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EXTENDING REGULAR FOLIATIONS (*)

by J. Wolfgang SMITH

1. Introduction.

In this paper M shall denote an open orientable differentiable n -manifold. To fix our ideas, we take « differentiable » to mean C^∞ throughout, and we suppose M to be Hausdorff and have empty boundary ⁽¹⁾. Let F denote a p -dimensional differentiable foliation ⁽²⁾ on M , i.e. a completely integrable smooth p -dimensional differential system on M with $0 < p < n$. Thus F assigns to every $x \in M$ a p -dimensional subspace of the tangent vectorspace M_x , and moreover, every point of M lies on a unique p -dimensional maximal integral manifold of F (in the sense of Chevalley [1]). These integral manifolds will be referred to as the *leaves* of F , and we let $\pi: M \rightarrow M/F$ denote the natural projection of M onto the quotient space M/F obtained by identifying points belonging to the same leaf. The foliation is called *regular* if π admits local cross-sections. For a regular foliation F the quotient M/F can be regarded as a differentiable m -manifold (with $m = n - p$), and π will then be differentiable. However, M/F will not in general be Hausdorff. The manifold M being orientable, we can define orientability for F by the condition that M/F be orientable, and this will henceforth be assumed. The tangent bundle $\tau(M/F)$ has then an Euler class ⁽³⁾ χ_F , whose algebraic sign depends

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⁽¹⁾ None of these suppositions is in fact crucial.

⁽²⁾ For basic terminology and results relating to foliations we refer to Palais [5], Chapter 1.

⁽³⁾ Milnor [4]. Although Milnor takes the base space to be Hausdorff, his construction of the Euler class does not depend upon this assumption.

upon a choice of orientation. Assuming such a choice (or alternatively taking χ_F to be determined modulo algebraic sign) we shall refer to χ_F as the *Euler class* of F . It may be remarked that the notion of an Euler class applies to nonregular foliations as well, but it cannot in general be defined in terms of a bundle over the quotient space ⁽⁴⁾.

Let us suppose, for the moment, that M/F is Hausdorff. Then it is known to be triangulable, and by the classical obstruction theory ⁽⁵⁾ it will admit a nonzero vectorfield (or a direction field) if and only if χ_F is zero. Moreover, a direction field on M/F pulls back under π to a $(p + 1)$ — dimensional orientable foliation \hat{F} on M such that $F \subset \hat{F}$ in the obvious sense. Such a foliation \hat{F} will be called an *extension* of F . Conversely one sees that an extension of F gives rise to a direction field on M/F (whose algebraic sign is determined by a choice of orientations). When M/F is not Hausdorff it is still true that the existence of direction fields on M/F is equivalent to the existence of extensions of F , and it is also true that the vanishing of χ_F constitutes a necessary condition for the existence of these structures. The question of sufficiency, however, appears to be open. The main result of this paper asserts sufficiency in the following weakened sense :

THEOREM A. — *An orientable regular foliation on M with vanishing Euler class extends on relatively compact subsets of M .*

Thus when $\chi_F = 0$ an extension \hat{F} of F exists at least on all relatively compact subsets $D \subset M$. It may be noted that no corresponding solution of the direction field problem for arbitrary non-Hausdorff manifolds can be envisaged.

At this point one is naturally interested to find geometric conditions on M and F which imply that $\chi_F = 0$. The following leads to one such set of conditions.

LEMMA B. — *Let F be a regular 1-dimensional foliation on M without compact leaves. Then $\pi : M \rightarrow M/F$ induces an isomorphism between the respective singular homology groups.*

⁽⁴⁾ Smith [6].

⁽⁵⁾ Steenrod [9].

