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ON THE MEAN CURVATURE FLOW OF GRAIN BOUNDARIES

by Lami KIM & Yoshihiro TONEGAWA (*)

Abstract. — Suppose that $\Gamma_0 \subset \mathbb{R}^{n+1}$ is a closed countably n-rectifiable set whose complement $\mathbb{R}^{n+1} \setminus \Gamma_0$ consists of more than one connected component. Assume that the n-dimensional Hausdorff measure of Γ_0 is finite or grows at most exponentially near infinity. Under these assumptions, we prove a global-in-time existence of mean curvature flow in the sense of Brakke starting from Γ_0. There exists a finite family of open sets which move continuously with respect to the Lebesgue measure, and whose boundaries coincide with the space-time support of the mean curvature flow.

Résumé. — Supposons que $\Gamma_0 \subset \mathbb{R}^{n+1}$ est un ensemble dénombrable fermé n-rectifiable dont le complément $\mathbb{R}^{n+1} \setminus \Gamma_0$ n'est pas connexe. Nous assumons que la mesure de Hausdorff n-dimensionnelle de Γ_0 est finie ou sa croissance est au plus exponentielle. Nous prouvons l'existence globale du flot de la courbure moyenne au sens de Brakke au départ de Γ_0. Il existe une famille finie d'ensembles ouverts qui se déplacent d'une manière continue par rapport à la mesure de Lebesgue et dont les bords coïncident avec le support du flot de la courbure moyenne.

1. Introduction

A family of n-dimensional surfaces $\{\Gamma(t)\}_{t \geqslant 0}$ in $\mathbb{R}^{n+1}$ is called the mean curvature flow (hereafter abbreviated by MCF) if the velocity is equal to its mean curvature at each point and time. Since the 1970's, the MCF has been studied by numerous researchers as it is one of the fundamental geometric evolution problems (see [5, 14, 15, 24, 37] for the overview and references related to the MCF) appearing in fields such as differential geometry, general relativity, image processing and materials science. Given a smooth surface

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Γ_0, one can find a smoothly moving MCF starting from Γ_0 until some singularities such as vanishing or pinching occur. The theory of MCF inclusive of such occurrence of singularities started with the pioneering work of Brakke in his seminal work [8]. He formulated a notion of MCF in the setting of geometric measure theory and discovered a number of striking measure-theoretic properties in this general setting. It is often called the Brakke flow and we call the flow by this name hereafter. It is a family of varifolds representing generalized surfaces which satisfy the motion law of MCF in a distributional sense. His aim was to allow a broad class of singular surfaces to move by the MCF which can undergo topological changes. Quoting from [8, p. 1]: "A physical system exhibiting this behavior is the motion of grain boundaries in an annealing pure metal [...] It is experimentally observed that these grain boundaries move with a velocity proportional to their mean curvature." One of Brakke's major achievements is his general existence theorem [8, Chapter 4]. Given a general integral varifold as an initial data, he proved a global-in-time existence of Brakke flow with an ingenious approximation scheme and delicate compactness-type theorems on varifolds. One serious uncertainty on his existence theorem, however, is that there is no guarantee that the MCF he obtained is nontrivial. That is, since the definition of Brakke flow is flexible enough to allow sudden loss of measure at any time, whatever the initial Γ_0 is, setting $\Gamma(t) = \emptyset$ for all $t > 0$, we obtain a Brakke flow satisfying the definition trivially. The proof of existence in [8] does not preclude the unpleasant possibility of getting this trivial flow when one takes the limit of approximate sequence. The idea of such "instantaneous vanishing" may appear unlikely, but the very presence of singularities of Γ_0 may potentially cause such catastrophe in his approximation scheme. For this reason, rigorous global-in-time existence of MCF of grain boundaries has been considered completely open among the specialists.

In this regard, we have two aims in this paper. The first aim is to reformulate and modify the approximation scheme so that we always obtain a nontrivial MCF even if Γ_0 is singular. We prove for the first time a rigorous global-in-time existence theorem of the MCF of grain boundaries which was not known even for the 1-dimensional case. The main existence theorem of the present paper may be stated roughly as follows.

THEOREM 1.1. — *Let n be a natural number and suppose that $\Gamma_0 \subset \mathbb{R}^{n+1}$ is a closed countably n-rectifiable set whose complement $\mathbb{R}^{n+1} \setminus \Gamma_0$ is not connected. Assume that the n-dimensional Hausdorff measure of Γ_0 is finite or grows at most exponentially near infinity. Let $E_{0,1}, \ldots, E_{0,N} \subset$*

$\mathbb{R}^{n+1}$ be mutually disjoint non-empty open sets with $N \geqslant 2$ such that $\mathbb{R}^{n+1} \setminus \Gamma_0 = \cup_{i=1}^N E_{0,i}$. Then, for each $i = 1, \ldots, N$, there exists a family of open sets $\{E_i(t)\}_{t \geqslant 0}$ with $E_i(0) = E_{0,i}$ such that $E_1(t), \ldots E_N(t)$ are mutually disjoint for each $t \geqslant 0$ and $\Gamma(t) := \mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_i(t)$ is a MCF with $\Gamma(0) = \Gamma_0$, in the sense that $\Gamma(t)$ coincides with the space-time support of a Brakke flow starting from Γ_0. Each $E_i(t)$ moves continuously in time with respect to the Lebesgue measure.

We may regard each $E_i(t) \subset \mathbb{R}^{n+1}$ as a region of "i-th grain" at time t, and $\Gamma(t)$ as the "grain boundaries" which move by their mean curvature. Some of $E_i(t)$ shrink and vanish, and some may grow and may even occupy the whole $\mathbb{R}^{n+1}$ in finite time. We may also consider a periodic setting, and in that case, a typical phenomenon is a grain coarsening. As a framework, loosely speaking, instead of working only with varifolds as Brakke did, we perceive the varifolds as boundaries of a finite number of open sets $E_i(t)$ at each time. The open sets are designed to move continuously with respect to the Lebesgue measure, so that the boundaries do not vanish instantaneously at $t = 0$. Sudden loss of measure may still occur when some "interior boundaries" inside $E_i(t)$ appear, but otherwise, one cannot vanish certain portion of boundaries arbitrarily. The resulting MCF as boundaries of open sets is more or less in accordance with the MCF of physical grain boundaries originally envisioned by Brakke. If $\mathbb{R}^{n+1} \setminus \Gamma_0$ consists of N connected components, we naturally define them to be $E_{0,1}, \ldots, E_{0,N}$. If there are infinitely many connected components, we need to group them to be finitely many mutually disjoint open sets $E_{0,1}, \ldots, E_{0,N}$ for some arbitrary $N \geqslant 2$, hence there is already non-uniqueness of grouping at this point in our scheme. Even if they are finitely many, simple examples indicate that the flow is non-unique in general, even though it is not clear how generically the non-uniqueness prevails.

The second aim of the paper is to clarify the content of [8] with a number of modifications and simplifications. Despite the potential importance of the claim, there have been no review on the existence theory of [8] so far. Also, we need to provide different definitions and proofs working in the framework of sets of boundaries. Here, we present a mostly self-contained proof which should be accessible to interested researchers versed in the basics of geometric measure theory. A good working knowledge on rectifiability [3, 18, 19] and basics on the theory of varifolds in [1, 41] are assumed.

Next, we briefly describe and compare the known results on the existence of Brakke flow to that of the present paper. Given a smooth compact embedded hypersurface in general dimension, one has a smooth MCF until the first time singularities occur. For $n = 1$, it is well known that the curves stay embedded until they become convex and shrink to a point by the results due to Gage-Hamilton [22] and Grayson [25] (see also [4] for the elegant and short proof). For general dimensions, one has the notion of viscosity solution [13, 16] which gives a family of closed sets as a unique weak solution of the MCF even after the occurrence of singularities. It is possible that the closed set may develop nontrivial interior afterwords, a phenomenon called fattening, and it is not clear if the set is Brakke flow after singularities appear in general. On the other hand, Evans and Spruck proved that almost all level sets of viscosity solution are unit density Brakke flows [17]. As a different track, there are other methods such as elliptic regularization [29] and phase field approximation via the Allen–Cahn equation [28, 44] to obtain rigorous global-in-time existence results of Brakke flow. All of the above results use the ansatz that the MCF is represented as a boundary of a single time-parametrized set, so that it is not possible to handle grain boundaries with more than two grains in general. For more general cases such as triple junction figure on a plane and the higher dimensional analogues, all known results up to this point are based more or less on a certain parametrized framework and the existence results cannot be extended past topological changes in general. For three regular curves meeting at a triple junction of 120 degrees, Bronsard and Reitich [10] proved short-time existence and uniqueness using a theory of system of parabolic PDE [42]. There are numerous results studying existence, uniqueness (or non-uniqueness) and stability under various boundary conditions as well as studies on the self-similar shrinking/expanding solutions, and we mention [6, 11, 12, 20, 21, 23, 27, 31, 33, 35, 36, 39, 40]. Compared to the above known results, our existence theorem does not require any parametrization and there is no restriction on the dimension or configuration. The regularity assumption put on the closed set Γ_0 is countable n-rectifiability, which allows wide variety of singularities, and the solution can undergo past topological changes. In this sense, even the results for the 1-dimensional case are new in an essential way.

On the computational side of the MCF of grain boundaries, there are enormous number of works on the simulations and algorithms, which are far beyond the scope of this paper. Here we simply mention for a point of reference that Brakke developed an interactive software *Surface Evolver* [9] which handles variety of geometric flow problems including the MCF. See video clips of 1-dimensional MCF of grain boundaries of as many as $N = 10000$ in Brakke's home page [7].

We end the introduction by describing the organization of the paper. In Section 2, we state our basic notation and present preliminary materials from geometric measure theory. In Section 3, we state the main existence results and give an overview of the proof. Section 4 introduces notions of open partition and a certain class of admissible functions as well as some preliminary materials concerning varifold smoothing. Section 5 contains a number of estimates on the approximation of smoothed mean curvature vector essential to the construction of approximate solutions. Section 6 gives the actual construction of approximate solutions with good estimates derived in Section 5. Section 7 and 8 are mostly independent from the previous sections and prove compactness-type theorems for rectifiability and integrality, respectively, of the limit varifold. Gathering all the results up to this point, Section 9 proves that the family of limit measures is a Brakke flow. Section 10 proves a certain continuity property of domains of "grains". Section 11 gives additional comments on the property of the solution.

2. Notation and preliminaries

2.1. Basic notation

$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ are the sets of natural numbers, rational numbers, real numbers, respectively. We set $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geqslant 0\}$. We reserve $n \in \mathbb{N}$ for the dimension of hypersurface and $\mathbb{R}^{n+1}$ is the $n+1$-dimensional Euclidean space. For $r \in (0,\infty)$ and $a \in \mathbb{R}^{n+1}$ define

$$B_r(a) := \{x \in \mathbb{R}^{n+1} : |x-a| \leqslant r\}, \quad B^n_r(a) := \{x \in \mathbb{R}^n : |x-a| \leqslant r\},$$
$$U_r(a) := \{x \in \mathbb{R}^{n+1} : |x-a| < r\}, \quad U^n_r(a) := \{x \in \mathbb{R}^n : |x-a| < r\}$$

and when $a = 0$ define $B_r := B_r(0)$, $B^n_r := B^n_r(0)$, $U_r := U_r(0)$ and $U^n_r := U^n_r(0)$. For a subset $A \subset \mathbb{R}^{n+1}$, $\operatorname{int} A$ is the set of interior points of A, and $\operatorname{clos} A$ denotes the closure of A. $\operatorname{diam} A$ is the diameter of A. For two subsets $A, B \subset \mathbb{R}^{n+1}$, define $A \triangle B := (A \setminus B) \cup (B \setminus A)$. For an open subset $U \subset \mathbb{R}^{n+1}$ let $C_c(U)$ be the set of all compactly supported continuous functions defined on U and let $C_c(U;\mathbb{R}^{n+1})$ be the set of all compactly supported continuous vector fields. Indices l of $C^l_c(U)$ and $C^l_c(U;\mathbb{R}^{n+1})$ indicate continuous l-th order differentiability. For $g \in C^1(U;\mathbb{R}^{n+1})$, we regard $\nabla g(x)$ as an element of $\operatorname{Hom}(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$. Similarly for $g \in C^2(U)$, we regard the Hessian matrix $\nabla^2 g(x)$ as an element of $\mathrm{Hom}(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$. For a Lipschitz function f, $\mathrm{Lip}\,(f)$ is the Lipschitz constant.

2.2. Notation related to measures

$\mathcal{L}^{n+1}$ denotes the Lebesgue measure on $\mathbb{R}^{n+1}$ and $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on $\mathbb{R}^{n+1}$. $\mathcal{H}^0$ denotes the counting measure. We use $\omega_n := \mathcal{H}^n(U_1^n)$ and $\omega_{n+1} := \mathcal{L}^{n+1}(U_1)$. The restriction of $\mathcal{H}^n$ to a set A is denoted by $\mathcal{H}^n\lfloor_A$, and when f is a $\mathcal{H}^n$ measurable function defined on $\mathbb{R}^{n+1}$, $\mathcal{H}^n\lfloor_f$ is the weighted $\mathcal{H}^n$ by f. Let $\mathbf{B}_{n+1}$ be the constant appearing in Besicovitch's covering theorem (see [18, §1.5.2]) on $\mathbb{R}^{n+1}$.

For a Radon measure μ on $\mathbb{R}^{n+1}$ and $\phi \in C_c(\mathbb{R}^{n+1})$, we often write $\mu(\phi)$ for $\int_{\mathbb{R}^{n+1}} \phi \, d\mu$. Let $\mathrm{spt}\,\mu$ be the support of μ, i.e., $\mathrm{spt}\,\mu := \{x \in \mathbb{R}^{n+1} : \mu(B_r(x)) > 0 \text{ for all } r > 0\}$. By definition, it is a closed set. Let $\theta^{*n}(\mu, x)$ be defined by $\limsup_{r\to 0+} \mu(B_r(x))/(\omega_n r^n)$ and let $\theta^n(\mu, x)$ be defined as $\lim_{r\to 0+} \mu(B_r(x))/(\omega_n r^n)$ when the limit exists. The set of μ measurable and (locally) square integrable functions as well as vector fields is denoted by $L^2(\mu)$ ($L^2_{loc}(\mu)$). For a set $A \subset \mathbb{R}^{n+1}$, χ_A is the characteristic function of A. If A is a set of finite perimeter, $\|\nabla\chi_A\|$ is the total variation measure of the distributional derivative $\nabla\chi_A$.

2.3. The Grassmann manifold and varifold

Let $\mathbf{G}(n+1, n)$ be the space of n-dimensional subspaces of $\mathbb{R}^{n+1}$. For $S \in \mathbf{G}(n+1, n)$, we identify S with the corresponding orthogonal projection of $\mathbb{R}^{n+1}$ onto S. Let $S^\perp \in \mathbf{G}(n+1, 1)$ be the orthogonal complement of S. For two elements A and B of $\mathrm{Hom}\,(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$, define a scalar product $A \cdot B := \mathrm{trace}\,(A^\top \circ B)$ where $A^\top$ is the transpose of A and $\circ$ indicates the composition. The identity of $\mathrm{Hom}\,(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ is denoted by I. Let $a \otimes b \in \mathrm{Hom}\,(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ be the tensor product of a, $b \in \mathbb{R}^{n+1}$, i.e., as an $(n+1)\times(n+1)$ matrix, the (i,j)-component is given by $a_i b_j$ where $a = (a_1, \ldots, a_{n+1})$ and similarly for b. For $A \in \mathrm{Hom}\,(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ define

$$|A| := \sqrt{A \cdot A}, \qquad \|A\| := \sup\{|A(x)| : x \in \mathbb{R}^{n+1}, |x| = 1\}.$$

For $A \in \mathrm{Hom}\,(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ and $S \in \mathbf{G}(n+1, n)$, let $|\Lambda_n(A \circ S)|$ be the absolute value of the Jacobian of the map $A\lfloor_S$. If S is spanned by a set of orthonormal basis $v_1, \ldots, v_n$, then $|\Lambda_n(A \circ S)|$ is the n-dimensional volume of the parallelepiped formed by $A(v_1), \ldots, A(v_n)$. If we form a $(n+1)\times n$

matrix B with these vectors as the columns, we may compute $|\Lambda_n(A \circ S)|$ as the square root of the sum of the squares of the determinants of the $n \times n$ submarices of B, or we may compute it as $\sqrt{\det(B^\top \circ B)}$.

We recall some notions related to varifolds and refer to [1, 41] for more details. Define $\mathbf{G}_n(\mathbb{R}^{n+1}) := \mathbb{R}^{n+1} \times \mathbf{G}(n+1,n)$. For any subset $C \subset \mathbb{R}^{n+1}$, we similarly define $\mathbf{G}_n(C) := C \times \mathbf{G}(n+1,n)$. A general n-varifold in $\mathbb{R}^{n+1}$ is a Radon measure on $\mathbf{G}_n(\mathbb{R}^{n+1})$. The set of all general n-varifolds in $\mathbb{R}^{n+1}$ is denoted by $\mathbf{V}_n(\mathbb{R}^{n+1})$. For $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$, let $\|V\|$ be the weight measure of V, namely, for all $\phi \in C_c(\mathbb{R}^{n+1})$,

$$\|V\|(\phi) := \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} \phi(x)\, dV(x,S).$$

For a proper map $f \in C^1(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ define $f_\# V$ as the push-forward of varifold $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ (see [1, §3.2] for the definition). Given any $\mathcal{H}^n$ measurable countably n-rectifiable set $\Gamma \subset \mathbb{R}^{n+1}$ with locally finite $\mathcal{H}^n$ measure, there is a natural n-varifold $|\Gamma| \in \mathbf{V}_n(\mathbb{R}^{n+1})$ defined by

$$|\Gamma|(\phi) := \int_\Gamma \phi(x, \mathrm{Tan}^n(\Gamma,x))\, d\mathcal{H}^n(x)$$

for all $\phi \in C_c(\mathbf{G}_n(\mathbb{R}^{n+1}))$. Here, $\mathrm{Tan}^n(\Gamma,x) \in \mathbf{G}(n+1,n)$ is the approximate tangent space which exists $\mathcal{H}^n$ a.e. on Γ (see [3, §2.2.11]). In this case, $\||\Gamma|\| = \mathcal{H}^n \lfloor_\Gamma$.

We say $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ is *rectifiable* if for all $\phi \in C_c(\mathbf{G}_n(\mathbb{R}^{n+1}))$,

$$V(\phi) = \int_\Gamma \phi(x, \mathrm{Tan}^n(\Gamma,x))\theta(x)\, d\mathcal{H}^n(x)$$

for some $\mathcal{H}^n$ measurable countably n-rectifiable set $\Gamma \subset \mathbb{R}^{n+1}$ and locally $\mathcal{H}^n$ integrable non-negative function θ defined on Γ. The set of all rectifiable n-varifolds is denoted by $\mathbf{RV}_n(\mathbb{R}^{n+1})$. Note that for such varifold, $\theta^n(\|V\|,x) = \theta(x)$, approximate tangent space as varifold exists and is equal to $\mathrm{Tan}^n(\Gamma,x)$, $\mathcal{H}^n$ a.e. on Γ. The approximate tangent space is denoted by $\mathrm{Tan}^n(\|V\|,x)$. In addition, if $\theta(x) \in \mathbb{N}$ for $\mathcal{H}^n$ a.e. on Γ, we say V is *integral*. The set of all integral n-varifolds in $\mathbb{R}^{n+1}$ is denoted by $\mathbf{IV}_n(\mathbb{R}^{n+1})$. We say V is a *unit density* n-varifold if V is integral and $\theta = 1$ $\mathcal{H}^n$ a.e. on Γ, i.e., $V = |\Gamma|$.

2.4. First variation and generalized mean curvature

For $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ let δV be the first variation of V, namely,

$$\delta V(g) := \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} \nabla g(x) \cdot S\, dV(x,S) \tag{2.1}$$

for $g \in C_c^1(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$. Let $\|\delta V\|$ be the total variation measure when it exists. If $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, by the Radon–Nikodym theorem, we have for some $\|V\|$ measurable vector field $h(\cdot,V)$

$$\delta V(g) = -\int_{\mathbb{R}^{n+1}} g(x)\cdot h(x,V)\,d\|V\|(x). \tag{2.2}$$

The vector field $h(\cdot,V)$ is called the *generalized mean curvature* of V. For any $V \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ with bounded first variation (so in particular when $h(x,V)$ exists), Brakke's perpendicularity theorem of generalized mean curvature [8, Chapter 5] says that we have for V a.e. $(x,S) \in \mathbf{G}_n(\mathbb{R}^{n+1})$

$$S^\perp(h(x,V)) = h(x,V). \tag{2.3}$$

One may also understand this property in connection with C^2 rectifiability of varifold established in [38].

2.5. The right-hand side of the MCF equation

For any $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$, $\phi \in C_c^1(\mathbb{R}^{n+1};\mathbb{R}^+)$ and $g \in C_c^1(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$, define

$$\begin{aligned}\delta(V,\phi)(g)\\ := \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} \phi(x)\nabla g(x)\cdot S\,dV(x,S) + \int_{\mathbb{R}^{n+1}} g(x)\cdot\nabla\phi(x)\,d\|V\|(x).\end{aligned} \tag{2.4}$$

As explained in [8, §2.10], $\delta(V,\phi)(g)$ may be considered as a ϕ-weighted first variation of V in the direction of g. Using $\phi\nabla g = \nabla(\phi g) - g\otimes\nabla\phi$ and (2.1), we have

$$\begin{aligned}\delta(V,\phi)(g) &= \delta V(\phi g) + \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} g(x)\cdot(I-S)(\nabla\phi(x))\,dV(x,S)\\ &= \delta V(\phi g) + \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} g(x)\cdot S^\perp(\nabla\phi(x))\,dV(x,S).\end{aligned} \tag{2.5}$$

When $\|\delta V\|$ is locally finite and absolutely continuous with respect to $\|V\|$, (2.2) and (2.5) show

$$\begin{aligned}\delta(V,\phi)(g) = &-\int_{\mathbb{R}^{n+1}} \phi(x)g(x)\cdot h(x,V)\,d\|V\|(x)\\ &+ \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} g(x)\cdot S^\perp(\nabla\phi(x))\,dV(x,S).\end{aligned} \tag{2.6}$$

Furthermore, if $V \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ with $h(\cdot, V) \in L^2_{loc}(\|V\|)$, by approximating each component of $h(\cdot, V)$ by a sequence of smooth functions, we may naturally define

$$\delta(V,\phi)(h(\cdot,V)) := \int_{\mathbb{R}^{n+1}} -\phi(x)|h(x,V)|^2 + h(x,V)\cdot\nabla\phi(x)\, d\|V\|(x). \tag{2.7}$$

Here, we also used (2.3). It is convenient to define $\delta(V,\phi)(h(\cdot,V))$ when some of the conditions above are not satisfied. Thus, we define (even if $h(\cdot, V)$ does not exist)

$$\delta(V,\phi)(h(\cdot,V)) := -\infty \tag{2.8}$$

unless $\|\delta V\|$ is locally finite, absolutely continuous with respect to $\|V\|$, $V \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ and $h(\cdot, V) \in L^2_{loc}(\|V\|)$. Formally, if a family of smooth n-dimensional surfaces $\{\Gamma(t)\}_{t\in\mathbb{R}^+}$ moves by the velocity equal to the mean curvature, then one can check that $V_t = |\Gamma(t)|$ satisfies

$$\frac{d}{dt}\|V_t\|(\phi(\cdot,t)) \leqslant \delta(V_t,\phi(\cdot,t))(h(\cdot,V_t)) + \|V_t\|\left(\frac{\partial\phi}{\partial t}(\cdot,t)\right) \tag{2.9}$$

for all $\phi = \phi(x,t) \in C^1_c(\mathbb{R}^{n+1}\times\mathbb{R}^+;\mathbb{R}^+)$. In fact, (2.9) holds with equality. Conversely, if (2.9) is satisfied for all such ϕ, then one can prove that the velocity of motion is equal to the mean curvature. The inequality in (2.9) allows the sudden loss of measure and it is the source of general non-uniqueness of Brakke's formulation.

3. Main results

3.1. Weight function Ω

To include unbounded sets which may have infinite measures in $\mathbb{R}^{n+1}$, we choose a weight function $\Omega \in C^2(\mathbb{R}^{n+1})$ satisfying

$$0 < \Omega(x) \leqslant 1,\ |\nabla\Omega(x)| \leqslant c_1\Omega(x),\ \|\nabla^2\Omega(x)\| \leqslant c_1\Omega(x) \tag{3.1}$$

for all $x \in \mathbb{R}^{n+1}$ where $c_1 \in \mathbb{R}^+$ is a constant depending on the choice of Ω. If one is interested in sets of finite $\mathcal{H}^n$ measure, one may choose

$$\Omega(x) = 1$$

and $c_1 = 0$ in this case. Another example is

$$\Omega(x) = e^{-\sqrt{1+|x|^2}}.$$

Note that the second condition of (3.1) restricts the behavior of Ω at infinity in the sense that $e^{-c_1|x|}\Omega(0) \leqslant \Omega(x)$ with c_1 as in (3.1). Thus we cannot choose arbitrarily fast decaying Ω. Depending on the choice of Ω, we may have different solutions in the end. Note that we are not so concerned with the uniqueness of the flow in this paper.

We often use the following

Lemma 3.1. — *Let c_1 be as in* (3.1). *Then for $x, y \in \mathbb{R}^{n+1}$, we have*

$$\Omega(x) \leqslant \Omega(y) \exp(c_1|x-y|). \tag{3.2}$$

3.2. Main existence theorems

The first theorem states that there exists a Brakke flow starting from Γ_0. The nontriviality is described subsequently.

Theorem 3.2. — *Suppose that $\Gamma_0 \subset \mathbb{R}^{n+1}$ is a closed countably n-rectifiable set whose complement $\mathbb{R}^{n+1} \setminus \Gamma_0$ consists of more than one connected component and suppose*

$$\mathcal{H}^n\lfloor_\Omega(\Gamma_0) \left(= \int_{\Gamma_0} \Omega(x)\, d\mathcal{H}^n(x)\right) < \infty. \tag{3.3}$$

For some $N \geqslant 2$, choose a finite collection of non-empty open sets $\{E_{0,i}\}_{i=1}^N$ such that they are disjoint and $\cup_{i=1}^N E_{0,i} = \mathbb{R}^{n+1} \setminus \Gamma_0$. Then there exists a family $\{V_t\}_{t\in\mathbb{R}^+} \subset \mathbf{V}_n(\mathbb{R}^{n+1})$ with the following property.

(1) $V_0 = |\Gamma_0|$.
(2) *For $\mathcal{L}^1$ a.e. $t \in \mathbb{R}^+$, $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ and $h(\cdot, V_t) \in L^2(\|V_t\|\lfloor_\Omega)$.*
(3) *For all $t > 0$, $\|V_t\|(\Omega) \leqslant \mathcal{H}^n\lfloor_\Omega(\Gamma_0)\exp(c_1^2 t/2)$ and $\int_0^t \int_{\mathbb{R}^{n+1}} |h(\cdot, V_s)|^2 \Omega\, d\|V_s\| ds < \infty$.*
(4) *For any $0 \leqslant t_1 < t_2 < \infty$ and $\phi \in C_c^1(\mathbb{R}^{n+1} \times \mathbb{R}^+; \mathbb{R}^+)$, we have*

$$\|V_t\|(\phi(\cdot,t))\Big|_{t=t_1}^{t_2} \leqslant \int_{t_1}^{t_2} \delta(V_t, \phi(\cdot,t))(h(\cdot,V_t)) + \|V_t\|\left(\frac{\partial\phi}{\partial t}(\cdot,t)\right) dt. \tag{3.4}$$

The choice of $E_{0,1}, \ldots, E_{0,N}$ appears irrelevant here but there are more properties as explained in Theorem 3.5. The assumption (3.3) allows various possibilities for the choice of Γ_0. If $\mathcal{H}^n(\Gamma_0) < \infty$, then, we may work with $\Omega = 1$ and $c_1 = 0$ as stated before. If $\mathcal{H}^n(\Gamma_0 \cap B_r) \leqslant ce^r$ for some $c > 0$ and for all $r > 0$, we may choose $\Omega(x) = e^{-2\sqrt{1+|x|^2}}$ with a suitable $c_1 > 0$ and we may satisfy (3.3). By (2), for a.e. t, $\delta(V_t, \phi(\cdot,t))(h(\cdot,V_t))$ in (3.4) is expressed by (2.7). Note that (3.4) is an integral version of (2.9).

For above $\{V_t\}_{t\in\mathbb{R}^+}$, we define the corresponding space-time Radon measure μ:

DEFINITION 3.3. — *Define a Radon measure μ on $\mathbb{R}^{n+1} \times \mathbb{R}^+$ by $d\mu = d\|V_t\|dt$, i.e., for $\phi \in C_c(\mathbb{R}^{n+1} \times \mathbb{R}^+)$,*

$$\int_{\mathbb{R}^{n+1}\times\mathbb{R}^+} \phi(x,t)\, d\mu(x,t) = \int_{\mathbb{R}^+} \int_{\mathbb{R}^{n+1}} \phi(x,t)\, d\|V_t\|(x)dt.$$

We have the following relations between $\|V_t\|$ and μ as well as a finiteness of the support.

PROPOSITION 3.4. — *For all $t > 0$ and $r > 0$,*

$$\operatorname{spt}\|V_t\| \subset \{x : (x,t) \in \operatorname{spt}\mu\} \text{ and } \mathcal{H}^n(B_r \cap \{x : (x,t) \in \operatorname{spt}\mu\}) < \infty. \tag{3.5}$$

We next state the existence of open complements, which may be considered as moving grains and which prevent arbitrary loss of measure of $\|V_t\|$.

THEOREM 3.5. — *Under the same assumptions of Theorem 3.2, there exists a family of open sets $\{E_i(t)\}_{t\in\mathbb{R}^+}$ for each $i = 1, \ldots, N$ with the following property. Define $\Gamma(t) := \cup_{i=1}^N \partial E_i(t)$.*

1. *$E_i(0) = E_{0,i}$ for $i = 1, \ldots, N$ and $\Gamma_0 = \Gamma(0)$.*
2. *$E_1(t), \ldots, E_N(t)$ are disjoint for each $t \in \mathbb{R}^+$.*
3. *$\{x : (x,t) \in \operatorname{spt}\mu\} = \mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_i(t) = \Gamma(t)$ for each $t > 0$.*
4. *$\|V_t\| \geqslant \|\nabla\chi_{E_i(t)}\|$ for each $t \in \mathbb{R}^+$ and $i = 1, \ldots, N$.*
5. *$S(i) := \{(x,t) : x \in E_i(t), t \in \mathbb{R}^+\}$ is open in $\mathbb{R}^{n+1} \times \mathbb{R}^+$ for each $i = 1, \ldots, N$.*
6. *Fix $i = 1, \ldots, N$, $t \in \mathbb{R}^+$, $x \in \mathbb{R}^{n+1}$ and $r > 0$, and define*
$$g(s) := \mathcal{L}^{n+1}((E_i(t) \triangle E_i(s)) \cap B_r(x))$$
for $s \in [0,\infty)$. Then $g \in C^{0,\frac{1}{2}}((0,\infty)) \cap C([0,\infty))$.

Since the Lebesgue measure of $E_i(t)$ changes locally continuously by (6), and the boundary measure bounds $\|V_t\|$ from below by (4), one may conclude that $\|V_t\|$ remains non-zero at least for some positive time. If Γ_0 is compact, $\|V_t\|$ will vanish in finite time. If unbounded, it may stay non-zero for all time.

We say that $\{V_t\}_{t\in\mathbb{R}^+}$ is a *unit density flow* if V_t is a unit density varifold for a.e. $t \in \mathbb{R}^+$. Under this unit density assumption, the results of partial regularity theory of [8, 32, 45] (see also [34]) apply to this flow.

THEOREM 3.6. — *Let $\{V_t\}_{t\in\mathbb{R}^+}$ be as in Theorem 3.2 and additionally assume that it is a unit density flow. Then, for a.e. $t \in \mathbb{R}^+$, there exists a closed set $S_t \subset \mathbb{R}^{n+1}$ with the following property. We have $\mathcal{H}^n(S_t) = 0$, and for any $x \in \mathbb{R}^{n+1} \setminus S_t$, there exists a space-time neighborhood $O_{(x,t)}$*

of (x,t) *such that, either* $\operatorname{spt}\mu \cap O_{(x,t)} = \emptyset$ *or* $\operatorname{spt}\mu$ *is a* C^∞ *embedded* n*-dimensional MCF in* $O_{(x,t)}$.

For further properties of $\{V_t\}_{t\in\mathbb{R}^+}$, see Section 11. In particular, under a mild measure-theoretic condition on Γ_0 (see Section 11.2), Theorem 3.6 is always applicable for an initial short time interval. Such general short-time existence of partially regular flow is also new in all dimensions.

3.3. Heuristic description of the proof

It is worthwhile to summarize the main steps to prove the existence of Brakke flow at this point. The proof may be roughly divided into two phases, the first is the construction of sequence of time-discrete approximate flows, and the second is to prove that the limit satisfies the desired properties of Brakke flow.

