(\ell, it)$ is given by (2.1). Since the original form is of level q and L will be at the end a small power of q, we cannot choose the standard coefficients $b_\ell = \overline{\lambda_f(\ell)}$ for ℓ a prime $p \sim L$, but rather the less obvious amplifier found by Iwaniec in [10],

(5.1) $$b_\ell = \begin{cases} \lambda_f(p)\overline{\omega}(p) & \text{if } \ell = p \sim L^{1/2} \text{ and } (p, 2q) = 1, \\ -\overline{\omega}(p) & \text{if } \ell = p^2 \sim L \text{ and } (p, 2q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since we will apply the trace formula, it is also better to consider the Fourier coefficients $\rho_g(n)$ instead of the Hecke eigenvalues $\lambda_g(n)$ in the definition of $\mathcal{S}_V(g, k; q)$. For this, we define

$$\widetilde{\mathcal{S}}_V(g, K; q) = \sum_n \rho_g(n) K(n) V\left(\frac{n}{q}\right)$$

and note that for g primitive, it is related to the original sum $\mathcal{S}_V(g, K; q)$ by the simple relation (cf. (2.10)),

$$(5.2) \qquad \widetilde{\mathcal{S}}_V(g, K; q) = \rho_g(1)\mathcal{S}_V(g, K; q).$$

We then let

$$(5.3) \quad M(L) := \sum_{\substack{k \equiv \kappa \pmod 2 \\ k > \kappa}} \dot{\phi}(k)(k-1)M(L; k)$$

$$+ \sum_{g \in \mathcal{B}(2q, \omega)} \widetilde{\phi}(t_g)\frac{4\pi}{\cosh(\pi t_g)}|B(g)|^2 \left|\widetilde{\mathcal{S}}_V(g, K; q)\right|^2$$

$$+ \sum_{\mathfrak{a}} \int_{-\infty}^{\infty} \widetilde{\phi}(t)\frac{1}{\cosh(\pi t)}|B(\mathfrak{a}, it)|^2 \left|\widetilde{\mathcal{S}}_V(E_{\mathfrak{a}}(\,\cdot\,, 1/2 + it), K; q)\right|^2 dt,$$

where for any $k \equiv \kappa \pmod 2$ with $k > \kappa$,

$$(5.4) \qquad M(L; k) := \frac{(k-2)!}{\pi(4\pi)^{k-1}} \sum_{g \in \mathcal{B}_k(2q, \omega)} |B(g)|^2 \left|\widetilde{\mathcal{S}}_V(g, K; q)\right|^2,$$

and we refer to [14, Section 3.2] or [2, (2.9)] for the choice and properties of the test function $\phi = \phi_{a,b}$. The key Proposition is the following [14, Proposition 4.1].

PROPOSITION 5.1. — *Let $K : \mathbf{F}_q \to \mathbf{C}$ and V as in Theorem 1.6. Let (b_ℓ) be the sequence of complex numbers defined by (5.1). Then for any $\varepsilon > 0$, there exists $k(\varepsilon) > \kappa$ such that for any $k \geqslant k(\varepsilon)$ and any integers $a > b > 2$ satisfying*

$$a - b \geqslant k(\varepsilon), \quad a - b \equiv \kappa \pmod 2,$$

we have the bound

$$(5.5) \quad M(L), \ M(L; k)$$

$$\ll \mathrm{cond}(K)^s \left\{ q^{1+\varepsilon}L^{1/2}P(P + Q) + q^{1/2}L^2PQ^2(P + Q) \right\},$$

for some absolute constant $s \geqslant 1$, provided

$$(5.6) \qquad q^{\varepsilon}LQ < q^{1/4}$$

and where the implied constant depends on C, ε, a, b, k and polynomially on the archimedean parameter of f.

Theorem 1.6 can be deduced from Proposition 5.1 exactly in the same way as in [14, Section 4.2]. The only changes is to use (5.2) to pass from $|B(f)|^2|\mathcal{S}_V(f, K; q)|^2$ to $|B(f)|^2|\widetilde{\mathcal{S}}_V(f, K; q)|^2$ and then (2.11) for the upper

bound on $|\rho_f(1)|^{-2}$. Finally, since for any prime p different from q we have the elementary relation

$$\lambda_f(p)^2 - \lambda_f(p^2) = \omega(p),$$

we obtain the lower bound

$$B(f) \gg \frac{L^{1/2}}{\log L},$$

simply using the prime number Theorem. Hence it remains to prove Proposition 5.1.

Expanding the square in $|B(g)|^2$ and $|\widetilde{\mathcal{S}}_V(g, K; q)|$ (choosing the variables ℓ_1, ℓ_2 for those comming from the amplifier), we get a first decomposition of $M(L)$ and $M(L; k)$

$$(5.7) \quad M(L) = M_d(L) + M_{nd}(L) \quad \text{and} \quad M(L; k) = M_d(L; k) + M_{nd}(L; k),$$

depending on weither $(\ell_1, \ell_2) > 1$ (the diagonal term) or not. For the diagonal term, we have the following lemma which is the analogous of [14, Lemma 5.1].

LEMMA 5.2. — *For any $\varepsilon > 0$, we have*

$$M_d(L), M_d(L; k) \ll \text{cond}(K)^2 q^{1+\varepsilon} L^{1/2} P(P+1),$$

where the implied constant depends only on ε.

Proof. — We consider $M_d(L)$ which decomposes as a sum of the holomorphic, Maass and Eisenstein contributions

$$M_d(L) = M_{d,\text{hol}}(L) + M_{d,\text{Maass}}(L) + M_{d,\text{Eis}}(L).$$

We treat only $M_{d,\text{Maass}}(L)$ since the others contributions are the same and even simpler. For instance, we have

$$M_{d,\text{Maass}}(L) = \sum_{g \in \mathcal{B}(2q,\omega)} \widetilde{\phi}(t_g) \frac{4\pi}{\cosh(\pi t_g)} \mathscr{C}(g, L) \left| \sum_n K(n) \rho_g(n) V\left(\frac{n}{q}\right) \right|^2,$$

with

$$\mathscr{C}(g, L) := \sum_{(\ell_1, \ell_2) > 1} b_{\ell_1} \overline{b_{\ell_2}} \lambda_g(\ell_1) \overline{\lambda_g(\ell_2)}.$$

By definition of the coefficients b_ℓ (cf. (5.1)), the case $(\ell_1, \ell_2) > 1$ appears when $\ell_1 = \ell_2 = p \sim L^{1/2}$, $\ell_1 = p^2 = \ell_2^2 \sim L$ (or the inverse) and $\ell_1 = \ell_2 = p^2 \sim L$. We write $\mathscr{C}(g, L) = \mathscr{C}_1(g, L) + \mathscr{C}_2(g, L) + \mathscr{C}_3(g, L)$ according to the

different possibilities and we estimate the three quantities individually. We first have by Cauchy–Schwarz inequality and (2.8),

$$\mathscr{C}_1(g, L) = \sum_{\substack{p \sim L^{1/2} \\ p \text{ prime}}} |\lambda_f(p)|^2 |\lambda_g(p)|^2$$

$$\leqslant \left(\sum_{p \sim L^{1/2}} |\lambda_f(p)|^4 \right)^{1/2} \left(\sum_{p \sim L^{1/2}} |\lambda_g(p)|^4 \right)^{1/2}$$

$$\ll (qL(1 + |t_f|)(1 + t_g))^\varepsilon L^{1/2},$$

where the implied constant only depends on ε. For the second case, we have using $|\lambda(p^2)| \leqslant 1 + |\lambda(p)|^2$ (cf. (2.2)), Hölder and again (2.8),

$$|\mathscr{C}_2(g, L)| \leqslant \sum_{\substack{p \sim L^{1/2} \\ p \text{ prime}}} |\lambda_f(p)| \|\lambda_g(p)\| \lambda_g(p^2)|$$

$$\leqslant \sum_{\substack{p \sim L^{1/2} \\ p \text{ prime}}} |\lambda_f(p)| |\lambda_g(p)| (1 + |\lambda_g(p)|^2)$$

$$\leqslant \left(\sum_{p \sim L^{1/2}} |\lambda_f(p)|^4 \right)^{1/4} \left(\sum_{p \sim L^{1/2}} |\lambda_g(p)|^4 \right)^{1/4}$$

$$\times \left(\sum_{p \sim L^{1/2}} (1 + |\lambda_g(p)|^2)^2 \right)^{1/2}$$

$$\ll (qL(1 + |t_f|)(1 + t_g))^\varepsilon L^{1/2}.$$

Using the inequality $|\lambda_g(p^2)|^2 \leqslant 2(1 + |\lambda_g(p)|^4)$, we treat in the same way $\mathscr{C}_3(g, L)$. The rest of the proof is exaclty the same as [14, Lemma 5.1], except that we must use Proposition 2.5 for the spectral large sieve (possible since the conductor of ω is either 1 or a prime q) instead of the original version of Deshouillers–Iwaniec [7, Theorem 2, (1.29)]. $\square$

Now comes the contribution of the ℓ_1, ℓ_2 such that $(\ell_1, \ell_2) = 1$. We first change the complex conjugate $\overline{\lambda_g(\ell_2)} = \overline{\omega}(\ell_2)\lambda_g(\ell_2)$ in $M_{nd}(L)$ and $M_{nd}(L; k)$ appearing in the decomposition (5.7) (cf. (2.3)). By the primiality condition, we use the multiplicativity of the Hecke eigenvalues (2.2) followed by the relation (2.9) to obtain

$$\lambda_g(\ell_1 \ell_2) \rho_g(n_1) = \sum_{d | (\ell_1 \ell_2, n_1)} \omega(d) \rho_g \left(\frac{\ell_1 \ell_2 n_1}{d^2} \right).$$

Once we have done this, we apply the Petersson trace formula (2.15) to $M_{nd}(L; k)$ in (5.7), obtaining

$$\pi M_{nd}(L; k) = M_1(L; k) + M_2(L; k),$$

where $M_1(L; k)$ corresponds to the diagonal term $\delta(\ell_1 \ell_2 n_1 d^{-2}, n_2)$. Similarily, we apply Kuznetsov formula (2.16) to $M_{nd}(L)$ and since there is no diagonal term, we write $M_{nd}(L) = M_2(L)$. The treatment of the diagonal term $M_1(L; k)$ is contained in [14, Lemma 5.3], with the appropriate changes using (2.7) for the coefficients of the amplifier,

$$(5.8) \qquad M_1(L; k) \ll \text{cond}(K)^2 (q(1 + |t_f|))^\varepsilon q L^{1/2} P,$$

with an implied constant depending only on ε.

5.1.2. The off-diagonal terms

This is the most important case of $M_2(L)$ and $M_2(L; k)$ and thus we write explicitly the quantities to study. For ϕ an arbitrary function, we write

$$(5.9) \quad M_2[\phi] = \frac{1}{2q} \sum_{(\ell_1, \ell_2)=1} b_{\ell_1} \overline{b_{\ell_2}} \overline{\omega}(\ell_2) \sum_{d | \ell_1 \ell_2} \omega(d)$$
$$\times \sum_{\substack{n_1, n_2 \\ d | n_1}} K(n_1) \overline{K(n_2)} V\left(\frac{n_1}{q}\right) V\left(\frac{n_2}{q}\right)$$
$$\times \sum_{c \geqslant 1} c^{-1} S_\omega(\ell_1 \ell_2 n_1 d^{-2}, n_2; 2cq) \phi\left(\frac{4\pi}{2cq} \sqrt{\frac{\ell_1 \ell_2 n_1 n_2}{d^2}}\right),$$

in order to have

$$M_2(L) = M_2[\phi_{a,b}] \quad \text{and} \quad M_2(L; k) = M_2[\phi_k]$$

where $\phi_k = 2\pi i^{-k} J_{k-1}$ is the Bessel function. We transform the sum as

$$(5.10) \qquad M_2[\phi] = \sum_{(\ell_1, \ell_2)=1} b_{\ell_1} \overline{b_{\ell_2}} \overline{\omega}(\ell_2) \sum_{de=\ell_1 \ell_2} \omega(d) M_2[\phi; d, e],$$

where

$$M_2[\phi; d, e] = \frac{1}{2q} \sum_{c \geqslant 1} c^{-1} \mathcal{E}_\phi(c, d, e)$$

and

$$\mathcal{E}_\phi(c, d, e) = \sum_{n_1, n_2} S_\omega(en_1, n_2; 2cq) K(dn_1) \overline{K(n_2)} H_\phi(n_1, n_2),$$

with

(5.11)
$$H_\phi(x, y) = \phi\left(\frac{4\pi\sqrt{exy}}{2cq}\right) V\left(\frac{dx}{q}\right) V\left(\frac{y}{q}\right).$$

As in [14, Section 5.4], we truncate the parameter c in $M_2[\phi; d, e]$ by writing $M_2[\phi; d, e] = M_{2,C}[\phi; d, e] + M_3[\phi; d, e]$ where $M_{2,C}[\phi; d, e]$ denotes the contribution of ther terms with $c > C$ for some $C = C(d, e) \geqslant 1/2$ and correspondingly

(5.12)
$$M_2[\phi] = M_{2,\text{tail}}[\phi] + M_3[\phi].$$

It turns out that with the choice

(5.13)
$$C = \max\left(\frac{1}{2}, q^\delta P\sqrt{\frac{e}{d}}\right) \ll q^\delta LP,$$

the contribution of $c > C$ is negligible (see [14, (5.9)]), so we focus on the complementary sum, which is given by

(5.14)
$$M_3[\phi; d, e] = \frac{1}{2q} \sum_{1 \leqslant c \leqslant C} c^{-1} \mathcal{E}_\phi(c, d, e).$$

In particular, the above expression is zero if $C < 1$.