We note that the Euler class of a q -plane bundle has order 2 when q is odd ⁽⁶⁾. The notation $H^q(M) = Q$ will signify that the q -dimensional singular integral cohomology of M vanishes or has no torsion of order 2, depending on whether q is even or odd, respectively. Combining Theorem A with Lemma B thus gives :

THEOREM B. — *Let F be a regular orientable 1-dimensional foliation on M without compact leaves. If $H^{n-1}(M) = Q$, then F extends on relatively compact subsets of M .*

This result can be sharpened if one replaces F by a corresponding vectorfield. A nonzero differentiable vectorfield on M will be called *nonrecurrent* if the induced foliation is regular and admits no compact leaves. Using Theorem B we establish the following results in [8] :

THEOREM C. — *Let X be a nonrecurrent vectorfield on M and let $D \subset M$ be relatively compact. If $H^{n-1}(M) = Q$, there exists a vectorfield Y on D such that X, Y are linearly independent and commute.*

THEOREM D. — *If $H^{n-1}(M) = Q$, then every relatively compact subset of M submerges in the plane.*

The present paper is devoted to a proof of Theorem A and Lemma B. An essential ingredient in our proof of Theorem A is a triangulation theorem which may also be of independent interest. We are greatly indebted to J. R. Munkres for having contributed the Appendix to this paper, setting forth a proof of this result.

2. Equivariant vectorfields.

Let E_0, \dots, E_s denote differentiable m -manifolds. For each pair (i, j) of indices, let

$$\varphi_{ij} : (U_{ij}, A_{ij}) \rightarrow (U_{ji}, A_{ji})$$

denote a diffeomorphism, where $U_{ij} \subset E_i$ is open and $A_{ij} \subset U_{ij}$

⁽⁶⁾ Milnor [4], p. 41. This again does not involve the assumption of a Hausdorff base space.

is compact. We shall assume that this family of diffeomorphisms satisfies the pseudo-group conditions

$$\begin{aligned} \varphi_{ii} &= \text{identity} \\ \varphi_{ik} &= \varphi_{jk} \circ \varphi_{ij} \quad (\text{whenever the composition is defined}). \end{aligned}$$

Let E denote the disjoint union of the manifolds E_i . A vectorfield X on E will be called A -equivariant provided

$$\begin{array}{ccc} TA_{ij} & \xrightarrow{d\varphi_{ij}} & TA_{ji} \\ X \uparrow & & \uparrow X \\ A_{ij} & \xrightarrow{\varphi_{ij}} & A_{ji} \end{array}$$

commutes for every index pair (i, j) . Here TA_{ij} denotes the total space of the tangent bundle $\tau(E)$ restricted to A_{ij} , and $d\varphi_{ij}$ denotes the differential of φ_{ij} .

THEOREM 1. — *Let E be oriented and the φ_{ij} orientation preserving. Let N be an orientable differentiable m -manifold (not necessarily Hausdorff) with vanishing Euler class, and $\pi: E \rightarrow N$ an immersion such that*

$$(2.1) \quad \begin{array}{ccc} U_{ij} & \xrightarrow{\varphi_{ij}} & U_{ji} \\ \pi \searrow & & \swarrow \pi \\ & N & \end{array}$$

commutes for every index pair (i, j) . Then E admits a differentiable nonzero A -equivariant vectorfield.

By the triangulation theorem of J. R. Munkres (see Appendix) there exist triangulations K_i of E_i and finite subcomplexes L_{ij} of K_i such that

- i) $A_{ij} \subset |L_{ij}| \subset U_{ij}$
- ii) φ_{ij} maps $|L_{ij}|$ simplicially onto $|L_{ji}|$.

We shall establish the existence of a differentiable nonzero vectorfield X on E such that

$$(2.2) \quad \begin{array}{ccc} T|L_{ij}| & \xrightarrow{d\varphi_{ij}} & T|L_{ji}| \\ X \uparrow & & \uparrow X \\ |L_{ij}| & \xrightarrow{\varphi_{ij}} & |L_{ji}| \end{array}$$

commutes for all (i, j) . Since $A_{ij} \subset |L_{ij}|$, X will then be A -equivariant.