3.3.1. Construction of approximate flows

Starting from $\{E_{0,i}\}_{i=1}^N$ where $\Gamma_0 = \cup_{i=1}^N \partial E_{0,i}$, time-discrete approximate flows are constructed by alternating two steps. Let Δt_j be a small time grid size which goes to 0 as $j \to \infty$. The very first step is to map $\{E_{0,i}\}_{i=1}^N$ by a Lipschitz map so that the image under this map almost minimizes n-dimensional measure of boundaries in a small length scale of order j^{-2} but at the same time, keeping the structure of "Ω- finite open partition" (Definition 4.1). We introduce a certain admissible class of Lipschitz functions called $\mathcal{E}$-admissible functions for this purpose (Definition 4.3). This "Lipschitz deformation step" (1st step) has a regularization effect in a small length scale, which is essential to prove the rectifiability and integrality of the limit flow. The map should also have an effect of de-singularizing certain unstable singularities, even though we do not know how to utilize it so far. After this first step, we next move the open partition by a smooth approximate mean curvature which is computed by smoothing the varifold. The length scale of smoothing is much smaller than that of Lipschitz deformation, and the time step Δt_j is even much smaller than the smoothing length scale, so that the motion of this step remains very small and the map is a diffeomorphism. We need to estimate how close the approximations are for various quantities and this takes up all of Section 5. We obtain a number of estimates which are expected to hold for the limit flow and this is a general guideline to keep in mind. After this "mean curvature motion step" (2nd step), we go back and do the 1st step, and then the 2nd step and we keep moving open partitions by repeating these two steps alternatingly. We make sure that we have the right estimates by an inductive argument (Proposition 6.1).

3.3.2. Proof of properties of Brakke flow

Once we have a sequence of approximate flows with proper estimates, such as the time semi-decreasing property and approximate motion law, we see that there exists a subsequence which converges as measures on $\mathbb{R}^{n+1}$ (not as varifolds at this point) for all $t \in \mathbb{R}^+$ (Proposition 6.4). We then proceed to prove that the limit measures are rectifiable first (Section 7), and then integral next (Section 8), for a.e. t. Because of the way they are constructed, for a.e. t, we know that the approximate mean curvatures are L^2 bounded and they are almost minimizing in a small length scale. The latter gives a uniform lower density ratio bound for the limit measure (Proposition 7.2), and since the L^2 norm of mean curvature is lower-semicontinuous under measure convergence, we are in a setting where Allard's rectifiability theorem applies. This gives rectifiability of the limit measure. Once this is done, we can focus on generic points where the approximate tangent space exists. Since we only have a control of L^2 norms of approximate mean curvature, not the exact mean curvature, some extra information on a small length scale has to come in. This is provided by small tilt excess and almost minimizing properties, which show that the hypersurfaces look like a finite number of layered hyperplanes in term of measure in a small length scale (Lemma 8.1). This combined with some argument of Allard's compactness theorem of integral varifold shows that the density of the limit flow is integer-valued wherever the approximate tangent space exists. Since an approximate motion law is available, we show the limit flow satisfies the exact motion law of Brakke flow (Section 9). We in addition need to analyze the behavior of open partitions using Huisken's monotonicity formula and the relative isoperimetric inequality in the end to make sure that the desired properties in Theorem 3.5 hold (Section 10).

4. Further preliminaries for construction of approximate flows

4.1. Ω-finite open partition

Definition 4.1. — *A finite and ordered collection of sets $\mathcal{E} = \{E_i\}_{i=1}^N$ in $\mathbb{R}^{n+1}$ is called an Ω-finite open partition of N elements if*

(a) $E_1, \ldots, E_N$ *are open and mutually disjoint,*
(b) $\mathcal{H}^n\lfloor_\Omega(\mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_i) < \infty$,
(c) $\cup_{i=1}^N \partial E_i$ *is countably* n*-rectifiable.*

The set of all Ω*-finite open partitions of* N *elements is denoted by* $\mathcal{OP}_\Omega^N$*.*

Remark 4.2. — Since $\Omega(x) \geqslant e^{-c_1|x|}\Omega(0)$, (b) implies that, for any compact set $K \subset \mathbb{R}^{n+1}$, we have $\mathcal{H}^n\lfloor_K(\mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_i) < \infty$. Also, this implies

$$\mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_i = \cup_{i=1}^N \partial E_i. \tag{4.1}$$

Any open set $E \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(\partial E) < \infty$ has finite perimeter and $\|\nabla\chi_E\| \leqslant \mathcal{H}^n\lfloor_{\partial E}$ (see [3, Proposition 3.62]). By De Giorgi's theorem, the reduced boundary of E is countably n-rectifiable. On the other hand, it may differ from the topological boundary ∂E in general and the assumption (c) is not redundant.

Given $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}_\Omega^N$, we define

$$\partial\mathcal{E} := |\cup_{i=1}^N \partial E_i| \in \mathbf{IV}_n(\mathbb{R}^{n+1}) \tag{4.2}$$

which is a unit density varifold induced naturally from the countably n-rectifiable set $\cup_{i=1}^N \partial E_i$. By (b), (4.1) and (4.2), we have $\|\partial\mathcal{E}\|(\Omega) < \infty$ for $\mathcal{E} \in \mathcal{OP}_\Omega^N$.

4.2. $\mathcal{E}$-admissible function and its push-forward map $f_\star$

Definition 4.3. — *For* $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}_\Omega^N$*, a function* $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *is called* $\mathcal{E}$*-admissible if it is Lipschitz and satisfies the following. Define* $\tilde{E}_i := \mathrm{int}\,(f(E_i))$ *for each* i*. Then*

(a) $\{\tilde{E}_i\}_{i=1}^N$ *are mutually disjoint,*
(b) $\mathbb{R}^{n+1} \setminus \cup_{i=1}^N \tilde{E}_i \subset f(\cup_{i=1}^N \partial E_i)$,
(c) $\sup_{x\in\mathbb{R}^{n+1}} |f(x) - x| < \infty$.

Lemma 4.4. — *For* $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}_\Omega^N$*, suppose that* f *is* $\mathcal{E}$*-admissible. Define* $\tilde{\mathcal{E}} := \{\tilde{E}_i\}_{i=1}^N$ *with* $\tilde{E}_i := \mathrm{int}\,(f(E_i))$*. Then we have* $\tilde{\mathcal{E}} \in \mathcal{OP}_\Omega^N$*.*

Proof. — We need to check that $\tilde{\mathcal{E}}$ satisfies Definition 4.1 (a)–(c). $\{\tilde{E}_i\}_{i=1}^N$ are open and mutually disjoint by Definition 4.3(a). By Definition 4.3(b), we have

$$
\begin{aligned}
(4.3)\quad \mathcal{H}^n\lfloor_\Omega(\mathbb{R}^{n+1}\setminus\cup_{i=1}^N\tilde{E}_i)
&\leqslant \mathcal{H}^n\lfloor_\Omega(f(\cup_{i=1}^N\partial E_i))\\
&\leqslant (\mathrm{Lip}(f))^n\int_{\cup_{i=1}^N\partial E_i}\Omega(f(y))\,d\mathcal{H}^n(y)\\
&\leqslant (\mathrm{Lip}(f))^n\exp(c_1\sup_{y\in\mathbb{R}^{n+1}}|f(y)-y|)\,\mathcal{H}^n\lfloor_\Omega(\cup_{i=1}^N\partial E_i),
\end{aligned}
$$

where we used (3.2). The last quantity is finite due to Definition 4.1(b) and (4.1) for $\mathcal{E}$ and Definition 4.3(c) for f. Since $\cup_{i=1}^N\partial E_i$ is countably n-rectifiable, so is the Lipschitz image $f(\cup_{i=1}^N\partial E_i)$. Any subset of countably n-rectifiable set is again countably n-rectifiable, thus by Definition 4.3(b) and (4.1) for $\tilde{\mathcal{E}}$, $\tilde{\mathcal{E}}$ satisfies Definition 4.1(c) as well. This concludes the proof. □

Definition 4.5. — *For $\mathcal{E}=\{E_i\}_{i=1}^N\in\mathcal{OP}_\Omega^N$ and $\mathcal{E}$-admissible function f, let $\tilde{\mathcal{E}}$ be defined as in Lemma 4.4. We define $\tilde{\mathcal{E}}$ to be the push-forward of $\mathcal{E}$ by f and define*

$$f_\star\mathcal{E}:=\tilde{\mathcal{E}}\in\mathcal{OP}_\Omega^N.$$

Under the definition of $f_\star$, the unit density varifold $\partial f_\star\mathcal{E}$ (cf. (4.2)) is $|\cup_{i=1}^N\partial\tilde{E}_i|$ and is in general different from the usual push-forward of varifold $f_\sharp\partial\mathcal{E}=f_\sharp|\cup_{i=1}^N\partial E_i|$ in that it does not count the multiplicity of image under the map. Moreover, $\partial f_\star\mathcal{E}$ is not defined as the varifold induced from the set $f(\cup_{i=1}^N\partial E_i)$ in general. For example, if $(\mathrm{int}\,f(E_i))\cap f(\partial E_i)$ is non-empty (whose possibility is not excluded by Definition 4.3), it does not belong to $\cup_{i=1}^N\partial\tilde{E}_i$ and thus $f(\cup_{i=1}^N\partial E_i)\neq\cup_{i=1}^N\partial\tilde{E}_i$ in this case.

4.3. Examples of $f_\star\mathcal{E}$

It is worthwhile to see some simple examples of $\mathcal{E}$ and $\mathcal{E}$-admissible functions to see what to expect. The choice of this particular admissible class characterizes general tendencies of what would happen to singularities. As we explain in Section 4.5, we are interested in maps which reduce $\mathcal{H}^n$ measure of boundaries.

4.3.1. Two lines crossing with four different open sets

Consider the following Figure 4.1, where two lines are intersecting, and $\mathcal{E}$ consists of four open sectors as shown. To reduce length of boundaries, one may consider a Lipschitz map f which vertically crushes triangle areas of E_1 and E_3 to a horizontal line segment, shrinks the neighboring areas next to them, and stretches some portion of E_2 and E_4 so that the map is Lipschitz. The map reduces the length of boundaries, and also $\mathcal{E}$-admissible since $f(\cup_{i=1}^4 \partial E_i) = \cup_{i=1}^4 \partial \tilde{E}_i$. This example indicates that, if we choose f which locally reduces boundary measure, junctions of more than three curves are likely to break up into triple junctions.

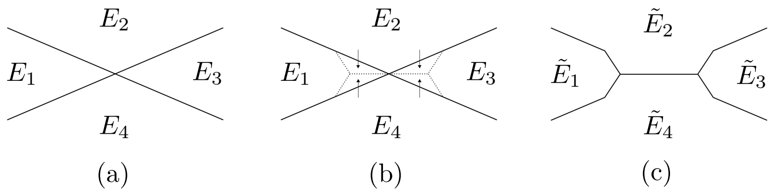

Figure 4.1.

4.3.2. Interior boundary

Suppose that we have only E_1 as shown in Figure 4.2, which is the complement of x-axis. For f, we may take a smooth map such that the dotted region of the second figure is stretched downwards to hang over the lower half. Then the portion of x-axis covered by this map will be interior points of the image of E_1 under f, thus we have $\tilde{E}_1$ as shown in the third figure. By considering such "stretching map", we may even eliminate the whole x-axis with arbitrarily small deformation. This example indicates that interior boundary is likely to be eliminated under measure reducing f. For this reason, as illustrated in Figure 4.3, if (a) is the initial data, the line segment connecting two circles is likely to vanish instantly.

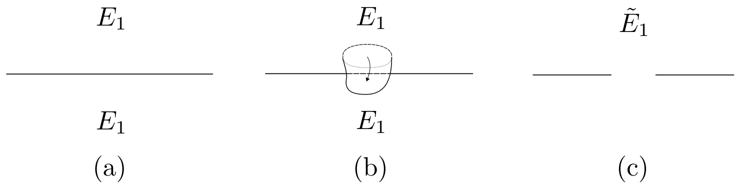

Figure 4.2.

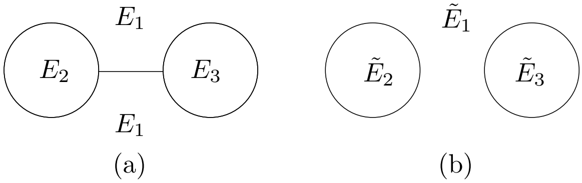

Figure 4.3.

4.3.3. Two lines crossing with two different open sets

The next example is similar to 4.3.1, but labeling is different as shown in Figure 4.4. By using a Lipschitz map of 4.3.1 and then composing a map of 4.3.2 to eliminate the horizontal line segment appearing in Figure 4.1(c), we can obtain Figure 4.4(b). Thus, depending on the combination of domains, we expect to have different behaviors.

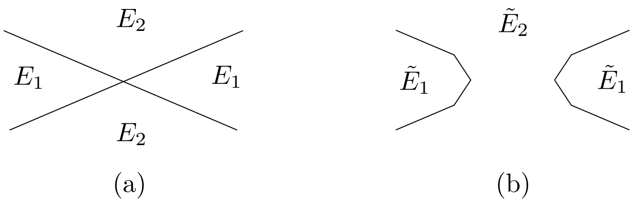

Figure 4.4.

4.3.4. Radial projection

As in Figure 4.5, consider a Lipschitz map which radially projects the annular region bounded by two dotted circles to the larger circle, with the trace of map being a radial line emanating from x_0. f expands the smaller disc to fill the larger disc one-to-one. Outside the larger disc, f is identity. This map is $\mathcal{E}$-admissible since the new boundary is in $f(\partial E_1)$. Note that some portion of $f(\partial E_1)$ does not become part of $\partial \tilde{E}_1$ because it is mapped to the interior of $\tilde{E}_2$. Depending on how much length there is inside the disc, the map reduces the length. This type of projection map is used when we prove the rectifiability and integrality of the limit flow.

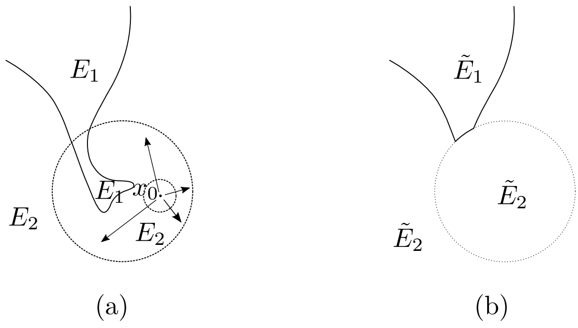

Figure 4.5.

4.4. Families $\mathcal{A}_j$ and $\mathcal{B}_j$ of test functions and vector fields

We define sets of test functions $\mathcal{A}_j$ and vector fields $\mathcal{B}_j$ for $j \in \mathbb{N}$ as

$$(4.4) \quad \mathcal{A}_j := \{\phi \in C^2(\mathbb{R}^{n+1}; \mathbb{R}^+) \,:\, \phi(x) \leqslant \Omega(x),\, |\nabla \phi(x)| \leqslant j\,\phi(x), \\ \|\nabla^2 \phi(x)\| \leqslant j\,\phi(x) \text{ for all } x \in \mathbb{R}^{n+1}\},$$

$$(4.5) \quad \mathcal{B}_j := \{g \in C^2(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}) \,:\, |g(x)| \leqslant j\Omega(x),\, \|\nabla g(x)\| \leqslant j\Omega(x), \\ \|\nabla^2 g(x)\| \leqslant j\Omega(x) \text{ for all } x \in \mathbb{R}^{n+1} \text{ and } \|\Omega^{-1} g\|_{L^2(\mathbb{R}^{n+1})} \leqslant j\}.$$

Note that $\Omega \in \mathcal{A}_j$ if $j \geqslant \max\{1, c_1\}$. Elements of $\mathcal{A}_j$ are strictly positive on $\mathbb{R}^{n+1}$ unless identically equal to 0. For $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ with $\|V\|(\Omega) < \infty$, we have $\|V\|(\phi) < \infty$ for $\phi \in \mathcal{A}_j$ since $\phi \leqslant \Omega$ from (4.4). For $g \in \mathcal{B}_j$, we naturally define $\delta V(g)$ as

$$\delta V(g) := \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} S \cdot \nabla g(x)\, dV(x, S)$$

which is finite and well-defined due to $\|\nabla g\| \leqslant j\Omega$ of (4.5).

Using (4.4), (3.2) and (4.5), the following can be seen easily.

Lemma 4.6. — *For all $x, y \in \mathbb{R}^{n+1}$, $j \in \mathbb{N}$ and $\phi \in \mathcal{A}_j$, we have*

$$(4.6) \qquad \phi(x) \leqslant \phi(y) \exp(j|x-y|),$$

$$(4.7) \qquad |\phi(x) - \phi(y)| \leqslant j|x-y|\phi(x) \exp(j|x-y|),$$

$$(4.8) \qquad |\phi(x) - \phi(y) - \nabla\phi(y) \cdot (x-y)| \leqslant j|x-y|^2 \phi(y) \exp(j|x-y|).$$

LEMMA 4.7. — *Let c_1 be as in (3.1). Then for all $x, y \in \mathbb{R}^{n+1}$, $j \in \mathbb{N}$ and $g \in \mathcal{B}_j$, we have*

$$|g(x) - g(y)| \leqslant j|x-y|\Omega(x)\exp(c_1|x-y|). \tag{4.9}$$

As these inequalities indicate, within a small distance of order $1/j$, minimum and maximum values of ϕ are compatible up to some fixed constant and this fact is used quite heavily in the following.

4.5. Area reducing Lipschitz deformation

DEFINITION 4.8. — *For $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}_\Omega^N$ and $j \in \mathbb{N}$, define $\mathbf{E}(\mathcal{E}, j)$ to be the set of all $\mathcal{E}$-admissible functions $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that*

(a) $|f(x) - x| \leqslant 1/j^2$ *for all* $x \in \mathbb{R}^{n+1}$,
(b) $\mathcal{L}^{n+1}(\tilde{E}_i \triangle E_i) \leqslant 1/j$ *for all* $i = 1, \ldots, N$ *and where* $\{\tilde{E}_i\}_{i=1}^N = f_\star \mathcal{E}$,
(c) $\|\partial f_\star \mathcal{E}\|(\phi) \leqslant \|\partial \mathcal{E}\|(\phi)$ *for all* $\phi \in \mathcal{A}_j$.

$\mathbf{E}(\mathcal{E}, j)$ includes the identity map $f(x) = x$, thus it is not empty. We are interested in this class with large j, so that (a) and (b) restrict f to be a very small deformation. Since $\Omega \in \mathcal{A}_j$ for all $j \geqslant \max\{1, c_1\}$, if $f \in \mathbf{E}(\mathcal{E}, j)$ with $j \geqslant \max\{1, c_1\}$, then we have

$$\|\partial f_\star \mathcal{E}\|(\Omega) \leqslant \|\partial \mathcal{E}\|(\Omega). \tag{4.10}$$

DEFINITION 4.9. — *For $\mathcal{E} \in \mathcal{OP}_\Omega^N$ and $j \in \mathbb{N}$, we define*

$$\Delta_j \|\partial \mathcal{E}\|(\Omega) := \inf_{f \in \mathbf{E}(\mathcal{E}, j)} (\|\partial f_\star \mathcal{E}\|(\Omega) - \|\partial \mathcal{E}\|(\Omega)). \tag{4.11}$$

In addition, for localized deformations, we define for a compact set $C \subset \mathbb{R}^{n+1}$

$$\mathbf{E}(\mathcal{E}, C, j) := \{f \in \mathbf{E}(\mathcal{E}, j) : \{x : f(x) \neq x\} \cup \{f(x) : f(x) \neq x\} \subset C\} \tag{4.12}$$

and

$$\Delta_j \|\partial \mathcal{E}\|(C) := \inf_{f \in \mathbf{E}(\mathcal{E}, C, j)} (\|\partial f_\star \mathcal{E}\|(C) - \|\partial \mathcal{E}\|(C)). \tag{4.13}$$

Since the identity map is in $\mathbf{E}(\mathcal{E}, j)$ and $\mathbf{E}(\mathcal{E}, C, j)$, $\Delta_j \|\partial \mathcal{E}\|(\Omega)$ and $\Delta_j \|\partial \mathcal{E}\|(C)$ are always non-positive. They measure the extent to which $\|\partial \mathcal{E}\|$ can be reduced under the Lipschitz deformation in the $\mathcal{E}$-admissible class. For $\mathcal{E} \in \mathcal{OP}_\Omega^N$ and $j \in \mathbb{N}$, we state their basic properties.

Lemma 4.10. — *For compact sets $C \subset \tilde{C}$, we have*

$$\Delta_j \|\partial\mathcal{E}\|(\tilde{C}) \leqslant \Delta_j \|\partial\mathcal{E}\|(C) \tag{4.14}$$

and

$$\Delta_j \|\partial\mathcal{E}\|(\Omega) \leqslant (\max_C \Omega)\{\Delta_j \|\partial\mathcal{E}\|(C) + (1-\exp(-c_1 \mathrm{diam}\, C))\|\partial\mathcal{E}\|(C)\}. \tag{4.15}$$

Proof. — By (4.12), $\mathbf{E}(\mathcal{E}, C, j) \subset \mathbf{E}(\mathcal{E}, \tilde{C}, j)$. For any $f \in \mathbf{E}(\mathcal{E}, C, j)$, $\|\partial f_\star \mathcal{E}\|(\tilde{C}) - \|\partial\mathcal{E}\|(\tilde{C}) = \|\partial f_\star \mathcal{E}\|(C) - \|\partial\mathcal{E}\|(C)$ since $f\lfloor_{\tilde{C}\setminus C}$ is identity and $f(C) \subset C$. Then (4.14) follows from (4.13). For (4.15), take arbitrary $f \in \mathbf{E}(\mathcal{E}, C, j)$ and since $f \in \mathbf{E}(\mathcal{E}, j)$, (4.11) and (4.12) give

$$\begin{aligned} &\Delta_j \|\partial\mathcal{E}\|(\Omega) \\ &\leqslant \|\partial f_\star \mathcal{E}\|(\Omega) - \|\partial\mathcal{E}\|(\Omega) = \|\partial f_\star \mathcal{E}\|\lfloor_C(\Omega) - \|\partial\mathcal{E}\|\lfloor_C(\Omega) \\ &\leqslant (\max_C \Omega)\|\partial f_\star \mathcal{E}\|(C) - (\min_C \Omega)\|\partial\mathcal{E}\|(C) \\ &\leqslant (\max_C \Omega)\{\|\partial f_\star \mathcal{E}\|(C) - \|\partial\mathcal{E}\|(C) + (1-\exp(-c_1 \mathrm{diam}\, C))\|\partial\mathcal{E}\|(C)\} \end{aligned} \tag{4.16}$$

where we used $(\min_C \Omega)/(\max_C \Omega) \geqslant \exp(-c_1 \mathrm{diam}\, C)$ which follows from (3.2). By taking inf over $\mathbf{E}(\mathcal{E}, C, j)$, we obtain (4.15). □

Lemma 4.11. — *Suppose that $\{C_i\}_{i=1}^\infty$ is a sequence of compact sets which are mutually disjoint and suppose that C is a compact set with $\cup_{i=1}^\infty C_i \subset C$ and $\mathcal{L}^{n+1}(C) < 1/j$. Then*

$$\Delta_j \|\partial\mathcal{E}\|(C) \leqslant \sum_{i=1}^\infty \Delta_j \|\partial\mathcal{E}\|(C_i). \tag{4.17}$$

Proof. — By Lemma 4.10, if $\Delta_j \|\partial\mathcal{E}\|(C) > -\infty$, then $\Delta_j \|\partial\mathcal{E}\|(C_i) > -\infty$ for all i. Let $m \in \mathbb{N}$ and $\varepsilon \in (0,1)$ be arbitrary. For all $i \leqslant m$, choose $f_i \in \mathbf{E}(\mathcal{E}, C_i, j)$ such that $\Delta_j \|\partial\mathcal{E}\|(C_i) + \varepsilon \geqslant \|\partial (f_i)_\star \mathcal{E}\|(C_i) - \|\partial\mathcal{E}\|(C_i)$. We define a map $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by setting $f\lfloor_{C_i}(x) = (f_i)\lfloor_{C_i}(x)$ and $f\lfloor_{\mathbb{R}^{n+1}\setminus \cup_{i=1}^m C_i}(x) = x$. Since $\{C_i\}_{i=1}^m$ are disjoint, f is well-defined, Lipschitz and $\mathcal{E}$-admissible. Using $\mathcal{L}^{n+1}(C) < 1/j$, one checks that $f \in \mathbf{E}(\mathcal{E}, C, j)$. Thus we have

$$\begin{aligned} \Delta_j \|\partial\mathcal{E}\|(C) &\leqslant \|\partial f_\star \mathcal{E}\|(C) - \|\partial\mathcal{E}\|(C) \\ &= \sum_{i=1}^m \|\partial (f_i)_\star \mathcal{E}\|(C_i) - \|\partial\mathcal{E}\|(C_i) \\ &\leqslant m\varepsilon + \sum_{i=1}^m \Delta_j \|\partial\mathcal{E}\|(C_i). \end{aligned} \tag{4.18}$$

By letting $\varepsilon \to 0$ first and letting $m \to \infty$, we obtain (4.17). □

Lemma 4.12 ([8, §4.10]). — *If $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}_\Omega^N$, $j \in \mathbb{N}$, C is a compact set of $\mathbb{R}^{n+1}$, $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a $\mathcal{E}$-admissible function such that*

(a) $\{x : f(x) \neq x\} \cup \{f(x) : f(x) \neq x\} \subset C$,
(b) $|f(x) - x| \leqslant 1/j^2$ *for all* $x \in \mathbb{R}^{n+1}$,
(c) $\mathcal{L}^{n+1}(\tilde{E}_i \triangle E_i) \leqslant 1/j$ *for all* $i = 1, \ldots, N$ *and where* $\{\tilde{E}_i\}_{i=1}^N = f_\star\mathcal{E}$,
(d) $\|\partial f_\star \mathcal{E}\|(C) \leqslant \exp(-j \operatorname{diam} C)\, \|\partial \mathcal{E}\|(C)$.

Then we have $f \in \mathbf{E}(\mathcal{E}, C, j)$.

Proof. — We only need to check Definition 4.8(c). By condition (a), $\|\partial f_\star \mathcal{E}\| \lfloor_{\mathbb{R}^{n+1}\setminus C} = \|\partial \mathcal{E}\| \lfloor_{\mathbb{R}^{n+1}\setminus C}$. Suppose $\phi \in \mathcal{A}_j$. Then by (4.6)

$$\begin{aligned}\|\partial f_\star \mathcal{E}\|(\phi) - \|\partial \mathcal{E}\|(\phi) &= \|\partial f_\star \mathcal{E}\| \lfloor_C(\phi) - \|\partial \mathcal{E}\| \lfloor_C(\phi)\\ &\leqslant \max_C \phi \, \|\partial f_\star \mathcal{E}\|(C) - \min_C \phi \, \|\partial \mathcal{E}\|(C)\\ &\leqslant \min_C \phi \ (\exp(j \operatorname{diam} C) \|\partial f_\star \mathcal{E}\|(C) - \|\partial \mathcal{E}\|(C)) \leqslant 0\end{aligned}$$

where (d) is used in the last line. □

4.6. Smoothing function Φ_ε

Let $\psi \in C^\infty(\mathbb{R}^{n+1})$ be a radially symmetric function such that

$$\begin{aligned}&\psi(x) = 1 \text{ for } |x| \leqslant 1/2, \quad \psi(x) = 0 \text{ for } |x| \geqslant 1,\\ &0 \leqslant \psi(x) \leqslant 1, \ |\nabla \psi(x)| \leqslant 3, \quad \|\nabla^2 \psi(x)\| \leqslant 9 \text{ for all } x \in \mathbb{R}^{n+1}.\end{aligned} \tag{4.19}$$

Define for each $\varepsilon \in (0,1)$

$$\hat{\Phi}_\varepsilon(x) := \frac{1}{(2\pi\varepsilon^2)^{\frac{n+1}{2}}} \exp\left(-\frac{|x|^2}{2\varepsilon^2}\right), \quad \Phi_\varepsilon(x) := c(\varepsilon)\psi(x)\hat{\Phi}_\varepsilon(x), \tag{4.20}$$

where the constant $c(\varepsilon)$ is chosen so that

$$\int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(x)\, dx = 1. \tag{4.21}$$

Since $\int_{\mathbb{R}^{n+1}} \hat{\Phi}_\varepsilon(x)\, dx = 1$ for any $\varepsilon > 0$ and $\hat{\Phi}_\varepsilon$ converges to the delta function as $\varepsilon \to 0+$, there exists a constant $c(n)$ depending only on n such that

$$1 < c(\varepsilon) \leqslant c(n) \text{ for } \varepsilon \in (0,1) \text{ and } \lim_{\varepsilon \to 0+} c(\varepsilon) = 1. \tag{4.22}$$

From the definitions of ψ and Φ_ε, we have the following estimates.

Lemma 4.13. — *There exists a constant c depending only on n such that, for $\varepsilon \in (0,1)$, we have*

$$|\nabla\Phi_\varepsilon(x)| \leqslant \frac{|x|}{\varepsilon^2}\Phi_\varepsilon(x) + c\chi_{B_1\setminus B_{1/2}}(x)\exp(-\varepsilon^{-1}), \tag{4.23}$$

$$\|\nabla^2\Phi_\varepsilon(x)\| \leqslant \frac{|x|^2}{\varepsilon^4}\Phi_\varepsilon(x) + \frac{c}{\varepsilon^2}\Phi_\varepsilon(x) + c\chi_{B_1\setminus B_{1/2}}(x)\exp(-\varepsilon^{-1}). \tag{4.24}$$

Lemma 4.14. — *With $c(\varepsilon)$ as in (4.20), we have*

$$x\,\Phi_\varepsilon(x) + \varepsilon^2\nabla\Phi_\varepsilon(x) = \varepsilon^2 c(\varepsilon)\nabla\psi(x)\hat{\Phi}_\varepsilon(x). \tag{4.25}$$

The exponential smallness of the right-hand side of (4.25) will be of critical importance in Proposition 5.4.

4.7. Smoothing of varifold [8, §4.3]

In this subsection, we consider a smoothing of varifold and derive various estimates. For general distribution T, there is a notion of smoothing of T using a duality, i.e., $\Phi_\varepsilon * T(\phi) = T(\Phi_\varepsilon * \phi)$ for any $\phi \in C_c^\infty(\mathbb{R}^{n+1})$. Here, given a varifold $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$, we smooth out with respect to only the space variables and not the Grassmannian part.

Definition 4.15. — *For $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$, we define $\Phi_\varepsilon * V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ through*

$$\begin{aligned}(\Phi_\varepsilon * V)(\phi) &:= V(\Phi_\varepsilon * \phi)\\ &= \int_{\mathbf{G}_n(\mathbb{R}^{n+1})}\int_{\mathbb{R}^{n+1}} \phi(x-y,S)\Phi_\varepsilon(y)\,dy\,dV(x,S)\end{aligned} \tag{4.26}$$

for $\phi \in C_c(\mathbf{G}_n(\mathbb{R}^{n+1}))$.

If $\|V\|(\Omega) < \infty$, we have $\|\Phi_\varepsilon * V\|(\Omega) < \infty$ since

$$\begin{aligned}\|\Phi_\varepsilon * V\|(\Omega) &\leqslant \int_{\mathbf{G}_n(\mathbb{R}^{n+1})}\int_{\mathbb{R}^{n+1}} e^{c_1}\Omega(x)\Phi_\varepsilon(y)\,dy\,dV(x,S)\\ &= e^{c_1}\|V\|(\Omega)\end{aligned} \tag{4.27}$$

by (3.2) and (4.21). Thus we have $\|\Phi_\varepsilon * V\|(\phi) < \infty$ for $\phi \in \mathcal{A}_j$ as well. For a general Radon measure μ on $\mathbb{R}^{n+1}$, we similarly define a Radon measure $\Phi_\varepsilon * \mu$. $\Phi_\varepsilon * \mu$ may be identified with a smooth function on $\mathbb{R}^{n+1}$ via the

L^2 inner product, because, for $\phi \in C_c(\mathbb{R}^{n+1})$,

$$(4.28)\qquad \begin{aligned}(\Phi_\varepsilon * \mu)(\phi) &= \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \phi(y)\Phi_\varepsilon(x-y)\,dy\,d\mu(x) \\ &= \int_{\mathbb{R}^{n+1}} \phi(y) \int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(x-y)\,d\mu(x)\,dy \\ &= <\Phi_\varepsilon * \mu, \phi>_{L^2(\mathbb{R}^{n+1})},\end{aligned}$$

and we may identify $\Phi_\varepsilon * \mu \in C^\infty(\mathbb{R}^{n+1})$ with

$$(4.29)\qquad (\Phi_\varepsilon * \mu)(x) = \int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(y-x)\,d\mu(y).$$

In a similar way, for general $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$, we may define $\Phi_\varepsilon * \delta V$ as a C^∞ vector field as follows. Note that V may not have a bounded first variation in general. For $g \in C_c^1(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$, $\Phi_\varepsilon * \delta V$ should be defined to satisfy

$$(4.30)\qquad \begin{aligned}&\int_{\mathbb{R}^{n+1}} (\Phi_\varepsilon * \delta V)(x)\cdot g(x)\,dx \\ &\quad = \delta V(\Phi_\varepsilon * g) \\ &\quad = \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} S\cdot((\nabla\Phi_\varepsilon * g)(x))\,dV(x,S) \\ &\quad = \int_{\mathbb{R}^{n+1}} g(y)\cdot \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} S(\nabla\Phi_\varepsilon(x-y))\,dV(x,S)dy.\end{aligned}$$

The equality (4.30) motivates the definition of $\Phi_\varepsilon * \delta V$ as a C^∞ vector field

$$(4.31)\qquad (\Phi_\varepsilon * \delta V)(x) := \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} S(\nabla\Phi_\varepsilon(y-x))\,dV(y,S).$$

Lemma 4.16. — *For $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$, we have*

$$(4.32)\qquad \Phi_\varepsilon * \|V\| = \|\Phi_\varepsilon * V\|,$$

$$(4.33)\qquad \Phi_\varepsilon * \delta V = \delta(\Phi_\varepsilon * V).$$

Proof. — For $\phi \in C_c(\mathbb{R}^{n+1})$, we have

$$\begin{aligned}(4.34)\qquad \int_{\mathbb{R}^{n+1}} \phi\, d\|\Phi_\varepsilon * V\| &= \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} \phi(x)\, d(\Phi_\varepsilon * V)(x,S)\\ &= \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} (\Phi_\varepsilon * \phi)(x)\, dV(x,S) \ \text{ (by (4.26))}\\ &= \int_{\mathbb{R}^{n+1}} \phi(y) \int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(x-y)\, d\|V\|(x)\, dy\\ &= \int_{\mathbb{R}^{n+1}} \phi\, d(\Phi_\varepsilon * \|V\|) \ \text{ (by (4.28))}.\end{aligned}$$

Thus we proved (4.32). For $g \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$, by (4.30),

$$(4.35)\qquad (\Phi_\varepsilon * \delta V)(g) = \delta V(\Phi_\varepsilon * g) = \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} S \cdot (\Phi_\varepsilon * \nabla g)(x)\, dV(x,S)$$

while by (4.26),

$$(4.36)\qquad \delta(\Phi_\varepsilon * V)(g) = \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} \Phi_\varepsilon * (S \cdot \nabla g)(x)\, dV(x,S).$$

Since $\Phi_\varepsilon *$ commutes with $S\,\cdot$, (4.35) and (4.36) prove (4.33). □

The following is used when we need to deal with error terms in the next section.

