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Miyuki KOISO, Paolo PICCIONE & Toshihiro SHODA

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ON BIFURCATION AND LOCAL RIGIDITY OF TRIPLY PERIODIC MINIMAL SURFACES IN $\mathbb{R}^3$

by Miyuki KOISO, Paolo PICCIONE & Toshihiro SHODA (*)

ABSTRACT. — We study the space of triply periodic minimal surfaces in $\mathbb{R}^3$, giving a result on the local rigidity and a result on the existence of bifurcation. We prove that, near a triply periodic minimal surface with nullity three, the space of triply periodic minimal surfaces consists of a smooth five-parameter family of pairwise non-homothetic surfaces. On the other hand, if there is a smooth one-parameter family of triply periodic minimal surfaces $\{X_t\}_t$ containing X_0 where the Morse index jumps by an odd integer, it will be proved the existence of a bifurcating branch issuing from X_0. We also apply these results to several known examples.

RÉSUMÉ. — Nous étudions l'espace des surfaces minimales triplement périodiques dans $\mathbb{R}^3$, obtenant un résultat sur la rigidité locale ainsi que sur l'existence de bifurcation. Nous démontrons que, près d'une surface minimale triplement périodique de nullité 3, l'espace des surfaces minimales triplement périodiques est une famille lisse à cinq paramètres de surfaces deux à deux non homothétiques. D'autre part, s'il y a une famille lisse à un paramètre de surfaces minimales triplement périodiques $\{X_t\}_t$ contenant X_0, dont l'indice de Morse saute d'un entier impair, ceci démontrera l'existence d'une branche bifurquant depuis X_0. Nous appliquons aussi ces résultats à plusieurs exemples connus.

1. Introduction

Construction of examples and classification of triply periodic minimal surfaces (TPMS) in $\mathbb{R}^3$ constitute a very active field of research in Differential Geometry. Such surfaces correspond, via universal covering, to

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minimal embeddings of closed orientable surfaces into a flat torus $\mathbb{T}^3$. As originally conjectured by Meeks, see [20], it is known that every closed surface of genus greater than or equal to 3 admits a minimal embedding in any flat torus $\mathbb{T}^3$, see [30]. The purpose of this paper is to give one of the first steps of the study of the structure of the space of TPMS's. We will prove that, near a TPMS with nullity three, the space of all TPMS's in $\mathbb{R}^3$ consists of a smooth five-parameter family of pairwise non-homothetic surfaces (Proposition 3.3). Note that the dimension of this space is related to the dimension of the space of (non-isometric) unit volume flat metrics on a 3-torus $\mathbb{T}^3$. Moreover, using bifurcation techniques, we discuss the existence of singularities near a TPMS whose nullity is greater than three. Roughly speaking, if there is a smooth one-parameter family of TPMS's $\{X_t\}_t$ containing a given TPMS X_0 where the Morse index has an odd jump, then the space of TPMS's contains an infinite set that lies outside of the five-parameter family, and that accumulates on X_0 (Theorem 5.4).

Interestingly enough, the question of stability is a central issue in the theory of minimal surfaces, since the pioneering work of Plateau on experimental and theoretical research on the equilibrium shapes of a liquid mass without gravity. Bifurcation phenomena, defined in terms of *spontaneous transformation* of families of constant mean curvature surfaces, are described in the historical paper [24]. TPMS's appear naturally in several applied sciences, including physics, chemistry, and crystallography, see for instance [2, 8, 9, 28]. G. E. Schröder-Turk, A. Fogden, and S. T. Hyde [10] studied one-parameter families of triply periodic minimal surfaces in $\mathbb{R}^3$. These families which are called H-family, rPD-family, tP-family, tD-family, tCLP-family, rG-family, and tG-family contain many classical examples (Schwarz P-surface, Schwarz D-surface, Schwarz H-surface, Schwarz CLP-surface, and Alan Schoen's gyroid). All TPMS's in these families are of genus three. On the other hand, the genus of any orientable stable TPMS in $\mathbb{R}^3$ is three, since the Morse index of any orientable stable TPMS is 1 (see §5 for the definition of the Morse index) and since closed orientable minimal surfaces with Morse index 1 immersed in an orientable 3-manifold with nonnegative Ricci curvature have genus $\leqslant 3$ (Ros [25]). Moreover, the lowest genus of orientable TPMS's is three (Meeks [20, Corollay 3.1, Theorem 3.1]). In view of these facts, it is interesting to study the space of all TPMS's with genus three. It is proved that some of the families of the above mentioned examples intersect (N. Ejiri and T. Shoda [7]. See §6). We will discuss the structure of the space of TPMS's near these families (Conjecture B, Corollary C and Theorem D. See also §8).

Since the pioneering work of Plateau, many of the subsequent works on bifurcation for minimal or CMC surfaces have relied mostly on experimental techniques. Recently, theoretical results have been obtained using perturbation techniques, and in particular variational bifurcation theory; the aim of this paper is to apply these techniques to the theory of TPMS. Bifurcation theory has been successfully employed to determine multiplicity of solutions for geometric variational problem, see for instance [3] and the references therein for applications to the Yamabe problem. Thus, employing bifurcation theoretical techniques in the context of TPMS's seems extremely promising, and in the present paper we take the first steps towards this goal. In order to apply our results to concrete examples (§7), we will try to use some recent results of N. Ejiri and T. Shoda, see [7, 6], who prove a finite dimensional reduction to compute the nullity and the Morse index for the above families of minimal embeddings. Let us describe more precisely the results discussed in this paper.

1.1. Bifurcation

Given a one-parameter family $]a_0-\varepsilon, a_0+\varepsilon[\ \ni a \mapsto g_a$ of (unit volume) flat metrics in $\mathbb{T}^3$, and a one-parameter family $a \mapsto x_a$ of g_a-minimal embeddings $x_a : \Sigma \to \mathbb{T}^3$, then a *bifurcating branch of minimal embeddings converging to* x_{a_0} for the family $(x_a)_a$ consists of:

- a sequence a_n tending to a_0 as $n \to \infty$;
- a sequence $x_n \colon \Sigma \to \mathbb{T}^3$ of embeddings, where x_n is g_{a_n}-minimal for all n,

such that

(1) $\lim_{n\to\infty} x_n = x_{a_0}$ in some suitable C^k-topology, with $k \geqslant 2$;
(2) x_n is not *congruent* to x_{a_n} for all n.

Let us also recall that two embeddings $y_1, y_2 : \Sigma \to \mathbb{T}^3$ are congruent if one is obtained from the other by a change of parameterization of Σ and by a translation of $\mathbb{T}^3$, i.e., if there exists a diffeomorphism ψ of Σ and $\mathfrak{t} \in \mathbb{T}^3$ such that $y_1 = y_2 \circ \psi + \mathfrak{t}$.

In the situation above, we say that a_0 is a *bifurcation instant* for the family $(x_a)_a$. An important related notion is that of *local rigidity* for a family of minimal embeddings, based on the notion of *equivariant nondegeneracy*, see Section 1.2 below. If $(x_a)_a$ is a continuous family of equivariantly nondegenerate g_a-minimal embeddings, then the *Morse index* of x_a (as a critical point of the area functional) is constant.

The abstract result proved in the paper gives a sufficient condition for the local rigidity and for the existence of a bifurcation instant for a one-parameter family of minimal embeddings into flat tori.

Theorem A. — *Let Σ be a closed orientable surface of genus greater than 1, let $(g_a)_{a\in]a_0-\varepsilon,a_0+\varepsilon[}$ be a continuous 1-parameter family of unit volume flat metrics on $\mathbb{T}^3$, and let $x_a:\Sigma\to\mathbb{R}^3$, be a continuous family of embeddings such that x_a is g_a-minimal for all a.*

(1) *Assume the following:*
- *x_a is equivariantly nondegenerate for all $a\neq a_0$;*
- *given $a'\in\,]a_0-\varepsilon,a_0[$ and $a''\in\,]a_0,a_0+\varepsilon[$, the difference between the Morse indices of $x_{a'}$ and of $x_{a''}$ is an odd integer.*

Then, a_0 is a bifurcation instant for the family $(x_a)_a$.

(2) *If x_a is equivariantly nondegenerate for all $a\in\,]a_0-\varepsilon,a_0+\varepsilon[$, then the family $(x_a)_a$ is locally rigid. This implies that near x_0, all triply periodic minimal surfaces in $\mathbb{R}^3$ consist of six-parameter family of surfaces. If we restrict ourselves to the unit volume lattice, then, near x_0, all triply periodic minimal surfaces consist of a five-parameter family of pairwise non-homothetic surfaces.*

The two statements of Theorem A will be proved under slightly more general assumptions, see Proposition 3.3 for the local rigidity statement, and Theorem 5.4 for the bifurcation statement.

We will recall in Section 6 the definitions of the H-family, the rPD-family, the tP-family, the tD-family and the tCLP-family of triply periodic minimal embeddings. Applications of part (1) of Theorem A to the above families yields the following conjecture which is very plausible (see §7):

Conjecture B. — *There is one bifurcation instant for the H-family and two bifurcation instants for the rPD-family, the tP-family, and the tD-family.*

In order to apply Theorem A to these families of TPMS's, we will employ some recent results obtained in [7, 6] for the computation of Morse index and nullity of minimal embeddings in the above families. Let us recall that the main result in [7, 6] is a finite dimensional reduction for the computation of the nullity and the Morse index of triply periodic minimal surfaces. This reduction, which is proved on a theoretical ground, allows to compute nullity and index in terms of finite dimensional (9×9, 18×18) hermitian/symmetric matrices. Once this theoretical result has been established, some numerical methods have been used in [7] to obtain approximations for the bifurcation instants for each of the families above. As to the jump of Morse index, the numerical algorithms employed in [7] to determine the sign of eigenvalues of symmetric matrices offer quite a solid basis for Conjecture B.

Theorem A and Conjecture B can be interpreted in terms of the *moduli space* $\mathcal{M}(\Sigma, \mathbb{T}^3)$ of minimal embeddings of a given compact surface Σ into a flat 3-torus. Let Σ be a closed orientable surface $\mathcal{M}(\Sigma)$ of genus equal to 3, and define $\mathcal{M}(\Sigma, \mathbb{T}^3)$ as the space of minimal embeddings of Σ into some flat 3-torus, modulo congruence and homotheties. The notion of equivariant nondegeneracy can be extended in an obvious way to the points of $\mathcal{M}(\Sigma, \mathbb{T}^3)$; this is clearly an open condition. The families tP, tD, and rPD contain only at most a countable set of equivariantly degenerate surfaces. This follows easily from the fact that these families are real-analytic, and so are the eigenvalues of their Jacobi operators. Note that tP and rPD families contain Schwarz P-surface, tD and rPD families contain Schwarz D-surface, and P and D surfaces have nullity three (see [26]).

Observe that equivariant degeneracy is equivalent to the (simultaneous) vanishing of at least 4 eigenvalues, counted with muliplicity, of the Jacobi operator. Since each one of these families contains at least one equivariantly nondegenerate surface, and the zeros of nonzero real-analytic functions are isolated, the set of equivariantly degenerate surfaces belonging to one of the families above is at most countable. Also every minimal surface in the tCLP-family has nullity equal to 3 (see Remark 1.2). Thus, Theorem A says that $\mathcal{M}(\Sigma, \mathbb{T}^3)$ contains a large open subset which has the structure of a 5-dimensional real analytic manifold, but Conjecture B suggests that $\mathcal{M}(\Sigma, \mathbb{T}^3)$ also admits points where nontrivial singularities occur.