Recall that we factored the product $\ell_1\ell_2$ as $de = \ell_1\ell_2$. Since we allow ℓ_1 and ℓ_2 to be square of primes, there are more different type of factorization to consider. We distinguish three types.

- Type I (balanced case) : this is when both d and e are $\neq 1$ and $d/e \sim 1$, so d and e are either primes $\sim L^{1/2}$ (type $(L^{1/2}, L^{1/2})$) or square of primes $\sim L$ (type (L, L)) with $(d, e) = 1$ in each case.
- Type II (unbalanced case) : this is when (d, e) satisfies $e/d \gg L^{1/2}$, i.e. is of type $(1, L), (1, L^{3/2}), (1, L^2), (L^{1/2}, L)$ and $(L^{1/2}, L^{3/2})$.
- Type III (unbalanced case) : this is when (d, e) satisfies $d/e \gg L^{1/2}$, so is of type $(L, 1), (L^{3/2}, 1), (L^2, 1), (L, L^{1/2})$ and $(L^{3/2}, L^{1/2})$.

Assuming the harmless condition

(5.15)
$$q^\delta P \ll L^{1/2},$$

we obtain by (5.13) :

LEMMA 5.3. — Suppose that (d, e) is of Type III and that (5.15) is satisfied. Then we have the equality

$$M_3[\phi; d, e] = 0.$$

It remains to deal with the Types I and II. The goal now is to transform the sums $\mathcal{E}_\phi(c, d, e)$ to connect them with the correlation sums $\mathcal{C}(\widehat{K}, \omega; \gamma)$ of the Fourier transform of K defined in (3.9) for suitable matrices γ. This is

the content of [14, Section 5.5] and it is achieved using principally twisted multiplicativity of the Kloosterman sums and Poisson summation formula. The only difference here is the appearance of the nebentypus ω when we open the Kloosterman sum. We also mention that it is in this treatment that we use the fact that the level is $2q$ and not q. The result is that for any $c \geqslant 1$, we have the identity

$$(5.16) \quad \mathcal{E}_\phi(c, d, e) = \frac{\omega(-d)}{q} \sum_{\substack{n_1 n_2 \neq 0, \ (n_2, 2c)=1 \\ n_1 n_2 \equiv e (\text{mod } 2c)}} \widehat{H}_\phi\left(\frac{n_1}{2cq}, \frac{n_2}{2cq}\right) \mathcal{C}(\widehat{K}, \omega; \gamma),$$

where $\widehat{H}_\phi$ is the Fourier transform of H_ϕ and

$$(5.17) \qquad \gamma = \gamma(c, d, e, n_1, n_2) := \begin{pmatrix} n_1 & \frac{n_1 n_2 - e}{2c} \\ 2cd & dn_2 \end{pmatrix} \in \mathrm{M}_2(\mathbf{Z}) \cap \mathrm{GL}_2(\mathbf{Q}).$$

Remark 5.4. — Observe that $\det(\gamma) = de$ which is coprime with q. Hence the reduction of γ modulo q provides a well defined element of $\mathrm{PGL}_2(\mathbf{F}_q)$.

5.1.3. Analysis of $\mathcal{E}_\phi(c, d, e)$

The first step in the analysis of (5.16) passes by the study of the Fourier transform of $H_\phi(x, y)$. This is the content of [14, Sections 5.6–5.7] and it is contained in Lemmas 5.7 and 5.9 therein. One of the consequences is that it allows to truncate the n_1, n_2-sum in $\mathcal{E}_\phi(c, d, e)$ to

$$(5.18) \qquad 0 \neq |n_1| \leqslant \mathcal{N}_1 := q^\varepsilon cd \frac{(Q + \frac{P}{2c}\sqrt{\frac{e}{d}})}{P}, \quad 0 \neq |n_2| \leqslant \mathcal{N}_2 := \frac{\mathcal{N}_1}{d},$$

(see [14, (5.21)]). The final strategy is to separate the terms in (5.16) (with the restriction (5.18) on n_1, n_2) according to whether the reduction of γ modulo q belongs to $\mathbf{G}_{\widehat{\mathcal{F}}, \omega}$ or not (see Definition 3.11). In the first case, we use the bound (see (3.13) and (3.8))

$$|\mathcal{C}(\widehat{K}, \omega; \gamma)| \leqslant \mathrm{cond}(\widehat{K})^2 q \leqslant 100 \, \mathrm{cond}(K)^4 q,$$

while for γ not in $\mathbf{G}_{\widehat{F}, \omega}$, we have by Proposition 3.15

$$|\mathcal{C}(\widehat{K}, \omega; \gamma)| \leqslant A(\mathrm{cond}(\widehat{K}))^s q^{1/2} \ll \mathrm{cond}(K)^{2s} q^{1/2}.$$

We thus write

$$\mathcal{E}_\phi(c, d, e) = \mathcal{E}_\phi^c(c, d, e) + \mathcal{E}_\phi^n(c, d, e),$$

where $\mathcal{E}_\phi^c(c, d, e)$ is the subsum of (5.16) where we restrict to the variables n_1, n_2 such that the reduction modulo q of $\gamma(c, d, e, n_1, n_2)$ belongs to

$\mathbf{G}_{\widehat{F},\omega}$ and $\mathcal{E}_\phi^n(c,d,e)$ is the contribution of the remaining terms. According to (5.14), (5.12) and (5.10), we also write

$$M_3[\phi; d, e] = \frac{1}{2q} \sum_{c \leqslant C} c^{-1} \left(\mathcal{E}_\phi^c(c,d,e) + \mathcal{E}_\phi^n(c,d,e) \right)$$

$$= M_3^c[\phi; d, e] + M_3^n[\phi; d, e],$$

and

$$M_3[\phi] = \sum_{(\ell_1, \ell_2) = 1} b_{\ell_1} \overline{b_{\ell_2}} \overline{\omega}(\ell_2) \sum_{de = \ell_1 \ell_2} \omega(d) \left(M_3^c[\phi; d, e] + M_3^n[\phi; d, e] \right)$$

$$= M_3^c[\phi] + M_3^n[\phi].$$

LEMMA 5.5. — *With the above notations, we have*

$$M_3^n[\phi_{a,b}] \ll \operatorname{cond}(K)^4 q^{1/2+\varepsilon} L^2 P Q^2 (P + Q)$$

and

$$M_3^n[\phi_k] \ll \operatorname{cond}(K)^4 k^3 q^{1/2+\varepsilon} L^2 P Q^2 (P + Q),$$

for any $\varepsilon > 0$ where the implied constant depends on ε, a, b for $\phi = \phi_{a,b}$ and on ε for $\phi = \phi_k$.

Proof. — This is the content of [14, p. 625–626], with minimal changes due to the different nature of pairs (d, e) of Type I and II. □

To conclude the proof of Proposition 5.1, it remains to evaluate the contribution $M_3^c[\phi; d, e]$ corresponding to the matrices whose reduction modulo q is in $\mathbf{G}_{\widehat{F},\omega}$. The final lemma is the following:

LEMMA 5.6. — *Under the assumption*

(5.19) $$q^{3\varepsilon} L Q < q^{1/4},$$

we have

$$M_3^c[\phi_k] \ll \operatorname{cond}(K)^{2s} k^3 q^{1+\varepsilon} L^{1/2} P Q$$

and

$$M_3^c[\phi_{a,b}] \ll \operatorname{cond}(K)^{2s} q^{1+\varepsilon} L^{1/2} P Q,$$

where $s \geqslant 2$ and the implied constant depends on ε, a, b.

Proof. — The proof is [14, Sections 6.1, 6.3 and 6.5] (recall that here there are no parabolic elements by Proposition 3.17). Various arguments use the fact that the discriminant of certain binary quadratic form is not zero. For example, if $\gamma = \gamma(c, d, e, n_1, n_2)$ is a toric matrix, then we need to have $(n_1 + d n_2)^2 - 4de \neq 0$ and we cannot say that $de = \ell_1 \ell_2$ is squarefree

since we allow square of primes in the amplifier. This is not a problem here because if $(n_1 + dn_2)^2 = 4de$, then we would get (see (5.17))

$$\mathrm{Tr}(\gamma)^2 - 4\det(\gamma) = 0 \text{ in } \mathbf{F}_q.$$

This means that γ has only one distinct eigenvalue, so its is necessarily scalar since not parabolic by assumption. But for γ scalar, we have $cd \equiv 0 \pmod{q}$, wich is not possible by (5.19) and (5.13).

Another argument uses the fact that $dn_2^2 - e \neq 0$ in a situation where $n_1 - dn_2 = 0$ (cf. [14, Section 6.3, p. 629]). Again, d and e are not necessarily coprime here so we cannot argue in the same way. However, if $dn_2^2 = e$, then since $n_1 = dn_2$, we obtain $n_1 n_2 - e = dn_2^2 - e = 0$ and the matrix $\gamma(c, d, e, n_1, n_2)$ takes the form

$$\gamma(c, d, e, n_1, n_2) = \begin{pmatrix} n_1 & 0 \\ 2cd & dn_2 \end{pmatrix}.$$

Since $dn_2 = n_1$, this matrix is parabolic with single fixed point $z = 0$, which contradicts the fact that $\mathbf{G}_{k,\omega,M}$ does not contain parabolic elements. $\square$

5.2. The Eisenstein case

We recall the notations from Section 1. For $q > 2$ prime, ω a Dirichlet character modulo q and $t \in \mathbf{R}$, we set

$$\lambda_\omega(n, it) = \sum_{ab=n} \omega(a) \left(\frac{a}{b}\right)^{it},$$

and

$$\mathcal{S}_V(\omega, it, K; q) = \sum_{n \geqslant 1} \lambda_\omega(n, it) K(n) V\left(\frac{n}{q}\right),$$

for K an isotypic trace function such that its Fourier transform is not ω-exceptional (see Definition 3.16) and V satisfying condition $V(C, P, Q)$. Since $\lambda_\omega(n, it)$ appears as Hecke eigenvalues (for $(n, q) = 1$) of the Eisenstein series $E_1(\,\cdot\,, 1/2+it)$ (the cusp $\mathfrak{a} = 1$) lying in the continuous spectrum of the Laplacian on the space of modular forms of level q (and thus also of level $2q$ after a normalization) and nebentypus ω (see Section 2.1.1 and Remark 2.1), we may estimate an amplified second moment of $\mathcal{S}_V(\omega, it, K; q)$ by embedding in the Eisentsein spectrum and using Kuznetsov trace formula as in the cuspidal case.

For $\tau \in \mathbf{R}$, we define as in (5.1)

$$(5.20) \qquad b_\ell(\tau) := \begin{cases} \lambda_\omega(\ell, i\tau)\overline{\omega}(\ell) & \text{if } \ell = p \sim L^{1/2} \text{ and } (p, 2q) = 1 \\ -\overline{\omega}(\ell) & \text{if } \ell = p^2 \sim L \text{ and } (p, 2q) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and for g a cuspidal form, we set

$$B_\tau(g) = \sum_{1 \leqslant \ell \leqslant 2L} b_\ell(\tau)\lambda_g(\ell).$$

For an Eisenstein series $E_\mathfrak{a}(\,\cdot\,, 1/2 + it)$, we let

$$B_\tau(\mathfrak{a}, it) = \sum_{1 \leqslant \ell \leqslant 2L} b_\ell(\tau)\lambda_\mathfrak{a}(\ell, it)$$

and we also write $B_\tau(\omega, it)$ so that it corresponds to the Eisenstein series $E_1(\,\cdot\,, 1/2 + it)$ having $\lambda_\omega(n, it)$ as Hecke eigenvalues. Since the trace formula involves Fourier coefficients instead of Hecke eigenvalues, we define as in Section 5.1

$$\widetilde{\mathcal{S}}_V(\omega, it, K; q) = \sum_{n \geqslant 1} \rho_\omega(n, it)K(n)V\left(\frac{n}{q}\right),$$

with the relation

$$(5.21) \qquad \mathcal{S}_V(\omega, it, K; q) = \rho_\omega(1, it)^{-1}\widetilde{\mathcal{S}}_V(\omega, it, K; q).$$

Remark 5.7. — Actually, the relation (5.21) is true if we restrict the n-summation in $\mathcal{S}_V(\omega, it, K; q)$ to $(n, q) = 1$. However, we could consider directly this restriction at the beginning since the error to pass from one to the other is given by

$$\mathcal{S}_V(\omega, it, K; q) = \sum_{(n,q)=1} \lambda_\omega(n, it)K(n)V\left(\frac{n}{q}\right) + O\left(q^\varepsilon M(P+1)\right).$$

Using the lower bound for $\rho_\omega(1, it)$ given by (2.12) and $\widetilde{\phi}_{a,b}(t) \asymp (1 + |t|)^{\kappa - 2b - 2}$ (cf. [2, (2.21)]), we obtain exactly as in Section 5.1 (see

Proposition 5.1),

$$(5.22) \quad \int_{\mathbf{R}} \frac{|\mathcal{S}_V(\omega, it, K; q)|^2}{(1 + |t|)^{2b+2-\varepsilon}} |B_\tau(\omega, it)|^2 \, dt$$

$$\ll q^{1+\varepsilon} \int_{\mathbf{R}} \frac{(1 + |t|)^{\kappa-2b-2}}{\cosh(\pi t)} \left| \widetilde{\mathcal{S}}_V(\omega, it, K; q) \right|^2 |B_\tau(\omega, it)|^2 \, dt$$

$$\ll q^{1+\varepsilon} \int_{\mathbf{R}} \frac{\widetilde{\phi_{a,b}}(t)}{\cosh(\pi t)} \left| \widetilde{\mathcal{S}}_V(\omega, it, K; q) \right|^2 |B_\tau(\omega, it)|^2 \, dt$$

$$\ll \mathrm{cond}(K)^{2s} \left\{ q^{2+\varepsilon} L^{1/2} P(P+Q) + q^{3/2} L^2 P Q^2 (P+Q) \right\}.$$

In order to apply (5.22), the following Lemma gets a suitable lower bound for the amplifier $B_\tau(\omega, it)$ when τ is close enough to t (see [13, Lemma 2.4]).