LEMMA 4.17. — *For $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ with $\|V\|(\Omega) < \infty$ and for all $x \in \mathbb{R}^{n+1}$ and $r > 0$, we have*

$$(4.37)\qquad \Omega(x)\|V\|(B_r(x)) \leqslant e^{c_1 r}\|V\|(\Omega),$$

$$(4.38)\qquad \int_{\mathbb{R}^{n+1}} \Omega(x)\|V\|(B_r(x))\, dx \leqslant \omega_{n+1} e^{c_1 r} r^{n+1}\|V\|(\Omega).$$

Proof. — By (3.2), for $y \in B_r(x)$, we have $\Omega(x) \leqslant \Omega(y)e^{c_1 r}$, thus

$$\Omega(x)\|V\|(B_r(x)) \leqslant \int_{B_r(x)} \Omega(y) e^{c_1 r}\, d\|V\|(y) \leqslant e^{c_1 r}\|V\|(\Omega),$$

proving (4.37). Similarly, since $\chi_{B_r(x)}(y) = \chi_{B_r(y)}(x)$,

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} \Omega(x)\|V\|(B_r(x))\, dx &= \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \Omega(x)\chi_{B_r(x)}(y)\, dx\, d\|V\|(y)\\ &= \int_{\mathbb{R}^{n+1}} \int_{B_r(y)} \Omega(x)\, dx\, d\|V\|(y)\\ &\leqslant \omega_{n+1} e^{c_1 r} r^{n+1} \int_{\mathbb{R}^{n+1}} \Omega(y)\, d\|V\|(y)\\ &= \omega_{n+1} e^{c_1 r} r^{n+1}\|V\|(\Omega),\end{aligned}$$

proving (4.38). □

5. Smoothed mean curvature vector $h_\varepsilon(\cdot, V)$

Given $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$, if the first variation δV is bounded and absolutely continuous with respect to $\|V\|$, the Radon–Nikodym derivative $h(\cdot, V) = -\delta V/\|V\|$ defines the generalized mean curvature vector of V as in (2.2). Here, even for V with unbounded first variation, we want to have a smooth analogue of $h(\cdot, V)$ to construct an approximate mean curvature flow. Thus we define a *smoothed mean curvature vector* $h_\varepsilon(\cdot, V)$ for $\varepsilon \in (0, 1)$ by

$$h_\varepsilon(\cdot, V) := -\Phi_\varepsilon * \left(\frac{\Phi_\varepsilon * \delta V}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\right). \tag{5.1}$$

We may often write $h_\varepsilon(\cdot, V)$ as h_ε for simplicity. Note that this is a well-defined smooth vector field; since $\Omega^{-1} \geqslant 1$ by (3.1), the denominator is strictly positive. Formally, as $\varepsilon \to 0+$, h_ε will be more and more concentrated around $\operatorname{spt}\|V\|$ and we expect that $h_\varepsilon(\cdot, V)$ converges in a suitable sense to $h(\cdot, V)$, as long as there are some suitable bounds. The term "smoothed mean curvature vector" is used in [8], but we should warn the reader that it may happen that the generalized mean curvature $h(\cdot, V)$ may not exist in general while $h_\varepsilon(\cdot, V)$ is always well-defined. We also point out that there is a difference from [8] that we have the extra $\varepsilon\Omega^{-1}$ term to avoid division by 0 (see [8, p. 39]). In [8], $\Phi_\varepsilon * \|V\|$ (with a different and more complicated Φ_ε, see [8, p. 37]) is prepared so that it is everywhere positive on $\mathbb{R}^{n+1}$ unless $\|V\|(\Omega) = 0$. Though it is a simple modification, various computations are clearly tractable compared to [8]. After some reading, one must admit that the corresponding computations in [8] are discouragingly difficult to follow in the original form. In the following, we also use the notation

$$\tilde{h}_\varepsilon := -\frac{\Phi_\varepsilon * \delta V}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}} \tag{5.2}$$

for simplicity and note that $h_\varepsilon = \Phi_\varepsilon * \tilde{h}_\varepsilon$.

5.1. Rough pointwise estimates on $h_\varepsilon(\cdot, V)$

LEMMA 5.1. — *There exists a constant $\epsilon_1 \in (0, 1)$ depending only on n, c_1 and M with the following property. Suppose $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ with $\|V\|(\Omega) \leqslant M$ and $\varepsilon \in (0, \epsilon_1)$. Then, for all $x \in \mathbb{R}^{n+1}$, we have*

$$|\tilde{h}_\varepsilon(x, V)| \leqslant 2\varepsilon^{-2}, \qquad |h_\varepsilon(x, V)| \leqslant 2\varepsilon^{-2}, \tag{5.3}$$

$$\|\nabla h_\varepsilon(x, V)\| \leqslant 2\varepsilon^{-4}, \tag{5.4}$$

$$\|\nabla^2 h_\varepsilon(x, V)\| \leqslant 2\varepsilon^{-6}. \tag{5.5}$$

Proof. — First by (4.31) and (4.23), we have

$$(5.6)\quad |(\Phi_\varepsilon * \delta V)(x)| \leqslant \int_{B_1(x)} \frac{|y-x|}{\varepsilon^2}\Phi_\varepsilon(y-x) + c(n)\exp(-\varepsilon^{-1})\, d\|V\|(y)$$
$$\leqslant \varepsilon^{-2}(\Phi_\varepsilon * \|V\|)(x) + c(n)\exp(-\varepsilon^{-1})\|V\|(B_1(x)),$$

where $c(n)$ is as in Lemma 4.13. Combining (5.6) and (4.37), we obtain

$$(5.7)\qquad \frac{|\Phi_\varepsilon * \delta V|}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}} \leqslant \varepsilon^{-2} + c(n)M\varepsilon^{-1}\exp(c_1 - \varepsilon^{-1}).$$

Choose ϵ_1 so that $c(n)M\varepsilon\exp(c_1-\varepsilon^{-1}) \leqslant 1$ if $\varepsilon \in (0,\epsilon_1)$. Now recalling $\Phi_\varepsilon * 1 = 1$ and (5.1), we obtain (5.3) from (5.7). For (5.4), we note that $|\nabla\Phi_\varepsilon| * 1 \leqslant \varepsilon^{-2} + c(n)\exp(-\varepsilon^{-1})\omega_n$ by (4.23). Thus using (5.7) and choosing an appropriate ϵ_1, we obtain (5.4). Using (4.24), we similarly obtain (5.5). □

The following quantity plays the role of Ω-weighted "approximate L^2-norm" of smoothed mean curvature vector. The reason is that, roughly speaking, we expect that

$$\int |h_\varepsilon(\cdot,V)|^2\, d\|V\| \approx \int \frac{|\Phi_\varepsilon * \delta V|^2}{(\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1})^2}\, d(\Phi_\varepsilon * \|V\|)$$
$$\approx \int \frac{|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dx.$$

Lemma 5.2. — *For $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ with $\|V\|(\Omega) < \infty$ and $\varepsilon \in (0,\epsilon_1)$,*

$$\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta V|^2\Omega}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dx < \infty.$$

Proof. — The claim follows from (5.7), (5.6), (4.38), (4.27) and (4.32). □

5.2. L^2 approximations

This subsection establishes various error estimates of approximations.

Proposition 5.3. — *There exists a constant $\epsilon_2 \in (0,1)$ depending only on n, c_1 and M such that, for any $g \in \mathcal{B}_j$, $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ with $\|V\|(\Omega) \leqslant M$, $j \in \mathbb{N}$, $\varepsilon \in (0,\epsilon_2)$ with*

$$(5.8)\qquad j \leqslant \frac{1}{2}\varepsilon^{-\frac{1}{6}},$$

we have

$$\left|\int_{\mathbb{R}^{n+1}} h_\varepsilon \cdot g\, d\|V\| + \int_{\mathbb{R}^{n+1}} (\Phi_\varepsilon * \delta V)\cdot g\, dy\right| \leqslant \varepsilon^{\frac14}\left(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta V|^2\Omega}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy\right)^{\frac12}. \tag{5.9}$$

Note that one can draw an analogy between (5.9) and (2.2).

Proof. — By (5.1) and (5.2), we have

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} h_\varepsilon \cdot g\, d\|V\| &= \int_{\mathbb{R}^{n+1}} (\Phi_\varepsilon * \tilde h_\varepsilon)\cdot g\, d\|V\| \\ &= \int_{\mathbb{R}^{n+1}} \tilde h_\varepsilon(y)\cdot \int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(\cdot - y) g(\cdot)\, d\|V\| dy.\end{aligned} \tag{5.10}$$

We may also rewrite using the notation (5.2)

$$\int_{\mathbb{R}^{n+1}} (\Phi_\varepsilon * \delta V)\cdot g\, dy = -\int_{\mathbb{R}^{n+1}} \tilde h_\varepsilon (\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1})\cdot g\, dy. \tag{5.11}$$

Summing (5.10) and (5.11), we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^{n+1}} h_\varepsilon \cdot g\, d\|V\| + \int_{\mathbb{R}^{n+1}} (\Phi_\varepsilon * \delta V)\cdot g\, dy\right| \\ &\quad\leqslant \int_{\mathbb{R}^{n+1}} |g(y)||\tilde h_\varepsilon(y,V)|\varepsilon\Omega^{-1}(y)\, dy + \int_{\mathbb{R}^{n+1}} |\tilde h_\varepsilon(y,V)| \\ &\qquad\times\left|\int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(x-y) g(x)\, d\|V\|(x) - (\Phi_\varepsilon * \|V\|)(y) g(y)\right| dy \\ &\quad =: I_1 + I_2.\end{aligned} \tag{5.12}$$

By Hölder's inequality and (4.5),

$$I_1 \leqslant \varepsilon\left(\int_{\mathbb{R}^{n+1}} |g|^2\Omega^{-2}\, dy\right)^{\frac12}\left(\int_{\mathbb{R}^{n+1}} |\tilde h_\varepsilon|^2\, dy\right)^{\frac12} \leqslant j\varepsilon\left(\int_{\mathbb{R}^{n+1}} |\tilde h_\varepsilon|^2\, dy\right)^{\frac12}. \tag{5.13}$$

Recalling (5.2), (5.13) in particular gives

$$I_1 \leqslant j\varepsilon^{\frac12}\left(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta V|^2\Omega}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy\right)^{\frac12}. \tag{5.14}$$

For I_2, using (4.9) for $g \in \mathcal{B}_j$,

$$\begin{aligned}(5.15)\quad \left|\int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(\cdot - y) g(\cdot)\, d\|V\| - (\Phi_\varepsilon * \|V\|) g\right| &= \left|\int_{\mathbb{R}^{n+1}} (g(x)-g(y))\Phi_\varepsilon(x-y)\, d\|V\|(x)\right| \\ &\leqslant j e^{c_1}\Omega(y)\int_{B_1(y)} |x-y|\Phi_\varepsilon(x-y)\, d\|V\|(x).\end{aligned}$$

Using the property of Φ_ε being exponentially small away from the origin, we have

$$(5.16)\quad \sup_{x\in B_1(y)\setminus B_{\sqrt{\varepsilon}}(y)} |x-y|\Phi_\varepsilon(x-y) \leqslant c(n)\varepsilon^{-n-1}\exp(-(2\varepsilon)^{-1}) =: c_\varepsilon.$$

Thus (5.15) and (5.16) give

$$\begin{aligned}(5.17)\quad I_2 \leqslant j e^{c_1}\varepsilon^{\frac{1}{2}} &\int_{\mathbb{R}^{n+1}} \Omega\, |\tilde{h}_\varepsilon| (\Phi_\varepsilon * \|V\|)\, dy \\ &+ j e^{c_1} c_\varepsilon \int_{\mathbb{R}^{n+1}} \Omega\, |\tilde{h}_\varepsilon|\, \|V\|(B_1(y))\, dy =: I_{2,a} + I_{2,b}.\end{aligned}$$

For $I_{2,a}$, use Hölder's inequality to obtain

$$(5.18)\quad I_{2,a} \leqslant j e^{c_1}\varepsilon^{\frac{1}{2}} \left(\int_{\mathbb{R}^{n+1}} |\tilde{h}_\varepsilon|^2 (\Phi_\varepsilon * \|V\|)\Omega\, dy\right)^{\frac{1}{2}} ((\Phi_\varepsilon * \|V\|)(\Omega))^{\frac{1}{2}}.$$

Substitution of (4.27) (with (4.32)) into (5.18) gives

$$(5.19)\quad I_{2,a} \leqslant j e^{2c_1}\varepsilon^{\frac{1}{2}} \left(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta V|^2 \Omega}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy\right)^{\frac{1}{2}} M^{\frac{1}{2}}.$$

For $I_{2,b}$, by Hölder's inequality,

$$(5.20)\quad I_{2,b} \leqslant j e^{c_1} c_\varepsilon \left(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta V|^2 \Omega}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{n+1}} \frac{\|V\|(B_1(y))^2 \Omega}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy\right)^{\frac{1}{2}}.$$

Using (4.37), we have

$$(5.21)\quad \int_{\mathbb{R}^{n+1}} \frac{\|V\|(B_1(y))^2 \Omega}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy \leqslant \varepsilon^{-1} e^{c_1} M \int_{\mathbb{R}^{n+1}} \|V\|(B_1(y))\Omega\, dy.$$

Then (5.20), (5.21) and (4.38) prove

$$(5.22)\quad I_{2,b} \leqslant j e^{2c_1} c_\varepsilon \varepsilon^{-\frac{1}{2}} \omega_{n+1}^{\frac{1}{2}} M \left(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta V|^2 \Omega}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy\right)^{\frac{1}{2}}.$$

Combining (5.12), (5.14), (5.17), (5.19), (5.22), (5.8) and choosing ϵ_2 appropriately depending only on n, c_1 and M, we obtain (5.9). □

Proposition 5.4. — *There exists a constant $\epsilon_3 \in (0,1)$ depending only on n, c_1 and M with the following property. For $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ with $\|V\|(\Omega) \leqslant M$, $j \in \mathbb{N}$, $\phi \in \mathcal{A}_j$ and $\varepsilon \in (0, \epsilon_3)$ with (5.8), we have*

$$(5.23)\quad |\delta V(\phi h_\varepsilon) + \int_{\mathbb{R}^{n+1}} \frac{\phi|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dx| \leqslant \varepsilon^{\frac14}\left(\int_{\mathbb{R}^{n+1}} \frac{\phi|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dx + 1\right)$$

and

$$(5.24)\quad \int_{\mathbb{R}^{n+1}} |h_\varepsilon|^2 \phi \, d\|V\| \leqslant \int_{\mathbb{R}^{n+1}} \frac{\phi|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}} (1 + \varepsilon^{\frac14})\, dx + \varepsilon^{\frac14}.$$

Note that (5.23) measures a deviation from $\delta V(\phi h) = -\int \phi |h|^2 \, d\|V\|$, which is (2.2) with $g = \phi h$ if all quantities are well-defined. We use (5.24) when we prove the lower semicontinuity of L^2-norm of mean curvature vector.

Proof. — From the definition of the first variation, we have

$$(5.25)\quad \begin{aligned} \delta V(\phi h_\varepsilon) &= \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} \nabla(\phi h_\varepsilon) \cdot S \, dV(\cdot, S) \\ &= \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} (\phi \nabla h_\varepsilon + \nabla\phi \otimes h_\varepsilon) \cdot S\, dV(\cdot, S) \\ &= \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} \int_{\mathbb{R}^{n+1}} (\phi(x)\nabla\Phi_\varepsilon(x-y) \\ &\quad + \nabla\phi(x)\Phi_\varepsilon(x-y)) \otimes \tilde{h}_\varepsilon(y) \cdot S \, dy dV(x, S) \end{aligned}$$

and by (4.31),

$$(5.26)\quad \begin{aligned} &\int_{\mathbb{R}^{n+1}} \frac{\phi|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dx \\ &\quad = -\int_{\mathbb{R}^{n+1}} \phi\tilde{h}_\varepsilon \cdot (\Phi_\varepsilon * \delta V)\, dy \\ &\quad = -\int_{\mathbb{R}^{n+1}} \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} \phi(y) S(\nabla\Phi_\varepsilon(x-y)) \cdot \tilde{h}_\varepsilon(y)\, dV(x,S) dy. \end{aligned}$$

By summing (5.25) and (5.26), we obtain

$$(5.27)\quad \begin{aligned} &\delta V(\phi h_\varepsilon) + \int_{\mathbb{R}^{n+1}} \frac{\phi|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dx \\ &\quad = \int_{\mathbb{R}^{n+1}} \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} \big((\phi(x) - \phi(y)) S(\nabla\Phi_\varepsilon(x-y)) \\ &\qquad + \Phi_\varepsilon(x-y) S(\nabla\phi(x))\big)\, dV(x,S) \cdot \tilde{h}_\varepsilon(y)\, dy. \end{aligned}$$

To continue, we carry out a second order approximation of ϕ and interpolate the right-hand side of (5.27) by defining (all integrations are over $\mathbb{R}^{n+1} \times \mathbf{G}_n(\mathbb{R}^{n+1})$)

$$\begin{aligned}
I_1 &:= \iint (\phi(x) - \phi(y) - \nabla\phi(y)\cdot(x-y)) \\
&\qquad S(\nabla\Phi_\varepsilon(x-y))\, dV(x,S)\cdot \tilde{h}_\varepsilon(y)\, dy, \\
I_2 &:= \iint \Phi_\varepsilon(x-y) S(\nabla\phi(x) - \nabla\phi(y))\, dV(x,S)\cdot \tilde{h}_\varepsilon(y)\, dy, \\
I_3 &:= \iint \nabla\phi(y)\cdot(x-y) S(\nabla\Phi_\varepsilon(x-y)) \\
&\qquad + \Phi_\varepsilon(x-y) S(\nabla\phi(y))\, dV(x,S)\cdot \tilde{h}_\varepsilon(y)\, dy
\end{aligned} \tag{5.28}$$

so that $I_1 + I_2 + I_3$ equals to (5.27). In addition, we define

$$I_4 := -\varepsilon^2 \iint S[\nabla_x(\nabla\phi(y)\cdot\nabla\Phi_\varepsilon(x-y))]\, dV(x,S)\cdot \tilde{h}_\varepsilon(y)\, dy, \tag{5.29}$$

where ∇_x indicates (for clarity) that the differentiation is with respect to x variables. In the following, we estimate I_1, I_2, $I_3 - I_4$ and I_4.

Estimate of I_1. — We use (4.8) to squeeze out a $|x-y|^2$ term to deal with ε^{-2} term coming from $\nabla\Phi_\varepsilon$. Then we separate the domain of integration to $B_{\varepsilon^{\frac{5}{6}}}(y)$ and the complement. On the latter, $\Phi_\varepsilon(\cdot - y)$ is exponentially small with respect to ε. With this in mind, we have by (4.8) and (4.23) that

$$\begin{aligned}
|I_1| &\leqslant j \int (|\tilde{h}_\varepsilon|\phi)(y) \int e^{j|\cdot-y|}|\cdot - y|^2 \\
&\qquad \left(\frac{|\cdot - y|}{\varepsilon^2}\Phi_\varepsilon(\cdot - y) + c(n)e^{-\varepsilon^{-1}}\chi_{B_1(y)}\right) d\|V\|dy \\
&\leqslant je^{j\varepsilon^{\frac{5}{6}}}\varepsilon^{\frac{1}{2}} \int (|\tilde{h}_\varepsilon|\phi)(y) \int \Phi_\varepsilon(\cdot - y)\, d\|V\|dy \\
&\qquad \left(\frac{|x-y|^3}{\varepsilon^2} \leqslant \varepsilon^{\frac{1}{2}} \text{ on } B_{\varepsilon^{\frac{5}{6}}}(y) \text{ is used}\right) \\
&\quad + je^j c(n)\varepsilon^{-n-3} e^{-\frac{\varepsilon^{-\frac{1}{3}}}{2}} \int_{\mathbb{R}^{n+1}} \|V\|(B_1(y))|\tilde{h}_\varepsilon(y)|\Omega(y)\, dy \\
&\quad + je^j c(n) e^{-\varepsilon^{-1}} \int_{\mathbb{R}^{n+1}} \|V\|(B_1(y))|\tilde{h}_\varepsilon(y)|\Omega(y)\, dy.
\end{aligned} \tag{5.30}$$

The integration of the first term of (5.30) may be estimated as

$$
\begin{aligned}
(5.31)\quad \int |\tilde{h}_\varepsilon|\phi \int \Phi_\varepsilon(\cdot - y)\, d\|V\| dy
&= \int_{\mathbb{R}^{n+1}} (\Phi_\varepsilon * \|V\|)|\tilde{h}_\varepsilon|\phi\, dy \\
&\leqslant ((\Phi_\varepsilon * \|V\|)(\Omega))^{\frac{1}{2}} \left(\int_{\mathbb{R}^{n+1}} (\Phi_\varepsilon * \|V\|)|\tilde{h}_\varepsilon|^2 \phi\, dy \right)^{\frac{1}{2}} \\
&\leqslant (e^{c_1} M)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{n+1}} (\Phi_\varepsilon * \|V\|)|\tilde{h}_\varepsilon|^2 \phi\, dy \right)^{\frac{1}{2}}
\end{aligned}
$$

where we used (4.27) and (4.32). Use (5.3) and (4.38) for the second and third terms of (5.30). Combined with (5.31), then, we have some c depending only on c_1, M and n such that

$$
\begin{aligned}
(5.32)\quad |I_1| &\leqslant j e^{j\varepsilon^{\frac{5}{6}}} \varepsilon^{\frac{1}{2}} (e^{c_1} M)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{n+1}} (\Phi_\varepsilon * \|V\|)|\tilde{h}_\varepsilon|^2 \phi\, dy \right)^{\frac{1}{2}} + jce^{j-\varepsilon^{-\frac{1}{6}}} \\
&\leqslant j\varepsilon^{\frac{1}{2}} \int_{\mathbb{R}^{n+1}} \frac{\phi|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy + jc\varepsilon^{\frac{1}{2}} + jce^{-\frac{1}{2}\varepsilon^{-\frac{1}{6}}},
\end{aligned}
$$

where we also used (5.8).

Estimate of I_2. — By the similar manner, we estimate I_2. Note that $\nabla\Phi_\varepsilon$ is not present while we have only $|\nabla\phi(x) - \nabla\phi(y)| \leqslant j|x-y|\phi(x)e^{j|x-y|}$ this time. We separate the domain of integration to $B_{\varepsilon^{\frac{1}{2}}}(y)$ and the complement, and estimate just like I_1 to obtain (5.32) for I_2 in place of I_1. We omit the detail since it is repetitive.

Estimate of $I_3 - I_4$. — The first point is that the integrand with respect to V of I_3 can be expressed as

$$
\begin{aligned}
(5.33)\quad \nabla\phi(y)\cdot(x-y)S(\nabla\Phi_\varepsilon(x-y)) + \Phi_\varepsilon(x-y)S(\nabla\phi(y))
&= S[\nabla\phi(y)\Phi_\varepsilon(x-y) + \nabla\phi(y)\cdot(x-y)\nabla\Phi_\varepsilon(x-y)] \\
&= S[\nabla_x((x-y)\cdot\nabla\phi(y)\Phi_\varepsilon(x-y))].
\end{aligned}
$$

The function $(x-y)\Phi_\varepsilon(x-y)$ may be replaced by $-\varepsilon^2\nabla\Phi_\varepsilon(x-y)$ with exponentially small error due to (4.25). So we first check that this replacement produces small error indeed. By (5.33),

$$
(5.34)\quad I_3 - I_4 = \iint S[\nabla_x(\nabla\phi(y)\cdot c(\varepsilon)\varepsilon^2\nabla\psi(x-y)\hat{\Phi}_\varepsilon(x-y))]\, dV(x,S)\cdot\tilde{h}_\varepsilon(y)\, dy.
$$

On the support of $\nabla\psi$, $\hat{\Phi}_\varepsilon$ is of the order of $e^{-\varepsilon^{-2}}$, thus estimating as in the second and third terms of (5.30), we obtain from (5.34) and (4.4) that

$$|I_3 - I_4| \leqslant jc(n, c_1, M)e^{-\varepsilon^{-1}}. \tag{5.35}$$

Estimate of I_4. — To be clear about the indices, the i-th component of the integrand of I_4 with respect to V is (the same indices imply summation over 1 to $n+1$)

$$S_{ij}\nabla_{x_j}(\nabla_{y_l}\phi(y)\nabla_{x_l}\Phi_\varepsilon(x-y)) = -\nabla_{y_l}\phi(y)\nabla_{y_l}(S_{ij}\nabla_{x_j}\Phi_\varepsilon(x-y)). \tag{5.36}$$

Recalling (4.31) and writing the i-th component of $\Phi_\varepsilon * \delta V$ as $(\Phi_\varepsilon * \delta V)_i$, (5.36) shows

$$\begin{aligned} I_4 &= \varepsilon^2 \int_{\mathbb{R}^{n+1}} \nabla\phi \cdot \nabla(\Phi_\varepsilon * \delta V)_i(\tilde{h}_\varepsilon)_i \, dy \\ &= -\frac{\varepsilon^2}{2}\int_{\mathbb{R}^{n+1}} \frac{\nabla\phi\cdot\nabla|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy. \end{aligned} \tag{5.37}$$

Here, we want to carry out one integration by parts for I_4. Let ψ_r be a cut-off function such that $\psi_r(x) = 1$ for $x \in B_{r/2}$, $\psi_r(x) = 0$ for $x \in \mathbb{R}^{n+1} \setminus B_r$ and $|\nabla\psi_r(x)| \leqslant 3/r$. For example, with ψ defined in (4.19), we may set $\psi_r(x) := \psi(x/r)$. Then we have

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}} \frac{\nabla\phi\cdot\nabla|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy \\ &\qquad = \lim_{r\to\infty}\int_{\mathbb{R}^{n+1}} \psi_r \frac{\nabla\phi\cdot\nabla|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy \\ &\qquad = -\int_{\mathbb{R}^{n+1}} \nabla\cdot\left(\frac{\nabla\phi}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\right)|\Phi_\varepsilon * \delta V|^2\, dy \\ &\qquad\quad - \lim_{r\to\infty}\int_{\mathbb{R}^{n+1}} \frac{(\nabla\psi_r\cdot\nabla\phi)|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy. \end{aligned} \tag{5.38}$$

For the second term of (5.38), we use (5.2), (5.3) and (4.4) to obtain

$$\left|\int_{\mathbb{R}^{n+1}} \frac{(\nabla\psi_r\cdot\nabla\phi)|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dy\right| \leqslant 2j\varepsilon^{-2}\int_{\mathbb{R}^{n+1}} |\nabla\psi_r||\Phi_\varepsilon * \delta V|\Omega\, dy. \tag{5.39}$$

By (5.6) and also noticing $(\Phi_\varepsilon * \|V\|)(x) \leqslant c(n,\varepsilon)\|V\|(B_1(x))$, with a suitable constant $c(n,\varepsilon)$, we have

$$\int_{\mathbb{R}^{n+1}} |\nabla\psi_r||\Phi_\varepsilon * \delta V|\Omega\, dy \leqslant \frac{c(n,\varepsilon)}{r}\int_{B_r\setminus B_{r/2}} \|V\|(B_1(x))\Omega(x)\, dx. \tag{5.40}$$

By (5.38)–(5.40) and (4.38), we may justify the integration by parts for I_4 on $\mathbb{R}^{n+1}$. Hence,

$$\tag{5.41}
\begin{aligned}
|I_4| &= \left| \frac{\varepsilon^2}{2} \int_{\mathbb{R}^{n+1}} \nabla \cdot \left(\frac{\nabla \phi}{\Phi_\varepsilon * \|V\| + \varepsilon \Omega^{-1}} \right) |\Phi_\varepsilon * \delta V|^2 \, dy \right| \\
&\leqslant \frac{\varepsilon^2}{2} \int_{\mathbb{R}^{n+1}} \left(\frac{((n+1)j + c_1 j)\phi}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}} + \frac{j\phi |\nabla \Phi_\varepsilon * \|V\||}{(\Phi_\varepsilon * \|V\| + \varepsilon \Omega^{-1})^2} \right) |\Phi_\varepsilon * \delta V|^2 \, dy,
\end{aligned}$$

where we also used $|\Delta\phi| \leqslant (n+1)j\phi$ and $\varepsilon\Omega^{-2}|\nabla\phi \cdot \nabla\Omega|(\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1})^{-1} \leqslant c_1 j\phi$ due to (3.1) and (4.4). To estimate the second term of (5.41), we have

$$\tag{5.42}
\begin{aligned}
|\nabla \Phi_\varepsilon * \|V\|(y)| &\leqslant \int_{\mathbb{R}^{n+1}} |\nabla \Phi_\varepsilon(x-y)| \, d\|V\|(x) \\
&\leqslant \int_{\mathbb{R}^{n+1}} \frac{|x-y|}{\varepsilon^2} \Phi_\varepsilon(x-y) \, d\|V\|(x) + c e^{-\varepsilon^{-1}} \|V\|(B_1(y)) \\
&\leqslant \varepsilon^{-\frac{3}{2}} \Phi_\varepsilon * \|V\|(y) + c e^{-\varepsilon^{-\frac{1}{2}}} \|V\|(B_1(y))
\end{aligned}$$

where we split the integration of the first term into $B_{\varepsilon^{\frac{1}{2}}}(y)$ and the complement as in the case of I_1, and also used (4.23). By substituting (5.42) into (5.41) and recalling estimates (4.38) and (5.7), with a suitable constant c depending only on c_1, M and n, we obtain

$$\tag{5.43} |I_4| \leqslant c j \varepsilon^{\frac{1}{2}} \int_{\mathbb{R}^{n+1}} \frac{\phi \, |\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon \Omega^{-1}} \, dy + c j e^{-\varepsilon^{-\frac{1}{6}}}.$$

Combining (5.32), remark for the estimate of I_2, (5.35), (5.43) and (5.8), we obtain (5.23) by suitably restricting ϵ_3.

For the proof of (5.24), by (5.2) and $h_\varepsilon = \Phi_\varepsilon * \tilde{h}_\varepsilon$, we have

$$\tag{5.44}
\begin{aligned}
\int_{\mathbb{R}^{n+1}} |h_\varepsilon|^2 \phi \, d\|V\| &= \int_{\mathbb{R}^{n+1}} |\Phi_\varepsilon * \tilde{h}_\varepsilon|^2 \phi \, d\|V\| \\
&\leqslant \int_{\mathbb{R}^{n+1}} \phi \, (\Phi_\varepsilon * |\tilde{h}_\varepsilon|^2) \, d\|V\| \\
&= \int_{\mathbb{R}^{n+1}} |\tilde{h}_\varepsilon(y)|^2 \int_{\mathbb{R}^{n+1}} \phi(x) \Phi_\varepsilon(x-y) \, d\|V\|(x) dy.
\end{aligned}$$

We then use (4.7) to conclude

$$\begin{aligned}(5.45)\quad &\int_{\mathbb{R}^{n+1}} \phi(x)\Phi_\varepsilon(x-y)\,d\|V\|(x)\\ &\leqslant \phi(y)(\Phi_\varepsilon*\|V\|)(y)+j\phi(y)\int_{\mathbb{R}^{n+1}} e^{j|x-y|}|x-y|\Phi_\varepsilon(x-y)\,d\|V\|(x)\end{aligned}$$

while the last term of (5.45) may be estimated by separating the integration over $B_{\varepsilon^{\frac12}}(y)$ and the complement as

$$\begin{aligned}(5.46)\quad &\int_{\mathbb{R}^{n+1}} e^{j|x-y|}|x-y|\Phi_\varepsilon(x-y)\,d\|V\|(x)\\ &\qquad\leqslant \varepsilon^{\frac12}e^{j\varepsilon^{\frac12}}(\Phi_\varepsilon*\|V\|)(y)+c(n)e^{j-\varepsilon^{-\frac12}}\|V\|(B_1(y)).\end{aligned}$$

Substitutions of (5.45) and (5.46) into (5.44) (and use (4.4) and (5.8)) give

$$\begin{aligned}(5.47)\quad \int_{\mathbb{R}^{n+1}}|h_\varepsilon|^2\phi\,d\|V\| \leqslant \int_{\mathbb{R}^{n+1}}|\tilde h_\varepsilon|^2\big\{(\Phi_\varepsilon*\|V\|)\phi(1+je\varepsilon^{\frac12})\\ +jc(n)e^{-\frac12\varepsilon^{-\frac12}}\Omega(y)\|V\|(B_1(y))\big\}\,dy.\end{aligned}$$

Since $|\tilde h_\varepsilon|^2\leqslant 4\varepsilon^{-4}$ by (5.3), the last term of (5.47) may be bounded by $jc(n,c_1,M)\varepsilon^{-4}e^{-\frac12\varepsilon^{-\frac12}}$, also using (4.38). By choosing an appropriate ϵ_3 depending only on n, c_1 and M, and again using (5.8), we obtain (5.24). □

Proposition 5.5. — *There exists $\epsilon_4\in(0,1)$ depending only on n, c_1 and M with the following property. Suppose $V\in\mathbf{V}_n(\mathbb{R}^{n+1})$ with $\|V\|(\Omega)\leqslant M$, $\varepsilon\in(0,\epsilon_4)$, $g\in\mathcal{B}_j$ and $j\in\mathbb{N}$ satisfying* (5.8). *Then we have*

$$(5.48)\quad \left|\int_{\mathbb{R}^{n+1}} h_\varepsilon\cdot g\,d\|V\|+\delta V(g)\right|\leqslant \varepsilon^{\frac14}+\varepsilon^{\frac14}\left(\int_{\mathbb{R}^{n+1}}\frac{|\Phi_\varepsilon*\delta V|^2\Omega}{\Phi_\varepsilon*\|V\|+\varepsilon\Omega^{-1}}\,dx\right)^{\frac12}.$$

Proof. — By (4.30) and a similar estimate as (4.9) for ∇g, we have

$$\begin{aligned}(5.49)\quad &\left|\int_{\mathbb{R}^{n+1}}(\Phi_\varepsilon*\delta V)\cdot g\,dy-\delta V(g)\right|\\ &\qquad= |\delta V(\Phi_\varepsilon*g)-\delta V(g)|\\ &\qquad\leqslant \int_{\mathbb{R}^{n+1}}|\nabla(\Phi_\varepsilon*g)-\nabla g|\,d\|V\|\\ &\qquad\leqslant cj\int_{\mathbb{R}^{n+1}}\int_{\mathbb{R}^{n+1}}|x-y|\Phi_\varepsilon(x-y)\Omega(x)\,d\|V\|(x)dy\\ &\qquad\leqslant cj\varepsilon^{\frac12}\|V\|(\Omega),\end{aligned}$$

where we estimated as in (5.17) and c is a constant depending only on n and c_1. Combining (5.9), (5.49), (5.8) and restricting $\epsilon_4 \leqslant \epsilon_2$ depending only on n, c_1 and M further, we obtain (5.48). □

5.3. Curvature of limit

By the estimates in the previous subsection, we obtain the following

Proposition 5.6. — *Suppose that we have $\{V_j\}_{j=1}^{\infty} \subset \mathbf{V}_n(\mathbb{R}^{n+1})$ with*

(1) $\sup_j \|V_j\|(\Omega) < \infty$,
(2) $\liminf_{j\to\infty} \int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_j} * \delta V_j|^2 \Omega}{\Phi_{\varepsilon_j} * \|V_j\| + \varepsilon_j \Omega^{-1}} \, dx < \infty$,
(3) $\lim_{j\to\infty} \varepsilon_j = 0$.

