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GROWTH OF HOMOLOGY TORSION IN FINITE COVERINGS AND HYPERBOLIC VOLUME

by Thang T. Q. LÊ (*)

Abstract. — We give an upper bound for the growth of homology torsions of finite coverings of irreducible oriented 3-manifolds in terms of the hyperbolic volume.

Résumé. — Nous donnons une limite supérieure pour la croissance des torsions homologiques de revêtements finis de 3-variétés orientées irréductibles en termes du volume hyperbolique.

1. Introduction

1.1. Growth of homology torsions in finite coverings

Suppose X is a connected finite CW-complex with fundamental group $\Pi = \pi_1(X)$. For a subgroup $\Gamma < \Pi$ of finite index let X_Γ be the covering of X corresponding to Γ, and $t_j(\Gamma)$ be the size of the $\mathbb{Z}$-torsion part of $H_j(X_\Gamma, \mathbb{Z})$. We want to study the growth of $t_j(\Gamma)$.

A sequence of subgroups $(\Gamma_k)_{k=1}^\infty$ of Π is *nested* if $\Gamma_{k+1} < \Gamma_k$ and it is *exhaustive* if $\bigcap_k \Gamma_k = \{e\}$, where e is the unit of Π. It is known that the fundamental group of any compact 3-manifold is residually finite [14], i.e. it has an exhaustive nested sequence of normal subgroups of finite index.

We will prove the following result and some generalizations of it.

Theorem 1.1. — *Suppose X is an orientable irreducible compact 3-manifold whose boundary ∂X is either an empty set or a collection of*

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tori. Let $(\Gamma_k)_{k=1}^{\infty}$ *be an exhaustive nested sequence of normal subgroups of* $\Pi = \pi_1(X)$ *of finite index. Then*

$$\limsup_{k\to\infty} \frac{\ln t_1(\Gamma_k)}{[\Pi : \Gamma_k]} \leqslant \frac{\mathrm{vol}(X)}{6\pi}\,.$$

Here $\mathrm{vol}(X)$ is defined as follows. If the fundamental group of X is finite, let $\mathrm{vol}(X) = 0$. Suppose the fundamental group of X is infinite. By the Geometrization Conjecture of W. Thurston proved by G. Perelman (see e.g. [6, 7]), every piece in the Jaco–Shalen–Johansson decomposition of X is either Seifert fibered or hyperbolic. Define $\mathrm{vol}(X)$ as the sum of the volumes of all hyperbolic pieces.

1.2. More general limit: trace limit

For any group G with unit e define a trace function $\mathrm{tr}_G : G \to \mathbb{Z}$ by $\mathrm{tr}_G(g) = 1$ if $g = e$ and $\mathrm{tr}_G(g) = 0$ if $g \neq e$. This extends to a $\mathbb{C}$-linear map $\mathrm{tr}_G : \mathbb{C}[G] \to \mathbb{C}$. Here $\mathbb{C}[G]$ is the group ring of G with complex coefficients.

Assume X is a 3-manifold satisfying the assumption of Theorem 1.1, with $\Pi = \pi_1(X)$. Let $\mathcal{G}$ be the set of all finite index subgroups of Π. For any $\Gamma \in \mathcal{G}$, Π acts on the right on the finite set $\Gamma\backslash\Pi$ of right cosets of Γ, and the action gives rise to an action of Π on $\mathbb{C}[\Gamma\backslash\Pi]$, the $\mathbb{C}$-vector space with base $\Gamma\backslash\Pi$. For $g \in \Pi$, define

$$\mathrm{tr}_{\Gamma\backslash\Pi}(g) = \frac{\mathrm{tr}(g, \mathbb{C}[\Gamma\backslash\Pi])}{[\Pi : \Gamma]}\,,$$

where $\mathrm{tr}(g, \mathbb{C}[\Gamma\backslash\Pi])$ is the trace of the operator g acting on the vector space $\mathbb{C}[\Gamma\backslash\Pi]$. Note that when Γ is a normal subgroup of Π, then $G := \Gamma\backslash\Pi$ is a group, and $\mathrm{tr}_{\Gamma\backslash\Pi}(g)$ coincides with the above definition of $\mathrm{tr}_G(p(g))$, where $p(g)$ is the image of g in the quotient group G.

Suppose $(\Gamma_k)_{k=1}^{\infty}$ is a sequence of subgroups of Π of finite index, i.e. $\Gamma_k \in \mathcal{G}$. We say $\Gamma_k \xrightarrow{\mathrm{tr}} 1$ if for any $g \in \Pi$,

$$\lim_{k\to\infty} \mathrm{tr}_{\Gamma_k\backslash\Pi}(g) = \mathrm{tr}_{\Pi}(g)\,.$$

For example, if $(\Gamma_k)_{k=1}^{\infty}$ is a nested exhaustive sequence of normal subgroups of Π of finite index, then $\Gamma_k \xrightarrow{\mathrm{tr}} 1$.

We have the following stronger version of Theorem 1.1.

Theorem 1.2. — *Suppose X is an orientable irreducible compact 3-manifold whose boundary ∂X is either an empty set or a collection of tori.*

Then

$$\limsup_{\mathcal{G}\ni\Gamma\to 1} \frac{\ln t_1(\Gamma)}{[\Pi:\Gamma]} \leqslant \frac{\mathrm{vol}(X)}{6\pi}. \tag{1.1}$$

Here $\limsup_{\mathcal{G}\ni\Gamma\to 1} f(\Gamma)$, for a function $f : \mathcal{G} \to \mathbb{R}$, is defined to be the infimum of the set of all values L such that for any sequence $\Gamma_k \in \mathcal{G}$ with $\Gamma_k \xrightarrow{\mathrm{tr}} 1$, one has $\limsup_{k\to\infty} f(\Gamma_k) \leqslant L$.

Conjecture 1.3. — *Suppose X is a an orientable irreducible compact 3-manifold whose boundary ∂X is either an empty set or a collection of tori.*

(1) *Let $\mathcal{G}^N$ be the set of all normal subgroups of Π of finite index. One has*
$$\limsup_{\mathcal{G}^N\ni\Gamma\to 1} \frac{\ln t_1(\Gamma)}{[\Pi:\Gamma]} = \frac{\mathrm{vol}(X)}{6\pi}.$$
(2) *(Weaker version) One has*
$$\limsup_{\mathcal{G}\ni\Gamma\to 1} \frac{\ln t_1(\Gamma)}{[\Pi:\Gamma]} = \frac{\mathrm{vol}(X)}{6\pi}.$$

Similar conjectures were formulated independently by Bergeron and Venkatesh [5] and Lück [21].

It is clear that the left hand side of (1.1) is non-negative. Hence we have the following result.

Corollary 1.4. — *The strong version of Conjecture 1.3 holds true for 3-manifolds X satisfying the assumption of Theorem 1.2 with* $\mathrm{vol}(X) = 0$.

For example, if X is the complement in S^3 of a tubular neighborhood of a torus link, then the strong conjecture holds for X. A 3-manifold X satisfying the assumption of Theorem 1.2 has $\mathrm{vol}(X) = 0$ if and only if X is spherical or a graph manifold.

A statement similar to Corollary 1.4, applicable to a large class of manifolds, was proved in [21].

1.3. Branched covering

We can extend Theorem 1.2 in the following direction. Suppose K is a knot in S^3 and $X = S^3 \setminus N(K)$, where $N(K)$ is an open tubular neighborhood of K. For a subgroup $\Gamma \leqslant \Pi = \pi_1(X)$ let $\hat{X}_\Gamma$ be the branched Γ-covering of S^3, branched along K. This means, $\hat{X}_\Gamma$ is obtained from X_Γ by attaching solid tori to ∂X_Γ in such a way that any lift of a meridian of K bounds a disk in $\hat{X}_\Gamma$.

THEOREM 1.5. — *Suppose K is a knot in S^3. For a subgroup $\Gamma \leqslant \Pi = \pi_1(S^3 \setminus K)$ let $\hat{X}_\Gamma$ be the branched Γ-covering of S^3, branched over K. Then*

$$\limsup_{\mathcal{G} \ni \Gamma \to 1} \frac{\ln |\operatorname{Tor}_{\mathbb{Z}} H_1(\hat{X}_\Gamma, \mathbb{Z})|}{[\Pi : \Gamma]} \leqslant \frac{\operatorname{vol}(X)}{6\pi}.$$

Here $|\operatorname{Tor}_{\mathbb{Z}} H_1(\hat{X}_\Gamma, \mathbb{Z})|$ is the size of the $\mathbb{Z}$-torsion part of $H_1(\hat{X}_\Gamma, \mathbb{Z})$.

1.4. On trace convergence

In the case when $(\Gamma_k)_{k=1}^\infty$ is a nested sequence of subgroups of Π of finite index, the definition of $\Gamma_k \xrightarrow{\text{tr}} 1$ was introduced in [10]. Even when $(\Gamma_k)_{k=1}^\infty$ is not nested, $\Gamma_k \xrightarrow{\text{tr}} 1$ if and only if the representations $\rho_k : \Pi \to \mathbb{C}[\Gamma\backslash\Pi]$ with $k = 1, 2, \ldots$, form an *arithmetic approximation of* Π in the sense of [10, Definition 9.1].

For a discrete group Π, there exists a sequence $(\Gamma_k)_{k=1}^\infty$ of subgroups of finite index such that $\Gamma_k \xrightarrow{\text{tr}} 1$ if and only if Π is residually finite.

It turns out that the limit $\Gamma_k \xrightarrow{\text{tr}} 1$, for not necessarily nested sequences, is closely related to known limits in the literature.

The case of normal subgroups. — When each of Γ_k is a normal subgroup of Π of finite index, i.e. $\Gamma_k \in \mathcal{G}^N$, the definition of $\Gamma_k \xrightarrow{\text{tr}} 1$ simplifies. Suppose $(\Gamma_k)_{k=1}^\infty$ is a nested sequence of normal subgroups of Π of finite index. Then $\Gamma_k \xrightarrow{\text{tr}} 1$ if and only if $\bigcap \Gamma_k = \{e\}$, i.e. $(\Gamma_k)_{k=1}^\infty$ is exhaustive.

More generally, suppose $(\Gamma_k)_{k=1}^\infty$ is a not necessarily nested sequence of normal subgroups of Π of finite index. Then $\Gamma_k \xrightarrow{\text{tr}} 1$ if and only if $\bigcap_{m=1}^\infty \bigcup_{k=m}^\infty \Gamma_k = \{e\}$. This means, $\Gamma_k \xrightarrow{\text{tr}} 1$ if and only if for every non-trivial $g \in \Pi$, g eventually does not belong to Γ_k.

Relation to sofic approximation. — Suppose Γ_k is a sequence of subgroups of finite index of Π. Then $\Gamma_k \xrightarrow{\text{tr}} 1$ if and only if $\Sigma = \{\sigma_k\}_{k=1}^\infty$, where σ_k is the map from Π to the permutation group of $\Gamma\backslash\Pi$, is a *sofic approximation* to Π. For the definition of sofic approximation see [8]. Sofic approximation has been important in dynamical system theory. Sofic groups, which are groups having a sofic approximation (not necessarily coming from actions on cosets), were introduced by Gromov in 1999.

Relation to Benjamini–Schramm convergence. — Suppose X is a hyperbolic 3-manifold. Raimbault observed that $\Gamma_k \xrightarrow{\text{tr}} 1$ if and only if X_{Γ_k} Benjamini–Schramm (BS) converge to the hyperbolic space $\mathbb{H}^3$ in the sense

of [1]. The notion of BS convergence was introduced in [1] for more general sequences of manifolds, not necessarily coverings of a fixed manifold.

1.5. Growth of Betti numbers

Suppose X is a connected finite CW-complex with fundamental group Π. For a subgroup $\Gamma \leqslant \Pi$ of finite index let $b_j(\Gamma)$ be the j-th Betti number of X_Γ.

If $(\Gamma_k)_{k=1}^{\infty}$ is an exhaustive nested sequence of normal subgroups of Π, then Kazhdan [15, 12] showed that

$$\limsup_{k\to\infty} \frac{b_j(X_{\Gamma_k})}{[\Pi:\Gamma_k]} \leqslant b_j^{(2)}(\tilde{X}), \tag{1.2}$$

where $b_j^{(2)}(\tilde{X})$ is the L^2-Betti number of the universal covering $\tilde{X}$. For the definition of L^2-Betti number and L^2-invariants in general, see [20]. Lück [18] then showed that one actually has a much stronger result

$$\lim_{k\to\infty} \frac{b_j(X_{\Gamma_k})}{[\Pi:\Gamma_k]} = b_j^{(2)}(\tilde{X}). \tag{1.3}$$

Farber extended Lück's result (1.3) to the trace convergence, $\Gamma_k \xrightarrow{\mathrm{tr}} 1$ (no normal subgroups). The following is a special case of [10, Theorem 9.2]: If $(\Gamma_k)_{k=1}^{\infty}$ is a sequence of subgroups of Π of finite index such that $\Gamma_k \xrightarrow{\mathrm{tr}} 1$, then

$$\lim_{k\to\infty} \frac{b_j(X_{\Gamma_k})}{[\Pi:\Gamma_k]} = b_j^{(2)}(\tilde{X}). \tag{1.4}$$

Theorem 1.2 is an analog of Kazhdan's inequality for the growth of the torsion part of the homology, and Conjecture 1.3 asks for analogs of Lück's and Farber's equalities. In general, the question about the growth of the torsion part of the homology is considered much more difficult than the similar question for the free part of the homology; it is closely related to the difficult question of approximating the Fuglede-Kadison determinant by finite determinants, see [20, Question 13.52] and the discussion there. There has been little progress in this direction. At the moment of this writing, there is even no known example of a hyperbolic 3-manifold (closed or cusped) and a sequence $\Gamma_k \xrightarrow{\mathrm{tr}} 1$ such that $\lim_{k\to\infty} \frac{\ln t_1(\Gamma_k)}{[\Pi:\Gamma_k]} > 0$.