1.2. Equivariant nondegeneracy and local rigidity

The first important issue is the question of *degeneracy* of the minimal embeddings in the above families, caused by the symmetries of the ambient space $\mathbb{T}^3$. Every minimal embedding admits a three dimensional space of (nontrivial) Jacobi fields, coming from the Killing fields of the ambient, which implies that each one of the embeddings is a degenerate critical point of the area functional. Recall that standard variational bifurcation assumptions require nondegeneracy at the endpoints of the path, which fails to hold in this situation. The central technical part of the paper consists in the construction of an alternative functional framework, suitable to handle such degenerate situation. To this aim, we use an idea originally introduced by Kapouleas to prove an implicit function theorem for constant mean curvature embeddings, which consists in considering a "perturbed" mean curvature function $\widetilde{\mathcal{H}}$ of an embedding, see Section 2.2 for details. Such a function vanishes identically exactly at minimal embeddings (Proposition 2.1), nevertheless maintains its differential surjective at possibly degenerate minimal embeddings, provided that the degeneracy arises exclusively from the ambient symmetries (Proposition 4.1). We call *equivariantly nondegenerate* a minimal embedding along which every Jacobi field arises from a Killing field of the ambient space, see Definition 3.2. A proof of statement (2) in Theorem A is obtained as a direct application of an equivariant implicit function theorem to the equation $\widetilde{\mathcal{H}} = 0$, see Proposition 3.3. This yields the following (§7):

Corollary C. — *In the H-family, the tP-family, the tD-family, the rPD-family, and the tCLP-family, every surface whose nullity is equal to* 3 *belongs to a (unique up to homotheties) smooth locally rigid* 5*-parameter family of pairwise non-homothetic triply periodic minimal surfaces.*

In other words, the minimal surfaces in the statement of Corollary C belong to the regular part of the moduli space $\mathcal{M}(\Sigma, \mathbb{T}^3)$.

Remark 1.1. — It is interesting to observe that W. Meeks proved in [20, Theorem 7.1] that every triply periodic minimal surface of genus 3 for which the ramified values of the Gauss map consist of 4 antipodal pairs in the 2-sphere, belongs to a real 5-dimensional family of triply periodic minimal surfaces of genus 3. Every surface in the tCLP-family, the tP-family, the tD-family, and the rPD family belongs to the class of surfaces to which [20, Theorem 7.1] applies. Thus, for each surface $x_0 : \Sigma \to \mathbb{T}^3$ with nullity three in these families, the 5-parameter family of triply periodic minimal surfaces that contains x_0 given in Corollary C coincides with Meeks' family. For these examples, the new information provided by our results (Theorem A (2)), besides a different approach to the proof, is the local rigidity property of these families around surfaces with nullity equal to 3, and the lack of local rigidity around surfaces with nullity larger than 3 corresponding to even jumps of the Morse index.

Remark 1.2. — Every minimal surface in the tCLP-family has nullity equal 3 and index equal to 3, see [21, Corollary 15]. An alternative proof by numerical methods is given in [7].

Remark 1.3. — The existence of a 5-parameter family of triply periodic minimal surfaces containing a given one is obtained in Proposition 3.3, more generally, near each embedded triply periodic minimal surface with genus greater than one and with nullity equal to three. The classical example of Schoen's gyroid does not satisfy the assumptions of [20, Theorem 7.1] (see [20, Remark 7.2]). However, since it belongs to the associate family of P and D-surfaces (Example 6.5), it has nullity equal to 3 (see [26]), and Proposition 3.3 applies in this situation (§7).

THEOREM D. — *Schoen's gyroid belongs to a unique (up to homotheties) locally rigid* 5*-parameter smooth family of pairwise non-homothetic triply periodic minimal surfaces.*

We remark that the existence of a family of deformations for the triply periodic minimal surfaces considered in Corollary C and Theorem D above can also be deduced from Ejiri's results in [5]. Actually, while we obtain the results via an equivariant implicit function theorem, [5] uses a different approach, and it gives a more explicit description of the deformation space.

1.3. Some technical aspects

As an undesired drawback in our bifurcation setup, we need to observe that the PDE:

$$\widetilde{\mathcal{H}} = 0,$$

defined in the space of "unparameterized embeddings" of Σ into $\mathbb{T}^3$, is *not* variational, i.e., it is not the Euler–Lagrange equation of some variational problem (recall that the standard mean curvature function is the gradient of the area functional). This entails that, in order to carry out our project, we have to resort to more general bifurcation theory for Fredholm operators (see Appendix A.2), which provides results somewhat weaker than variational bifurcation theorems. More specifically, rather than the classical "jump of Morse index" assumption, in the nonvariational case one has to postulate the less general (and intuitive) "odd crossing number" condition for the eigenvalues of the linearized problem. In particular, we can only infer the existence of bifurcating branches at those instants at which the jump of the Morse index is an *odd* integer. We should remark that, for the H-family, it is conjectured that there exists a degeneracy instant ($a_1 \approx 0.71479$) where the jump of Morse index is equal to 2, see Section 7.

In Section 8, we will present an analysis of the type of bifurcation occurring at the bifurcation instants stated in Conjecture B, mostly based on numerical computations. We will give numerical evidences to show that for the H-family, the degeneracy instant corresponds to a *transcritical bifurcation*, which does not produce *essentially new* triply periodic minimal surfaces. This is due to the fact that, around the degenerate instant where bifurcation occurs, the homothety class of the flat metric on the torus does not depend bijectively on the parameter of the family, see Remark 5.2. The same situation occurs at one of the two bifurcation instants of the families rPD, tP and tD. On the other hand, the second bifurcation instant of each of these three families is *genuine*, in the sense that the bifurcation branch that issues from these instants consists of triply periodic minimal surfaces that are not homothetic to any other member of the family.

1.4. Future developments

As we mentioned above, in order to obtain Conjecture B (§7) and to study geometry of triply periodic minimal surfaces in the bifurcation branches (§8), we use some numerical computations. It is a future subject to estimate the error terms to ensure that those phenomena are true. Numerical verification methods which are based on the use of interval arithmetic (cf. [27]) may be useful to this purpose.

As a final remark, we would like to observe that a full-fledged bifurcation theory for triply periodic minimal surfaces in $\mathbb{R}^3$ will require a further development of the results exposed here. In first place, it would be interesting to extend the existence result to degeneracy instants corresponding to *even* jumps of the Morse index, which is very likely a matter of applying finer bifurcation results. The second point would be to study the topology of the bifurcating branches, like connectedness, cardinality, regularity, etc., which ultimately depends on the behavior of the eigenvalues of the Jacobi operator near zero (derivative, transversal crossing). An important question to assess is establishing the pitchfork picture of the bifurcation set, and the stability/instability of minimal surfaces in the bifurcating branches. Note that a triply periodic minimal surface that divides $\mathbb{T}^3$ into two parts is said to be stable if the second variation of the area is nonnegative for all volume-preserving variations as a compact surface in $\mathbb{T}^3$ with the corresponding metric. Here, by volume it is meant the volume of each part of $\mathbb{T}^3$ divided by the surface. For instance, the Schwarz P and D surfaces and Schoen's gyroid are stable (Ross [26]).

Finally, it would be very interesting to study the geometry of the new triply periodic minimal surfaces in the bifurcating branches issuing from the genuine bifurcation instants along the rPD, the tP and the tD family whose existence is suggested in Conjecture B. These topics constitute the subject of an ongoing research project by the authors.

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2. The functional setup

2.1. Notations and terminology

We will denote by Λ a generic lattice in $\mathbb{R}^3$. The quotient $\mathbb{R}^3/\Lambda$ is diffeomorphic to the 3-torus $\mathbb{T}^3$, the quotient map $\mathbb{R}^3 \to \mathbb{R}^3/\Lambda$ will be denoted by π_Λ and the induced flat metric will be denoted by g_Λ. The identity connected component of the isometry group of $(\mathbb{R}^3/\Lambda, g_\Lambda)$ consists of translations $\mathfrak{t} \mapsto \mathfrak{t} + \mathfrak{t}_0$, $\mathfrak{t}, \mathfrak{t}_0 \in \mathbb{R}^3/\Lambda$.

The symbol $\mathcal{T}(\mathbb{T}^3)$ will denote(1) the set of flat metrics on $\mathbb{T}^3$ modulo isometries, or, equivalently, the set of isometry classes of lattices of $\mathbb{R}^3$. The isometry class of a flat metric g will be denoted by $[g]$, and the isometry class of a lattice Λ will be denoted by $[\Lambda]$. The *volume* of a lattice Λ is the volume of the metric g_Λ; by $\mathcal{T}_1(\mathbb{T}^3)$ we will denote the isometry classes of unit volume lattices of $\mathbb{R}^3$.

Let Σ be a closed surface; in our main applications, Σ will be a closed orientable surface of genus 3. For $k \in \mathbb{N}\bigcup\{0\}$ and $\alpha \in \,]0,1[$, the symbol $C^{k,\alpha}(\Sigma)$ will denote the Banach space of $C^{k,\alpha}$ real functions on Σ.

Let Λ_0 be a fixed lattice of $\mathbb{R}^3$, $g_0 = g_{\Lambda_0}$ be the corresponding flat metric on $\mathbb{T}^3$, and let us assume that $x_0 \colon \Sigma \to \mathbb{T}^3$ is a fixed g_0-minimal embedding, which is transversally oriented. Given $[\Lambda]$ sufficiently close to $[\Lambda_0]$ and $\varphi \in C^{2,\alpha}$ near 0, let us denote by $x_{\varphi,\Lambda} \colon \Sigma \to \mathbb{T}^3$ the embedding:

$$x_{\varphi,\Lambda}(p) = \exp^{g_\Lambda}_{x_0(p)}\big(\varphi(p)\cdot \vec{n}^{g_\Lambda}_{x_0(p)}\big), \quad p \in \Sigma,$$

where $\exp^{g_\Lambda}$ is the exponential map of the metric g_Λ, and $\vec{n}^{g_\Lambda}_{x_0}$ is the positively oriented g_Λ-unit normal vector along x_0. It is well known that, for Λ fixed, the map $\varphi \mapsto x_{\varphi,\Lambda}$ gives a bijection between a neighborhood of 0 in $C^{2,\alpha}(\Sigma)$ and a neighborhood of x_0 in the space of *unparameterized*

(1) We have a surjective map from $\mathrm{GL}(3) = \mathrm{GL}(3,\mathbb{R})$ to the set of lattices of $\mathbb{R}^3$: given $A \in \mathrm{GL}(3)$, one associates the lattice $\Lambda_A = \mathrm{span}_\mathbb{Z}\{Ae_1, Ae_2, Ae_3\}$, where e_1, e_2, e_3 is the canonical basis of $\mathbb{R}^3$. Given $A, A' \in \mathrm{GL}(3)$, then the lattices Λ_A and $\Lambda_{A'}$ are isometric if and only if there exists $U \in \mathrm{O}(3)$ such that $A' = UA$. Thus, $\mathcal{T}(\mathbb{T}^3)$ is identified with the quotient space $\mathrm{O}(3)\backslash\mathrm{GL}(3)$. This is a 6-dimensional manifold; the set $\mathcal{T}_1(\mathbb{T}^3)$ of isometry classes of lattices having volume 1 has dimension equal to 5.

embeddings (i.e., embeddings modulo reparameterizations) of Σ into $\mathbb{T}^3$. Details of this construction can be found, for instance, in reference [1].