Let R_L denote the equivalence relation on E generated by all pairs $(x, y) \in |L_{ij}| \times |L_{ji}|$ such that $y = \varphi_{ij}(x)$. Let \bar{E} denote the quotient space E/R_L and $\alpha: E \rightarrow \bar{E}$ the projection. We observe that the given triangulation K of E induces a triangulation of \bar{E} . Moreover, by the usual clutching construction ⁽⁷⁾ the bundle $\tau(E)$ induces an orientable m -plane bundle ξ over \bar{E} , together with a bundle map $h: \tau(E) \rightarrow \xi$ over α . In other words, for every pair (i, j) were are « glueing together » the bundle spaces $T|L_{ij}|$ and $T|L_{ji}|$ via $d\varphi_{ij}$. The total space $T\xi$ of ξ is thus a quotient of TE , and $h: TE \rightarrow T\xi$ a projection. Now let

$$(x, \bar{x}, \varpi) \in E \times \bar{E} \times N$$

such that $\bar{x} = \pi(x)$ and $\varpi = \alpha(x)$. Let $h_x: E_x \rightarrow \xi_x$, $d\pi_x: E_x \rightarrow N_w$ denote the fibre isomorphism induced by the bundle maps h and $d\pi$, respectively, and let $g_x = d\pi_x \circ (h_x)^{-1}$. Commutativity of the diagrams (2.1) implies that g_x depends only on \bar{x} , and one obtains thus a function $g: T\xi \rightarrow TN$ which restricts to an isomorphism on the fibres. Since $d\pi = g \circ h$ and h is a projection, g is continuous because $d\pi$ is continuous. Hence $g: \xi \rightarrow \tau(N)$ is a bundle map.

It now follows by naturality that ξ has vanishing Euler class. Moreover, since \bar{E} is triangulable, one obtains thus a nonzero cross-section $\bar{X}: \bar{E} \rightarrow T\xi$ by classical obstruction theory ⁽⁸⁾ and this pulls back under h to a nonzero vectorfield X^0 on E . But our construction clearly implies commutativity of the diagrams (2.2).

However, the construction does not guarantee differentiability of X^0 . We will complete the argument by showing that X^0 can be approximated by a smooth vectorfield X without losing the commutativity conditions (2.2). Let $A \subset E$ denote an open relatively compact subset containing $|L_{ij}|$ for all (i, j) , and let $B = E - A$. It is not difficult to see that there exists then a nonzero differentiable vectorfield X^*

⁽⁷⁾ Husemoller [3], p. 123.

⁽⁸⁾ Steenrod [9], § 39.6.

on E which agrees with X^0 on every $|L_{ij}|$ and is differentiable on B . For every index k we let

$$V_k = \bigcup_{i \leq k} E_i, \quad W_k = \bigcup_{i \leq k} |L_{ki}|;$$

and we let \mathcal{X}_k denote the set of all nonzero differentiable vectorfields X defined on $B \cup V_k$ such that i) X agrees with X^* on B , ii) the diagram (2.2) commutes for all pairs (i, j) with $i, j \leq k$. We observe that every $X \in \mathcal{X}_{k-1}$ determines a nonzero differentiable vectorfield \hat{X} on $B \cup W_k$ (in an obvious way). We would like to argue that if X is « sufficiently close » to X^* , \hat{X} will be near enough to X^* to extend to a nonzero vectorfield on $B \cup E_k$. By a well known result ⁽⁹⁾ this would imply that \hat{X} extends to a nonzero *differentiable* vectorfield on $B \cup E_k$, and consequently that X extends to a vectorfield in \mathcal{X}_k . To make this precise, let ρ denote a Riemannian metric on $\tau(E)$. For every $S \subset E$ and vectorfield X defined on S , let

$$\mu(X|S) = \text{l.u.b.}_{x \in S} \{\rho(X_x, X_x^*)\},$$

where X_x, X_x^* denote the respective tangent vectors at x . Compactness of the spaces W_k and relative compactness of A permit us to make the following observations:

1) There exists a constant $\lambda > 1$ such that

$$\mu(X|W_k) < \lambda \mu(X|V_{k-1})$$

for all $k > 0$ and $X \in \mathcal{X}_{k-1}$.

2) There exists an $\varepsilon > 0$ such that for every $k > 0$ and $X \in \mathcal{X}_{k-1}$ with $\mu(X|W_k) < \delta < \varepsilon$, X extends to a vectorfield $Y \in \mathcal{X}_k$ with $\mu(Y|E_k) < \delta$.