1.6. Abelian covering

In the abelian case, results on the growth of the torsion part of the homology are much more satisfactory, as below. Proofs of results on the growth of the torsion part of the homology are more complicated than those on the growth of the Betti numbers.

Suppose $\tilde{X} \to X$ is a regular covering with the group of deck transformations equal to $\mathbb{Z}^n$. Here X is a finite CW-complex. We choose generators $z_1, \dots, z_n$ of $\mathbb{Z}^n$ and identify $\mathbb{Z}[\mathbb{Z}^n] \equiv \mathbb{Z}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$.

Let $\mathcal{G}$ in this case be the set of all subgroups $\Gamma \leqslant \mathbb{Z}^n$ of finite index. Again X_Γ is the corresponding finite covering, and $b_j(\Gamma)$ and $t_j(\Gamma)$, for $\Gamma \in \mathcal{G}$, are respectively the rank and the size of the $\mathbb{Z}$-torsion part of $H_j(X_\Gamma, \mathbb{Z})$. In this abelian case, $\Gamma_k \xrightarrow{\text{tr}} 1$ is equivalent to $\lim_{k\to\infty} \langle \Gamma_k \rangle = \infty$, where $\langle \Gamma \rangle$, for $\Gamma \leqslant \mathbb{Z}^n$, is the smallest among norms of non-zero elements in Γ.

In this case, we have a positive answer to an analog of Conjecture 1.3, as follows. For any $j \geqslant 0$,

$$\limsup_{\mathbb{Z}^n \supset \Gamma \to 1} \frac{\ln t_j(X_\Gamma)}{[\mathbb{Z}^n : \Gamma]} = \mu(\Delta(H_j(\tilde{X}, \mathbb{Z}))). \tag{1.5}$$

Here $\mu(f)$, for a non-zero $f \in \mathbb{Z}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$, is the Mahler measure of f, and $\Delta(H_j(\tilde{X}, \mathbb{Z}))$ is the first non-trivial Alexander polynomial of $H_j(\tilde{X}, \mathbb{Z})$, considered as a $\mathbb{Z}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$-module. For details and discussions of related results, see [17]. The proof of (1.5) in [17] uses tools from algebraic geometry, commutative algebra, and also results from algebraic dynamics and cannot be generalized to non-commutative cases.

For an arbitrary finite CW-complex with fundamental group Π and a fixed index j, we don't know what would be the upper limit of the left hand side of (1.5), with $\mathbb{Z}^n$ replaced by Π. Only when X has some geometric structure, like 3-dimensional manifolds, do we have results like Theorem 1.2.

1.7. Related results

The growth of torsion parts of homology of coverings or more generally bundles over a fixed manifold has attracted a lot of attention lately. For related results see e.g. [1, 4, 5, 9, 23, 24, 25]. Probably C. Gordon [11] was the first to ask about the asymptotic growth of torsion of homology in finite, albeit abelian, coverings. Recently R. Sauer [27] gave an upper bound for the growth rate of $t_j(\Gamma)$, for closed aspherical Riemannian manifolds, but the bound is not expected to be sharp like in the one in Theorem 1.2.

1.8. Organization

In Section 2 we recall the definition of geometric determinant, introduce the notion of metric abelian groups, and use them to get an estimate for the torsion parts in exact sequences. In Section 3 we recall the definition of the Fuglede-Kadison determinant and prove an upper bound for growth of geometric determinants by Fuglede-Kadison determinant. Finally in Section 4 we give proofs of the main results.

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2. Geometric determinant, lattices, and volume in inner product spaces

In this section we recall the definition of the geometric determinant, introduce the notion of metric abelian groups, and establish certain results on upper bounds for torsion parts in exact sequences. For a finitely-generated abelian group A let $\mathrm{rk}(A)$ and $t(A)$ be respectively the rank of A and the size of the $\mathbb{Z}$-torsion part of A. For a ring R let $\mathrm{Mat}(n \times m, R)$ denote the R-module of all $n \times m$ matrices with entries in R.

2.1. Geometric determinant

For a linear map $f : V_1 \to V_2$, where each V_i is a finite-dimensional inner product space the *geometric determinant* $\det'(f)$ is defined to be the product of all non-zero singular values of f. Recall that $x \in \mathbb{R}$ is singular

value of f if $x \geqslant 0$ and x^2 is an eigenvalue of f^*f. By convention $\det'(f) = 1$ if f is the zero map. We always have $\det'(f) > 0$. It is easy to show that $\det'(f) = \det'(f^*)$.

Since the maximal singular value of f is the norm $||f||$, we have

$$\det'(f) \leqslant ||f||^{\mathrm{rk}(f)} \qquad \text{with convention } 0^0 = 1. \tag{2.1}$$

If $M \in \mathrm{Mat}(n \times m, \mathbb{C})$, we define $\det'(M) = \det'(f)$, where $f : \mathbb{C}^n \to \mathbb{C}^m$ is the linear operator defined by M, and $\mathbb{C}^n$ and $\mathbb{C}^m$ are standard Hermitian spaces.

Remark 2.1. — The geometric meaning of $\det' f$ is the following. The map f restricts to a linear isomorphism f' from $\mathrm{Im}(f^*)$ to $\mathrm{Im}(f)$, each is an inner product space. Then $\det' f = |\det(f')|$, where the ordinary determinant $\det(f')$ is calculated using orthonomal bases of the inner product spaces.

LEMMA 2.2. — *Suppose g is a generator of a cyclic group A of order l. Define an inner product structure on $\mathbb{C}[A]$ such that A is an orthonormal basis. Let A act on $\mathbb{C}[A]$ by left multiplication. Then* $\det'(1-g) = l$.

Proof. — Let $f = 1 - g$. Then $f^* = 1 - g^{-1}$, and $f^*f = 2 - g - g^{-1}$. Let $\zeta = \exp(2\pi i/l)$. Since ζ^k, with $k = 0, 1, \ldots, l-1$, are all eigenvalues of g, the eigenvalues of f^*f are $2 - \zeta^k - \zeta^{-k} = |1 - \zeta^k|^2$, with $k = 0, 1, \ldots, l-1$. Excluding the 0 value, we have

$$\det'(f) = \left| \prod_{j=1}^{l-1} (1 - \zeta^l) \right| = l\,,$$

where the last identity follows since for any complex number z one has

$$\prod_{j=1}^{l-1} (z - \zeta^k) = \sum_{k=0}^{l-1} z^k = \frac{z^l - 1}{z - 1}\,.$$

(The above holds since ζ^k, with $k = 1, \ldots, l-1$, are roots of $\frac{z^l-1}{z-1}$.) □

2.2. Volume of lattices

We recall here some well-known facts about volumes of lattices in inner product spaces.

Suppose V is a finite-dimensional inner product space. In this paper, any discrete finite-rank abelian subgroup $\Lambda \subset V$ will be called a *lattice* in V.

Note that we don't require Λ to be of maximal rank as in many texts. For a lattice $\Lambda \subset V$ with $\mathbb{Z}$-basis $v_1, \dots, v_l$ define

$$\operatorname{vol}(\Lambda) = |\det\left((v_i, v_j)_{i,j=1}^{l}\right)|^{1/2} > 0\,.$$

That is, $\operatorname{vol}(\Lambda)$ is the volume of the parallelepiped spanned by a set of basis vectors. By convention, the volume of the zero space is 1.

It is clear that if $\Lambda_1 \subset \Lambda_2$ are lattices in V of the same rank, then

$$|\Lambda_2/\Lambda_1| = \frac{\operatorname{vol}(\Lambda_1)}{\operatorname{vol}(\Lambda_2)}\,. \tag{2.2}$$

Let V and W be finite-dimensional inner product spaces. Suppose Λ is a lattice in V of rank equal the dimension of V and $f : \Lambda \to W$ is an abelian group homomorphism such that the image $\operatorname{Im}(f)$ is discrete in W. The kernel $\ker(f)$ and the image $\operatorname{Im}(f)$ are lattices in respectively V and W. Note that f extends uniquely to a linear map $\tilde{f} : V \to W$, and we put $\det'(f) = \det'(\tilde{f})$. We have

$$\operatorname{vol}(\ker f)\, \operatorname{vol}(\operatorname{Im}(f)) = \det'(f)\, \operatorname{vol}(\Lambda)\,. \tag{2.3}$$

2.3. Metric abelian groups

A *metric abelian group* is a finitely generated free abelian group A equipped with an inner product on $A \otimes_{\mathbb{Z}} \mathbb{R}$. An *integral metric abelian group* is a metric abelian group A such that the inner product of any two elements $x, y \in A$ is an integer.

Suppose A is a metric abelian group. Then any subgroup $B \leqslant A$ inherits a metric from A. Besides, if A is an integral metric abelian, then the induced metric on B is also integral.

What we will use is the following obvious property of an integral metric group: If A is an integral metric group and $B \leqslant A$, then $\operatorname{vol}(B) \geqslant 1$.

The following is a basic example. Suppose X is a finite CW-complex. The cellular $\mathbb{Z}$-complex $\mathcal{C}(X)$ of X consists of free abelian groups $\mathcal{C}_j(X)$, which are free abelian groups with bases the sets of cells (of corresponding dimensions) of X. We equip $\mathcal{C}_j(X)$ with a metric such that the mentioned basis is an orthonormal basis. All these metrics are integral.

Suppose $\alpha : A \to B$ is a group homomorphism between metric abelian groups. Then $\alpha \otimes \mathrm{id} : A \otimes_{\mathbb{Z}} \mathbb{R} \to B \otimes_{\mathbb{Z}} \mathbb{R}$ is a linear map between finite-dimensional inner product spaces. We define

$$\det'(\alpha) := \det'(\alpha \otimes_{\mathbb{Z}} \mathrm{id})\,.$$

2.4. Torsion estimate

Recall that for a finitely-generated abelian group A, $t(A)$ is the size of its $\mathbb{Z}$-torsion part.

LEMMA 2.3. — *Suppose F_1, F_2 are integral metric abelian groups.*

(1) *If $\alpha : F_1 \to F_2$ is a group homomorphism, then*

$$t(\operatorname{coker}(\alpha)) \leqslant \operatorname{vol}(\operatorname{Im}(\alpha)) \leqslant \det{}'(\alpha) \operatorname{vol}(F_1)\,. \tag{2.4}$$

$$t(\operatorname{coker}(\alpha)) \operatorname{vol}(\ker(\alpha)) \leqslant \det{}'(\alpha) \operatorname{vol}(F_1)\,. \tag{2.5}$$

(2) *If $F_1 \xrightarrow{\beta} F_2 \to A \to B$ is an exact sequence, where A and B are finitely-generated abelian groups, then*

$$t(A) \leqslant t(B)\ \det{}'(\beta)\ \operatorname{vol}(F_1)\,.$$

Proof.

(1). — Let $\overline{\operatorname{Im}\alpha} = ((\operatorname{Im}\alpha) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap F_2$, where both $(\operatorname{Im}\alpha) \otimes_{\mathbb{Z}} \mathbb{Q}$ and F_2 are considered as subsets of $F_2 \otimes_{\mathbb{Z}} \mathbb{Q}$. Then (2.2) shows that

$$t(\operatorname{coker}(\alpha)) = \frac{\operatorname{vol}(\operatorname{Im}(\alpha))}{\operatorname{vol}(\overline{\operatorname{Im}\alpha})} \leqslant \operatorname{vol}(\operatorname{Im}(\alpha))\,, \tag{2.6}$$

where the last inequality follows since $\operatorname{vol}(\overline{\operatorname{Im}\alpha}) \geqslant 1$ due to the integrality of the metric. By (2.3),

$$\operatorname{vol}(\operatorname{Im}(\alpha)) = \frac{\det{}'(\alpha) \operatorname{vol}(F_1)}{\operatorname{vol}(\ker(\alpha))} \leqslant \det{}'(\alpha) \operatorname{vol}(F_1)\,,$$

which, together with (2.6), proves (2.4).