Finally, for fixed Λ and $i = 1, 2, 3$, set $K_i^\Lambda = (\pi_\Lambda)_*(e_i)$, where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$. The K_i^Λ, $i = 1, 2, 3$, form a basis of Killing vector fields of $(\mathbb{T}^3, g_\Lambda)$. For $\varphi \in C^{2,\alpha}$ near 0 and $i = 1, 2, 3$, define $f_i^{\varphi,\Lambda} \colon \Sigma \to \mathbb{R}$ by:

$$f_i^{\varphi,\Lambda} = g_\Lambda\big(K_i^\Lambda, \vec{n}^{g_\Lambda}_{x_{\varphi,\Lambda}}\big).$$

Here, $\vec{n}^{g_\Lambda}_{x_{\varphi,\Lambda}}$ denotes the g_Λ-unit normal field along the embedding $x_{\varphi,\Lambda}$. We will also denote by $K_i^{\varphi,\Lambda}$ the vector fields on Σ obtained by g_Λ-orthogonal projection to $x_{\varphi,\Lambda}$ of K_i^Λ:

$$K_i^{\varphi,\Lambda} = K_i^\Lambda - f_i^{\varphi,\Lambda} \cdot \vec{n}^{g_\Lambda}_{x_{\varphi,\Lambda}}. \tag{2.1}$$

2.2. The functional framework

For a lattice $\Lambda \subset \mathbb{R}^3$ and an embedding $x : \Sigma \to \mathbb{T}^3$, let us denote by $\mathcal{H}^\Lambda(x) \colon \Sigma \to \mathbb{R}$ the mean curvature function of the embedding x relative to the metric g_Λ. Let us now fix a lattice Λ_0 and let us consider the function:

$$\widetilde{\mathcal{H}} \colon \mathfrak{U}_0 \times \mathbb{R}^3 \times \mathfrak{V}_0 \longrightarrow C^{0,\alpha}(\Sigma),$$

where $\mathfrak{U}_0$ is a neighborhood of 0 in the Banach space $C^{2,\alpha}(\Sigma)$ and $\mathfrak{V}_0$ is a neighborhood of Λ_0 in the set(2) of lattices of $\mathbb{R}^3$, defined by:

$$\widetilde{\mathcal{H}}\big(\varphi, a_1, a_2, a_3, \Lambda\big) = 2\mathcal{H}^\Lambda(x_{\varphi,\Lambda}) + \sum_{i=1}^{3} a_i f_i^{\varphi,\Lambda}. \tag{2.2}$$

For a lattice $\Lambda \subset \mathbb{R}^3$, we will also use the notation:

$$\widetilde{\mathcal{H}}_\Lambda \colon \mathfrak{U}_0 \times \mathbb{R}^3 \longrightarrow C^{0,\alpha}(\Sigma)$$

for the map:

$$\widetilde{\mathcal{H}}_\Lambda(\varphi, a_1, a_2, a_3) = \widetilde{\mathcal{H}}\big(\varphi, a_1, a_2, a_3, \Lambda\big). \tag{2.3}$$

The following result is based on an idea of N. Kapouleas [11, 12], which was then also employed by R. Mazzeo, F. Pacard and D. Pollack [19], R. Mazzeo and F. Pacard [18], B. White [31, §3], J. Pérez and A. Ros [23, Thm 6.7], and, finally in [4]. Let **0** denote the zero function on Σ.

(2) The set of lattices of $\mathbb{R}^3$ can be identified with $\mathrm{GL}(3, \mathbb{R})$.

PROPOSITION 2.1. — *Assume that the functions* $f_i^{\mathbf{0},\Lambda_0}$, $i = 1,2,3$, *are linearly independent*(3)*. Then, for sufficiently small neighborhoods* $\mathfrak{U}_0$ *and* $\mathfrak{V}_0$*:*

$$\widetilde{\mathcal{H}}^{-1}(\mathbf{0}) = \{(\varphi,0,0,0,\Lambda) : x_{\varphi,\Lambda} \text{ is } g_\Lambda\text{-minimal}\}. \tag{2.4}$$

Proof. — First, we choose $\mathfrak{U}_0$ and $\mathfrak{V}_0$ small enough so that the functions $f_i^{\varphi,\Lambda}$, $i = 1,2,3$, are linearly independent for all $(\varphi,\Lambda) \in \mathfrak{U}_0 \times \mathfrak{V}_0$. Now, let $(\varphi, a_1, a_2, a_3, \Lambda)$ be such that:

$$\mathcal{H}^\Lambda(x_{\varphi,\Lambda}) + \sum_{i=1}^{3} a_i f_i^{\varphi,\Lambda} = 0.$$

In order to prove (2.4), we need to show that from the above equality it follows $a_1 = a_2 = a_3 = 0$. Multiplying both sides of the above equality by $\sum_{i=1}^3 a_i f_i^{\varphi,\Lambda}$ we get:

$$\mathcal{H}^\Lambda(x_{\varphi,\Lambda}) \sum_{i=1}^{3} a_i f_i^{\varphi,\Lambda} + \left(\sum_{i=1}^{3} a_i f_i^{\varphi,\Lambda}\right)^2 = 0. \tag{2.5}$$

We claim that for all $i = 1,2,3$ we have:

$$\int_\Sigma \mathcal{H}^\Lambda(x_{\varphi,\Lambda}) f_i^{\varphi,\Lambda} \, \mathrm{d}\Sigma_{\varphi,\Lambda} = 0, \tag{2.6}$$

where $\mathrm{d}\Sigma_{\varphi,\Lambda}$ is the volume element of the pull-back by $x_{\varphi,\Lambda}$ of g_Λ. This follows from Stokes' Theorem, observing that:

$$\mathcal{H}^\Lambda(x_{\varphi,\Lambda}) f_i^{\varphi,\Lambda} = \operatorname{div}(K_i^{\varphi,\Lambda}), \tag{2.7}$$

where the $K_i^{\varphi,\Lambda}$'s are defined in (2.1), see Lemma A.1. Using (2.5) and (2.6) we get:

$$\int_\Sigma \left(\sum_{i=1}^{3} a_i f_i^{\varphi,\Lambda}\right)^2 \mathrm{d}\Sigma_{\varphi,\Lambda} = 0,$$

which gives $\sum_{i=1}^3 a_i f_i^{\varphi,\Lambda} = 0$. Since the $f_i^{\varphi,\Lambda}$'s are linearly independent, we obtain $a_1 = a_2 = a_3 = 0$, which proves our result. □

3. Local rigidity

Fix $[\Lambda] \in \mathcal{T}(\mathbb{T}^3)$; two embeddings $x_1, x_2 \colon \Sigma \to \mathbb{R}^3/\Lambda$ will be called Λ-*congruent* if there exists a diffeomorphism $\psi \colon \Sigma \to \Sigma$ and an element $\mathfrak{t}_0 \in$

(3) The linear independence assumption is always satisfied when the genus of Σ is greater than 1, see Remark 5.5.

$\mathbb{R}^3/\Lambda$ such that $x_2 = \mathfrak{t}_0 + (x_1 \circ \psi)$. Observe that if x_1 and x_2 are Λ-congruent, and x_1 is g_Λ-minimal, then also x_2 is g_Λ-minimal.

Let $\mathfrak{V}$ be a neighborhood of $[\Lambda_0]$ in $\mathcal{T}(\mathbb{T}^3)$, and let $\mathfrak{V} \ni [\Lambda] \mapsto \varphi_\Lambda \in \mathfrak{U}_0$ be a continuous map such that $x_{\varphi_\Lambda,\Lambda}$ is g_Λ-minimal for all $[\Lambda] \in \mathfrak{V}$.

Definition 3.1. — *The family of minimal embeddings $(x_{\varphi_\Lambda,\Lambda})_{[\Lambda]\in\mathfrak{V}}$ is said to be* locally rigid at Λ_0 *if for every $[\Lambda] \in \mathfrak{V}$ and every g_Λ-minimal embedding $y\colon \Sigma \to \mathbb{T}^3$ sufficiently close to x_0, y is Λ-congruent to $x_{\varphi_\Lambda,\Lambda}$.*

In order to formulate a rigidity criterion, let us introduce a suitable notion of nondegeneracy for minimal embeddings in $(\mathbb{T}^3, g_\Lambda)$.

Definition 3.2. — *Assume that $x_{\varphi,\Lambda}$ is a g_Λ-minimal embedding, and let $J_{\varphi,\Lambda}\colon C^{2,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma)$ denote its Jacobi*[4] *operator. We say that $x_{\varphi,\Lambda}$ is* equivariantly nondegenerate *if* $\mathrm{Ker}(J_{\varphi,\Lambda}) = \mathrm{span}\big\{f_1^{\varphi,\Lambda}, f_2^{\varphi,\Lambda}, f_3^{\varphi,\Lambda}\big\}$.

The span of $\big\{f_1^{\varphi,\Lambda}, f_2^{\varphi,\Lambda}, f_3^{\varphi,\Lambda}\big\}$ is the space of the so-called *Killing–Jacobi* fields along $x_{\varphi,\Lambda}$. Thus, an equivalent way of characterizing equivariant nondegeneracy is the fact that every Jacobi field along $x_{\varphi,\Lambda}$ is a Killing–Jacobi field.

A direct application of the equivariant implicit function theorem proved in [4] gives the following:

Proposition 3.3. — *Assume that the functions $f_i^{\mathbf{0},\Lambda_0}$, $i = 1, 2, 3$, are linearly independent*[5]*, and that x_0 is equivariantly nondegenerate. Then, there exists a smooth function $\mathfrak{V} \ni \Lambda \mapsto \varphi_\Lambda \in \mathfrak{U}_0$ defined in a neighborhood $\mathfrak{V}$ of $[\Lambda_0]$ in $\mathcal{T}(\mathbb{T}^3)$, such that:*

(1) $\varphi_{\Lambda_0} = \mathbf{0}$;
(2) *$x_{\varphi_\Lambda,\Lambda}$ is a minimal g_Λ embedding for all $[\Lambda] \in \mathfrak{V}$;*
(3) *the family $(x_{\varphi_\Lambda,\Lambda})_{[\Lambda]\in\mathfrak{V}}$ is locally rigid at Λ_0.*

Therefore, near x_0, all triply periodic minimal surfaces in $\mathbb{R}^3$ consist of six-parameter family of surfaces. If we restrict ourselves to the unit volume lattice, then, near x_0, all triply periodic minimal surfaces consist of a five-parameter family of pairwise non-homothetic surfaces.

---

(4) Denote by $\Delta_{\varphi,\Lambda}$ the Laplacian of the pull-back of the metric g_Λ by $x_{\varphi,\Lambda}$, that is, for the euclidean metric $ds^2 = \sum_{i,j} \delta_{ij} du^i du^j$, $\Delta f = f_{u^1u^1} + f_{u^2u^2}$. $J_{\varphi,\Lambda}$ is the elliptic operator on Σ given by $\Delta_{\varphi,\Lambda} + \|\mathcal{S}_{\varphi,\Lambda}\|^2$, where $\mathcal{S}_{\varphi,\Lambda}$ is the second fundamental form of $x_{\varphi,\Lambda}$, and $\|\cdot\|$ is the Hilbert–Schmidt norm.

(5) A statement similar to that of Theorem 3.3 holds without the linear independence assumption, with suitable modifications of the function $\widetilde{\mathcal{H}}$ in (2.2). However, we observe that such assumption is always satisfied when the genus of Σ is greater than 1, see Remark 5.5.

Proof. — (1)–(3) follow from [4, Theorem 5.2]. Then, we know the dimension of the space of triply periodic minimal surfaces near x_0 from the dimension of the isometry class in the flat 3-torus. □

Proposition 3.3 proves statement (2) in Theorem A.

4. Linearization

In order to study the lack of local rigidity for a family of minimal g_Λ-embeddings, we study the linearization of the map $\widetilde{\mathcal{H}}_\Lambda$, given in (2.3), at one of its zeros, described in Proposition 2.1. Let $(\varphi, \Lambda) \in \mathfrak{U}_0 \times \mathfrak{V}_0$ be such that $\widetilde{\mathcal{H}}_\Lambda(\varphi, 0, 0, 0) = 0$. Let us denote by:

$$T_{\varphi,\Lambda} : C^{2,\alpha}(\Sigma) \times \mathbb{R}^3 \longrightarrow C^{0,\alpha}(\Sigma) \tag{4.1}$$

the bounded linear operator:

$$T_{\varphi,\Lambda} = \mathrm{d}\widetilde{\mathcal{H}}_\Lambda(\varphi, 0, 0, 0).$$

Proposition 4.1. — *The operator $T_{\varphi,\Lambda}$ is given by:*

$$T_{\varphi,\Lambda}(\psi, b_1, b_2, b_3) = J_{\varphi,\Lambda}(\psi) + \sum_{i=1}^{3} b_i f_i^{\varphi,\Lambda}, \tag{4.2}$$

for all $(\psi, b_1, b_2, b_3) \in C^{2,\alpha}(\Sigma) \times \mathbb{R}^3$, where $J_{\varphi,\Lambda}$ is the Jacobi operator of the g_Λ-minimal embedding $x_{\varphi,\Lambda}$. This is a Fredholm operator of index equal to 3. If the $f_i^{\varphi,\Lambda}$'s are linearly independent, then $T_{\varphi,\Lambda}$ is surjective if and only if $x_{\varphi,\Lambda}$ is an equivariantly nondegenerate g_Λ-minimal embedding.