But this does the trick. For we can choose $X_0 \in \mathcal{X}_0$ such that

$$\mu(X_0|E_0) < \frac{\varepsilon}{\lambda^s}.$$

This implies by 1) that

$$\mu(\hat{X}_0|W_1) < \frac{\varepsilon}{\lambda^{s-1}} < \varepsilon$$

⁽⁹⁾ Steenrod [9], § 6.7.

and hence by 2) that X_0 extends to $X_1 \in \mathcal{K}_1$ with

$$\mu(X_1|V_1) < \frac{\varepsilon}{\lambda^{x-1}}$$

By induction one thus obtains a vectorfield $X_s \in \mathcal{K}_s$.

3. Proof of Theorem A.

For $r > 0$ and q a positive integer let J_r^q denote the open cube in \mathbb{R}^q given by

$$\sum_{i=1}^q |t_i| < r,$$

where the t_i denote natural coordinates in \mathbb{R}^p . A differentiable chart $\psi: J_r^m \times J_s^p \rightarrow M$ will be called *flat* (with respect to the foliation F) if for every $u \in J_r^m$ the points $\{\psi(u, v) | v \in J_s^p\}$ lie on a single leaf of F . A flat chart ψ is *regular* if every leaf of F meets $\psi(J_r^m \times J_s^p)$ in at most one connected component. Since F is regular, every point of M is covered by a flat regular chart.

Let D be a relatively compact subset of M . There exists then a finite family of flat regular charts $\psi_j: J_2^m \times J_1^p \rightarrow M$ such that $\{\psi_j(J_2^m \times J_1^p)\}$ constitutes a covering of D . Let

$$E_j = \psi_j(J_2^m \times 0), \quad B_j = \psi_j(J_1^m \times 0) \\ V = \bigcup_j \psi_j(J_2^m \times J_1^p), \quad W = \bigcup_j \psi_j(J_1^m \times J_1^p);$$

and let \bar{B}_i, \bar{W} denote the respective closures. Thus every E_j is diffeomorphic to an open m -cube and $\bar{B}_j \subset E_j$ is compact. Moreover, W is an open subset of M containing D . For any subset $S \subset M$, let R_S denote the equivalence relation on S consisting of all pairs (x, y) such that x and y are connected by a curve in S lying in a single leaf of F . For every index pair (i, j) let U_{ij} denote the set of all $x \in E_i$ such that $(x, y) \in R_V$ for some $y \in E_j$. We note that this point y is uniquely determined by x (regularity of ψ_j), so that one obtains functions $\varphi_{ij}: U_{ij} \rightarrow U_{ji}$. Similarly, for every pair (i, j) let A_{ij} denote the set of all $x \in \bar{B}_i$ such that $(x, y) \in R_{\bar{W}}$ for some $y \in \bar{B}_j$. The following assertions are easily verified.

LEMMA 3.1. — *For every index pair (i, j) , $U_{ij} \subset E_i$ is open and $A_{ij} \subset U_{ij}$ compact. Each φ_{ij} constitutes a diffeomorphism of U_{ij} onto U_{ji} and maps A_{ij} onto A_{ji} . The family of these diffeomorphisms satisfies the pseudogroup conditions (as given in Theorem 1).*

The disjoint union E of the spaces E_j constitutes a differentiable m -manifold, and we note that E can be oriented so as to render every φ_{ij} orientation preserving. Moreover, the natural projection $\pi: E \rightarrow M/F$ commutes with every φ_{ij} . By Lemma 3.1 and Theorem 1 one concludes that E admits a differentiable nonzero A -equivariant vectorfield X . Since $D \subset W$, it will suffice to prove:

LEMMA 3.2. — *X induces an extension of F on W .*

Let F_0 denote the restriction of F to W and $\beta: W \rightarrow W/F_0$ the natural projection. Thus W/F_0 constitutes a differentiable m -manifold (not necessarily Hausdorff), and β maps each B_j diffeomorphically onto a subset $V_j \subset W/F_0$. The restriction of X to B_j consequently induces a nonzero differentiable vectorfield Y_j on V_j . Moreover, A -equivariance of X implies that Y_i, Y_j agree on $V_i \cap V_j$ for every index pair (i, j) . For if $v \in V_i \cap V_j$ and x, y denote the corresponding points in B_i and B_j , respectively, then $(x, y) \in R_W \subset R_{\bar{W}}$. Hence $y = \varphi_{ij}(x)$ and $X_y = d\varphi_{ij}(X_x)$ by A -equivariance. Since β commutes with φ_{ij} , it follows that $d\beta(X_x) = d\beta(X_y)$, as claimed. But the subsets $\{V_j\}$ cover W/F_0 , and one obtains thus a nonzero differentiable vectorfield Y on W/F_0 , which in turn determines a 1-dimensional foliation H . Finally, H pulls back under $\beta: W \rightarrow W/F_0$ to an orientable $(p+1)$ -dimensional foliation ⁽¹⁰⁾ on W which extends F .