(2). — First we observe that if

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

is an exact sequence of finitely generated abelian groups, then

$$t(A_2) \leqslant t(A_1) t(A_3)\,.$$

From the assumption we have the following exact sequence

$$0 \to \operatorname{coker}(\beta) \to A \to B' \to 0\,,$$

where $B' \leqslant B$. By the above observation,

$$t(A) \leqslant t(B')\, t(\operatorname{coker}\beta) \leqslant t(B)\, t(\operatorname{coker}\beta) \leqslant t(B)\ \det{}'(\beta)\ \operatorname{vol}(F_1)\,,$$

where the last inequality follows from (2.5). This completes the proof of the lemma. □

2.5. Torsion of a matrix

For a matrix $M \in \mathrm{Mat}(m \times n, \mathbb{Z})$ let $t(M) := t(\operatorname{coker} M)$, where M is also considered as the homomorphism $\mathbb{Z}^m \to \mathbb{Z}^n$ given by $x \to xM$.

PROPOSITION 2.4. — *Suppose $M \in \mathrm{Mat}(m \times n, \mathbb{Z})$.*

(1) *Let M^* be the transpose of M, then $t(M) = t(M^*)$.*
(2) *One has $t(M) \leqslant \det'(M)$.*
(3) *Suppose $(\mathcal{C}, \partial)$ is a chain complex of free finitely generated $\mathbb{Z}$-modules. Then $t(H_i(\mathcal{C})) = t(\operatorname{coker} \partial_{i+1})$.*

Proof. — We first prove (1). Note that $t(M)$ is equal to the greatest common divisor of all the minors of M of size r where $r = \mathrm{rk}(M)$, see [28, Section 4.2]. From here we have $t(M) = t(M^*)$.

(2) is a special case of (2.4). For (3), notice that the torsion part of either $H_i(\mathcal{C})$ or $\operatorname{coker} \partial_{i+1}$ is equal to $\overline{\operatorname{Im} \partial_{i+1}} / \operatorname{Im} \partial_{i+1}$. □

3. Fuglede-Kadison determinant

We recall here the definition and establish some properties of the Fuglede-Kadison determinant. We will introduce the Fuglede-Kadison determinant only for a class of operators which we will need in this paper. For a detailed treatment of the Fuglede-Kadison determinant, the reader should consult the book [20]. We prove that the Fuglede-Kadison determinant of a matrix with entries in $\mathbb{Z}[\Pi]$ serves as an upper bound for the growth of the geometric determinants of a sequence of finite matrices which approximate the original matrix well enough. This extends a result of Lück.

Recall that Π is the fundamental group of a finite CW-complex.

3.1. Fuglede-Kadison determinant of a density function

For simplicity we use the following definition which is more restrictive than the one in [18].

DEFINITION 3.1.

(1) *A right continuous function*

$$F : [0, \infty) \to [0, \infty)$$

is called a density function *if*

(a) *F is increasing (i.e. $F(\lambda) \leqslant F(\lambda')$ if $\lambda \leqslant \lambda'$), and*
(b) *There is a constant K such that F is constant on interval $[K, \infty)$.*

(2) *A density function F is said to be in the determinantal class if the integral $\int_{0^+}^{\infty} \ln(\lambda) \mathrm{d}F$ exists as a real number. If F is in the determinant class, define its* determinant *by*

$$\det(F) := \exp\left(\int_{0^+}^{\infty} \ln(\lambda)\, \mathrm{d}F\right).$$

If F is not in the determinantal class, define $\det(F) = 0$.

(In probability theory, a density function is also known as a cumulative distribution function.)

Let K be the number in Condition (1b) of Definition 3.1. If F is a density function in the determinantal class, then one has (see [20, Lemma 3.15])

$$\ln \det(F) = (F(K) - F(0)) \ln(K) - \int_{0^+}^{K} \frac{F(\lambda) - F(0)}{\lambda}\, \mathrm{d}\lambda. \tag{3.1}$$

For an increasing function F we define

$$F^+(\lambda) = \lim_{\varepsilon \to 0^+} F(\lambda + \varepsilon).$$

If $F : [0, \infty) \to [0, \infty)$ is a not necessarily right continuous function satisfying conditions (1a)–(1b), then F^+, which is F made right continuous, is a density function.

3.2. Von Neumann algebra of a group and trace function

Let $\ell^2(\Pi)$ be the Hilbert space with orthonormal basis Π. In other words, $\ell^2(\Pi)$ is the set of all formal sums $\sum_{g \in \Pi} c_g g$, with $c_g \in \mathbb{C}$ and $\sum_{g \in \Pi} |c_g|^2 < \infty$, with inner product $\langle g, g' \rangle = \delta_{g,g'}$. For every positive integer n, $(\ell^2(\Pi))^n$ inherits a Hilbert structure, where

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_{j=1}^{n} \langle x_j, y_j \rangle.$$

We will consider $(\ell^2(\Pi))^n$ as a left Π-module by the left multiplication.

By definition, the von Neumann algebra $\mathcal{N}(\Pi)$ of Π is the $\mathbb{C}$-algebra of bounded Π-equivariant operators from $\ell^2(\Pi)$ to $\ell^2(\Pi)$.

For $f \in \mathcal{N}(\Pi)$ its trace is defined by

$$\mathrm{tr}_{\Pi}(f) = \langle e, f(e) \rangle,$$

where e is the unit of Π. More generally, suppose $f : (\ell^2(\Pi))^n \to (\ell^2(\Pi))^n$ is a bounded Π-equivariant operator, define its trace by

$$\mathrm{tr}(f) = \sum_{j=1}^{n} \langle e_j, f e_j \rangle ,$$

where $e_j = 0^{j-1} \times e \times 0^{n-j} \in (\ell^2(\Pi))^n$.

3.3. Fuglede-Kadison determinants

Suppose B is an $n \times k$ matrix with entries in $\mathbb{C}[\Pi]$. Let $\mathcal{R}_B : (\ell^2(\Pi))^n \to (\ell^2(\Pi))^k$ be the bounded $\mathbb{Z}[\Pi]$-linear operator defined by $x \to xB$. Then $(\mathcal{R}_B)^* \mathcal{R}_B = \mathcal{R}_{BB^*}$, where $(\mathcal{R}_B)^*$ is the adjoint operator, and B^* is obtained from B by the transpose followed by the conjugation map on $\mathbb{C}[\Pi]$ given by $\sum c_i g_i \to \sum \bar{c_i} g_i^{-1}$, with $\bar{c_i}$ being the complex conjugation of c_i. We define the norm $\|B\| = \|\mathcal{R}_B\|$.

Let $\{P(\lambda), \lambda \in [0, \infty)\}$ be the right continuous spectral family of the positive operator $\mathcal{R}_{BB^*} : (\ell^2(\Pi))^n \to (\ell^2(\Pi))^n, x \to xBB^*$. *The spectral density function of* B is defined by

$$F(\lambda) = F_B(\lambda) := \mathrm{tr}_\Pi(P(\lambda)) .$$

Then $F : [0, \infty) \to [0, \infty)$ is a density function (Definition 3.1). We say that B is *in the determinantal class* if F is in the determinantal class. Define

$$\det_\Pi(B) = \sqrt{\det(F)} .$$

When B is in the determinantal class $\det_\Pi(B)$ is a positive real number.

3.4. Relation between Fuglede-Kadison determinant and geometric determinant

Any matrix $B \in \mathrm{Mat}(n \times k, \mathbb{C})$ (with complex entries) can be considered as an element of $\mathrm{Mat}(n \times k, \mathbb{C}[\Pi])$, where Π is the trivial group, since $\mathbb{C}[\Pi] = \mathbb{C}$. In this case, tr_Π is the usual trace, B is always in the determinantal class, and (see [20, Example 3.12])

$$\det_\Pi(B) = \det'(B) . \tag{3.2}$$

For $\lambda \geqslant 0$ the spectral density function $F_B(\lambda)$ is the number of eigenvalues of $\mathcal{R}_{BB^*}$ which are less than or equal to λ, counted with multiplicity. The density function $F_B : [0, \infty) \to [0, \infty)$ is a bounded, right continuous, step function, and $F_B(\lambda) = \mathrm{nr}(B)$, which is the number of rows of B, if $\lambda \geqslant \|B\|^2$. Besides, $F_B(0) = \dim \ker(\mathcal{R}_B)$.

3.5. Universal bound for norm

Suppose $B \in \mathrm{Mat}(n \times k, \mathbb{C}[\Pi])$. For any $\Gamma \in \mathcal{G}$ let $B_\Gamma : \mathbb{C}[\Gamma\backslash\Pi]^n \to \mathbb{C}[\Gamma\backslash\Pi]^k$ be the induced $\mathbb{C}$-homomorphism defined by $x \to xB$. We always equip $\mathbb{C}[\Gamma\backslash\Pi]$ (and consequently $\mathbb{C}[\Gamma\backslash\Pi]^n$) with the Hermitian structure in which $\Gamma\backslash\Pi$ is an orthonormal basis.

One can easily find a universal upper bound for the norm of all operators induced from one acting on $\mathbb{Z}[\Pi]^n$.

LEMMA 3.2. — *Suppose $B \in \mathrm{Mat}(n \times k, \mathbb{C}[\Pi])$. There is a constant K (depending on B) such that $\|B\| < K$ and $\|B_\Gamma\| < K$ for any $\Gamma \in \mathcal{G}$.*

Proof. — When Γ is a normal subgroup, the statement was proved in [18] with $K = nk \max_{i,j} |B_{ij}|_1$, where for an element $x = \sum c_g g \in \mathbb{C}[\Pi]$ one sets $|x|_1 = \sum |c_g|$. The easy proof in [18] also works for our more general case. □

3.6. Upper limit of growth of determinants

Suppose $B \in \mathrm{Mat}(n \times k, \mathbb{Z}[\Pi])$. In [18], Lück shows that if $(\Gamma_m)_{m \geqslant 1}$ is an exhaustive nested sequence of normal subgroups of Π, then

$$\limsup_{m\to\infty} \frac{\ln \det'(B_{\Gamma_m})}{[\Pi : \Gamma_m]} \leqslant \ln \det_\Pi(B)\,.$$

We will show that a similar result holds if the sequence $(B_{\Gamma_m})_{m \geqslant 0}$ is replaced by a sequence of matrices that approximates well enough the matrix B.

DEFINITION 3.3. — *Let B be a matrix with entries in $\mathbb{Z}[\Pi]$. A sequence $(B_m, N_m)_{m \geqslant 1}$ is said to tracely approximate B, if each B_m is a matrix with complex entries, each N_m is a positive number, and there exists $K > 0$ such that all the following conditions are satisfied.*

1. $\|B\|, \|B_m\| < K$.
2. *For every polynomial $p(z) \in \mathbb{C}[z]$ we have* $\mathrm{tr}_\Pi(p(BB^*)) = \lim_{m\to\infty} \frac{\mathrm{tr}(p(B_m B_m^*))}{N_m}$.
3. *If F is the spectral density of B and F_m is the spectral density of B_m, then* $F(0) = \lim_{m\to\infty} \frac{F_m(0)}{N_m}$.

THEOREM 3.4. — *Suppose a sequence $(B_m, N_m)_{m \geqslant 1}$ tracely approximates a matrix of determinantal class $B \in \mathrm{Mat}(n \times k, \mathbb{Z}[\Pi])$. Then*

$$\limsup_{m\to\infty} \frac{\ln \det'(B_m)}{N_m} \leqslant \ln \det_\Pi(B)\,.$$

The proof is a simple modification of Lück's proof and will be given in Appendix A. Actually, the definition of trace approximation is constructed so that Lück's proof works.

As a corollary, we get the following special case.

THEOREM 3.5. — *Let $B \in \mathrm{Mat}(n \times k, \mathbb{Z}[\Pi])$ be of determinantal class and $(\Gamma_m)_{m\geqslant 1}$ be a sequence of subgroups of finite index such that $\Gamma_m \xrightarrow{\mathrm{tr}} 1$. Then $(B_{\Gamma_m}, [\Pi : \Gamma_m])_{m\geqslant 1}$ tracely approximates B. Consequently,*

$$\limsup_{m\to\infty} \frac{\ln \det'(B_{\Gamma_m})}{[\Pi : \Gamma]} \leqslant \ln \det_{\Pi}(B)\,.$$

Proof. — Let K be the number appeared in Lemma 3.2, then we have (1) of Definition 3.3.

Let $p(z) \in \mathbb{C}[z]$. Then $(p(BB^*))_{\Gamma_m} = p(B_m B_m^*)$. Hence we have (2) by the definition of $\Gamma_m \xrightarrow{\mathrm{tr}} 1$.

The conclusion of (3), which is a generalization of [18, Theorem 0.1], is a special case of [10, Theorem 9.2]. See Equation (1.4). In [10], the results are formulated for the case of a chain complex coming from universal covering of a finite CW complex, but the proof there works for the general case when the boundary operators have entries in $\mathbb{Z}[\Pi]$. □

3.7. Perturbation of a sequence of matrices

The following statement shows that in many cases, a small perturbation of a sequence $(B_m, N_m)_{m\geqslant 1}$ tracely approximating B gives another sequence which also tracely approximates B. For a matrix B let $\mathrm{nc}(B)$ and $\mathrm{nr}(B)$ denote respectively the number of columns and the number of rows of B.