Proof. — It is well known that the differential of the mean curvature map $\varphi \mapsto \mathcal{H}(x_{\varphi,\Lambda})$ at a minimal embedding is given by the Jacobi operator $J_{\varphi,\Lambda}$. Equality (4.2) follows easily, observing that:

- the map $\mathbb{R}^3 \ni (a_1, a_2, a_3) \longmapsto \sum_{i=1}^3 a_i f_i^{\varphi,\Lambda} \in C^{0,\alpha}(\Sigma)$ is linear;
- the differential of the map $\varphi \mapsto f_i^{\varphi,\Lambda}$ is not involved in formula (4.2), since $\mathrm{d}\widetilde{\mathcal{H}}_\Lambda$ is computed at $a_1 = a_2 = a_3 = 0$.

As to the Fredholmness, it is well known that $J_{\varphi,\Lambda} : C^{2,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma)$ is Fredholm, and it has index 0 (it is an elliptic differential operator), and so the operator $C^{2,\alpha}(\Sigma) \times \mathbb{R}^3 \ni (\psi, b_1, b_2, b_3) \mapsto J_{\varphi,\Lambda}(\psi) \in C^{0,\alpha}(\Sigma)$ is Fredholm of index 3. Clearly, $T_{\varphi,\Lambda}$ is a finite rank perturbation of such operator, and therefore it is also a Fredholm operator of index 3.

As to the last statement, note that $J_{\varphi,\Lambda}$ is symmetric with respect to the L^2-pairing (using the volume element of g_Λ), and that its image is the L^2-orthogonal of its (finite dimensional) kernel. Such kernel contains the span of the $f_i^{\varphi,\Lambda}$'s, and it is equal to this span when $x_{\varphi,\Lambda}$ is equivariantly nondegenerate. Clearly:

$$\mathrm{Range}(T_{\varphi,\Lambda}) = \mathrm{Range}(J_{\varphi,\Lambda}) + \mathrm{span}\big\{f_1^{\varphi,\Lambda}, f_2^{\varphi,\Lambda}, f_3^{\varphi,\Lambda}\big\},$$

and the conclusion follows easily. □

5. Bifurcation

Let us now assume that $[-\varepsilon,\varepsilon] \ni s \mapsto \big(\varphi_s, \Lambda_{(s)}\big) \in \mathfrak{U}_0 \times \mathfrak{V}_0$ is a continuous map such that:

(1) $x_{\varphi_s,\Lambda_{(s)}}$ is a $g_{\Lambda_{(s)}}$-minimal embedding for all s;
(2) $\varphi_0 = \mathbf{0}$ and $\Lambda_{(0)} = \Lambda_0$.

Definition 5.1. — *We say that $s = 0$ is a bifurcation instant for the path $s \mapsto x_{\varphi_s,\Lambda_{(s)}}$ if there exists a sequence $(s_n)_{n\in\mathbb{N}} \subset \,]-\varepsilon,\varepsilon[$ and a sequence $x_n \colon \Sigma \to \mathbb{T}^3$ of embeddings such that:*

- *$\lim_{n\to\infty} s_n = 0$ and $\lim_{n\to\infty} x_n = x_0$ (in the $C^{2,\alpha}$-topology);*
- *x_n is $g_{\Lambda_{(s_n)}}$-minimal for all n;*
- *x_n is* not *$\Lambda_{(s_n)}$-congruent to $x_{\varphi_{s_n},\Lambda_{(s_n)}}$ for all n.*

In particular, if $s = 0$ is a bifurcation instant, then no family $(x_{\varphi_\Lambda,\Lambda})_{[\Lambda]\in\mathfrak{V}}$ that contains the path $s \mapsto \big(\varphi_s, \Lambda_{(s)}\big)$ is locally rigid at Λ_0. Thus, by Proposition 3.3, bifurcation can occur at $s = 0$ only if x_0 is a Λ_0-minimal equivariantly degenerate embedding. In the situation above, the sequence x_n possibly belongs to a continuous set of minimal embeddings, which is usually called the *bifurcating branch* issuing from $x_{\varphi_0,\Lambda_{(0)}}$, while the family $s \mapsto x_{\varphi_s,\Lambda_{(s)}}$ is called the *trivial branch*.

Remark 5.2. — Let us observe that in the definition of bifurcation given above, it is not required that, near $s = 0$, the metrics $g_{\Lambda_{(s)}}$ should be pairwise non-homothetic. Under this additional hypothesis, a stronger conclusion about the bifurcation branch can be drawn. Namely, if the flat metrics $g_{\Lambda(s)}$ are pairwise non-homothetic near $s = 0$, then every embedding x_n in the bifurcating branch is not homothetic to any of the minimal surfaces in the trivial branch.

The notion of Morse index is central in Bifurcation Theory. Let $x_{\varphi,\Lambda}$ be a g_Λ-minimal embedding.

Definition 5.3. — *The* Morse index $\mathrm{i}_{\mathit{Morse}}(\varphi,\Lambda)$ *of $x_{\varphi,\Lambda}$ is the number of negative eigenvalues of the Jacobi operator $J_{\varphi,\Lambda}$, counted with multiplicity. The nullity of $x_{\varphi,\Lambda}$ is the dimension of* $\mathrm{Ker}(J_{\varphi,\Lambda})$.

The number $\mathrm{i}_{\mathrm{Morse}}(\varphi,\Lambda)$ is in fact the Morse index of $x_{\varphi,\Lambda}$ as a critical point of the g_Λ-area functional defined in the space of embeddings of Σ into $\mathbb{T}^3$. A sufficient condition for variational bifurcation is given in terms of jumps of the Morse index. Here we cannot employ directly variational techniques, in that our equation $\widetilde{\mathcal{H}}=0$ is not variational, and we have to resort to a weaker bifurcation result for general Fredholm operators. This requires a certain parity change in the negative spectrum of the path of operators, which corresponds to an *odd* jump of the Morse index.

Theorem 5.4. — *Let* $[-\varepsilon,\varepsilon]\ni s\mapsto\big(\varphi_s,\Lambda_{(s)}\big)\in\mathfrak{U}_0\times\mathfrak{V}_0$ *be a continuous map satisfying* (1) *and* (2) *above. Assume the following:*

(1) $x_{\varphi_s,\Lambda_{(s)}}$ *is equivariantly nondegenerate for all* $s\neq 0$*;*
(2) *the functions* $f_i^{\mathbf{0},\Lambda_0}$, $i=1,2,3$*, are linearly independent;*
(3) $\mathrm{i}_{Morse}(\varphi_{-\varepsilon},\Lambda_{(-\varepsilon)})-\mathrm{i}_{Morse}(\varphi_{\varepsilon},\Lambda_{(\varepsilon)})$ *is an odd integer.*

Then, $s=0$ *is a bifurcation instant for the path* $s\mapsto\big(\varphi_s,\Lambda_{(s)}\big)$.

Proof. — A precise statement of the bifurcation theorem employed in this proof is given in Appendix A, Section A.2. For the reader's convenience, we will refer to the assumptions of this result throughout the proof.

By (1), the integer valued function $s\mapsto\mathrm{i}_{\mathrm{Morse}}(\varphi_s,\Lambda_{-s})$ is constant on $[-\varepsilon,0[$ and on $]0,\varepsilon]$. Thus, we can choose arbitrarily small values of ε and reduce the size of $\mathfrak{U}_0$ and $\mathfrak{V}_0$ when needed, maintaining the validity of assumption (3).

First, by continuity, we can assume that the functions $f_i^{\varphi,\Lambda}$, $i=1,2,3$, are linearly independent for all fixed $\big(\varphi,[\Lambda]\big)\in\mathfrak{U}_0\times\mathfrak{V}_0$. Second, we choose a codimension 3 closed subspace X_0 of $C^{2,\alpha}(\Sigma)$ which is transversal to $Y_0:=\mathrm{span}\big\{f_1^{\mathbf{0},\Lambda_0},f_2^{\mathbf{0},\Lambda_0},f_3^{\mathbf{0},\Lambda_0}\big\}$; for instance, we can choose X_0 to be the L^2-orthogonal of Y_0 relatively to the volume element of g_{Λ_0}. By continuity, we can also assume that X_0 is transversal[(6)] to $Y_{\varphi,\Lambda}:=\mathrm{span}\big\{f_1^{\varphi,\Lambda},f_2^{\varphi,\Lambda},f_3^{\varphi,\Lambda}\big\}$, for all $\big(\varphi,[\Lambda]\big)\in\mathfrak{U}_0\times\mathfrak{V}_0$ with φ smooth.

For all Λ, the group $G_\Lambda=\mathbb{R}^3/\Lambda$ acts isometrically on $(\mathbb{T}^3,g_{\Lambda_{(s)}})$ by translation, and this defines a smooth action on the set of embeddings of Σ into $\mathbb{T}^3$. Passing to the quotient by the action of the diffeomorphism group of Σ, we have a continuous action on the set of unparameterized embeddings, and therefore a local action[(7)] on the open set $\mathfrak{U}_0$. The G_Λ-orbit of

(6) Note that transversality, i.e., $X_0+Y_{\varphi,\Lambda}=C^{2,\alpha}(\Sigma)$ for φ smooth, also implies $X_0\cap Y_{\varphi,\Lambda}=\{0\}$, by a dimension argument.
(7) More precisely, the definition of the local action of G_Λ on $\mathfrak{U}_0$ is as follows. For $\mathfrak{t}\in\mathbb{R}^3/\Lambda$ close to 0, and $\varphi\in\mathfrak{U}_0$, consider the embedding $y=\mathfrak{t}+x_{\varphi,\Lambda}$. There exists a unique $\varphi'\in\mathfrak{U}_0$ such that $x_{\varphi',\Lambda}$ is a reparameterization of y. Then, $\mathfrak{t}\cdot\varphi=\varphi'$.

every smooth embedding, and in particular, of any minimal embedding, is a smooth submanifold of $\mathfrak{U}_0$; details of the proof of this fact can be found in [1]. Note that the G_Λ-orbit of an (unparameterized) embedding x is precisely the set of (unparameterized) embeddings that are Λ-congruent to x. Using the fact that transversality is an open condition in the C^1-topology, by taking $\mathfrak{U}_0$ and $\mathfrak{V}_0$ small enough, we can assume that for all $[\Lambda] \in \mathfrak{V}_0$ and any smooth function $\varphi \in \mathfrak{U}_0$, there is a unique intersection point φ^Λ between the orbit $G_\Lambda \cdot \varphi$ and $X_0 \cap \mathfrak{U}_0$. Again by transversality, the path $s \mapsto \varphi_s^{\Lambda_{(s)}}$ is continuous, and up to replacing φ_s with $\varphi_s^{\Lambda_{(s)}}$, we can therefore assume that $\varphi_s \in X_0 \cap \mathfrak{U}_0$ for all s. This settles assumption (B) in Section A.2.