Then there exists a converging subsequence $\{V_{j_l}\}_{l=1}^{\infty}$, and the limit $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ has a generalized mean curvature $h(\cdot, V)$ with

$$\int_{\mathbb{R}^{n+1}} |h(\cdot, V)|^2 \phi \, d\|V\| \leqslant \liminf_{l\to\infty} \int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_{j_l}} * \delta V_{j_l}|^2 \phi}{\Phi_{\varepsilon_{j_l}} * \|V_{j_l}\| + \varepsilon_{j_l} \Omega^{-1}} \, dx \tag{5.50}$$

for any $\phi \in \cup_{i\in\mathbb{N}} \mathcal{A}_i$.

Proof. — By (1), we may choose a subsequence $\{V_{j_l}\}_{l=1}^{\infty}$ converging to a limit $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ and so that the integrals in (2) are uniformly bounded for this subsequence as well. Fix $\phi \in \mathcal{A}_i$ and consider a Hilbert space

$$X_\phi := \left\{ g = (g_1, \ldots, g_{n+1}); g \in L^2_{loc}(\|V\|), \int_{\mathbb{R}^{n+1}} |g|^2 \phi^{-1} \, d\|V\| < \infty \right\}$$

equipped with inner product $(f, g)_{X_\phi} := \int_{\mathbb{R}^{n+1}} f \cdot g \, \phi^{-1} \, d\|V\|$. Recall that $\phi > 0$ on $\mathbb{R}^{n+1}$, and $C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ is a dense subspace in X_ϕ. Fix arbitrary $g \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$. Corresponding to g, there exists $j' \in \mathbb{N}$ such that $g \in \mathcal{B}_{j'}$. By Proposition 5.5 with $j = j'$ and combined with (1) and (2), we have

$$\lim_{l\to\infty} \delta V_{j_l}(g) = -\lim_{l\to\infty} \int_{\mathbb{R}^{n+1}} h_{\varepsilon_{j_l}}(\cdot, V_{j_l}) \cdot g \, d\|V_{j_l}\|. \tag{5.51}$$

The left-hand side is equal to $\delta V(g)$ by the varifold convergence. For $\phi \in \mathcal{A}_i$, we have by (5.24) (with $j = i$) and (2) that, writing $h_{\varepsilon_{j_l}} = h_{\varepsilon_{j_l}}(\cdot, V_{j_l})$,

$$(5.52)\quad -\lim_{l\to\infty}\int_{\mathbb{R}^{n+1}} h_{\varepsilon_{j_l}}\cdot g\, d\|V_{j_l}\| \\ \leqslant \liminf_{l\to\infty}\left(\int_{\mathbb{R}^{n+1}}|h_{\varepsilon_{j_l}}|^2\phi\, d\|V_{j_l}\|\right)^{\frac12}\left(\int_{\mathbb{R}^{n+1}}|g|^2\phi^{-1}\, d\|V_{j_l}\|\right)^{\frac12} \\ \leqslant \left(\liminf_{l\to\infty}\int_{\mathbb{R}^{n+1}}\frac{\phi|\Phi_{\varepsilon_{j_l}}*\delta V_{j_l}|^2}{\Phi_{\varepsilon_{j_l}}*\|V_{j_l}\|+\varepsilon_{j_l}\Omega^{-1}}\, dx\right)^{\frac12}\left(\int_{\mathbb{R}^{n+1}}|g|^2\phi^{-1}\, d\|V\|\right)^{\frac12}.$$

Writing the first term on the right-hand side of (5.52) as C_0, (5.51) and (5.52) show

$$(5.53)\qquad \delta V(g) \leqslant C_0\|g\|_{X_\phi}$$

for any $g \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$. By a density argument, δV may be uniquely extended as a bounded linear functional on X_ϕ. By the Riesz representation theorem, there exists a unique $f \in X_\phi$ with $\|f\|_{X_\phi} \leqslant C_0$ such that $\delta V(g) = (f, g)_{X_\phi}$ for all $g \in X_\phi$. Then, note that $-f\phi^{-1}$ is the generalized mean curvature $h(\cdot, V)$, and (5.50) is equivalent to $\|f\|_{X_\phi} \leqslant C_0$. □

5.4. Motion by smoothed mean curvature

This subsection establishes an approximate motion law when a varifold is moved by the smoothed mean curvature vector.

Proposition 5.7. — *There exists $\epsilon_5 \in (0,1)$ depending only on n, c_1 and M with the following. Suppose $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ with $\|V\|(\Omega) \leqslant M$, $j \in \mathbb{N}$, $\phi \in \mathcal{A}_j$, $\varepsilon \in (0, \epsilon_5)$ with (5.8), $\Delta t \in (2^{-1}\varepsilon^{c_2}, \varepsilon^{c_2}]$, where we set*

$$(5.54)\qquad c_2 := 3n + 20.$$

Define

$$f(x) := x + h_\varepsilon(x, V)\,\Delta t.$$

Then we have

$$(5.55)\qquad \left|\frac{\|f_\sharp V\|(\phi) - \|V\|(\phi)}{\Delta t} - \delta(V,\phi)(h_\varepsilon(\cdot, V))\right| \leqslant \varepsilon^{c_2-10},$$

$$(5.56)\qquad \frac{\|f_\sharp V\|(\Omega) - \|V\|(\Omega)}{\Delta t} + \frac14\int_{\mathbb{R}^{n+1}}\frac{|\Phi_\varepsilon * \delta V|^2\,\Omega}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dx \\ \leqslant 3\varepsilon^{\frac14} + \frac{c_1^2}{2}\|V\|(\Omega).$$

Moreover, if $\|f_\sharp V\|(\Omega) \leqslant M$*, then we have*

$$|\delta(V,\phi)(h_\varepsilon(\cdot,V)) - \delta(f_\sharp V,\phi)(h_\varepsilon(\cdot,f_\sharp V))| \leqslant \varepsilon^{c_2-2n-19}, \tag{5.57}$$

$$\left| \int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta V|^2\Omega}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\,dx - \int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta(f_\sharp V)|^2\Omega}{\Phi_\varepsilon * \|f_\sharp V\| + \varepsilon\Omega^{-1}}\,dx \right| \leqslant \varepsilon^{c_2-3n-18}. \tag{5.58}$$

Proof. — For simplicity, write $F(x) := f(x) - x = h_\varepsilon(x,V)\Delta t$. We have

$$|F(x)| = |h_\varepsilon(x,V)|\Delta t \leqslant 2\varepsilon^{c_2-2} \tag{5.59}$$

by (5.3),

$$\|\nabla F(x)\| = \Delta t\|\nabla h_\varepsilon(x,V)\| \leqslant 2\varepsilon^{c_2-4} \tag{5.60}$$

by (5.4),

$$|\phi(f(x)) - \phi(x)| \leqslant j\Omega(x)\exp(j|F(x)|)|F(x)| \leqslant \varepsilon^{c_2-3}\Omega(x) \tag{5.61}$$

by (4.7), (4.4), (5.59), (5.8) and restricting ε,

$$||\Lambda_n \nabla f(x) \circ S| - 1| \leqslant c(n)\|\nabla F(x)\| \leqslant \frac{1}{2}\varepsilon^{c_2-5} \leqslant \varepsilon^{-5}\Delta t \tag{5.62}$$

by (5.60) and restricting ε depending only on n,

$$\begin{aligned} |\phi(f(x)) - \phi(x) - F(x)\cdot\nabla\phi(x)| &\leqslant j|F(x)|^2\Omega(x)\exp(j|F(x)|) \\ &\leqslant \frac{1}{2}\varepsilon^{2c_2-5}\Omega(x) \leqslant \varepsilon^{c_2-5}\Omega(x)\Delta t \end{aligned} \tag{5.63}$$

by (4.8), (5.59), (5.8) and by restricting ε,

$$\begin{aligned} ||\Lambda_n \nabla f(x) \circ S| &- 1 - \nabla F(x)\cdot S| \\ &\leqslant c(n)\|\nabla F(x)\|^2 \leqslant 4c(n)\varepsilon^{2c_2-8} \leqslant \varepsilon^{c_2-9}\Delta t \end{aligned} \tag{5.64}$$

by (5.60) and restricting ε depending only on n. Now recalling the definition of push-forward of varifold and (2.4), we have

$$\begin{aligned} \|f_\sharp V\|(\phi) &- \|V\|(\phi) - \delta(V,\phi)(h_\varepsilon(\cdot,V))\Delta t \\ &= \|f_\sharp V\|(\phi) - \|V\|(\phi) - \delta(V,\phi)(F) \\ &= \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} (\phi(f(x))|\Lambda_n \nabla f(x)\circ S| - \phi(x))\,dV(x,S) \\ &\quad - \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} (\nabla F(x)\cdot S\,\phi(x) + F(x)\cdot\nabla\phi(x))\,dV(x,S). \end{aligned} \tag{5.65}$$

We then interpolate (5.65) and use (5.61)–(5.64) as

$$
\begin{aligned}
&|(5.65)| \\
&\leqslant \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} \Big|(\phi(f(x)) - \phi(x))|\Lambda_n \nabla f(x)\circ S| + (|\Lambda_n \nabla f(x)\circ S| - 1)\phi(x) \\
&\qquad - \nabla F(x)\cdot S\,\phi(x) - F(x)\cdot \nabla\phi(x)\Big|\, dV(x,S) \\
&= \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} \Big|(\phi(f(x)) - \phi(x))(|\Lambda_n \nabla f(x)\circ S| - 1) + (\phi(f(x)) - \phi(x) \\
&\qquad - F(x)\cdot\nabla\phi(x)) + (|\Lambda_n \nabla f(x)\circ S| - 1 - \nabla F(x)\cdot S)\phi(x)\Big|\, dV(x,S) \\
&\leqslant (\varepsilon^{c_2-8} + \varepsilon^{c_2-5} + \varepsilon^{c_2-9})\|V\|(\Omega)\Delta t
\end{aligned}
$$

where we also used $\phi \leqslant \Omega$ for the last step. By restricting ε so that $3\varepsilon M \leqslant 1$, we obtain (5.55). For (5.56), using (5.23) and (5.24) with $\phi = \Omega$, $j \in [c_1+1, c_1+2)$ and restricting ε depending on c_1, we have

$$
\begin{aligned}
(5.66)\quad &\delta(V,\Omega)(h_\varepsilon) \\
&= \delta V(\Omega h_\varepsilon) + \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} h_\varepsilon \cdot S^\perp(\nabla\Omega)\, dV(\cdot,S) \\
&\leqslant \delta V(\Omega h_\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^{n+1}} |h_\varepsilon|^2\Omega + |\nabla\Omega|^2\Omega^{-1}\, d\|V\| \\
&\leqslant -\frac{1}{2}(1 - 3\varepsilon^{\frac14})\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta V|^2\Omega}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\, dx + 2\varepsilon^{\frac14} + \frac{c_1^2}{2}\|V\|(\Omega)
\end{aligned}
$$

where we also used (3.1). Restrict ϵ_5 so that $1 - 3\varepsilon^{\frac14} > \frac12$. Then (5.66) and (5.55) give (5.56).

For (5.57) and (5.58), for short, write $\hat V := f_\sharp V$. Due to the assumption that $\|f_\sharp V\|(\Omega) = \|\hat V\|(\Omega) \leqslant M$, we have (5.3)–(5.5) for $h_\varepsilon(\cdot,\hat V)$ as well. We first estimate $\Phi_\varepsilon * \|\hat V\| - \Phi_\varepsilon * \|V\|$ and $\Phi_\varepsilon * \delta\hat V - \Phi_\varepsilon * \delta V$, which lead to estimates of $h_\varepsilon(\cdot,V) - h_\varepsilon(\cdot,\hat V)$. We have

$$
\begin{aligned}
(5.67)\quad &|\Phi_\varepsilon * \|\hat V\|(x) - \Phi_\varepsilon * \|V\|(x)| \\
&= \left|\int \Phi_\varepsilon(z-x)\, d\|\hat V\|(z) - \int \Phi_\varepsilon(y-x)\, d\|V\|(y)\right| \\
&= \left|\int \Phi_\varepsilon(f(y)-x)|\Lambda_n\nabla f(y)\circ S| - \Phi_\varepsilon(y-x)\, dV(y,S)\right| \\
&\leqslant \int |\Phi_\varepsilon(f(y)-x) - \Phi_\varepsilon(y-x)||\Lambda_n\nabla f(y)\circ S|\, dV(y,S) \\
&\qquad + \int \Phi_\varepsilon(y-x)||\Lambda_n\nabla f(y)\circ S| - 1|\, dV(y,S).
\end{aligned}
$$

By (5.59) and (4.23), for some $\hat{y}$ lying on the line segment connecting $y-x$ and $f(y)-x$,

$$\begin{aligned}|\Phi_\varepsilon(f(y)-x)-\Phi_\varepsilon(y-x)| &\leqslant |F(y)||\nabla\Phi_\varepsilon(\hat{y})| \\ &\leqslant c(n)\varepsilon^{c_2-n-5}\chi_{B_2(x)}(y).\end{aligned} \tag{5.68}$$

By (5.62),

$$\Phi_\varepsilon(y-x)||\Lambda_n\nabla f(y)\circ S|-1| \leqslant \varepsilon^{c_2-n-6}\chi_{B_1(x)}(y). \tag{5.69}$$

Combining (5.67)–(5.69), we obtain

$$|\Phi_\varepsilon * \|\hat{V}\|(x)-\Phi_\varepsilon * \|V\|(x)| \leqslant \varepsilon^{c_2-n-7}\|V\|(B_2(x)). \tag{5.70}$$

Next, by (4.31),

$$\begin{aligned}&|\Phi_\varepsilon * \delta\hat{V}(x)-\Phi_\varepsilon * \delta V(x)| \\ &\quad= \left|\int T\left(\nabla\Phi_\varepsilon(z-x)\right)\,d\hat{V}(z,T)-\int S\left(\nabla\Phi_\varepsilon(y-x)\right)\,dV(y,S)\right| \\ &\quad= \left|\int \big\{(\nabla f(y)\circ S)(\nabla\Phi_\varepsilon(f(y)-x))|\Lambda_n\nabla f(y)\circ S|\right. \\ &\qquad\qquad \left. - S(\nabla\Phi_\varepsilon(y-x))\big\}\,dV(y,S)\right|.\end{aligned} \tag{5.71}$$

By estimating $\nabla f(y)-I$ using (5.60) and using similar estimates as in (5.68) and (5.69) (where Φ_ε is replaced by $\nabla\Phi_\varepsilon$, causing a multiplication by ε^{-2}), we obtain

$$|\Phi_\varepsilon * \delta\hat{V}(x)-\Phi_\varepsilon * \delta V(x)| \leqslant \varepsilon^{c_2-n-9}\|V\|(B_2(x)) \tag{5.72}$$

from (5.71) by the similar interpolations. We also have rough estimates of

$$|\Phi_\varepsilon * \delta V(x)|,\ |\Phi_\varepsilon * \delta\hat{V}(x)| \leqslant \varepsilon^{-n-4}\|V\|(B_2(x)). \tag{5.73}$$

Using (5.70), (5.72) and (5.73), we have

$$\begin{aligned}&\left|\frac{\Phi_\varepsilon * \delta\hat{V}}{\Phi_\varepsilon * \|\hat{V}\|+\varepsilon\Omega^{-1}}-\frac{\Phi_\varepsilon * \delta V}{\Phi_\varepsilon * \|V\|+\varepsilon\Omega^{-1}}\right| \\ &\qquad\leqslant \frac{|\Phi_\varepsilon * \delta\hat{V}-\Phi_\varepsilon * \delta V|}{\varepsilon\Omega^{-1}}+\frac{|\Phi_\varepsilon * \delta V||\Phi_\varepsilon * \|\hat{V}\|-\Phi_\varepsilon * \|V\||}{\varepsilon^2\Omega^{-2}} \\ &\qquad\leqslant \varepsilon^{c_2-n-10}\Omega(x)\|V\|(B_2(x))+\varepsilon^{c_2-2n-13}\Omega(x)^2\|V\|(B_2(x))^2\end{aligned} \tag{5.74}$$

and similarly

$$\left|\frac{|\Phi_\varepsilon * \delta\hat{V}|^2\Omega}{\Phi_\varepsilon * \|\hat{V}\| + \varepsilon\Omega^{-1}} - \frac{|\Phi_\varepsilon * \delta V|^2\Omega}{\Phi_\varepsilon * \|V\| + \varepsilon\Omega^{-1}}\right| \leqslant \varepsilon^{c_2-2n-15}\Omega(x)^2\|V\|(B_2(x))^2 + \varepsilon^{c_2-3n-17}\Omega(x)^3\|V\|(B_2(x))^3. \tag{5.75}$$

By Lemma 4.17 with $r = 2$, we obtain (5.58) from (5.75). Recalling the definition (5.1), from (5.74) and with (4.37), we obtain (writing $h_\varepsilon(\cdot, V)$ as $h_\varepsilon(V)$)

$$\begin{aligned} |h_\varepsilon(V) - h_\varepsilon(\hat{V})| &\leqslant \varepsilon^{c_2-2n-14}(M + M^2), \\ \|\nabla^l h_\varepsilon(V) - \nabla^l h_\varepsilon(\hat{V})\| &\leqslant \varepsilon^{c_2-2n-14-2l}(M + M^2) \end{aligned} \tag{5.76}$$

for $l = 1, 2$. Finally, we have

$$\begin{aligned} |\delta(V,\phi)(h_\varepsilon(V)) &- \delta(\hat{V},\phi)(h_\varepsilon(\hat{V}))| \\ &= \bigg| \int (\nabla h_\varepsilon(V)\cdot S\phi + h_\varepsilon(V)\cdot\nabla\phi)\, dV \\ &\quad - \int \big\{(\nabla h_\varepsilon(\hat{V})\circ f)\cdot(\nabla f\circ S)(\phi\circ f) \\ &\quad + (h_\varepsilon(\hat{V})\circ f)\cdot(\nabla\phi\circ f)\big\}|\Lambda_n \nabla f\circ S|\, dV\bigg|. \end{aligned} \tag{5.77}$$

Using (5.76) as well as (5.59)–(5.62) and (4.4), estimates by interpolations on (5.77) give (5.57). □

6. Existence of limit measures

Proposition 6.1. — *Given any $\mathcal{E}_0 \in \mathcal{OP}^N_\Omega$ and $j \in \mathbb{N}$ with $j \geqslant \max\{1, c_1\}$, there exist $\varepsilon_j \in (0, j^{-6})$, $p_j \in \mathbb{N}$, a family $\mathcal{E}_{j,l} \in \mathcal{OP}^N_\Omega$ ($l = 0, 1, 2, \ldots, j\,2^{p_j}$) with the following property.*

$$\mathcal{E}_{j,0} = \mathcal{E}_0 \ \text{ for all } j \in \mathbb{N} \tag{6.1}$$

and with the notation of

$$\Delta t_j := \frac{1}{2^{p_j}}, \tag{6.2}$$

we have

$$\|\partial\mathcal{E}_{j,l}\|(\Omega) \leqslant \|\partial\mathcal{E}_0\|(\Omega)\exp\left(\frac{c_1^2 l}{2}\Delta t_j\right) + \frac{2\varepsilon_j^{\frac{1}{8}}}{c_1^2}\left(\exp\left(\frac{c_1^2 l}{2}\Delta t_j\right) - 1\right), \tag{6.3}$$

$$\frac{\|\partial\mathcal{E}_{j,l}\|(\Omega)-\|\partial\mathcal{E}_{j,l-1}\|(\Omega)}{\Delta t_j}+\frac{1}{4}\int_{\mathbb{R}^{n+1}}\frac{|\Phi_{\varepsilon_j}*\delta(\partial\mathcal{E}_{j,l})|^2\Omega}{\Phi_{\varepsilon_j}*\|\partial\mathcal{E}_{j,l}\|+\varepsilon_j\Omega^{-1}}\,dx \\ -\frac{(1-j^{-5})}{\Delta t_j}\Delta_j\|\partial\mathcal{E}_{j,l-1}\|(\Omega)\leqslant\varepsilon_j^{\frac{1}{8}}+\frac{c_1^2}{2}\|\partial\mathcal{E}_{j,l-1}\|(\Omega), \tag{6.4}$$

$$\frac{\|\partial\mathcal{E}_{j,l}\|(\phi)-\|\partial\mathcal{E}_{j,l-1}\|(\phi)}{\Delta t_j}\leqslant\delta(\partial\mathcal{E}_{j,l},\phi)(h_{\varepsilon_j}(\cdot,\partial\mathcal{E}_{j,l}))+\varepsilon_j^{\frac{1}{8}} \tag{6.5}$$

for $l=1,2,\ldots,j\,2^{p_j}$ *and* $\phi\in\mathcal{A}_j$. *When* $c_1=0$, *the right-hand side of* (6.3) *should be understood as the limit* $c_1\to 0+$.

Proof. — Given $\mathcal{E}_0\in\mathcal{OP}^N_\Omega$ and $j\in\mathbb{N}$ with $j\geqslant\max\{1,c_1\}$, define

$$M_j:=\|\partial\mathcal{E}_0\|(\Omega)\exp\left(\frac{c_1^2 j}{2}\right)+1. \tag{6.6}$$

Let $\epsilon_1,\ldots,\epsilon_5$ be chosen in the previous section corresponding to M_j as M, then we choose ε_j so that $\varepsilon_j\leqslant\min\{\epsilon_1,\ldots,\epsilon_5\}$,

$$\frac{2\varepsilon_j^{\frac{1}{8}}}{c_1^2}\left(\exp\left(\frac{c_1^2 j}{2}\right)-1\right)<1,\qquad 3\varepsilon_j^{\frac{1}{4}}+\varepsilon_j^{c_2-3n-18}<\varepsilon_j^{\frac{1}{8}} \tag{6.7}$$

and (5.8) hold. Let c_2 be as in (5.54), and choose $p_j\in\mathbb{N}$ so that

$$\frac{1}{2^{p_j}}\in(2^{-1}\varepsilon_j^{c_2},\varepsilon_j^{c_2}]. \tag{6.8}$$

Define Δt_j as in (6.2). We proceed with inductive argument. Set $\mathcal{E}_{j,0}=\mathcal{E}_0$. Assume that up to $k=l\in\{0,1,\ldots,j\,2^{p_j}-1\}$, $\mathcal{E}_{j,k}$ is determined with the estimates (6.3)–(6.5). We will define $\mathcal{E}_{j,l+1}$ satisfying the estimates. Choose $f_1\in\mathbf{E}(\mathcal{E}_{j,l},j)$ (cf. Definition 4.8) such that

$$\|\partial(f_1)_\star\mathcal{E}_{j,l}\|(\Omega)-\|\partial\mathcal{E}_{j,l}\|(\Omega)\leqslant(1-j^{-5})\Delta_j\|\partial\mathcal{E}_{j,l}\|(\Omega) \tag{6.9}$$

and define

$$\mathcal{E}^*_{j,l+1}:=(f_1)_\star\mathcal{E}_{j,l}\in\mathcal{OP}^N_\Omega. \tag{6.10}$$

We note that

$$\|\partial\mathcal{E}^*_{j,l+1}\|(\Omega)\leqslant\|\partial\mathcal{E}_{j,l}\|(\Omega)\leqslant M_j \tag{6.11}$$

by (6.9), (6.3), (6.7) and (6.6). We next define a smooth function $f_2:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ by

$$f_2(x):=x+\Delta t_j\,h_{\varepsilon_j}(x,\partial\mathcal{E}^*_{j,l+1}). \tag{6.12}$$

By the choice of ε_j and Δt_j, and by (5.3) and (5.4), we have

$$|\Delta t_j\,h_{\varepsilon_j}(x,\partial\mathcal{E}^*_{j,l+1})|\leqslant 2\varepsilon_j^{c_2-2},\quad \|\nabla(\Delta t_j\,h_{\varepsilon_j}(x,\partial\mathcal{E}^*_{j,l+1}))\|\leqslant 2\varepsilon_j^{c_2-4}, \tag{6.13}$$

thus f_2 is a diffeomorphism and $\mathcal{E}^*_{j,l+1}$-admissible in particular. We then define

$$\mathcal{E}_{j,l+1} := (f_2)_\star \mathcal{E}^*_{j,l+1} \in \mathcal{OP}^N_\Omega. \tag{6.14}$$

Note that, since f_2 is a diffeomorphism, if we write $\mathcal{E}^*_{j,l+1} = \{E_i\}_{i=1}^N$, then we have $\mathcal{E}_{j,l+1} = \{f_2(E_i)\}_{i=1}^N$. Furthermore, we have

$$(f_2)_\sharp \partial \mathcal{E}^*_{j,l+1} = (f_2)_\sharp |\cup_{i=1}^N \partial E_i| = |\cup_{i=1}^N \partial (f_2(E_i))| = \partial \mathcal{E}_{j,l+1}. \tag{6.15}$$

To close the inductive argument, we need to check (6.3)–(6.5) with l replaced by $l+1$. To prove (6.3), we use (5.56) with $M = M_j$, $V = \partial \mathcal{E}^*_{j,l+1}$ as well as $3\varepsilon_j^{\frac{1}{4}} < \varepsilon_j^{\frac{1}{8}}$ of (6.7) to obtain

$$\begin{aligned} \|(f_2)_\sharp \partial \mathcal{E}^*_{j,l+1}\|(\Omega) &\leqslant \|\partial \mathcal{E}^*_{j,l+1}\|(\Omega) + \Delta t_j (\varepsilon_j^{\frac{1}{8}} + \frac{c_1^2}{2} \|\partial \mathcal{E}^*_{j,l+1}\|(\Omega)) \\ &\leqslant \|\partial \mathcal{E}_{j,l}\|(\Omega) + \Delta t_j (\varepsilon_j^{\frac{1}{8}} + \frac{c_1^2}{2} \|\partial \mathcal{E}_{j,l}\|(\Omega)), \end{aligned} \tag{6.16}$$

the last inequality due to (6.11). By (6.16) and (6.3), a direct computation using $e^{(x+s)} \geqslant (1+s)e^x$ for $s \geqslant 0$ proves (6.3) with l replaced by $l+1$. In particular, this proves that $\|\partial \mathcal{E}_{j,l+1}\|(\Omega) \leqslant M_j$, giving the validity of (5.57) and (5.58) for the pair $V = \partial \mathcal{E}^*_{j,l+1}$ and $f_\sharp V = \partial \mathcal{E}_{j,l+1}$. From (5.56), (6.11), (5.58), (6.9) and (6.7), we obtain (6.4) for $l+1$ in place of l. From (5.55), (5.57), (6.7) and $f_1 \in \mathbf{E}(\mathcal{E}_{j,l}, j)$, we obtain (6.5) for $l+1$ in place of l. This closes the inductive step, showing (6.3)–(6.5) up to $l = j\, 2^{p_j}$. □

Remark 6.2. — Due to the choice of ε_j, each $\partial \mathcal{E}_{j,l}$ satisfies various estimates obtained in Section 5 with $V = \partial \mathcal{E}_{j,l}$, $\varepsilon = \varepsilon_j$.

