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q-ANALOGUES OF LAPLACE AND BOREL TRANSFORMS BY MEANS OF q-EXPONENTIALS

by Hidetoshi TAHARA (*)

ABSTRACT. — The article discusses certain q-analogues of Laplace and Borel transforms, and shows a new inversion formula between q-Laplace and q-Borel transforms. q-Analogues of Watson type lemma and convolution operators are also discussed. These results give a new framework of the summability of formal power series solutions of q-difference equations.

RÉSUMÉ. — Nous considérons certaines q-analogues des transformées de Laplace et Borel et montrons une nouvelle formule d'inversion entre les transformées de q-Laplace et de q-Borel. Des q-analogues des lemmes de type Watson et des opérateurs de convolution sont aussi discutés. Ces résultats donnent un nouveau cadre pour la sommabilité des séries formelles qui sont solutions d'équations aux q-différences.

1. Introduction

The classical Laplace transform of a function $f(t)$ is given by

$$L[f](s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t.$$

To define a q-analogue of the Laplace transform has been a very interesting thema, and many authors have defined its q-analogue in various forms. The first person who introduced q-Laplace transform would be Hahn in 1949: in the paper [6], Hahn gave two definitions of q-Laplace transforms:

$$ {}_qL_s f(t) = \frac{1}{1-q}\int_0^{s^{-1}} f(t)E_q(-qst)\,\mathrm{d}_q t, \tag{1.1}$$

$$ {}_q\mathscr{L}_s f(t) = \frac{1}{1-q}\int_0^{\infty} f(t)e_q(-st)\,\mathrm{d}_q t \tag{1.2}$$

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in the case $0 < q < 1$, where

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} z^n}{(q;q)_n} = \prod_{m=0}^{\infty} (1 + q^m z), \quad z \in \mathbb{C},$$

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{\prod_{m=0}^{\infty}(1 - q^m z)}, \quad |z| < 1$$

(with $(q;q)_0 = 1$ and $(q;q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ for $n \geqslant 1$) and the integrals are taken as the Jackson integrals (see Jackson [7], Kac–Cheung [8]). Abdi [1, 2] studied certain properties of these q-Laplace transforms, and applied them to solve q-difference equations with constant coefficients.

Later, Ramis [16] had suggested in 1993 the use of the Gaussian function to formulate a q-analogue of the Laplace transform; this allowed to find in Zhang [19] a q-analogue of the summability of formal power series by using the q-Laplace transform

$$\mathcal{L}_q[f](s) = \frac{q^{-1/8}}{\sqrt{2\pi \log q}} \int_0^{\infty} f(t) q^{-\log_q(st)(\log_q(st)+1)/2} \, \frac{\mathrm{d}t}{t} \tag{1.3}$$

where $q > 1$. Marotte–Zhang [12] applied it to show the multi-summability of formal power series solutions of linear q-difference equations with variable coefficients.

In 2000, Zhang [20] defined another q-analogue of the Laplace transform by means of Jacobi theta function:

$$\mathcal{L}_q[f](s) = \frac{\prod_{m=0}^{\infty}(1 - q^{-m-1})}{\log q} \int_0^{\infty} \frac{f(t)}{\vartheta_q(st)} \, \mathrm{d}t \tag{1.4}$$

where $q > 1$, and $\vartheta_q(z)$ denotes the Jacobi theta function:

$$\vartheta_q(z) = \sum_{n=-\infty}^{\infty} \frac{z^n}{q^{n(n-1)/2}} = \prod_{m=0}^{\infty} (1 - q^{-m-1})(1 + q^{-m} z)(1 + q^{-m-1} z^{-1})$$

(for the properties of $\vartheta_q(z)$, see Subsection 2.2). This q-Laplace transform was used in Lastra–Malek [9] in the study of singularly perturbed q-difference-differential equations.

Further, Ramis–Zhang [17] defined in 2002 another discrete q-analogue of the Laplace transform by

$$\mathcal{L}_q[f](s) = \sum_{i=-\infty}^{\infty} \frac{f(q^i)}{\vartheta_q(sq^i)}, \tag{1.5}$$

where $q > 1$. This definition was used in Zhang [21], Malek [11], Lastra–Malek–Sanz [10] and Tahara–Yamazawa [18] in the study of q-summability of formal power series solutions of various q-difference (and q-difference-differential) equations.

Also, in connection with the study of confluence (as $q \longrightarrow 1$), Di Vizio–Zhang [3] introduced in 2009 four q-Laplace transforms:

$$\begin{aligned}
L_q^{(1)}[f](s) &= \frac{q}{1-p}\int_0^\infty \frac{f(t)}{\vartheta_q(qst)}\,\mathrm{d}_p t,\\
L_q^{(2)}[f](s) &= \frac{q}{1-p}\int_0^\infty \frac{f(t/(1-p))}{\exp_q(qst/(1-p))}\,\mathrm{d}_p t,\\
L_q^{(3)}[f](s) &= \frac{q}{\log q}\int_0^\infty \frac{f(t)}{\vartheta_q(qst)}\,\mathrm{d}t,\\
L_q^{(4)}[f](s) &= \frac{q}{\log q}\int_0^\infty \frac{f(t)}{\exp_q(qst)}\,\mathrm{d}t
\end{aligned}$$

in the case $q > 1$, where $p = 1/q$, the integrals in $L_q^{(1)}$ and $L_q^{(2)}$ are taken as p-Jackson integrals, and

$$\exp_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{m=0}^{\infty} (1 + q^{-m-1}(q-1)z)$$

(for the properties of the q-exponential $\exp_q(z)$, see Subsection 2.1), where

$$\begin{aligned}
&[0]_q = 0 \quad \text{and} \quad [n]_q = \frac{q^n-1}{q-1} = 1 + q + \cdots + q^{n-1} \ \ (\text{for } n = 1, 2 \ldots),\\
&[0]_q! = 1 \quad \text{and} \quad [n]_q! = [1]_q[2]_q \cdots [n]_q \ \ (\text{for } n = 1, 2 \ldots).
\end{aligned}$$

The first one $L_q^{(1)}$ is essentially the same as (1.5), and the third one $L_q^{(3)}$ is the same as (1.4).

In this paper, we let $q > 1$ and we will use

$$\mathscr{L}_q[f](s) = \int_0^\infty f(t)\,\mathrm{Exp}_q(-qst)\,\mathrm{d}_q t \tag{1.6}$$

as the definition of q-Laplace transform, where

$$\mathrm{Exp}_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} z^n}{[n]_q!} = \frac{1}{\prod_{m=0}^{\infty}(1 - q^{-m-1}(q-1)z)}$$

(for the properties of the q-exponential $\mathrm{Exp}_q(z)$, see Subsection 2.1). Since $\mathrm{Exp}_q(-z)\exp_q(z) = 1$ holds (see Subsection 2.1), our q-Laplace transform (1.6) is essentially the same as the second one $L_q^{(2)}$ of Di Vizio–Zhang [3].

The paper is organized as follows. In the next Section 2, we recall some basic results in q-calculus, and then in Section 3 we study our q-Laplace transform (1.6). In Section 4 we define a q-analogue of the Borel transform, and in Section 5 we show the inversion formulas between our q-Laplace transform and q-Borel transform. Further, in Section 6 we define a q-analogue of the convolution and study the relation of our q-convolution and our q-Laplace transform. In Section 7 we show some results on Watson type lemma for our q-Laplace transform. These results give a new framework of the summability of formal power series solutions of linear q-difference equations. In the last Section 8, we explain how to apply the properties and theorems of this paper to q-difference equations.

In Sections 2–8, we always let $q > 1$ and set $p = 1/q$. For a function $f(z)$ we define the q-difference $D_q(f(z))$ (or $D_q(f)(z)$) of $f(z)$ by

$$D_q(f(z))\,(= D_q(f)(z)) = \frac{f(qz) - f(z)}{qz - z}.$$

This operator D_q is called q-difference operator, or q-derivative operator.

2. Preliminaries

In this section, we summarize basics of q-calculus and give some preliminary results which are needed in the discussion in this paper. For the details of the topics and the proofs of some results, readers can refer to Jackson [7], Kac–Cheung [8], Gasper–Rahman [5], Olde Daalhuis [13], and Ramis[15].

2.1. q-Exponentials

As q-analogues of the exponential function $e^z = \exp(z)$, the following two functions are well-known:

$$\exp_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{((q-1)z)^n}{(q-1)(q^2-1)\cdots(q^n-1)},$$

$$\mathrm{Exp}_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} z^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}((q-1)z)^n}{(q-1)(q^2-1)\cdots(q^n-1)}.$$

We have the following properties:

(1) $\exp_q(z)$ is an entire function on $\mathbb{C}$.

(2) $\mathrm{Exp}_q(z)$ is a holomorphic function on $\{z \in \mathbb{C}\,;\, |z| < q/(q-1)\}$.

(3) $\exp_{q^{-1}}(z) = \mathrm{Exp}_q(z)$ and $\mathrm{Exp}_{q^{-1}}(z) = \exp_q(z)$.

(4) $D_q(\exp_q(z)) = \exp_q(z)$, and $\exp_q(z)$ is a unique holomorphic solution of $D_q(f(z)) = f(z)$, $f(0) = 1$ in a neighborhood of $z = 0$.
(5) $D_q(\mathrm{Exp}_q(z)) = \mathrm{Exp}_q(qz)$, and $\mathrm{Exp}_q(z)$ is a unique holomorphic solution of $D_q(f(z)) = f(qz)$, $f(0) = 1$ in a neighborhood of $z = 0$.

We set $p = 1/q$: by (3) we have $\mathrm{Exp}_q(z) = \exp_p(z)$ and $\exp_q(z) = \mathrm{Exp}_p(z)$. Since $0 < p < 1$ holds, we can use infinite product expressions of $\exp_p(z)$ and $\mathrm{Exp}_p(z)$ (see [8, §9]): we have

$$\exp_q(z) = \prod_{m=0}^{\infty} (1 + q^{-m-1}(q-1)z),$$
$$\mathrm{Exp}_q(z) = \frac{1}{\prod_{m=0}^{\infty}(1 - q^{-m-1}(q-1)z)}.$$

This shows that $\exp_q(z)$ has a simple zero at $z = -q^{m+1}/(q-1)$ $(m = 0, 1, 2 \ldots)$, and that $\mathrm{Exp}_q(z)$ can be considered as a meromorphic function on $\mathbb{C}$ having a simple pole at $z = q^{m+1}/(q-1)$ $(m = 0, 1, 2 \ldots)$. Moreover, we have the identity: $\exp_q(z)\,\mathrm{Exp}_q(-z) = 1$.

As to the asymptotic behavior we have

Proposition 2.1. — *For any $\delta > 0$ sufficiently small, we have*

$$\log\left[\mathrm{Exp}_q(z)\sin\Big(\frac{\pi\log((q-1)z)}{\log q}\Big)\right] = -\frac{(\log z)^2}{2\log q} + \Big(\frac{1}{2} - \frac{\log(q-1)}{\log q}\Big)\log z + O(1) \quad (\text{ as } z \longrightarrow \infty) \tag{2.1}$$

uniformly on $\{z \in \mathcal{R}(\mathbb{C}\setminus\{0\})\,;\, |\arg z| \leqslant 2\pi - \delta\}$, where $\mathcal{R}(\mathbb{C}\setminus\{0\})$ denotes the universal covering space of $\mathbb{C}\setminus\{0\}$.

Proof. — Since $p = 1/q$, we have $0 < p < 1$ and $\mathrm{Exp}_q(z) = \exp_p(z) = e_p((1-p)z)$ with

$$e_p(z) = \sum_{n=0}^{\infty} \frac{z^n}{(p;p)_n} = \frac{1}{\prod_{m=0}^{\infty}(1 - p^m z)}.$$

Therefore, (2.1) follows from [13, formula (3.13)]:

$$\frac{1}{e_p(z)} = \frac{1}{\prod_{m=0}^{\infty}(1 - p^{m+1}/z)} \times 2\sin\Big(\frac{\pi\log z}{\log p}\Big) \times \exp\left[\frac{\log z}{2} - \frac{1}{\log p}\Big(-\frac{\pi^2}{3} + \frac{(\log z)^2}{2}\Big) - \frac{\log p}{12} + \sum_{k\geqslant 1}\frac{\cos(2\pi k\log z/(\log p))}{k\times\sinh(2\pi^2 k/(\log p))} \times \exp\Big(\frac{2\pi^2 k}{\log p}\Big)\right]$$

on $\{z \in \mathcal{R}(\mathbb{C} \setminus \{0\})\,;\, |\arg z| \leqslant 2\pi\}$. □

For simplicity we set:

$$\begin{aligned}
W_0 &= \{z \in \mathbb{C}\,;\, |z| \leqslant q^{1/2}/(q-1)\}\,,\\
\mathscr{Z} &= \{z = -q^{m+1}/(q-1)\,;\, m = 0,1,2\ldots\}\,,\\
B_{m,\epsilon} &= \{z \in \mathbb{C}\,;\, |z + q^{m+1}/(q-1)| \leqslant \epsilon q^{m+1}/(q-1)\}\,,\\
\mathscr{Z}_\epsilon &= \bigcup_{m=0}^{\infty} B_{m,\epsilon}\,.
\end{aligned}$$

We note that $\mathscr{Z}$ is the set of zeros of the function $\exp_q(z)$ and that $\mathrm{Exp}_q(-z)$ has a simple pole at any point in $\mathscr{Z}$. If $\epsilon > 0$ is sufficiently small, the set $\mathscr{Z}_\epsilon$ is a disjoint union of closed balls $B_{m,\epsilon}$ ($m \in \mathbb{N}$ ($= \{0,1,2\ldots\}$)).

Proposition 2.2. — *In the above situation we have the following results.*

(1) *There is a* $c_0 > 0$ *such that* $|\mathrm{Exp}_q(-z)| \leqslant c_0$ *on* W_0.