Finally, there is a correspondence between zeros of the function $\widetilde{\mathcal{H}}$ in $\mathfrak{U}_0 \times \mathbb{R}^3 \times \mathfrak{V}_0$, defined in (2.2), and its restriction to $(X_0 \cap \mathfrak{U}_0) \times \mathbb{R}^3 \times \mathfrak{V}_0$: if $\big(\varphi, [\Lambda]\big) \in \mathfrak{U}_0 \times \mathfrak{V}_0$ is such that $\widetilde{\mathcal{H}}(\varphi, 0, 0, 0, \Lambda) = 0$, i.e., $x_{\varphi,\Lambda}$ is a (smooth) g_Λ-minimal embedding, then also $\widetilde{\mathcal{H}}(\varphi^\Lambda, 0, 0, 0, \Lambda) = 0$.

In conclusion, the argument above shows that Λ-congruence classes of g_Λ-minimal embeddings of Σ into $\mathbb{T}^3$ are into 1-1 correspondence with zeros of the function $\widetilde{\mathcal{H}}$ in $(X_0 \cap \mathfrak{U}_0) \times \mathbb{R}^3 \times \mathfrak{V}_0$. The aimed bifurcation result will then be proved in this context, and it will be obtained as a direct application of a classical bifurcation theorem for Fredholm operators, see [15, Theorem II.4.4, p. 212]. A precise statement of this theorem is recalled in Appendix A.

If $s \in [-\varepsilon, \varepsilon]$ is such that $\widetilde{\mathcal{H}}_{\Lambda_{(s)}}(\varphi_s, 0, 0, 0) = 0$, consider the restriction of $T_s = \mathrm{d}\widetilde{\mathcal{H}}_{\Lambda_{(s)}}(\varphi_s, 0, 0, 0)$ to $X_0 \times \mathbb{R}^3$, which will be denoted by $\overline{T}_s$. Recalling (4.2), an explicit formula for $\overline{T}_s$ is given by:

$$\overline{T}_s(\psi, b_1, b_2, b_3) = J_s\psi + \sum_{i=1}^{3} b_i f_i^s, \quad \psi \in X_0,\ b_i \in \mathbb{R}, \tag{5.1}$$

where

$$J_s := J_{\varphi_s,\Lambda_{(s)}}, \quad \text{and} \quad f_i^s := f_i^{\varphi_s,\Lambda_{(s)}}, \quad i = 1, 2, 3.$$

Then, $\overline{T}_s$ is a Fredholm operator of index 0, which settles assumption (C1) in Section A.2. This follows easily from Proposition 4.1, since X_0 is transversal to $Y_s\colon = Y_{\varphi_s,\Lambda_{(s)}}$, which is a 3-dimensional subspace of $\mathrm{Ker}(T_s)$. By the same argument, Proposition 4.1 says that $x_{\varphi_s,\Lambda_{(s)}}$ is an equivariantly nondegenerate $g_{\Lambda_{(s)}}$-minimal embedding if and only if $\overline{T}_s$ is an isomorphism. Thus, assumption (1) implies that, for $s \neq 0$, $\overline{T}_s$ is nonsingular, which settles assumption (BT1) in Section A.2.

Let us consider the injective continuous linear map

$$X_0 \oplus \mathbb{R}^3 \ni (\psi, b_1, b_2, b_3) \longmapsto \psi + \sum_{i=1}^{3} b_i f_i^0 \in C^{2,\alpha}(\Sigma) \tag{5.2}$$

to identify $X_0 \oplus \mathbb{R}^3$ with a subspace(8) of $C^{2,\alpha}(\Sigma)$, as in assumption (A) in Section A.2. Since $\overline{T}_0$ is Fredholm, then 0 is an isolated eigenvalue of $\overline{T}_0$ (assumption (D) in Section A.2), and it has finite multiplicity, given by $m = \dim\big(\mathrm{Ker}(T_0)\big) - 3 > 0$. Notice that $\overline{T}_0$ is diagonalizable: it coincides with the Jacobi operator J_0 on X_0, which is J_0-invariant, and it is the identity on $\mathbb{R}^3$. In particular, the generalized 0-eigenspace of T_0, i.e., $E_0 := \bigcup_{k\geqslant 1} \mathrm{Ker}(\overline{T}_0^k)$, coincides with the kernel of $\overline{T}_0$, given by $\mathrm{Ker}(J_0) \cap X_0$.

Using the identification (5.2), the operators $\overline{T}_s$ can be seen as unbounded linear operators on $C^{0,\alpha}(\Sigma)$, with domain $C^{2,\alpha}(\Sigma)$. As such, they are *closed* operators, i.e., they have closed graphs. This follows easily observing that they are finite rank perturbations(9) of the self-adjoint elliptic operators of second order J_s, which are closed (see for instance [15, Section III.1]). This settles assumption (C2) in Section A.2.

Let us now show that the path of Fredholm operators $\overline{T}_s$ has an odd crossing number at $s = 0$ using assumption (3).

To this aim, let us consider the continuous path of Fredholm operators $\overline{T}'_s = J_s + P_s$, where $P_s : C^{2,\alpha}(\Sigma) \cong X_s \oplus Y_s \to Y_s$ is the projection, and X_s is the L^2-orthogonal complement of Y_s relatively to the metric $g_{\Lambda_{(s)}}$. Observe that:

(i) $\overline{T}'_0 = \overline{T}_0$;
(ii) $\overline{T}'_s$ is invertible for all $s \neq 0$;
(iii) $\overline{T}'_s$ is diagonalizable with real eigenvalues, and for $s \neq 0$, its spectrum $\mathrm{spec}(\overline{T}'_s)$ coincides with $\big(\mathrm{spec}(J_s) \setminus \{0\}\big) \bigcup \{1\}$;
(iv) $\overline{T}'_s$ has an odd crossing number at $s = 0$, by assumption (3).

Statement (iv) follows easily from (iii). In order to conclude that also the family $\overline{T}_s$ has an odd crossing number at $s = 0$, it suffices to show that for r arbitrarily small there exists $s = s(r)$ such that the isomorphisms $\overline{T}_s$ and $\overline{T}'_s$ are endpoints of a continuous path of invertible operators that remain inside the ball $B(\overline{T}_0, r)$ of radius r centered at $\overline{T}_0$, see Remark A.2.

Using (5.1), the difference $\overline{T}'_s - \overline{T}_s$ is equal to:

- P_s on X_0;

(8) Using the identification (5.2), the operator $\overline{T}_0 \colon C^{2,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma)$ is given by $J_{\mathbf{0},\Lambda_0} + P_0$, where $P_0 \colon X_0 \oplus Y_0 \to Y_0$ is the projection.
(9) The sum of a closed and a bounded operator is closed.

- $J_s + (P_s - \mathrm{I}_s) = (J_s - J_0) + (P_s - \mathrm{I}_s)$ on Y_0,

where $I_s\colon Y_0 \to Y_s$ is the isomorphism defined by $\mathrm{I}_s\big(f_i^0\big) = f_i^s$, $i = 1, 2, 3$. For the second formula above, observe that $\overline{T}_s\big|_{Y_0} = \mathrm{I}_s$, as it follows easily from (5.1); keep also in mind that J_0 vanishes on Y_0.

Since $\lim_{s\to 0} P_s = P_0$ and $\lim_{s\to 0} J_s = J_0$ in the operator norm,

$$\lim_{s\to 0} \|P_s|_{X_0}\| = 0, \quad \text{and} \quad \lim_{s\to 0} \|[(J_s - J_0) + (P_s - \mathrm{I}_s)]|_{Y_0}\| = 0.$$

Observe that P_s is an operator of rank 3, and Y_0 has dimension 3. In other words, $\overline{T}_s'$ is the sum $\overline{T}_s + R_s$, with R_s a finite rank operator such that $\lim_{s\to 0} \|R_s\| = 0$. This implies easily that, given any $r > 0$, there exists $s = s(r)$ such that both $\overline{T}_s$ and $\overline{T}_s'$ belong to the ball $B(\overline{T}_0, r)$, and that there exists a continuous path of invertible operators in $B(\overline{T}_0, r)$ joining $\overline{T}_s$ and $\overline{T}_s'$. This shows that the family $\overline{T}_s$ has an odd crossing number at $s = 0$ (assumption (BT2) in Section A.2), and concludes the proof. □

Remark 5.5. — Assumption (2) in Theorem 5.4 is always satisfied when the genus of Σ is greater than 1. There is a number of ways to prove this fact, here we propose the most elementary one. Recalling the definition of the Jacobi field $f_i^{\mathbf{0},\Lambda_0}$ in Section 2.1, observe that they are linearly dependent if and only if some nonzero constant (i.e., translation invariant) vector field of $\mathbb{R}^3/\Lambda_0$ is everywhere tangent to $S = x_0(\Sigma)$.

Lemma 5.6. — *Let S be an embedded submanifold of $\mathbb{T}^3$ which is diffeomorphic to a closed orientable surface of genus $\mathrm{gen}(S) > 1$. Then, no nontrivial constant vector field of $\mathbb{T}^3$ is everywhere tangent to S.*

Proof. — Nontrivial constant vector fields are never vanishing. But the Euler characteristic of S is $2 - 2\,\mathrm{gen}(S) < 0$, so there are no never vanishing vector fields on S by the Poincaré–Hopf theorem. □

Theorem 5.4 proves statement (1) in Theorem A. More on the geometry of minimal submanifolds in flat tori can be found in reference [22].

6. Examples of triply periodic minimal surfaces

In this section we give definitions of several known examples of triply periodic minimal surfaces. The Morse index of each surface in these families are given in [7] (see the remark at the beginning of §7). Figure 6.1 shows seven one-parameter families of triply periodic minimal surfaces: tCLP, tD, tP, rPD, H, rG, and tG-family. The existence of crossing points in Black lines are proved rigorously in [7], while the crossings with green lines are conjectural (proved by numerical methods).

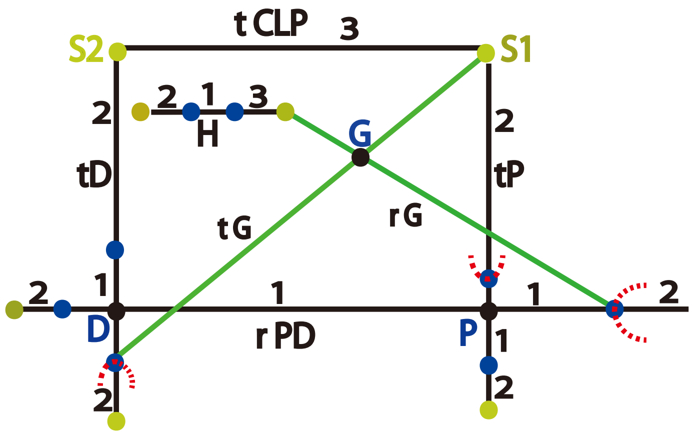


Figure 6.1. Each point represents a triply periodic minimal surface. The numbers indicate the Morse indices of the surfaces. Dotted curves indicate expected bifurcation branches, where each surface is not homothetic to any surface in the black line near the bifurcation point. P, D, G represents the Schwarz P-surface, Schwarz D-surface, and Alan Schoen's gyroid, respectively. S1 is the "limit" of the tCLP family and the tP family, and it is the singly periodic Scherk surface (Scherk's second surface) ([10, p. 513])*. S2 is the "limit" of the tCLP family and the tD family, and it is the doubly periodic Scherk surface (Scherk's first surface)* ([10, p. 513])*.*

Example 6.1 (H-family – Figure 6.5). — For $a \in \,]0,1[$, let M_a be a hyperelliptic Riemann surface of genus 3 defined by $w^2 = z(z^3 - a^3)\left(z^3 - \frac{1}{a^3}\right)$ and f a conformal minimal immersion given by

$$f(p) = \Re \int_{p_0}^{p} i\big(1 - z^2,\, i(1+z^2),\, 2z\big)^{\mathrm{t}} \frac{\mathrm{d}z}{w}.$$

$f(M_a)$ is called H-family.