LEMMA 5.8. — *For L large enough, we have*

$$B_\tau(\omega, it) \gg \frac{L^{1/2}}{\log L},$$

uniformly in $t, \tau \in \mathbf{R}$ satisfying

$$|t - \tau| \leqslant \frac{1}{\log^2 L}.$$

Proof. — Observe that since ℓ has at most three divisors, we have $|b_\ell(\tau)| \leqslant 3$ and thus

$$|B_\tau(\omega, it) - B_\tau(\omega, i\tau)| \leqslant \sum_\ell |b_\ell(\tau)| |\lambda_\omega(\ell, it) - \lambda_\omega(\ell, i\tau)|$$

$$\leqslant 3 \sum_{\substack{p \sim L^{1/2} \\ p \text{ prime}}} \left\{ |\lambda_\omega(p, it) - \lambda_\omega(p, i\tau)| + |\lambda_\omega(p^2, it) - \lambda_\omega(p^2, i\tau)| \right\}$$

$$\leqslant 6 \sum_{\substack{p \sim L^{1/2} \\ p \text{ prime}}} \left\{ |p^{it} - p^{i\tau}| + |p^{2it} - p^{2i\tau}| \right\}$$

$$\leqslant 36|t - \tau| \sum_{\substack{p \sim L^{1/2} \\ p \text{ prime}}} \log(p) \ll \frac{L^{1/2}}{\log^2 L}.$$

It is therefore enough to prove the result for $t = \tau$. But this is a consequence of the elementary relation

$$\lambda_\omega(p, it)^2 \overline{\omega}(p) - \overline{\omega}(p) \lambda_\omega(p^2, it) = 1,$$

valid for $(p, q) = 1$, and the prime number Theorem.   □

The above Lemma combining with the average bound (5.22) allows us to deduce a first upper-bound for short averages of twists of Eisenstein series. For this, we introduce the notation

$$I(\tau, q) := \left\{ t \in \mathbf{R} \,\middle|\, |t - \tau| \leqslant \frac{1}{\log^2 q} \right\}.$$

and

$$\mathcal{K}(P, Q; q) := \operatorname{cond}(K)^s q^{1 - \frac{1}{16}} (PQ)^{1/2} (P + Q)^{1/2},$$

so that Theorem 1.6 claims that

$$\mathcal{S}_V(\omega, it, K; q) \ll_\varepsilon q^\varepsilon (1 + |t|)^A \mathcal{K}(P, Q; q)$$

for any $\varepsilon > 0$ and some $A \geqslant 1$ depending on ε.

PROPOSITION 5.9. — For any $\varepsilon > 0$, there exists $B \geqslant 1$, depending only on ε, such that for any $\tau \in \mathbf{R}$ we have

$$(5.23) \qquad \int_{I(\tau, q)} |\mathcal{S}_V(\omega, it, K; q)|^2 \, dt \ll_\varepsilon q^\varepsilon (1 + |\tau|)^B \mathcal{K}(P, Q; q)^2,$$

where the implied constant depends only on ε.

Proof. — Using Lemma 5.8 and (5.22), we obtain

$$\frac{L}{\log^2 L} \int_{I(\tau, q)} |\mathcal{S}_V(\omega, it, K; q)|^2 \, dt$$

$$\ll \int_{I(\tau, q)} \frac{(1 + |\tau|)^{2b+2}}{(1 + |t|)^{2b+2-\varepsilon}} |\mathcal{S}_V(\omega, it, K; q)|^2 |B_\tau(\omega, it)|^2 \, dt$$

$$\ll (1 + |\tau|)^{2b+2} \operatorname{cond}(K)^{2s} \left\{ q^{2+\varepsilon} L^{1/2} P(P + Q) + q^{3/2} L^2 PQ^2(P + Q) \right\},$$

and we conclude as in [14, Section 4.2] by choosing

$$L = \frac{1}{2} q^{1/4-\varepsilon} Q^{-1},$$

and $B = 2b + 2$ which depends on ε. $\qquad\square$

The last step is to derive a pointwise bound for $\mathcal{S}_V(\omega, it, K; q)$. For this, we separate the variables n, m in the twisted divisor function $\lambda_\omega(n, it)$ and using a partition of unity, we can decompose $\mathcal{S}_V(\omega, it, K; q)$ into $O(\log Pq)$ sums of the shape

$$\mathcal{S}_{V,M,N}(\omega, it, K; q)$$

$$= \sum_{n, m \geqslant 1} K(mn) \omega(m) \left(\frac{m}{n} \right)^{it} W_1 \left(\frac{m}{M} \right) W_2 \left(\frac{n}{N} \right) V \left(\frac{nm}{q} \right),$$

where the parameters (M, N) belongs to the set

$$(5.24) \qquad \mathbf{P} := \left\{ (M, N) \,\middle|\, \frac{Pq}{4} \leqslant NM \leqslant 4Pq \, , \ 1 \leqslant N, M \right\}$$

and W_1, W_2 are smooth and compactly supported functions on $[-1/2, 2]$ satisfying $x^j W_i^{(j)}(x) \ll_j 1$ for every $j \geqslant 0$. It follows that

$$(5.25) \qquad \mathcal{S}_V(\omega, it, K; q) \ll \log(Pq) \max_{(M,N) \in \mathbf{P}} |\mathcal{S}_{V,M,N}(\omega, it, K; q)| \, .$$

The relation between $\mathcal{S}_{V,M,N}(\omega, it, K; q)$ and an average of $\mathcal{S}_V(\omega, it, K; q)$ is given through the Mellin transform (see [13, Lemma 2.1]).

LEMMA 5.10. — *Given $s \in \mathbf{C}$ and $x > 0$, we define*

$$V_s(x) := V(x)x^{-s}.$$

Then for every $\varepsilon > 0$, we have

$$(5.26) \quad \mathcal{S}_{V,M,N}(\omega, it, K; q)$$
$$\ll_\varepsilon \iint\limits_{|t_1|,|t_2| \leqslant q^\varepsilon} \left| \mathcal{S}_{V_{it_1}}(\omega, it_2 + it, K; q) \right| \mathrm{d}t_1 \mathrm{d}t_2 + O\left(q^{-100}\right).$$

Proof. — Using Mellin inversion formula for W_1 and W_2, we can write

$$\mathcal{S}_{V,M,N}(\omega, it, K; q) = \frac{1}{(2\pi i)^2} \iint\limits_{(0)\,(0)} \widehat{W_1}(s_1)\widehat{W_2}(s_2)\mathcal{T}_V(s_1, s_2)M^{s_1}N^{s_2}\mathrm{d}s_1\mathrm{d}s_2,$$

where $\widehat{W_1}, \widehat{W_2}$ denote the Mellin transform of the smooth functions W_1, W_2 and

$$\mathcal{T}_V(s_1, s_2) = \sum_{n,m \geqslant 1} K(nm)\omega(m)m^{it-s_1}n^{-it-s_2}V\left(\frac{nm}{q}\right).$$

Note that this sum can be expressed as a twist of Eisenstein series, namely

$$\mathcal{T}_V(s_1, s_2) = q^{-\theta_1}\mathcal{S}_{V_{\theta_1}}(\omega, \theta_2 + it, K; q),$$

with

$$\theta_1 = \frac{s_1 + s_2}{2}, \quad \theta_2 = \frac{-s_1 + s_2}{2}.$$

For $\Re e(\theta_1)$, the smooth function V_{θ_1} satisfies condition $V(C, P, Q(\theta_1))$ with

$$(5.27) \qquad\qquad Q(\theta_1) \ll Q + |\theta_1|,$$

where the implied constant is absolute. Thus by a change of variables, we get

$$(5.28) \quad \mathcal{S}_{V,M,N}(\omega, it, K; q)$$

$$= \frac{1}{(2\pi i)^2} \int\limits_{(0)} \int\limits_{(0)} \widehat{W}_1(\theta_1 - \theta_2)\widehat{W}_2(\theta_1 + \theta_2) \left(\frac{M}{N}\right)^{\theta_2}$$

$$\times \left(\frac{MN}{q}\right)^{\theta_1} \mathcal{S}_{V_{\theta_1}}(\omega, \theta_2 + it, K; q)\mathrm{d}\theta_1\mathrm{d}\theta_2.$$

Because we have the estimations

$$\widehat{W}_1(s), \widehat{W}_2(s) \ll \frac{1}{(1 + |s|)^C},$$

with an implied constant depending on C and $\Re e(s)$, we can truncate the integral (5.28) to $|\theta_1| \leqslant q^\varepsilon$, $|\theta_2| \leqslant q^\varepsilon$ for a cost of $O(q^{-100})$ by taking C large enough in term of ε and using the trivial bound for $\Re e(\theta_1) = \Re e(\theta_2) = 0$

$$\mathcal{S}_{V_{\theta_1}}(\omega, \theta_2 + it, K; q) \ll MPq \log q. \qquad \square$$

5.2.1. Conclusion

We are now in position to obtain the conclusion of Theorem 1.6 in the Eisenstein case. Indeed, fix $\varepsilon > 0$ and take $B = B(\varepsilon)$ as in Proposition 5.9. By (5.25), it is enough to estimate $\mathcal{S}_{V,M,N}(\omega, it, K; q)$ for $(M, N) \in \mathbf{P}$. Now let $\varepsilon' = \varepsilon/B$ such that we have the estimate (5.26) of Lemma 5.10. We thus get

$$\mathcal{S}_{V,M,N}(\omega, it, K; q)$$
$$\ll_{\varepsilon'} q^{\varepsilon'} \max_{|t_1| \leqslant q^{\varepsilon'}} \int_{|t_2| \leqslant q^{\varepsilon'}} \left|\mathcal{S}_{V_{it_1}}(\omega, it_2 + it, K; q)\right| \mathrm{d}t_1\mathrm{d}t_2 + O\left(q^{-100}\right).$$

We split the above integral into $O(q^{\varepsilon'})$ integrals over intervals of length $\log^{-2} q$. For such interval I centered at τ, we obtain by Proposition 5.9, the value (5.27) and Cauchy–Schwarz inequality, the bound

$$\mathcal{S}_{V,M,N}(\omega, it, K; q) \ll q^\varepsilon (1 + |\tau|)^{B/2} \mathcal{K}(P, Q + q^{\varepsilon'}; q),$$

(the function $Q \mapsto \mathcal{K}(P, Q; q)$ is increasing). Finally, taking the maximal value $|\tau| \leqslant |t| + q^{\varepsilon'}$ yields the desire result.

6. Proof of Theorem 1.3

6.1. The Eisenstein case

It is natural to separate the sum in (1.1) into even and odd primitives characters because they have different gamma factors in their functional equations. We will treat only the case of even characters since the odd case is completely similar. We therefore consider

$$(6.1) \quad \mathcal{T}^3_{\text{even}}(\omega_1, \omega_2, \ell; q) := \frac{2}{q-1} \sum_{\substack{\chi \pmod{q} \\ \chi \neq \omega_0, \overline{\omega}_1, \overline{\omega}_2}}^{+} L\left(\chi, \tfrac{1}{2}\right) L\left(\chi\omega_1, \tfrac{1}{2}\right) L\left(\chi\omega_2, \tfrac{1}{2}\right) \chi(\ell),$$

where $\sum^{+}$ means that we sum over even characters.

6.1.1. Applying the approximate functional equation

Applying the approximate function equation provided by Lemma 2.6, we decompose (6.1) into two terms

$$\mathcal{T}^3_{\text{even}}(\omega_1, \omega_2, \ell; q) = \mathcal{S}_1(\omega_1, \omega_2, \ell; q) + i^{\kappa_1 + \kappa_2} \mathcal{S}_2(\omega_1, \omega_2, \ell; q),$$

with

$$(6.2) \quad \mathcal{S}_1(\omega_1, \omega_2, \ell; q) := \frac{2}{q-1} \sum_{\substack{\chi \pmod{q} \\ \chi \neq \omega_0, \overline{\omega}_1, \overline{\omega}_2}}^{+} \sum_{n_0, n_1, n_2 \geqslant 1}^{*}$$

$$\times \frac{\chi(n_0 n_1 n_2 \ell)\omega_1(n_1)\omega_2(n_2)}{(n_0 n_1 n_2)^{1/2}} \mathbf{V}_\chi\left(\frac{n_0 n_1 n_2}{q^{3/2}}\right),$$

and

$$(6.3) \quad \mathcal{S}_2(\omega_1, \omega_2, \ell; q) := \frac{2}{q-1} \sum_{\substack{\chi \pmod{q} \\ \chi \neq \omega_0, \overline{\omega}_1, \overline{\omega}_2}}^{+} \sum_{n_0, n_1, n_2 \geqslant 1}^{*} \frac{\chi(\overline{n_0 n_1 n_2}\ell)\overline{\omega}_1(n_1)\overline{\omega}_2(n_2)}{(n_0 n_1 n_2)^{1/2}}$$

$$\times \varepsilon(\chi)\varepsilon(\chi\omega_1)\varepsilon(\chi\omega_2)\mathbf{V}_\chi\left(\frac{n_0 n_1 n_2}{q^{3/2}}\right),$$

where the symbol * over the $n_i's$ sum means that we restrict to $(n_0 n_1 n_2, q) = 1$ and the function $\mathbf{V}_\chi$ is defined in (2.21). In particular, since we sum over even characters, this function is constant on the average and we write $\mathbf{V}$ instead of $\mathbf{V}_\chi$.