(2) $\mathrm{Exp}_q(-z)$ *is rapidly decreasing* (*as* $|z| \longrightarrow \infty$ *in* $\mathbb{C} \setminus \mathscr{Z}_\epsilon$). *In addition, there is a* $K_0 > 0$ *such that for any sufficiently small* $\epsilon > 0$ *we have*

$$|\mathrm{Exp}_q(-z)| \leqslant \frac{K_0}{\epsilon} \exp(-\mu(|z|)) \quad \text{on } \mathbb{C} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$$

where

$$\mu(|z|) = \frac{(\log|z|)^2}{2\log q} + \left(-\frac{1}{2} + \frac{\log(q-1)}{\log q}\right)\log|z|\,.$$

(3) *There is a* $c_1 > 0$ *such that* $|\exp_q(z)| \leqslant c_1$ *on* W_0.

(4) $\exp_q(z)$ *is rapidly increasing* (*as* $|z| \longrightarrow \infty$ *in* $\mathbb{C} \setminus \mathscr{Z}_\epsilon$). *In addition, there is a* $K_1 > 0$ *such that* $|\exp_q(z)| \leqslant K_1 \exp(\mu(|z|))$ *on* $\mathbb{C} \setminus W_0$.

Proof. — Since $\mathrm{Exp}_q(-z)$ and $\exp_q(z)$ are holomorphic in a neighborhood of W_0, the results (1) and (3) are clear. Let us show (2). For $\delta > 0$ we write $D_\delta = \{z \in \mathbb{C}\,;\, |z| < \delta\}$: it is easy to see that there is an $A_0 > 0$ such that

$$|\sin z| \geqslant A_0\delta \quad \text{on } \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}} (D_\delta + k\pi)$$

holds for any $\delta > 0$ sufficiently small. By using this fact, we can see that there is an $A_1 > 0$ such that

$$\left|\sin\left(\frac{\pi\log((q-1)z)}{\log q}\right)\right| \geqslant A_1\epsilon \quad \text{on } \mathbb{C} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$$

holds for any $\epsilon > 0$ sufficiently small. Thus, by applying this to Proposition 2.1 we have the result (2). The result (4) follows from Proposition 2.1, the condition $\mathrm{Exp}_q(-z)\exp_q(z)=1$ and the fact that $|\sin z|$ is bounded on the strip region $\{z=x+\sqrt{-1}y\in\mathbb{C}\,;\,|y|<d\}$ for any $d>0$. $\square$

Corollary 2.3. — *For any $\beta\in\mathbb{R}$ there is a $\gamma_\beta>0$ such that for any sufficiently small $\epsilon>0$ we have*

$$|z|^\beta|\,\mathrm{Exp}_q(-z)|\leqslant\frac{K_0}{\epsilon}e^{\gamma_\beta}\quad\text{on } \mathbb{C}\setminus(W_0\cup\mathscr{Z}_\epsilon)\,. \tag{2.2}$$

Precisely, the constant e^{γ_β} is given by

$$e^{\gamma_\beta}=\frac{q^{\beta(\beta+1)/2}}{(q-1)^\beta}\times e^{c_2}\quad\text{with } c_2=\frac{\log q}{2}\left(\frac{1}{2}-\frac{\log(q-1)}{\log q}\right)^2. \tag{2.3}$$

Proof. — By a calculation we have

$$\begin{aligned}-\mu(|z|)+\beta\log|z|&=-\frac{1}{2\log q}\left(\log|z|-\log q\left(\beta+\frac{1}{2}-\frac{\log(q-1)}{\log q}\right)\right)^2\\&\qquad+\frac{\log q}{2}\left(\beta+\frac{1}{2}-\frac{\log(q-1)}{\log q}\right)^2\\&\leqslant\frac{\log q}{2}\left(\beta+\frac{1}{2}-\frac{\log(q-1)}{\log q}\right)^2=\gamma_\beta\,.\end{aligned}$$

Therefore, by (2) of Proposition 2.2 we have (2.2). $\square$

2.2. Jacobi theta function

The following function $\vartheta_q(z)$ is called Jacobi theta function:

$$\vartheta_q(z)=\sum_{n=-\infty}^{\infty}\frac{z^n}{q^{n(n-1)/2}}\,,$$

and the following properties are known (for example, see [8] and [17]):

(1) $\vartheta_q(z)$ is a holomorphic function on $\mathbb{C}\setminus\{0\}$.
(2) (Jacobi triple product formula). We have

$$\vartheta_q(z)=\prod_{m=0}^{\infty}(1-q^{-m-1})(1+q^{-m}z)(1+q^{-m-1}z^{-1})\,.$$

(3) $\vartheta_q(z)$ has a simple zero at $z=-q^m$ $(m\in\mathbb{Z})$.
(4) $\vartheta_q(q^mz)=q^{m(m+1)/2}z^m\vartheta_q(z)$ holds for any $m\in\mathbb{Z}$.

We set $p=1/q$ and $(p;p)_\infty=\lim_{n\to\infty}(p;p)_n$. By considering $\vartheta_q((1-1/q)z)$ and by using the infinite product expressions of $\vartheta_q(z)$, $\exp_q(z)$ and $\mathrm{Exp}_q(z)$ we have

PROPOSITION 2.4. — *We have the equalities:*

$$\exp_q(z) = \frac{\vartheta_q((1-1/q)z)}{(p;p)_\infty} \operatorname{Exp}_q\left(\frac{-q}{(q-1)^2 z}\right), \quad |z| > \frac{1}{q-1}, \tag{2.4}$$

$$\operatorname{Exp}_q(-z) = \frac{(p;p)_\infty}{\vartheta_q((1-1/q)z)} \exp_q\left(\frac{q}{(q-1)^2 z}\right), \quad |z| > 0. \tag{2.5}$$

The first equality is considered as an equality of holomorphic functions on $\{z \in \mathbb{C}\,;\, |z| > 1/(q-1)\}$*, while the second equality is considered as an equality of meromorphic functions on* $\mathbb{C} \setminus \{0\}$*.*

The following result is very important:

PROPOSITION 2.5. — *There are* $K_2 > 0$ *and* $R > 0$ *such that*

$$|\vartheta_q((q-1)z)| \leqslant K_2 \exp\left(\frac{(\log|z|)^2}{2\log q} + \left(\frac{1}{2} + \frac{\log(q-1)}{\log q}\right)\log|z|\right) \tag{2.6}$$

holds on $\{z \in \mathbb{C}\,;\, |z| > R\}$*.*

Proof. — By (2.5) and Proposition 2.1 we have

$$\begin{aligned}
\log\left[\frac{\vartheta_q((q-1)(-z))}{\sin\left(\frac{\pi\log((q-1)qz)}{\log q}\right)}\right] \tag{2.7}
&= \log\left[\frac{(p;p)_\infty}{\operatorname{Exp}_q(qz)\sin\left(\frac{\pi\log((q-1)qz)}{\log q}\right)} \exp_q\left(\frac{-1}{(q-1)^2 z}\right)\right] \\
&= \frac{(\log(qz))^2}{2\log q} + \left(-\frac{1}{2} + \frac{\log(q-1)}{\log q}\right)\log(qz) + O(1) \\
&= \frac{(\log z)^2}{2\log q} + \left(\frac{1}{2} + \frac{\log(q-1)}{\log q}\right)\log z + O(1) \quad (\text{as } z \longrightarrow \infty)
\end{aligned}$$

uniformly on $\{z \in \mathcal{R}(\mathbb{C} \setminus \{0\})\,;\, |\arg z| \leqslant 2\pi - \delta\}$ for any $\delta > 0$. This yields the estimate (2.6). □

2.3. Entire functions of q-exponential growth

The following result is proved in [15, Prop. 2.1]:

Proposition 2.6. — *Let $\hat{f}(z) = \sum_{n\geqslant 0} a_n z^n \in \mathbb{C}[\![z]\!]$. The following two conditions are equivalent:*

(1) *There are $A > 0$ and $H > 0$ such that*

$$|a_n| \leqslant \frac{AH^n}{[n]_q!}\,, \quad n = 0, 1, 2 \ldots \tag{2.8}$$

(2) *$\hat{f}(z)$ is the Taylor expansion at $t = 0$ of an entire function $f(z)$ satisfying the estimate*

$$|f(z)| \leqslant M \exp\left(\frac{(\log|z|)^2}{2\log q} + \alpha \log|z|\right) \quad \text{on } \mathbb{C}\setminus\{0\} \tag{2.9}$$

for some $M > 0$ and $\alpha \in \mathbb{R}$.

Since $q^{m(m-1)/2} \leqslant [m]_q!$ holds, by setting $t = q^m$ in (2.9) we have

$$\begin{aligned} |f(q^m)| &\leqslant M q^{m^2/2+\alpha m} = M(q^{\alpha+1/2})^m q^{m(m-1)/2} \\ &\leqslant M(q^{\alpha+1/2})^m [m]_q!\,, \quad m = 0, 1, 2 \ldots \end{aligned}$$

Hence, we have:

Corollary 2.7. — *If a_n $(n = 0, 1, 2 \ldots)$ satisfy the estimates (2.8) for some $A > 0$ and $H > 0$, we have*

$$\sum_{n\geqslant 0} |a_n|(q^m)^n \leqslant C h^m [m]_q!\,, \quad m = 0, 1, 2 \ldots$$

for some $C > 0$ and $h > 0$.

2.4. p-Jackson integrals

Since $p = 1/q$, we have $0 < p < 1$. In this case, the p-Jackson integral of $f(t)$ on $(0, a]$ is defined by

$$\int_0^a f(t)\,\mathrm{d}_p t = \sum_{i=0}^{\infty} f(ap^i)(ap^i - ap^{i+1}) = (1-p)a \sum_{i=0}^{\infty} f(ap^i)p^i\,.$$

A p-antiderivative of $f(t)$ is a function $F(t)$ which satisfies $D_p(F)(t) = f(t)$. By [8, Thm. 19.1] we have

Proposition 2.8. — *Let $f(t)$ be a function defined on $(0, A]$. If $f(t)t^\alpha$ is bounded on $(0, A]$ for some $0 \leqslant \alpha < 1$, the p-integral*

$$F(t) = \int_0^t f(y)\,\mathrm{d}_p y \tag{2.10}$$

converges to a function on $(0, A]$ which is a p-antiderivative of $f(t)$. Moreover, $F(t)$ is continuous at $t = 0$ with $F(0) = 0$.

We denote by $D_{p,t}$ the p-difference operator with respect to t. By the definition of p-integral, we have

Lemma 2.9. — *Let $f(t,y)$ be a function on $(0,A]\times(0,A]$ which satisfies the following properties: $f(t,y)y^\alpha$ and $D_{p,t}(f)(t,y)y^\alpha$ are bounded on $(0,A]\times(0,A]$ for some $0\leqslant\alpha<1$. Set*

$$F(t)=\int_0^t f(t,y)\,\mathrm{d}_p y:$$

then we have

$$D_p(F)(t)=f(pt,t)+\int_0^t D_{p,t}(f)(t,y)\,\mathrm{d}_p y\,.$$

2.5. q-Antiderivatives in the case $q>1$

By Proposition 2.8 we know that a p-antiderivative of $f(t)$ is given by (2.10). In this subsection, we will consider q-antiderivatives (in the case $q>1$).

Suppose that $D_q(F)(t)=f(t)$ holds; then we have

$$f(t)=D_q(F)(t)=\frac{F(qt)-F(t)}{qt-t}=\frac{F(qt)-F(pqt)}{qt-pqt}=D_p(F)(qt)$$

and so by setting $x=qt$ we have $D_p(F)(x)=f(x/q)$. By Proposition 2.8 (and [8, Thms. 18.1 and 19.1]) we have:

Lemma 2.10. — *Let $q>1$ and set $p=1/q$. Let $f(t)$ be a function on $(0,A]$. If $f(t)t^\alpha$ is bounded on $(0,A]$ for some $0\leqslant\alpha<1$, the p-integral*

$$F(t)=\int_0^t f(q^{-1}y)\,\mathrm{d}_p y\,,\quad 0<t\leqslant qA$$

gives a unique q-antiderivative of $f(t)$ satisfying $F(t)\longrightarrow 0$ (as $t\longrightarrow 0$).

Proof. — By the definition of $F(t)$ we see that $F(t)$ is a function on $(0,qA]$. By Proposition 2.8 we have $D_p(F)(t)=f(q^{-1}t)$, and so $D_q(F)(t)=D_p(F)(qt)=f(t)$. This shows that $F(t)$ is a q-antiderivative of $f(t)$. Since $|f(t)|\leqslant Mt^{-\alpha}$ holds on $(0,A]$ for some $M>0$, we have

$$\begin{aligned}|F(t)|&=\Big|(1-p)t\sum_{i=0}^\infty f(q^{-1}tp^i)p^i\Big|\\&\leqslant(1-p)t\sum_{i=0}^\infty M(q^{-1}tp^i)^{-\alpha}p^i=\frac{M(1-p)q^\alpha t^{1-\alpha}}{1-p^{1-\alpha}}\,.\end{aligned}$$

Since $1-\alpha>0$ holds, we have $F(t) \longrightarrow 0$ (as $t \longrightarrow 0$).

Let us show the uniqueness. Let $F_1(t)$ and $F_2(t)$ be two q-antiderivatives of $f(t)$ satisfying $F_i(t) \longrightarrow 0$ (as $t \longrightarrow 0$) for $i=1,2$. Set $\phi(t)=F_1(t)-F_2(t)$: then we have $D_q(\phi)(t)=0$ and $\phi(t) \longrightarrow 0$ (as $t \longrightarrow 0$). Our purpose is to prove the condition $\phi(t) \equiv 0$ on $(0,qA]$.