Remark 6.3. — It is convenient to define approximate solutions for all $t \geqslant 0$ instead of discrete times. For each $j \in \mathbb{N}$ with $j \geqslant \max\{1, c_1\}$, define a family $\mathcal{E}_j(t) \in \mathcal{OP}^N_\Omega$ for $t \in [0, j]$ by

$$\mathcal{E}_j(t) := \mathcal{E}_{j,l} \text{ if } t \in ((l-1)\Delta t_j, l\Delta t_j]. \tag{6.17}$$

Proposition 6.4. — *Under the assumptions of Proposition 6.1, there exist a subsequence $\{j_l\}_{l=1}^\infty$ and a family of Radon measures $\{\mu_t\}_{t\in\mathbb{R}^+}$ on $\mathbb{R}^{n+1}$ such that*

$$\lim_{l\to\infty} \|\partial \mathcal{E}_{j_l}(t)\|(\phi) = \mu_t(\phi) \tag{6.18}$$

for all $\phi \in C_c(\mathbb{R}^{n+1})$ *and for all* $t \in \mathbb{R}^+$. *For all* $T < \infty$, *we have*

$$(6.19)\quad \limsup_{l\to\infty} \int_0^T \left(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_{j_l}} * \delta(\partial\mathcal{E}_{j_l}(t))|^2 \Omega}{\Phi_{\varepsilon_{j_l}} * \|\partial\mathcal{E}_{j_l}(t)\| + \varepsilon_{j_l}\Omega^{-1}}\, dx - \frac{1}{\Delta t_{j_l}} \Delta_{j_l} \|\partial\mathcal{E}_{j_l}(t)\|(\Omega) \right) dt < \infty,$$

and for a.e. $t \in \mathbb{R}^+$, *we have*

$$(6.20)\quad \lim_{l\to\infty} j_l^{2(n+1)} \Delta_{j_l} \|\partial\mathcal{E}_{j_l}(t)\|(\Omega) = 0.$$

Proof. — Let $2_{\mathbb{Q}}$ be the set of all non-negative numbers of the form $\frac{i}{2^j}$ for some $i, j \in \mathbb{N} \cup \{0\}$. $2_{\mathbb{Q}}$ is dense in $\mathbb{R}^+$ and countable. For each fixed $J \in \mathbb{N}$, $\limsup_{j\to\infty} (\sup_{t\in[0,J]} \|\partial\mathcal{E}_j(t)\|(\Omega)) \leqslant \|\partial\mathcal{E}_0\|(\Omega) \exp(c_1^2 J/2)$ by (6.3). Thus, by diagonal argument, we may choose a subsequence and a family of Radon measures $\{\mu_t\}_{t\in 2_{\mathbb{Q}}}$ on $\mathbb{R}^{n+1}$ such that

$$(6.21)\quad \lim_{l\to\infty} \|\partial\mathcal{E}_{j_l}(t)\|(\phi) = \mu_t(\phi)$$

for all $\phi \in C_c(\mathbb{R}^{n+1})$ and $t \in 2_{\mathbb{Q}}$. We also have

$$(6.22)\quad \mu_t(\Omega) \leqslant \|\partial\mathcal{E}_0\|(\Omega) \exp(c_1^2 t/2)$$

for all $t \in 2_{\mathbb{Q}}$. Next, let $Z := \{\phi_q\}_{q\in\mathbb{N}}$ be a countable subset of $C_c^2(\mathbb{R}^{n+1}; \mathbb{R}^+)$ which is dense in $C_c(\mathbb{R}^{n+1}; \mathbb{R}^+)$ with respect to the supremum norm. We claim that, for any given $J \in \mathbb{N}$,

$$(6.23)\quad g_{q,J}(t) := \mu_t(\phi_q) - 2t\|\nabla^2\phi_q\|_\infty \big(\min_{x\in\operatorname{spt}\phi_q} \Omega(x)\big)^{-1} \|\partial\mathcal{E}_0\|(\Omega) \exp(c_1^2 J/2)$$

is a monotone decreasing function of $t \in [0, J] \cap 2_{\mathbb{Q}}$. Since ϕ_q has a compact support and due to the linear dependence of (6.23) on ϕ_q, we may assume $\phi_q < \Omega$ without loss of generality. To prove (6.23), just like (5.66), using (5.23) and (5.24), we have

$$(6.24)\quad \delta(\partial\mathcal{E}_j(t), \phi)(h_{\varepsilon_j}(\cdot, \partial\mathcal{E}_j(t))) \leqslant 2\varepsilon_j^{\frac14} + \frac12 \int_{\mathbb{R}^{n+1}} \frac{|\nabla\phi|^2}{\phi}\, d\|\partial\mathcal{E}_j(t)\|$$

for $\phi \in \mathcal{A}_j$ and $t \in [0, j]$. For any $\phi_q \in Z$ with $\phi_q < \Omega$ and sufficiently large $i \in \mathbb{N}$, choose $j_0 \in \mathbb{N}$ so that $\phi_q + i^{-1}\Omega \in \mathcal{A}_{j_0}$ holds and $j_0 \geqslant J$. For any $t_1, t_2 \in [0, J] \cap 2_{\mathbb{Q}}$ with $t_2 > t_1$ fixed, choose a larger j_0 so that t_1 and t_2 are integer-multiples of $1/2^{j_0}$. Then, by (6.5) and (6.24), we have

$$(6.25)\quad \|\partial\mathcal{E}_{j_l}(t_2)\|(\phi_q + i^{-1}\Omega) - \|\partial\mathcal{E}_{j_l}(t_1)\|(\phi_q + i^{-1}\Omega) \leqslant (\varepsilon_{j_l}^{\frac18} + 2\varepsilon_{j_l}^{\frac14})(t_2 - t_1) + \frac12 \int_{t_1}^{t_2} \int_{\mathbb{R}^{n+1}} \frac{|\nabla(\phi_q + i^{-1}\Omega)|^2}{\phi_q + i^{-1}\Omega}\, d\|\partial\mathcal{E}_{j_l}(t)\| dt$$

for all $j_l \geqslant j_0$. As $l \to \infty$, the left-hand side of (6.25) may be bounded from below using (6.21) and (6.3) as

$$\geqslant \mu_{t_2}(\phi_q) - \mu_{t_1}(\phi_q) - i^{-1}\|\partial\mathcal{E}_0\|(\Omega)\exp(c_1^2 J/2). \tag{6.26}$$

To estimate the right-hand side of (6.25), note that

$$\begin{aligned} \frac{|\nabla(\phi_q + i^{-1}\Omega)|^2}{\phi_q + i^{-1}\Omega} &\leqslant 2\frac{|\nabla\phi_q|^2}{\phi_q} + 2i^{-1}\frac{|\nabla\Omega|^2}{\Omega} \\ &\leqslant 4\|\nabla^2\phi_q\|_\infty(\min_{x\in\operatorname{spt}\phi_q}\Omega(x))^{-1}\Omega + 2i^{-1}c_1^2\Omega. \end{aligned} \tag{6.27}$$

Now, using (6.25)–(6.27), and then letting $i \to \infty$, we obtain

$$\begin{aligned} &\mu_{t_2}(\phi_q) - \mu_{t_1}(\phi_q) \\ &\qquad \leqslant 2\|\nabla^2\phi_q\|_\infty(\min_{x\in\operatorname{spt}\phi_q}\Omega(x))^{-1}\|\partial\mathcal{E}_0\|(\Omega)\exp(c_1^2 J/2)(t_2 - t_1). \end{aligned} \tag{6.28}$$

Then (6.28) proves that $g_{q,J}(t)$ defined in (6.23) is monotone decreasing. Define

$$D := \cup_{J\in\mathbb{N}}\{t \in (0, J) : \text{for some } q \in \mathbb{N}, \lim_{s\to t-} g_{q,J}(s) > \lim_{s\to t+} g_{q,J}(s)\}. \tag{6.29}$$

By the monotone property of $g_{q,J}$, D is a countable set on $\mathbb{R}^+$, and $\mu_t(\phi_q)$ may be defined continuously on the complement of D uniquely from the values on $2_{\mathbb{Q}}$. For any $t \in \mathbb{R}^+ \setminus (D \cup 2_{\mathbb{Q}})$ and $\phi_q \in Z$, we claim that

$$\lim_{l\to\infty}\|\partial\mathcal{E}_{j_l}(t)\|(\phi_q) = \mu_t(\phi_q). \tag{6.30}$$

Due to the definition of $\partial\mathcal{E}_{j_l}(t)$, there exists a sequence $\{t_l \in 2_{\mathbb{Q}}\}_{l=1}^\infty$ such that $\partial\mathcal{E}_{j_l}(t_l) = \partial\mathcal{E}_{j_l}(t)$ and that $\lim_{l\to\infty} t_l = t+$. For any $s > t$ with $s \in 2_{\mathbb{Q}}$ and for all sufficiently large l, (6.25) shows

$$\|\partial\mathcal{E}_{j_l}(s)\|(\phi_q + i^{-1}\Omega) \leqslant \|\partial\mathcal{E}_{j_l}(t_l)\|(\phi_q + i^{-1}\Omega) + O(s - t). \tag{6.31}$$

Taking $\liminf_{l\to\infty}$ and taking $i \to \infty$ on both sides of (6.31), we have

$$\mu_s(\phi_q) \leqslant \liminf_{l\to\infty}\|\partial\mathcal{E}_{j_l}(t_l)\|(\phi_q) + O(s - t). \tag{6.32}$$

By letting $s \to t+$, $\partial\mathcal{E}_{j_l}(t_l) = \partial\mathcal{E}_{j_l}(t)$, (6.32) and the continuity of $\mu_s(\phi_q)$ at $s = t$ imply

$$\mu_t(\phi_q) \leqslant \liminf_{l\to\infty}\|\partial\mathcal{E}_{j_l}(t)\|(\phi_q). \tag{6.33}$$

For any $s < t$ with $s \in 2_{\mathbb{Q}}$, we also have

$$\|\partial\mathcal{E}_{j_l}(t_l)\|(\phi_q + i^{-1}\Omega) \leqslant \|\partial\mathcal{E}_{j_l}(s)\|(\phi_q + i^{-1}\Omega) + O(t_l - s). \tag{6.34}$$

Take $\limsup_{l\to\infty}$, then let $i \to \infty$ to obtain from (6.34)

$$\limsup_{l\to\infty}\|\partial\mathcal{E}_{j_l}(t)\|(\phi_q) \leqslant \mu_s(\phi_q) + O(t - s). \tag{6.35}$$

By letting $s \to t-$ and by the continuity of $\mu_s(\phi_q)$, we have

$$\limsup_{l\to\infty} \|\partial\mathcal{E}_{j_l}(t)\|(\phi_q) \leqslant \mu_t(\phi_q). \tag{6.36}$$

(6.33) and (6.36) prove (6.30) for all $\phi_q \in Z$. Since Z is dense in $C_c(\mathbb{R}^{n+1};\mathbb{R}^+)$, (6.30) determines the limit measure uniquely and the convergence also holds in general for $\phi \in C_c(\mathbb{R}^{n+1})$. For $t \in D$, since D is countable, we may choose a further subsequence by a diagonal argument so that a further subsequence of $\{\|\partial\mathcal{E}_{j_l}(t)\|\}_{l=1}^\infty$ converges for all $t \in \mathbb{R}^+$ to a Radon measure μ_t. Finally (6.19) follows from (6.4). Since $\Delta t_{j_l} \leqslant \varepsilon_{j_l}^{c_2} \ll j_l^{-2(n+1)}$ by (6.8), (5.54) and (5.8), we have $\lim_{l\to\infty} \int_0^T -j_l^{2(n+1)} \Delta_{j_l} \|\partial\mathcal{E}_{j_l}(t)\|(\Omega)\, dt \leqslant \lim_{l\to\infty} \Delta t_{j_l} j_l^{2(n+1)} = 0$. Thus there exists a further subsequence such that the integrand converges pointwise to 0 for a.e. on $[0,T]$. As $T \to \infty$ and carrying out a diagonal argument, we may conclude (6.20) holds for a.e. $t \in \mathbb{R}^+$ for a subsequence. □

Remark 6.5. — In (6.9), we choose $f_1 \in \mathbf{E}(\mathcal{E}_{j,l}, j)$ so that f_1 nearly achieves inf among $\mathbf{E}(\mathcal{E}_{j,l}, j)$. The choice of factor $1 - j^{-5}$ can be different, on the other hand. In fact, all we need is (6.20) (which is needed to obtain integrality later) and we may replace $1 - j^{-5}$ by any fixed number in $(0,1)$, or even a sequence of numbers α_j as long as $\lim_{j\to\infty} j^{2(n+1)}\alpha_j^{-1}\Delta t_j = 0$ is satisfied. Such choice would give a different estimate in (6.4) with different factor instead of $1-j^{-5}$ but otherwise, the proof is identical. Since Δt_j goes to 0 very fast ($\Delta t_j \leqslant \varepsilon_j^{c_2} = \varepsilon_j^{3n+20}$ and $\varepsilon_j < j^{-6}$), we may make a choice so that α_j goes to 0 very fast. This means that, if we wish, we may choose $f_1 \in \mathbf{E}(\mathcal{E}_{j,l}, j)$ which only achieves a "tiny fraction" of inf in $\Delta_j \|\partial\mathcal{E}_{j,l}\|(\Omega)$, and asymptotically doing almost no apparent area reducing as $j \to \infty$. The choice should be reflected upon the singularities of the limiting V_t but we do not know how to characterize this aspect.

7. Rectifiability theorem

The main result of this section is Theorem 7.3, which is analogous to Allard's rectifiability theorem [1, §5.5(1)] but with an added difficulty of having only a control of smoothed mean curvature vector up to the length scale of $O(1/j^2)$ and a certain area minimizing property in a smaller length scale. Except for using the notions introduced in Section 4 such as $\mathcal{E}$-admissible functions and $\Delta_j \|\partial\mathcal{E}\|(\Omega)$, the content of Section 7 and 8 are more or less independent of Section 5 and 6, and they can be of independent interests.

We first recall a formula usually referred to as the monotonicity formula from [1, §5.1(3)]:

Lemma 7.1. — *Suppose* $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$, $0 < r_1 < r_2 < \infty$, $x \in \mathbb{R}^{n+1}$, *and for* $0 \leqslant s < \infty$,

$$\|\delta V\|(B_r(x)) \leqslant s\|V\|(B_r(x)) \tag{7.1}$$

whenever $r_1 < r < r_2$. *Then*

$$(\exp(sr))r^{-n}\|V\|(B_r(x)) \tag{7.2}$$

is nondecreasing in r *for* $r_1 < r < r_2$.

The following Proposition 7.2 is essential to prove the rectifiability of the limit measure. For the similar purpose in [8], Brakke cites a result in [2] of Almgren. The proof by Almgren requires extensive tools involving varifold slicing and piecewise smooth Lipschitz deformation to cubical complexes. On the other hand, his proof does not provide a deformation with $\mathcal{E}$-admissibility or volume estimate (Proposition 7.2(4)) which are essential in our proof. For codimension 1 case, we provide a more direct proof using radial projection as follows.

Proposition 7.2. — *There exist* $c_3, c_4 \in (0,\infty)$ *depending only on* n *with the following property. For* $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}_\Omega^N$, *suppose* $0 \in \operatorname{spt}\|\partial\mathcal{E}\|$ *and* $\|\partial\mathcal{E}\|(B_R) \leqslant c_3 R^n$. *Then there exist a* $\mathcal{E}$*-admissible function* f *and* $r \in [\frac{R}{2}, R]$ *such that*

(1) $f(x) = x$ *for* $x \in \mathbb{R}^{n+1} \setminus U_r$,
(2) $f(x) \in B_r$ *for* $x \in B_r$,
(3) $\|\partial f_\star\mathcal{E}\|(B_r) \leqslant \frac{1}{2}\|\partial\mathcal{E}\|(B_r)$,
(4) $\mathcal{L}^{n+1}(E_i \triangle \tilde{E}_i) \leqslant c_4(\|\partial\mathcal{E}\|(B_r))^{\frac{n+1}{n}}$ *for all* i, *where* $\{\tilde{E}_i\}_{i=1}^N = f_\star\mathcal{E}$.

Proof. — For $r > 0$ let $\nu(r) := \|\partial\mathcal{E}\|(B_r) = \mathcal{H}^n(B_r \cap \cup_{i=1}^N \partial E_i)$. Since $0 \in \operatorname{spt}\|\partial\mathcal{E}\|$, we have $\nu(r) > 0$ for $r > 0$ and $\nu(r)$ is a monotone increasing function which is differentiable a.e.. We also have

$$\mathcal{H}^{n-1}(\partial B_r \cap \cup_{i=1}^N \partial E_i) \leqslant \nu'(r) < \infty \tag{7.3}$$

whenever ν is differentiable. By the relative isoperimetric inequality [3, p. 152], there exists c_4 depending only on n such that

$$\min\{\mathcal{L}^{n+1}(U_R \cap E_i), \mathcal{L}^{n+1}(U_R \setminus E_i)\} \leqslant c_4(\mathcal{H}^n(U_R \cap \partial E_i))^{\frac{n+1}{n}}. \tag{7.4}$$

We assume

$$\nu(R) \leqslant \left(\frac{\mathcal{L}^{n+1}(U_R)}{2^{n+2}c_4}\right)^{\frac{n}{n+1}}, \tag{7.5}$$

and we further restrict $\nu(R)$ in the following. Since $\mathcal{H}^n(U_R \cap \partial E_i) \leqslant \nu(R)$, (7.4) and (7.5) imply that there is a unique $i_0 \in \{1, \ldots, N\}$ such that

$$\mathcal{L}^{n+1}(U_R \setminus E_{i_0}) \leqslant c_4(\nu(R))^{\frac{n+1}{n}} \leqslant \frac{1}{2^{n+2}} \mathcal{L}^{n+1}(U_R), \tag{7.6}$$

i.e., E_{i_0} takes up a major part of U_R. The reason for the existence of such i_0 is as follows. Otherwise, all E_i would have a small measure in U_R. Since $U_R \cap \cup_{i=1}^N E_i$ is a full measure set, there exists a combination $E_{i_1}, \ldots, E_{i_J}$ such that $(\mathcal{L}^{n+1}(U_R))^{-1}\mathcal{L}^{n+1}(\cup_{k=1}^J E_{i_k}) \in (1/4, 3/4)$. The relative isoperimetric inequality applied to $\hat{E} := \cup_{k=1}^J E_{i_k}$ gives a lower bound $c_4(\|\nabla \chi_{\hat{E}}\|(U_R))^{\frac{n+1}{n}} \geqslant \mathcal{L}^{n+1}(U_R)/4$ while we have $\|\nabla \chi_{\hat{E}}\|(U_R) \leqslant \mathcal{H}^n(U_R \cap \cup_{i=1}^N \partial E_i)$. This gives a contradiction to (7.5).

For all $r \in [\frac{R}{2}, R]$, (7.6) also gives $\mathcal{L}^{n+1}(U_r \setminus E_{i_0}) \leqslant \frac{1}{2}\mathcal{L}^{n+1}(U_r)$, thus (7.4) with R replaced by r shows

$$\mathcal{L}^{n+1}(U_r \setminus E_{i_0}) \leqslant c_4(\mathcal{H}^n(U_r \cap \partial E_{i_0}))^{\frac{n+1}{n}} \tag{7.7}$$

for all $r \in [\frac{R}{2}, R]$. Next, let $\tilde{A} := \{r \in [\frac{R}{2}, R] : \mathcal{H}^n(\partial B_r \setminus E_{i_0}) > \frac{1}{2}\mathcal{H}^n(\partial B_r)\}$ and $A := [\frac{R}{2}, R] \setminus \tilde{A}$. Since

$$\mathcal{L}^{n+1}((U_R \setminus B_{\frac{R}{2}}) \setminus E_{i_0}) = \int_{\frac{R}{2}}^R \mathcal{H}^n(\partial B_r \setminus E_{i_0})\, dr \geqslant \frac{1}{2}\mathcal{L}^1(\tilde{A})\mathcal{H}^n(\partial B_{\frac{R}{2}}), \tag{7.8}$$

(7.6) and (7.8) show

$$\mathcal{L}^1(\tilde{A}) \leqslant \frac{R}{2(n+1)} \quad \text{and} \quad \mathcal{L}^1(A) \geqslant (\frac{1}{2} - \frac{1}{2(n+1)})R \geqslant \frac{R}{4}. \tag{7.9}$$

In particular, (7.9) proves that

$$\mathcal{H}^n(\partial B_r \setminus E_{i_0}) \leqslant \frac{1}{2}\mathcal{H}^n(\partial B_r) \text{ for } r \in A \subset [\frac{R}{2}, R] \text{ with } \mathcal{L}^1(A) \geqslant \frac{R}{4}. \tag{7.10}$$

Next, fix arbitrary $r \in A$ which also satisfies (7.3), and let $G_i := E_i \cap \partial B_r$. Each G_i is open with respect to the topology on ∂B_r and $\partial G_i \subset \partial B_r \cap \partial E_i$. Note also that $\partial B_r \setminus E_i = \partial B_r \setminus G_i$. By the relative isoperimetric inequality on ∂B_r and (7.10), there exists c_5 depending only on n such that

$$\begin{aligned} \mathcal{H}^n(\partial B_r \setminus G_{i_0}) &= \min\{\mathcal{H}^n(G_{i_0}), \mathcal{H}^n(\partial B_r \setminus G_{i_0})\} \\ &\leqslant c_5(\mathcal{H}^{n-1}(\partial G_{i_0}))^{\frac{n}{n-1}}. \end{aligned} \tag{7.11}$$

Now we choose $B_{2r_0}(x_0) \subset U_r \cap E_{i_0}$ and choose a Lipschitz map f as follows. $f(x) = x$ if $x \in \mathbb{R}^{n+1} \setminus U_r$, f maps $B_{r_0}(x_0)$ to B_r bijectively, and $B_r \setminus U_{r_0}(x_0)$ onto ∂B_r by radial projection centered at x_0. See Figure 4.5 for a general idea of the map. We claim that such f is $\mathcal{E}$-admissible. Let $\tilde{E}_i := \mathrm{int}(f(E_i))$. For $i \neq i_0$, $\tilde{E}_i = E_i \setminus B_r$, because f is identity on $\mathbb{R}^{n+1} \setminus B_r$ and $f(E_i \cap B_r) \subset \partial B_r$. On the other hand, $\tilde{E}_{i_0} = E_{i_0} \cup U_r$ since

$U_r = f(U_{r_0}(x_0))$ and $U_{r_0}(x_0) \subset E_{i_0}$, and any $x \in \partial B_r \cap E_{i_0}$ is in $E_{i_0} \cup U_r$. For two open sets A and B, we have $\partial(A \cap B) \subset (\partial A \cap \operatorname{clos} B) \cup (\partial B \cap A)$ and $\partial(A \cup B) \subset (\partial A \setminus \operatorname{clos} B) \cup (\partial B \setminus A)$. So

$$\begin{aligned} \partial \tilde{E}_i &= \partial(E_i \cap (\mathbb{R}^{n+1} \setminus B_r)) \\ &\subset (\partial E_i \cap \operatorname{clos}(\mathbb{R}^{n+1} \setminus B_r)) \cup (\partial B_r \cap E_i) \\ &= (\partial E_i \setminus U_r) \cup G_i \end{aligned} \tag{7.12}$$

for $i \neq i_0$ while

$$\begin{aligned} \partial \tilde{E}_{i_0} &= \partial(E_{i_0} \cup U_r) \subset (\partial E_{i_0} \setminus B_r) \cup (\partial B_r \setminus E_{i_0}) \\ &= (\partial E_{i_0} \setminus B_r) \cup (\partial B_r \setminus G_{i_0}). \end{aligned} \tag{7.13}$$

We need to check $\mathbb{R}^{n+1} \setminus \cup_{i=1}^N \tilde{E}_i \subset f(\cup_{i=1}^N \partial E_i)$. Since $\mathbb{R}^{n+1} \setminus \cup_{i=1}^N \tilde{E}_i$ does not have any interior point, it is enough to prove $\cup_{i=1}^N \partial \tilde{E}_i \subset f(\cup_{i=1}^N \partial E_i)$. For $i \neq i_0$, $\partial E_i \setminus U_r \subset f(\partial E_i)$ since f is identity on $\mathbb{R}^{n+1} \setminus U_r$. For any $x \in G_i$, consider a line segment I with two ends, x_0 and x. Since $x \in G_i = \partial B_r \cap E_i$, there is some neighborhood of x of I belonging to E_i. On the other hand, we have $B_{r_0}(x_0) \subset E_{i_0}$, thus there must be some point $\hat{x} \in I \cap \partial E_{i_0}$. Since f on $B_r \setminus B_{r_0}(x_0)$ is a radial projection to ∂B_r, $f(\hat{x}) = x$. This proves that $G_i \subset f(\partial E_{i_0})$. Then (7.12) shows $\partial \tilde{E}_i \subset f(\partial E_i \cup \partial E_{i_0})$ for $i \neq i_0$. For $i = i_0$, $\partial E_{i_0} \setminus B_r = f(\partial E_{i_0} \setminus B_r)$ since f is identity there. For any $x \in \partial B_r \setminus G_{i_0} = \partial B_r \setminus E_{i_0}$, either $x \in \partial E_i$ for some i (including $i = i_0$), or $x \in E_i$ for some $i \neq i_0$. In the former case, since f is identity on ∂B_r, $x \in f(\partial E_i)$. In the latter case, the line segment connecting x_0 and x contains $\hat{x} \in \partial E_{i_0}$ just as before, hence $x \in f(\partial E_{i_0})$. Thus by (7.13), we have $\partial \tilde{E}_{i_0} \subset f(\cup_{i=1}^N \partial E_i)$. In all, we have proved that $\cup_{i=1}^N \partial \tilde{E}_i \subset f(\cup_{i=1}^N \partial E_i)$, and this proves that f is $\mathcal{E}$-admissible. With $\tilde{\mathcal{E}} = f_\star \mathcal{E} = \{\tilde{E}_i\}_{i=1}^N$, we have from (7.12), (7.13) and $\cup_{i \neq i_0} G_i \subset \partial B_r \setminus G_{i_0}$ that

$$\begin{aligned} \|\partial \tilde{\mathcal{E}}\|(B_r) &= \mathcal{H}^n(\cup_{i=1}^N \partial \tilde{E}_i \cap B_r) \\ &\leqslant \mathcal{H}^n(\partial B_r \setminus G_{i_0}) + \sum_{i \neq i_0} \mathcal{H}^n(\partial E_i \cap \partial B_r) \\ &= \mathcal{H}^n(\partial B_r \setminus G_{i_0}), \end{aligned} \tag{7.14}$$

the last equality due to (7.3). We next note that $E_i \triangle \tilde{E}_i = E_i \cap B_r$ for $i \neq i_0$ and $= U_r \setminus E_{i_0}$ for $i = i_0$. Since both are included in $B_r \setminus E_{i_0}$, (7.7) shows that the condition (4) is satisfied with this c_4. Thus we conclude that $\mathcal{E}$-admissible function f satisfies conditions (1), (2), (4) so far.

If the conclusion were not true, then, we must have

$$\|\partial \tilde{\mathcal{E}}\|(B_r) > \frac{1}{2}\|\partial \mathcal{E}\|(B_r) = \frac{1}{2}\nu(r)$$

if $r \in A$ with (7.3). Combining (7.14), (7.11) and (7.3), we obtain

$$\frac{1}{2}\nu(r) \leqslant c_5(\nu'(r))^{\frac{n}{n-1}}. \tag{7.15}$$

Since we have $\mathcal{L}^1(A) \geqslant \frac{R}{4}$ by (7.10),

$$\nu^{\frac{1}{n}}(R) \geqslant \int_A (\nu^{\frac{1}{n}}(r))'\,dr \geqslant n^{-1}(2c_5)^{\frac{1-n}{n}}\frac{R}{4}. \tag{7.16}$$

We would obtain a contradiction to $\|\partial\mathcal{E}\|(B_R) = \nu(R) \leqslant c_3 R^n$ by choosing an appropriately small c_3 depending only on n. □

Theorem 7.3 (cf. [8, p. 78]). — *Suppose that* $\{\mathcal{E}_j\}_{j=1}^\infty \subset \mathcal{OP}_\Omega^N$ *and* $\{\varepsilon_j\}_{j=1}^\infty \subset (0,1)$ *satisfy*

(1) $\lim_{j\to\infty} j^4\varepsilon_j = 0$,
(2) $\sup_j \|\partial\mathcal{E}_j\|(\Omega) < \infty$,
(3) $\liminf_{j\to\infty} \int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_j} * \delta(\partial\mathcal{E}_j)|^2\Omega}{\Phi_{\varepsilon_j} * \|\partial\mathcal{E}_j\| + \varepsilon_j\Omega^{-1}}\,dx < \infty$,
(4) $\lim_{j\to\infty} \Delta_j\|\partial\mathcal{E}_j\|(\Omega) = 0$.

Then there exists a converging subsequence $\{\partial\mathcal{E}_{j_l}\}_{l=1}^\infty$ *whose limit* $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ *satisfies*

$$\theta^{*n}(\|V\|, x) \geqslant \frac{c_3}{16\omega_n} \text{ for } \|V\| \text{ a.e. } x. \tag{7.17}$$

Furthermore, $V \in \mathbf{RV}_n(\mathbb{R}^{n+1})$.

Proof. — The existence of converging subsequence $\{\partial\mathcal{E}_{j_l}\}_{l=1}^\infty$ and the limit V with

$$\|V\|(\Omega) \leqslant \sup_l \|\partial\mathcal{E}_{j_l}\|(\Omega) \leqslant M \tag{7.18}$$

for some $M \in (0,\infty)$ follows from the compactness of Radon measures. We may also assume that the quantities in (3) are uniformly bounded also by M for this subsequence. Fix $R \in (0,1)$ and $x_0 \in \mathbb{R}^{n+1}$ and define

$$F_R := \{x \in B_1(x_0) : R^{-n}\|V\|(B_R(x)) < c_3/16\}, \tag{7.19}$$

where c_3 is the constant given by Proposition 7.2. We will prove that $\lim_{R\to 0}\|V\|(F_R) = 0$ which proves (7.17) in $B_1(x_0)$. Since x_0 is arbitrary, we have (7.17) on $\mathbb{R}^{n+1}$.

For $x \in F_R$, we may choose $\phi \in C_c^\infty(\mathbb{R}^{n+1})$ approximating $\chi_{B_R(x)}$ such that $\phi = 1$ on $B_R(x)$, $\phi = 0$ outside $B_{2R}(x)$ and $0 \leqslant \phi \leqslant 1$ with $R^{-n}\|V\|(\phi) < c_3/16$. Since $\lim_{l\to\infty}\|\partial\mathcal{E}_{j_l}\| = \|V\|$, for all sufficiently large l depending on x, we have

$$R^{-n}\|\partial\mathcal{E}_{j_l}\|(\phi) < c_3/16. \tag{7.20}$$

Since $\Phi_{\varepsilon_{j_l}} * \phi$ converges uniformly to ϕ on $B_{2R+1}(x)$ by (1) and is equal to 0 outside,

$$\begin{aligned}(7.21)\quad \big|\|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(\phi) - \|\partial\mathcal{E}_{j_l}\|(\phi)\big| &= \big|\|\partial\mathcal{E}_{j_l}\|(\Phi_{\varepsilon_{j_l}} * \phi - \phi)\big| \\ &\leqslant \sup_{B_{2R+1}(x)} (|\Phi_{\varepsilon_{j_l}} * \phi - \phi|\Omega^{-1})M\end{aligned}$$

converges to 0. Thus, by (7.20) and (7.21), for $x \in F_R$ there exists $m_x \in \mathbb{N}$ such that $R^{-n}\|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_R(x)) < c_3/16$ for all $l \geqslant m_x$. Thus, if we define

$$(7.22)\quad F_{R,m} := \{x \in F_R : R^{-n}\|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_R(x)) < c_3/16 \text{ for all } l \geqslant m\},$$

$F_{R,m} \subset F_{R,m+1}$ for all $m \in \mathbb{N}$ with $\cup_{m\in\mathbb{N}} F_{R,m} = F_R$. Hence we may choose $m_1 \in \mathbb{N}$ with

$$(7.23)\quad \|V\|\lfloor_\Omega(F_{R,m_1}) \geqslant \frac{1}{2}\|V\|\lfloor_\Omega(F_R).$$

Next, define

$$(7.24)\quad G_R := \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, F_{R,m_1}) < (1 - 2^{-\frac{1}{n}})R\}.$$

By definition, G_R is open, and for any $x \in G_R$, there exists $y \in F_{R,m_1}$ with $|x - y| < (1 - 2^{-\frac{1}{n}})R$. By (7.22),

$$\begin{aligned}(7.25)\quad (2^{-\frac{1}{n}}R)^{-n}&\|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_{2^{-\frac{1}{n}}R}(x)) \\ &\leqslant 2R^{-n}\|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_R(y)) < c_3/8\end{aligned}$$

for all $l \geqslant m_1$ and $x \in G_R$. Since G_R is open, we may choose $m_2 \in \mathbb{N}$ with $m_2 \geqslant m_1$ such that

$$(7.26)\quad \|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(G_R) \geqslant \frac{1}{2}\|V\|\lfloor_\Omega(G_R)$$

for all $l \geqslant m_2$. Since $F_{R,m_1} \subset G_R$, (7.26) and (7.23) show

$$(7.27)\quad \|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(G_R) \geqslant \frac{1}{4}\|V\|\lfloor_\Omega(F_R)$$

for all $l \geqslant m_2$. Choose $m_3 \in \mathbb{N}$ such that $m_3 \geqslant m_2$ and

$$(7.28)\quad \frac{1}{2j_{m_3}^2} < \frac{R}{2}.$$

Define

$$\begin{aligned}(7.29)\quad G_{R,j_l,1} := \{x \in G_R :\ &\theta^n(\|\partial\mathcal{E}_{j_l}\|, x) = 1 \\ &\text{and } (2j_l^2)^n\|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_{\frac{1}{2j_l^2}}(x)) > c_3/4\}\end{aligned}$$

and

$$(7.30)\quad G_{R,j_l,2} := \{x \in G_R : \ \theta^n(\|\partial\mathcal{E}_{j_l}\|, x) = 1 \text{ and } (2j_l^2)^n \|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_{\frac{1}{2j_l^2}}(x)) \leqslant c_3/4\}.$$

Since $\theta^n(\|\partial\mathcal{E}_{j_l}\|, x) = 1$ for $\|\partial\mathcal{E}_{j_l}\|$ a.e. x, we have

$$(7.31)\qquad \|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(G_{R,j_l,1} \cup G_{R,j_l,2}) = \|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(G_R).$$

First we consider the case $x \in G_{R,j_l,1}$ with $l \geqslant m_3$. We use $r_1 = \frac{1}{2j_l^2} < 2^{-\frac{1}{n}}R = r_2$ in Lemma 7.1. Here, the inequality follows from (7.28). If (7.1) holds with $s := (2^{-\frac{1}{n}}R - \frac{1}{2j_l^2})^{-1}(\ln 2)$, then we would have a contradiction to (7.25) and (7.29). Thus there exists $\frac{1}{2j_l^2} < r_x < 2^{-\frac{1}{n}}R$ such that (7.1) does not hold, i.e.,

$$(7.32)\qquad \begin{aligned}\|\delta(\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l})\|(B_{r_x}(x)) &> s\|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_{r_x}(x))\\ &\geqslant \frac{1}{2R}\|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_{r_x}(x)),\end{aligned}$$

where the last inequality holds from the definition of s. Since $\varepsilon_{j_l} \leqslant j_l^{-4} < j_l^{-2} < 2r_x$ by (1) for all large l, $\Phi_{\varepsilon_{j_l}} * \chi_{B_{r_x}(x)} \geqslant \frac{1}{4}$ on $B_{r_x}(x)$. Thus we have

$$(7.33)\quad \|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_{r_x}(x)) = \|\partial\mathcal{E}_{j_l}\|(\Phi_{\varepsilon_{j_l}} * \chi_{B_{r_x}(x)}) \geqslant \frac{1}{4}\|\partial\mathcal{E}_{j_l}\|(B_{r_x}(x)).$$

By (4.33), (3.2), (7.32) and (7.33), we have

$$(7.34)\quad \begin{aligned}&\|\Phi_{\varepsilon_{j_l}} * \delta(\partial\mathcal{E}_{j_l})\|\lfloor_\Omega(B_{r_x}(x))\\ &= \|\delta(\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l})\|\lfloor_\Omega(B_{r_x}(x))\\ &\geqslant \Omega(x)\exp(-2c_1R)\|\delta(\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l})\|(B_{r_x}(x))\\ &\geqslant \frac{1}{8R}\Omega(x)\exp(-2c_1R)\|\partial\mathcal{E}_{j_l}\|(B_{r_x}(x))\\ &\geqslant \frac{1}{8R}\exp(-4c_1R)\|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(B_{r_x}(x)).\end{aligned}$$

Let $\mathcal{C} := \{B_{r_x}(x) : x \in G_{R,j_l,1}\}$, where r_x is as above. By the Besicovitch covering theorem, there exists a collection of subfamilies $\mathcal{C}_1, \dots, \mathcal{C}_{\mathbf{B}_{n+1}}$, each of them consisting of mutually disjoint balls and such that

$$(7.35)\qquad G_{R,j_l,1} \subset \cup_{i=1}^{\mathbf{B}_{n+1}} \cup_{B_{r_x}(x)\in\mathcal{C}_i} B_{r_x}(x).$$