Remark 6.1. — The function $\mathbf{V}$ has rapid decay at infinity by Remark 2.7, so the sums (6.2)–(6.3) are essentially supported on $1 \leqslant n_0 n_1 n_2 \leqslant q^{3/2+\varepsilon}$. It follows that the sum over n_0, n_1, n_2 is trivially bounded by $O(q^{3/4+\varepsilon})$, so we can remove as it suits us the contribution of $\chi = \omega_0, \overline{\omega}_1$ or $\chi = \overline{\omega}_2$ for an error of size $O(q^{-1/4+\varepsilon})$.

6.1.2. Average over the primitive and even characters

We need to average the sum over the characters in (6.2)–(6.3). For this, we use some orthogonality relations asserting that for any prime $q > 2$ and any integer a corpime with q, we have (cf. [20, (3.2)–(3.4)])

$$(6.4) \qquad \sideset{}{^+}\sum_{\substack{\chi \pmod q \\ \chi \neq 1}} \chi(a) = \frac{q-1}{2}\delta_{a \equiv \pm 1(q)} - 1,$$

and for $\kappa \in \{0, 1\}$,

$$(6.5) \qquad \sideset{}{^\kappa}\sum_{\substack{\chi \pmod q \\ \chi \neq 1}} \chi(m)\varepsilon(\chi) = \frac{q-1}{2q^{1/2}} \sum_{\pm} (\pm 1)^\kappa \left(e\left(\pm \frac{\overline{m}}{q} \right) + \frac{1}{(q-1)} \right),$$

where the superscript κ means that we sum over χ such that $\chi(-1) = (-1)^\kappa$. In (6.2), we remove the contribution of $\chi = \overline{\omega}_1, \overline{\omega}_2$ (see Remark 6.1) and after applying (6.4), we get $\mathcal{S}_1 = \mathcal{S}_1^+ + \mathcal{S}_1^- + O(q^{-1/4+\varepsilon})$ with

$$(6.6) \quad \mathcal{S}_1^\pm(\omega_1, \omega_2, \ell; q) = \sum_{\substack{n_0, n_1, n_2 \geqslant 1 \\ n_0 n_1 n_2 \ell \equiv \pm 1 \pmod q}} \sum \sum \frac{\omega_1(n_1)\omega_2(n_2)}{(n_0 n_1 n_2)^{1/2}} \mathbf{V}\left(\frac{n_0 n_1 n_2}{q^{3/2}} \right).$$

For (6.3), we remove the contribution of $\chi = \omega_0, \overline{\omega}_2$ and note that for $(m, q) = 1$ we have, opening the Gauss sum $\varepsilon(\chi\omega_2)$ and using (6.5),

$$(6.7) \quad \frac{2}{q-1} \sideset{}{^+}\sum_{\chi \neq \overline{\omega}_1} \chi(m)\varepsilon(\chi\omega_1)\varepsilon(\chi\omega_2)$$

$$= \frac{1}{q^{1/2}} \sum_{a \in \mathbf{F}_q^\times} \omega_2(a) e\left(\frac{a}{q} \right) \left(\frac{2}{q-1} \sideset{}{^+}\sum_{\chi \neq \overline{\omega}_1} \chi(am)\varepsilon(\chi\omega_1) \right)$$

$$= \frac{\overline{\omega}_1(m)}{q^{1/2}} \sum_{a \in \mathbf{F}_q^\times} \overline{\omega}_1\omega_2(a) e\left(\frac{a}{q} \right) \left(\frac{2}{q-1} \sideset{}{^{\kappa_1}}\sum_{\chi \neq 1} \chi(am)\varepsilon(\chi) \right)$$

$$= \frac{1}{q^{1/2}} \sum_{\pm} \overline{\omega}_1(\pm m) \left(\frac{1}{q^{1/2}} \sum_{a \in \mathbf{F}_q^\times} \overline{\omega}_1\omega_2(a) e\left(\frac{a}{q} \right) \left(e\left(\frac{\pm \overline{am}}{q} \right) + \frac{1}{q-1} \right) \right).$$

The second expression in the above parenthesis is easily computed as a Gauss sum. For the first term, we have

$$\frac{1}{q^{1/2}} \sum_{a\in\mathbf{F}_q^\times} \overline{\omega}_1\chi_2(a)e\left(\frac{a}{q}\right)e\left(\frac{\pm\overline{am}}{q}\right)$$

$$= \omega_1\overline{\omega}_2(\pm m)\frac{1}{q^{1/2}} \sum_{a\in\mathbf{F}_q^\times} \overline{\omega}_1\omega_2(a)e\left(\frac{\overline{a}}{q}\right)e\left(\frac{\pm\overline{m}a}{q}\right)$$

$$= \omega_1(\pm m)\mathrm{Kl}_2(\pm\overline{m},\omega_1,\omega_2;q),$$

where the twisted Kloosterman sum is defined by (3.9) (see also (3.11)). Hence we see that (6.7) equals

$$(6.8)\quad \frac{2}{q-1}\sum_{\chi\neq\overline{\omega}_1}^{+} \chi(m)\varepsilon(\chi\omega_1)\varepsilon(\chi\omega_2)$$

$$= \frac{1}{q^{1/2}}\mathrm{Kl}_2(\pm\overline{m},\omega_1,\omega_2;q) + \frac{\varepsilon(\overline{\omega}_1\omega_2)\overline{\omega}_1(m)(1+(-1)^{\kappa_1})}{q^{1/2}(q-1)}.$$

Now opening the Gauss sum $\varepsilon(\chi)$ and using (6.8), we obtain for every $(m,q)=1$,

$$(6.9)\quad \frac{2}{q-1}\sum_{\chi\neq\overline{\omega}_1}^{+} \chi(m)\varepsilon(\chi)\varepsilon(\chi\omega_1)\varepsilon(\chi\omega_2)$$

$$= \frac{1}{q^{1/2}} \sum_{a\in\mathbf{F}_q^\times} e\left(\frac{a}{q}\right)\left(\frac{2}{q-1}\sum_{\chi\neq\overline{\omega}_1}^{+}\chi(am)\varepsilon(\chi\omega_1)\varepsilon(\chi\omega_2)\right)$$

$$= \frac{1}{q} \sum_{a\in\mathbf{F}_q^\times} \mathrm{Kl}_2(\pm\overline{am},\omega_1,\omega_2;q)e\left(\frac{a}{q}\right) + O\left(q^{-3/2}\right)$$

$$= \frac{1}{q^{1/2}}\mathrm{Kl}_3(\pm\overline{m},\omega_1,\omega_2,1;q) + O\left(q^{-3/2}\right).$$

Finally, applying (6.9) in (6.3) with $m = \overline{n_0 n_1 n_2}\ell$ yields $\mathcal{S}_2 = \mathcal{S}_2^+ + \mathcal{S}_2^- + O(q^{-1/4+\varepsilon})$ (recall Remark 6.1) with

$$(6.10)\quad \mathcal{S}_2^\pm(\omega_1,\omega_2,\ell;q) = \frac{1}{q^{1/2}} \sum_{n_0,n_1,n_2\geqslant 1}^{*} \frac{\overline{\omega}_1(n_1)\overline{\omega}_2(n_2)}{(n_0 n_1 n_2)^{1/2}}$$

$$\times \mathrm{Kl}_3(\pm n_0 n_1 n_2\overline{\ell},\omega_1,\omega_2,1;q)\mathbf{V}\left(\frac{n_0 n_1 n_2}{q^{3/2}}\right).$$

We will evaluate each of these two terms ((6.6)–(6.10)) separately and find that a main term appears only in $\mathcal{S}_1^+(\omega,\omega_2,\ell;q)$ when $\ell = 1$. The others will contribute as an error term.

6.1.3. The Main Term

The main contribution comes from $n_0 = n_1 = n_2 = \ell = 1$ in (6.6). Indeed, assuming $n_0 n_1 n_2 \ell = 1$, we obtain by the Remark 2.7

$$\mathbf{V}\left(\frac{1}{q^{3/2}}\right) = 1 + O\left(q^{-3/4+\varepsilon}\right).$$

When $n_0 n_1 n_2 \ell \equiv \pm 1 \pmod{q}$ with $n_0 n_1 n_2 \ell \neq 1$, we write the congruence equation in the form $n_0 n_1 n_2 \ell = \pm 1 + kq$ with $1 \leqslant k \leqslant \ell q^{1/2+\varepsilon} + 1$. Therefore, we get that the contribution of $n_0 n_1 n_2 \ell \neq 1$ is at most

$$\ell^{1/2} q^{\varepsilon - 1/2} \sum_{1 \leqslant k \leqslant \ell q^{1/2+\varepsilon}+1} \frac{1}{k^{1/2}} \ll \ell q^{-1/4+\varepsilon}.$$

We conclude with

$$\mathcal{S}_1^+(\omega_1, \omega_2, \ell; q) = \delta_{\ell=1} + O\left(\ell q^{-1/4+\varepsilon}\right)$$

$$\mathcal{S}_1^-(\omega_1, \omega_2, \ell; q) = O\left(\ell q^{-1/4+\varepsilon}\right),$$

which gives the desired main term of Theorem 1.3 provided

$$(6.11) \qquad \ell \leqslant q^{\frac{1}{4} - \frac{1}{64}} = q^{\frac{15}{64}}.$$

6.1.4. The error term

In this section, we analyze the expression (6.10) and will find that it contributes as an error term. Applying a partition of unity to $[1, \infty)$ for each variable in order to locate n_0, n_1, n_2 and we obtain $\mathcal{S}_2^\pm(\omega_1, \omega_2, \ell; q) = \sum_{N_0, N_1, N_2} \mathcal{S}_2^\pm(\ell, N_0, N_1, N_2; q)$ with

$$(6.12) \quad \mathcal{S}_2^\pm(\ell, N_0, N_1, N_2; q)$$

$$= \frac{1}{(qN_0 N_1 N_2)^{1/2}} \times \sideset{}{^*}\sum_{n_0, n_1, n_2 \in \mathbf{Z}} \overline{\omega}_1(n_1)\overline{\omega}_2(n_2) f_1\left(\frac{n_1}{N_1}\right) f_2\left(\frac{n_2}{N_2}\right)$$

$$\times \mathrm{Kl}_3(\pm n_0 n_1 n_2 \overline{\ell}, \omega_1, \omega_2, 1; q) f_0\left(\frac{n_0}{N_0}\right) \mathbf{V}\left(\frac{n_0 n_1 n_2}{q^{3/2}}\right),$$

where the functions f_i are smooth and compactly supported on $(1/2, 2)$ and the N_i's runs over real numbers of the form 2^i, $i \geqslant 0$. By the fast decay at infinity of $\mathbf{V}$, we can assume that

$$1 \leqslant N_0, N_1, N_2 \quad \text{and} \quad N_0 N_1 N_2 \leqslant q^{3/2+\varepsilon}.$$

Hence it remains to bound $O(\log^3 q)$ sums of the shape (6.12). It is also convenient to separate the variables $n_0 n_1 n_2$ in the test function $\mathbf{V}$. This

can be done using its integral representation (2.21) and we refer to [37, Section 4.1] for more details. We keep the same notation $\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q)$, but with the factor $\mathbf{V}$ removed, and also for the functions f_i, i.e.

$$(6.13) \quad \mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q)$$

$$= \frac{1}{(qN_0N_1N_2)^{1/2}} \times \sum_{n_0, n_1, n_2 \in \mathbf{Z}}^{*} \overline{\omega}_1(n_1)\overline{\omega}_2(n_2)\mathrm{Kl}_3(\pm n_0 n_1 n_2 \overline{\ell}, \omega_1, \omega_2, 1; q)$$

$$\times f_0\left(\frac{n_0}{N_0}\right) f_1\left(\frac{n_1}{N_1}\right) f_2\left(\frac{n_2}{N_2}\right),$$

with

$$(6.14) \qquad\qquad x^j f_i^{(j)}(x) \ll_j q^{\varepsilon j}.$$

Note finally that the trivial estimate is

$$(6.15) \qquad\qquad \mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll \left(\frac{N_0 N_1 N_2}{q}\right)^{1/2}.$$

6.1.5. Polyá–Vinogradov bound

We show here that (6.13) is very small if we assume that one of the parameters N_i is greater than q. Indeed, since the argument is the same, we suppose that $N_1 \geqslant q/2$. In this case, for fixed $(n_0 n_2, q) = 1$ we focus on the n_1-sum

$$\mathcal{P}(N_1; q) = \sum_{n_1 \in \mathbf{Z}}^{*} \overline{\omega}(n_1)\mathrm{Kl}_3(\pm n_0 n_1 n_2 \overline{\ell}, \omega_1, \omega_2, 1; q) f_1\left(\frac{n_1}{N_1}\right).$$

By Remark 3.8, we can add the contribution of $q|n_1$ for an error of size $O(N_1/q)$ (since $N_1 \geqslant q/2$). Hence, applying Proposition 4.1 with the Fourier trace sheaf

$$\mathcal{L}_{\overline{\omega}} \otimes \left[\times \left(\pm n_0 n_2 \overline{\ell}\right)\right]^{*} \mathcal{K}\ell_3(\omega_1, \omega_2, 1; q),$$

we get

$$\mathcal{P}(N_1; q) = O\left(q^{\varepsilon} \frac{N_1}{q^{1/2}} + \frac{N_1}{q}\right) = O\left(q^{\varepsilon} \frac{N_1}{q^{1/2}}\right).$$

Finally, averaging trivially over n_0 and n_2 in (6.13) yields

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll q^{\varepsilon}\left(\frac{N_0 N_1 N_2}{q^2}\right)^{1/2}.$$

Since $N_0 N_1 N_2 \leqslant q^{3/2+\varepsilon}$, we obtain

PROPOSITION 6.2. — *Assume that $N_i \geqslant q/2$ for some $i \in \{0,1,2\}$. Then for any $\varepsilon > 0$, we have*

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) = O\left(q^{-1/4+\varepsilon}\right),$$

with an implied constant depending only on ε.