Suppose that $\phi(t) \not\equiv 0$ on $(0,qA]$. Then, we have $\phi(t_0)=c$ for some $t_0 \in (0,qA]$ and some $c \neq 0$. Since $\phi(t)$ satisfies $\phi(t)=\phi(q^{-1}t)$ on $(0,qA]$, we have $\phi(q^{-n}t_0)=\phi(t_0)=c$ for any $n \in \mathbb{N}$. Since $c \neq 0$ and $q^{-n}t_0 \longrightarrow 0$ (as $n \longrightarrow \infty$), this contradicts the condition $\phi(t) \longrightarrow 0$ (as $t \longrightarrow 0$). Thus, we have shown that $\phi(t) \equiv 0$ holds on $(0,qA]$, that is, $F_1(t) \equiv F_2(t)$ holds on $(0,qA]$. □

Next, let $n \in \mathbb{N}^*$ $(=\{1,2\ldots\})$, and let us consider the Cauchy problem

$$\begin{cases} D_q^n(F)(t)=f(t)\,, \\ D_q^i(F)(+0)=0 \quad \text{for } i=0,1,\ldots,n-1\,. \end{cases} \tag{2.11}$$

We have:

Proposition 2.11. — *Suppose $q>1$, and set $p=1/q$. Let $f(t)$ be a function on $(0,A]$. If $f(t)t^{\alpha}$ is bounded on $(0,A]$ for some $0 \leqslant \alpha < 1$, the equation (2.11) has a unique solution $F(t)$ on $(0,q^nA]$ satisfying $D_q^i(F)(t) \longrightarrow 0$ (as $t \longrightarrow 0$) for $i=0,1,\ldots,n-1$. Moreover, the unique solution is given by*

$$F_n(t)=\frac{1}{q^{n-1}[n-1]_q!}\int_0^t f(q^{-n}y)(t-py)_p^{n-1}\,\mathrm{d}_p y\,, \tag{2.12}$$

where $(t-py)_p^0=1$ and

$$(t-py)_p^{n-1}=(t-py)(t-p^2y)\cdots(t-p^{n-1}y)\,, \quad n \geqslant 2\,.$$

Proof. — The former half is verified by using Lemma 2.10 n-times. Let us show that $F_n(t)$ defined by (2.12) gives the unique solution of (2.11). The case $n=1$ is already proved in Lemma 2.10.

Let us show the case $n \geqslant 2$. Since $D_{p,t}((t-py)_p^{n-1})=[n-1]_p(t-py)_p^{n-2}$ holds, by Lemma 2.9, the equality $(pt-pt)_p^{n-1}=0$ and the relation $[n-1]_p=[n-1]_q/q^{n-2}$ we have

$$D_p(F_n)(t)=\frac{[n-1]_q/q^{n-2}}{q^{n-1}[n-1]_q!}\int_0^t f(q^{-n}y)(t-py)_p^{n-2}\,\mathrm{d}_p y\,.$$

Since

$$\int_0^{qt} f(q^{-n}y)(qt-py)_p^{n-2}\,\mathrm{d}_p y=q^{n-1}\int_0^t f(q^{-n+1}y)(t-py)_p^{n-2}\,\mathrm{d}_p y$$

holds, by using $D_q(F_n)(t) = D_p(F_n)(qt)$ we have the result $D_q(F_n)(t) = F_{n-1}(t)$.

By repeating the same argument we have $D_q^n(F_n)(t) = D_q(F_1)(t) = f(t)$: in the last equality we have used Lemma 2.10.

Moreover, since $|f(t)| \leqslant Mt^{-\alpha}$ holds on $(0, A]$ for some $M > 0$ and since $|(t-py)_p^{k-1}| \leqslant t^{k-1}$ holds, by the same argument as in the proof of Lemma 2.10 we have

$$\begin{aligned} |F_k(t)| &= \Big| \frac{1}{q^{k-1}[k-1]_q!} \int_0^t f(q^{-k}y)(t-py)_p^{k-1} \, \mathrm{d}_p y \Big| \\ &\leqslant \frac{1}{q^{k-1}[k-1]_q!} \frac{M(1-p)q^{k\alpha} t^{k-\alpha}}{1-p^{1-\alpha}} \quad \longrightarrow 0 \quad (\text{as } t \longrightarrow 0) \end{aligned}$$

for any $k \geqslant 1$. Therefore, we have $D_q^i(F_n)(t) = F_{n-i}(t) \longrightarrow 0$ (as $t \longrightarrow 0$) for any $i = 0, 1, \ldots, n-1$. Thus, by the uniqueness of the solution we have the condition that $F_n(t)$ in (2.12) gives the unique solution of (2.11). □

In the case $f(t) = t^m$ $(m \in \mathbb{N})$, the unique solution of (2.11) is given by

$$F(t) = \frac{t^{m+n}}{[m+n]_q \cdots [m+1]_q} = \frac{[m]_q! \, t^{m+n}}{[m+n]_q!}.$$

Therefore, by the uniqueness of the solution we have the equality

$$\frac{1}{q^{n-1}[n-1]_q!} \int_0^t (q^{-n}y)^m (t-py)_p^{n-1} \, \mathrm{d}_p y = \frac{[m]_q! \, t^{m+n}}{[m+n]_q!}$$

which is equivalent to the following well-known result on p-beta function:

$$\int_0^1 x^m (1-px)_p^{n-1} \, \mathrm{d}_p x = B_p(m+1, n) = \frac{[m]_p! [n-1]_p!}{[m+n]_p!} \tag{2.13}$$

(see [8, §21]).

2.6. q-Improper integral on $(0, \infty)$

In the case $q > 1$, for a function $f(t)$ on $(0, \infty)$ we define the q-improper integral of $f(t)$ from 0 to ∞ by

$$\int_0^\infty f(t) \, \mathrm{d}_q t = \sum_{i=-\infty}^{\infty} f(q^i)(q^{i+1} - q^i) = (q-1) \sum_{i=-\infty}^{\infty} f(q^i) q^i. \tag{2.14}$$

If we set $p = 1/q$, we have $0 < p < 1$ and the above integral is written as a p-improper integral:

$$\int_0^\infty f(t) \, \mathrm{d}_q t = \int_0^\infty f(px) \, \mathrm{d}_p x = (1-p) \sum_{i=-\infty}^{\infty} f(p^{i+1}) p^i.$$

As to p-improper integrals, basic properties are discussed in [8, §19]: in the case (2.14) we have also

Lemma 2.12.

(1) *The integral* (2.14) *is convergent, if* $f(t)$ *satisfies* $f(t) = O(1/t^{\alpha})$ (*as* $t \longrightarrow +0$) *for some* $0 \leqslant \alpha < 1$, *and* $f(t) = O(1/t^{\beta})$ (*as* $t \longrightarrow \infty$) *for some* $\beta > 1$.

(2) *We have*

$$\int_0^{\infty} D_q(f)(t)\,\mathrm{d}_q t = \lim_{n\to\infty} f(q^n) - \lim_{m\to\infty} f(q^{-m}).$$

For simplicity we denote by $[f(t)]_0^{\infty}$ *the right-hand side of the above formula.*

(3) (*Integration by parts*). *We have*

$$\int_0^{\infty} f(t)D_q(g)(t)\,\mathrm{d}_q t = \big[f(t)g(t)\big]_0^{\infty} - \int_0^{\infty} D_q(f)(t)g(qt)\,\mathrm{d}_q t.$$

3. q-Analogue of Laplace transform

For a function $f(t)$ on $\{t = q^n\,;\, n \in \mathbb{Z}\}$ we define a *q-analogue* $\mathscr{L}_q[f](s)$ *of Laplace transform of* $f(t)$ by

$$\mathscr{L}_q[f](s) = \int_0^{\infty} f(t)\,\mathrm{Exp}_q(-qst)\,\mathrm{d}_q t. \tag{3.1}$$

Example 3.1. — $\mathscr{L}_q[t^k] = [k]_q!/s^{k+1}$ for $k = 0, 1, 2\ldots$

Proof. — Since $\mathrm{Exp}_q(-qst) = (-1/s)D_q(\mathrm{Exp}_q(-st))$ holds, we have

$$\mathscr{L}_q[t^k] = \int_0^{\infty} t^k\ \mathrm{Exp}_q(-qst)\,\mathrm{d}_q t = \frac{1}{-s}\int_0^{\infty} t^k\, D_q(\mathrm{Exp}_q(-st))\,\mathrm{d}_q t.$$

Therefore, the case $k = 0$ is verfied by (2) of Lemma 2.12. If $k \geqslant 1$, by the integration by parts ((3) of Lemma 2.12) we can show $\mathscr{L}_q[t^k] = ([k]_q/s)\mathscr{L}_q[t^{k-1}]$. Hence, the general case is verified by induction on k. □

We let W_0, $\mathscr{X}$, $B_{m,\epsilon}$ ($m \in \mathbb{N}$ and $\epsilon > 0$) and $\mathscr{X}_{\epsilon}$ ($\epsilon > 0$ sufficiently small) be as in Subsection 2.1. About the convergence of the integral (3.1) we have

Proposition 3.2. — *Let* $f(t)$ *be a function on* $\{t = q^n\,;\, n \in \mathbb{Z}\}$ *satisfying the estimates*

$$|f(q^n)| \leqslant Ch^n[n]_q! \qquad \textit{for any } n = 0, 1, 2\ldots \tag{3.2}$$

$$|f(q^{-m})| \leqslant AB^m \qquad \textit{for any } m = 1, 2\ldots \tag{3.3}$$

for some $C > 0$, $h > 0$, $A > 0$ *and* $0 < B < q$.

Then, we have:

(1) *$\mathscr{L}_q[f](s)$ is well-defined as a holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > hq/(q-1)^2\} \setminus (W_0 \cup \mathscr{Z})$. Moreover, $\mathscr{L}_q[f](s)$ has at most a simple pole at any point in $\mathscr{Z} \cap (\{s \in \mathbb{C}\,;\, |s| > hq/(q-1)^2\} \setminus W_0)$.*

(2) *There are $H > 0$ and $R > 0$ such that*

$$|\mathscr{L}_q[f](s)| \leqslant \frac{H}{\epsilon\,|s|^{\alpha}} \quad \text{on } \{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}_\epsilon \tag{3.4}$$

holds for any sufficiently small $\epsilon > 0$, where

$$\alpha = \frac{\log q - \log B}{\log q} > 0\,. \tag{3.5}$$

Proof. — Let us show (1): to do so, it is sufficient to show that

$$I_1 = (q-1)\sum_{n\geqslant 0} f(q^n)\,\mathrm{Exp}_q(-sq^{n+1}) \times q^n,$$

$$\text{and} \quad I_2 = (q-1)\sum_{m\geqslant 1} f(q^{-m})\,\mathrm{Exp}_q(-sq^{-m+1}) \times q^{-m}$$

are convergent uniformly on any compact subset of $\{s \in \mathbb{C}\ ;\ |s| > hq/(q-1)^2\} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$ for any sufficiently small $\epsilon > 0$.

Let $\epsilon > 0$ be sufficiently small. Take any $s \in \mathbb{C} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$: then we have $sq^{n+1} \in \mathbb{C} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$ $(n = 0, 1, 2 \ldots)$. Therefore, by (2) of Proposition 2.2, the assumption (3.2) and the fact $[n]_q! \leqslant q^{n(n+1)/2}/(q-1)^n$ we have

$$\begin{aligned} |I_1| &\leqslant (q-1)\sum_{n\geqslant 0} Ch^n [n]_q! \times \frac{K_0}{\epsilon}\exp(-\mu(|s|q^{n+1})) \times q^n \\ &= \frac{CK_0\exp(-\mu(|s|))}{\epsilon|s|}\sum_{n\geqslant 0}\frac{[n]_q!}{q^{n(n+1)/2}}\Big(\frac{hq}{|s|(q-1)}\Big)^n \\ &\leqslant \frac{CK_0\exp(-\mu(|s|))}{\epsilon|s|}\sum_{n\geqslant 0}\Big(\frac{hq}{|s|(q-1)^2}\Big)^n. \end{aligned} \tag{3.6}$$

This shows that I_1 is convergent uniformly on any compact subset of $\{s \in \mathbb{C}\,;\, |s| > hq/(q-1)^2\} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$.

Similarly, take any $s \in \mathbb{C} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$ and any $m \in \mathbb{N}^*$: then we can see that $sq^{-m+1} \in W_0$ or $sq^{-m+1} \in \mathbb{C} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$ holds. If $sq^{-m+1} \in W_0$ we have $|\,\mathrm{Exp}_q(-sq^{-m+1})| \leqslant c_0$ (by (1) of Proposition 2.2), and if $sq^{-m+1} \in \mathbb{C} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$ we have $|\,\mathrm{Exp}_q(-sq^{-m+1})| \leqslant K_0 e^{\gamma_0}/\epsilon$ (by Corollary 2.3). Therefore, if we set $C_1 = \max\{c_0, K_0 e^{\gamma_0}/\epsilon\}$, by the assumption (3.3) we have

$$|I_2| \leqslant (q-1)\sum_{m\geqslant 1} AB^m C_1 \times q^{-m} = (q-1)AC_1 \times \frac{B/q}{1-B/q}\,.$$

This shows that I_2 is convergent uniformly on $\mathbb{C} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$.

Thus we have seen that $\mathscr{L}_q[f](s)$ is well-defined as a holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > hq/(q-1)^2\} \setminus (W_0 \cup \mathscr{Z})$. Since the function $\mathrm{Exp}_q(-sq^{n+1})$ (resp. $\mathrm{Exp}_q(-sq^{-m+1})$) has a simple pole at any point in $q^{-n-1}\mathscr{Z}$ (resp. in $q^{m-1}\mathscr{Z}$), the latter half of (1) is clear.

Next, let us show (2). Since $\exp(-\mu(|s|))$ is rapidly decreasing (as $|s| \longrightarrow \infty$), by (3.6) we see that there are $H_1 > 0$ and $R > 0$ such that

$$|I_1| \leqslant \frac{H_1}{\epsilon\,|s|^\alpha} \quad \text{on } \{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}_\epsilon \tag{3.7}$$

for any $\epsilon > 0$ sufficiently small. We may suppose that $R > q^{1/2}/(q-1)$ holds.