Example 6.2 (rPD family, Karcher's TT surface, see ref. [13] *– Figure 6.2).* For $a \in \,]0,\infty[$, let M_a be a hyperelliptic Riemann surface of genus 3 defined by $w^2 = z(z^3 - a^3)\left(z^3 + \frac{1}{a^3}\right)$ and f a conformal minimal immersion given by

$$f(p) = \Re \int_{p_0}^{p} \big(1 - z^2,\, i(1+z^2),\, 2z\big)^{\mathrm{t}} \frac{\mathrm{d}z}{w}.$$

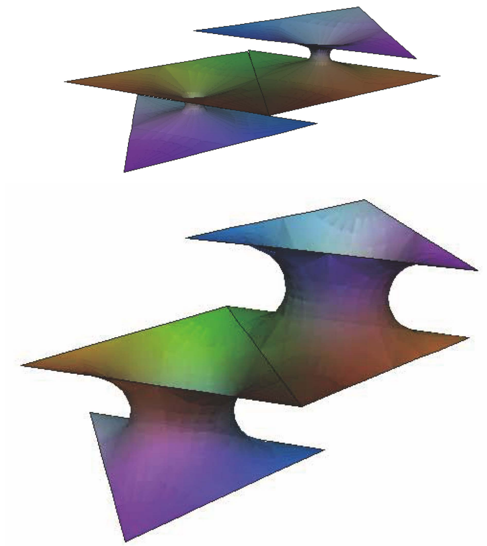


Figure 6.2. Surfaces M_a of the rPD-family, with $a = 0.1$ (Morse index 2), and $a = 0.5$ (Morse index 1).

$f(M_a)$ is called rPD family or Karcher's TT surface. $M_{\sqrt{2}}$ gives the so-called Schwarz Primitive surface (Schwarz P surface), and $M_{1/\sqrt{2}}$ gives the so-called Schwarz Diamond surface (Schwarz D surface).

Example 6.3 (tP-family – Figure 6.3, tD-family – Figure 6.4). — For $a \in \,]2, +\infty[$, let M_a be a hyperelliptic Riemann surface of genus 3 defined by $w^2 = z^8 + az^4 + 1$. Let f be a conformal minimal immersion given by

$$f(p) = \Re \int_{p_0}^{p} \left(1 - z^2,\, i(1 + z^2),\, 2z\right)^{\mathrm{t}} \frac{\mathrm{d}z}{w}$$

and f'

$$f'(p) = \Re \int_{p_0}^{p} i\left(1 - z^2,\, i(1 + z^2),\, 2z\right)^{\mathrm{t}} \frac{\mathrm{d}z}{w}.$$

$f(M_a)$ is called tP-family and $f'(M_a)$ is called tD-family. $f(M_{14})$ gives the Schwarz P surface, and $f'(M_{14})$ gives the Schwarz D surface.

Example 6.4 (tCLP-family – Figure 6.6). — For $a \in \,]-2, 2[$, let M_a be a hyperelliptic Riemann surface of genus 3 defined by $w^2 = z^8 + az^4 + 1$. Let f be a conformal minimal immersion given by

$$f(p) = \Re \int_{p_0}^{p} \left(1 - z^2,\, i(1 + z^2),\, 2z\right)^{\mathrm{t}} \frac{\mathrm{d}z}{w}.$$

Figure 6.3. Surfaces M_a of the tP-family, with $a = 0.285$ (Morse index 2), and $a = 14$ (Morse index 1). The surface M_{14} of the tP-family is also called Schwarz Primitive *surface, or* P-surface.

$f(M_a)$ is called tCLP family. [7, Numerical Result 4] suggests by using numerical computation that for all a, the minimal surface M_a has constant nullity equal to 3 and Morse index equal to 3.

Example 6.5 (associate family of Schwarz P-surface). — Let M be a hyperelliptic Riemann surface of genus 3 defined by $w^2 = z^8 + 14z^4 + 1$. Then the Schwarz P surface is given by

$$p \longmapsto \Re \int_{p_0}^{p} (1 - z^2,\, i(1 + z^2),\, 2z)^t \frac{\mathrm{d}z}{w},$$

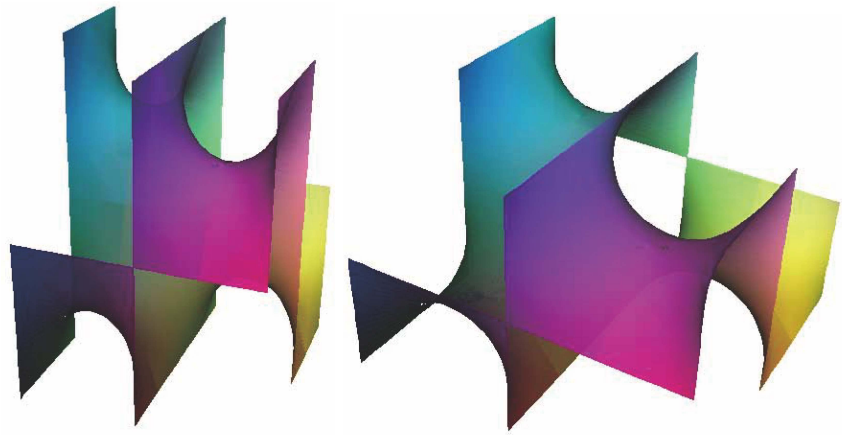


Figure 6.4. Surfaces M_a of the tD-family, with $a = 2.85$ (Morse index 2), and $a = 14$ (Morse index 1).

Figure 6.5. Surfaces M_a of the H-family, with $a = 0.1$ (Morse index 2), $a = 0.5$ (Morse index 1), and $a = 0.9$ (Morse index 3).

and the Schwarz D surface is given by

$$p \longmapsto \Re \int_{p_0}^{p} i(1 - z^2,\, i(1 + z^2),\, 2z)^t \frac{\mathrm{d}z}{w}.$$

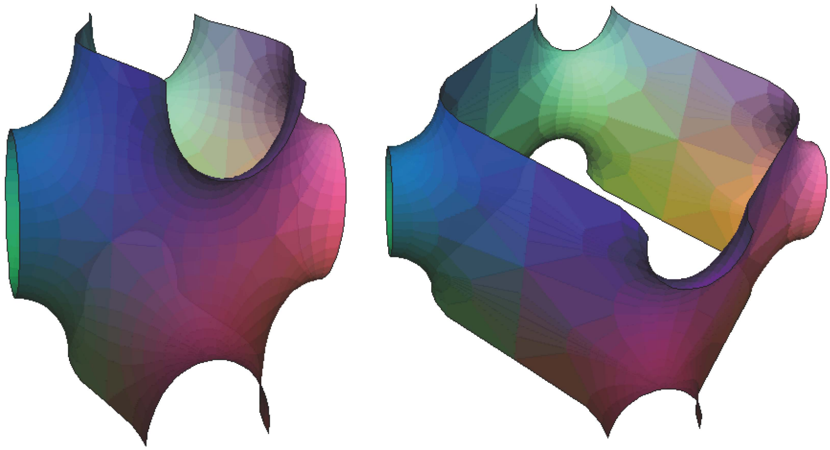


Figure 6.6. Surfaces M_a of the tCLP-family, with $a = 0$ (also called the Schwarz CLP-surface*) and with $a = 1.96$. Numerical computations suggest that all the surfaces of this family have Morse index equal to* 3 *and nullity equal to* 3*.*

The general associate surface of the P surface is given by

$$p \longmapsto \Re \int_{p_0}^{p} e^{i\theta}(1 - z^2,\, i(1 + z^2),\, 2z)^t \frac{\mathrm{d}z}{w}, \qquad \theta \in \mathbb{R}. \tag{6.1}$$

It is known that there exists $\theta \in\,]0, \pi/2[$ such that (6.1) gives a triply periodic minimal surface in $\mathbb{R}^3$. Actually, $\theta \approx 0.907313 (= 51.9852^\circ)$ gives this surface, which is called the Schoen's gyroid ([29]). Since these three surfaces have the same Riemann metric:

$$\mathrm{d}s^2 = \frac{(1 + |z|^2)^2}{|w|^2} |\mathrm{d}z|^2,$$

and thus the same Jacobi operator:

$$J = \Delta - 2K = \frac{4|w|^2}{(1 + |z|^2)^4} \left((1 + |z|^2)^2 \frac{\partial^2}{\partial z \partial \overline{z}} + 2 \right), \tag{6.2}$$

they also have the same nullity and the same Morse index. Since Schwarz P and D surfaces have nullity 3 and Morse index 1 because they are volume-preserving stable (Ross [26]), the Schoen's gyroid also has nullity 3 and Morse index 1.

7. Remarks about Conjecture B. Proofs of Corollary C and Theorem D

We will explain how to apply Theorem 5.4 and Proposition 3.3 to Conjecture B, Corollary C, and Theorem D. Let us recall that in references [7, 6] it is proved the existence of a finite dimensional reduction to compute the nullity and the Morse index of triply periodic minimal surfaces. More precisely, the nullity is shown to be equal to the multiplicity of the 0 eigenvector of a 9×9 complex Hermitian matrix, and the Morse index is shown to be equal to the number of negative eigenvalues of an 18×18 real symmetric matrix plus 1. Both matrices have entries defined by elliptic integrals, that have a real analytic dependence on the parameter a of the family. In particular, singularities of these matrices, that correspond to equivariantly degenerate embeddings of the family, are isolated. In [6, 7], this method is applied to several families of concrete examples to obtain the Morse indices with help of numerical computation by Mathematica. In Figure 6.1 the expected Morse indices are indicated. The validity of this result depends on an estimate the error term of the numerical approximation, which is the object of a future research project.

Observation about Conjecture B. — For the H-family $(M_a)_{a\in]0,1[}$, see Example 6.1, a numerical result given in [7, Numerical Result 1] suggests the existence of exactly two instants a_0 and a_1 corresponding to equivariantly degenerate surfaces. A numerical approximation shows that $a_0 \approx 0.49701$ and $a_1 \approx 0.71479$. The Morse index of M_a is computed numerically to be equal to 2 for $a \in]0, a_0[$, it is equal to 1 for $a \in]a_0, a_1[$, and it is equal to 3 for $a \in]a_1, 1[$. Thus, it is plausible that there is an odd jump at a_0, which must be a bifurcation instant. We cannot infer the existence of bifurcation at a_1, where the Morse index seems to have an even jump.

As to the rPD-family $(M_a)_{a\in]0,\infty[}$, see Example 6.2 and [7, Numerical Result 2], it is plausible that there are two odd jumps of the Morse index at instants a_1 and a_2. Numerical computations give the approximations $a_1 \approx 0.494722$, $a_2 = (a_1)^{-1} \approx 2.02133$. The nullity at every other instant seems to be equal to 3.

For the tP-family and the tD-family, see Example 6.3 and [7, Numerical Result 3], it is computed numerically that there are two odd jumps of the Morse index: $a_1, a_2 \in]2, +\infty[$. Numerical approximations provide the estimates $a_1 \approx 7.40284$ and $a_2 \approx 28.7783$. The nullity at every other instant seems to be equal to 3.

Observe that assumption (2) of Theorem 5.4 is satisfied in all cases, because all the minimal surfaces have genus equal to 3, see Remark 5.5. □

As to the local rigidity, we can apply Proposition 3.3 to the above families. We have seen that there seem to be exactly two surfaces that are not equivariantly nondegenerate in each of the rPD-family, the H-family, the tP-family and the tD-family. There is a fifth family of triply periodic minimal surfaces, called the tCLP-family (see Example 6.4).