4. Proof of Lemma B.

Let F be a regular 1-dimensional foliation on M without compact leaves, and let $\pi: M \rightarrow N$ denote the natural projection, where $N = M/F$. Neither M nor F are required

⁽¹⁰⁾ By Palais [5], Chapter I, Theorem XIII.

to be orientable. It will be shown that the induced map $\pi_{\#}: C_{\#}(M) \rightarrow C_{\#}(N)$ between the respective singular chain complexes constitutes a chain homotopy equivalence.

We observe that this assertion is quite trivial in case N is Hausdorff. Choosing a complete Riemannian metric on M determines ⁽¹¹⁾ a bundle structure for $\pi: M \rightarrow N$ with fibre R (the real line) and structure group G consisting of all transformations of the form $t \rightarrow (\pm t + a)$, with $a \in R$. The fibre being contractible and N being a Hausdorff manifold implies ⁽¹²⁾ that there exists a cross-section $s: N \rightarrow M$. The restriction of π to $s(N)$ is then a homeomorphism, and $s(N)$ is clearly a deformation retract of M . Thus one obtains the desired conclusion. On the other hand, if N is not Hausdorff, a cross-section of π may not exist. Consider M , for example, to be a punctured plane foliated by a parallel family of straight lines. The leaf space N is then the real line with a single point doubled, and it is clear that a cross-section $s: N \rightarrow M$ does not exist.

To prove Lemma B, we choose a complete Riemannian metric on M and an open covering \mathcal{V} of N such that every $V \in \mathcal{V}$ admits a local cross-section $s_V: V \rightarrow M$. The metric, together with s_V , permits us to define a projection $p_V: \pi^{-1}(V) \rightarrow R$, and this gives a homeomorphism $\theta_V: \pi^{-1}(V) \rightarrow V \times R$ by setting $\theta_V(x) = (\pi(x), p_V(x))$.

Let $C_{\#}(N, \mathcal{V})$ denote the subcomplex of $C_{\#}(N)$ generated by singular simplexes subordinate to \mathcal{V} . The inclusion $C_{\#}(N, \mathcal{V}) \rightarrow C_{\#}(N)$ is then a chain homotopy equivalence ⁽¹³⁾. Similarly we let $\mathcal{W} = \pi^{-1}(\mathcal{V})$ and observe that the inclusion $C_{\#}(M, \mathcal{W}) \rightarrow C_{\#}(M)$ is likewise a chain homotopy equivalence. It will therefore clearly suffice to show that

$$\pi_{\#}: C_{\#}(M, \mathcal{W}) \rightarrow C_{\#}(N, \mathcal{V})$$

is a chain homotopy equivalence. This will be accomplished by constructing a chain map $\tau: C_{\#}(N, \mathcal{V}) \rightarrow C_{\#}(M, \mathcal{W})$ which preserves singular simplexes and satisfies $\pi_{\#} \circ \tau = 1$. In other words, instead of constructing a cross-section $s: N \rightarrow M$ (which may not exist), we construct a chain cross-section τ

⁽¹¹⁾ Smith [8].

⁽¹²⁾ Steenrod [9], § 12.2.