6.1.6. Applying Poisson summation in the three variables

In this section, we obtain an estimate for $\mathcal{S}_2^{\pm}$ which is satisfactory if the product of the three variables $N_0 N_1 N_2$ is greater that q. This can be done using successive applications of Poisson summation formula. Before this, we just note that by Proposition 6.2, we can assume that $N_i < q/2$ for $i = 0, 1, 2$, which allows us to ignore the primality condition $(n_0 n_1 n_2, q) = 1$ in (6.13) since we also have $N_i \geqslant 1$.

We begin with the n_0-variable. In (6.13), we write the Kloosterman sum as the Fourier transform of the function

$$\mathbf{F}_q \ni x \mapsto \mathbf{K}_2(x) := \mathbf{K}_2(x, \omega_1, \omega_2; q)$$

defined in (3.12). Hence, an application of Poisson summation in n_0 and Fourier inversion formula gives (recall that $(n_1 n_2, q) = 1$)

$$\sum_{n_0 \in \mathbf{Z}} \widehat{\mathbf{K}_2}(\pm n_0 n_1 n_2 \bar{\ell}) f_0\left(\frac{n_0}{N_0}\right) = \frac{N_0}{q^{1/2}} \sum_{n_0 \in \mathbf{Z}} \widehat{\widehat{\mathbf{K}_2}}(\pm n_0 \overline{n_1 n_2} \ell) \widehat{f_0}\left(\frac{n_0 N_0}{q}\right)$$

$$= \frac{N_0}{q^{1/2}} \sum_{n_0 \in \mathbf{Z}} \mathbf{K}_2(\mp n_0 \overline{n_1 n_2} \ell) \widehat{f_0}\left(\frac{n_0 N_0}{q}\right).$$

Since by Definition $\mathbf{K}_2(x) = 0$ for $q | x$, we obtain

$$(6.16) \quad \mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q)$$

$$= \frac{N_0^{1/2}}{q(N_1 N_2)^{1/2}} \times \sum_{\substack{n_0, n_1, n_2 \in \mathbf{Z} \\ (n_0, q) = 1}} \overline{\omega}_1(n_1) \overline{\omega}_2(n_2) \mathrm{Kl}_2(\mp n_1 n_2 \overline{n_0 \ell}, \omega_1, \omega_2; q)$$

$$\times \widehat{f_0}\left(\frac{n_0 N_0}{q}\right) f_1\left(\frac{n_1}{N_1}\right) f_2\left(\frac{n_2}{N_2}\right).$$

We continue with the n_2-variable. As before, since the argument of Kl_2 is non zero modulo q, we can express the Kloosterman sum as suitable Fourier transform, namely (see (3.12))

$$(6.17) \quad \mathrm{Kl}_2(\mp n_1 n_2 \bar{\ell}, \omega_1, \omega_2; q) = \omega_2(\mp n_1 n_2 \overline{n_0 \ell})[\omega_2 \widehat{\mathbf{K}_1}(\omega_1); q](\mp n_1 n_2 \overline{n_0 \ell}).$$

Using exactly the same argument as before, after replacing Kl_2 by (6.17) in (6.16), we get

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q)$$

$$= \frac{\overline{\omega}_2(\mp\ell)N_0^{1/2}}{q(N_1 N_2)^{1/2}} \times \sum_{\substack{n_0,n_1,n_2\in\mathbf{Z} \\ (n_0 n_2, q)=1}} \overline{\omega}_1\chi_2(n_1)\overline{\omega}_2(n_0)[\omega_2\widehat{\mathbf{K}_1(\omega_1)}](\mp n_1 n_2\overline{n_0}\ell)$$

$$\times \widehat{f}_0\left(\frac{n_0 N_0}{q}\right) f_1\left(\frac{n_1}{N_1}\right) f_2\left(\frac{n_2}{N_2}\right).$$

Applying Poisson in the n_2-variable yields

$$\sum_{n_2\in\mathbf{Z}} \omega_2\widehat{\mathbf{K}_1(\omega_1)}(\mp n_1 n_2\overline{n_0}\ell)f_2\left(\frac{n_2}{N_2}\right)$$

$$= \frac{N_2}{q^{1/2}} \sum_{n_2\in\mathbf{Z}} \omega_2\widehat{\widehat{\mathbf{K}_1(\omega_1)}}(\mp n_2\overline{n_1}n_0\ell)\widehat{f}_2\left(\frac{n_2 N_2}{q}\right)$$

$$= \frac{N_2}{q^{1/2}} \sum_{n_2\in\mathbf{Z}} \omega_2(\pm n_2\overline{n_1}n_0\ell)\mathbf{K}_1(\pm n_2\overline{n_1}n_0\ell, \omega_1; q)\widehat{f}_2\left(\frac{n_2 N_2}{q}\right)$$

$$= \frac{N_2}{q^{1/2}} \sum_{(n_2,q)=1} \overline{\omega}_1\omega_2(\pm n_2\overline{n_1}n_0\ell)e\left(\frac{\pm n_1\overline{n_2 n_0\ell}}{q}\right)\widehat{f}_2\left(\frac{n_2 N_2}{q}\right).$$

Hence

$$(6.18) \quad \mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) = \overline{\omega}_1(\pm\ell)\omega_2(-1)\left(\frac{N_0 N_2}{q^3 N_1}\right)^{1/2}$$

$$\times \sum_{\substack{n_0,n_1,n_2\in\mathbf{Z} \\ (n_0 n_2, q)=1}} \overline{\omega}_1(n_2 n_0)\omega_2(n_2)e\left(\frac{\pm n_1\overline{n_2 n_0\ell}}{q}\right)$$

$$\times f_1\left(\frac{n_1}{N_1}\right)\widehat{f}_0\left(\frac{n_0 N_0}{q}\right)\widehat{f}_2\left(\frac{n_2 N_2}{q}\right).$$

It remains to do Poisson in the n_1-variable. Let $a \in \mathbf{F}_q$, we denote by $\boldsymbol{\delta}_a$ the Dirac function on $\mathbf{F}_q$ defined by $\boldsymbol{\delta}_a(x) = 1$ if $x = a$ and zero else. Then the exponential map

$$n_1 \mapsto e\left(\frac{\pm n_1\overline{n_0 n_2\ell}}{q}\right)$$

is the additive Fourier transform of the Dirac function $x \mapsto q^{1/2}\boldsymbol{\delta}_{\pm\overline{n_0 n_2\ell}}(x)$. It follows that the n_1 sum in (6.18) equals

$$\sum_{n_1\in\mathbf{Z}} q^{1/2}\widehat{\boldsymbol{\delta}_{\pm\overline{n_0 n_2\ell}}}(n_1)f_1\left(\frac{n_1}{N_1}\right) = N_1 \sum_{n_1\in\mathbf{Z}} \boldsymbol{\delta}_{\pm\overline{n_0 n_2\ell}}(-n_1)\widehat{f}_1\left(\frac{n_1 N_1}{q}\right).$$

Summarizing all the previous computations yields the bound

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q)$$

$$\ll \left(\frac{N_0 N_1 N_2}{q^3}\right)^{1/2} \times \sum_{n_0 n_1 n_2 \ell \equiv \mp 1 (\mathrm{mod}\ q)} \left| \widehat{f_0}\left(\frac{n_0 N_0}{q}\right) \widehat{f_1}\left(\frac{n_1 N_1}{q}\right) \widehat{f_2}\left(\frac{n_2 N_2}{q}\right) \right|.$$

Finally, using the fact that all these Fourier transform have fast decay at infinity, we see that the above sum is essentially supported on $|n_i| \leqslant \frac{q^{1+\varepsilon}}{N_i}$ (recall that $N_i < q/2$) and thus, a trivial estimate leads to

$$(6.19) \quad \begin{aligned} \mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) &\ll q^\varepsilon \left(\frac{N_0 N_1 N_2}{q^3}\right)^{1/2} \left(\frac{q^2}{N_0 N_1 N_2} + 1\right) \\ &\ll q^\varepsilon \left(\frac{q}{N_0 N_1 N_2}\right)^{1/2}. \end{aligned}$$

6.1.7. Estimation of $\mathcal{S}_2^{\pm}$ using Theorems 1.6 and 4.2

We return to expression (6.13). The combination of (6.15) and (6.19) shows that it remains to deal with the case where the product $N_0 N_1 N_2$ is of length about q. The strategy is the following : if one of the variables N_i is very small, then we factorize the two others to form a new long variable and apply Theorem 1.6 for the twist of Eisenstein series. If all N_i are not too small, then it is possible to factorize two variables and form a bilinear sum in such a way that an application of Theorem 4.2 is beneficial.

We prove in this section :

PROPOSITION 6.3. — Let $N = \max(N_0, N_1, N_2)$, $M = \min(N_0, N_1, N_2)$ and write D for the remaining parameter, i.e. $M \leqslant D \leqslant N$. Then for every $\varepsilon > 0$, we have

$$(6.20) \quad \mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q)$$

$$\ll_\varepsilon q^\varepsilon \left(\frac{N_0 N_1 N_2}{q}\right)^{1/2} \begin{cases} \left(1 + \frac{qM}{N_0 N_1 N_2}\right)^{1/2} q^{-1/16} \\ \frac{1}{q^{1/4}} + \frac{1}{D^{1/2}} + \frac{q^{1/4}}{(NM)^{1/2}}. \end{cases}$$

Proof. — To fix the ideas, we assume that

$$M = N_0 \leqslant N_1 = D \leqslant N_2 = N,$$

and we leave it to the reader to ensure that the other cases treated with minimal changes.

We first focus on the n_1, n_2-sum in (6.13) and write it in the form (recall that $N_1, N_2 < q/2$ so the primality condition is satisfied)

$$(6.21) \quad \sum_{n_1, n_2 \geqslant 1} \overline{\omega}_1 \omega_2(n_1) \overline{\omega}_2(n_1 n_2) \mathrm{Kl}_3(n_0 n_1 n_2 \overline{\ell}, \omega_1, \omega_2; q) f_1\left(\frac{n_1}{N_1}\right) f_2\left(\frac{n_2}{N_2}\right).$$

We show now how to transform this expression in order to obtain the same as in Corollary 3.2. To simplify notations, we define

$$(6.22) \qquad K(n) := \overline{\omega}_2(n) \mathrm{Kl}_3(\pm n n_0 \overline{\ell}, \omega_1, \omega_2; q).$$

Using Mellin inversion on f_1 and f_2 in (6.21) leads to

$$\frac{1}{(2\pi i)^2} \int_{(0)} \int_{(0)} \widetilde{f}_1(s_1) \widetilde{f}_2(s_2) N_1^{s_1} N_2^{s_2}$$

$$\times \sum_{n_1, n_2 \geqslant 1} \overline{\omega}_1 \omega_2(n_1) K(n_1 n_2) n_1^{-s_1} n_2^{-s_2} \mathrm{d}s_1 \mathrm{d}s_2.$$

Making the change of variables

$$\theta_1 = \frac{s_1 + s_2}{2} \ , \quad \theta_2 = \frac{-s_1 + s_2}{2},$$

and we see that the above integral takes the form

$$(6.23) \quad \frac{2}{(2\pi i)^2} \int_{(0)} \int_{(0)} \widetilde{f}_1(\theta_1 - \theta_2) \widetilde{f}_2(\theta_1 + \theta_2) \left(\frac{N_2}{N_1}\right)^{\theta_2}$$

$$\times \sum_{n_1, n_2 \geqslant 1} \overline{\omega}_1 \omega_2(n_1) \left(\frac{n_1}{n_2}\right)^{\theta_2} K(n_1 n_2) \left(\frac{N_1 N_2}{n_1 n_2}\right)^{\theta_1} \mathrm{d}\theta_1 \mathrm{d}\theta_2$$

$$= \frac{2}{(2\pi i)^2} \int_{(0)} \left(\frac{N_2}{N_1}\right)^{\theta_2} \sum_{n \geqslant 1} \lambda_{\overline{\omega}_1 \omega_2}(n, \theta_2) K(n) V\left(\frac{n}{N_1 N_2}, \theta_2\right) \mathrm{d}\theta_2,$$

where for any $x \geqslant 0$ and $\Re e(\theta_2) = 0$, we defined

$$(6.24) \qquad V(x, \theta_2) := \int_{(0)} \widetilde{f}_1(\theta_1 - \theta_2) \widetilde{f}_2(\theta_1 + \theta_2) x^{-\theta_1} \mathrm{d}\theta_1.$$