Proof of Corollary C. — As we have observed in Remark 5.5, the linear independence assumption of Proposition 3.3 is always satisfied in the case of all minimal surfaces of the given families. This implies that equivariantly nondegenerate surfaces correspond precisely to minimal surfaces with nullity equal to 3. Each such surface belongs to a unique smooth family of triply periodic minimal surfaces, parameterized by isometry classes of flat metrics in $\mathbb{T}^3$ (a 6-dimensional space), by Proposition 3.3. If we consider isometry classes of flat metrics with fixed volume, which are thus pairwise non-homothetic, we get a smooth 5-parameter family of pairwise non-homothetic triply periodic minimal surfaces. □

Remark 7.1. — By the proof of Theorem A, Examples 6.2 and 6.3, and Conjecture B, we can make the following observation. Since Schwarz P-surface is stable ([26]), it has nullity equal to 3, and it is contained both in the rPD-family and in the tP-family, which are two one-parameter families of triply periodic minimal surfaces. Near the P-surface, these two families consist of surfaces that are pairwise non-homothetic. On the other hand, by Corollary C, the P-surface belongs to a (unique up to homotheties) smooth locally rigid 5-parameter family of pairwise non-homothetic triply periodic minimal surfaces. By the local rigidity, such 5-parameter family must contain (a portion of) the rPD-family and the tP-family. By the proof of our rigidity theorem, each family corresponds to a variation of the lattice from the cube which is the lattice corresponding to the P surface. This means that each minimal surface in the family loses symmetries according to the loss of symmetries of the lattice: symmetry with respect to the planes $\{x = 1/2\}$, $\{y = 1/2\}$, $\{z = 1/2\}$, $\{x = y\}$, $\{y = z\}$, $\{z = x\}$, $\pi/2$-rotational symmetry with respect to the vertical line, etc., here we assume that the original cube is spanned by $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. An analogous situation occurs for the Schwarz D-surface. Since it is stable ([26]), it has nullity equal to 3, and it is contained both in the rPD-family and in the tD-family.

Proof of Theorem D. — The local rigidity statement given in Theorem D follows readily from Proposition 3.3, since for Schoen's gyroid has nullity equal to 3 (Example 6.5). □

8. Remarks on the geometry of triply periodic minimal surfaces in the bifurcation branches

In general, Theorem 5.4 does not imply(10) the existence of essentially new triply periodic minimal surfaces, because there we do not assume that the flat metrics $g_{\Lambda_{(s)}}$ are pairwise non-homothetic near $s = 0$, see Remark 5.2.

Based on some numerical computations, in this section we will discuss this question at each bifurcation instant suggested in Conjecture B, and we will give a conjecture that determines at which bifurcations instants one obtains the existence of new examples of triply periodic minimal surfaces. Again, an estimate of the error term to ensure the observations in this section is the subject of a future investigation. Every bifurcation instant suggested in Conjecture B corresponds to an equivariantly degenerate minimal embedding whose Jacobi operator has kernel spanned by the three Killing–Jacobi fields (Definition 3.2), and one additional Jacobi field which is not Killing. Such a Jacobi field gives a first order approximation of the bifurcating branch, and it will be used in our discussion. The exact equations of these Jacobi fields and how to derive them will be discussed in a forthcoming paper, and will be omitted here.

8.1. rPD-family

First, let us look at the bifurcation instants along the rPD-family M_a, $a \in \,]0, +\infty[$ (Example 6.2). Set

$$A\colon = A(a)\colon = \frac{1}{\sqrt{3}a}\int_0^1 \frac{1+a^2t^2}{\sqrt{t(1-t^3)(a^3t^3+\frac{1}{a^3})}}\,\mathrm{d}t,$$

$$C\colon = C(a)\colon = 4\int_0^1 \frac{t}{\sqrt{t(1-t^3)(a^3+\frac{t^3}{a^3})}}\,\mathrm{d}t.$$

Then, the lattice is

$$\Lambda = \begin{pmatrix} 3A & 3A & 4A \\ \sqrt{3}A & -\sqrt{3}A & 0 \\ 0 & 0 & C \end{pmatrix}.$$

(10) An instructive example of bifurcation by Morse index jump in geometric variational problems that does not produce new solutions is discussed in [16, Section 2.6], in the context of constant mean curvature surfaces.

C is the height of the lattice, and A is a certain fixed constant times the length of the edge of the triangle (see Figure 6.2). Hence, the ratio A/C determines the lattice (up to homothety). Figure 8.1 represents the ratio A/C as a function of a, and $a = a_1 \approx 0.494722$ gives the minimum. It shows that there exist positive constants ϵ_1 and ϵ_2, a strictly monotone-increasing function $\delta\colon [0, \epsilon_1[\to [0, \epsilon_2[$ with $\delta(0) = 0$ and such that the lattice $\Lambda_{(a_1-\epsilon)}$ is a homothety of the lattice $\Lambda_{(a_1+\delta(\epsilon))}$.

Denote by $X(c)$ the surface $f(M_{a_1+c})$. Since $A(a)$ and $C(a)$ are increasing functions of a near the bifurcation instant $a = a_1 \approx 0.494722$ (see Figure 8.2), the surfaces $X(c)$ are like the pictures in the upper row in Figure 8.3.

Now, for $\epsilon \in]0, \epsilon_1[$, reduce the surface $X(\delta(\epsilon))$ to $\frac{C(a_1-\epsilon)}{C(a_1+\delta(\epsilon))}$, and denote the new surface by $Y(-\epsilon)$. Then the lattice of $Y(-\epsilon)$ is the same as the lattice of $X(-\epsilon)$, but the surfaces $X(-\epsilon)$ and $Y(-\epsilon)$ are not congruent to each other (see Figure 8.3).

Similarly, for $\epsilon \in]0, \epsilon_1[$, expand the surface $X(-\epsilon)$ to $\frac{C(a_1+\delta(\epsilon))}{C(a_1-\epsilon)}$, and denote the new surface by $Y(\delta(\epsilon))$. Then the lattice of $Y(\delta(\epsilon))$ is the same as the lattice of $X(\delta(\epsilon))$, but the surfaces $X(\delta(\epsilon))$ and $Y(\delta(\epsilon))$ are not congruent to each other (see Figure 8.3).

One can show the nodal lines of the zero-eigenfunction at $a = a_1$ are exactly the boundary triangles in Figure 8.3, and since the "essential" dimension of the zero eigenspace is one, it seems that the surfaces $Y(c)$ give the bifurcation branch from the instant $a = a_1$. However, they are homotheties of the original surfaces in the rPD family.

For $\epsilon \in]0, \epsilon_1[$, $X(-\epsilon)$ has index 2 and nullity 3, $X(\delta(\epsilon))$ has index 1 and nullity 3, $Y(-\epsilon)$ has index 1 and nullity 3, $Y(\delta(\epsilon))$ has index 2 and nullity 3. Hence, this bifurcation is a transcritical bifurcation.

On the other hand, at $a = a_2 = (a_1)^{-1} \approx 2.02133$, the ratio A/C is strictly monotone (Figure 8.1). This implies that the bifurcation branch contains triply periodic minimal surfaces that are not homothetic to any other surface in the five families given in §7 (Examples 6.1–6.4). Moreover, we can show that the nodal lines of the zero eigenfunctions at $a = a_2$ are planar geodesics that connect each vertex of each triangle with the middle point of a side of a triangle (Figure 8.4, the left picture in Figure 8.5). Remarkably, the sides of the triangles are not nodal lines, which suggests that near $a = a_2$, our new triply periodic minimal surfaces are like the right picture in Figure 8.5. It would be interesting to determine the symmetries of the new surfaces.

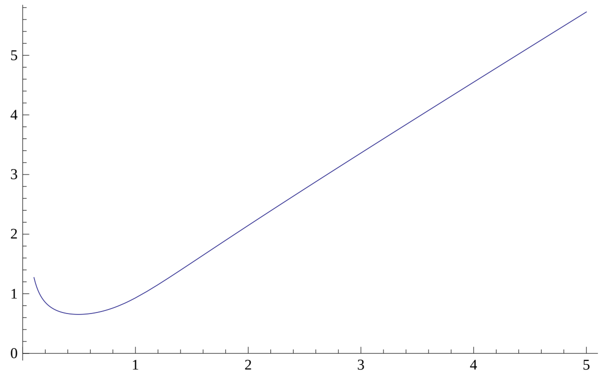


Figure 8.1. The horizontal axis represents a, while the vertical axis indicates the ratio A/C for the rPD family. The minimum of A/C is attained at $a = a_1 \approx 0.494722$.

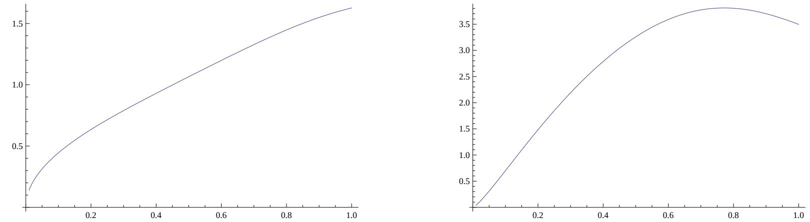


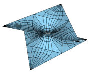
Figure 8.2. The horizontal axis represents a, while the vertical axis indicates A (left) and C (right) for the rPD family.

8.2. H-family

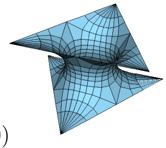
By a similar fashion, we find the lattice of the H-family. Set

$$B := \sqrt{3}\int_0^1 \frac{1-t^2}{\sqrt{t(t^3+a^3)(t^3+\frac{1}{a^3})}}\mathrm{d}t + 4\int_{\frac{1}{2}}^1 \frac{x}{\sqrt{(a^3+\frac{1}{a^3}+6x-8x^3)(1-x^2)}}\mathrm{d}x,$$

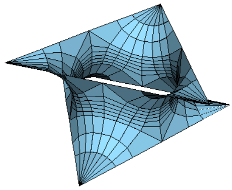
$$D := 8\int_0^1 \frac{t}{\sqrt{t(t^3+a^3)(t^3+\frac{1}{a^3})}}\mathrm{d}t.$$



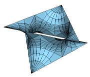
(a) $X(-\epsilon)$



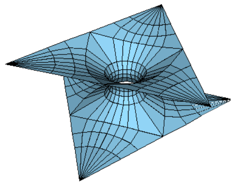
(b) $X(0)$



(c) $X(\delta(\epsilon))$



(d) $Y(-\epsilon)$



(e) $Y(\delta(\epsilon))$

Figure 8.3. Bifurcation at $a = a_1$ of the rPD family. Each picture shows a half period of the corresponding triply periodic minimal surface. The surfaces in the upper row belong to the rPD family. The surfaces in the lower row belong to the bifurcation branch, and they are homothetic to surfaces in the rPD family.

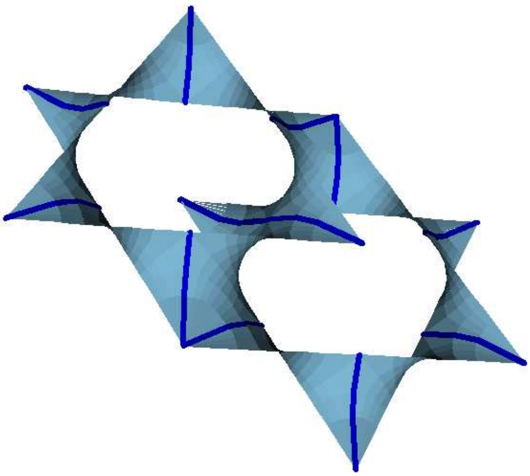


Figure 8.4. One period of a surface in the rPD-family with the planar geodesics.