⁽¹³⁾ Eilenberg and Steenrod [2], Theorem 8.2.

for $\pi_{\#}: C_{\#}(M, \mathcal{W}) \rightarrow C_{\#}(N, \mathcal{V})$. But τ determines a chain homotopy $D: C_{\#}(M, \mathcal{W}) \rightarrow C_{\#}(M, \mathcal{W})$ by the following construction: Let $\sigma: \Delta_q \rightarrow M$ be a singular q -simplex subordinate to \mathcal{W} and let $\sigma_0 = \tau \circ \pi_{\#}(\sigma)$. For every $y \in \Delta_q$, the points $\sigma_0(y)$ and $\sigma(y)$ belong to the same leaf of F . Since every leaf of F is homeomorphic to \mathbb{R} , the two points determine a homeomorph $[\sigma_0(y), \sigma(y)]$ of a directed line segment. We can therefore define a singular prism $P_{\sigma}: \Delta_q \times I \rightarrow M$ (I denotes the unit interval) by letting $P_{\sigma}(y, t)$ be the point in $[\sigma_0(y), \sigma(y)]$ which divides this segment in the ration $t: 1$, this being understood in terms of the distance function on $[\sigma_0(y), \sigma(y)]$ induced by our Riemannian metric. Continuity of P_{σ} is immediate, and by the usual process ⁽¹⁴⁾ the correspondance $\sigma \rightarrow P_{\sigma}$ determines a chain homotopy D . Moreover, one can verify by an easy calculation that

$$\partial D_q + D_{q-1} \partial = 1 - \tau \circ \pi_{\#},$$

where ∂ and 1 denote the boundary operator and identity map of $C_{\#}(M, \mathcal{W})$, respectively. It remains, therefore, to establish the existence of τ .

To this end we make the inductive hypothesis that $\tau_q: C_q(N, \mathcal{V}) \rightarrow C_q(M, \mathcal{W})$ has been defined for all $q < r$, subject to the conditions

$$(4.1) \quad \tau_{q-1} \circ \partial_q = \partial_q \circ \tau_q.$$

More precisely, for every singular q -simplex $\sigma: \Delta_q \rightarrow N$ subordinate to \mathcal{V} , $\tau_q(\sigma)$ is assumed to be a singular q -simplex $\bar{\sigma}: \Delta_q \rightarrow M$ such that $\pi \circ \bar{\sigma} = \sigma$. Now let $\sigma: \Delta_r \rightarrow V$ denote a singular r -simplex, with $V \in \mathcal{V}$. The function τ_{r-1} determines then a map $h_{\sigma}: \dot{\Delta}_r \rightarrow M$ by virtue of condition (4.1), where $\dot{\Delta}_r$ denotes the boundary of Δ_r . This defines a map $p_V \circ h_{\sigma}: \dot{\Delta}_r \rightarrow \mathbb{R}$, which can be extended to a map $\varphi_{\sigma}: \Delta_r \rightarrow \mathbb{R}$. Let $g_{\sigma}: \Delta_r \rightarrow M$ be defined by setting $g_{\sigma}(y) = \theta_{\varphi}^{-1}(\sigma(y), \varphi_{\sigma}(y))$. One now has a commutative diagram

$$\begin{array}{ccc} & & \pi^{-1}(V) \\ & \nearrow h_{\sigma} & \nearrow g_{\sigma} \\ \dot{\Delta}_r & \rightarrow \Delta_r & \xrightarrow{\sigma} V \\ & & \downarrow \pi \end{array}$$

⁽¹⁴⁾ Eilenberg and Steenrod [2], Chapter VII, § 6.

Setting $\tau_r(\sigma) = g_\sigma$ defines τ_r on the generators of $C_r(N, \mathcal{V})$, and we extend by linearity. It is obvious that τ_r is a simplex preserving cross-section of $\pi_\#$, and commutativity of (4.2) implies condition (4.1) with $q = r$. This establishes the existence of τ .

Appendix.

(This appendix was written by J. R. Munkres.)

DEFINITION. — Let $f_i: |K_i| \rightarrow R^m$ be a homeomorphism where K_i is a finite complex and $i = 1, \dots, n$. We say that $(K_1, f_1), \dots, (K_n, f_n)$ intersect in subcomplexes if for each (i, j) , $f_i^{-1}(f_i(|K_i|) \cap f_j(|K_j|))$ is the polytope of a subcomplex L_{ij} of K_i and if $f_j^{-1}f_i$ is a linear isomorphism of L_{ij} with L_{ji} . They are said to intersect in full subcomplexes if each L_{ij} is full in K_i . (This means that a simplex of K_i belongs to L_{ij} if all its vertices are in L_{ij} .) It is easy to see that if $(K_1, f_1), \dots, (K_n, f_n)$ intersect in subcomplexes, then $(K'_1, f_1), \dots, (K'_n, f_n)$ intersect in full subcomplexes, where K'_i is the first barycentric subdivision of K_i .