Then for some $i_0 \in \{1, \dots, \mathbf{B}_{n+1}\}$, we have

$$
\begin{aligned}
(7.36)\quad & \|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(G_{R,j_l,1}) \\
&\leqslant \mathbf{B}_{n+1} \sum_{B_{r_x}(x)\in\mathcal{C}_{i_0}} \|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(B_{r_x}(x)) \\
&\leqslant 8R\exp(4c_1R)\mathbf{B}_{n+1} \sum_{B_{r_x}(x)\in\mathcal{C}_{i_0}} \|\Phi_{\varepsilon_{j_l}} * \delta(\partial\mathcal{E}_{j_l})\|\lfloor_\Omega(B_{r_x}(x)) \\
&\leqslant 8R\exp(4c_1R)\mathbf{B}_{n+1}\|\Phi_{\varepsilon_{j_l}} * \delta(\partial\mathcal{E}_{j_l})\|\lfloor_\Omega(B_{1+2R}(x_0))
\end{aligned}
$$

by (7.34) and $G_R \subset B_{1+R}(x_0)$. In addition, by (4.31) and the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
(7.37)\quad & \|\Phi_{\varepsilon_{j_l}} * \delta(\partial\mathcal{E}_{j_l})\|\lfloor_\Omega(B_{1+2R}(x_0)) \\
&= \int_{B_{1+2R}(x_0)} \Omega|\Phi_{\varepsilon_{j_l}} * \delta(\partial\mathcal{E}_{j_l})|\,dx \\
&\leqslant \left(\int_{\mathbb{R}^{n+1}} \frac{\Omega|\Phi_{\varepsilon_{j_l}} * \delta(\partial\mathcal{E}_{j_l})|^2}{\Phi_{\varepsilon_{j_l}} * \|\partial\mathcal{E}_{j_l}\| + \varepsilon_{j_l}\Omega^{-1}}\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{B_{1+2R}(x_0)} \Omega(\Phi_{\varepsilon_{j_l}} * \|\partial\mathcal{E}_{j_l}\| + \varepsilon_{j_l}\Omega^{-1})\right)^{\frac{1}{2}} \\
&\leqslant M^{\frac{1}{2}}(M + c(n)\varepsilon_{j_l})^{\frac{1}{2}}.
\end{aligned}
$$

(7.36) and (7.37) prove that, for all fixed $0 < R < 1$,

$$
(7.38)\qquad \limsup_{l\to\infty}\|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(G_{R,j_l,1}) \leqslant 8R\exp(4c_1R)\mathbf{B}_{n+1}M.
$$

Next, suppose that $x \in G_{R,j_l,2}$. From (7.30) and (7.33) (where r_x may be replaced by $(2j_l^2)^{-1}$ for the same reason), we have

$$
(7.39)\qquad (2j_l^2)^n\|\partial\mathcal{E}_{j_l}\|(B_{\frac{1}{2j_l^2}}(x)) \leqslant c_3.
$$

Note that $x \in \operatorname{spt}\|\partial\mathcal{E}_{j_l}\|$. Then, Proposition 7.2 shows the existence of $r_x \in [\frac{1}{4j_l^2}, \frac{1}{2j_l^2}]$ and a $\mathcal{E}_{j_l}$-admissible function f_x such that

(i) $f_x(y) = y$ for $y \in \mathbb{R}^{n+1} \setminus U_{r_x}(x)$,
(ii) $f_x(y) \in B_{r_x}(x)$ for $y \in B_{r_x}(x)$,
(iii) $\|\partial(f_x)_\star\mathcal{E}_{j_l}\|(B_{r_x}(x)) \leqslant \frac{1}{2}\|\partial\mathcal{E}_{j_l}\|(B_{r_x}(x))$,
(iv) $\mathcal{L}^{n+1}(E_i \triangle \tilde{E}_{x,i}) \leqslant c_4(\|\partial\mathcal{E}_{j_l}\|(B_{r_x}(x)))^{\frac{n+1}{n}}$ for all $i = 1, \dots, N$,

where $\{E_i\}_{i=1}^N = \mathcal{E}_{j_l}$ and $\{\tilde{E}_{x,i}\}_{i=1}^N = (f_x)_\star\mathcal{E}_{j_l}$. By (3.2), (iii) may be replaced by

$$
(7.40)\qquad \|\partial(f_x)_\star\mathcal{E}_{j_l}\|\lfloor_\Omega(B_{r_x}(x)) \leqslant 2^{-\frac{1}{2}}\|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(B_{r_x}(x))
$$

for all sufficiently large l depending only on c_1. Applying the Besicovitch covering theorem to the family $\{B_{r_x}(x)\}_{x\in G_{R,j_l,2}}$, we have a finite set $\{x_k\}_{k=1}^{\Lambda}$ such that $\{B_{r_{x_k}}(x_k)\}_{k=1}^{\Lambda}$ is mutually disjoint and (writing $B_{r_{x_k}}(x_k)$ as $B(k)$)

$$\|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(\cup_{k=1}^{\Lambda}B(k)) \geqslant \mathbf{B}_{n+1}^{-1}\|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(G_{R,j_l,2}). \tag{7.41}$$

Note that the finiteness of Λ follows from $r_x \geqslant \frac{1}{4j_l^2}$ and $G_R \subset B_{1+R}(x_0)$. With this choice, define $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$f(x) := \begin{cases} f_{x_k}(x) & \text{if } x \in B(k) \text{ for some } k \in \{1,\dots,\Lambda\}, \\ x & \text{otherwise.} \end{cases} \tag{7.42}$$

Since f_{x_k} is $\mathcal{E}_{j_l}$-admissible, due to the disjointness of $\{B(k)\}_{k=1}^{\Lambda}$, so is f. In addition, f belongs to $\mathbf{E}(\mathcal{E}_{j_l}, j_l)$. For this, we need to check the conditions of Definition 4.8 (a)-(c). (a) is satisfied since $\max|f(x)-x| \leqslant \max(\operatorname{diam} B(k)) \leqslant \frac{1}{j_l^2}$. For (b), write $f_\star\mathcal{E}_{j_l} =: \{\tilde{E}_i\}_{i=1}^N$. Then we have $E_i\triangle\tilde{E}_i = \cup_{k=1}^{\Lambda}E_i\triangle\tilde{E}_{x_k,i}$ and (iv) and (7.39) give

$$\begin{aligned}\mathcal{L}^{n+1}(E_i\triangle\tilde{E}_i) &\leqslant c_4\sum_{k=1}^{\Lambda}(\|\partial\mathcal{E}_{j_l}\|(B(k)))^{\frac{n+1}{n}} \\ &\leqslant \frac{c_4 c_3^{\frac{1}{n}}}{2j_l^2}\|\partial\mathcal{E}_{j_l}\|(\cup_{k=1}^{\Lambda}B(k)) \\ &\leqslant c(n)(\min_{B_3(x_0)}\Omega)^{-1}\frac{M}{j_l^2}.\end{aligned} \tag{7.43}$$

Thus, for all sufficiently large l, we have (b). For (c), using $\operatorname{diam} B(k) \leqslant 1/j_l^2$ and arguing as in the proof of Lemma 4.12 with (iii), we may prove

$$\begin{aligned}&\|\partial f_\star\mathcal{E}_{j_l}\|(\phi) - \|\partial\mathcal{E}_{j_l}\|(\phi) \\ &\qquad = \sum_{k=1}^{\Lambda}(\|\partial(f_{x_k})_\star\mathcal{E}_{j_l}\|\lfloor_\phi(B(k)) - \|\partial\mathcal{E}_{j_l}\|\lfloor_\phi(B(k))) \leqslant 0\end{aligned} \tag{7.44}$$

for $\phi \in \mathcal{A}_{j_l}$ for all sufficiently large l. Thus we proved $f \in \mathbf{E}(\mathcal{E}_{j_l}, j_l)$. By (4.11), (7.40) and (7.41), then, we have

$$\begin{aligned}(7.45)\quad \Delta_{j_l}\|\partial\mathcal{E}_{j_l}\|(\Omega) &\leqslant \|\partial f_\star\mathcal{E}_{j_l}\|(\Omega) - \|\partial\mathcal{E}_{j_l}\|(\Omega)\\ &= \sum_{k=1}^{\Lambda}(\|\partial(f_{x_k})_\star\mathcal{E}_{j_l}\|\lfloor_\Omega(B(k)) - \|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(B(k)))\\ &\leqslant (2^{-\frac{1}{2}}-1)\sum_{k=1}^{\Lambda}\|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(B(k))\\ &\leqslant (2^{-\frac{1}{2}}-1)\mathbf{B}_{n+1}^{-1}\|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(G_{R,j_l,2}).\end{aligned}$$

(7.45) and (4) prove

$$(7.46)\qquad \lim_{l\to\infty}\|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(G_{R,j_l,2}) = 0,$$

and (7.31), (7.38) and (7.46) prove

$$(7.47)\qquad \limsup_{l\to\infty}\|\partial\mathcal{E}_{j_l}\|\lfloor_\Omega(G_R) \leqslant 8R\exp(4c_1R)\mathbf{B}_{n+1}M.$$

Recalling (7.27), (7.47) proves $\lim_{R\to0}\|V\|\lfloor_\Omega(F_R) = 0$, which proves (7.17). From Proposition 5.6, $\|\delta V\|$ is a Radon measure and applying Allard's rectifiability theorem [1, §5.5(1)], V is rectifiable. □

8. Integrality theorem

In the following, we write $T \in \mathbf{G}(n+1,n)$ as the subspace corresponding to $\{x_{n+1} = 0\}$ and $T^\perp \in \mathbf{G}(n+1,1)$ as the orthogonal complement $\{x_1 = \cdots = x_n = 0\}$. As usual, they are identified with the $(n+1)\times(n+1)$ matrices representing the orthogonal projections to these subspaces. Given a set $Y \subset T^\perp$ and $r_1, r_2 \in (0,\infty)$, define a closed set

$$(8.1)\qquad E(r_1,r_2) := \{x \in \mathbb{R}^{n+1} : |T(x)| \leqslant r_1,\ \mathrm{dist}\,(T^\perp(x), Y) \leqslant r_2\}.$$

Lemma 8.1 ([8, §4.20]). — *Corresponding to $n,\nu \in \mathbb{N}$, $\alpha \in (0,1)$ and $\zeta \in (0,1)$, there exist $\gamma \in (0,1)$ and $j_0 \in \mathbb{N}$ with the following property. Assume*

(1) $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}_\Omega^N$, $j \in \mathbb{N}$ *with* $j \geqslant j_0$, $R \in (0, \frac{1}{2}j^{-2})$, $\rho \in (0, \frac{1}{2}j^{-2})$,
(2) $\rho \geqslant \alpha R$,
(3) $Y \subset T^\perp$ *has no more than* ν *elements,* $\mathrm{diam}\, Y < j^{-2}$ *and* $\theta^n(\|\partial\mathcal{E}\|, y) = 1$ *for all* $y \in Y$*, and writing* $E^*(r) := E(r, (1+R^{-1}r)\rho)$ *for short, assume further that*

(4) $\int_{\mathbf{G}_n(E^*(r))} \|S - T\| \, d(\partial\mathcal{E})(x, S) \leqslant \gamma \|\partial\mathcal{E}\|(E^*(r))$ *for all* $r \in (0, R)$,
(5) $\Delta_j \|\partial\mathcal{E}\|(E^*(r)) \geqslant -\gamma \|\partial\mathcal{E}\|(E^*(r))$ *for all* $r \in (0, R)$.

Then we have

$$\|\partial\mathcal{E}\|(E(R, 2\rho)) \geqslant (\mathcal{H}^0(Y) - \zeta)\omega_n R^n. \tag{8.2}$$

Remark 8.2. — We note that conditions (3), (4) and (5) are different from Brakke's. The differences are essential to complete the proof of integrality.

Proof. — We may assume that

$$\mathcal{H}^0(Y) = \nu \tag{8.3}$$

since the lesser cases $\mathcal{H}^0(Y) \in \{1, \ldots, \nu - 1\}$ can be equally proved and we may simply choose the smallest γ and the largest j_0 among them. We choose and fix a large $j_0 \in \mathbb{N}$ so that

$$\left(\nu - 2^{-1}(1 + \zeta)\right)(\nu - \zeta)^{-1} < \exp(-4j_0^{-1}) \tag{8.4}$$

which depends only on ν and ζ. In the following, we assume that $\mathcal{E}$, j, R, ρ and Y satisfy (1)–(5). Next we set

$$r_1 := \inf\{\lambda > 0 : \|\partial\mathcal{E}\|(E(\lambda, (1 + R^{-1}\lambda)\rho)) \leqslant (\nu - \zeta)\omega_n \lambda^n\}. \tag{8.5}$$

Since $\cup_{y \in Y} U_\lambda(y) \subset E(\lambda, (1 + R^{-1}\lambda)\rho)$ for $\lambda < \rho$,

$$\liminf_{\lambda \to 0} (\omega_n \lambda^n)^{-1} \|\partial\mathcal{E}\|(E(\lambda, (1 + R^{-1}\lambda)\rho)) \geqslant \sum_{y \in Y} \theta^n(\|\partial\mathcal{E}\|, y) = \nu \tag{8.6}$$

by (3) and (8.3). Thus, (8.6) shows $r_1 > 0$. If $r_1 \geqslant R$, then, we would have the opposite inequality in (8.5) for all $\lambda < R$. By letting $\lambda \nearrow R$, we would obtain (8.2). In the following, we assume that $r_1 < R$, and look for a contradiction to (5), with an appropriate choice of γ. For the repeated use, we define

$$\rho_1 := (1 + R^{-1}r_1)\rho \tag{8.7}$$

and note that

$$\|\partial\mathcal{E}\|(E(r_1, \rho_1)) = (\nu - \zeta)\omega_n r_1^n. \tag{8.8}$$

This is because, considering the inequality for $\lambda < r_1$ and letting $\lambda \nearrow r_1$, we have $\geqslant$. On the other hand, there exists a sequence $\lambda_i \geqslant r_1$ satisfying the inequality in (8.5) and letting $i \to \infty$, we obtain $\leqslant$. Combined with (4) and (5), (8.8) gives

$$\int_{\mathbf{G}_n(E(r_1, \rho_1))} \|S - T\| \, d(\partial\mathcal{E})(x, S) \leqslant \gamma(\nu - \zeta)\omega_n r_1^n \tag{8.9}$$

and

$$\Delta_j \|\partial\mathcal{E}\|(E(r_1,\rho_1)) \geqslant -\gamma(\nu-\zeta)\omega_n r_1^n. \tag{8.10}$$

Next, define

$$V := \partial\mathcal{E}\lfloor_{\mathbf{G}_n(E(r_1,\rho_1))}(= |E(r_1,\rho_1)\cap \cup_{i=1}^N \partial E_i|) \tag{8.11}$$

and consider $T_\sharp V$, the usual push-forward of varifold counting multiplicities. One notes that

$$T_\sharp V(\phi) = \int_T \phi(x,T)\mathcal{H}^0(T^{-1}(x)\cap \left(\cup_{i=1}^N \partial E_i\right)\cap E(r_1,\rho_1))\, d\mathcal{H}^n(x) \tag{8.12}$$

for $\phi \in C_c(\mathbf{G}_n(\mathbb{R}^{n+1}))$ and $\theta^n(\|T_\sharp V\|, x) = \mathcal{H}^0(T^{-1}(x)\cap \left(\cup_{i=1}^N \partial E_i\right)\cap E(r_1,\rho_1))$ for $\mathcal{H}^n$ a.e. $x\in T$. Define

$$A_0 := \{x\in U_{r_1}^n,\ \theta^n(\|T_\sharp V\|,x)\leqslant \nu-1\}. \tag{8.13}$$

For $\mathcal{H}^n$ a.e. $x\in U^n_{r_1}\setminus A_0$, we have $\theta^n(\|T_\sharp V\|,x)\geqslant\nu$. Thus,

$$\begin{aligned}\nu(\omega_n r_1^n - \mathcal{H}^n(A_0)) \leqslant \|T_\sharp V\|(U^n_{r_1}) &= \int_{\mathbf{G}_n(E(r_1,\rho_1))} |\Lambda_n T\circ S|\, dV(x,S) \\ &\leqslant \|V\|(E(r_1,\rho_1)) = (\nu-\zeta)\omega_n r_1^n,\end{aligned} \tag{8.14}$$

where we used (8.8) and (8.11) in the last line. By (8.14) we obtain

$$\mathcal{H}^n(A_0)\geqslant \nu^{-1}\zeta\omega_n r_1^n. \tag{8.15}$$

We next set

$$\eta := \frac{1-\zeta}{8} \tag{8.16}$$

in the following. We then choose $s\in(0,1/2)$ depending only on ν,ζ and n so that $\mathcal{H}^n(U_1^n\setminus U^n_{1-2s})\leqslant \eta(2\nu)^{-1}\zeta\omega_n$. This implies from (8.15) that

$$\mathcal{H}^n(A_0\cap U^n_{r_1(1-2s)})\geqslant (1-2^{-1}\eta)\nu^{-1}\zeta\omega_n r_1^n. \tag{8.17}$$

We then claim that there exist

$$\delta\in(0,sr_1) \text{ and } A\subset A_0 \tag{8.18}$$

such that

$$A\subset U^n_{r_1(1-2s)} \text{ and } \mathcal{H}^n(A)\geqslant (1-\eta)\nu^{-1}\zeta\omega_n r_1^n, \tag{8.19}$$

and for each $a\in A$, we have

$$\int_{\mathbf{G}_n(C(T,a,\delta))} |\Lambda_n T\circ S|\, dV(x,S)\leqslant (\nu-1+\eta)\omega_n\delta^n \tag{8.20}$$

and

$$\|V\|(C(T,a,\delta))\leqslant \eta\omega_n\delta^{n-1}r_1. \tag{8.21}$$

Here, $C(T,a,\delta) := \{x \in \mathbb{R}^{n+1} : |T(x)-a| \leqslant \delta\}$. The reason for the existence of A and δ is as follows. Since $\theta^n(\|T_\sharp V\|,\cdot) \leqslant \nu - 1$ on A_0, we have

$$\lim_{r\to 0} \frac{1}{\omega_n r^n} \int_{B_r^n(x)} \theta^n(\|T_\sharp V\|, y)\, d\mathcal{H}^n(y) \leqslant \nu - 1 \tag{8.22}$$

for a.e. $x \in A_0$ by the Lebesgue theorem. On the other hand,

$$\begin{aligned}\int_{B_r^n(x)} \theta^n(\|T_\sharp V\|, y)\, d\mathcal{H}^n(y) &= \|T_\sharp V\|(B_r^n(x)) \\ &= \int_{\mathbf{G}_n(C(T,x,r))} |\Lambda_n T \circ S|\, dV(y,S).\end{aligned} \tag{8.23}$$

Combining (8.22) and (8.23), one may argue that for sufficiently small δ, (8.20) is satisfied for a set in A_0 whose complement can be arbitrarily small in measure. For (8.21), define $A_{0,\delta} := \{a \in A_0 : \|V\|(C(T,a,\delta)) \geqslant \eta\omega_n\delta^{n-1}r_1\}$. By the Besicovitch covering theorem, there exists a disjoint family $\{B_\delta^n(x_i)\}_{i=1}^m$ such that

$$\begin{aligned}\mathcal{H}^n(A_{0,\delta}) \leqslant \mathbf{B}_n m\omega_n\delta^n &\leqslant \mathbf{B}_n\delta(\eta r_1)^{-1}\sum_{i=1}^m \|V\|(C(T,x_i,\delta)) \\ &\leqslant \mathbf{B}_n\delta(\eta r_1)^{-1}(\nu-\zeta)\omega_n r_1^n,\end{aligned} \tag{8.24}$$

where we also used (8.8) and (8.11). Thus (8.24) shows that we may choose δ sufficiently small so that the measure of $A_{0,\delta}$ is small. On the complement of $A_{0,\delta}$, we have (8.21). Comparing (8.15), (8.17) and (8.19), we may thus choose δ and $A \subset A_0$ so that (8.19)–(8.21) are satisfied. We should emphasize that the choice of s is solely determined by ζ, ν and n while δ may depend additionally on other quantities.

Let $\xi \in (0, \frac{\rho_1 r_1}{R})$ be arbitrary and for each $a \in A$, define

$$a^* := \frac{r_1 a}{r_1 - \delta}, \tag{8.25}$$

$$E_1(a) := \{x \in C(T,a,\delta) : |T(x) - a^*| \leqslant 2\delta\xi^{-1}(\rho_1 - \operatorname{dist}(T^\perp(x), Y))\}, \tag{8.26}$$

$$\begin{aligned}E_2(a) := \{x \in C(T,0,r_1) \setminus E_1(a) : |T(x) - a^*| \\ \leqslant 2r_1\xi^{-1}(\rho_1 - \operatorname{dist}(T^\perp(x), Y))\}.\end{aligned} \tag{8.27}$$

We give a few remarks on the definitions (8.25)–(8.27). We have

$$|a - a^*| = \frac{\delta}{r_1-\delta}|a| < \frac{\delta r_1}{r_1 - \delta}(1-2s) < \delta < r_1 s \tag{8.28}$$

by $a \in A$, (8.18) and (8.19), so in particular

$$a^* \in U^n_{r_1(1-s)} \cap U^n_\delta(a). \tag{8.29}$$

The choice of a^* is made so that the radial expansion centered at $T^{-1}(a^*)$ by the factor of r_1/δ maps $E_1(a)$ to $E_1(a)\cup E_2(a)$ one-to-one. More precisely, let $F_a:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ be defined by

$$F_a(x) := T^{\perp}(x) + \frac{r_1}{\delta}(T(x)-a^*) + a^*. \tag{8.30}$$

Then, one can check that $|T(x)-a|\leqslant\delta$ if and only if $|T(F_a(x))|\leqslant r_1$ using (8.25). The latter conditions involving $|T(x)-a^*|$ on $E_1(a)$ and $E_2(a)$ are also equivalent for x and $F_a(x)$. Thus we have a one-to-one correspondence between $E_1(a)$ and $E_1(a)\cup E_2(a)$ by F_a, i.e.,

$$F_a(E_1(a)) = E_1(a)\cup E_2(a). \tag{8.31}$$

By the definition of $E(r_1,\rho_1)$, one can check that $E_i(a)\subset E(r_1,\rho_1)$ for $i=1,2$.

With these sets defined, let $f_a:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ be a Lipschitz map such that $f_a(x)=x$ on $\mathbb{R}^{n+1}\setminus(E_1(a)\cup E_2(a))$, $f_a\lfloor_{E_1(a)}= F_a\lfloor_{E_1(a)}$, and f_a radially projects $E_2(a)$ onto $\partial(E_1(a)\cup E_2(a))$ from $T^{-1}(a^*)$. By (8.31), f_a expands $E_1(a)$ to $E_1(a)\cup E_2(a)$ and "crushes" $E_2(a)$ to the boundary $\partial(E_1(a)\cup E_2(a))$. It is not difficult to check that f_a is $\mathcal{E}$-admissible. Write $\tilde{E}_i:=\mathrm{int}\,(f_a(E_i))$. We need to check (a)–(c) of Definition 4.3. (c) is trivial. (a) follows from the bijective nature between $E_1(a)$ and $E_1(a)\cup E_2(a)$. For (b), suppose $x\in\partial(E_1(a)\cup E_2(a))\setminus\cup_{i=1}^N\tilde{E}_i$. If $x\in\partial E_i$ for some i, then $x\in f_a(\partial E_i)$ since f_a is identity there. If $x\notin\partial E_i$ for all i, then there exists some i such that $x\in E_i$ due to (4.1). $f_a^{-1}(x)$ is a closed line segment or a point. If this set is all included in E_i, then, we would have $x\in\mathrm{int}(f_a(E_i))=\tilde{E}_i$, a contradiction. Thus there is some $y\in\partial E_i\cap f_a^{-1}(x)$ and this shows $x\in f_a(\partial E_i)$. Other case when $x\notin\partial(E_1(a)\cup E_2(a))$ is easily handled to conclude that (b) holds. Thus f_a is $\mathcal{E}$-admissible.

To separate $E_2(a)$ into two parts, we next define

$$E_3(a) := \{x\in E_2(a) : f_a(x)\in\partial C(T,0,r_1)\}, \tag{8.32}$$

$$E_4(a) := E_2(a)\setminus E_3(a). \tag{8.33}$$

Note that $\partial(E_1(a)\cup E_2(a))$ consists of the sets in a cylinder $\partial C(T,0,r_1)$ and cones of type $\{x : |T(x)-a^*| = 2r_1\xi^{-1}(\rho_1-\mathrm{dist}\,(T^{\perp}(x),Y))\}$ (see Figure 8.1 for $n=1$). The set $E_3(a)$ thus is the one mapped to the cylinder by f_a and $E_4(a)$ is the one to the cones.

We note that

$$E_4(a)\subset\{x\in E_2(a) : \mathrm{dist}\,(T^{\perp}(x),Y)\geqslant\rho_1-\xi\} \tag{8.34}$$

and

$$E(r_1,\rho_1-\xi)\subset E_1(a)\cup E_2(a). \tag{8.35}$$

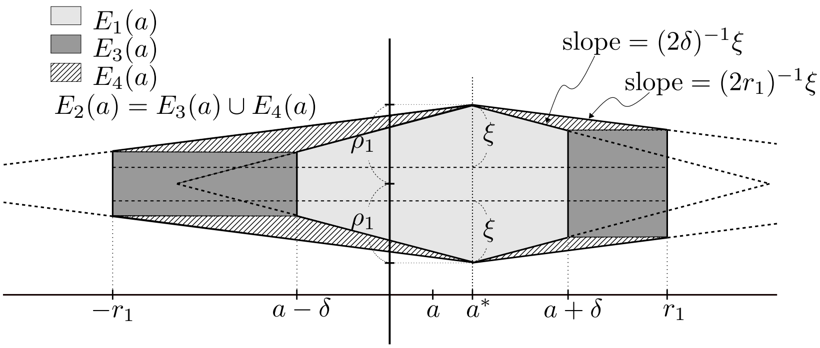

Figure 8.1.

To see these, for $x \in E_4(a)$, since $f_a(x)$ is a point on the cone, we have $|T(f_a(x)) - a^*| = 2r_1\xi^{-1}(\rho_1 - \operatorname{dist}(T^\perp(f_a(x)), Y))$. Since $f_a(x), a^* \in C(T,0,r_1)$, $|T(f_a(x)) - a^*| \leqslant 2r_1$. By the definition of f_a, we have $T^\perp(f_a(x)) = T^\perp(x)$. These considerations show (8.34). If $x \in E(r_1, \rho_1 - \xi)$, by (8.1), $|T(x)| \leqslant r_1$ and $\operatorname{dist}(T^\perp(x), Y) \leqslant \rho_1 - \xi$. Then we have $|T(x) - a^*| \leqslant 2r_1 \leqslant 2r_1\xi^{-1}(\rho_1 - \operatorname{dist}(T^\perp(x), Y))$ and (8.35) follows.

For a given $x \in E_1(a) \cup E_2(a)$, let $v_1, \dots, v_{n+1}$ be a set of orthonormal vectors such that $v_1 = \frac{T(x)-a^*}{|T(x)-a^*|}$, $v_2, \dots, v_n \in T$ and $v_{n+1} \in T^\perp$. Direct computations show

$$\nabla_{v_i} f_a(x) = \frac{r_1}{\delta} v_i \text{ if } 1 \leqslant i \leqslant n \text{ and } \nabla_{v_{n+1}} f_a(x) = v_{n+1} \text{ on } E_1(a), \tag{8.36}$$

$$\nabla_{v_1} f_a(x) = 0 \text{ on } E_2(a), \tag{8.37}$$

$$\nabla_{v_{n+1}} f_a(x) = v_{n+1} \text{ on } E_3(a), \tag{8.38}$$

$$\nabla_{v_i} f_a(x) \in T \text{ and } |\nabla_{v_i} f_a(x)| \leqslant \frac{4r_1}{|T(x) - a^*|\sqrt{s}} \quad \text{if } 2 \leqslant i \leqslant n \text{ on } E_3(a), \tag{8.39}$$

$$\nabla_{v_{n+1}} f_a(x) = v_{n+1} \pm 2r_1\xi^{-1} v_1 \text{ on } E_4(a), \tag{8.40}$$

$$\nabla_{v_i} f_a(x) \parallel v_i \text{ and } |\nabla_{v_i} f_a(x)| \leqslant \frac{2r_1}{|T(x) - a^*|} \text{ if } 2 \leqslant i \leqslant n \text{ on } E_4(a). \tag{8.41}$$

Above computations are all valid whenever $\nabla_{v_i} f_a(x)$ is defined. On $E_1(a)$, (8.30) gives (8.36). On $E_2(a)$, since f_a is a radial projection in the direction of v_1 to $\partial(E_1(a) \cup E_2(a))$, we have (8.37).

For $x \in E_3(a)$ more precisely, f_a is a radial projection from $T^{-1}(a^*)$ of $C(T,0,r_1) \setminus C(T,a,\delta)$ to $\partial C(T,0,r_1)$. Thus, it is clear that we have (8.38). One may express the formula of f_a implicitly by introducing a "stretching factor” $t = t(x) > 0$ as

$$f_a(x) = T^\perp(x) + t(T(x) - a^*) + a^*, \ |t(T(x) - a^*) + a^*|^2 = r_1^2. \tag{8.42}$$

Differentiating both identities of (8.42) with respect to v_i ($i = 2, \dots, n$), we obtain

$$\nabla_{v_i} f_a(x) = \nabla_{v_i} t(T(x) - a^*) + tv_i, \ f_a(x) \cdot (\nabla_{v_i} t(T(x) - a^*) + tv_i) = 0$$

and

$$\nabla_{v_i} f_a(x) = tv_i - t\frac{f_a(x)\cdot v_i}{f_a(x)\cdot(T(x) - a^*)}(T(x) - a^*) = tv_i - t\frac{f_a(x)\cdot v_i}{f_a(x)\cdot v_1}v_1. \tag{8.43}$$

We need a lower bound of $|f_a(x) \cdot v_1|$ to estimate (8.43). From (8.42) and by the definition of $v_2, \dots, v_n$, we have $f_a(x) \cdot v_i = a^* \cdot v_i$ for $i = 2, \dots, n$. Then, we have

$$\begin{aligned} |f_a(x) \cdot v_1|^2 = |T(f_a(x))|^2 - \sum_{i=2}^n |T(f_a(x)) \cdot v_i|^2 &= r_1^2 - |a^*|^2 \\ &\geqslant r_1^2 - (1 - s)^2 r_1^2 \end{aligned} \tag{8.44}$$

where we used $|T(f_a(x))| = r_1$ and $|a^*| < r_1(1 - s)$ from (8.29). Thus we have from (8.43) and (8.44) that

$$|\nabla_{v_i} f_a(x)| \leqslant t\left(1 + \frac{1}{\sqrt{2s - s^2}}\right) \leqslant \frac{4r_1}{|T(x) - a^*|\sqrt{s}}. \tag{8.45}$$

The last inequality is due to $|t(T(x) - a^*)| \leqslant |t(T(x) - a^*) + a^*| + |a^*| \leqslant 2r_1$ and $s < 1/2$. Combined with the expression of (8.43), this proves (8.39).

For $x \in E_4(a)$, one can check that

$$f_a(x) = T^\perp(x) + \frac{T(x) - a^*}{|T(x) - a^*|} 2r_1\xi^{-1}(\rho_1 - \operatorname{dist}(T^\perp(x), Y)) + a^*. \tag{8.46}$$

We have $\nabla_{v_{n+1}} f_a(x) = v_{n+1} \pm \frac{T(x) - a^*}{|T(x) - a^*|} 2r_1\xi^{-1}$, which gives (8.40). For $i = 2, \dots, n$, we have $\nabla_{v_i} f_a(x) = \frac{2r_1\xi^{-1}(\rho_1 - \operatorname{dist}(T^\perp(x), Y))}{|T(x) - a^*|} v_i$ since $T(x) - a^* \parallel v_1$ and $v_1 \perp v_i$. Using (8.34), we obtain (8.41).

We next need to compute the Jacobian $|\Lambda_n \nabla f_a(x) \circ S|$ for arbitrary $S \in \mathbf{G}(n + 1, n)$ to compute $\|(f_a)_\sharp V\|$. As we will check, we may estimate as

$$|\Lambda_n \nabla f_a(x) \circ S| \leqslant \left(\frac{r_1}{\delta}\right)^n |\Lambda_n T \circ S| + \left(\frac{r_1}{\delta}\right)^{n-1} \text{ on } E_1(a), \tag{8.47}$$

$$|\Lambda_n \nabla f_a(x) \circ S| \leqslant \|S - T\| \left(\frac{4r_1}{|T(x) - a^*|\sqrt{s}}\right)^{n-1} \text{ on } E_3(a), \tag{8.48}$$

$$|\Lambda_n \nabla f_a(x) \circ S| \leqslant \|S - T\| \frac{\sqrt{4r_1^2 + \xi^2}}{\xi} \left(\frac{2r_1}{|T(x) - a^*|}\right)^{n-1} \text{ on } E_4(a). \tag{8.49}$$

To see (8.47)–(8.49), after an orthogonal rotation, we may consider that $v_1, \dots, v_{n+1}$ are parallel to coordinate axis of $x_1, \dots, x_{n+1}$, respectively, and let $u_1 = (u_{1,1}, \dots, u_{n+1,1})^\top$, ..., $u_{n+1} = (u_{1,n+1}, \dots, u_{n+1,n+1})^\top$ be a set of orthonormal vectors such that $u_1, \dots, u_n$ span S and $u_{n+1} \in S^\perp$. Then, $|\Lambda_n \nabla f_a(x) \circ S|$ is the volume of n-dimensional parallelepiped formed by $\nabla f_a(x) \circ u_1, \dots, \nabla f_a(x) \circ u_n \in \mathbb{R}^{n+1}$. Let $L = (L_{i,j})$ be the $(n+1) \times n$ matrix whose column vectors are formed by $\nabla f_a(x) \circ u_1, \dots, \nabla f_a(x) \circ u_n$. Then we have by the Binet–Cauchy formula ([18, Theorem 3.7])

$$|\Lambda_n \nabla f_a(x) \circ S|^2 = \det(L^\top \circ L) = \sum_{l=1}^{n+1} (\det[(L_{i,j})_{i \neq l, 1 \leqslant j \leqslant n}])^2. \tag{8.50}$$

Computation for (8.47). — On $E_1(a)$, due to (8.36), $\nabla f_a(x)$ is the $(n+1) \times (n+1)$ diagonal matrix whose first n diagonal elements are all r_1/δ and whose last diagonal element is 1. Then, the minor formed by eliminating the last row of L is $(u_{i,j})_{1 \leqslant i,j \leqslant n}$ times r_1/δ, and its determinant is $(r_1/\delta)^n$ times determinant of $(u_{i,j})_{1 \leqslant i,j \leqslant n}$. Note that the determinant of $(u_{i,j})_{1 \leqslant i,j \leqslant n}$ is precisely $|\Lambda_n T \circ S|$ since T now is the diagonal matrix whose first n diagonal elements are all 1 and whose $n+1$-th diagonal element is 0. For a minor formed by eliminating the l-th row of L, $1 \leqslant l \leqslant n$, the determinant is $(r_1/\delta)^{n-1}$ times the determinant of $(u_{i,j})_{i \neq l, 1 \leqslant j \leqslant n}$. Considering the orthogonality of the matrix $(u_{i,j})_{1 \leqslant i,j \leqslant n+1}$ and the formula for the inverse matrix, the determinant is given by $(-1)^{l+n+1} u_{l,n+1}$. Thus, from (8.50), we have

$$|\Lambda_n \nabla f_a(x) \circ S|^2 = \left(\frac{r_1}{\delta}\right)^{2n} |\Lambda_n T \circ S|^2 + \left(\frac{r_1}{\delta}\right)^{2(n-1)} \sum_{l=1}^{n} (u_{l,n+1})^2. \tag{8.51}$$

Since $|u_{n+1}| = 1$, (8.51) gives (8.47).