Then, the lattice is

$$\Lambda = \begin{pmatrix} \frac{\sqrt{3}}{2}B & 0 & 0 \\ \frac{B}{2} & B & 0 \\ 0 & 0 & D \end{pmatrix}.$$

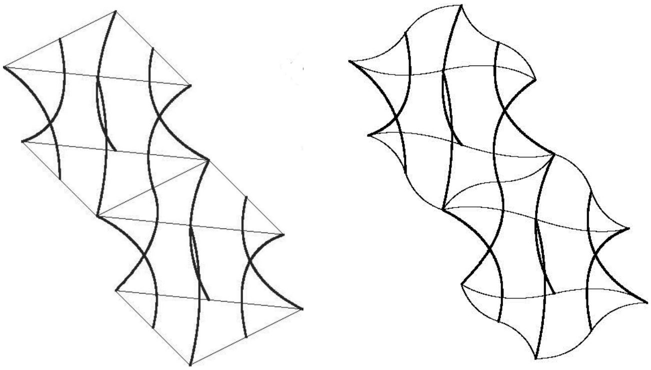


Figure 8.5. Left: one period of a surface in the rPD-family with the planar geodesics, Right: variation from a surface ($a = a_2$) in the rPD family with the zero eigenfunction as variation vector filed.

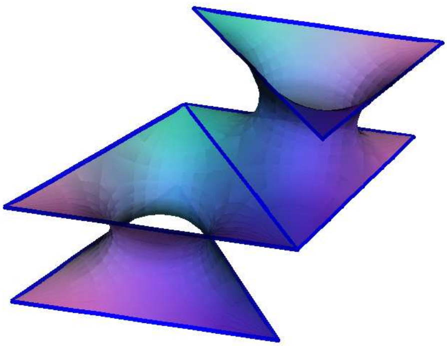


Figure 8.6. One period of a surface in the H-family.

The ratio B/D determines the lattice (up to homothety). Figure 8.7 represents the ratio B/D as a function of a, and $a = a_0 \approx 0.49701$ gives the minimum of B/D. Moreover, we can show that the nodal lines of the zero eigenfunctions at $a = a_0$ are exactly the triangles indicated in Figure 8.6. And so, arguing as the case of the rPD-family, we conjecture that the bifurcation we obtained in Conjecture B at $a = a_0$ for the H-family gives only homotheties of the surfaces in the original H-family.

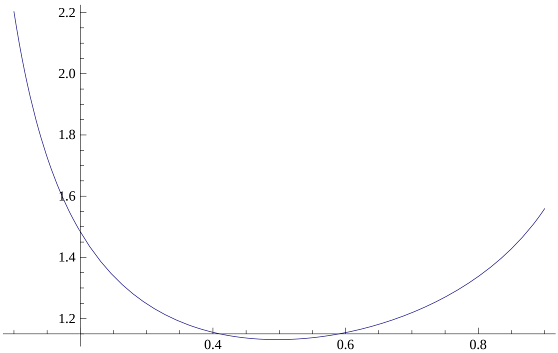


Figure 8.7. The horizontal axis represents a, while the vertical axis indicates the ratio B/D for the H-family. The minimum of B/D is attained at $a = a_0 \approx 0.49701$.

8.3. tP-family and tD-family

By a similar way, we find the lattice of the tP-family. Set

$$E = 2\int_0^1 \frac{1-t^2}{\sqrt{t^8+at^4+1}}\mathrm{d}t + 4\int_0^1 \frac{\mathrm{d}t}{\sqrt{16t^4-16t^2+2+a}},$$
$$F = 8\int_0^1 \frac{t}{\sqrt{t^8+at^4+1}}\mathrm{d}t.$$

Then, the lattice is

$$\Lambda = \begin{pmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & F \end{pmatrix}.$$

The ratio E/F determines the lattice (up to homothety). Figure 8.8 represents E/F as a function of a. By the same reason as the case of the rPD-family, we conjecture that the bifurcation we obtained in Conjecture B at $a = a_2 \approx 28.7783$ for the tP-family gives only homotheties of the surfaces in the original tP-family. However, we conclude that the bifurcation at $a = a_1 \approx 7.40284$ for the tP-family give triply periodic minimal surfaces that are not homothetic to any other surface in the five families given in §7 (Examples 6.1–6.4). As for the tD-family, the situation is totally analogous.

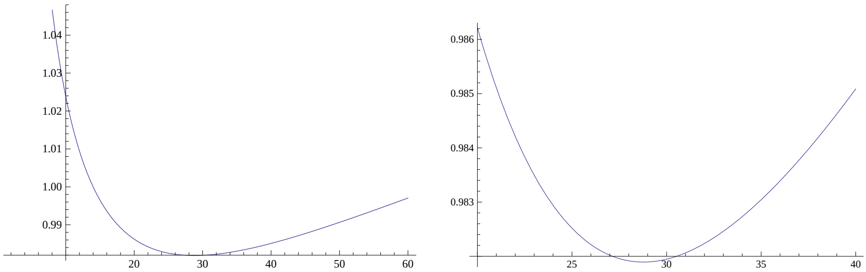


Figure 8.8. The horizontal axis represents a, while the vertical axis indicates the ratio E/F for the tP-family. The minimum of E/F is attained at $a = a_2 \approx 28.7783$.

Appendix A. Auxiliary results

A.1. A divergence formula

Let us recall here a standard formula of Riemannian geometry used in (2.7) for the proof of Proposition 2.1.

Lemma A.1. — *Let $(\overline{M}, \overline{\mathbf{g}})$ be a Riemannian manifold, let $M \subset \overline{M}$ be a compact submanifold (without boundary), with mean curvature vector field $\vec{H}$, and let $K \in \mathfrak{X}(\overline{M})$ be a Killing field in $\overline{M}$. Denote by $K_M \in \mathfrak{X}(M)$ the vector field on M obtained by orthogonal projection of K. Then, $\mathrm{div}_M(K_M) = \overline{\mathbf{g}}(K, \vec{H})$.*

Proof. — See for instance [17, Section 3]. □

A.2. On bifurcation for families of Fredholm operators

Let us recall briefly the precise statement of a well known bifurcation result for solutions of an equation of the form $F(\mathfrak{x}, \mu) = 0$, with $\mu \in [\mu_0 - \delta, \mu_0 + \delta]$, $\delta > 0$, a real parameter and $F(\cdot, \mu) : \mathfrak{X} \to \mathfrak{Z}$ is a continuous family of smooth maps from the (open subset of a) Banach space $\mathfrak{X}$ to a Banach space $\mathfrak{Z}$. Our basic references are the books [14] and [15].

Assume that:

(A) $\mathfrak{X}$ is continuously embedded into $\mathfrak{Z}$, i.e., there exists a continuous injective linear map $\mathfrak{i} \colon \mathfrak{X} \hookrightarrow \mathfrak{Z}$ (we will implicitely consider $\mathfrak{X} \subset \mathfrak{Z}$);

(B) $[\mu_0 - \delta, \mu_0 + \delta] \ni \mu \mapsto \mathfrak{x}_\mu \in \mathfrak{X}$ is a continuous map such that $F(\mathfrak{x}_\mu, \mu) = 0$ for all μ;

(C) setting $A_\mu = \mathrm{D}_{\mathfrak{x}} F(\mathfrak{x}_\mu, \mu) : \mathfrak{X} \to \mathfrak{Z}$, then for all μ:
  (C1) A_μ is a Fredholm operator of index 0;
  (C2) $A_\mu \colon \mathfrak{Z} \to \mathfrak{Z}$ is closed as an unbounded linear operator with domain $\mathfrak{X}$;

(D) 0 is an isolated eigenvalue of A_{μ_0}.

Assumption (C1) implies that $\mathrm{Ker}(A_{\mu_0})$ is finite dimensional, while assumption (D) implies that the generalized eigenspace $E_{\mu_0} = \bigcup_{k\geqslant 1} \mathrm{Ker}(A_{\mu_0}^k)$ is also finite dimensional, see [14, Section IV.5.4]. For the spectral theory, one considers a complexification of the space $\mathfrak{X}$.

Deep results from perturbation theory, see [14, Sections II.5.1 and III.6.4], imply that there exists $\delta' \in \,]0, \delta]$ and a continuous map of finite dimensional subspaces $[\mu_0 - \delta', \mu_0 + \delta'] \ni \mu \mapsto E_\mu \subset \mathfrak{X}$ such that for all μ:

- $\dim(E_\mu) = \dim(E_{\mu_0})$;
- E_μ is invariant by A_μ.

Denote by $\overline{A}_\mu$ the restriction of A_μ to E_μ, and ϵ_μ denote the sign of the determinant of $\overline{A}_\mu$:

$$\epsilon_\mu = \begin{cases} +1, & \text{if } \det(\overline{A}_\mu) > 0; \\ 0, & \text{if } \det(\overline{A}_\mu) = 0; \\ -1, & \text{if } \det(\overline{A}_\mu) < 0. \end{cases} \tag{A.1}$$

Note that $\epsilon_\mu = 0$ only if A_μ is singular, because $\mathrm{Ker}(\overline{A}_\mu) = \mathrm{Ker}(A_\mu) \cap E_\mu$.

Bifurcation Theorem for Fredholm Operators. — *In the above situation, assume:*

(BT1) *for* $\mu \in [\mu_0 - \delta, \mu_0[\,\bigcup\,]\mu_0, \mu_0 + \delta]$, *the operator* A_μ *is nonsingular*[(11)];

(BT2) $\epsilon_{\mu_0 - \delta} \neq \epsilon_{\mu_0 + \delta}$.

Then, $(\mathfrak{x}_{\mu_0}, \mu_0)$ *is a bifurcation point for the equation* $F(\mathfrak{x}, \mu) = 0$, *i.e., the closure of the set* $\{(\mathfrak{x}, \mu) : \mathfrak{x} \neq \mathfrak{x}_\mu,\ F(\mathfrak{x}, \mu) = 0\}$ *contains* $(\mathfrak{x}_{\mu_0}, \mu_0)$.

Proof. — See [15, Theorem II.4.4]. □

Condition (BT2) in the above theorem is usually referred to by saying that A_μ has an *odd crossing number* at $\mu = \mu_0$.

The bifurcation result for minimal embeddings proved in Theorem 5.4 employs the above bifurcation criterion for Fredholm operators. Usually, the odd crossing number assumption (BT2) is hard to verify, in that one has no explicit description of the perturbed eigenspaces E_μ. However, in some cases the following elementary observation simplifies the task.

(11) i.e., $A_\mu \colon \mathfrak{X} \to \mathfrak{Z}$ is an isomorphism

Remark A.2. — Assume that $[\mu_0-\delta,\mu_0+\delta]\ni\mu\mapsto A_\mu, A'_\mu$ are continuous paths of Fredholm operators of index 0 from $\mathfrak{X}$ to $\mathfrak{Z}$, with $A_{\mu_0}=A'_{\mu_0}$, and assume that for all $\mu\in[\mu_0-\delta,\mu_0[\bigcup]\mu_0,\mu_0+\delta]$, both A_μ and A'_μ are nonsingular. Assume that for all $r>0$ there exists $\delta'>0$ and two continuous paths of invertible operators in the ball[12] $B(A_{\mu_0},r)$ centered at A_{μ_0} and of radius r joining $A_{\mu_0-\delta'}$ with $A'_{\mu_0-\delta'}$ and $A_{\mu_0+\delta'}$ with $A'_{\mu_0+\delta'}$ respectively. Then A_μ has an odd crossing number at μ_0 if and only if A'_μ has an odd crossing number at μ_0. This follows easily from the fact that the sign function ϵ, which is defined in a sufficiently small neighborhood of A_{μ_0}, is constant along continuous paths of invertible operators. This is because the sign of the determinant does not change along continous paths of invertible linear maps.

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Miyuki KOISO
Institute of Mathematics for Industry
Kyushu University
744, Motooka Nishi-ku
Fukuoka 819-0395 (Japan)
koiso@math.kyushu-u.ac.jp

Paolo PICCIONE
Departamento de Matemática
Universidade de São Paulo
Rua do Matão 1010
CEP 05508-900, São Paulo, SP (Brazil)
piccione.p@gmail.com

Toshihiro SHODA
Faculty of Education
Saga University
Saga, SAGA 840-8502 (Japan)
tshoda@cc.saga-u.ac.jp