Let us give a sharp estimate of I_2. We note that by (3.5) we have $B = q^{1-\alpha}$, and so by (3.3) we have $|f(q^{-m})| \leqslant A(q^{1-\alpha})^m$ for $m = 1, 2\ldots$

Take any $s \in \{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}_\epsilon$ and fix it: then, we can take a positive integer N such that

$$\frac{q^{N-1/2}}{q-1} < |s| \leqslant \frac{q^{N+1/2}}{q-1}\,.$$

If $m \geqslant N+1$, we have $|s|q^{-m+1} \leqslant |s|q^{-N} \leqslant q^{1/2}/(q-1)$ which implies $-sq^{-m+1} \in W_0$. If $1 \leqslant m \leqslant N$, we have $|s|q^{-m+1} \geqslant |s|q^{-N+1} > q^{1/2}/(q-1)$ which implies $-sq^{-m+1} \in \mathbb{C} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$. Thus, by (1) of Proposition 2.2 and Corollary 2.3 we have

$$|\,\mathrm{Exp}_q(-sq^{-m+1})| \leqslant c_0 \qquad \text{for } m \geqslant N+1, \tag{3.8}$$

$$(|s|q^{-m+1})^{\alpha+1}|\,\mathrm{Exp}_q(-sq^{-m+1})| \leqslant \frac{K_0}{\epsilon}e^{\gamma_{\alpha+1}} \qquad \text{for } 1 \leqslant m \leqslant N. \tag{3.9}$$

By using these facts, let us estimate $I_2 = I_{2,1} + I_{2,2}$ with

$$\begin{aligned} I_{2,1} &= (q-1)\sum_{m \geqslant N+1} f(q^{-m})\,\mathrm{Exp}_q(-sq^{-m+1}) \times q^{-m}\,,\\ \text{and} \quad I_{2,2} &= (q-1)\sum_{1 \leqslant m \leqslant N} f(q^{-m})\,\mathrm{Exp}_q(-sq^{-m+1}) \times q^{-m}\,. \end{aligned}$$

In the case $I_{2,1}$, by the conditions $B = q^{1-\alpha}$, (3.8) and $|s| \leqslant q^{N+1/2}/(q-1)$ we have

$$
\begin{aligned}
(3.10) \qquad |I_{2,1}| &\leqslant (q-1) \sum_{m \geqslant N+1} A(q^{1-\alpha})^m\, c_0\, q^{-m} \\
&= \frac{(q-1)Ac_0}{|s|^\alpha} \sum_{m \geqslant N+1} (|s|q^{-m})^\alpha \\
&\leqslant \frac{(q-1)Ac_0}{|s|^\alpha} \sum_{m \geqslant N+1} \Big(\frac{q^{1/2}}{q-1}\Big)^\alpha (q^{N-m})^\alpha \\
&= \frac{(q-1)Ac_0}{|s|^\alpha} \frac{q^{\alpha/2}}{(q-1)^\alpha} \frac{1}{(q^\alpha - 1)}.
\end{aligned}
$$

Similarly, under the setting $\gamma = \gamma_{\alpha+1}$, by using the conditions (3.9) and $|s| > q^{N-1/2}/(q-1)$ we have

$$
\begin{aligned}
(3.11) \qquad |I_{2,2}| &\leqslant (q-1) \sum_{1 \leqslant m \leqslant N} A(q^{1-\alpha})^m \times \frac{(K_0/\epsilon)e^\gamma}{(|s|q^{-m+1})^{\alpha+1}} \times q^{-m} \\
&= \frac{(q-1)AK_0e^\gamma}{\epsilon|s|^\alpha q^\alpha} \sum_{1 \leqslant m \leqslant N} \frac{1}{|s|q^{-m+1}} \\
&\leqslant \frac{(q-1)AK_0e^\gamma}{\epsilon|s|^\alpha q^\alpha} \sum_{1 \leqslant m \leqslant N} \frac{q-1}{q^{N-m+1/2}} \\
&\leqslant \frac{(q-1)^2AK_0e^\gamma}{\epsilon|s|^\alpha q^{\alpha+1/2}} \sum_{k \geqslant 0} \frac{1}{q^k} = \frac{(q-1)^2AK_0e^\gamma}{\epsilon|s|^\alpha q^{\alpha+1/2}} \frac{1}{(1-1/q)}.
\end{aligned}
$$

Thus, by (3.7), (3.10) and (3.11) we have the estimate (3.4). □

Lastly, let us see the case where $f(t)$ is an entire function. We have

Proposition 3.3. — *Let $f(t)$ be an entire function with the estimate*

$$
|f(t)| \leqslant M \exp\left(\frac{(\log|t|)^2}{2\log q} + \beta \log|t|\right) \quad \textit{on } \mathbb{C} \setminus \{0\}
$$

for some $M > 0$ and $\beta \in \mathbb{R}$, and let $f(t) = \sum_{k \geqslant 0} a_k t^k$ be the Taylor expansion of $f(t)$ at $t = 0$. Then, $F(s) = \mathscr{L}_q[f](s)$ is a bounded holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > R\}$ for some $R > 0$, and its Taylor expansion at $s = \infty$ is given by

$$
F(s) = \sum_{k \geqslant 0} a_k \frac{[k]_q!}{s^{k+1}}.
$$

Proof. — As is seen in Subsection 2.3 we have $|f(q^n)| \leqslant M(q^{\beta+1/2})^n [n]_q!$ for any $n = 0, 1, 2\dots$ Since $f(t)$ is an entire function, we have $|f(t)| \leqslant A$

on $\{t \in \mathbb{C}\,;\, |t| \leqslant 1\}$ for some $A > 0$: this means that $|f(q^{-m})| \leqslant A$ for any $m = 1, 2\dots$ Thus, by Proposition 3.2 we see that the q-Laplace transform $F(s) = \mathscr{L}_q[f](s)$ of $f(t)$ is well-defined as a holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > q^{\beta+3/2}/(q-1)^2\} \setminus \mathscr{Z}$ and it has at most simple poles on the set $\mathscr{Z}$.

Let $f(t) = \sum_{k\geqslant 0} a_k t^k$ be the Taylor expansion of $f(t)$ at $t = 0$: by Proposition 2.6 we see that there are $A > 0$ and $H > 0$ such that $|a_k| \leqslant AH^k/[k]_q!$ for $k = 0, 1, 2\dots$ Therefore, the function

$$F_1(s) = \sum_{k\geqslant 0} a_k \frac{[k]_q!}{s^{k+1}}$$

defines a holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > H\}$. By Fubini's theorem we have

$$\begin{aligned} F(s) &= \int_0^\infty f(t)\,\mathrm{Exp}_q(-qst)\,\mathrm{d}_q t = \int_0^\infty \Big(\sum_{k\geqslant 0} a_k t^k\Big)\,\mathrm{Exp}_q(-qst)\,\mathrm{d}_q t \\ &= \sum_{k\geqslant 0} a_k \int_0^\infty t^k\,\mathrm{Exp}_q(-qst)\,\mathrm{d}_q t = \sum_{k\geqslant 0} a_k \frac{[k]_q!}{s^{k+1}} = F_1(s)\,. \end{aligned}$$

We note that every series appearing in the above equalities is absolutely convergent and so the discussion makes sense. This proves Proposition 3.3. □

By this result, for

$$\hat{f}(t) = \sum_{k\geqslant 0} a_k t^k \in \mathbb{C}[\![t]\!]$$

we may define a q-analogue of formal Laplace transform of $\hat{f}(t)$ by

$$\hat{\mathscr{L}}_q[\hat{f}](s) = \sum_{k\geqslant 0} a_k \frac{[k]_q!}{s^{k+1}}\,.$$

4. q-Analogue of Borel transform

Let $R > 0$: for a holomorphic function $F(s)$ on $\{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}$ having at most simple poles on the set $\mathscr{Z} \cap \{s \in \mathbb{C}\,;\, |s| > R\}$, we define a *$q$-analogue $\mathscr{B}_q[F](t)$ of Borel transform of $F(s)$* by

$$\mathscr{B}_q[F](t) = \frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho_t} F(s)\exp_q(st)\,\mathrm{d}s\,, \quad t = q^n \ \ (n \in \mathbb{Z})\,, \tag{4.1}$$

where the integration is taken as a contour integration along the circle $\{s \in \mathbb{C}\,;\, |s| = \rho_t\}$ in the complex plane, and $\rho_t > 0$ is taken as follows:

$$\begin{cases} \rho_t > \max\{R, q^{1/2}/(q-1)\}, & \text{if } t = q^n \text{ with } n \geqslant 0, \\ \rho_t > \max\{R, q^{|n|}/(q-1)\}, & \text{if } t = q^n \text{ with } n < 0. \end{cases} \tag{4.2}$$

We note that $\mathscr{B}_q[F](t)$ is regarded as a function on $\{t = q^n\,;\, n \in \mathbb{Z}\}$. The choice of ρ_t in (4.2) comes from the fact that the integrand $F(s)\exp_q(sq^n)$ is holomorphic on $\{s \in \mathbb{C}\,;\, |s| > \max\{R, q^{1/2}/(q-1)\}\}$ if $n \geqslant 0$, and on $\{s \in \mathbb{C}\,;\, |s| > \max\{R, q^{|n|}/(q-1)\}\}$ if $n < 0$: we note that the simple pole of $F(s)$ is canceled by the zero of $\exp_q(sq^n)$.

Example 4.1. — We have $\mathscr{B}_q[1/s^{k+1}](t) = t^k/[k]_q!$ for $k = 0, 1, 2 \ldots$

Proof. — We have

$$\begin{aligned} \mathscr{B}_q\Big[\frac{1}{s^{k+1}}\Big](t) &= \frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho_t} \frac{1}{s^{k+1}} \sum_{m \geqslant 0} \frac{(st)^m}{[m]_q!} ds \\ &= \sum_{m \geqslant 0} \frac{t^m}{[m]_q!} \frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho_t} \frac{s^m}{s^{k+1}} ds = \frac{t^k}{[k]_q!}\,. \end{aligned}$$

□

As is seen in (2) of Proposition 3.2, the q-Laplace transform $F(s) = \mathscr{L}_q[f](s)$ of $f(t)$ satisfies the estiamte

$$|F(s)| \leqslant \frac{H}{\epsilon\,|s|^{\alpha}} \quad \text{on } \{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}_\epsilon \tag{4.3}$$

(for any sufficiently small $\epsilon > 0$) for some $H > 0$, $\alpha > 0$ and $R > 0$. Under this condition we have

Proposition 4.2. — *Let $F(s)$ be a holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}$ having at most simple poles on the set $\mathscr{Z} \cap \{s \in \mathbb{C}\,;\, |s| > R\}$. Suppose that there are $H > 0$ and $\alpha > 0$ such that the estimate (4.3) holds for any sufficiently small $\epsilon > 0$. Then, the q-Borel transform $f(t) = \mathscr{B}_q[F](t)$ of $F(s)$ is well-defined as a function on $\{t = q^n\,;\, n \in \mathbb{Z}\}$ and it satisfies the following estimates:*

(1) *For any $h > (q-1) \times \max\{R, q^{1/2}/(q-1)\}$ there is a $C = C_h > 0$ such that*

$$|f(q^n)| \leqslant C h^n [n]_q!\,, \quad n = 0, 1, 2 \ldots$$

(2) *There is an $A > 0$ such that*

$$|f(q^{-m})| \leqslant A (q^{1-\alpha})^m\,, \quad m = 1, 2 \ldots$$

Proof. — Take any $h > (q-1)\times\max\{R, q^{1/2}/(q-1)\}$ and fix it. Set $\rho_0 = h/(q-1)$: we have $\rho_0 > \max\{R, q^{1/2}/(q-1)\}$. Set $M_0 = \max_{|s|=\rho_0} |F(s)|$. Then, by (4) of Proposition 2.2, for any $n \geqslant 0$ we have

$$\begin{aligned}(4.4)\qquad |f(q^n)| = |\mathscr{B}_q[F](q^n)| &= \left|\frac{1}{2\pi\sqrt{-1}}\int_{|s|=\rho_0} F(s)\exp_q(sq^n)\,\mathrm{d}s\right| \\ &\leqslant \frac{1}{2\pi} M_0 K_1 \exp(\mu(\rho_0 q^n)) \times 2\pi\rho_0 \\ &= \rho_0 M_0 K_1 \times q^{n^2/2} \times \left(\frac{\rho_0(q-1)}{q^{1/2}}\right)^n e^d\end{aligned}$$

with

$$d = \frac{(\log\rho_0)^2}{2\log q} + \left(-\frac{1}{2} + \frac{\log(q-1)}{\log q}\right)\log\rho_0\,.$$

Since $q^{n^2/2} = q^{n/2}q^{n(n-1)/2} \leqslant q^{n/2}[n]_q!$ hold, by (4.4) we have

$$\begin{aligned}|f(q^n)| &\leqslant \rho_0 M_0 K_1 \times q^{n/2}[n]_q! \times \left(\frac{\rho_0(q-1)}{q^{1/2}}\right)^n e^d \\ &= \rho_0 M_0 K_1 e^d \times (\rho_0(q-1))^n \times [n]_q!\,.\end{aligned}$$

Thus, by the condition $\rho_0(q-1) = h$ we have the result (1).