PROPOSITION 3.6. — *Suppose $(B_m, N_m)_{m\geqslant 1}$ tracely approximates a matrix B with entries in $\mathbb{Z}[\Pi]$. For each $m \geqslant 1$, assume d_m is a positive integer, and B_m is an upper left corner submatrix of a matrix A_m which has integer entries (i.e. B_m is obtained from an integer matrix A_m by removing several last columns and several last rows). Also assume that*

- *there exists $L > 0$ such that $\|A_m\| < L$ for all m,*
- *$\mathrm{nc}(A_m) - \mathrm{nc}(B_m) < d_m$ and $\mathrm{nr}(A_m) - \mathrm{nr}(B_m) < d_m$, and*

$$\lim_{m\to\infty} \frac{d_m}{N_m} = 0\,. \tag{3.3}$$

Then the sequence $(A_m, N_m)_{m\geqslant 1}$ also tracely approximates B. Consequently,

$$\limsup_{m\to\infty} \frac{\ln \det'(A_m)}{N_m} \leqslant \ln \det_\Pi(B)\,. \tag{3.4}$$

Proof. — Replacing L by a bigger number, we can assume that $K < L$, where K is the number appeared in Definition 3.3 of (B_m, N_m). We will prove (1)–(3) of Definition 3.3 hold for (A_m, N_m), with K replaced by L.

(1) clearly holds, since $\|B\| < K < L$ and $\|A_m\| < L$.

Let us prove (3). For any matrix A with complex entries,

$$F_A(0) = \dim \ker(\mathcal{R}_A) = \operatorname{nr}(A) - \operatorname{rk}(A)\,. \tag{3.5}$$

Let F_m, G_m be respectively the spectral density function of B_m, A_m. Since B_m is a submatrix of A_m and the size difference is $< d_m$, we have

$$|\operatorname{nr}(B_m) - \operatorname{nr}(A_m)| < d_m, \quad 0 \leqslant \operatorname{rk}(A_m) - \operatorname{rk}(B_m) < 2d_m\,.$$

From the above inequalities and (3.5), we have

$$|G_m(0) - F_m(0)| < 3d_m\,.$$

Combining with (3.3), we have $\lim_{m\to\infty} \frac{G_m(0)}{N_m} = \lim_{m\to\infty} \frac{F_m(0)}{N_m} = F(0)$, proving (3).

Let us prove (2). The idea is as follows. Since entries are integers and $\|A_m\| \leqslant L$, on each row or each column of A_m there cannot be more L^2 non-zero entries. From here we will show that for all indices i except for a small set, $(p(B_m B_m^*))_{ii} = (p(A_m A_m^*))_{ii}$. This will show $|\operatorname{tr}(p(B_m B_m^*)) - \operatorname{tr}(p(A_m A_m^*))|$ is small compared to N_m.

Let go to the details. Let $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$ and let $\operatorname{Mat}_0(\mathbb{Z}_+ \times \mathbb{Z}_+, \mathbb{C})$ be the set of all $\mathbb{Z}_+ \times \mathbb{Z}_+$ matrix (with complex entries) with finite support, i.e. all entries are 0 except for a finite number of them. Each matrix $\alpha \in \operatorname{Mat}_0(\mathbb{Z}_+ \times \mathbb{Z}_+, \mathbb{C})$ is a linear endomorphism of the standard Hilbert space ℓ^2, and hence one can define its norm $\|\alpha\|$. We define $\operatorname{tr}(\alpha) = \sum_i \alpha_{ii}$, which is finite due to the finite support. Let $R_i(\alpha)$ and $C_j(\alpha)$ be respectively the i-th row of α and the j-th column of α.

Suppose $\alpha, \alpha' \in \operatorname{Mat}_0(\mathbb{Z}_+ \times \mathbb{Z}_+, \mathbb{C})$. Let

$$\operatorname{Diff}(\alpha, \alpha') = \{i \in \mathbb{Z}_+ \mid R_i(\alpha) \neq R_i(\alpha') \text{ or } \ C_i(\alpha) \neq C_i(\alpha')\}\,.$$

We write $\rho(\alpha, \alpha') \leqslant (d, L)$ if $\|\alpha\|, \|\alpha'\| < L$ and $|\operatorname{Diff}(\alpha, \alpha')| < d$.

LEMMA 3.7. — *Suppose $\alpha, \beta, \alpha', \beta' \in \operatorname{Mat}_0(\mathbb{Z}_+ \times \mathbb{Z}_+, \mathbb{C})$ and $\rho(\alpha, \alpha') \leqslant (d, L)$, $\rho(\beta, \beta') \leqslant (d, L)$.*

(a) *For $c_1, c_2 \in \mathbb{C}$ one has $\rho(c_1\alpha + c_2\beta, c_1\alpha' + c_2\beta') \leqslant (C_1 d, C_1)$, where $C_1 = C_1(L, c_1, c_2)$ is a constant depending only on L, c_1, c_2.*

(b) *If the entries of $\alpha, \alpha', \beta, \beta'$ are integers, then $\rho(\alpha\beta, \alpha'\beta') \leqslant (C_2 d, C_2)$, where $C_2 = C_2(L)$ is a constant depending only on L.*

Proof.

(a). — Let $C_1 = \max(2, (|c_1| + |c_2|)L))$. It clear that $\|c_1\alpha + c_2\beta\| < (|c_1| + |c_2|)L) \leqslant C_1$. Since

$$\operatorname{Diff}(c_1\alpha + c_2\beta, c_1\alpha' + c_2\beta') \subset \operatorname{Diff}(\alpha, \alpha') \cup \operatorname{Diff}(\beta, \beta'),$$

we have $|\operatorname{Diff}(c_1\alpha + c_2\beta, c_1\alpha' + c_2\beta')| < 2d \leqslant C_1 d$. Part (a) is proved.

(b). — Let $C_2 = 4L^4 + 2$. It is clear that $\|\alpha\beta\| < L^2 \leqslant C_2$. Similarly $\|\alpha'\beta'\| < C_2$.

Let $I_1 = \operatorname{Diff}(\alpha, \alpha'), J_1 = \operatorname{Diff}(\beta, \beta')$. By assumption, $|I_1|, |J_1| < d$. For a subset $S \subset \mathbb{Z}_+$ denote $S^c = \mathbb{Z}_+ \setminus S$. If $i \in (I_1)^c$ and $j \in (J_1)^c$ then ij-entry of $\alpha\beta$ is

$$(\alpha\beta)_{ij} = R_i(\alpha) \cdot C_j(\beta) = R_i(\alpha') \cdot C_j(\beta') = (\alpha'\beta')_{ij}\,.$$

This shows $\alpha\beta$ and $\alpha'\beta'$ have the same $(I_1)^c \times (J_1)^c$ submatrix, which is denoted by γ.

Since $\|\alpha\beta\| < L^2$, the norm of each row or each column is $< L^2$, which implies on each row or on each column there are at most L^4 non-zero entries. It follows that in any collection of d rows (or d columns) of $\alpha\beta$, all the entries are 0 except for at most dL^4 of them. From the $(I_1)^c \times (J_1)^c$ submatrix γ we can recover the full matrix $\alpha\beta$ by adding back less than d columns and less than d rows. In these d columns there are at most dL^4 non-zero rows, and let I_2 be the set of indexes of those none-zero rows. We have $|I_2| < dL^4$. Now if $i \notin (I_1 \cup I_2)$ then $R_i(\alpha\beta)$ is the 0-extension of $R_i(\gamma)$. Similarly, there is a subset $I_2' \subset \mathbb{Z}_+$ with $|I_2'| < dL^4$ such that if $i \notin (I_1 \cup I_2')$, then $R_i(\alpha'\beta')$ is the 0-extension of $R_i(\gamma)$. Hence if $i \notin (I_1 \cup I_2 \cup I_2')$, then $R_i(\alpha\beta) = R_i(\alpha'\beta')$.

Similarly, there are subsets $J_2, J_2' \subset \mathbb{Z}_+$ with $|J_2|, |J_2'| < dL^4$ such that if $j \notin (J_1 \cup J_2 \cup J_2')$, then $C_j(\alpha\beta) = C_j(\alpha'\beta')$.

Let $I = I_1 \cup I_2 \cup I_2' \cup J_1 \cup J_2 \cup J_2'$. Then $|I| \leqslant 2d + 4dL^4 \leqslant dC_2$. We have $R_i(\alpha\beta) = R_i(\alpha'\beta')$ and $C_i(\alpha\beta) = C_i(\alpha'\beta')$ if $i \notin I$. This implies $\operatorname{Diff}(\alpha\beta, \alpha'\beta') \subset I$, and

$$|\operatorname{Diff}(\alpha\beta, \alpha'\beta')| \leqslant |I| < dC_2\,.$$

This completes the proof of the lemma. □

Let us continue with the proof of (2). For a finite matrix α let $\hat{\alpha} \in \operatorname{Mat}_0(\mathbb{Z}_+ \times \mathbb{Z}_+, \mathbb{C})$ be its 0-extension, i.e. $(\hat{\alpha})_{ij} = \alpha_{ij}$ if α_{ij} exists, otherwise $(\hat{\alpha})_{ij} = 0$. The map $\alpha \to \hat{\alpha}$ is linear and preserves the trace, the norm, and the matrix product. Since A_m is obtained from B_m by adding less than

d_m rows and then less than d_m columns, and on each row or each column there are no more than L^2 non-zero elements, we have

$$|\operatorname{Diff}(\widehat{B_m}, \widehat{A_m})| < d_m + 2L^2 d_m = d_m(2L^2+1)\,.$$

It follows that

$$\rho(\widehat{B_m}, \widehat{A_m}) \leqslant (d_m(2L^2+1), L), \quad \rho(\widehat{B_m^*}, \widehat{A_m^*}) \leqslant (d_m(2L^2+1), L)\,.$$

Applying part (b) of Lemma 3.7 repeatedly to products of $\widehat{B_m}, \widehat{B_m^*}, \widehat{A_m}, \widehat{A_m^*}$, then applying part (a) of Lemma 3.7, we see that there is a constant $C = C(L,p)$ depending only on L and the polynomial p, such that

$$\rho(p(\widehat{B_m B_m^*}), p(\widehat{A_m A_m^*})) \leqslant (Cd_m, C)\,.$$

The absolute value of each entry of either $p(\widehat{B_m B_m^*})$ or $p(\widehat{A_m A_m^*})$ is less than C since C is an upper bound for the norm.

From the definition, if $i \notin \operatorname{Diff}(p(\widehat{B_m B_m^*}), p(\widehat{A_m A_m^*}))$, then $(p(B_m B_m^*))_{ii} = (p(A_m A_m^*))_{ii}$. Since $|\operatorname{Diff}(p(\widehat{B_m B_m^*}), p(\widehat{A_m A_m^*}))| < Cd_m$ and the absolute value of each entry is less than C, we have

$$|\operatorname{tr}(p(\widehat{B_m B_m^*})) - \operatorname{tr}(p(\widehat{A_m A_m^*}))| < 2C(Cd_m) = 2C^2 d_m\,.$$

Since $\lim_{m\to\infty} d_m/N_m = 0$ and $\operatorname{tr}(\alpha) = \operatorname{tr}(\hat{\alpha})$, we conclude that

$$\lim_{m\to\infty} \frac{\operatorname{tr}(p(B_m B_m^*))}{N_m} = \lim_{m\to\infty} \frac{\operatorname{tr}(p(A_m A_m^*))}{N_m} = \operatorname{tr}_\Pi(p(BB^*))\,,$$

which completes the proof of (2). □

Remark 3.8. — Since $\det'(A_m)$ does not change if one permutes the rows or the columns of A_m, Inequality (3.4) still holds if in Proposition 3.6 we replace the assumption "B_m is an upper left corner submatrix of A_m" by the weaker assumption "B_m is a submatrix of A_m". (Actually, one can also prove that Proposition 3.6 holds under this weaker assumption.)

3.8. Hyperbolic volume and Fuglede-Kadison determinant

Suppose X is an irreducible orientable compact 3-manifold with infinite fundamental group and with boundary ∂X either empty or a collection of tori. We define now *good presentations* of $\Pi = \pi_1(X)$ and their *reduced Jacobians*.

First assume that $\partial X \neq \emptyset$. Then X is homotopic to a 2-dimensional finite CW-complex Y which has one zero-cell. Suppose there are n two-cells. Since the Euler characteristic is 0, there must be $n+1$ one-cells, denoted by $a_1, \dots, a_{n+1}$. Then $\Pi := \Pi_1(X) = \Pi_1(Y)$ has a presentation

$$\Pi = \langle a_1, \dots, a_n, a_{n+1} \mid r_1, \dots, r_n \rangle , \tag{3.6}$$

where r_j is the boundary of the j-th two-cell. We call such a presentation a *good presentation for* Π if a_{n+1} is an element of infinite order in Π. For a good presentation (3.6), define its *reduced Jacobian* to be the square matrix

$$J = \left(\frac{\partial r_i}{\partial a_j} \right)_{i,j=1}^{n} \in \mathrm{Mat}(n \times n, \mathbb{Z}[\Pi]) ,$$

where $\frac{\partial r_i}{\partial a_j}$ is the Fox derivative.