If $(K_1, f_1), \dots, (K_n, f_n)$ intersect in full subcomplexes, then there exists a complex K and a homeomorphism $f: K \rightarrow R^m$ such that $f(|K|) = \bigcup_j f_j(|K_j|)$ and such that $f^{-1}f_j$ is a linear isomorphism of K_j with a subcomplex of K for each j . Furthermore, (K, f) is unique up to linear isomorphism. It is called the *union* of $(K_1, f_1), \dots, (K_n, f_n)$. (Compare 10.1 of [EDT].)

Now suppose that $(K_1, f_1), \dots, (K_n, f_n)$ intersect in full subcomplexes and that each $f_i: K_i \rightarrow R^m$ is a smooth imbedding, in the sense of 8.3 of [EDT]. This means not only that it is a topological imbedding which is smooth on each simplex of K_i , but also that the differential is one-to-one. The union (K, f) will not be an imbedding except under additional hypotheses. (See 10.1 of [EDT].) However, one can say the following:

LEMMA 1. — Let M_i be a subcomplex of K_i such that $f_i(|M_i|) \subset \text{Int } f_i(|K_i|)$. Then the union of $(M_1, f_1), \dots, (M_n, f_n)$ is a smooth imbedding.

Proof. — Let (K, f) be the union of $\{(K_i, f_i)\}$; the union M of $\{(M_i, f_i)\}$ may be taken as a subcomplex of K . Let x be a point of M_i . Then $f_i: K_i \rightarrow \mathbb{R}^m$ triangulates a neighborhood of $f_i(x)$, and so does $f: K \rightarrow \mathbb{R}^m$, so that $f^{-1}f_i: K_i \rightarrow K$ is a homeomorphism of $\overline{\text{St}}(x, K_i)$ with $\overline{\text{St}}(x, K)$. Since $f_i: \overline{\text{St}}(x, K_i) \rightarrow \mathbb{R}^m$ is an imbedding, $d(f_i)_x$ is 1-1, and hence so is df_x .

LEMMA 2. — *Let A be a closed subset of the differentiable manifold M . Let $f: K \rightarrow M$ be a smooth imbedding such that $A \subset \text{Int } f(|K|)$. If there is a subcomplex K_0 of K such that $f|K_0$ triangulates A , then $f|K_0$ may be extended to a triangulation of M .*

This lemma is problem 10.7 of [EDT]. It can be proved by straightforward application of the triangulation techniques of J. H. C. Whitehead expounded there.

THEOREM. — *Let E_0, \dots, E_n be differentiable m -manifolds. Suppose that for each pair (i, j) of indices, we are given a diffeomorphism*

$$\varphi_{ij}: (U_{ij}, A_{ij}) \rightarrow (U_{ji}, A_{ji}),$$

where U_{ij} is an open subset of E_i and A_{ij} is a compact subset of U_{ij} . Furthermore, φ_{ii} is the identity and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ whenever the composition is defined. (This implies $U_{ik} \supset \text{domain}(\varphi_{jk} \circ \varphi_{ij})$.)

Then there are smooth triangulations K_i of E_i , and finite subcomplexes L_{ij} of K_i , such that

- i) $A_{ij} \subset |L_{ij}| \subset U_{ij}$
- ii) φ_{ij} maps $|L_{ij}|$ simplicially onto $|L_{ji}|$.

Proof. — We proceed by induction on n . The theorem is trivial for $n = 0$. Suppose it is true for $n - 1$.

Choose compact sets B_{0j} ($j = 1, \dots, n$) such that

$$A_{0j} \subset \text{Int } B_{0j} \quad \text{and} \quad B_{0j} \subset U_{0j}.$$

Then for $1 \leq i < j \leq n$, choose a compact set $B_{ij} \subset U_{ij}$ such that

$$A_{ij} \subset B_{ij} \quad \text{and} \quad \varphi_{0i}(B_{0i} \cap B_{0j}) \subset B_{ij}.$$

This makes sense because

$$B_{0i} \cap B_{0j} \subset U_{0i} \cap U_{0j} = \varphi_{i0} (\text{domain} (\varphi_{0j} \circ \varphi_{i0})) \subset \varphi_{i0}(U_{ij}),$$

so that $\varphi_{0i}(B_{0i} \cap B_{0j}) \subset U_{ij}$. Finally, for $0 \leq i < j \leq n$, set $B_{ji} = \varphi_{ij}(B_{ij})$.