Because the Mellin transforms satisfy (cf. (6.14))

$$(6.25) \qquad \widetilde{f}_1(s), \widetilde{f}_2(s) \ll \left(\frac{q^\varepsilon}{1 + |s|}\right)^B,$$

with an implied constant depending on ε, B and $\Re e(s)$, the function $V(x, \theta_2)$ satisfies

$$V(x, \theta_2) \ll_B \frac{1}{(1 + x)^B} \quad \text{and} \quad x^\nu V^{(\nu)}(x, \theta_2) \ll_{\nu, \varepsilon} q^{\nu \varepsilon},$$

uniformly in $\Re e(\theta_2) = 0$. Since we want to estimate the inner sum in (6.23) using Theorem 1.6 and then average trivially over the θ_2-integral, we also need to control the function $V(x, \theta_2)$ with respect to the θ_2-variable. By (6.25), for any $B \geqslant 1$, we have uniformly on $x > 0$ and with an implied constant depending only on B,

$$V(x, \theta_2) \ll \int_{(0)} \left(\frac{q^\varepsilon}{(1 + |\theta_1 - \theta_2|)(1 + |\theta_1 + \theta_2|)} \right)^B d\theta_1.$$

Note the identity

$$(1 + |\theta_1 - \theta_2|)(1 + |\theta_1 + \theta_2|) = 1 + |\theta_1^2 - \theta_2^2| + |\theta_1 - \theta_2| + |\theta_1 + \theta_2|$$
$$\geqslant 1 + |\theta_1^2 - \theta_2^2| + 2\max(|\theta_1|, |\theta_2|).$$

Hence, splitting the integral depending on whether $|\theta_1| \leqslant |\theta_2|$ or not and we get

$$V(x, \theta_2) \ll \int_{\substack{\Re e(\theta_1) = 0 \\ |\theta_1| \geqslant |\theta_2|}} \left(\frac{q^\varepsilon}{1 + |\theta_1^2 - \theta_2^2| + 2|\theta_1|} \right)^B d\theta_1$$

$$+ \int_{\substack{\Re e(\theta_1) = 0 \\ |\theta_1| \leqslant |\theta_2|}} \left(\frac{q^\varepsilon}{1 + |\theta_1^2 - \theta_2^2| + 2|\theta_2|} \right)^B d\theta_1$$

$$\leqslant \int_{|t| \geqslant |\theta_2|} \left(\frac{q^\varepsilon}{1 + 2|t|} \right)^B dt + \int_{|t| \leqslant |\theta_2|} \left(\frac{q^\varepsilon}{1 + 2|\theta_2|} \right)^B dt$$

$$\ll \left(\frac{q^\varepsilon}{1 + |\theta_2|} \right)^{B-1}.$$

Therefore, for any $\varepsilon' > 0$, we obtain that (6.23) is bounded, up to a constant which depends only on ε', by

$$(6.26) \qquad q^{\varepsilon'} \max_{\substack{|\theta_2| \leqslant q^{\varepsilon'} \\ \Re e(\theta_2) = 0}} \left| \sum_{n \geqslant 1} \lambda_{\overline{\omega}_1 \omega_2}(n, \theta_2) K(n) V\left(\frac{n}{N_1 N_2}, \theta_2 \right) \right|.$$

We now apply Corollary 3.2 with the Schwartz function $V(x, \theta)$ and with the sheaf

$$\mathcal{F} := \mathcal{L}_{\overline{\omega}_2} \otimes [\pm n_0 \overline{\ell}]^* \mathcal{K}\ell_3(\omega_1, \omega_2, 1; q)$$

having trace function (6.22). Note that since $\mathrm{Kl}_3(\cdot, \omega_1, \omega_2, 1; q)$ is invariant under permutation of the triple $(\omega_1, \omega_2, 1)$, we have by (3.11) a geometric isomorphism

$$\mathcal{F} \simeq [\times(\pm n_0 \overline{\ell})]^* \mathrm{FT} \left(\mathcal{L}_{\omega_2} \otimes [x \mapsto x^{-1}]^* \mathcal{K}\ell_2(\omega_1, 1; q) \right)$$

and hence $\mathcal{F}$ is not Fourier-exceptional since by Fourier inversion, its ℓ-adic Fourier transform is a rank 2 irreducible sheaf. It follows that for any $\varepsilon > 0$, we can estimate (6.26) by

$$q^{\varepsilon'} \max_{|\theta_2| \leqslant q^{\varepsilon'}} (qN_1N_2)^{\varepsilon}(1+|\theta_2|)^A N_1 N_2 \left(1 + \frac{q}{N_1N_2}\right)^{1/2} q^{-1/16}.$$

Choosing $\varepsilon' = \varepsilon/A$, maximizing the above quantity by setting $\theta_2 = q^{\varepsilon'}$, replacing the obtained bound in (6.21) and finally, averaging trivially over n_0 in (6.13) yields the first estimate of (6.20)

$$(6.27) \quad \mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll_{\varepsilon} q^{\varepsilon}\left(\frac{N_0N_1N_2}{q}\right)^{1/2}\left(1 + \frac{q}{N_1N_2}\right)^{1/2} q^{-1/16}.$$

For the second bound, we group together the variables $n_0n_2 = m$ in (6.13) and we obtain

$$(6.28) \quad \mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q)$$
$$= \frac{1}{(qN_0N_1N_2)^{1/2}} \sum_{n,n_1} \alpha_m \beta_{n_1} \mathrm{Kl}_3(\pm nn_1\bar{\ell}, \omega_1, \omega_2, 1; q),$$

with

$$\alpha_m := \sum_{n_0n_2=m} \overline{\omega}_2(n_2) f_0\left(\frac{n_0}{N_0}\right) f_2\left(\frac{n_2}{N_2}\right) \quad \text{and} \quad \beta_{n_1} := \overline{\omega}_1(n_1) f_1\left(\frac{n_1}{N_1}\right).$$

Applying Theorem 4.2 (1) with $N = N_0N_2$ and $M = N_2$ gives

$$(6.29) \quad \mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q)$$
$$\ll q^{\varepsilon}\left(\frac{N_0N_1N_2}{q}\right)^{1/2}\left(\frac{1}{q^{1/4}} + \frac{1}{N_1^{1/2}} + \frac{q^{1/4}}{(N_0N_2)^{1/2}}\right),$$

as wishes. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.1.8. Conclusion of the Eisenstein case

Write $N_i = q^{\mu_i}$ with $\mu_i \geqslant 0$ and let $\eta > 0$ be a parameter. If $\mu_0 + \mu_1 + \mu_2 < 1 - 2\eta$ or $\mu_0 + \mu_1 + \mu_2 > 1 + 2\eta$, we use the trivial bound (6.15) or the estimate (6.19) to obtain

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) = O\left(q^{-\eta+\varepsilon}\right).$$

We therefore assume that we are in the range

$$(6.30) \qquad\qquad 1 - 2\eta \leqslant \mu_0 + \mu_1 + \mu_2 \leqslant 1 + 2\eta.$$

Let $\delta > 0$ be an auxiliary parameter. As we already see, there is no loose of generality assuming that $\mu_0 \leqslant \mu_1 \leqslant \mu_2$. Suppose first that

(6.31) $$\mu_0 \leqslant \delta.$$

In this case, we apply (6.27) which, combining with (6.30) and (6.31) gives

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll_\varepsilon q^\varepsilon \left(q^{\eta - \frac{1}{16}} + q^{\frac{\delta}{2} - \frac{1}{16}} \right) \ll_\varepsilon q^{-\eta + \varepsilon},$$

provided

(6.32) $$\eta \leqslant \frac{1}{32} \quad \text{and} \quad \delta \leqslant \frac{1}{8} - 2\eta,$$

which condition we henceforth assume to hold.

Suppose now that we are in the case

(6.33) $$\mu_0 \geqslant \delta.$$

The estimate (6.29) leads to

$$\mathcal{S}_2^{\pm}(\ell, N_0, N_1, N_2; q) \ll q^\varepsilon \left(q^{\eta - \frac{1}{4}} + q^{\frac{1}{2}(\mu_0 + \mu_2 - 1)} + q^{\frac{1}{2}(\mu_1 - \frac{1}{2})} \right).$$

The first term is clearly smaller than $q^{-\eta + \varepsilon}$ by (6.32). For the second, note that $\mu_1 \geqslant \mu_0 \geqslant \delta$ and thus, by (6.30)

$$\mu_0 + \mu_2 - 1 \leqslant 2\eta - \delta.$$

It follows that

$$q^{\varepsilon + \frac{1}{2}(\mu_0 + \mu_2 - 1)} \leqslant q^{\varepsilon + \eta - \frac{\delta}{2}} \leqslant q^{-\eta + \varepsilon},$$

under the assumption that

(6.34) $$\delta \geqslant 4\eta.$$

Finally, the combination of (6.30), $\mu_1 \leqslant \mu_2$ and (6.33) gives

$$\mu_1 \leqslant \frac{1}{2} + \eta - \frac{\delta}{2}$$

and hence

$$q^{\varepsilon + \frac{1}{2}(\mu_1 - \frac{1}{2})} \leqslant q^{\varepsilon + \frac{\eta}{2} - \frac{\delta}{4}} \leqslant q^{-\eta + \varepsilon},$$

provided

(6.35) $$\delta \geqslant 6\eta,$$

which is more restrictive than (6.34). To finalize the computations, we just note that the second condition in (6.32) and (6.35) are simultaneously satisfied as long as $\eta \leqslant \frac{1}{64}$, which gives the correct exponent of the error term in Theorem 1.3.

6.2. The cuspidal case

We consider as in Section 6.1 the average over primitive and even characters (recall that the nebentypus is trivial)

$$(6.36) \qquad \mathscr{T}_{\mathrm{even}}^3(f, \ell; q) := \frac{2}{q-1} \sideset{}{^+}\sum_{\substack{\chi \pmod q \\ \chi \neq \omega_0}} L\left(f \otimes \chi, \tfrac{1}{2}\right) L\left(\chi, \tfrac{1}{2}\right) \chi(\ell).$$

6.2.1. Applying the approximate functional equation

Using Proposition 2.10, we can write (6.36) in the form

$$\mathscr{T}_{\mathrm{even}}^3(f, \ell; q) = \mathcal{C}_1(f, \ell; q) + \varepsilon_\infty(f, +1)\mathcal{C}_2(f, \ell; q)$$

with

$$\mathcal{C}_1(f, \ell; q) = \frac{2}{q-1} \sideset{}{^+}\sum_{\substack{\chi \pmod q \\ \chi \neq \omega_0}} \sum_{n,m \geqslant 1} \frac{\lambda_f(n)\chi(nm\ell)}{(nm)^{1/2}} \mathbf{V}_{f,\chi}\left(\frac{nm}{q^{3/2}}\right),$$

$$\mathcal{C}_2(f, \ell; q) = \frac{2}{q-1} \sideset{}{^+}\sum_{\substack{\chi \pmod q \\ \chi \neq \omega_0}} \sum_{n,m \geqslant 1} \frac{\overline{\lambda_f(n)}\overline{\chi}(nm)\chi(\ell)}{(nm)^{1/2}} \varepsilon(\chi)^3 \mathbf{V}_{f,\chi}\left(\frac{nm}{q^{3/2}}\right),$$

where we recall that $\mathbf{V}_{f,\chi}$ depends on χ only through its parity. Since we assume that f satisfies the Ramanujan–Petersson conjecture, we have $|\lambda_f(n)| \leqslant \tau(n)$. Hence, proceeding as in Section 6.1.2 for the average over the characters and writing $\mathbf{V} = \mathbf{V}_{f,\chi}$, we find

$$\mathcal{C}_i(f, \ell; q) = \sum_{\pm} \mathcal{C}_i^{\pm}(f, \ell; q) + O\left(q^{-1/4+\varepsilon}\right),$$

where

$$(6.37) \qquad \mathcal{C}_1^{\pm}(f, \ell; q) = \sideset{}{^*}\sum_{nm\ell \equiv \pm 1 \pmod q} \frac{\lambda_f(n)}{(nm)^{1/2}} \mathbf{V}\left(\frac{nm}{q^{3/2}}\right),$$

and

$$(6.38) \qquad \mathcal{C}_2^{\pm}(f, \ell; q) = \frac{1}{q^{1/2}} \sideset{}{^*}\sum_{n,m \geqslant 1} \frac{\overline{\lambda_f(n)}}{(nm)^{1/2}} \mathrm{Kl}_3(\pm nm\overline{\ell}; q)\mathbf{V}\left(\frac{nm}{q^{3/2}}\right).$$

6.2.2. The main term

The extraction of the main term is done is a similar way as in Section 6.1.3. We just conclude with

$$\mathcal{C}_1^+(f,\ell;q) = \delta_{\ell=1} + O\left(\ell q^{-1/4+\varepsilon}\right), \quad \mathcal{C}_1^-(f,\ell;q) = O\left(\ell q^{-1/4+\varepsilon}\right).$$

Note that the error terms are $O(q^{-\frac{1}{52}+\varepsilon})$ (cf. Theorem 1.3) if we assume that

$$(6.39) \qquad \ell \leqslant q^{\frac{1}{4}-\frac{1}{52}} = q^{\frac{3}{13}}.$$

6.2.3. The error term

Applying a partition of unity to (6.38), removing the test function $\mathbf{V}$ using its integral representation (see Section 6.1.4) and we are reduced to analyze $O(\log^2 q)$ sums of the shape

$$(6.40) \quad \mathcal{C}_2^\pm(f,N,M;q)$$
$$:= \frac{1}{(qNM)^{1/2}} \sideset{}{^*}\sum_{n,m\in\mathbf{Z}} \overline{\lambda_f(n)} \mathrm{Kl}_3(\pm nm\overline{\ell};q) W_1\left(\frac{m}{M}\right) W_2\left(\frac{n}{N}\right),$$

where W_i are smooth and compactly supported functions on $(1/2,2)$ such that $x^j W_i^{(j)}(x) \ll_{\varepsilon,j} q^{\varepsilon j}$ for all $j \geqslant 0$ and M, N are reals numbers with the standard restriction due to the fast decay of $\mathbf{V}$ at infinity

$$1 \leqslant M, N \quad\text{and}\quad NM \leqslant q^{3/2+\varepsilon}.$$

Note that the trivial bound is

$$(6.41) \qquad \mathcal{C}_2^\pm(f,N,M;q) \ll \left(\frac{NM}{q}\right)^{1/2}.$$

Moreover, if $M \geqslant q/2$, then an application of Polyá–Vinogradov method in the m-variable (see Proposition 4.1 and Section 6.1.5) leads to

$$\mathcal{C}_2^\pm(f,N,M;q) \ll_\varepsilon q^\varepsilon \left(\frac{NM}{q^2}\right)^{1/2} \ll q^{-1/4+\varepsilon}.$$

Hence we can suppose from now on that $M < q/2$ in such a way that the condition $(m,q) = 1$ under the summation in (6.38) is automatically satisfied.