Now assume that X is a closed oriented 3-manifold. Assume $X = H \cup H'$ is a Heegaard splitting of X, where each of H and H' is a handlebody of genus g. There is a graph $G \subset H$ with 1 vertex and g loop-edges $a_1, \dots, a_g$ such that H is a regular neighborhood of G. Similarly there is a graph $G' \subset H'$ with 1 vertex and g loop-edges $a'_1, \dots, a'_g$ such that H' is a regular neighborhood of G'. There is a collection $\{D'_1, \dots, D'_g\}$ of properly embedded disks in H' which cuts H' into balls such that D'_i meets a'_i transversally at 1 point and does not meet a'_j for $j \neq i$. We assume that a_g and a'_g are non-trivial in $\pi_1(X)$. A presentation of $\Pi = \pi_1(X)$ can be given by

$$\Pi = \langle a_1, \dots, a_g \mid r_1, \dots, r_g \rangle , \tag{3.7}$$

where r_i is given by the boundary of D'_i, see e.g. [13]. We call such a presentation a *good presentation* of $\pi_1(X)$, and define its *reduced Jacobian* to be the square matrix

$$J = \left(\frac{\partial r_i}{\partial a_j} \right)_{i,j=1}^{g-1} \in \mathrm{Mat}((g-1) \times (g-1), \mathbb{Z}[\Pi]) .$$

We quote here an important result which follows from a result of Lück and Schick [22] (appeared as Theorem 4.3 in [20]) and Lück [20, Theorem 4.9].

THEOREM 3.9. — *Suppose X is an irreducible orientable compact 3-manifold with infinite fundamental group and with boundary either empty or a collection of tori. Let J be the reduced Jacobian of a good presentation of $\Pi = \pi_1(X)$. Then*

$$\ln \det{}_{\Pi}(J) = \mathrm{vol}(X)/6\pi .$$

Proof. — In the proof of Theorem 4.9 of [20] appeared in [19] it was explicitly shown that $\ln \det_\Pi(J) = -\rho^{(2)}(\tilde{X})$, where $\rho^{(2)}(\tilde{X})$ is the (additive) L^2-torsion of the universal covering of X.

By [20, Theorem 4.3], $\rho^{(2)}(\tilde{X}) = -\operatorname{vol}(X)/6\pi$. Hence $\ln \det_\Pi(J) = \operatorname{vol}(X)/6\pi$. □

Remark 3.10. — In the formulation of [20, Theorem 4.3] there is an assumption that X satisfies the conclusion of Thurston Geometrization Conjecture, which is redundant now due Perelman's celebrated result. Besides, there is a requirement that the boundary of X be incompressible. But if a torus component of X is compressible, then X must be a solid torus, for which all the results are trivial.

4. Proofs of main results

4.1. Growth of functions

Recall that $\mathcal{G}$ is the set of all subgroups of Π of finite index. Suppose $f, g : \mathcal{G} \to \mathbb{R}_{>0}$ are functions on $\mathcal{G}$ with positive values. We say that f has *negligible growth* if

$$\limsup_{\mathcal{G} \ni \Gamma \to 1} (f(\Gamma))^{1/[\Pi:\Gamma]} \leqslant 1 .$$

We will write $f \trianglelefteq g$ if f/g has negligible growth.

4.2. Chain complexes of coverings

Suppose X is a connected finite CW-complex with fundamental group Π, and $\tilde{X}$ is its universal covering. Then $\tilde{X}$ inherits a CW-complex structure from X, where the cells of $\tilde{X}$ are lifts of cells of X. The action of Π on $\tilde{X}$ preserves the CW-structure and commutes with the boundary operators. Let $\mathcal{C}(\tilde{X})$ be the chain $\mathbb{Z}$-complex of the CW-structure of $\tilde{X}$. Then $\mathcal{C}_j(\tilde{X})$ is the free $\mathbb{Z}$-module with basis the set of all j-cells of $\tilde{X}$. We will identify $\mathcal{C}_j(\tilde{X})$ with $\mathbb{Z}[\Pi]^{n_j}$, a free $\mathbb{Z}[\Pi]$-module, as follows.

We assume that

(i) X has only one 0-cell e^0, and

(ii) for any j-cell e of X with $j \geqslant 1$, one has $e^0 = \chi_e((1, 0, \dots, 0))$, where $\chi_e : D^j \to X$ is the characteristic map of e. Here $D^j = \{x \in \mathbb{R}^j, \|x\| \leqslant 1\}$ is the standard unit j-disk.

Choose a lift $\tilde{e}^0$ of e^0. Condition (i) shows that for every j-cell e there is a unique lift $\tilde{e}$ defined by using the lift of the characteristic map which sends $(1, 0, \dots, 0)$ to $\tilde{e}^0$. Let $e^j_1, \dots, e^j_{n_j}$ be an ordered set of all j-cells of X, then

$$\mathcal{C}_j(\tilde{X}) = \bigoplus_{l=1}^{n_j} \mathbb{Z}[\Pi] \cdot \tilde{e}^j_l \,,$$

and we use the above equality to identify $\mathcal{C}_j(\tilde{X})$ with $\mathbb{Z}[\Pi]^{n_j}$.

We will write an element $x \in \mathbb{Z}[\Pi]^n$ as a row vector $x = (x_1, \dots, x_n)$ where each $x_j \in \mathbb{Z}[\Pi]$. Note that $\mathbb{Z}[\Pi]^n$ can be considered as a left $\mathbb{Z}[\Pi]$-module or a right $\mathbb{Z}[\Pi]$-module. We will consider $\mathbb{Z}[\Pi]^n$ as a left $\mathbb{Z}[\Pi]$-module unless otherwise stated. If $B \in \mathrm{Mat}(n \times m, \mathbb{Z}[\Pi])$ is an $n \times m$ matrix with entries in $\mathbb{Z}[\Pi]$, then the right multiplication by B defines a $\mathbb{Z}[\Pi]$-linear map from $\mathbb{Z}[\Pi]^n$ to $\mathbb{Z}[\Pi]^m$, and every $\mathbb{Z}[\Pi]$-linear map $\mathbb{Z}[\Pi]^n \to \mathbb{Z}[\Pi]^m$ arises in this way.

The boundary operator $\partial_j : \mathcal{C}_j(\tilde{X}) \to \mathcal{C}_{j-1}(\tilde{X})$ is given by a $n_j \times n_{j-1}$ matrix with entries in $\mathbb{Z}[\Pi]$; by abusing notation we also use ∂_j to denote this matrix. The chain complex $\mathcal{C}(\tilde{X})$ has the form

$$\mathcal{C}(\tilde{X}) = \Big(\cdots \to \mathbb{Z}[\Pi]^{n_{j+1}} \xrightarrow{\partial_{j+1}} \mathbb{Z}[\Pi]^{n_j} \xrightarrow{\partial_j} \mathbb{Z}[\Pi]^{n_{j-1}} \to \cdots \\ \cdots \to \mathbb{Z}[\Pi]^{n_1} \xrightarrow{\partial_1} \mathbb{Z}[\Pi]^{n_0} \xrightarrow{\partial_0} 0 \Big).$$

The 2-skeleton of X gives a presentation of the fundamental group

$$\pi_1(X) = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle, \tag{4.1}$$

where a_i is the represented by the 1-cell e^1_i, and r_j is the boundary of the 2-cell e^2_j, written as a product of a_i's. In this case, ∂_1 is the $n \times 1$ matrix whose i-entry is $1 - a_i$, and ∂_2 is the $m \times n$ matrix whose ij-entry is $\frac{\partial r_i}{\partial a_j}$, see [28, Claim 16.6].

Suppose $\Gamma \leqslant \Pi$ is a subgroup and X_Γ is the corresponding covering. Then X_Γ inherits a CW-structure from X, and its chain $\mathbb{Z}$-complex is exactly $\mathbb{Z}[\Gamma\backslash\Pi] \otimes_{\mathbb{Z}[\Pi]} \mathcal{C}(\tilde{X})$. Here we consider $\mathbb{Z}[\Gamma\backslash\Pi]$ as a right $\mathbb{Z}[\Pi]$-module.

In general, if $\mathcal{C}$ is a chain complex over $\mathbb{Z}[\Pi]$ of left $\mathbb{Z}[\Pi]$-modules and $\Gamma \leqslant \Pi$, then we denote by $\mathcal{C}_\Gamma$ the chain $\mathbb{Z}$-complex $\mathbb{Z}[\Gamma\backslash\Pi] \otimes_{\mathbb{Z}[\Pi]} \mathcal{C}$.

4.3. Circle complex

Fix a non-trivial element a of a residually finite group Π. Let $\mathcal{S}$ be the following chain $\mathbb{Z}[\Pi]$-complex

$$0 \to \mathbb{Z}[\Pi] \xrightarrow{\partial_1} \mathbb{Z}[\Pi] \xrightarrow{\partial_0} 0\,,$$

where $\partial_1 = 1-a$. For every $\Gamma \in \mathcal{G}$, one has the $\mathbb{Z}$-complex $\mathcal{S}_\Gamma = \mathbb{Z}[\Gamma\backslash\Pi]\otimes_{\mathbb{Z}[\Pi]} \mathcal{S}$. We provide $\mathbb{Z}[\Gamma\backslash\Pi]$ with a metric such that $\Gamma\backslash\Pi$ is an orthogonal lattice. Recall that $b_j(\mathcal{S}_\Gamma)$ is the rank of $H_j(\mathcal{S}_\Gamma)$.

LEMMA 4.1. — *Suppose a has infinite order as an element of Π.*

(1) *One has*

$$\limsup_{\mathcal{G}\ni\Gamma\to 1} \frac{b_0(\mathcal{S}_\Gamma)}{[\Pi:\Gamma]} = \limsup_{\mathcal{G}\ni\Gamma\to 1} \frac{b_1(\mathcal{S}_\Gamma)}{[\Pi:\Gamma]} = 0\,. \tag{4.2}$$

(2) *The function $\Gamma \to \det'(1-a_\Gamma)$ is negligible.*

Proof.

(1). — Since the order of a is infinite, the group $\langle a\rangle$ is an infinite cyclic subgroup of Π. By decomposing Π as the disjoint union of cosets of $\langle a\rangle$, it is easy to see that the L^2-Betti numbers of $\mathcal{S}$ are 0. Alternatively, the L^2-Betti numbers of the circle are all 0, and hence the L^2-Betti numbers of $\mathcal{S}$ are 0 by the induction theorem, see [20, Theorem 1.35(10)]. Hence Lemma 4.1(1) is a special case of the main result of Farber [10].

Actually, (1) is much simpler than the full result of [10] due to the simple nature of $\mathcal{S}$, and here is a direct proof (supplied by J. Meumertzheim). Let a_Γ denote the action of a on $\mathbb{Z}[\Gamma\backslash\Pi]$. The chain complex $\mathcal{S}_\Gamma$ has the form

$$0 \to \mathbb{Z}[\Gamma\backslash\Pi] \xrightarrow{1-a_\Gamma} \mathbb{Z}[\Gamma\backslash\Pi] \longrightarrow 0\,.$$

From the Euler characteristic consideration, one has $b_1(\mathcal{S}_\Gamma) = b_0(\mathcal{S}_\Gamma)$.

The element a_Γ acts on $\mathbb{C}[\Gamma\backslash\Pi]$ by permuting the basis $\Gamma\backslash\Pi$. Suppose as a permutation, a_Γ has $d_n(a_\Gamma)$ cycles of length n, with total $d = d(\Gamma) := \sum_n d_n(a_\Gamma)$ cycles. From $H_1(\mathcal{C}_\Gamma) = \ker(1-a_\Gamma)$ one can easily see that $b_1(\mathcal{S}_\Gamma) = d$. We have

$$[\Pi:\Gamma] = \sum_n n d_n(a_\Gamma)\,. \tag{4.3}$$

The trace of a_Γ is equal to the number of cycles of length 1, i.e.

$$[\Pi:\Gamma]\,\mathrm{tr}_{\Gamma\backslash\Pi}(a) = \mathrm{tr}(a_\Gamma, \mathbb{C}[\Gamma\backslash\Pi]) = d_1(a_\Gamma)\,. \tag{4.4}$$

Using (4.4) and counting the number of cycles of length 1 of $(a^m)_\Gamma$, one has

$$[\Pi : \Gamma] \operatorname{tr}_{\Gamma\backslash\Pi}(a^m) = d_1((a^m)_\Gamma) = \sum_{n|m, n\geqslant 1} n d_n(a_\Gamma).$$

In particular,

$$d_n(a_\Gamma) \leqslant \frac{1}{n} [\Pi : \Gamma] \operatorname{tr}_{\Gamma\backslash\Pi}(a^n). \tag{4.5}$$

For all positive integers l we have

$$\begin{aligned}
\frac{b_1(\mathcal{S}_\Gamma)}{[\Pi : \Gamma]} &= \frac{\sum_n d_n(a_\Gamma)}{[\Pi : \Gamma]} \\
&= \frac{\sum_{n=1}^{l} d_n(a_\Gamma)}{[\Pi : \Gamma]} + \frac{\sum_{n\geqslant l+1} d_n(a_\Gamma)}{[\Pi : \Gamma]} \\
&\leqslant \sum_{n=1}^{l} \frac{\operatorname{tr}_{\Gamma\backslash\Pi}(a^n)}{n} + \frac{\sum_{n\geqslant l+1} d_n(a_\Gamma)}{\sum_n n d_n(a_\Gamma)} \qquad \text{by (4.3) and (4.5)} \\
&\leqslant \sum_{n=1}^{l} \frac{\operatorname{tr}_{\Gamma\backslash\Pi}(a^n)}{n} + \frac{1}{l+1}.
\end{aligned}$$

Taking the limit, using the fact that $\limsup_{\mathcal{G}\ni\Gamma\to 1} \frac{\operatorname{tr}_{\Gamma\backslash\Pi}(a^n)}{[\Pi:\Gamma]} = \operatorname{tr}_\Gamma(a^n) = 0$ (since $a^n \neq 1$), we have

$$\limsup_{\mathcal{G}\ni\Gamma\to 1} \frac{b_1(\mathcal{S}_\Gamma)}{[\Pi : \Gamma]} \leqslant \frac{1}{l+1}.$$

Since this is true for all l, we have (4.2), which proves (1).