Computation for (8.48) *and* (8.49). — Here let us write $\nabla f_a(x)$ as ∇f for short and the (i,j)-element of ∇f as $\nabla f_{i,j}$. From (8.37), we have $\nabla f_{i,1} = 0$ for all $1 \leqslant i \leqslant n+1$. Then, from (8.50), we have

$$\begin{aligned} |\Lambda_n \nabla f \circ S|^2 &= \det\left[(u_1, \dots, u_n)^\top \circ (\nabla f)^\top \circ \nabla f \circ (u_1, \dots, u_n)\right] \\ &= (\det\left[(u_{i,j})_{2 \leqslant i \leqslant n+1, 1 \leqslant j \leqslant n}\right])^2 \det\left[((\nabla f)^\top \circ \nabla f)_{2 \leqslant i,j \leqslant n+1}\right]. \end{aligned} \tag{8.52}$$

By the orthogonality again, we have $\det\left[(u_{i,j})_{2 \leqslant i \leqslant n+1, 1 \leqslant j \leqslant n}\right] = (-1)^n u_{1,n+1}$. Note that $|u_{1,n+1}| \leqslant \left(\sum_{i=1}^n (u_{i,n+1})^2\right)^{\frac{1}{2}} \leqslant |(T-S) \circ u_{n+1}|$, so that $|u_{1,n+1}| \leqslant \|T-S\|$. Also, considering the fact that $\det\left[((\nabla f)^\top \circ \nabla f)_{2 \leqslant i,j \leqslant n+1}\right]$ is the square of n-dimensional volume of parallelepiped formed by vectors $(\nabla f_{1,j},\dots,\nabla f_{n+1,j})^\top$, $j=2,\dots,n+1$, it is bounded by $\prod_{j=2}^{n+1}|\nabla_{v_j} f|^2$. These considerations combined with (8.52), (8.38) and (8.39) give (8.48). Similarly using (8.40), (8.41) and (8.52), we obtain (8.49).

We next calculate the mass of $(f_a)_\sharp V$. For later use, we note the following. Since $\cup_{i=1}^N \partial\tilde{E}_i \subset f(\cup_{i=1}^N \partial E_i)$ and the varifold push-forward counts the multiplicities of the image, we have

$$\begin{aligned}(8.53)\quad \|\partial(f_a)_\star\mathcal{E}\|(E(r_1,\rho_1)) &= \mathcal{H}^n(\cup_{i=1}^N \partial\tilde{E}_i\cap E(r_1,\rho_1)) \\ &\leqslant \|(f_a)_\sharp V\|(E(r_1,\rho_1)).\end{aligned}$$

Now, using (8.47)–(8.49), we have

$$\begin{aligned}(8.54)\quad &\|(f_a)_\sharp V\|(E(r_1,\rho_1)) \\ &= \int_{\mathbf{G}_n(E(r_1,\rho_1))} |\Lambda_n \nabla f_a(x)\circ S|\, dV(x,S) \\ &\leqslant \int_{\mathbf{G}_n(E_1(a))} r_1^n\delta^{-n}|\Lambda_n T\circ S| + r_1^{n-1}\delta^{1-n}\, dV(x,S) \\ &\quad + \int_{\mathbf{G}_n(E_3(a))} \|S-T\| \left(\frac{4r_1}{|T(x)-a^*|\sqrt{s}}\right)^{n-1} dV(x,S) \\ &\quad + \int_{\mathbf{G}_n(E_4(a))} \|S-T\|\xi^{-1}\sqrt{4r_1^2+\xi^2}\left(\frac{2r_1}{|T(x)-a^*|}\right)^{n-1} dV(x,S) \\ &\quad + \|V\|(E(r_1,\rho_1)\setminus(E_1(a)\cup E_2(a))) \\ &=: I_1+\ldots+I_4.\end{aligned}$$

Since $E_1(a)\subset C(T,a,\delta)$, and by (8.20) and (8.21), we have

$$(8.55)\qquad I_1 \leqslant (\nu-1+\eta)\omega_n r_1^n + \eta\omega_n r_1^n = (\nu-1+2\eta)\omega_n r_1^n.$$

By defining

$$(8.56)\qquad c(r_1\xi^{-1}) := \max\{4^n s^{\frac{1-n}{2}}, 2^n(2r_1\xi^{-1}+1)\},$$

we have

$$(8.57)\quad I_2+I_3 \leqslant c(r_1\xi^{-1})\int_{\mathbf{G}_n(E(r_1,\rho_1))} \|S-T\|\left(\frac{r_1}{|T(x)-a^*|}\right)^{n-1} dV(x,S).$$

By (8.35), we have

$$\begin{aligned}(8.58)\quad I_4 &\leqslant \|V\|(E(r_1,\rho_1)\setminus E(r_1,\rho_1-\xi)) \\ &= \|V\|(E(r_1,\rho_1)) - \|V\|(E(r_1,\rho_1-\xi)).\end{aligned}$$

On the other hand, due to (8.5), we have $\|V\|(E(\lambda,(1+\lambda R^{-1})\rho)) > (\nu-\zeta)\omega_n\lambda^n$ for $\lambda < r_1$. Hence, for $\lambda_* := r_1 - R\xi\rho^{-1}$ which solves $\rho_1 - \xi = (1+\lambda_* R^{-1})\rho$, we have

$$\|V\|(E(r_1,\rho_1-\xi)) \geqslant \|V\|(E(\lambda_*,(1+\lambda_*/R)\rho)) > (\nu-\zeta)\omega_n\lambda_*^n. \tag{8.59}$$

By Bernoulli's inequality, we have $\lambda_*^n = (r_1 - R\xi\rho^{-1})^n \geqslant r_1^n - nr_1^{n-1}R\xi\rho^{-1}$, and (8.58), (8.59) and (8.8) show

$$I_4 \leqslant (\nu-\zeta)\omega_n n r_1^{n-1}R\xi\rho^{-1} \leqslant \nu n\omega_n\alpha^{-1}(\xi r_1^{-1})r_1^n, \tag{8.60}$$

where we used (2) ($\rho \geqslant \alpha R$) in the last inequality. The estimates so far hold for any $a \in A$. To estimate $I_2 + I_3$, we integrate the right-hand side of (8.57) with respect to a. For any fixed $x \in E(r_1,\rho_1)$, by (8.25),

$$\begin{aligned}\int_A \left(\frac{r_1}{|T(x)-a^*|}\right)^{n-1} d\mathcal{H}^n(a) &= \left(\frac{r_1-\delta}{r_1}\right)^{n-1}\int_A\left(\frac{r_1}{|\frac{r_1-\delta}{r_1}T(x)-a|}\right)^{n-1} d\mathcal{H}^n(a)\\ &\leqslant \int_{B^n_{2r_1}}\left(\frac{r_1}{|y|}\right)^{n-1} d\mathcal{H}^n(y) = 2n\omega_n r_1^n\end{aligned} \tag{8.61}$$

after a change of variable $y = \frac{r_1-\delta}{r_1}T(x) - a$ and using $\{y : \frac{r_1-\delta}{r_1}T(x) - y \in A\} \subset B^n_{2r_1}$. Then, by Fubini's theorem and (8.61),

$$\begin{aligned}\int_A d\mathcal{H}^n(a)\int_{\mathbf{G}_n(E(r_1,\rho_1))}\|S-T\|\left(\frac{r_1}{|T(x)-a^*|}\right)^{n-1} dV(x,S)\\ \leqslant 2n\omega_n r_1^n\int_{\mathbf{G}_n(E(r_1,\rho_1))}\|S-T\|\,dV(x,S) \leqslant 2n\omega_n^2\nu r_1^{2n}\gamma\end{aligned} \tag{8.62}$$

where (8.9) is used. By (8.19) and (8.62), there exists $a \in A$ such that we have

$$\begin{aligned}\int_{\mathbf{G}_n(E(r_1,\rho_1))}\|S-T\|\left(\frac{r_1}{|T(x)-a^*|}\right)^{n-1} dV(x,S)\\ \leqslant 2n(1-\eta)^{-1}\nu^2\zeta^{-1}\omega_n\gamma r_1^n.\end{aligned} \tag{8.63}$$

With this choice of a, (8.54), (8.55), (8.57), (8.60) and (8.63) show

$$\begin{aligned}&\|(f_a)_\sharp V\|(E(r_1,\rho_1))\\ &\leqslant \{\nu - 1 + 2\eta + c(r_1\xi^{-1})2n(1-\eta)^{-1}\nu^2\zeta^{-1}\gamma + \nu n\alpha^{-1}\xi r_1^{-1}\}\omega_n r_1^n.\end{aligned} \tag{8.64}$$

Up to this point, $\xi \in (0, \frac{\rho_1 r_1}{R})$ is arbitrary. Fix ξ so that $\nu n\alpha^{-1}\xi r_1^{-1} = \eta$. Since $\rho_1 > \rho$ and $\rho \geqslant \alpha R$, one can check that $\xi \in (0, \rho_1 r_1/R)$. The choice of

ξr_1^{-1} depends only on ν, ζ, n, α. This fixes $c(r_1\xi^{-1})$ in (8.56), and $c(r_1\xi^{-1})$ depends only on ν, ζ, n, α. We then restrict γ so that

$$c(r_1\xi^{-1})2n(1-\eta)^{-1}\nu^2\zeta^{-1}\gamma \leqslant \eta,$$

which again depends only on the same constants. Then we have from (8.64) and (8.16) that

$$\|(f_a)_\sharp V\|(E(r_1,\rho_1)) \leqslant (\nu - 1 + 4\eta)\omega_n r_1^n = \big(\nu - 1 + 2^{-1}(1-\zeta)\big)\,\omega_n r_1^n. \tag{8.65}$$

We next check that $f_a \in \mathbf{E}(\mathcal{E}, E(r_1,\rho_1), j)$ by using Lemma 4.12. We have already seen that f_a is $\mathcal{E}$-admissible. We may take $C = E(r_1,\rho_1)$ in Lemma 4.12 and (a) is satisfied. Since $T^\perp(f_a(x)-x) = 0$, $f_a(x) \in C(T,0,r_1)$ for $x \in E(r_1,\rho_1)$ and $r_1 < R < \frac{1}{2}j^{-2}$ (by (1)), we have $|f_a(x)-x| \leqslant 2r_1 < j^{-2}$ so we have (b) satisfied. For (c), we have $\tilde{E}_i \triangle E_i \subset E(r_1,\rho_1)$ and due to (1) and (3), $\operatorname{diam} E(r_1,\rho_1) < 4j^{-2}$ (note (8.7)). Thus for suitably restricted j depending on n, we have (c). For (d), by (8.53), (8.65), (8.8) and (8.4) we have

$$\|\partial (f_a)_\star \mathcal{E}\|(E(r_1,\rho_1)) \leqslant \exp(-4j_0^{-1})\|\partial\mathcal{E}\|(E(r_1,\rho_1)). \tag{8.66}$$

Since $\operatorname{diam} E(r_1,\rho_1) < 4j^{-2}$, we have (d), and $f_a \in \mathbf{E}(\mathcal{E}, E(r_1,\rho_1), j)$. Finally, consider $\Delta_j\|\partial\mathcal{E}\|(E(r_1,\rho_1))$. By (8.10), (4.13), (8.8) and (8.65), we have

$$\begin{aligned} -\gamma(\nu-\zeta)\omega_n r_1^n &\leqslant \Delta_j\|\partial\mathcal{E}\|(E(r_1,\rho_1)) \\ &\leqslant \|\partial(f_a)_\star\mathcal{E}\|(E(r_1,\rho_1)) - \|\partial\mathcal{E}\|(E(r_1,\rho_1)) \\ &\leqslant -2^{-1}(1-\zeta)\omega_n r_1^n. \end{aligned} \tag{8.67}$$

By restricting γ further depending only on ζ and ν, (8.67) is a contradiction. This concludes the proof. □

For large length scale ($\geqslant j^{-2}$), we use the following.

Lemma 8.3 ([8, §4.21]). — *Suppose*

(1) $\nu \in \mathbb{N}$, $\xi \in (0,1)$, $M \in (1,\infty)$, $0 < r_0 < R < \infty$, $T \in \mathbf{G}(n+1,n)$ *and* $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$,
(2) $Y \subset T^\perp$ *has no more than* $\nu + 1$ *elements,*
(3) $(M+1)\operatorname{diam} Y \leqslant R$,
(4) $r_0 < (3\nu)^{-1}\operatorname{diam} Y$,
(5) $R\|\delta V\|(B_r(y)) \leqslant \xi\|V\|(B_r(y))$ *for all* $y \in Y$ *and* $r \in (r_0, R)$,
(6) $\int_{\mathbf{G}_n(B_r(y))}\|S - T\|\,dV(x,S) \leqslant \xi\|V\|(B_r(y))$ *for all* $y \in Y$ *and* $r \in (r_0, R)$.

Then there are $V_1, V_2 \in \mathbf{V}_n(\mathbb{R}^{n+1})$ *and a partition of* Y *into subsets* Y_0, Y_1, Y_2 *such that*

$$V \geqslant V_1 + V_2, \tag{8.68}$$

$$\text{neither } Y_1 \text{ nor } Y_2 \text{ has more than } \nu \text{ elements}, \tag{8.69}$$

$$\begin{aligned}(M \operatorname{diam} Y)\|\delta V_j\|(B_r(y)) &\leqslant 2M(\nu+1)(3\nu M)^{n+1}(\exp\xi)\xi\|V_j\|(B_r(y)) \\ &\quad \text{for all } y \in Y_j,\ r \in (r_0, M \operatorname{diam} Y) \text{ and } j = 1, 2,\end{aligned} \tag{8.70}$$

$$\begin{aligned}\int_{\mathbf{G}_n(B_r(y))} \|S - T\|\, dV_j(x,S) &\leqslant M(3\nu M)^n(\exp\xi)\xi\|V_j\|(B_r(y)) \\ &\quad \text{for all } y \in Y_j,\ r \in (r_0, M \operatorname{diam} Y) \text{ and } j = 1, 2,\end{aligned} \tag{8.71}$$

$$V_j \geqslant V \lfloor \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(T^\perp(x), Y_j) \leqslant r_0\} \times \mathbf{G}(n+1, n) \quad \text{for } j = 1, 2, \tag{8.72}$$

$$\begin{aligned}&\{(1+1/M)^n + (\nu+1)/M\}(\exp\xi)\frac{\|V\|(\{x : \operatorname{dist}(x, Y) \leqslant R\})}{\omega_n R^n} \\ &\geqslant \sum_{y \in Y_0} \frac{\|V\|(B_{r_0}(y))}{\omega_n r_0^n} + \sum_{j=1,2} \frac{\|V_j\|(\{x : \operatorname{dist}(x, Y_j) \leqslant M \operatorname{diam} Y\})}{\omega_n (M \operatorname{diam} Y)^n}.\end{aligned} \tag{8.73}$$

The proof of Lemma 8.3 is the same as [1, Lemma 6.1] except that $r_0 \to 0$ in [1] while it is stopped at a positive radius r_0 here.

Lemma 8.4. — *Corresponding to* $n, \nu \in \mathbb{N}$ *and* $\lambda \in (1,2)$, *there exist* $\gamma \in (0,1)$ *and* $\tilde{M} \in (1, \infty)$ *with the following property. Suppose*

(1) $0 < r_0 < R < \infty$, $T \in \mathbf{G}(n+1, n)$, $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$,
(2) $Y \subset T^\perp$ *has no more than* $\nu + 1$ *elements,*
(3) $\{(1+3\nu)^2 + \tilde{M}^2\}^{\frac{1}{2}} r_0 < R$,
(4) $\operatorname{diam} Y \leqslant \gamma R$,
(5) $R\|\delta V\|(B_r(y)) \leqslant \gamma\|V\|(B_r(y))$ *for all* $y \in Y$ *and* $r \in (r_0, R)$,
(6) $\int_{\mathbf{G}_n(B_r(y))} \|S - T\|\, dV(x,S) \leqslant \gamma\|V\|(B_r(y))$ *for all* $y \in Y$ *and* $r \in (r_0, R)$.

Then there exists a partition of Y *into subsets* $Y_0, Y_1, \ldots, Y_J$ *such that*

$$\operatorname{diam} Y_j \leqslant 3\nu r_0 \ \text{ for all } j \in \{1, \ldots, J\}, \tag{8.74}$$

$$\begin{aligned}\lambda\frac{\|V\|(\{x : \operatorname{dist}(x, Y) \leqslant R\})}{\omega_n R^n} &\geqslant \sum_{y \in Y_0} \frac{\|V\|(B_{r_0}(y))}{\omega_n r_0^n} \\ &\quad + \sum_{j=1}^{J} \frac{\|V\|(\{x : \operatorname{dist}(T^\perp(x), Y_j) \leqslant r_0,\ |T(x)| \leqslant \tilde{M} r_0\})}{\omega_n (\tilde{M} r_0)^n}.\end{aligned} \tag{8.75}$$

Proof. — We use Lemma 8.3 to partition Y into subsets whose diameters are all smaller than $3\nu r_0$. In the case Y consists of only one element, we may take $Y_0 := Y$ and Lemma 7.1 shows (8.75) by choosing an appropriately small γ in (5) depending only on λ. We do not need $\tilde{M}$ in this case. If Y consists of more than one element, we apply Lemma 8.3. We separate into two cases first.

First inductive step: Case 1. — Suppose (4) of Lemma 8.3 is not satisfied, i.e.,

$$\operatorname{diam} Y \leqslant 3\nu r_0. \tag{8.76}$$

In this case, we set $J = 1$, $Y_1 := Y$ and $Y_0 = \emptyset$. For any $y \in Y$, we have by (8.76)

$$\{x : \operatorname{dist}(T^{\perp}(x), Y_1) \leqslant r_0,\ |T(x)| \leqslant \tilde{M} r_0\} \subset B_{r_0((1+3\nu)^2+\tilde{M}^2)^{\frac{1}{2}}}(y). \tag{8.77}$$

We have

$$\begin{aligned} &\frac{\|V\|(B_{r_0((1+3\nu)^2+\tilde{M}^2)^{\frac{1}{2}}}(y))}{\omega_n (r_0\tilde{M})^n} \\ &\qquad = \frac{\|V\|(B_{r_0((1+3\nu)^2+\tilde{M}^2)^{\frac{1}{2}}}(y))}{\omega_n (r_0((1+3\nu)^2+\tilde{M}^2)^{\frac{1}{2}})^n}\left(1+\frac{(1+3\nu)^2}{\tilde{M}^2}\right)^{\frac{n}{2}}. \end{aligned} \tag{8.78}$$

By Lemma 7.1 with (5), (8.76), (3) and (8.78), we have

$$\begin{aligned} &\frac{\|V\|(B_{r_0((1+3\nu)^2+\tilde{M}^2)^{\frac{1}{2}}}(y))}{\omega_n (r_0\tilde{M})^n} \\ &\qquad \leqslant (\exp\gamma)\left(1+\frac{(1+3\nu)^2}{\tilde{M}^2}\right)^{\frac{n}{2}} \frac{\|V\|(B_R(y))}{\omega_n R^n}. \end{aligned} \tag{8.79}$$

Since $B_R(y) \subset \{x : \operatorname{dist}(x, Y) \leqslant R\}$, combining (8.77), (8.79), we choose large $\tilde{M}$ and small γ depending only on n, ν and λ so that (8.75) is satisfied.

First inductive step : Case 2. — . Suppose (4) of Lemma 8.3 is satisfied. With M satisfying (3) of Lemma 8.3 and $\xi = \gamma$, we apply Lemma 8.3. Thus we have a partition of Y into Y_0, Y_1, Y_2 with (8.68)–(8.73).

Second inductive step for Y_1 and Y_2. — We next proceed just like before for Y_1 and Y_2. That is, for each $j = 1, 2$, if $Y_j = \{y\}$, we use Lemma 7.1 with (8.70) to derive

$$\begin{aligned} &\frac{\|V\|(B_{r_0}(y))}{\omega_n r_0^n} \\ &\qquad \leqslant \exp\{2M(\nu+1)(3\nu M)^{n+1}(\exp\gamma)\gamma\} \frac{\|V_j\|(B_{M \operatorname{diam} Y}(y))}{\omega_n (M \operatorname{diam} Y)^n} \end{aligned} \tag{8.80}$$

where we have also used (8.72). Note that the right-hand side of (8.80) is bounded from above via (8.73). We add this Y_j to Y_0. Suppose Y_j consists of more than one point, and furthermore, (8.76) is satisfied with Y_j in place of Y. Note that (8.72) shows

$$(8.81)\quad \|V\|(\{x : \mathrm{dist}\,(T^\perp(x), Y_j) \leqslant r_0,\ |T(x)| \leqslant \tilde{M} r_0\}) \\ \leqslant \|V_j\|(\{x : \mathrm{dist}\,(T^\perp(x), Y_j) \leqslant r_0,\ |T(x)| \leqslant \tilde{M} r_0\}).$$

We then go through the same argument (8.77)–(8.79) for V_j in place of V and for $M \mathrm{diam}\, Y$ in place of R there. Note that we may apply Lemma 7.1 for V_j due to (8.70). For doing so, we may achieve $r_0((1+3\nu)^2 + \tilde{M}^2)^{\frac{1}{2}} < M\,\mathrm{diam}\, Y$ since $\mathrm{diam}\, Y > 3\nu r_0$ holds and since we may choose M greater than $\tilde{M}$ by a factor depending only on ν. If Y_j does not satisfy (8.76), then we again apply Lemma 8.3 to Y_j to obtain a partition. Since the number of elements of Y_j is strictly decreasing in each step, the process ends at most after ν times. Depending only on n, ν and λ, choose large $\tilde{M}$ and M, and then small γ. Note that we need not take the same M in this inductive step. If we take M in the first step, we may take $M-1$ as M in Lemma 8.3 in the next step so that (3) of Lemma 8.3 is automatically satisfied (since $((M-1)+1)\mathrm{diam}\, Y_1 \leqslant M\,\mathrm{diam}\, Y$, for example). □

Lemma 8.5. — *Corresponding to $n, \nu \in \mathbb{N}$ and $\lambda \in (1,2)$, there exist $\gamma, \eta \in (0,1)$, $\tilde{M} \in (1,\infty)$ and $j_0 \in \mathbb{N}$ with the following property. Suppose*

1. *$\mathcal{E} \in \mathcal{OP}^N_\Omega$, $j \in \mathbb{N}$ with $j \geqslant j_0$,*
2. *$\varepsilon \leqslant \gamma j^{-4}$,*
3. *$\eta j^{-2} < R$,*
4. *$Y \subset T^\perp$ has no more than ν elements and $\theta^n(\|\partial\mathcal{E}\|, y) = 1$ for each $y \in Y$,*
5. *$\mathrm{diam}\, Y \leqslant \gamma R$,*
6. *$R\|\delta(\Phi_\varepsilon * \partial\mathcal{E})\|(B_r(y)) \leqslant \gamma \|\Phi_\varepsilon * \partial\mathcal{E}\|(B_r(y))$ for all $y \in Y$ and $r \in (\eta^2 j^{-2}, R)$,*
7. *$\int_{\mathbf{G}_n(B_r(y))} \|S - T\|\, d(\Phi_\varepsilon * \partial\mathcal{E})(x,S) \leqslant \gamma \|\Phi_\varepsilon * \partial\mathcal{E}\|(B_r(y))$ for all $y \in Y$ and $r \in (\eta^2 j^{-2}, R)$, and writing*
   - (a) $\tilde{R}_1 := \eta^2 j^{-2} \lambda^{-\frac{1}{4n}}$,
   - (b) $\tilde{R}_2 := \tilde{M} \eta^2 j^{-2} \lambda^{-\frac{1}{4n}}$,
   - (c) $\rho := \frac{1}{2}\eta^2 j^{-2}(1 - \lambda^{-\frac{1}{4n}})$,

   and for any subset $Y' \subset Y$, define
   - (d) $E_1^*(r, Y') := \{x \in \mathbb{R}^{n+1} : |T(x)| \leqslant r, \mathrm{dist}(Y', T^\perp(x)) \leqslant (1 + \tilde{R}_1^{-1} r)\rho\}$,
   - (e) $E_2^*(r, Y') := \{x \in \mathbb{R}^{n+1} : |T(x)| \leqslant r, \mathrm{dist}(Y', T^\perp(x)) \leqslant (1 + \tilde{R}_2^{-1} r)\rho\}$,

and assume for all $Y' \subset Y$ *with* $\operatorname{diam} Y' < j^{-2}$, $i = 1, 2$ *and* $r \in (0, j^{-2})$ *that*

(8) $\int_{\mathbf{G}_n(E_i^*(r,Y'))} \|S - T\| \, d(\partial\mathcal{E})(x, S) \leqslant \gamma \|\partial\mathcal{E}\|(E_i^*(r, Y'))$,

(9) $\Delta_j \|\partial\mathcal{E}\|(E_i^*(r, Y')) \geqslant -\gamma \|\partial\mathcal{E}\|(E_i^*(r, Y'))$.

Then we have

$$\lambda \|\Phi_\varepsilon * \partial\mathcal{E}\|(\{x : \operatorname{dist}(x, Y) \leqslant R\}) \geqslant \omega_n R^n \mathcal{H}^0(Y). \tag{8.82}$$

Proof. — Given $\lambda \in (1, 2)$, we first use Lemma 8.4 with λ there replaced by $\lambda^{\frac{1}{4}}$ to obtain $\gamma_1 \in (0, 1)$ and $\tilde{M} \in (1, \infty)$ depending only on n, ν and λ with the stated property. Choose $\eta \in (0, 1)$ depending only on n, ν and λ so that

$$(2\tilde{M} + 3\nu)\eta < 1. \tag{8.83}$$

By setting

$$\alpha := \frac{1}{2\tilde{M}} \lambda^{\frac{1}{4n}} (1 - \lambda^{-\frac{1}{4n}}) \in (0, 1) \tag{8.84}$$

and fixing

$$\zeta := 1 - \lambda^{-\frac{1}{4}} \in (0, 1), \tag{8.85}$$

we use Lemma 8.1 to obtain $\gamma_2 \in (0, 1)$ and $j_0 \in \mathbb{N}$ depending only on n, ν and λ with the stated property. We assume that $\gamma \leqslant \min\{\gamma_1, \gamma_2\}$ and assume that we have (1)–(9). We set

$$r_0 := \eta^2 j^{-2} \tag{8.86}$$

in Lemma 8.4. We first check that the assumptions of Lemma 8.4 are satisfied, where V is replaced by $\Phi_\varepsilon * \partial\mathcal{E}$. By (3), we have $r_0 < R$. By (8.86), (8.83) and (3), we have $\{(1+3\nu)^2 + \tilde{M}^2\}^{\frac{1}{2}} r_0 \leqslant (2\tilde{M} + 3\nu)\eta^2 j^{-2} < \eta j^{-2} < R$. Thus we have (3) of Lemma 8.4. Note that (2), (4)–(6) of Lemma 8.4 follow from (4)–(7) of Lemma 8.5. Thus all the assumptions of Lemma 8.4 are satisfied, and there exists a partition of Y into $Y_0, Y_1, \ldots, Y_J$ such that

$$\operatorname{diam} Y_l \leqslant 3\nu\eta^2 j^{-2} < j^{-2} \text{ for all } l \in \{1, \ldots, J\}, \tag{8.87}$$

$$\begin{aligned} \lambda^{\frac{1}{4}} & \frac{\|\Phi_\varepsilon * \partial\mathcal{E}\|(\{x : \operatorname{dist}(x, Y) \leqslant R\})}{\omega_n R^n} \\ & \geqslant \sum_{y \in Y_0} \frac{\|\Phi_\varepsilon * \partial\mathcal{E}\|(B_{\eta^2 j^{-2}}(y))}{\omega_n (\eta^2 j^{-2})^n} \\ & \quad + \sum_{l=1}^{J} \frac{\|\Phi_\varepsilon * \partial\mathcal{E}\|(\{x : \operatorname{dist}(T^\perp(x), Y_l) \leqslant \eta^2 j^{-2}, |T(x)| \leqslant \tilde{M}\eta^2 j^{-2}\})}{\omega_n (\tilde{M}\eta^2 j^{-2})^n}. \end{aligned} \tag{8.88}$$

Depending only on n, ν and λ, there exists $\gamma_3 \in (0, \eta^8)$ such that, if $\varepsilon < \gamma_3 j^{-4}$,

$$\lambda^{\frac{1}{4}} \Phi_\varepsilon * \chi_{B_{\eta^2 j^{-2}}(y)} \geqslant 1 \quad \text{on } S_0(y) := \{x : |T^\perp(x) - y| \leqslant \eta^2 j^{-2}(1 - \lambda^{-\frac{1}{4n}}), |T(x)| \leqslant \eta^2 j^{-2} \lambda^{-\frac{1}{4n}}\}, \tag{8.89}$$

$$\lambda^{\frac{1}{4}} \Phi_\varepsilon * \chi_{\{x : \operatorname{dist}(T^\perp(x), Y_l) \leqslant \eta^2 j^{-2}, |T(x)| \leqslant \tilde{M}\eta^2 j^{-2}\}} \geqslant 1 \quad \text{on } S_l := \{x : \operatorname{dist}(T^\perp(x), Y_l) \leqslant \eta^2 j^{-2}(1 - \lambda^{-\frac{1}{4n}}), |T(x)| \leqslant \tilde{M}\eta^2 j^{-2} \lambda^{-\frac{1}{4n}}\}. \tag{8.90}$$

Since $\|\Phi_\varepsilon * \partial\mathcal{E}\|(B_{\eta^2 j^{-2}}(y)) = \|\partial\mathcal{E}\|(\Phi_\varepsilon * \chi_{B_{\eta^2 j^{-2}}(y)})$ and similarly for the other cases, (8.88)–(8.90) show

$$\lambda^{\frac{3}{4}} \frac{\|\Phi_\varepsilon * \partial\mathcal{E}\|(\{x : \operatorname{dist}(x, Y) \leqslant R\})}{\omega_n R^n} \geqslant \sum_{y \in Y_0} \frac{\|\partial\mathcal{E}\|(S_0(y))}{\omega_n (\eta^2 j^{-2} \lambda^{-\frac{1}{4n}})^n} + \sum_{l=1}^{J} \frac{\|\partial\mathcal{E}\|(S_l)}{\omega_n (\tilde{M}\eta^2 j^{-2} \lambda^{-\frac{1}{4n}})^n}. \tag{8.91}$$

We now use Lemma 8.1. For elements in Y_0, we let $R = \eta^2 j^{-2} \lambda^{-\frac{1}{4n}}$ (the reader should not confuse this R with R in the statement of the present Lemma) and $\rho = \frac{1}{2}\eta^2 j^{-2}(1 - \lambda^{-\frac{1}{4n}})$, and for $Y_1, \ldots, Y_J$, we let $R = \tilde{M}\eta^2 j^{-2} \lambda^{-\frac{1}{4n}}$ and the same ρ. Since they are similar, we only give the detail for Y_l, $l \in \{1, \ldots, J\}$. We check that the assumptions of Lemma 8.1 are satisfied first. We have already assumed $j \geqslant j_0$, and $\tilde{M}\eta^2 j^{-2} \lambda^{-\frac{1}{4n}} < \eta j^{-2} < \frac{1}{2} j^{-2}$ by (8.83). We also have $\frac{1}{2}\eta^2 j^{-2}(1 - \lambda^{-\frac{1}{4n}}) < \frac{1}{2} j^{-2}$, thus (1) is satisfied. For (2), note that $\frac{1}{2}\eta^2 j^{-2}(1 - \lambda^{-\frac{1}{4n}})/(\tilde{M}\eta^2 j^{-2} \lambda^{-\frac{1}{4n}}) = \alpha$ by (8.84), thus we have (2). (3) is satisfied due to (8.87). (4) and (5) are satisfied respectively due to (8) and (9) of Lemma 8.5. Thus the assumptions of Lemma 8.1 are all satisfied for Y_l, and we have

$$\frac{\|\partial\mathcal{E}\|(S_l)}{\omega_n (\tilde{M}\eta^2 j^{-2} \lambda^{-\frac{1}{4n}})^n} \geqslant \mathcal{H}^0(Y_l) - \zeta \geqslant \lambda^{-\frac{1}{4}} \mathcal{H}^0(Y_l) \tag{8.92}$$

where we used (8.85). The similar formula holds for $y \in Y_0$, and (8.91) and (8.92) show (8.82). Finally we let γ be re-defined as $\min\{\gamma_1, \gamma_2, \gamma_3\}$ if necessary. □

Theorem 8.6 ([8, §4.24]). — *Suppose that* $\{\mathcal{E}_j\}_{j=1}^\infty \subset \mathcal{OP}_\Omega^N$ *and* $\{\varepsilon_j\}_{j=1}^\infty \subset (0, 1)$ *satisfy*

(1) $\lim_{j \to \infty} j^4 \varepsilon_j = 0$,
(2) $\sup_j \|\partial\mathcal{E}_j\|(\Omega) < \infty$,

(3) $\liminf_{j\to\infty}\int_{\mathbb{R}^{n+1}}\frac{|\Phi_{\varepsilon_j}*\delta(\partial\mathcal{E}_j)|^2\Omega}{\Phi_{\varepsilon_j}*\|\partial\mathcal{E}_j\|+\varepsilon_j\Omega^{-1}}\,dx<\infty$,

(4) $\lim_{j\to\infty} j^{2(n+1)}\Delta_j\|\partial\mathcal{E}_j\|(\Omega)=0$.