Next, let us show (2). We take an $N \in \mathbb{N}^*$ such that $q^N/(q-1) \geqslant R$, and fix it. For $1 \leqslant m \leqslant N$ we set $\rho_m = q^{N+1/2}/(q-1)$, and for $m \geqslant N+1$ we set $\rho_m = q^{m+1/2}/(q-1)$. We take any $0 < \epsilon < 1 - q^{-1/2}$, and fix it. Then, for any $m = 1, 2\dots$ we have $\{s \in \mathbb{C}\,;\, |s| = \rho_m\} \subset \{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}_\epsilon$ and so by (4.3) we have

$$|F(s)| \leqslant \frac{H}{\epsilon\,(\rho_m)^\alpha} \quad \text{for } |s| = \rho_m,\ m = 1, 2\dots$$

Moreover, if $|s| = \rho_m$ we have

$$|sq^{-m}| = \begin{cases} q^{N-m+1/2}/(q-1)\,, & \text{if } 1 \leqslant m \leqslant N\,, \\ q^{1/2}/(q-1)\,, & \text{if } m \geqslant N+1\,: \end{cases}$$

since $\exp(z)$ is an entire function, we have a $C > 0$ such that $|\exp_q(sq^{-m})| \leqslant C$ for any $|s| = \rho_m$ and $m = 1, 2\dots$ Thus, for any $m = 1, 2\dots$ we have

$$\begin{aligned}|f(q^{-m})| = |\mathscr{B}_q[F](q^{-m})| &= \left|\frac{1}{2\pi\sqrt{-1}}\int_{|s|=\rho_m} F(s)\exp_q(sq^{-m})\,\mathrm{d}s\right| \\ &\leqslant \frac{1}{2\pi}\frac{H}{\epsilon\,(\rho_m)^\alpha} C \times 2\pi\rho_m = \frac{HC}{\epsilon}(\rho_m)^{1-\alpha}\,.\end{aligned}$$

Since

$$(\rho_m)^{1-\alpha} = \begin{cases} (q^{N+1/2}/(q-1))^{1-\alpha}, & \text{if } 1 \leqslant m \leqslant N, \\ (q^{1/2}/(q-1))^{1-\alpha} \times (q^{1-\alpha})^m, & \text{if } m \geqslant N+1 \end{cases}$$

we have the result (2). □

Let $F(s)$ be a holomorphic function on $\{s \in \mathbb{C}\,; |s| > R\}$ satisfying $F(s) = O(1/|s|)$ (as $|s| \longrightarrow \infty$): then we can expand this into Taylor series of the form

$$F(s) = \sum_{k \geqslant 0} \frac{c_k}{s^{k+1}}$$

and we have the following: for any $h > R$ there is an $M > 0$ such that $|c_k| \leqslant Mh^k$ holds for any $k = 0, 1, 2 \ldots$

Proposition 4.3. — *Let $F(s)$ be a holomorphic function on $\{s \in \mathbb{C}\,; |s| > R\}$ satisfying $F(s) = O(1/|s|)$ (as $|s| \longrightarrow \infty$). Then, $f(t) = \mathscr{B}_q[F](t)$ can be extended as an entire function satisfying the estimate*

$$|f(t)| \leqslant M \exp\Big(\frac{(\log|t|)^2}{2\log q} + \beta \log|t|\Big) \quad \text{on } \mathbb{C} \setminus \{0\} \tag{4.5}$$

for some $M > 0$ and $\beta \in \mathbb{R}$. Moreover, its Taylor expansion at $t = 0$ is given by

$$f(t) = \sum_{k \geqslant 0} \frac{c_k}{[k]_q!} t^k .$$

Proof. — Take any $\rho > 0$ sufficiently large. Then, we have

$$\begin{aligned} f(t) = \mathscr{B}_q[F](t) &= \frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho} \sum_{k \geqslant 0} \frac{c_k}{s^{k+1}} \times \sum_{m \geqslant 0} \frac{(st)^m}{[m]_q!} \,\mathrm{d}s \\ &= \sum_{k \geqslant 0} c_k \sum_{m \geqslant 0} \frac{t^m}{[m]_q!} \times \frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho} \frac{s^m}{s^{k+1}} \,\mathrm{d}s = \sum_{k \geqslant 0} c_k \frac{t^k}{[k]_q!} . \end{aligned}$$

Since $|c_k| \leqslant Mh^k$ $(k = 0, 1, 2 \ldots)$ holds, by Proposition 2.6 we have the estimate (4.5). □

By this result, for

$$\hat{F}(s) = \sum_{k \geqslant 0} \frac{b_k}{s^{k+1}} \in s^{-1}\mathbb{C}[\![s^{-1}]\!]$$

we may define a q-analogue of formal Borel transform of $\hat{F}(s)$ by

$$\hat{\mathscr{B}}_q[\hat{F}](t) = \sum_{k \geqslant 0} \frac{b_k t^k}{[k]_q!} .$$

5. Inversion formulas

In Sections 3 and 4, we have given q-analogues of Laplace and Borel transforms. In this section, we will show that one is the inverse of the other.

5.1. In the case $\mathscr{B}_q \circ \mathscr{L}_q$

As to the identity $\mathscr{B}_q \circ \mathscr{L}_q = \mathrm{id}$ we have:

THEOREM 5.1 (Inversion formula: $\mathscr{B}_q \circ \mathscr{L}_q = \mathrm{id}$). — *Let $f(t)$ be a function on $\{t = q^n\,; n \in \mathbb{Z}\}$, and suppose that it satisfies (3.2) and (3.3) for some $C > 0$, $h > 0$, $A > 0$ and $0 < B < q$. Then, we have*

$$f(t) = (\mathscr{B}_q \circ \mathscr{L}_q)[f](t), \quad t = q^n \ \ (n \in \mathbb{Z}). \tag{5.1}$$

Proof. — Set $F(s) = \mathscr{L}_q[f](s)$. By Propositions 3.2 and 4.2 we see that $g(t) = \mathscr{B}_q[F](t)$ is well-defined as a function on $\{t = q^k\,; k \in \mathbb{Z}\}$, and we have

$$\begin{aligned} g(q^k) &= \frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho_k} F(s) \exp_q(sq^k)\,\mathrm{d}s \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho_k} (q-1) \sum_{n\in\mathbb{Z}} f(q^n) \operatorname{Exp}_q(-sq^{n+1}) q^n \times \exp_q(sq^k)\,\mathrm{d}s \\ &= (q-1) \sum_{n\in\mathbb{Z}} f(q^n) q^n \times \frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho_k} \operatorname{Exp}_q(-sq^{n+1}) \exp_q(sq^k)\,\mathrm{d}s \end{aligned} \tag{5.2}$$

for any $k \in \mathbb{Z}$, where ρ_k is taken sufficiently large. Here, we note:

LEMMA 5.2. — *In the above situation, we have*

$$\frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho_k} \operatorname{Exp}_q(-sq^{n+1}) \exp_q(sq^k) ds = \frac{\delta_{n,k}}{q^k(q-1)},$$

where $\delta_{n,k}$ denotes the Kronecker's delta (that is, $\delta_{n,k} = 1$ if $n = k$ and $\delta_{n,k} = 0$ if $n \neq k$).

If we admit this lemma, by (5.2) we have

$$g(q^k) = (q-1) \sum_{n\in\mathbb{Z}} f(q^n) q^n \times \frac{\delta_{n,k}}{q^k(q-1)} = f(q^k).$$

This proves (5.1). □

Proof of Lemma 5.2. — We set $G(s) = \mathrm{Exp}_q(-sq^{n+1})\,\mathrm{exp}_q(sq^k)$: then we have

$$G(s) = \frac{\prod_{m=0}^{\infty}(1 + q^{k-m-1}(q-1)s)}{\prod_{m=0}^{\infty}(1 + q^{n-m}(q-1)s)}\,.$$

Therefore, we have:

(1) If $n \leqslant k-1$, $G(s)$ is a polynomial of degree $k-1-n$ in s.

(2) If $n = k$, $G(s)$ is given by

$$G(s) = \frac{1}{(1 + q^k(q-1)s)} = \frac{1}{q^k(q-1)(s + q^{-k}/(q-1))}\,.$$

(3) If $n \geqslant k+1$, $G(s)$ is given by

$$G(s) = \prod_{i=k}^{n} \frac{1}{(1 + q^i(q-1)s)}\,.$$

In the case (1), $G(s)$ is an entire function and so we have

$$\frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho_k} G(s)\,\mathrm{d}s = 0.$$

In the case (2), $G(s)$ has a simple pole at $s = -q^{-k}/(q-1)$. Since $\rho_k > 0$ is taken sufficiently large, the pole $s = -q^{-k}/(q-1)$ is located in the domain $\{s \in \mathbb{C}\,;\, |s| < \rho_k\}$, and we have

$$\frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho_k} G(s)\,\mathrm{d}s = \frac{1}{q^k(q-1)}\,.$$

In the case (3), the denominator is a polynomial in s of degree greater than or equal to 2, and it has simple poles at $s = -q^{-i}/(q-1)$ $(k \leqslant i \leqslant n)$. In this case, the property

$$\frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho_k} G(s)\,\mathrm{d}s = 0$$

follows from

Lemma 5.3. — *Let $P(s)$ be a polynomial of degree greater than or equal to 2. Take a positive constant $\rho_0 > 0$ so that all the roots of $P(s) = 0$ are located in the domain $\{s \in \mathbb{C}\,;\, |s| < \rho_0\}$. Then, we have*

$$\int_{|s|=\rho_0} \frac{1}{P(s)}\,\mathrm{d}s = 0\,.$$

Proof. — By the assumption, we have $|P(s)| \geqslant c|s|^{\mu}$ on $\{s \in \mathbb{C}\,;\, |s| > \rho_0\}$ for some $c > 0$ and $\mu = \deg_s P \geqslant 2$. Therefore,

$$\left|\int_{|s|=\rho_0} \frac{1}{P(s)}\,\mathrm{d}s\right| = \left|\int_{|s|=R} \frac{1}{P(s)}\,\mathrm{d}s\right| \leqslant 2\pi R \times \frac{1}{cR^{\mu}} = \frac{2\pi}{cR^{\mu-1}}$$

for any $R > \rho_0$. By letting $R \longrightarrow \infty$ we have the result in Lemma 5.3. $\square$

The proof of Lemma 5.2 is completed. $\square$

5.2. In the case $\mathscr{L}_q \circ \mathscr{B}_q$

As to the identity $\mathscr{L}_q \circ \mathscr{B}_q = \mathrm{id}$, we have:

THEOREM 5.4 (Inversion formula: $\mathscr{L}_q \circ \mathscr{B}_q = \mathrm{id}$). — *Let $F(s)$ be a holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}$ having at most simple poles on the set $\mathscr{Z} \cap \{s \in \mathbb{C}\,;\, |s| > R\}$. Suppose that there are $H > 0$ and $\alpha > 0$ such that the estimate*

$$|F(s)| \leqslant \frac{H}{\epsilon\, |s|^{\alpha}} \quad \text{on } \{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}_{\epsilon} \tag{5.3}$$

holds for any sufficiently small $\epsilon > 0$. Then, we have

$$F(s) = (\mathscr{L}_q \circ \mathscr{B}_q)[F](s) \quad \text{on } \{s \in \mathbb{C}\,;\, |s| > R_1\} \setminus \mathscr{Z} \tag{5.4}$$

for some $R_1 > 0$.

The proof given below is based on the following lemma which comes from the identity: $\mathrm{Exp}_q(z)\, \mathrm{exp}_q(-z) = 1$.

LEMMA 5.5. — *We have*

$$\sum_{m+n=d} \frac{q^{m(m-1)/2}}{[m]_q!} \times \frac{(-1)^n}{[n]_q!} = \begin{cases} 1, & \text{if } d = 0, \\ 0, & \text{if } d \neq 0. \end{cases}$$

Proof of Theorem 5.4. — Suppose that $F(s)$ satisfies the assumption in Theorem 5.4. By Propositions 4.2 and 3.2, the function

$$G(s) = (\mathscr{L}_q \circ \mathscr{B}_q)[F](s)$$

is well-defined as a holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > R_1\} \setminus \mathscr{Z}$ for some $R_1 > 0$, it has at most simple poles on the set $\mathscr{Z} \cap \{s \in \mathbb{C}\,;\, |s| > R_1\}$, and there is an $H_1 > 0$ such that the estimate

$$|G(s)| \leqslant \frac{H_1}{\epsilon\, |s|^{\alpha}} \quad \text{on } \{s \in \mathbb{C}\,;\, |s| > R_1\} \setminus \mathscr{Z}_{\epsilon}$$

holds for any sufficiently small $\epsilon > 0$.

Our purpose is to prove that $G(s) = F(s)$ holds on $\{s \in \mathbb{C}\,;\, |s| > R_2\} \setminus \mathscr{Z}$ for some $R_2 > 0$. To do so, we set

$$A(s) = F(s)\vartheta_q((q-1)s), \quad B(s) = G(s)\vartheta_q((q-1)s).$$

Since $\vartheta_q((q-1)s)$ has a simple zero at $s = -q^m/(q-1)$ ($m \in \mathbb{Z}$), the functions $A(s)$ and $B(s)$ are holomorphic functions on $\{s \in \mathbb{C}\,;\, |s| > \max\{R, R_1\}\}$, and so we can expand them into Laurent series

$$A(s) = \sum_{n\in\mathbb{Z}} \frac{A_n}{s^n}, \quad B(s) = \sum_{n\in\mathbb{Z}} \frac{B_n}{s^n}. \tag{5.5}$$

Then, to prove $G(s) = F(s)$ it is enough to show that $B_n = A_n$ holds for any $n \in \mathbb{Z}$. We will show this from now.

We set $f(t) = \mathscr{B}_q[F](t)$. For any $k \in \mathbb{Z}$ we take ρ_k as in (4.2): then by (2.4) we have

$$\begin{aligned} f(q^k) &= \frac{1}{2\pi\sqrt{-1}} \int_{|z|=\rho_k} F(z) \exp_q(zq^k)\,\mathrm{d}z \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{|z|=\rho_k} F(z) \frac{\vartheta_q(q^{k-1}(q-1)z)}{(p;p)_\infty} \operatorname{Exp}_q\left(\frac{-1}{q^{k-1}(q-1)^2 z}\right) \mathrm{d}z. \end{aligned}$$

Since $\vartheta_q(q^{k-1}(q-1)z) = q^{k(k-1)/2}((q-1)z)^{k-1}\vartheta_q((q-1)z)$ holds, by (5.5) and the expansion form of $\operatorname{Exp}_q(-1/(q^{k-1}(q-1)^2 z)$ we have

$$\begin{aligned} f(q^k) &= \frac{q^{k(k-1)/2}(q-1)^{k-1}}{(p;p)_\infty 2\pi\sqrt{-1}} \int_{|z|=\rho_k} A(z) z^{k-1} \operatorname{Exp}_q\left(\frac{-1}{q^{k-1}(q-1)^2 z}\right) \mathrm{d}z \\ &= \frac{q^{k(k-1)/2}(q-1)^{k-1}}{(p;p)_\infty} \times C_k \end{aligned} \tag{5.6}$$

with

$$C_k = \sum_{n=0}^{\infty} A_{k-n} \left(\frac{-1}{q^{k-1}(q-1)^2}\right)^n \frac{q^{n(n-1)/2}}{[n]_q!}, \quad k \in \mathbb{Z}. \tag{5.7}$$