6.2.4. Application of Poisson/Voronoi summation formula

The first step is to apply Voronoi summation formula in the n-variable. To get in a good position, we write the Kloosterman sum Kl_3 for $(a, q) = 1$ in the form

$$(6.42) \qquad \mathrm{Kl}_3(a; q) = \frac{1}{q^{1/2}} \sum_{x \in \mathbf{F}_q^\times} \mathrm{Kl}_2(\overline{x}; q) e\left(\frac{ax}{q}\right).$$

Note that this definition can be extended to $a = 0$ with the value

$$\mathrm{Kl}_3(0; q) = \frac{1}{q} \left(\sum_{x \in \mathbf{F}_q^\times} e\left(\frac{x}{q}\right)\right)^2 = \frac{1}{q}.$$

It follows that after writing $\mathrm{Kl}_3(\pm nm\overline{\ell}; q)$ in the form (6.42) and adding the contribution of $q | n$ for negligible error term (of size at most $q^{-3/4+\varepsilon}$), we get

$$\mathcal{C}_2^\pm(f, N, M; q) = \frac{1}{(qNM)^{1/2}} \frac{1}{q^{1/2}} \sum_{x \in \mathbf{F}_q^\times} \mathrm{Kl}_2(\overline{x}; q)$$

$$(6.43)$$

$$\times \sum_{m \in \mathbf{Z}} W_1\left(\frac{m}{M}\right) \sum_{n \geqslant 1} \overline{\lambda_f(n)} e\left(\frac{\pm nm\overline{\ell}x}{q}\right) W_2\left(\frac{n}{N}\right).$$

Assuming we are dealing with the plus case and applying Voronoi formula (cf. Proposition 2.13) to the inner sum in (6.43), we obtain

$$\mathcal{C}_2^+(f, N, M; q) = \left(\frac{N}{q^3 M}\right)^{1/2} \frac{1}{q^{1/2}} \sum_{\pm} \sum_{x \in \mathbf{F}_q^\times} \mathrm{Kl}_2(\overline{x}; q) \sum_{n \geqslant 1} \overline{\lambda_f(n)} W_2^\pm\left(\frac{nN}{q^2}\right)$$

$$\times \sum_{m \in \mathbf{Z}} e\left(\frac{\mp n\overline{m}\overline{x}\ell}{q}\right) W_1\left(\frac{m}{M}\right).$$

Changing the order of summation, making the change of variable $\overline{x} \leftrightarrow xm$ (recall that $(m, q) = 1$) allows us to write

$$(6.44) \quad \mathcal{C}_2^+(f, N, M; q)$$

$$= \left(\frac{N}{q^3 M}\right)^{1/2} \frac{1}{q^{1/2}} \sum_{\pm} \sum_{x \in \mathbf{F}_q^\times} \sum_{n \geqslant 1} \overline{\lambda_f(n)} e\left(\frac{\mp nx\ell}{q}\right) W_2^\pm\left(\frac{nN}{q^2}\right)$$

$$\times \sum_{m \in \mathbf{Z}} \mathrm{Kl}_2(xm; q) W_1\left(\frac{m}{M}\right).$$

By Poisson formula and since Kl_2 is the Fourier transform of the function defined by (3.12), we see that the m-sum in (6.44) is equal to

$$\frac{M}{q^{1/2}} \sum_{(m,q)=1} e\left(-\frac{x\overline{m}}{q}\right) \widehat{W_1}\left(\frac{mM}{q}\right).$$

Replacing this identity in (6.44) yields

$$(6.45) \quad \mathcal{C}_2^+(f, N, M; q)$$

$$= \left(\frac{NM}{q^3}\right)^{1/2} \sum_{\pm} \sum_{n \geqslant 1} \sum_{(m,q)=1} \overline{\lambda_f(n)} \widehat{W_1}\left(\frac{mM}{q}\right) W_2^{\pm}\left(\frac{nN}{q^2}\right)$$

$$\times \frac{1}{q} \sum_{x \in \mathbf{F}_q^{\times}} e\left(x \frac{\mp n\ell - \overline{m}}{q}\right),$$

with the same expression for the minus case $\mathcal{C}_2^-$, but with $\mp$ replaced by $\pm$ in the exponential. Because of the fast decay of $\widehat{W_1}$ and $W_2^{\pm}$ at infinity (cf. Lemma 2.3), the n, m-sum (6.45) is essentially supported on $|m| \leqslant q^{1+\varepsilon}/M$ and $|n| \leqslant q^{2+\varepsilon}/N$. In this range, we use the estimate $|\lambda_f(n)| \leqslant \tau(n) \ll_\varepsilon n^\varepsilon$ and we apply Lemma 2.3 with $\vartheta = 0$ (recall that f satisfies R-P-C) to bound $W_2^{\pm}$ by q^ε. Adding the contribution of $x = 0$, estimating this extra factor trivially and executing the complete x-summation gives

$$\mathcal{C}_2^+(f, N, M; q) = \left(\frac{NM}{q^3}\right)^{1/2} \sum_{\pm} \sum_{nm\ell \equiv \mp 1 \ (\mathrm{mod}\ q)} \overline{\lambda_f(n)} \widehat{W_1}\left(\frac{mM}{q}\right)$$

$$\times W_2^{\pm}\left(\frac{nN}{q^2}\right) + O\left(q^\varepsilon \left(\frac{q}{NM}\right)^{1/2}\right).$$

Therefore, as in Section 6.1.6, we obtain

$$(6.46) \qquad \mathcal{C}_2^{\pm}(f, N, M; q) \ll_\varepsilon q^\varepsilon \left(\frac{q}{NM}\right)^{1/2},$$

uniformly on $\ell < q$.

6.2.5. Estimation of $\mathcal{C}_2$ using bounds for bilinear forms and Theorem 1.6

We finally state the analogous of Proposition 6.3 which is an immediate application of Theorem 4.2 (1)-(2), Theorem 4.3 and Corollary 3.2.

PROPOSITION 6.4. — *For any $\varepsilon > 0$, the quantity defined in (6.40) satisfies*

$$\mathcal{C}_2^{\pm}(f, N, M; q) \ll q^{\varepsilon} \left(\frac{NM}{q}\right)^{1/2} \begin{cases} \frac{1}{q^{1/4}} + \frac{1}{M^{1/2}} + \frac{q^{1/4}}{N^{1/2}} \\[2mm] \frac{1}{q^{1/2}} + \frac{q^{1/2}}{M} \\[2mm] \left(\frac{N^2 M^5}{q^3}\right)^{-1/12} \\[2mm] \left(1 + \frac{q}{N}\right)^{1/2} q^{-1/16}, \end{cases}$$

where the implied constant depends on ε and polynomially on t_f in the last bound and the third bound is valid in the case where $1 \leqslant N \leqslant M^2$, $M < q$ and $NM < q^{3/2}$ (cf. (4.1)).

6.2.6. Conclusion of the cuspidal case

Fix $\eta > 0$ a parameter and write $M = q^{\mu}$, $N = q^{\nu}$ with $\mu, \nu \geqslant 0$. By the trivial bound (6.41) and (6.46), we can assume that

(6.47) $$1 - 2\eta \leqslant \mu + \nu \leqslant 1 + 2\eta,$$

otherwise we get $\mathcal{C}_2^{\pm}(f, N, M; q) = O(q^{-\eta + \varepsilon})$. We now let $\delta_1, \delta_2, \delta_3 > 0$ be sufficiently small auxiliary parameters and we distinguish four cases :

(a) Assume that $\mu \leqslant \delta_1$. In this case we apply the fourth estimate of Proposition 6.4 and we get by (6.47)

$$\mathcal{C}_2^{\pm}(f, N, M; q) \ll_{\varepsilon, t_f} q^{\varepsilon} \left(q^{\eta - \frac{1}{16}} + q^{\frac{\delta_1}{2} - \frac{1}{16}}\right) \leqslant q^{-\eta + \varepsilon},$$

provided

(6.48) $$\eta \leqslant \frac{1}{32} \quad \text{and} \quad \delta_1 \leqslant \frac{1}{8} - 2\eta.$$

(b) If $\delta_1 < \mu \leqslant \frac{1}{2} - \delta_2$, the first bound of Proposition 6.4 yields

$$\mathcal{C}_2^{\pm}(f, N, M; q) \ll_{\varepsilon} q^{\varepsilon} \left(q^{\eta - \frac{1}{4}} + q^{\frac{1}{2}(\nu - 1)} + q^{\frac{1}{2}(\mu - \frac{1}{2})}\right).$$

The first term is less than $q^{\varepsilon - \eta}$ since $\eta \leqslant \frac{1}{32}$. For the second, we have $\nu - 1 \leqslant 2\eta - \delta_1$ (use (6.47) and $\mu \geqslant \delta_1$). Thus it is less than $q^{\varepsilon - \eta}$ under the assumption that

(6.49) $$\delta_1 \geqslant 4\eta.$$

The third term is at most $q^{-\delta_2/2} \leqslant q^{-\eta}$ if

(6.50) $$\delta_2 \geqslant 2\eta.$$

(c) Suppose that $\frac{1}{2} - \delta_2 < \mu \leqslant \frac{1}{2} + \delta_3$. In this configuration, we apply the third bound and we obtain

$$\mathcal{C}_2^{\pm}(f, N, M; q) \ll_{\varepsilon} q^{\varepsilon - \frac{1}{4} + \frac{\nu}{3} + \frac{\mu}{12}} = q^{\varepsilon - \frac{1}{4} + \frac{1}{12}(\mu + \nu) + \frac{\nu}{4}}.$$

Using (6.47) and $\nu \leqslant 1 + 2\eta - \mu \leqslant \frac{1}{2} + 2\eta + \delta_2$ allows us to bound the above expression by

$$q^{\varepsilon - \frac{1}{4} + \frac{1}{12}(1 + 2\eta) + \frac{1}{4}(\frac{1}{2} + 2\eta + \delta_2)} = q^{\varepsilon - \frac{1}{12}(\frac{1}{2} - 8\eta - 3\delta_2)} \leqslant q^{\varepsilon - \eta},$$

provided

(6.51) $$3\delta_2 \leqslant \frac{1}{2} - 20\eta.$$

(d) Assume that $\mu > \frac{1}{2} + \delta_3$ the second bound gives

$$\mathcal{C}_2^{\pm}(f, N, M; q) \ll_{\varepsilon} q^{\varepsilon} \left(q^{\eta - \frac{1}{2}} + q^{\eta + \frac{1}{2} - \mu} \right) \leqslant q^{\varepsilon - \eta} + q^{\varepsilon + \eta - \delta_3} \ll q^{\varepsilon - \eta},$$

if we assume that

$$\delta_3 \geqslant 2\eta.$$

Finally, the combination of conditions (6.48) and (6.49) forces $\eta \leqslant \frac{1}{48}$ and (6.50)–(6.51) are simultaneously satisfied as long as $\eta \leqslant \frac{1}{52}$, which gives the correct exponent of the error term in Theorem 1.3.

Remark 6.5. — The treatment carried out in Section 5.1 remains almost identical if f is level 1 Hecke cusp form. The only change we have to make is to replace the exponent $1/16$ by $1/8$ in the fourth bound of Proposition 6.4, which is due to the original Theorem [14, Theorem 1.2] for small level compared with q. However, it does not improve the final exponent $\frac{1}{52}$ since (6.50)–(6.51) is more restrictive and independent of (6.48)–(6.49).