(2). — Let $N = N(\Gamma) := [\Pi : \Gamma]$. Recall that $b_1(\mathcal{S}_\Gamma) = d(\Gamma)$, the number of cycles of a_Γ.

Claim 4.2. — The function $(N(\Gamma)/d(\Gamma))^{d(\Gamma)}$ on $\mathcal{G}$ is negligible.

Proof of Claim 4.2. — By part (1), $d/N \to 0$ as $\Gamma \xrightarrow{\text{tr}} 1$. It follows that $(N/d)^{(d/N)} \to 1$ as $\Gamma \xrightarrow{\text{tr}} 1$. This proves Claim 1.

Let a_Γ have d cycles of length $l_1, \ldots, l_d$. Lemma 2.2 shows that

$$\det{}'(1 - a_\Gamma) = \prod_{j=1}^{d} l_j.$$

Since $\sum_{j=1}^{d} l_j = N$, the arithmetic-geometric mean inequality implies

$$\det{}'(1 - a_\Gamma) = \prod_{j=1}^{d} l_j \leqslant (N/d)^d,$$

which, in light of Claim 1, proves that $\det'(1 - a_\Gamma)$ is negligible. □

Remark 4.3. — Concerning the assumption on the order of a, we have the following. Suppose X is an irreducible 3-manifold whose fundamental group Π is infinite. Then Π does not have any non-trivial torsion element (see e.g. [3, (C3) Section 3.2]), i.e. if $a \in \Pi$ is not the unit, then the order of a is infinite.

4.4. Proof of Theorem 1.2 for the case $\partial X \neq \emptyset$

Proof. — Suppose X satisfies the assumption of Theorem 1.2 and $\partial X \neq \emptyset$. As in Section 3.8, choose a finite 2-dimensional CW-complex Y homotopic to X which gives a good presentation for Π:

$$\Pi = \langle a_1, \dots, a_n, a_{n+1} \mid r_1, \dots, r_n \rangle .$$

Let $\mathcal{C} = \mathcal{C}(\tilde{Y})$ be the chain $\mathbb{Z}$-complex of the CW-structure of the universal covering $\tilde{Y}$ of Y. As in Section 4.2, $\mathcal{C}$ has the form

$$\mathcal{C} = \Big(0 \to \mathbb{Z}[\Pi]^n \xrightarrow{\partial_2} \mathbb{Z}[\Pi]^{n+1} \xrightarrow{\partial_1} \mathbb{Z}[\Pi] \to 0 \Big),$$

where $\partial_2 \in \mathrm{Mat}(n \times (n+1), \mathbb{Z}[\Pi])$ is the $n \times n+1$ matrix with entries $(\partial_2)_{ij} = \frac{\partial r_i}{\partial a_j}$ and $\partial_1 \in \mathrm{Mat}(n+1 \times 1, \mathbb{Z}[\Pi])$ is a column vector with entries $(\partial_1)_j = 1 - a_j$. The reduced Jacobian J, defined in Section 3.8, is the matrix obtained from ∂_2 be removing the last column, which will be denoted by c.

Then $\mathcal{C} = \mathcal{C}(\tilde{Y})$ is the middle row of the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}[\Pi] & \xrightarrow{1-a_{n+1}} & \mathbb{Z}[\Pi] & \xrightarrow{\partial_0} & 0 \\
\downarrow & & \downarrow & & \downarrow \iota & & \downarrow \mathrm{id} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z}[\Pi]^n & \xrightarrow{\partial_2} & \mathbb{Z}[\Pi]^{n+1} & \xrightarrow{\partial_1} & \mathbb{Z}[\Pi] & \xrightarrow{\partial_0} & 0 \\
\downarrow & & \downarrow \mathrm{id} & & \downarrow p & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{Z}[\Pi]^n & \xrightarrow{J} & \mathbb{Z}[\Pi]^n & \longrightarrow & 0 & \xrightarrow{\partial_0} & 0
\end{array}$$

where $\iota : \mathbb{Z}[\Pi] \to \mathbb{Z}[\Pi]^{n+1}$ is the embedding into the last component, and $p : \mathbb{Z}[\Pi]^{n+1} \to \mathbb{Z}[\Pi]^n$ is the projection onto the first n components. Each row is a chain $\mathbb{Z}[\Pi]$-complex. Denote the chain complex of the first row and the third row by respectively $\mathcal{K}$ and $\mathcal{Q}$. The operators ∂_0 on the diagram indicate how to index the components of the complexes. For example, $\mathcal{Q}_1 = \mathcal{Q}_2 = \mathbb{Z}[\Pi]^n$.

The sequence

$$0 \to \mathcal{K} \to \mathcal{C} \to \mathcal{Q} \to 0$$

is split exact in each degree. Hence, for every $\Gamma \in \mathcal{G}$, one has the following exact sequence of $\mathbb{Z}$-complexes

$$0 \to \mathcal{K}_\Gamma \to \mathcal{C}_\Gamma \to \mathcal{Q}_\Gamma \to 0\,. \tag{4.6}$$

Note that $\mathcal{C}_\Gamma$ is the chain $\mathbb{Z}$-complex of Y_Γ, and its homology groups are the homology groups $H_*(Y_\Gamma, \mathbb{Z}) = H_*(X_\Gamma, \mathbb{Z})$. We provide $\mathbb{Z}[\Gamma\backslash\Pi]$ with the metric in which $\Gamma\backslash\Pi$ is an orthonormal basis. It is clear that this metric is integral, see Section 2.3.

The short exact sequence (4.6) generates a long exact sequence, part of it is

$$H_2(\mathcal{Q}_\Gamma) \xrightarrow{\beta_\Gamma} H_1(\mathcal{K}_\Gamma) \to H_1(\mathcal{C}_\Gamma) \to H_1(\mathcal{Q}_\Gamma)\,.$$

Note that $H_2(\mathcal{Q}_\Gamma) \leqslant (\mathcal{Q}_\Gamma)_2 = \mathbb{Z}[\Gamma\backslash\Pi]^n$ inherits an integral metric from $\mathbb{Z}[\Gamma\backslash\Pi]^n$. Similarly, $H_1(\mathcal{K}_\Gamma) \leqslant (\mathcal{K}_\Gamma)_1 = \mathbb{Z}[\Gamma\backslash\Pi]$ inherits a integral metric. Applying Lemma 2.3 (2) to the above exact sequence, we get

$$t(H_1(\mathcal{C}_\Gamma)) \leqslant \det{}'(\beta_\Gamma)\, t(H_1(\mathcal{Q}_\Gamma))\, \mathrm{vol}(H_2(\mathcal{Q}_\Gamma))\,. \tag{4.7}$$

Since $H_1(\mathcal{Q}_\Gamma) = \mathrm{coker}(J_\Gamma)$ and $H_2(\mathcal{Q}_\Gamma) = \ker(J_\Gamma)$, by Lemma 2.3 (1), we have

$$t(H_1(\mathcal{Q}_\Gamma))\, \mathrm{vol}(H_2(\mathcal{Q}_\Gamma)) = t(\mathrm{coker}(J_\Gamma))\, \mathrm{vol}(\ker(J_\Gamma)) \leqslant \det{}'(J_\Gamma)\,.$$

Using the above inequality in (4.7), and $t_1(\Gamma) = t(H_1(\mathcal{C}_\Gamma))$, we get

$$t_1(\Gamma) \leqslant \det{}'(\beta_\Gamma)\, \det{}'(J_\Gamma)\,. \tag{4.8}$$

Claim 4.4. — $\det{}'(\beta_\Gamma)$ is a negligible function on $\mathcal{G}$.

Proof of Claim 4.4. — We estimate $\det{}'(\beta_\Gamma)$ by using upper bounds for the rank and the norm of β_Γ. First, the rank of β_Γ is less than or equal to rank of its codomain, which is $b_1(\mathcal{K}_\Gamma)$.

The connecting homomorphism β is the restriction of $\tilde{\beta} : \mathbb{Z}[\Pi]^n \to \mathbb{Z}[\Pi]$ given by $\tilde{\beta}(x) = x \cdot c$, where c is the last column of ∂_2. It follows that $\|\beta_\Gamma\| \leqslant \|\tilde{\beta}_\Gamma\| < \nu$ for some constant ν not depending on Γ (see Lemma 3.2). By (2.1),

$$\det{}'(\beta_\Gamma) \leqslant \nu^{b_1(\mathcal{K}_\Gamma)}\,.$$

By Lemma 4.1, $\lim b_1(\mathcal{K}_\Gamma)/[\Pi : \Gamma] = 0$. It follows that $\det{}'(\beta_\Gamma)$ is negligible. This completes the proof of the claim. □

From (4.8) and the above claim, we have

$$t_1(\Gamma) \trianglelefteq \det{}' J_\Gamma\,.$$

Hence

$$\limsup_{\mathcal{G} \ni \Gamma \to 1} \frac{\ln t_1(\Gamma)}{[\Pi : \Gamma]} \leqslant \limsup_{\mathcal{G} \ni \Gamma \to 1} \frac{\ln \det{}'(J_\Gamma)}{[\Pi : \Gamma]} \leqslant \ln \det{}_\Pi J = \frac{\mathrm{vol}(X)}{6\pi}\,,$$

where we use Theorem 3.5 in the second inequality and Theorem 3.9 in the last equality. □

4.5. Proof of Theorem 1.5

Let $X = S^3 \setminus N(K)$, where $N(K)$ is a small tubular open neighborhood of K. Then ∂X is a torus. We will use the notations of Section 4.4. We can further assume that a_{n+1} represents a meridian of K.

Suppose $\Gamma \in \mathcal{G}$. Let $p : X_\Gamma \to X$ be the covering map. Then $p^{-1}(\partial X)$ consists of tori, and $p^{-1}(a_{n+1})$ is a collection of simple closed curves $C_1, \dots C_l$ on $p^{-1}(\partial X)$. From the definition, $H_1(\hat{X}_\Gamma, \mathbb{Z}) = H_1(X_\Gamma, \mathbb{Z})/U$, where U is the subgroup of $H_1(X_\Gamma, \mathbb{Z})$ generated by $C_1, \dots, C_l$. The curves $C_1, \dots, C_l$ are made up from all the lifts of a_{n+1}.

The exact sequence (4.6) gives rise to a long exact sequence

$$\dots H_1(\mathcal{K}_\Gamma) \xrightarrow{\gamma} H_1(\mathcal{C}_\Gamma) \to H_1(\mathcal{Q}_\Gamma) \to H_0(\mathcal{K}_\Gamma) \dots \tag{4.9}$$

Recall that we use the identification $\mathcal{C}_1 \equiv \mathbb{Z}[\Pi]^{n+1}$ via $\mathcal{C}_1 = \bigoplus_{j=1}^{n+1} \mathbb{Z}[\Pi] \cdot \tilde{a}_j$. Correspondingly, the identification $\mathcal{K}_1 \equiv \mathbb{Z}[\Pi]$ is via $\mathcal{K}_1 = \mathbb{Z}[\Pi] \cdot \tilde{a}_{n+1}$. Under these identifications, one has $U \equiv \mathrm{Im}(\gamma)$. Hence, from the exact sequence (4.9), one has the following exact sequence

$$0 \to H_1(\hat{X}_\Gamma, \mathbb{Z}) \to H_1(\mathcal{Q}_\Gamma) \to H_0(\mathcal{K}_\Gamma) \dots$$

Since $H_0(\mathcal{K}_\Gamma)$ is a free abelian group, the above exact sequence implies that

$$\mathrm{Tor}_\mathbb{Z}(H_1(\hat{X}_\Gamma, \mathbb{Z})) = \mathrm{Tor}_\mathbb{Z}(H_1(\mathcal{Q}_\Gamma)) .$$

Applying (2.3) to the map $J_\Gamma : \mathbb{Z}[\Gamma\backslash\Pi]^n \to \mathbb{Z}[\Gamma\backslash\Pi]^n$, we get

$$\begin{aligned} |\operatorname{Tor}_\mathbb{Z}(H_1(\hat{X}_\Gamma, \mathbb{Z}))| &= |\operatorname{Tor}_\mathbb{Z}(H_1(\mathcal{Q}_\Gamma))| \\ &= \frac{\det'(J_\Gamma)}{\mathrm{vol}(\ker(J_\Gamma))\,\mathrm{vol}(\overline{\mathrm{Im}(J_\Gamma)})} \leqslant \det'(J_\Gamma) . \end{aligned}$$

Here $\overline{\mathrm{Im}(J_\Gamma)} := (\mathrm{Im}(J_\Gamma) \otimes_\mathbb{Z} \mathbb{Q}) \cap \mathbb{Z}[\Gamma\backslash\Pi]^n$.