Similarly, by (2.5) we have

$$\begin{aligned} G(s) = \mathscr{L}_q[f](s) &= (q-1) \sum_{k\in\mathbb{Z}} f(q^k) \operatorname{Exp}_q(-sq^{k+1}) q^k \\ &= (q-1) \sum_{k\in\mathbb{Z}} f(q^k) \frac{(p;p)_\infty}{\vartheta_q(q^k(q-1)s)} \exp_q\left(\frac{1}{q^k(q-1)^2 s}\right) \times q^k \\ &= \frac{(p;p)_\infty}{\vartheta_q((q-1)s)} \sum_{k\in\mathbb{Z}} \frac{f(q^k)}{q^{k(k-1)/2}(q-1)^{k-1}s^k} \exp_q\left(\frac{1}{q^k(q-1)^2 s}\right) \end{aligned} \tag{5.8}$$

in the last equality we have used: $\vartheta_q(q^k(q-1)s) = q^{k(k+1)/2}((q-1)s)^k \times \vartheta_q((q-1)s)$. By (5.6) and (5.8) we have

$$B(s) = \sum_{k\in\mathbb{Z}} \frac{C_k}{s^k} \exp_q\left(\frac{1}{q^k(q-1)^2 s}\right)$$

which leads us to

$$B_N=\sum_{m=0}^{\infty}\frac{C_{N-m}}{[m]_q!}\Big(\frac{1}{q^{N-m}(q-1)^2}\Big)^m,\quad N\in\mathbb{Z}. \tag{5.9}$$

Thus, the equality $B_N=A_N$ is verified in the following way: by (5.7), (5.9) and Lemma 5.5 we have

$$\begin{aligned}
B_N&=\sum_{m=0}^{\infty}\frac{1}{[m]_q!}\left(\frac{1}{q^{N-m}(q-1)^2}\right)^m\\
&\qquad\times\sum_{n=0}^{\infty}A_{N-m-n}\left(\frac{-1}{q^{N-m-1}(q-1)^2}\right)^n\frac{q^{n(n-1)/2}}{[n]_q!}\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{(N-m-n)(N-m-n-1)/2}A_{N-m-n}}{q^{N(N-1)/2}(q-1)^{2(m+n)}}\times\frac{q^{m(m-1)/2}}{[m]_q!}\times\frac{(-1)^n}{[n]_q!}\\
&=\frac{1}{q^{N(N-1)/2}}\sum_{d=0}^{\infty}\frac{q^{(N-d)(N-d-1)/2}A_{N-d}}{(q-1)^{2d}}\sum_{m+n=d}\frac{q^{m(m-1)/2}}{[m]_q!}\times\frac{(-1)^n}{[n]_q!}\\
&=\frac{1}{q^{N(N-1)/2}}\times\frac{q^{(N-d)(N-d-1)/2}A_{N-d}}{(q-1)^{2d}}\bigg|_{d=0}=A_N.
\end{aligned}$$

This completes the proof of Theorem 5.4. □

We note that every series appearing in the above proof is absolutely convergent and so the discussion makes sense: this can be verified by using the following lemma.

Lemma 5.6. — *There are $C>0$, $h>0$ and $A>0$ such that the following estimates hold:*

$$|A_n|\leqslant Ch^n,\quad n=0,1,2\dots \tag{5.10}$$

$$|A_{-m}|\leqslant\frac{A(q^{1-\alpha})^m}{[m]_q!},\quad m=1,2\dots \tag{5.11}$$

Proof. — Since $A(s)$ is a holomorphic function on $\{s\in\mathbb{C}\,;\,|s|>R\}$, the estimate (5.10) is clear. Let us show (5.11). Take any $\epsilon_0>0$ sufficiently small: then, by (5.3) and Proposition 2.5 we have

$$|A(s)|\leqslant\frac{H}{\epsilon_0|s|^\alpha}\times K_2\exp\left(\frac{(\log|s|)^2}{2\log q}+\left(\frac{1}{2}+\frac{\log(q-1)}{\log q}\right)\log|s|\right)$$

on $\{s\in\mathbb{C}\,;\,|s|>R\}\setminus\mathscr{Z}_{\epsilon_0}$. By the maximum principle, we see that the same estimate holds also on $\{s\in\mathbb{C}\,;\,|s|>R\}$.

Let $m \in \mathbb{N}$ be sufficiently large: then we have

$$\begin{aligned}|A_{-m}| &= \left| \frac{1}{2\pi\sqrt{-1}} \int_{|s|=r} \frac{A(s)}{s^{m+1}} \, \mathrm{d}s \right| \\ &\leqslant \frac{HK_2}{\epsilon_0} \exp\left(\frac{(\log r)^2}{2\log q} + \left(-m - \alpha + \frac{1}{2} + \frac{\log(q-1)}{\log q} \right) \log r \right)\end{aligned}$$

for any $r \in (R, \infty)$. Hence, by setting $r = q^{m+\alpha-1/2}/(q-1)$ we obtain

$$|A_{-m}| \leqslant \frac{HK_2K_3}{\epsilon_0} \times \frac{(q-1)^m}{q^{m(m-1)/2}(q^\alpha)^m} \tag{5.12}$$

where

$$K_3 = \exp\left[\frac{-\log q}{2} \left(\alpha - \frac{1}{2} - \frac{\log(q-1)}{\log q} \right)^2 \right].$$

Since $[m]_q! \leqslant q^{m(m+1)/2}/(q-1)^m$ holds, by combining this with (5.12) we have

$$|A_{-m}| \leqslant \frac{HK_2K_3}{\epsilon_0} \frac{q^m}{[m]_q!(q^\alpha)^m} = \frac{HK_2K_3}{\epsilon_0} \frac{(q^{1-\alpha})^m}{[m]_q!} .$$

This proves the result (5.11). □

6. q-Analogue of convolution

Let $a(t)$ be a holomorphic function in a neighborhood of $t = 0$, and let

$$a(t) = \sum_{k \geqslant 0} a_k t^k$$

be the Taylor expansion of $a(t)$ at $t = 0$. For a function $f(t)$, we define a q-analogue of the convolution of $a(t)$ and $f(t)$ by

$$(a *_q f)(t) = \sum_{k \geqslant 0} \frac{a_k}{q^k} \int_0^t (t - py)_p^k f(q^{-k-1}y) \, \mathrm{d}_p y \, , \tag{6.1}$$

where $p = 1/q$, $(t - py)_p^0 = 1$ and

$$(t - py)_p^k = (t - py)(t - p^2 y) \cdots (t - p^k y) , \quad k \geqslant 1 .$$

Example 6.1. — We have the formula:

$$t^m *_q t^n = \frac{[m]_q![n]_q!}{[m+n+1]_q!} t^{m+n+1} , \quad m, n \in \mathbb{N} . \tag{6.2}$$

Proof. — By the definition (6.1) and the formula (2.13) for q-beta function we have

$$\begin{aligned} t^m *_q t^n &= \frac{1}{q^m}\int_0^t (t-py)_p^m (q^{-m-1}y)^n \,\mathrm{d}_p y \\ &= \frac{t^{m+n+1}}{q^{mn+m+n}}\int_0^1 (1-px)_p^m x^n \,\mathrm{d}_p x = \frac{t^{m+n+1}}{q^{mn+m+n}}\frac{[m]_p![n]_p!}{[m+n+1]_p!}\,. \end{aligned}$$

Since $[m]_p! = [m]_q!/q^{m(m-1)/2}$ we have the result (6.2). □

In general we have

Proposition 6.2. — *If $a(t)$ and $f(t)$ are holomorphic on $\{t \in \mathbb{C}\,;\ |t| < r\}$ for some $r > 0$, the q-convolution $(a *_q f)(t)$ is well-defined as a holomorphic function on $\{t \in \mathbb{C}\,; |t| < rq\}$, and its Taylor expansion is given by*

$$(a *_q f)(t) = \sum_{n\geqslant 0}\left(\sum_{i+k=n} a_i f_k \frac{[i]_q![k]_q!}{[i+k+1]_q!}\right) t^{n+1}$$

where f_k $(k \geqslant 0)$ are the coefficients of the Taylor expansion $f(t) = \sum_{k\geqslant 0} f_k t^k$.

Proof. — The former half is clear from the definition. The latter half follows from Example 6.1. □

By this result, we may define a formal q-convolution $(\hat{a}\hat{*}_q\hat{f})(t)$ of $\hat{a}(t) = \sum_{i\geqslant 0} a_i t^i$ and $\hat{f}(t) = \sum_{k\geqslant 0} a_k t^k$ by

$$(\hat{a}\hat{*}_q\hat{f})(t) = \sum_{n\geqslant 0}\left(\sum_{i+k=n} a_i f_k \frac{[i]_q![k]_q!}{[i+k+1]_q!}\right) t^{n+1}\,. \tag{6.3}$$

Now, let us consider the case where $f(t)$ is a function on $\{t = q^n\,;\, n \in \mathbb{Z}\}$. Our purpose is to prove the following convolution theorem for our q-Laplace transforms.

Theorem 6.3 (Convolution theorem for $\mathscr{L}_q$). — *Let $f(t)$ be a function on $\{t = q^n\,;\, n \in \mathbb{Z}\}$ satisfying the estimates (3.2) and (3.3) for some $C > 0$, $h > 0$, $A > 0$ and $0 < B < q$, and let $a(t)$ be an entire function with the estimate*

$$|a(t)| \leqslant M \exp\left(\frac{(\log|t|)^2}{2\log q} + \alpha\log|t|\right) \quad \text{on } \mathbb{C}\setminus\{0\}$$

for some $M > 0$ and $\alpha \in \mathbb{R}$. Then, we have

$$\mathscr{L}_q[a *_q f](s) = \mathscr{L}_q[a](s) \times \mathscr{L}_q[f](s) \quad \text{on } \{s \in \mathbb{C}\,; |s| > R\}\setminus\mathscr{X} \tag{6.4}$$

for some $R > 0$.

By Propositions 3.2 and 3.3, we know that $\mathscr{L}_q[a](s)$ and $\mathscr{L}_q[f](s)$ are well-defined as holomorphic functions on $\{s \in \mathbb{C}\,; |s| > R\} \setminus \mathscr{Z}$. The well-definedness of $\mathscr{L}_q[a *_q f](s)$ is guaranteed by the following

Proposition 6.4. — *Under the conditions in Theorem 6.3 we see that $(a *_q f)(t)$ is well-defined as a function on $\{t = q^n\,; n \in \mathbb{Z}\}$ and satisfies*

$$|(a *_q f)(q^n)| \leqslant C_1 {h_1}^n [n]_q! \quad \text{for any } n = 0, 1, 2 \ldots \tag{6.5}$$

$$|(a *_q f)(q^{-m})| \leqslant A_1 (B/q)^m \quad \text{for any } m = 1, 2 \ldots \tag{6.6}$$

for some $C_1 > 0$, $h_1 > 0$ and $A_1 > 0$ (where $B > 0$ is the same one as in Theorem 6.3).

The rest part of this section is organized as follows. In the next Subsection 6.1 we give a proof of Proposition 6.4, and in Subsection 6.2 we prove Theorem 6.3. In Subsection 6.3, by applying the inversion formula to Theorem 6.3 we show the equality: $\mathscr{B}_q[A \times F] = \mathscr{B}_q[A] *_q \mathscr{B}_q[F]$.

6.1. Proof of Proposition 6.4

For $k \in \mathbb{N}$ we set $\phi_k(t) = t^k$: then we have

$$\begin{aligned} |(\phi_k *_q f)(t)| &= \left| \frac{(1-p)}{q^k} t^{k+1} \sum_{i \geqslant 0} (1 - p^{i+1})_p^k f(q^{-k-1-i} t) q^{-i} \right| \\ &\leqslant \frac{(1-p)}{q^k} |t|^{k+1} \sum_{i \geqslant 0} |f(q^{-k-1-i} t)| q^{-i}. \end{aligned}$$

By using this, we can easily show

Lemma 6.5. — *There are $A_2 > 0$ and $C_2 > 0$ which satisfy the following:*

$$|(\phi_k *_q f)(q^n)| \leqslant C_2 (1+qh)^{n-k-1} q^{nk} [n-k-1]_q! \quad \text{for } n \geqslant k+1. \tag{6.7}$$

$$|(\phi_k *_q f)(q^n)| \leqslant A_2 q^{nk} (B/q)^{k-n} \quad \text{for } 0 \leqslant n \leqslant k, \tag{6.8}$$

$$|(\phi_k *_q f)(q^{-m})| \leqslant A_2 (B/q)^{m+k} \quad \text{for any } m \geqslant 0. \tag{6.9}$$

Now, let us prove Proposition 6.4. Let $a(t) = \sum_{k \geqslant 0} a_k t^k$ be the Taylor expansion of $a(t)$: by Proposition 2.6 we see that $|a_k| \leqslant A_3 {h_3}^k / [k]_q!$ ($k = 0, 1, 2 \ldots$) hold for some $A_3 > 0$ and $h_3 > 0$.

Let $n \geqslant 0$ and let us estimate $|(a *_q f)(q^n)|$: we divide it into

$$(a *_q f)(q^n) = \sum_{0 \leqslant k \leqslant n-1} a_k (\phi_k *_q f)(q^n) + \sum_{k \geqslant n} a_k (\phi_k *_q f)(q^n) = I_1 + I_2.$$

Since $q^{j(j-1)/2} \leqslant [j]_q! \leqslant q^{j(j+1)/2}/(q-1)^j$ holds for any $j \geqslant 0$, by (6.7) we have

$$
\begin{aligned}
(6.10)\quad |I_1| &\leqslant \sum_{0\leqslant k\leqslant n-1} \frac{A_3 {h_3}^k}{[k]_q!} \times C_2(1+qh)^{n-k-1} q^{nk}[n-k-1]_q! \\
&\leqslant A_3C_2 \sum_{0\leqslant k\leqslant n-1} \frac{{h_3}^k}{q^{k(k-1)/2}} \times \frac{(1+qh)^{n-k-1}q^{nk}q^{(n-k-1)(n-k)/2}}{(q-1)^{n-k-1}} \\
&= A_3C_2q^{n(n-1)/2} \sum_{0\leqslant k\leqslant n-1} q^k{h_3}^k \times \frac{(1+qh)^{n-k-1}}{(q-1)^{n-k-1}} \\
&\leqslant A_3C_2q^{n(n-1)/2}\Big(qh_3 + \frac{1+qh}{q-1}\Big)^{n-1}.
\end{aligned}
$$

Similarly, by (6.8) we have

$$
\begin{aligned}
(6.11)\qquad |I_2| &\leqslant \sum_{k\geqslant n} \frac{A_3{h_3}^k}{[k]_q!} \times A_2q^{nk}(B/q)^{k-n} \\
&\leqslant A_2A_3 \sum_{k\geqslant n} \frac{{h_3}^k}{q^{k(k-1)/2}} \times q^{nk}(B/q)^{k-n} \\
&= A_2A_3(qh_3)^n q^{n(n-1)/2} \sum_{k\geqslant n} \frac{{h_3}^{k-n}}{q^{(k-n)(k-n-1)/2}} \times (B/q)^{k-n}.
\end{aligned}
$$

Since $q^{n(n-1)/2} \leqslant [n]_q!$ holds, by (6.10) and (6.11) we have the result (6.5).