7. Proof of Theorems 1.1 and 1.2

7.1. The mollification method

We show here how to derive Theorems 1.1 and 1.2 from Theorem 1.3. Let $1 < L < q$ be a real number such that $\log L \asymp \log q$. For any multiplicative character $\chi \pmod{q}$, we define the short linear form

(7.1) $$M(\chi; L) := \sum_{\ell \leqslant L} \frac{\chi(\ell)\mu(\ell)}{\ell^{1/2}} \left(\frac{\log L/\ell}{\log L} \right)^2,$$

where μ is the Möbius function. Let $\{\lambda_f(n)\}_{n \geqslant 1}$ denotes the sequence of Hecke eigenvalues of a Hecke cusp form of level one and $\mu_f(n)$ be the convolution inverse of $\lambda_f(n)$ given by

$$L(f, s)^{-1} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}} \right) = \sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^s} \ , \ \Re e(s) > 1.$$

For $1 < L' < q$ with $\log L' \asymp \log q$, we also define

$$(7.2) \qquad M(f \otimes \chi; L') := \sum_{\ell \leqslant L'} \frac{\chi(\ell)\mu_f(\ell)}{\ell^{1/2}} \left(\frac{\log L'/\ell}{\log L'} \right).$$

We finally consider the two mollified cubic moments

$$(7.3) \qquad \mathscr{M}^3(\omega_1, \omega_2; q) := \frac{1}{q-1} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq 1, \overline{\omega}_1, \overline{\omega}_2}} \prod_{i=0}^{2} L\left(\chi\omega_i, \tfrac{1}{2} \right) M(\chi\omega_i; L),$$

and

$$(7.4) \ \mathscr{M}^3(f; q) := \frac{1}{q-1} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq 1}} L\left(f \otimes \chi, \tfrac{1}{2} \right) M(f \otimes \chi; L') L\left(\chi, \tfrac{1}{2} \right) M(\chi; L).$$

Note that (7.3) and (7.4) can be written in the form

$$\mathscr{M}^3(\omega_1, \omega_2; q) = \sum_{\ell_1, \ell_2, \ell_3 \leqslant L} \frac{\boldsymbol{x}(\ell_1)\boldsymbol{x}(\ell_2)\boldsymbol{x}(\ell_3)}{(\ell_1\ell_2\ell_3)^{1/2}} \mathscr{T}^3(\omega_1, \omega_2, \ell_1\ell_2\ell_3; q),$$

$$\mathscr{M}^3(f; q) = \sum_{\ell \leqslant L, \ell' \leqslant L'} \frac{\boldsymbol{x}_f(\ell)\boldsymbol{x}(\ell')}{(\ell\ell')^{1/2}} \mathscr{T}^3(f, \ell\ell'; q),$$

with

$$\boldsymbol{x}(\ell) := \mu(\ell) \left(\frac{\log L/\ell}{\log L} \right)^2 \quad \text{and} \quad \boldsymbol{x}_f(\ell') := \mu_f(\ell') \left(\frac{\log L/\ell'}{\log L} \right).$$

Since f satisfies the Ramanujan–Petersson conjecture, we have for $1 \leqslant \ell \leqslant L$ and $1 \leqslant \ell' \leqslant L'$,

$$|\boldsymbol{x}(\ell)| \leqslant 1 \quad \text{and} \quad |\boldsymbol{x}_f(\ell')| \leqslant \tau(\ell'),$$

where $\tau(n) = \sum_{d|n} 1$ is the divisor function. Hence an immediate consequence of Theorem 1.3 is the following corollary :

COROLLARY 7.1. — For any $\varepsilon > 0$, the mollified cubic moments (7.3) and (7.4) satisfies

$$\mathscr{M}^3(\omega_1, \omega_2; q) = 1 + O\left(L^{3/2} q^{-\frac{1}{64} + \varepsilon} \right), \ \mathscr{M}^3(f; q) = 1 + O\left((L'L)^{1/2} q^{-\frac{1}{52} + \varepsilon} \right),$$

where the implied constant only depends on $\varepsilon > 0$ and polynomially on t_f in the second expression.

In [37, Theorem 1.2], we established an asymptotic formula for a mollified fourth moment of Dirichlet L-functions : for $L = q^\lambda$ with $0 < \lambda < \frac{11}{8064}$, we obtained

$$(7.5) \quad \mathscr{M}^4(q) := \frac{1}{q-1} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \neq \omega_0}} \left| L\left(\chi, \tfrac{1}{2}\right) M(\chi; L) \right|^4 = P(\lambda^{-1}) + o_\lambda(1),$$

where $P(X) \in \mathbf{R}[X]$ is a degree four polynomial with calculable coefficients.

Similarly, in a paper of preparation *The second moment theory of families of L-functions : The case of twisted Hecke L-functions* by Blomer, Fouvry, Kowalski, Michel, Milićević and Sawin, they obtained

$$
\begin{aligned}
\mathscr{M}^4(f; q) :&= \frac{1}{q-1} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \neq \omega_0}} \left| L\left(f \otimes \chi, \tfrac{1}{2}\right) M(f \otimes \chi; L') \right|^2 \\
&= 2\eta \left(1 + \frac{2}{\lambda'} \right) + o_{\lambda', t_f}(1),
\end{aligned}
$$

(7.6)

for $L' = q^{\lambda'}$ with $0 < \lambda' < \frac{1}{360}$ and η is an absolute constant satisfying $\eta \leqslant \zeta(3/2)$. Hence, combining Corollary 7.1 with (7.5) and (7.6) yields

Proof of Theorems 1.1 and 1.2.

The Dirichlet characters case. — We first present the proof of Theorem 1.1. For any $\chi \,(\mathrm{mod}\ q)$, we define the characteristic function

$$\mathbf{1}(\chi) := \delta_{|L(\chi, 1/2)| \geqslant \frac{1}{\log q}} \delta_{|L(\chi\omega_1, 1/2)| \geqslant \frac{1}{\log q}} \delta_{|L(\chi\omega_2, 1/2)| \geqslant \frac{1}{\log q}}.$$

Using the generalized Hölder's inequality, we infer

$$
\left| \frac{1}{q-1} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \neq \omega_0, \overline{\omega}_1, \overline{\omega}_2}} \mathbf{1}(\chi) \prod_{i=0}^{2} L\left(\chi\omega_i, \tfrac{1}{2}\right) M(\chi\omega_i; L) \right|
$$

$$
\leqslant \left(\frac{1}{q-1} \sum_{\chi \,(\mathrm{mod}\ q)} \mathbf{1}(\chi) \right)^{1/4} \left(\mathscr{M}^4(q) \right)^{3/4}.
$$

On the other hand, we have the lower bound for the left handside

$$
\left| \frac{1}{q-1} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \neq \omega_0, \overline{\omega}_1, \overline{\omega}_2}} \mathbf{1}(\chi) \prod_{i=0}^{2} L\left(\chi\omega_i, \tfrac{1}{2}\right) M(\chi\omega_i; L) \right| \geqslant \left| \mathscr{M}^3(\omega_1, \omega_2; q) \right| - \mathscr{D},
$$

where $\mathscr{M}^3(\omega_1, \omega_2; q)$ is defined in (7.3) and

$$\mathscr{D} := \frac{1}{q-1} \sum_{\substack{\chi \neq \omega_0, \overline{\omega}_1, \overline{\omega}_2 \\ \mathbf{1}(\chi)=0}} \left| \prod_{i=0}^{2} L\left(\chi\omega_i, \tfrac{1}{2}\right) M(\chi\omega_i; L) \right|.$$

To estimate $\mathscr{D}$, note that the condition $\mathbf{1}(\chi) = 0$ means that one of the central values is less than $\log(q)^{-1}$. Therefore, if for $i = 0, 1, 2$, $\mathscr{D}_i$ is the subsum of $\mathscr{D}$ restricted to χ such that $|L(\chi\omega_i, \tfrac{1}{2})| \leqslant \log(q)^{-1}$, we obtain, by positivity, $\mathscr{D} \leqslant \mathscr{D}_0 + \mathscr{D}_1 + \mathscr{D}_2$ with for each $i = 0, 1, 2$,

$$\mathscr{D}_i \leqslant \frac{1}{\log(q)} \left(\frac{1}{q-1} \sum_{\chi \,(\mathrm{mod}\ q)} |M(\chi; L)|^2 \right)^{1/2}$$

$$\times \left(\frac{1}{q-1} \sum_{\chi \,(\mathrm{mod}\ q)}^{*} \left| L\left(\chi, \tfrac{1}{2}\right) M(\chi; L) \right|^4 \right)^{1/2}$$

$$\ll_\lambda \frac{1}{\log(q)} \left(\frac{1}{q-1} \sum_{\chi \,(\mathrm{mod}\ q)} |M(\chi; L)|^2 \right)^{1/2},$$

using again twice Cauchy–Schwarz inequality and (7.5) (recall that $L = q^\lambda$). Moreover, opening the square in $|M(\chi; L)|^2$ and applying the orthogonality relation yields

$$\frac{1}{q-1} \sum_{\chi \,(\mathrm{mod}\ q)} |M(\chi; L)|^2 \leqslant \sum_{\substack{\ell \equiv \ell' \,(\mathrm{mod}\ q) \\ \ell, \ell' \leqslant L}} \frac{|\boldsymbol{x}(\ell)\boldsymbol{x}(\ell')|}{(\ell\ell')^{1/2}} \leqslant \sum_{\ell \leqslant L} \frac{1}{\ell} \ll \log L,$$

since $L < q$. Hence, assuming $L = q^\lambda$ with $0 < \lambda < \frac{11}{8064}$, we get

$$\frac{1}{q-1} \sum_{\chi \,(\mathrm{mod}\ q)} \mathbf{1}(\chi) \geqslant \frac{\left| \mathscr{M}^3(\omega_1, \omega_2; q) \right|^4}{\mathscr{M}^4(q)^3} + O_\lambda \left(\frac{1}{\log(q)^{1/2}} \right)$$

$$= \frac{1}{P(\lambda^{-1})^3} + o_\lambda(1).$$

If

$$c_1 := \max_{0 < \lambda \leqslant \frac{11}{8064}} P(\lambda^{-1})^{-3},$$

then for any $\varepsilon > 0$, there exists $0 < \widetilde{\lambda} < \frac{11}{8064}$ depending on ε satisfying $|P(\widetilde{\lambda}^{-1})^{-3} - c_1| \leqslant \varepsilon/2$. Finally, choosing $Q = Q(\varepsilon)$ large enough such that $|o_{\widetilde{\lambda}}(1)| \leqslant \varepsilon/2$ for $q \geqslant Q$ and the result follows.

The cuspidal case. — We proceed in a similar way. Setting

$$\mathbf{1}(\chi, f) := \delta_{|L(\chi, 1/2)| \geqslant \frac{1}{\log q}} \delta_{|L(f \otimes \chi, 1/2)| \geqslant \frac{1}{\log^2 q}},$$

we obtain

$$\left| \frac{1}{q-1} \sum_{\chi \neq 1} \mathbf{1}(\chi, f) L\left(f \otimes \chi, \tfrac{1}{2}\right) M(f \otimes \chi; L') L\left(\chi, \tfrac{1}{2}\right) M(\chi; L) \right|$$

$$\leqslant \left(\frac{1}{q-1} \sum_{\chi \pmod q} \mathbf{1}(\chi, f) \right)^{1/4} \left(\mathscr{M}^4(q)\right)^{1/4} \left(\mathscr{M}^4(f; q)\right)^{1/2},$$

where $\mathscr{M}^4(q)$ (resp. $\mathscr{M}^2(f; q)$) are defined by (7.5) (resp. by (7.6)). As in the previous part, the left handside admits the lower bound

$$\geqslant \left| \mathscr{M}^3(f; q) \right| - \mathscr{C},$$

where $\mathscr{C}$ is the same as $\mathscr{M}^3(f; q)$, but with the absolute values inside and with the restriction in the summation to χ such that $\mathbf{1}(\chi, f) = 0$. Writing $\mathscr{C}_1$ (resp. $\mathscr{C}_2$) for the contribution of $|L(\chi, \tfrac{1}{2})| \leqslant \frac{1}{\log q}$ (resp. $|L(f \otimes \chi, \tfrac{1}{2})| \leqslant \frac{1}{\log^2 q}$), we get $\mathscr{C} \leqslant \mathscr{C}_1 + \mathscr{C}_2$ with

$$\mathscr{C}_1 = O_\lambda \left(\frac{1}{\log(q)^{1/2}} \right).$$

Finally, we have

$$\mathscr{C}_2 \leqslant \frac{1}{\log^2(q)} \left(\frac{1}{q-1} \sum_{\chi \pmod q} |M(f \otimes \chi; L')|^2 \right)^{1/2}$$

$$\times \left(\frac{1}{q-1} \sum_{\chi \pmod q}^* \left| L\left(\chi, \tfrac{1}{2}\right) M(\chi; L) \right|^2 \right)^{1/2}$$

$$\ll_\lambda \frac{1}{\log^2(q)} \left(\frac{1}{q-1} \sum_{\chi \pmod q} |M(f \otimes \chi; L')|^2 \right)^{1/2},$$

with

$$\frac{1}{q-1} \sum_{\chi \pmod q} |M(f \otimes \chi; L')|^2 \leqslant \sum_{\substack{\ell \equiv \ell' \pmod q \\ \ell, \ell' \leqslant L'}} \frac{\tau(\ell) \tau(\ell')}{(\ell \ell')^{1/2}}$$

$$= \sum_{\ell \leqslant L'} \frac{\tau(\ell)^2}{\ell} \ll \log^3 L'.$$

Hence,

$$\mathscr{C}_2 = O_{\lambda'}\left(\frac{1}{\log(q)^{1/2}}\right),$$

and the rest of the proof is exactly the same as in the previous case. $\qquad\square$

Remark 7.2. — Let f be a primitive Hecke cusp form of prime level q satisfying the Ramanujan–Petersson conjecture. The formula (1.4) could be used to prove simultaneous non-vanishing for $L(f \otimes \chi, \frac{1}{2})L(\chi, \frac{1}{2})$ as χ runs over non trivial Dirichlet characters modulo q provided that it is possible to evaluate a second twisted moment of the form

$$\frac{1}{q-1}\sum_{\substack{\chi \pmod q \\ \chi \neq 1}} \left|L\left(f \otimes \chi, \tfrac{1}{2}\right)\right|^2 \chi(\ell_1)\overline{\chi}(\ell_2),$$

where $(\ell_1, \ell_2) = 1$ and are coprime with q. An asymptotic formula for this moment is given in [1] in the special case where the level is 1 and $\ell_1 = \ell_2 = 1$ and for general $(\ell_1, \ell_2) = 1$ in the paper of preparation mentioned above (also for level 1). The principal difficulty here is that since the level is q, we have the solve a shifted convolution problem of the shape

$$\sum\sum_{\ell_1 n - \ell_2 m = hq} \lambda_f(n)\lambda_f(m)W_1\left(\frac{n}{N}\right)W_2\left(\frac{m}{M}\right),$$

for Hecke eigenvalues $\lambda_f(n)$ of level q.

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