Theorem 3.5 and Theorem 3.9 show

$$\limsup_{\mathcal{G} \ni \Gamma \xrightarrow{\mathrm{tr}} 1} \frac{\ln |\operatorname{Tor}_\mathbb{Z}(H_1(\hat{X}_\Gamma, \mathbb{Z}))|}{[\Pi : \Gamma]} \leqslant \limsup_{\mathcal{G} \ni \Gamma \xrightarrow{\mathrm{tr}} 1} \frac{\ln \det'(J_\Gamma)}{[\Pi : \Gamma]} \leqslant \ln \det_\Pi(J) = \frac{\mathrm{vol}(X)}{6\pi} .$$

This completes the proof of Theorem 1.5.

4.6. Heegaard splitting and homology

Assume that X is an oriented connected closed 3-manifold and $X = H \cup H'$ is a Heegaard splitting of X. This means each of H and H' is a handlebody of genus g, and $H \cap H' = \partial H = \partial H'$. For generalities on Heegaard splittings the reader can consult [13].

A properly embedded disk $D \subset H$ is *essential* if its boundary does not bound a disk in $F = \partial H$. A *system of disks* of a handlebody H of genus g is a collection $\mathcal{D} = \{D_1, \dots, D_n\}$ of properly embedded, essential oriented disks in H which are disjoint and cut H into balls. Then $n \geqslant g$, and if $n = g$, we say that the system of disks $\mathcal{D}$ is *minimal*. A disk system $\mathcal{D}$ is minimal if and only if $H \setminus (\bigcup_{D \in \mathcal{D}} D)$ is connected, and any disk system has a subset which is a minimal disk system.

Suppose $\mathcal{D} = \{D_1, \dots, D_n\}$ is a disk system of H and $\mathcal{D}' = \{D'_1, \dots, D'_m\}$ is a disk system of H'. Let $\alpha_i = \partial D_i$ and $\alpha'_j = \partial D'_j$. The common boundary $F = \partial H = \partial H'$ inherits an orientation from H, and each curve α_i, α'_j inherits an orientation from D_i, D'_j. We will assume that α_i and α'_j are transversal in F for all pairs (i, j), and let $\mu(\alpha_i, \alpha'_j)$ be the intersection index of α_i and α'_j.

The disk system $\mathcal{D}$ generates a *dual graph* G embedded in H: to every connected component Q of $H \setminus (\bigcup_{i=1}^{n} D_i)$ there corresponds a vertex v_Q which is a point in the interior of Q, and to every disk D_i there corresponds an edge $a_i \subset H$. If D_i is in the closure of two connected components Q and Q', then a_i is an edge connecting v_Q and $v_{Q'}$. If D_i is in the closure of only one connected component Q, then a_i is a loop edge based at v_Q. In all cases, a_i intersects D_i transversely at exactly one point, and $a_i \cap D_j = \emptyset$ for $i \neq j$. We orient a_i so that the intersection index $\mu(a_i, D_i) = 1$. Note that G, known as a spine of H, is a deformation retract of H. Let $G' \subset H'$ be a dual graph of $\mathcal{D}'$, with edge a'_i dual to disk D'_i.

Although a_i and a'_j do not intersect, we will define

$$\mu(a'_i, a_j) = \mu(\alpha'_i, \alpha_j)\,. \tag{4.10}$$

Proposition 4.5. — *Let $M \in \mathrm{Mat}(m \times n, \mathbb{Z})$ be given by $M_{ij} = \mu(a'_i, a_j)$. Then $t(H_1(X, \mathbb{Z})) = t(M)$.*

Proof. — Without loss of generality we can assume that $\bar{\mathcal{D}} = \{D_1, \dots, D_g\}$ is a minimal disk system of H. Let $\bar{M}$ be the $m \times g$ submatrix of M consisting of the first g columns. The dual graph $\bar{G}$ of $\bar{\mathcal{D}}$ has only one vertex and g loop edges $\bar{a}_1, \dots, \bar{a}_g$ (dual to $D_1, \dots, D_g$). The group $H_1(H, \mathbb{Z})$ is free abelian with basis $\{\bar{a}_1, \dots, \bar{a}_g\}$. If $c \subset F = \partial H$ is an oriented closed curve,

then the homology class of c in $H_1(H,\mathbb{Z})$ is $\sum_{i=1}^{g}\mu(c,\alpha_i)\bar{a}_i$. To obtain X one glues H' to H, and this can be done in 2 steps. In the first one glues the disks D'_j to H, then in the second one glues in the complements (in H') of D'_i which are 3-balls. The second step does not change the homology group H_1. The first gluing shows that $H_1(X,\mathbb{Z}) = \operatorname{coker}\mathcal{R}_{\bar{M}}$, where $\mathcal{R}_{\bar{M}}(x) = x\bar{M}$.

Cutting $F = \partial H$ along $\alpha_1,\dots,\alpha_g$, we get a sphere with $2g$ disks removed. More precisely, F is obtained from F', a compact surface of genus 0 and $2g$ boundary components $c'_1, c''_1, c'_2, c''_2, \dots, c'_g, c''_g$, by gluing c'_i to c''_i, with the common image being α_i. Any simple closed curve in the interior of F' is separating (because F' has genus 0), and hence is homologically equal to a $\mathbb{Z}$-linear combination of boundary curves with coefficients ± 1. It follows that the homology class of any curve in F not meeting any of $\alpha_1,\dots,\alpha_g$ is in the $\mathbb{Z}$-linear span of $\alpha_1,\dots,\alpha_g$; this applies to the curves α_i with $i > g$. This implies any column of M is a $\mathbb{Z}$-linear combination of the first g columns. In other words, the images of $\mathcal{R}_{M^*}$ and $\mathcal{R}_{\bar{M}^*}$ are same. Hence $t(M^*) = t(\bar{M}^*)$. By Proposition 2.4, one has $t(M) = t(M^*)$. It follows that $t(H_1(X,\mathbb{Z})) = t(\bar{M}) = t(M)$. □

Here is another proof of Proposition 4.5. A Heegaard splitting $X = H \cup H'$ and disk systems $\mathcal{D}$ of H and $\mathcal{D}'$ of H' give rise to a CW-complex structure of X as follows. First, the handlebody H is obtained from zero-handles, each is a regular neighborhood of a vertex of the graph G dual to $\mathcal{D}$, by attaching one-handles whose cores are in the edges a_i of G. Then to H one glues two-handles whose cores are $D' \in \mathcal{D}$. Finally by gluing in three-handles one gets X. This handle decomposition gives rise to a CW-complex structure of X, see [26, Chapter 6]. The second boundary map of the associated chain complex is given by the matrix M. By Proposition 2.4(3), we get $t(H_1(X,\mathbb{Z})) = t(M)$.

Recall that a cycle of a graph is a closed walk in the graph which does not visit any vertex twice. We show here a way to simplify a disk system of a handlebody.

LEMMA 4.6. — *Suppose $c_1,\dots,c_r$ are disjoint cycles of G. In each cycle choose an edge, called the preferred edge of the cycle. Then the set $\mathcal{E} \subset \mathcal{D}$, consisting of all D_i such that either a_i is a preferred edge of a cycle or a_i does not belong to any cycle, is a system of disks of H.*

Proof. — A cycle of G either is one of $c_1,\dots,c_r$, or contains an edge which is not in any of $c_1,\dots,c_r$. This shows $\mathcal{E}$ cuts G into trees, implying each connected component of $H \setminus (\bigcup_{D\in\mathcal{E}} D)$ is contractible. Hence $\mathcal{E}$ is a disk system of H. □

4.7. Proof of Theorem 1.2 for the case when $\partial X = \emptyset$

Suppose X is an irreducible, connected, closed, oriented 3-manifolds. If $\Pi = \pi_1(X)$ is finite, then the statement of Theorem 1.2 is trivial. We will assume that Π is infinite. Then every non-trivial element of Π has infinite order.

Suppose $X = H \cup H'$ is a Heegaard splitting of X, where $F = \partial H = \partial H'$ has genus g. Suppose $\mathcal{D} = \{D_1, \dots, D_g\}$ is a minimal disk system of H and $\mathcal{D}' = \{D'_1, \dots, D'_g\}$ is a minimal disk system of H'. Let the dual graph G (resp. G') with the set of edges $\mathcal{A} = \{a_1, \dots, a_g\}$ (resp. $\mathcal{A}' = \{a'_1, \dots, a'_g\}$) be as defined in Subsection 4.6. The minimality of $\mathcal{D}$ and $\mathcal{D}'$ implies each of G and G' has one vertex, and each of a_i, a'_i is a loop-edge. Each of the sets $\{a_1, \dots, a_g\}$ and $\{a'_1, \dots, a'_g\}$ generates $\Pi = \pi_1(X)$. Hence by reordering, one can assume that a_g and a'_g are non-trivial in Π. Thus, the associated presentation of Π,

$$\Pi = \langle a_1, \dots, a_g \mid r_1, \dots, r_g \rangle, \tag{4.11}$$

where r_i is determined by the two-cell D'_i, is good.

Let $\tilde{J} \in \mathrm{Mat}(g \times g, \mathbb{Z}[\Pi])$ be the matrix whose ij-entry is given by $\tilde{J}_{ij} = \frac{\partial r_i}{\partial a_j}$. Then $\tilde{J}$ is the second boundary operator of the chain complex of the universal covering of X, see Subsection 4.2. The reduced Jacobian J is obtained from $\tilde{J}$ by removing the last row and the last column.

Let $(\Gamma_m)_{m \geqslant 0}$ be a sequence of subgroups of Π of finite index such that $\Gamma_m \xrightarrow{\mathrm{tr}} 1$. Fix an index m for now. Let $P_m : X_m \to X$ be the covering map corresponding to the subgroup Γ_m and $\tilde{H}, \tilde{H}', \tilde{G}, \tilde{G}'$ be respectively the preimage of H, H', G, G' under P_m.

For a disk D in $\mathcal{D}$ or $\mathcal{D}'$, the preimage $(P_m)^{-1}(D)$ is a collection of disjoint disks, each is called a *lift* of D. The collection $\tilde{\mathcal{D}}$ (resp. $\tilde{\mathcal{D}}'$) of all lifts of all disks in $\mathcal{D}$ (resp. $\mathcal{D}'$) is a disk system of $\tilde{H}$ (resp. $\tilde{H}'$). The CW-structure of X_m, corresponding to the Heegaard splitting $X_m = \tilde{H} \cup \tilde{H}'$ and the disk systems $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}'$, is exactly the lift of the CW-structure of X. Hence the boundary operator $\partial_2 : \mathcal{C}_2(X_m) \to \mathcal{C}_1(X_m)$ is equal to $\tilde{J}_m := \tilde{J}_{\Gamma_m}$.

Let $\tilde{\mathcal{A}}$ (resp. $\tilde{\mathcal{A}}'$) be the set of edges of $\tilde{G}$ (resp. $\tilde{G}'$). An edge $\tilde{a}$ of $\tilde{G}$ (resp. $\tilde{G}'$) is called a *lift* of an edge a of G (resp. G') if $P_m(\tilde{a}) = a$. As explained in Subsection 4.6, if we identify $\mathcal{C}_2(X_m)$ with the free $\mathbb{Z}$-module with basis $\tilde{\mathcal{A}}'$ and identify $\mathcal{C}_1(X_m)$ with the free $\mathbb{Z}$-module with basis $\tilde{\mathcal{A}}$, then $\tilde{J}_m = \partial_2 : \mathcal{C}_2(X_m) \to \mathcal{C}_1(X_m)$ is given by the $\tilde{\mathcal{A}}' \times \tilde{\mathcal{A}}$ matrix whose (a', a)-entry is $\mu(a', a)$.