By using (6.9) we can prove (6.6) in the same way. □

6.2. Proof of Theorem 6.3

As in Subsection 2.5 we set

$$
F_k(t) = \frac{1}{q^{k-1}[k-1]_q!} \int_0^t f(q^{-k}y)(t-py)_p^{k-1}\, \mathrm{d}_p y, \quad k=1,2\ldots
$$

Then, by Proposition 2.11 we have $D_q(F_1)(t) = f(t)$, $D_q(F_k)(t) = F_{k-1}(t)$ (for $k \geqslant 2$), and $F_k(t) \longrightarrow 0$ (as $t \longrightarrow 0$) for $k \geqslant 1$. Moreover, under the notation $\phi_k(t) = t^k$ we have

$$
(6.12)\qquad (\phi_k *_q f)(t) = [k]_q! \times F_{k+1}(t), \quad k \geqslant 0.
$$

We note:

Lemma 6.6. — *Let $\epsilon > 0$ be sufficiently small. Then, if $s \in \{s \in \mathbb{C}\,;|s| > (1+qh)/(q-1)^2\} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$, we have*

$$\lim_{m\to\infty} F_k(q^{-m})\,\mathrm{Exp}_q(-sq^{-m}) = 0\,, \qquad k \geqslant 1\,, \tag{6.13}$$

$$\lim_{m\to\infty} F_k(q^{m})\,\mathrm{Exp}_q(-sq^{m}) = 0\,, \qquad k \geqslant 1\,. \tag{6.14}$$

Proof. — Since $F_k(0) = 0$ and $\mathrm{Exp}_q(0) = 1$ hold, the result (6.13) is clear. Let us show (6.14). Take any $s \in \mathbb{C} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$ and fix it. Since $sq^m \in \mathbb{C} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$ holds for any $m \in \mathbb{N}$, by (2) of Proposition 2.2 we have

$$|\,\mathrm{Exp}_q(-sq^m)| \leqslant \frac{K_0}{\epsilon}\exp(-\mu(|sq^m|)) = \frac{K_0\exp(-\mu(|s|))}{\epsilon q^{m(m-1)/2}(|s|(q-1))^m}\,. \tag{6.15}$$

Since $F_k(t) = (1/[k-1]_q!)(\phi_{k-1} *_q f)(t)$ holds, by (6.7) we have

$$\begin{aligned} |F_k(q^m)| &\leqslant \frac{C_2}{[k-1]_q!} \times (1+qh)^{m-k} q^{m(k-1)}[m-k]_q! \\ &\leqslant \frac{C_2}{[k-1]_q!} \times (1+qh)^{m-k} q^{m(k-1)} \times \frac{q^{(m-k)(m-k+1)/2}}{(q-1)^{m-k}} \\ &= \frac{C_2}{[k-1]_q!} \times \frac{(1+qh)^{m-k} q^{m(m-1)/2} q^{k(k-1)/2}}{(q-1)^{m-k}}\,. \end{aligned} \tag{6.16}$$

Thus, by (6.15) and (6.16) we have

$$\begin{aligned} &|F_k(q^m)\,\mathrm{Exp}_q(-sq^m)| \\ &\qquad \leqslant \frac{C_2K_0\exp(-\mu(|s|))q^{k(k-1)/2}}{\epsilon[k-1]_q!} \times \frac{(1+qh)^{m-k}}{(q-1)^{m-k}}\frac{1}{(|s|(q-1))^m} \\ &\qquad\qquad \longrightarrow 0 \quad (\text{as } m \longrightarrow \infty), \quad \text{if } \frac{(1+qh)}{|s|(q-1)^2} < 1\,. \end{aligned}$$

This proves (6.14). $\square$

Now, let us prove Theorem 6.3 in the case $a(t) = \phi_k(t)$. Since $[n-k-1]_q! \leqslant [n]_q!$ holds, Lemma 6.5 and (6.12) show that $\mathscr{L}_q[F_{k+1}](s)$ is well-defined as a holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > (1+qh)q^2/(q-1)^2\} \setminus \mathscr{Z}$. By (6.12), Lemma 6.6 (with k replaced by $k+1$) and the condition

$D_q(F_{k+1}) = F_k(t)$ we have

$$\begin{aligned}
&\mathscr{L}_q[(\phi_k *_q f)](s)\\
&= \int_0^\infty (\phi_k *_q f)(t)\,\mathrm{Exp}_q(-qst)\,\mathrm{d}_q t = [k]_q! \int_0^\infty F_{k+1}(t)\,\mathrm{Exp}_q(-qst)\,\mathrm{d}_q t\\
&= -\frac{[k]_q!}{s}\int_0^\infty F_{k+1}(t) D_q(\mathrm{Exp}_q(-st))\,\mathrm{d}_q t\\
&= -\frac{[k]_q!}{s}\left(\Big[F_{k+1}(t)\,\mathrm{Exp}_q(-st)\Big]_0^\infty - \int_0^\infty D_q(F_{k+1})(t)\,\mathrm{Exp}_q(-qst)\,\mathrm{d}_q t\right)\\
&= \frac{[k]_q!}{s}\int_0^\infty F_k(t)\,\mathrm{Exp}_q(-qst)\,\mathrm{d}_q t\,.
\end{aligned}$$

Repeating the same argument we have

$$\mathscr{L}_q[(\phi_k *_q f)](s) = \frac{[k]_q!}{s^{k+1}}\int_0^\infty f(t)\,\mathrm{Exp}_q(-qst)\,\mathrm{d}_q t = \frac{[k]_q!}{s^{k+1}} \times \mathscr{L}_q[f](s)\,.$$

Since $\mathscr{L}_q[t^k](s) = [k]_q!/s^{k+1}$ holds, this proves (6.4) in the case $a(t) = \phi_k(t)$.

The general case is verified in the following way:

$$\begin{aligned}
\mathscr{L}_q[a *_q f](s) &= \sum_{k\geqslant 0} a_k \mathscr{L}_q[\phi_k *_q f](s) = \sum_{k\geqslant 0} a_k(\mathscr{L}_q[\phi_k](s) \times \mathscr{L}_q[f](s))\\
&= \sum_{k\geqslant 0} a_k \frac{[k]_q!}{s^{k+1}} \times \mathscr{L}_q[f](s) = \mathscr{L}_q[a](s) \times \mathscr{L}_q[f](s). \qquad \square
\end{aligned}$$

6.3. On $\mathscr{B}_q[A \times F]$

By applying the inversion formula to Theorem 6.3 we have

THEOREM 6.7 (Convolution theorem for $\mathscr{B}_q$). — *Let $A(s)$ be a holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > R\}$ satisfying $A(s) = O(1/|s|)$ (as $|s| \longrightarrow \infty$), and let $F(s)$ is a holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}$ having at most simple poles on the set $\mathscr{Z} \cap \{s \in \mathbb{C}\,;\, |s| > R\}$. Suppose that there are $H > 0$ and $\alpha > 0$ satisfying*

$$|F(s)| \leqslant \frac{H}{\epsilon\,|s|^\alpha} \quad \text{on } \{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}_\epsilon$$

for any $\epsilon > 0$ sufficiently small. Then, we have the equality

$$\mathscr{B}_q[A \times F](t) = (\mathscr{B}_q[A] *_q \mathscr{B}_q[F])(t), \quad t = q^n \ (n \in \mathbb{Z})\,. \tag{6.17}$$

Proof. — We set $a(t) = \mathscr{B}_q[A](t)$ and $f(t) = \mathscr{B}_q[F](s)$: then $a(t)$ and $f(t)$ satisfy the conditions in Theorem 6.3, and so we have the equality

$$\mathscr{L}_q[a *_q f](s) = \mathscr{L}_q[a](s) \times \mathscr{L}_q[f](s) = A(s) \times F(s)\,.$$

By applying $\mathscr{B}_q$ to the both side of the above equality we have

$$(\mathscr{B}_q \circ \mathscr{L}_q)[a *_q f](t) = \mathscr{B}_q[A \times F](t), \quad t = q^m \ (m \in \mathbb{Z}).$$

Since $(\mathscr{B}_q[A] *_q \mathscr{B}_q[F])(t) = (a *_q f)(t) = (\mathscr{B}_q \circ \mathscr{L}_q)[a *_q f](t)$ holds, we have the result (6.17). □

7. q-Analogue of Watson's lemma

The classical Watson's lemma says that if a function $f(t) \in C^0((0,\infty))$ satisfies $f(t) = O(e^{at})$ (as $t \longrightarrow \infty$) for some $a \in \mathbb{R}$ and $f(t) \sim \sum_{n \geqslant 0} a_n t^n$ (as $t \longrightarrow 0$) for some $a_n \in \mathbb{C}$ $(n = 0, 1, 2 \ldots)$ the Laplace transform $L[f](s)$ of $f(t)$ also has the asymptotic expansion $L[f](s) \sim \sum_{n \geqslant 0} a_n n!/s^{n+1}$ (as $s \longrightarrow \infty$). See, for example, Olver [14].

Some q-analogues of this lemma are obtained in Zhang [19], Ramis–Zhang [17] and Lastra–Malek–Sanz [10]. In this section we will show the following

Proposition 7.1 (Watson's lemma for $\mathscr{L}_q$). — *Let $f(t)$ be a function on $\{t = q^n\,;\, n \in \mathbb{Z}\}$, and let a_k $(k = 0, 1, 2 \ldots)$ a sequence of complex numbers. Suppose the condition (3.2) for some $C > 0$ and $h > 0$: in addition, we suppose that for any $N = 0, 1, 2 \ldots$ there is a constant $A_N > 0$ satisfying*

$$\left| f(t) - \sum_{k=0}^{N-1} a_k t^k \right| \leqslant A_N |t|^N \quad \text{on } \{t = q^{-m}\,;\, m \in \mathbb{N}^*\}. \tag{7.1}$$

Then, there are $R > 0$ and $C_N > 0$ $(N = 0, 1, 2 \ldots)$ such that

$$\left| \mathscr{L}_q[f](s) - \sum_{k=0}^{N-1} \frac{a_k [k]_q!}{s^{k+1}} \right| \leqslant \frac{C_N}{\epsilon |s|^{N+1}} \quad \text{on } \{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}_\epsilon \tag{7.2}$$

holds for any $N = 0, 1, 2 \ldots$ and $\epsilon > 0$.

Proof. — Since (7.1) implies that $|f(q^{-m})| \leqslant A_0$ holds for any $m \in \mathbb{N}^*$, we see that $f(t)$ satisfies the condition (3.3): therefore, $\mathscr{L}_q[f](s)$ is well-defined as a holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > hq/(q-1)^2\} \setminus (W_0 \cup \mathscr{Z})$.

Take any $N \in \mathbb{N}$, and set

$$g_N(s) = f(t) - \sum_{k=0}^{N-1} a_k t^k.$$

Then we have

$$\mathscr{L}_q[g_N](s) = \mathscr{L}_q[f](s) - \sum_{k=0}^{N-1} \frac{a_k [k]_q!}{s^{k+1}} \tag{7.3}$$

on $\{s \in \mathbb{C}\,;\,|s| > hq/(q-1)^2\} \setminus (W_0 \cup \mathscr{Z})$. By the definition of q-Laplace transform we have

$$\mathscr{L}_q[g_N](s) = F_1(s) - \sum_{k=0}^{N-1} a_k J_{1,k}(s) + G_N(s) \tag{7.4}$$

with

$$\begin{aligned} F_1(s) &= (q-1)\sum_{n\geqslant 0} f(q^n)\operatorname{Exp}_q(-sq^{n+1})q^n, \\ J_{1,k}(s) &= (q-1)\sum_{n\geqslant 0}(q^n)^k\operatorname{Exp}_q(-sq^{n+1})q^n \quad (0\leqslant k\leqslant N-1), \\ G_N(s) &= (q-1)\sum_{m\geqslant 1} g_N(q^{-m})\operatorname{Exp}_q(-sq^{-m+1})q^{-m}. \end{aligned}$$

Take any $\epsilon > 0$ and $\delta > 0$ sufficiently small: let us estimate $F_1(s)$, $J_{1,k}(s)$ and $G_N(s)$ on $\{s \in \mathbb{C}\,;\,|s| > (h+\delta)q/(q-1)^2\} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$.

The term $F_1(s)$ is estimated in the same way as (3.6) and we have

$$\begin{aligned} |F_1(s)| &\leqslant \frac{CK_0\exp(-\mu(|s|))}{\epsilon|s|}\sum_{n\geqslant 0}\left(\frac{hq}{|s|(q-1)^2}\right)^n \\ &\leqslant \frac{CK_0e^{\gamma_N}}{\epsilon|s|^{N+1}} \times \frac{1}{1-h/(h+\delta)} : \end{aligned} \tag{7.5}$$

in the above we have used $hq/(|s|(q-1)^2) \leqslant h/(h+\delta)$ and Corollary 2.3 (and its proof). If $n \in \mathbb{N}$ we have $sq^{n+1} \in \mathbb{C} \setminus (W_0 \cup \mathscr{Z}_\epsilon)$: by using Corollary 2.3 and the condition $q^{k+1}/q^{N+1} \leqslant 1/q$ (for $0 \leqslant k \leqslant N-1$) we have

$$\begin{aligned} |J_{1,k}| &\leqslant (q-1)\sum_{n\geqslant 0}\frac{(q^{k+1})^n}{(|s|q^{n+1})^{N+1}} \times (|s|q^{n+1})^{N+1}|\operatorname{Exp}_q(-sq^{n+1})| \\ &\leqslant \frac{(q-1)}{|s|^{N+1}q^{N+1}}\sum_{n\geqslant 0}\Big(\frac{q^{k+1}}{q^{N+1}}\Big)^n \times \frac{K_0e^{\gamma_{N+1}}}{\epsilon} \\ &\leqslant \frac{(q-1)}{|s|^{N+1}q^{N+1}}\frac{1}{(1-1/q)} \times \frac{K_0e^{\gamma_{N+1}}}{\epsilon}. \end{aligned} \tag{7.6}$$

Since $|g_N(q^{-m})| \leqslant A_N(q^{-m})^N = A_N(q^{-N})^m$ holds for $m = 1, 2\ldots$ By the same argument as in (3.10) and (3.11) with $B = q^{-N}$ and $\alpha = N+1$ we have

$$|G_N| \leqslant \frac{A_N}{|s|^{N+1}}\left(\frac{c_0(q-1)q^{(N+1)/2}}{(q-1)^{N+1}(q^{N+1}-1)} + \frac{(q-1)^2K_0e^{\gamma_{N+2}}}{\epsilon q^{N+3/2}(1-1/q)}\right) \tag{7.7}$$

where c_0 and K_0 are the same as in Proposition 2.2.