Now apply the induction hypothesis to the manifolds E_1, \dots, E_n , using

$$\varphi_{ij} : (U_{ij}, B_{ij}) \rightarrow (U_{ji}, B_{ji})$$

as the diffeomorphisms. We then have complexes K_i smoothly triangulating E_i , and subcomplexes L_{ij} of K_i ($1 \leq i, j \leq n$) such that $B_{ij} \subset |L_{ij}| \subset U_{ij}$ and φ_{ij} is a linear isomorphism of L_{ij} with L_{ji} .

We then proceed to triangulate E_0 . First, we may assume that *mesh* K_i is less than one-third the distance from A_{i0} to $E_i - B_{i0}$, for $i = 1, \dots, n$. (For this situation may be obtained by choosing a very large p and replacing each K_i and L_{ij} by its p th barycentric subdivision.) This means that for $i = 1, \dots, n$, we may choose subcomplexes L_{i0} , M_{i0} , and N_{i0} of K_i such that

$$A_{i0} \subset |L_{i0}| \subset \text{Int} |M_{i0}| \quad \text{and} \quad |M_{i0}| \subset \text{Int} |N_{i0}| \subset B_{i0}.$$

Consider the maps $\varphi_{i0} : N_{i0} \rightarrow E_0$. Because φ_{i0} is a diffeomorphism on U_{i0} and N_{i0} is a smoothly imbedded complex in E_i , this map is a smooth imbedding of N_{i0} in E_0 . We claim also these maps intersect in subcomplexes: For $\varphi_{i0}(N_{i0}) \cap \varphi_{j0}(N_{j0}) \subset B_{0i} \cap B_{0j}$, so that $\varphi_{i0}^{-1}(\varphi_{i0}(N_{i0}) \cap \varphi_{j0}(N_{j0}))$ is contained in $B_{ij} \subset L_{ij}$. This implies that

$$\varphi_{i0}^{-1}(\varphi_{i0}(N_{i0}) \cap \varphi_{j0}(N_{j0})) = N_{i0} \cap \varphi_{ji}(N_{j0} \cap L_{ji}),$$

which is clearly a subcomplex of K_i (since φ_{ji} is by assumption simplicial on L_{ji}). Furthermore, the map $\varphi_{j0}^{-1}\varphi_{i0}$ is a linear isomorphism of this subcomplex of K_i with a subcomplex of K_j , since the subcomplex is contained in L_{ij} and the map equals φ_{ij} there.

Without change of notation, let us replace each K_i , L_{i0} , M_{i0} , N_{i0} , and L_{ij} ($1 \leq i, j \leq n$) by its first barycentric subdivision. The maps $\varphi_{i0} : N_{i0} \rightarrow E_0$ are still smooth imbeddings but now they intersect in full subcomplexes.

By Lemma 1, the union

$$\varphi_0 : M_0 \rightarrow E_0 \quad \text{of} \quad (M_{10}, \varphi_{10}), \dots, (M_{n0}, \varphi_{n0})$$

is now an imbedding. The union of $(L_{10}, \varphi_{10}), \dots, (L_{n0}, \varphi_{n0})$ may be considered as a subcomplex L_0 of M_0 , and $\varphi_0(|L_0|)$ lies in the interior of $\varphi_0(|M_0|)$. By Lemma 2, $\varphi_0 : L_0 \rightarrow E_0$ may be extended to a smooth triangulation of E_0 . Said differently, there is a complex K_0 smoothly triangulating E_0 such that φ_0 is a linear isomorphism of L_0 with a subcomplex of K_0 . Then φ_{i0} is a linear isomorphism of L_{i0} with a subcomplex of K_0 which we denote by L_{0i} .

The proof of the theorem is now complete.

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