Suppose all the lifts of a_g form l_m cycles. In each cycle choose one lift of a_g, called a *preferred lift*. Similarly, all the lifts of a'_g form l'_m cycles. In

each cycle choose one lift of a'_g, called a *preferred lift* of a'_g. Let $\tilde{\mathcal{B}}$ (resp. $\tilde{\mathcal{B}}'$) be obtained from $\tilde{\mathcal{A}}$ (resp. $\tilde{\mathcal{A}}'$) by removing all the non-preferred lifts of a_g (resp. a'_g). Let J'_m be the $\tilde{\mathcal{B}}' \times \tilde{\mathcal{B}}$-submatrix of $\tilde{J}_m$. By Lemma 4.6 and Proposition 4.5, one has $t(H_1(X_m)) = t(J'_m)$, hence by Proposition 2.4,

$$t(H_1(X_m)) = t(J'_m) \leqslant \det'(J'_m)\,.$$

By Theorem 3.5, the sequence $(J_m, [\Pi : \Gamma_m])_{m\geqslant 1}$ tracely approximates J, where $J_m := J_{\Gamma_m}$. Note that J_m is obtained from J'_m by removing l_m rows corresponding to the l_m preferred lifts of a_g and l'_m columns corresponding to the l'_m preferred lifts of a'_g. Let $d_m = \max(l_m, l'_m)$. By (the proof of) Lemma 4.1, $\lim_{m\to\infty} \frac{d_m}{[\Pi:\Gamma_m]} = 0$. Besides, by Lemma 3.2, there is $L > 0$ such that $\|\tilde{J}_{\Gamma_m}\| < L$. As a submatrix of $\tilde{J}_{\Gamma_m}$, we also have $\|J'_m\| < \|\tilde{J}_{\Gamma_m}\| < L$. Now Proposition 3.6 (see Remark 3.8) shows the second inequality in the following

$$\limsup_{m\to\infty} \frac{\ln t(H_1(X_m))}{[\Pi : \Gamma_m]} \leqslant \limsup_{m\to\infty} \frac{\ln \det'(J'_m)}{[\Pi : \Gamma_m]} \leqslant \ln \det{}_\Pi J = \frac{\mathrm{vol}(X)}{6\pi}\,,$$

which proves Theorem 1.2.

Remark 4.7. — One can have a proof of Theorem 1.2 for the case when $\partial X \neq \emptyset$ similar to (and actually simpler than) the above proof for the case when $\partial X = \emptyset$.

Appendix A. Proof of Theorem 3.4

Assume $(B_m, N_m)_{m\geqslant 1}$ tracely approximates B, and K is the constant in Definition 3.3. Let F be the spectral density function of B, and F_m be the spectral density function of B_m. Define

$$G_m(\lambda) = \frac{F_m(\lambda)}{N_m},\ \ \underline{G}(\lambda) = \liminf_{m\to\infty} G_m(\lambda),\ \ \overline{G}(\lambda) = \limsup_{m\to\infty} G_m(\lambda)\,. \tag{A.1}$$

Recall that for any increasing function $h : \mathbb{R} \to \mathbb{R}$ one defines $h^+(\lambda) = \lim_{\varepsilon\to 0^+} h(\lambda + \varepsilon)$.

Lemma A.1. — *One has* $F = \underline{G}^+ = \overline{G}^+$.

Proof. — The proof is a slight modification of that of [18, Theorem 2.3].

Since $\|B\|, \|B_m B_m^*\| < K^2$, we have $F_m(\lambda) = \mathrm{nr}(B_m)$ and $F(\lambda) = \mathrm{nr}(B)$ for all $\lambda \in [K^2, \infty)$. Applying Condition (2) of Definition 3.3 to the constant polynomial $p = 1$ and $\lambda = K^2$, one has

$$\lim_{m\to\infty} \frac{\mathrm{nr}(B_m)}{N_m} = \mathrm{nr}(B)\,. \tag{A.2}$$

Fix a number $\lambda \in [0, K^2)$ and let $(p_n)_{n\geqslant 1}$ be the sequence of polynomials constructed in [18, Section 2]. These polynomials approximate the characteristic function of the interval $[0, \lambda]$ and have the following properties:

$$0 \leqslant p_n \text{ on } [0, K^2], \quad 1 \leqslant p_n \leqslant 1 + \frac{1}{n} \text{ on } [0, \lambda], \tag{A.3}$$

$$p_n \leqslant 1 + \frac{1}{n} \text{ on } [\lambda, \lambda + \frac{1}{n}], \quad p_n(\mu) \leqslant \frac{1}{n} \text{ on } [\lambda + \frac{1}{n}, K^2]. \tag{A.4}$$

$$\lim_{n\to\infty} \mathrm{tr}_\Pi(p_n(BB^*)) = F(\lambda). \tag{A.5}$$

Using the bounds of p_n given by (A.3), we have

$$\begin{aligned}\mathrm{tr}(p_n(B_m^* B_m)) &= \int_0^{K^2} p_n(\mu)\, \mathrm{d}F_m(\mu) \\ &\geqslant \int_0^{\lambda} p_n(\mu)\, \mathrm{d}F_m(\mu) \geqslant \int_0^{\lambda} \mathrm{d}F_m(\mu) = F_m(\lambda).\end{aligned}$$

Dividing by N_m,

$$G_m(\lambda) = \frac{F_m(\lambda)}{N_m} \leqslant \frac{\mathrm{tr}(p_n(B_m^* B_m))}{N_m}.$$

Take the superior limit as $m \to \infty$, then the limit as $n \to \infty$. Using Condition (2) of Definition 3.3 and (A.5), we have

$$\overline{G}(\lambda) \leqslant F(\lambda). \tag{A.6}$$

Use the bounds of p_n given by (A.4), we have

$$\begin{aligned}\mathrm{tr}(p_n(B_m^* B_m)) &= \int_0^{K^2} p_n(\mu)\, \mathrm{d}F_m(\mu) \\ &= \int_0^{\lambda+\frac{1}{n}} p_n(\mu)\, \mathrm{d}F_m(\mu) + \int_{(\lambda+\frac{1}{n})^+}^{K^2} p_n(\mu)\, \mathrm{d}F_m(\mu) \\ &\leqslant \left(1 + \frac{1}{n}\right) F_m\left(\lambda + \frac{1}{n}\right) + \frac{1}{n}\left(F_m(K^2) - F_m\left(\lambda + \frac{1}{n}\right)\right) \\ &= F_m\left(\lambda + \frac{1}{n}\right) + \frac{1}{n} F_m(K^2) = F_m\left(\lambda + \frac{1}{n}\right) + \frac{\mathrm{nr}(B_m)}{n}.\end{aligned}$$

Dividing by N_m, then taking the inferior limit as $m \to \infty$ and using (A.2), we have

$$\mathrm{tr}(p_n(B^* B) \leqslant \underline{G}(\lambda + \frac{1}{n}) + \frac{\mathrm{nr}(B)}{n}.$$

Taking the limit as $n \to \infty$, we have

$$F(\lambda) \leqslant \underline{G}^+(\lambda). \tag{A.7}$$

Using (A.7), (A.6) and the fact that $\underline{G}$ is increasing, for all $\varepsilon > 0$ we have

$$F(\lambda) \leqslant \underline{G}^+(\lambda) \leqslant \underline{G}(\lambda+\varepsilon) \leqslant \overline{G}(\lambda+\varepsilon) \leqslant F(\lambda+\varepsilon)\,.$$

Taking the limit $\varepsilon \to 0^+$ and noting that F is right continuous, i.e. $\lim_{\varepsilon\to 0+} F(\lambda+\varepsilon) = F(\lambda)$, we have $F(\lambda) = \underline{G}^+(\lambda) = \overline{G}^+(\lambda)$. □

Lemma A.2. — *One has*

$$\underline{G}^+(0) = \lim_{m\to\infty} G_m(0)\,, \tag{A.8}$$

$$\underline{G}^+(K^2) = \lim_{m\to\infty} G_m(K^2)\,. \tag{A.9}$$

Proof. — We have (A.8), since the left hand side is equal to $F(0)$ by Lemma A.1, while the right hand side is also equal to $F(0)$ by Condition (3) of Definition 3.3.

Since each G_m, like F_m, is a constant function on $[K^2,\infty)$, each of $\underline{G}$ and $\overline{G}$ is constant on $[K^2,\infty)$. It follows that $\underline{G} = \underline{G}^+$ and $\overline{G} = \overline{G}^+$ on $[K^2,\infty)$. Hence for $\lambda \in [K^2,\infty)$,

$$\underline{G}(\lambda) = \underline{G}^+(\lambda) = F(\lambda) = \overline{G}^+(\lambda) = \overline{G}(\lambda)\,.$$

This implies $\lim_{m\to\infty} G_m(\lambda) = \underline{G}(\lambda) = \underline{G}^+(\lambda)$ for $\lambda \in [K^2,\infty)$. In particular, we have (A.9). □

Let us prove Theorem 3.4. By definition and Lemma A.1, one has $(\det_\Pi(B))^2 = \det(F) = \det(\underline{G}^+)$. Using (3.1) then Lemma A.2, we have

$$\begin{aligned}
2\ln(\det_\Pi(B)) &= \ln\det(\underline{G}^+)\\
&= (\underline{G}^+(K^2) - \underline{G}^+(0))\ln(K^2) - \int_{0^+}^{K^2} \frac{\underline{G}^+(\lambda)-\underline{G}^+(0)}{\lambda}\,\mathrm{d}\lambda\\
&= \lim_{m\to\infty} (G_m(K^2) - G_m(0))\ln K^2 - \int_{0^+}^{K^2} \frac{\underline{G}^+(\lambda)-\underline{G}^+(0)}{\lambda}\,\mathrm{d}\lambda \qquad \text{(A.10)}
\end{aligned}$$

By [18, Lemma 3.2], we the first of the following identities

$$\begin{aligned}
\int_{0^+}^{K^2} \frac{\underline{G}^+(\lambda)-\underline{G}^+(0)}{\lambda}\,\mathrm{d}\lambda &= \int_{0^+}^{K^2} \frac{\underline{G}(\lambda)-\underline{G}^+(0)}{\lambda}\,\mathrm{d}\lambda\\
&= \int_{0^+}^{K^2} \liminf_{m\to\infty} \frac{G_m(\lambda)-G_m(0)}{\lambda}\,\mathrm{d}\lambda\\
&\leqslant \liminf_{m\to\infty} \int_{0^+}^{K^2} \frac{G_m(\lambda)-G_m(0)}{\lambda}\,\mathrm{d}\lambda, \qquad \text{(A.11)}
\end{aligned}$$

where the second identity follows from $\underline{G}(\lambda) = \liminf_{m\to\infty} G_m(\lambda)$ and (A.8), and the last inequality follows from Fatou's lemma. Using Inequality (A.11) in (A.10), we have

$$\begin{aligned}
&2\ln(\det_\Pi(B)) \\
&\quad\geqslant \lim_{m\to\infty}(G_m(K^2)-G_m(0))\ln K^2 - \liminf_{m\to\infty}\int_{0^+}^{K^2}\frac{G_m(\lambda)-G_m(0)}{\lambda}\,\mathrm{d}\lambda \\
&\quad= \limsup_{m\to\infty}\left\{(G_m(K^2)-G_m(0))\ln K^2 - \int_{0^+}^{K^2}\frac{G_m(\lambda)-G_m(0)}{\lambda}\,\mathrm{d}\lambda\right\} \\
&\quad= \limsup_{m\to\infty}\{2\ln(\det'(B_m))/N_m\},
\end{aligned}$$

where in the last equality we again use (3.1). This completes the proof of Theorem 3.4.

Remark A.3. — As observed by the referee, Theorem 3.4 follows from the portmanteau theorem in probability theory as follows (sketch). Let G_m, F be defined as in (A.1). Then $P_m := dG_m$ and $P := \mathrm{d}F$ are finite Borel measures on $\mathbb{R}$, with support in $[0, K^2]$. Condition (2) of Definition 3.3 implies that for any polynomial f,

$$\lim_{m\to\infty}\int_{[0,K^2]} f\,\mathrm{d}P_m = \int_{[0,K^2]} f\,\mathrm{d}P\,. \tag{A.12}$$

Every continuous function on $[0, K^2]$ can be uniformly approximated by polynomials. It follows that (A.12) holds for all bounded continuous functions. Hence, by definition (see [2]), the sequence of Borel measures P_m *converges weakly* (or *converges narrowly*) to P.

Let P'_m, P' be respectively the restriction of the measures P_m, P on $(0, K^2]$. Condition (3) of Definition 3.3 means that $\lim_{m\to\infty} P_m(0) = P(0)$. From here and the weak convergence of P_m to P, one can prove that the sequence P'_m weakly converges to P'. By a version of the pormanteau theorem (see [2, Section 5.1.1], in particular Formula (5.1.15) there), one has that if h is a function on $(0, K^2]$ which is continuous and bounded from above, then

$$\limsup_{m\to\infty}\int_{(0,K^2]} h\,\mathrm{d}P'_m \leqslant \int_{(0,K^2]} h\,\mathrm{d}P'.$$

Applying to $h = \ln(x)$, we get

$$\limsup_{m\to\infty}(\ln\det G_m) \leqslant \ln\det F,$$

which proves Theorem 3.4.

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