Thus, by (7.3)–(7.7) and by setting

$$C_N = \frac{CK_0 e^{\gamma_N}}{1-h/(h+\delta)} + \frac{(q-1)K_0 e^{\gamma_{N+1}}}{q^{N+1}(1-1/q)} \sum_{k=0}^{N-1} |a_k| + A_N\left(\frac{c_0(q-1)q^{(N+1)/2}}{(q-1)^{N+1}(q^{N+1}-1)} + \frac{(q-1)^2K_0 e^{\gamma_{N+2}}}{q^{N+3/2}(1-1/q)}\right) \tag{7.8}$$

we have the result (7.2). □

In the analytic case we have

Corollary 7.2 (Analytic case for $\mathscr{L}_q$). — *Let $f(t)$ be a function on $\{t = q^n\,;\, n \in \mathbb{Z}\}$, and let a_k $(k = 0, 1, 2 \ldots)$ a sequence of complex numbers. Suppose the condition* (3.2) *for some $C > 0$ and $h > 0$: in addition, we suppose that there are $A_1 > 0$ and $h_1 > 0$ satisfying $|a_k| \leqslant A_1 {h_1}^k (k = 0, 1, 2 \ldots)$ and*

$$\left| f(t) - \sum_{k=0}^{N-1} a_k t^k \right| \leqslant A_1 {h_1}^N |t|^N \quad \text{on } \{t = q^{-m}\,;\, m \in \mathbb{N}^*\} \tag{7.9}$$

for any $N = 0, 1, 2 \ldots$ Then, we can find $B > 0$, $H > 0$ and $R > 0$ such that

$$\left| \mathscr{L}_q[f](s) - \sum_{k=0}^{N-1} \frac{a_k [k]_q!}{s^{k+1}} \right| \leqslant \frac{BH^N}{\epsilon} \times \frac{[N]_q!}{|s|^{N+1}} \quad \text{on } \{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}_\epsilon \tag{7.10}$$

holds for any $N = 0, 1, 2 \ldots$ and $\epsilon > 0$.

Proof. — By (2.3) and the fact $q^{N(N-1)/2} \leqslant [N]_q!$ we have

$$e^{\gamma_{N+i}} = e^{c_2} \frac{q^{(N+i)(N+i+1)/2}}{(q-1)^{N+i}} \leqslant \frac{e^{c_2} q^{i(i+1)/2}}{(q-1)^i} \times \frac{(q^{i+1})^N [N]_q!}{(q-1)^N}, \quad i = 0, 1, 2.$$

Moreover, we have

$$\sum_{k=0}^{N-1} |a_k| \leqslant \sum_{k=0}^{N-1} A_1 {h_1}^k \leqslant A_1 (1 + h_1)^{N-1}.$$

Thus, by applying these estimates to (7.8) we can obtain the result (7.10). □

Similarly, in the case of q-Borel transform we have

Proposition 7.3 (Watson's lemma for $\mathscr{B}_q$). — *Let $F(s)$ be a holomorphic function on $\{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}$ (with $R > 0$) having at most simple* *poles on the set* $\mathscr{Z} \cap \{s \in \mathbb{C}\,;\, |s| > R\}$. *Suppose that there are* $H > 0$ *and* $\alpha > 0$ *such that the estimate*

$$|F(s)| \leqslant \frac{H}{\epsilon\,|s|^{\alpha}} \quad \text{on } \{s \in \mathbb{C}\,;\, |s| > R\} \setminus \mathscr{Z}_{\epsilon} \tag{7.11}$$

holds for any sufficiently small $\epsilon > 0$. *Let* c_k $(k = 0, 1, 2 \ldots)$ *be a sequence of complex numbers: we suppose that for any* $N = 0, 1, 2 \ldots$ *there is a constant* $C_N > 0$ *satisfying*

$$\left| F(s) - \sum_{k=0}^{N-1} \frac{c_k}{s^{k+1}} \right| \leqslant \frac{C_N}{\epsilon |s|^{N+1}} \quad \text{on } \{s \in \mathbb{C}\,;\, |s| > R\} \setminus \cup \mathscr{Z}_{\epsilon} \tag{7.12}$$

for any $\epsilon > 0$. *Then we have a constant* $M > 0$ *which is independent of* N *and* C_N *such that*

$$\left| \mathscr{B}_q[F](t) - \sum_{k=0}^{N-1} \frac{c_k}{[k]_q!} t^k \right| \leqslant \frac{MC_N}{[N]_q!} |t|^N \quad \text{on } \{t = q^{-m}, m \in \mathbb{N}^*\} \tag{7.13}$$

holds for any $N = 0, 1, 2 \ldots$

Proof. — We take any $0 < \epsilon < q^{-1/2}$ and fix it. We take also $L \in \mathbb{N}$ so that $q^L/(q-1) \geqslant R$, and we set

$$\rho_{N,m} = \max\{q^{L+1/2}/(q-1), q^{N+1/2+m}/(q-1)\}, \quad (N, m) \in \mathbb{N} \times \mathbb{N}^* :$$

we have $\{s \in \mathbb{C}\,;\, |s| = \rho_{N,m}\} \subset \{s \in \mathbb{C}\,;\, |s| > R\} \setminus \cup \mathscr{Z}_{\epsilon}$ for any $(N, m) \in \mathbb{N} \times \mathbb{N}^*$.

Take any $N \in \mathbb{N}$ and set

$$G_N(s) = F(s) - \sum_{k=0}^{N-1} \frac{c_k}{s^{k+1}} :$$

we have

$$\mathscr{B}_q[G_N](t) = \mathscr{B}_q[F](t) - \sum_{k=0}^{N-1} \frac{c_k}{[k]_q!} t^k, \quad t = q^{-m} \ (m \in \mathbb{N}^*). \tag{7.14}$$

By the definition of $\rho_{N,m}$ we have

$$\mathscr{B}_q[G_N](q^{-m}) = \frac{1}{2\pi\sqrt{-1}} \int_{|s|=\rho_{N,m}} G_N(s) \exp_q(sq^{-m})\, \mathrm{d}s, \quad m \in \mathbb{N}^* :$$

therefore, if we write $\rho = \rho_{N,m}$, by (7.12) and (4) of Proposition 2.2 we have

$$(7.15) \qquad \begin{aligned} |\mathscr{B}_q[G_N](q^{-m})| &\leqslant \frac{1}{2\pi}\int_{|s|=\rho} \frac{C_N}{\epsilon|s|^{N+1}} K_1 \exp(\mu(|s|q^{-m}))\,|\mathrm{d}s| \\ &= \frac{C_N}{\epsilon\rho^N} K_1 \exp(\mu(\rho q^{-m})) \\ &= \frac{C_N K_1 (q^{-m})^N}{\epsilon(\rho q^{-m})^N} \exp(\mu(\rho q^{-m})) \end{aligned}$$

where $K_1 > 0$ is the constant in Proposition 2.2. We note:

$$\begin{aligned} &\mu(\rho q^{-m}) - N\log(\rho q^{-m}) \\ &= \frac{1}{2\log q}\left[\log\left(\rho \times \frac{(q-1)}{q^{N+1/2+m}}\right)\right]^2 - \frac{((N+1/2)\log q - \log(q-1))^2}{2\log q}. \end{aligned}$$

If $N+m \geqslant L$, by the definition of $\rho = \rho_{N,m}$ we have $\rho = q^{N+1/2+m}/(q-1)$ and so $\rho \times (q-1)/q^{N+1/2+m} = 1$ which yields

$$\begin{aligned} \mu(\rho q^{-m}) - N\log(\rho q^{-m}) &= -\frac{((N+1/2)\log q - \log(q-1))^2}{2\log q} \\ &= -\frac{N(N+1)}{2}\log q + N\log(q-1) + M_1 \end{aligned}$$

with

$$M_1 = -\frac{1}{8}\log q + \frac{1}{2}\log(q-1) - \frac{(\log(q-1))^2}{2\log q}.$$

By applying this to (7.15) we have

$$(7.16) \qquad \begin{aligned} |\mathscr{B}_q[G_N](q^{-m})| &\leqslant \frac{C_N K_1 (q^{-m})^N (q-1)^N e^{M_1}}{\epsilon q^{N(N+1)/2}} \\ &\leqslant \frac{C_N K_1 (q^{-m})^N e^{M_1}}{\epsilon [N]_q!}. \end{aligned}$$

Thus, by (7.14) and (7.16) we have the result (7.13) for any $(N,m) \in \mathbb{N}\times\mathbb{N}^*$ satisfying $N+m \geqslant L$. Since the number of (N,m) satisfying $N+m < L$ is finite, this completes the proof of Proposition 7.3. $\square$

If we set $C_N = Ah^N$ $(N = 0,1,2\dots)$ for some $A > 0$ and $h > 0$, Proposition 7.3 gives the analytic case of Watson's lemma for $\mathscr{B}_q$.

8. Application to q-difference equations

In this section we will explain how to apply the properties and theorems of this paper to q-difference equations. We will explain only the strategy of the argument: the systematic study will be done in the forthcoming paper.

Let $x \in \mathbb{C}$ be the complex variable, let $m \in \mathbb{N}^*$ and let us consider the following typical model of q-difference equation:

$$\sum_{0 \leqslant j \leqslant m} A_j(x)(x^2 D_q)^j U = F(x) \tag{8.1}$$

with the unkown function $U = U(x)$, where $A_j(x)$ $(0 \leqslant j \leqslant m)$ and $F(x)$ are holomorphic functions in a neighborhood of $x = 0$. We suppose that (8.1) has a formal solution $\hat{U}(x) = \sum_{k \geqslant 1} c_k x^k \in \mathbb{C}[\![x]\!]$. Then, the basic problem in the summability theory is:

Problem. — *Can we give a concrete meaning to this formal solution?*

This problem has been solved by Marotte–Zhang [12] by the method of factorization. In the paper of Dreyfus [4] this equation is treated after the reduction to a first order system. In this section we will treat (8.1) directly in the single higher order form.

We suppose that the formal solution $\hat{U}(x) = \sum_{k \geqslant 1} c_k x^k$ satisfies

$$|c_k| \leqslant C h^k [k]_q!, \quad k = 0, 1, 2 \dots \tag{8.2}$$

for some $C > 0$ and $h > 0$. Then, we can define the formal q-Borel transform of the formal solution $\hat{U}(x)$ (under $s = 1/x$) by

$$u(t) = \hat{\mathscr{B}}_q[\hat{U}](t) = \sum_{k \geqslant 1} \frac{c_k t^{k-1}}{[k-1]_q!} .$$

By (8.2) we see that $u(t)$ is a holomorphic function in a neighborhood of $t = 0$ and by Proposition 6.2 we see that $u(t)$ satisfies the following q-convolution equation

$$P(t) u + \sum_{0 \leqslant j \leqslant m} (a_j *_q [t^j u])(t) = f(t) \tag{8.3}$$

in a neighborhood of $t = 0$, where

$$\begin{aligned} P(t) &= \sum_{0 \leqslant j \leqslant m} A_j(0) t^j, \\ a_j(t) &= \mathscr{B}_q[A_j(x) - A_j(0)](t) \ \text{(under } s = 1/x), \quad 0 \leqslant j \leqslant m, \\ f(t) &= \mathscr{B}_q[F(x)](t) \ \text{(under } s = 1/x). \end{aligned}$$

The sufficient condition for G_q-summability (see Zhang [19]) is:

(H) $u(t)$ is extended to a function on $\{t = q^n\,;\, n \in \mathbb{Z}\}$ so that it is a solution of (8.3) on $\{t = q^n\,;\, n \in \mathbb{Z}\}$ and that it satisfies $|u(q^n)| \leqslant C_1 {h_1}^n [n]_q!$ $(n = 0, 1, 2 \dots)$ for some $C_1 > 0$ and $h_1 > 0$.

We set

$$\mathscr{X}^0 = \{x = -(q-1)q^{-m-1}\,;\, m = 0,1,2\dots\},$$

$$\mathscr{X}^0_\epsilon = \bigcup_{m=0}^{\infty} \{x \in \mathbb{C}\,;\, |x + (q-1)q^{-m-1}| \leqslant \epsilon |x|\}.$$

Then we have the following result.

Proposition 8.1. — *If the conditions* (8.2) *and* (H) *are satisfied, by setting* $U(x) = \mathscr{L}_q[u](1/x)$ *we have a true holomorphic solution* $U(x)$ *of* (8.1) *satisfying the following conditions:* $U(x)$ *is a holomorphic function on* $\{x \in \mathbb{C}\,;\, 0 < |x| < r\} \setminus \mathscr{X}^0$ *for some* $r > 0$*, it has at most simple poles on* $\mathscr{X}^0 \cap \{x \in \mathbb{C}\,;\, 0 < |x| < r\}$*, and there are* $B > 0$ *and* $H > 0$ *such that*

$$\left| U(x) - \sum_{k=1}^{N-1} c_k x^k \right| \leqslant \frac{BH^N}{\epsilon} [N]_q! |x|^N \quad \text{on } \{x \in \mathbb{C}\,;\, 0 < |x| < r\} \setminus \mathscr{X}^0_\epsilon$$

holds for any $N = 0,1,2\dots$ *and any* $\epsilon > 0$.

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