Then there exists a converging subsequence $\{\partial\mathcal{E}_{j_l}\}_{l=1}^\infty$ *whose limit satisfies* $V\in\mathbf{IV}_n(\mathbb{R}^{n+1})$.

Proof. — We may choose a subsequence $\{j_l\}_{l=1}^\infty$ such that the quantities in (2) and (3) are uniformly bounded by M and the sequence $\{\partial\mathcal{E}_{j_l}\}_{l=1}^\infty$ converges to $V\in\mathbf{RV}_n(\mathbb{R}^{n+1})$ by Theorem 7.3. Without loss of generality, it is enough to prove that V is integral in U_1. For each pair of positive integers j and q, let $A_{j,q}$ be a set consisting of all $x\in B_1$ such that

$$\|\delta(\Phi_{\varepsilon_j}*\partial\mathcal{E}_j)\|(B_r(x))\leqslant q\|\Phi_{\varepsilon_j}*\partial\mathcal{E}_j\|(B_r(x)) \tag{8.93}$$

for all $r\in(j^{-2},1)$. For any $x\in B_1\setminus A_{j,q}$, we have

$$\|\delta(\Phi_{\varepsilon_j}*\partial\mathcal{E}_j)\|(B_r(x))> q\|\Phi_{\varepsilon_j}*\partial\mathcal{E}_j\|(B_r(x)) \tag{8.94}$$

for some $r\in(j^{-2},1)$. Since $\Phi_{\varepsilon_j}*\chi_{B_r(x)}\geqslant\frac14\chi_{B_r(x)}$ as long as $\varepsilon_j\ll r^2$, we have

$$\|\delta(\Phi_{\varepsilon_j}*\partial\mathcal{E}_j)\|(B_r(x))>\frac{q}{4}\|\partial\mathcal{E}_j\|(B_r(x)). \tag{8.95}$$

For sufficiently large j, (1) and $r\in(j^{-2},1)$ guarantee that $\varepsilon_j\ll r^2$. Applying the Besicovitch covering theorem to a collection of such balls covering $B_1\setminus A_{j,q}$, there exists a family $\mathcal{C}$ of disjoint balls such that

$$\|\partial\mathcal{E}_j\|(B_1\setminus A_{j,q})\leqslant\mathbf{B}_{n+1}\sum_{B_r(x)\in\mathcal{C}}\|\partial\mathcal{E}_j\|(B_r(x)). \tag{8.96}$$

Thus, with (8.96) and (8.95), we obtain

$$\|\partial\mathcal{E}_j\|(B_1\setminus A_{j,q})\leqslant\frac{4\mathbf{B}_{n+1}}{q}\|\delta(\Phi_{\varepsilon_j}*\partial\mathcal{E}_j)\|(B_2). \tag{8.97}$$

By the Cauchy–Schwarz inequality and (4.33),

$$\|\delta(\Phi_{\varepsilon_j}*\partial\mathcal{E}_j)\|(B_2)\leqslant\left(\int_{B_2}\frac{|\Phi_{\varepsilon_j}*\delta(\partial\mathcal{E}_j)|^2}{\Phi_{\varepsilon_j}*\|\partial\mathcal{E}_j\|+\varepsilon_j\Omega^{-1}}\,dx\right)^{\frac12}\times\left(\int_{B_2}\Phi_{\varepsilon_j}*\|\partial\mathcal{E}_j\|+\varepsilon_j\Omega^{-1}\,dx\right)^{\frac12}. \tag{8.98}$$

The right-hand side of (8.98) for j_l is bounded by $(\min_{B_3}\Omega)^{-1}M^{\frac12}(M^{\frac12}+2^{n+1}\omega_{n+1})$ for all l. Then (8.97) and (8.98) show

$$\|\partial\mathcal{E}_{j_l}\|(B_1\setminus A_{j_l,q})\leqslant\frac{c(n,\Omega,M)}{q} \tag{8.99}$$

for all $l, q \in \mathbb{N}$. Now for each $q \in \mathbb{N}$, set

(8.100)
$$A_q := \{x \in B_1 : \text{there exist } x_l \in A_{j_l,q} \text{ for infinitely many } l \text{ with } x_l \to x\}$$

and define

$$A := \cup_{q=1}^{\infty} A_q. \tag{8.101}$$

Then we have

$$\|V\|(U_1 \setminus A) = 0. \tag{8.102}$$

This can be seen as follows. Take arbitrary compact set $K \subset U_1 \setminus A$. For any $q \in \mathbb{N}$ we have $K \subset U_1 \setminus A_q$ by (8.101). For each point $x \in K$, by (8.100), there exists a neighborhood of x which does not intersect with $A_{j_l,q}$ for all sufficiently large l. Due to the compactness of K, there exist $l_0 \in \mathbb{N}$ and an open set $O_q \subset U_1$ such that $K \subset O_q$ and $O_q \cap A_{j_l,q} = \emptyset$ for all $l \geqslant l_0$. Let $\phi_q \in C_c(O_q; \mathbb{R}^+)$ be such that $0 \leqslant \phi_q \leqslant 1$ and $\phi_q = 1$ on K. Then

$$\begin{aligned} \|V\|(K) \leqslant \|V\|(\phi_q) &= \lim_{l\to\infty} \|\partial\mathcal{E}_{j_l}\|(\phi_q) = \lim_{l\to\infty} \|\partial\mathcal{E}_{j_l}\|\lfloor_{B_1\setminus A_{j_l,q}}(\phi_q) \\ &\leqslant \liminf_{l\to\infty} \|\partial\mathcal{E}_{j_l}\|(B_1 \setminus A_{j_l,q}) \leqslant \frac{c(n,\Omega,M)}{q} \end{aligned} \tag{8.103}$$

where we used (8.99). Since q is arbitrary, (8.103) gives $\|V\|(K) = 0$, proving (8.102).

Let A^* be a set of points in U_1 such that the approximate tangent space of V exists, i.e.,

$$\begin{aligned} A^* := \{x \in U_1 : \theta^n(\|V\|, x) \in (0,\infty), \mathrm{Tan}^n(\|V\|, x) \in \mathbf{G}(n+1,n), \\ \lim_{r\to 0+} (f_{(r)} \circ \tau_{(-x)})_\sharp V = \theta^n(\|V\|, x)|\mathrm{Tan}^n(\|V\|, x)|\}. \end{aligned} \tag{8.104}$$

Here, $f_{(r)}(y) := r^{-1}y$ and $\tau_{(-x)}(y) = y - x$ for $y \in \mathbb{R}^{n+1}$. Since $V \in \mathbf{RV}_n(\mathbb{R}^{n+1})$, we have $\|V\|(U_1 \setminus A^*) = 0$. Thus, for $\|V\|$ a.e. $x \in U_1$, we have $x \in A^* \cap A$. In the following, we fix x and prove that $\theta^n(\|V\|, x) \in \mathbb{N}$ for such x, which proves that $V \in \mathbf{IV}_n(\mathbb{R}^{n+1})$. For simplicity, we write

$$d := \theta^n(\|V\|, x),\ T := \mathrm{Tan}^n(\|V\|, x). \tag{8.105}$$

By an appropriate change of variables, we may assume that $x = 0$ and $T = \{x_{n+1} = 0\}$, with the understanding that all the relevant quantities are re-defined accordingly with no loss of generality. By (8.101), there exists $q \in \mathbb{N}$ such that $x = 0 \in A^* \cap A_q$, hence there exists a further subsequence of $\{j_l\}_{l=1}^{\infty}$ (denoted by the same index) such that $x_{j_l} \in A_{j_l,q}$ with $\lim_{l\to\infty} x_{j_l} = 0$. Set $r_l := l^{-1}$, and choose a further subsequence so that

$$\lim_{l\to\infty} (f_{(r_l)})_\sharp \partial\mathcal{E}_{j_l} = \lim_{l\to\infty} (f_{(r_l)})_\sharp (\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}) = d|T|, \tag{8.106}$$

$$\lim_{l\to\infty}\frac{x_{j_l}}{r_l}=0 \tag{8.107}$$

and

$$\lim_{l\to\infty}\frac{j_l^{-1}}{r_l}=\lim_{l\to\infty}\frac{l}{j_l}=0. \tag{8.108}$$

We define

$$V_{j_l}:=(f_{(r_l)})_\sharp\partial\mathcal{E}_{j_l},\qquad \tilde{V}_{j_l}:=(f_{(r_l)})_\sharp(\Phi_{\varepsilon_{j_l}}*\partial\mathcal{E}_{j_l}) \tag{8.109}$$

for simplicity in the following.

Suppose that ν is the smallest positive integer strictly greater than d, i.e.,

$$\nu\in\mathbb{N}\ \text{ and }\ \nu\in(d,d+1]. \tag{8.110}$$

Choose $\lambda\in(1,2)$ such that

$$\lambda^{n+1}d<\nu. \tag{8.111}$$

Corresponding to such λ and ν, we choose $\gamma,\eta\in(0,1)$, $\tilde{M}\in(1,\infty)$ and $j_0\in\mathbb{N}$ using Lemma 8.5. We use Lemma 8.5 with $R=r_l$ in the following. To do so, as a first step, we prove that the first variations of $\tilde{V}_{j_l}$ converge to 0, i.e.,

$$\lim_{l\to\infty}\|\delta\tilde{V}_{j_l}\|(B_s)=\lim_{l\to\infty}r_l^{1-n}\|\delta(\Phi_{\varepsilon_{j_l}}*\partial\mathcal{E}_{j_l})\|(B_{sr_l})=0 \tag{8.112}$$

for all $s>0$. To see this, note that we have $x_{j_l}\in A_{j_l,q}$, so that

$$\|\delta(\Phi_{\varepsilon_{j_l}}*\partial\mathcal{E}_{j_l})\|(B_{sr_l}(x_{j_l}))\leqslant q\|\Phi_{\varepsilon_{j_l}}*\partial\mathcal{E}_{j_l}\|(B_{sr_l}(x_{j_l})) \tag{8.113}$$

by (8.93), where we note that $sr_l\in(j_l^{-2},1)$ for all sufficiently large l due to (8.108). One can check that (8.113) is equivalent to

$$\|\delta\tilde{V}_{j_l}\|(B_s(r_l^{-1}x_{j_l}))\leqslant r_lq\|\tilde{V}_{j_l}\|(B_s(r_l^{-1}x_{j_l})). \tag{8.114}$$

By (8.107), $r_l^{-1}x_{j_l}\to 0$, and by (8.106), $\|\tilde{V}_{j_l}\|\to\|d|T|\|$. Since $r_l=l^{-1}$, by letting $l\to\infty$, (8.114) proves (8.112). We also need

$$\lim_{l\to\infty}\int_{\mathbf{G}_n(B_s)}\|S-T\|\,d\tilde{V}_{j_l}=\lim_{l\to\infty}r_l^{-n}\int_{\mathbf{G}_n(B_{sr_l})}\|S-T\|\,d(\Phi_{\varepsilon_{j_l}}*\partial\mathcal{E}_{j_l})=0 \tag{8.115}$$

and

$$\lim_{l\to\infty}\int_{\mathbf{G}_n(B_s)}\|S-T\|\,dV_{j_l}=\lim_{l\to\infty}r_l^{-n}\int_{\mathbf{G}_n(B_{sr_l})}\|S-T\|\,d(\partial\mathcal{E}_{j_l})=0 \tag{8.116}$$

for all $s>0$, but these follow directly from the varifold convergence of (8.106) to $d|T|$.

For each $l \in \mathbb{N}$ define

$$G_l := \big\{x \in B_{(\lambda-1)r_l} : r_l \|\delta(\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l})\|(B_s(x)) \leqslant \gamma \|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_s(x)) \text{ and } \int_{\mathbf{G}_n(B_s(x))} \|S - T\| \, d(\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}) \leqslant \gamma \|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_s(x)) \text{ for all } s \in (\eta^2 j_l^{-2}, r_l)\big\}. \tag{8.117}$$

By exactly the same line of argument as in (8.93)–(8.97), we have

$$\|\partial\mathcal{E}_{j_l}\|(B_{(\lambda-1)r_l} \setminus G_l) \leqslant 4\mathbf{B}_{n+1}\gamma^{-1}\bigg(r_l \|\delta(\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l})\|(B_{\lambda r_l}) + \int_{\mathbf{G}_n(B_{\lambda r_l})} \|S - T\| \, d(\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l})\bigg). \tag{8.118}$$

Then, (8.112), (8.115) and (8.118) show that

$$\lim_{l\to\infty} r_l^{-n} \|\partial\mathcal{E}_{j_l}\|(B_{(\lambda-1)r_l} \setminus G_l) = 0. \tag{8.119}$$

Define

$$G_l^* := \{x \in G_l : \theta^n(\|\partial\mathcal{E}_{j_l}\|, x) = 1\}. \tag{8.120}$$

Since $\partial\mathcal{E}_{j_l}$ is a unit density varifold,

$$\|\partial\mathcal{E}_{j_l}\|(G_l \setminus G_l^*) = 0. \tag{8.121}$$

We next define, as in Lemma 8.5 (a)–(c),

$$\tilde{R}_{1,l} := \eta^2 j_l^{-2} \lambda^{-\frac{1}{4n}}, \quad \tilde{R}_{2,l} := \tilde{M}\eta^2 j_l^{-2} \lambda^{-\frac{1}{4n}}, \quad \rho_l := \frac{1}{2}\eta^2 j_l^{-2}(1 - \lambda^{-\frac{1}{4n}}). \tag{8.122}$$

We wish to apply Lemma 8.5 and define $G_l^{**} \subset G_l^*$ as follows. For $x \in G_l^*$, take any arbitrary finite set $Y' = \{y_1, \ldots, y_m\} \subset G_l^*$ with $y_1 = x$, $T(x - y_i) = 0$ for $i \in \{2, \ldots, m\}$ and $\operatorname{diam} Y' < j_l^{-2}$. We do not exclude the possibility that $Y' = \{y_1\} = \{x\}$. Define

$$E_{i,l}^*(r, Y') := \{z \in \mathbb{R}^{n+1} : \ |T(z - x)| \leqslant r, \operatorname{dist}(T^\perp(Y'), T^\perp(z)) \leqslant (1 + \tilde{R}_{i,l}^{-1} r)\rho_l\} \tag{8.123}$$

for $i = 1, 2$. We define G_l^{**} as a set of point $x \in G_l^*$ such that, for arbitrary such Y' described above and for all $r \in (0, j_l^{-2})$ and $i = 1, 2$, we have

$$\int_{\mathbf{G}_n(E_{i,l}^*(r,Y'))} \|S - T\| \, d(\partial\mathcal{E}_{j_l}) \leqslant \gamma \|\partial\mathcal{E}_{j_l}\|(E_{i,l}^*(r, Y')) \quad \text{and} \quad \Delta_{j_l} \|\partial\mathcal{E}_{j_l}\|(E_{i,l}^*(r, Y')) \geqslant -\gamma \|\partial\mathcal{E}_{j_l}\|(E_{i,l}^*(r, Y')). \tag{8.124}$$

We wish to show that $\|\partial\mathcal{E}_{j_l}\|(G_l^*\setminus G_l^{**})$, which is a missed mass we cannot apply Lemma 8.5, is small. Whenever $x\in G_l^*\setminus G_l^{**}$, there exist a finite set $Y_x'=\{y_1,\dots,y_m\}\subset G_l^*$ with

$$y_1=x,\ T(x-y_i)=0 \text{ for } i\in\{2,\dots,m\}, \operatorname{diam} Y_x' < j_l^{-2}, \tag{8.125}$$

and $r_x\in(0,j_l^{-2})$ such that

$$\begin{aligned}\int_{\mathbf{G}_n(E_{i,l}^*(r_x,Y_x'))}\|S-T\|\,d(\partial\mathcal{E}_{j_l}) > \gamma\|\partial\mathcal{E}_{j_l}\|(E_{i,l}^*(r_x,Y_x'))\\ \text{for } i=1 \text{ or } i=2 \text{ or}\\ \Delta_{j_l}\|\partial\mathcal{E}_{j_l}\|(E_{i,l}^*(r_x,Y_x')) < -\gamma\|\partial\mathcal{E}_{j_l}\|(E_{i,l}^*(r_x,Y_x'))\\ \text{for } i=1 \text{ or } i=2.\end{aligned} \tag{8.126}$$

We separate $G_l^*\setminus G_l^{**}$ into four sets depending on the conditions in (8.126),

$$\begin{aligned}W_{i,l}:=\{x\in G_l^*\setminus G_l^{**}:\ \int_{\mathbf{G}_n(E_{i,l}^*(r_x,Y_x'))}\|S-T\|\,d(\partial\mathcal{E}_{j_l})\\ > \gamma\|\partial\mathcal{E}_{j_l}\|(E_{i,l}^*(r_x,Y_x'))\}\end{aligned} \tag{8.127}$$

for $i=1,2$ and

$$\begin{aligned}\tilde W_{i,l}:=\{x\in G_l^*\setminus G_l^{**}:\ \Delta_{j_l}\|\partial\mathcal{E}_{j_l}\|(E_{i,l}^*(r_x,Y_x'))\\ < -\gamma\|\partial\mathcal{E}_{j_l}\|(E_{i,l}^*(r_x,Y_x'))\}\end{aligned} \tag{8.128}$$

for $i=1,2$ so that

$$G_l^*\setminus G_l^{**}=\cup_{i=1}^2(W_{i,l}\cup\tilde W_{i,l}). \tag{8.129}$$

Typically, we would use the Besicovitch covering theorem to estimate the missed mass, but here, the elements of covering of $G_l^*\setminus G_l^{**}$ are $E_{i,l}^*(r_x,Y_x')$, which are not closed balls. Thus, direct use of the Besicovitch is not possible. On the other hand, note that at any point in $W_{i,l}$, the covering $E_{i,l}^*(r_x,Y_x')$ has always "height" bigger than ρ_l in $T^\perp$ direction, and ρ_l is $O(j_l^{-2})$. We take advantage of this property in the following. We estimate $\|\partial\mathcal{E}_{j_l}\|(W_{i,l})$ for $i=1,2$ first. We choose a finite set of points $\{w_{l,k}\}_{k=1}^{K_l}$ in $B_{(\lambda-1)r_l}$ so that

$$B_{(\lambda-1)r_l}\subset\cup_{k=1}^{K_l}B_{j_l^{-2}}(w_{l,k}) \tag{8.130}$$

and the number of intersection $\{k' : B_{4j_l^{-2}}(w_{l,k'})\cap B_{4j_l^{-2}}(w_{l,k})\neq\emptyset\}$ for each k is less than a constant $c(n)$ depending only on n. Such a set of points can be lattice points with width j_l^{-2} in $B_{(\lambda-1)r_l}$, for example. We then have

$$\sum_{k=1}^{K_l}\int_{\mathbf{G}_n(B_{4j_l^{-2}}(w_{l,k}))}\|S-T\|\,d(\partial\mathcal{E}_{j_l})(x,S) \leqslant c(n)\int_{\mathbf{G}_n(B_{\lambda r_l})}\|S-T\|\,d(\partial\mathcal{E}_{j_l})(x,S). \tag{8.131}$$

If we set for $k\in\{1,\dots,K_l\}$

$$W_{i,l,k} := W_{i,l}\cap B_{j_l^{-2}}(w_{l,k}), \tag{8.132}$$

by (8.130), we have

$$\cup_{k=1}^{K_l} W_{i,l,k} = W_{i,l}. \tag{8.133}$$

We next separate each $W_{i,l,k}$ into a stacked regions of width ρ_l in $T^\perp$ direction. Define for $m\in\mathbb{Z}$ with $|m|<j_l^{-2}\rho_l^{-1}+1$

$$W_{i,l,k,m} := W_{i,l,k}\cap\{x\in\mathbb{R}^{n+1} : m\rho_l < T^\perp(x-w_{l,k}) \leqslant (m+1)\rho_l\}. \tag{8.134}$$

Since $W_{i,l,k}\subset B_{j_l^{-2}}(w_{l,k})$, we have

$$W_{i,l,k} = \cup_{|m|<j_l^{-2}\rho_l^{-1}+1} W_{i,l,k,m} \tag{8.135}$$

and it is important to note that $j_l^{-2}\rho_l^{-1}+1$ is a constant depending only on η and λ, so ultimately only on n,ν and λ. For each $x\in W_{i,l,k,m}$, there exist $Y_x\subset G_l^*$ and $r_x\in(0,j_l^{-2})$ with the inequality of (8.127). Define

$$\mathcal{C}_{i,l,k,m} := \{B^n_{r_x}(T(x))\subset\mathbb{R}^n : x\in W_{i,l,k,m}\} \tag{8.136}$$

which is a covering of $T(W_{i,l,k,m})$. Observe that, if there is a subfamily $\hat{\mathcal{C}}_{i,l,k,m}\subset\mathcal{C}_{i,l,k,m}$ such that $T(W_{i,l,k,m})\subset\cup_{C\in\hat{\mathcal{C}}_{i,l,k,m}} C$, we have

$$W_{i,l,k,m}\subset\cup_{B^n_{r_x}(T(x))\in\hat{\mathcal{C}}_{i,l,k,m}}\{y : |T(x-y)|\leqslant r_x, |T^\perp(x-y)|\leqslant\rho_l\}. \tag{8.137}$$

This is because, for any $x'\in W_{i,l,k,m}$, we have some $B^n_{r_x}(T(x))\in\hat{\mathcal{C}}_{i,l,k,m}$ with $T(x')\in B^n_{r_x}(T(x))$. Since $x',x\in W_{i,l,k,m}$, $|T^\perp(x'-x)|<\rho_l$, so $x'\in\{y : |T(x-y)|\leqslant r_x, |T^\perp(x-y)|\leqslant\rho_l\}$, which proves (8.137). We apply the Besicovitch covering theorem to $\mathcal{C}_{i,l,k,m}$ and obtain a set of subfamilies $\mathcal{C}^{(1)}_{i,l,k,m},\dots,\mathcal{C}^{(L_{i,l,k,m})}_{i,l,k,m}\subset\mathcal{C}_{i,l,k,m}$ such that

$$L_{i,l,k,m}\leqslant\mathbf{B}_n, \tag{8.138}$$

each $\mathcal{C}^{(h)}$ $(h = 1, \ldots, L_{i,l,k,m})$ consists of disjoint sets and $T(W_{i,l,k,m}) \subset \cup_{h=1}^{L_{i,l,k,m}} \cup_{C \in \mathcal{C}^{(h)}_{i,l,k,m}} C$. Then (8.137) shows that we have

(8.139)
$$W_{i,l,k,m} \subset \cup_{h=1}^{L_{i,l,k,m}} \cup_{B^n_{r_x}(x) \in \mathcal{C}^{(h)}_{i,l,k,m}} \{y : |T(x-y)| \leqslant r_x, |T^\perp(x-y)| \leqslant \rho_l\}.$$

For $x \in W_{i,l,k,m}$,

$$\{y : |T(x-y)| \leqslant r_x, |T^\perp(x-y)| \leqslant \rho_l\} \subset E^*_{i,l}(r_x, Y'_x). \tag{8.140}$$

We note that if $B^n_{r_x}(x) \cap B^n_{r_{x'}}(x') = \emptyset$, then $E^*_{i,l}(r_x, Y'_x) \cap E^*_{i,l}(r_{x'}, Y'_{x'}) = \emptyset$ since their projections to T is $B^n_{r_x}(x) \cap B^n_{r_{x'}}(x')$. Also we note that

$$E^*_{i,l}(r_x, Y'_x) \subset B_{4j_l^{-2}}(w_{l,k}) \tag{8.141}$$

since $x \in B_{j_l^{-2}}(w_{l,k})$, $Y'_x \in T^\perp(B_{j_l^{-2}}(x))$ (by (8.125)), $r_x \in (0, j_l^{-2})$, $(1 + \tilde{R}^{-1}_{i,l} r_x)\rho_l \leqslant \rho_l + \frac{r_x}{2} < j_l^{-2}$ (by (8.122) and (8.123)). We have by (8.139), (8.140), (8.127), (8.138) and (8.141) that

$$\begin{aligned} \|\partial\mathcal{E}_{j_l}\|(W_{i,l,k,m}) &\leqslant \sum_{h=1}^{L_{i,l,k,m}} \sum_{B^n_{r_x}(x) \in \mathcal{C}^{(h)}_{i,l,k,m}} \|\partial\mathcal{E}_{j_l}\|(E^*_{i,l}(r_x, Y'_x)) \\ &\leqslant \sum_{h=1}^{L_{i,l,k,m}} \sum_{B^n_{r_x}(x) \in \mathcal{C}^{(h)}_{i,l,k,m}} \gamma^{-1} \int_{\mathbf{G}_n(E^*_{i,l}(r_x, Y'_x))} \|S - T\| \, d(\partial\mathcal{E}_{j_l})(x, S) \\ &\leqslant \gamma^{-1} \mathbf{B}_n \int_{\mathbf{G}_n(B_{4j_l^{-2}}(w_{l,k}))} \|S - T\| \, d(\partial\mathcal{E}_{j_l})(x, S). \end{aligned} \tag{8.142}$$

Now summing (8.142) over $|m| < j_l^{-2}\rho_l^{-1} + 1$ (note (8.135) and the following remark), we have

$$\|\partial\mathcal{E}_{j_l}\|(W_{i,l,k}) \leqslant \gamma^{-1} c(n, \nu, \lambda) \int_{\mathbf{G}_n(B_{4j_l^{-2}}(w_{l,k}))} \|S - T\| \, d(\partial\mathcal{E}_{j_l})(x, S). \tag{8.143}$$

Summing (8.143) over $k = 1, \ldots, K_l$ and by (8.133) and (8.131), we obtain

$$\|\partial\mathcal{E}_{j_l}\|(W_{i,l}) \leqslant \gamma^{-1} c(n, \nu, \lambda) \int_{\mathbf{G}_n(B_{\lambda r_l})} \|S - T\| \, d(\partial\mathcal{E}_{j_l})(x, S). \tag{8.144}$$

By (8.116) and (8.144), we obtain

$$\lim_{l \to \infty} r_l^{-n} \|\partial\mathcal{E}_{j_l}\|(W_{i,l}) = 0. \tag{8.145}$$

Next we estimate $\|\partial\mathcal{E}_{j_l}\|(\tilde{W}_{i,l})$ for $i = 1, 2$. The argument is identical up to the second line of (8.142) except that we use the covering satisfying the inequality of (8.128) in place of (8.127). By using the same notation, we obtain

$$(8.146)\ \|\partial\mathcal{E}_{j_l}\|(\tilde{W}_{i,l,k,m}) \leqslant -\sum_{h=1}^{L_{i,l,k,m}} \sum_{B^n_{r_x}(x)\in\mathcal{C}^{(h)}_{i,l,k,m}} \gamma^{-1}\Delta_{j_l}\|\partial\mathcal{E}_{j_l}\|(E^*_{i,l}(r_x,Y'_x)).$$

Recall that $\{E^*_{i,l}(r_x,Y'_x)\}_{B^n_{r_x}(x)\in\mathcal{C}^{(h)}_{i,l,k,m}}$ is disjoint and we have (8.141). Since $\mathcal{L}^{n+1}(B_{4j_l^{-2}}(w_{l,k})) < j_l^{-1}$ for large l, Lemma 4.11 shows

$$(8.147)\quad \Delta_{j_l}\|\partial\mathcal{E}_{j_l}\|(B_{4j_l^{-2}}(w_{l,k})) \leqslant \sum_{B^n_{r_x}(x)\in\mathcal{C}^{(h)}_{i,l,k,m}} \Delta_{j_l}\|\partial\mathcal{E}_{j_l}\|(E^*_{i,l}(r_x,Y'_x))$$

for each h. Hence (8.146), (8.147) and (8.138) show

$$(8.148)\qquad \|\partial\mathcal{E}_{j_l}\|(\tilde{W}_{i,l,k,m}) \leqslant -\mathbf{B}_n\gamma^{-1}\Delta_{j_l}\|\partial\mathcal{E}_{j_l}\|(B_{4j_l^{-2}}(w_{l,k}))$$

and summation over $|m| < j_l^{-2}\rho_l^{-1}+1$ gives

$$(8.149)\qquad \|\partial\mathcal{E}_{j_l}\|(\tilde{W}_{i,l,k}) \leqslant -\gamma^{-1}c(n,\nu,\lambda)\Delta_{j_l}\|\partial\mathcal{E}_{j_l}\|(B_{4j_l^{-2}}(w_{l,k})).$$

By Lemma 4.10, we have

$$(8.150)\ \ -\Delta_{j_l}\|\partial\mathcal{E}_{j_l}\|(B_{4j_l^{-2}}(w_{l,k})) \leqslant -(\max_{B_{4j_l^{-2}}(w_{l,k})}\Omega)^{-1}\Delta_{j_l}\|\partial\mathcal{E}_{j_l}\|(\Omega) \\ + (1-e^{-4c_1 j_l^{-2}})\|\partial\mathcal{E}_{j_l}\|(B_{4j_l^{-2}}(w_{l,k})).$$

Noticing that K_l in (8.130) satisfies $K_l \leqslant c(n)(r_l j_l^2)^{n+1}$, summation over k of (8.149) combined with (8.150) gives

$$(8.151)\quad \|\partial\mathcal{E}_{j_l}\|(\tilde{W}_{i,l}) \leqslant \gamma^{-1}c(n,\nu,\lambda,\Omega)\{-(r_l j_l^2)^{n+1}\Delta_{j_l}\|\partial\mathcal{E}_{j_l}\|(\Omega) \\ + (1-e^{-4c_1 j_l^{-2}})\|\partial\mathcal{E}_{j_l}\|(B_{\lambda r_l})\}.$$

With (4), (8.106) and (8.151), we conclude that

$$(8.152)\qquad \lim_{l\to\infty} r_l^{-n}\|\partial\mathcal{E}_{j_l}\|(\tilde{W}_{i,l}) = 0.$$

Now, by (8.129), (8.145) and (8.152) we have

$$(8.153)\qquad \lim_{l\to\infty} r_l^{-n}\|\partial\mathcal{E}_{j_l}\|(G^*_l\setminus G^{**}_l) = 0.$$

Combining (8.119), (8.121) and (8.153), we have

$$(8.154)\qquad \lim_{l\to\infty} r_l^{-n}\|\partial\mathcal{E}_{j_l}\|(B_{(\lambda-1)r_l}\setminus G^{**}_l) = 0.$$

Since $G^{**}_l \subset G^*_l \subset G_l$, $x\in G^{**}_l$ satisfies (8.117), (8.120) and (8.124). Given any $s\in(0,\frac{1}{4})$ and $x\in G^{**}_l$, we use Lemma 8.5 with $R = r_l s$ for $Y = \{T^\perp(x)\}$, a single element case. For all sufficiently large j_l, assumptions of

Lemma 8.5 are all satisfied: (1) is fine for large j_l, (2) from Theorem 8.6(1) for large j_l, (3) from (8.108) for large j_l, (4) from Y having single element and $x \in G_l^*$, (5) from $\operatorname{diam} Y = 0$, (6) and (7) from (8.117), (8) and (9) from (8.124). Thus we have (8.82), or

$$\lambda \|\Phi_{\varepsilon_{j_l}} * \partial \mathcal{E}_{j_l}\|(B_{r_l s}(x)) \geqslant \omega_n (r_l s)^n \tag{8.155}$$

for all large j_l. (8.155) implies

$$G_l^{**} \subset B_{(\lambda-1)r_l} \cap \{x : |T^\perp(x)| \leqslant 3 r_l s\} \tag{8.156}$$

for all sufficiently large j_l. This is because, if (8.156) were not true, then there would exist a subsequence (denoted by the same index) $x_{j_l} \in G_l^{**}$ with $|T^\perp(x_{j_l})| > 3 r_l s$ and we may assume that $r_l^{-1} x_{j_l} \in B_{\lambda-1}$ converges to $\bar{x} \in B_{\lambda-1} \cap \{x : |T^\perp(x)| \geqslant 3s\}$. By (8.106), since $B_{2s}(\bar{x}) \cap T = \emptyset$, we have

$$\begin{aligned} 0 &= \lim_{l\to\infty} \|(f_{(r_l)})_\sharp(\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l})\|(B_{2s}(\bar{x})) \\ &= \lim_{l\to\infty} r_l^{-n} \|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_{2 r_l s}(r_l \bar{x})). \end{aligned} \tag{8.157}$$

Since $\lim_{l\to\infty} r_l^{-1}|r_l\bar{x} - x_{j_l}| = 0$, for sufficiently large j_l, we have $B_{r_l s}(x_{j_l}) \subset B_{2r_l s}(r_l \bar{x})$. Hence, continuing from (8.157), we have

$$\geqslant \limsup_{l\to\infty} r_l^{-n} \|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_{r_l s}(x_{j_l})) \geqslant \lambda^{-1}\omega_n s^n \tag{8.158}$$

where (8.155) is used in the last step, and we have a contradiction. This proves (8.156). We next show that, for all sufficiently large j_l,

$$\mathcal{H}^0(\{x \in G_l^{**} \,:\, T(x) = a\}) \leqslant \nu - 1 \tag{8.159}$$

for all $a \in B_{(\lambda-1)r_l} \cap T$. For a contradiction, suppose we had some $a_l \in B_{(\lambda-1)r_l} \cap T$ such that (8.159) fails. Then there exists $Y_l \subset T^{-1}(\{x \in G_l^{**} : T(x) = a_l\})$ with $\mathcal{H}^0(Y_l) = \nu$. We use Lemma 8.5 to Y_l and $R = r_l$. One can check that the assumptions are all satisfied just as for the single element case, except for (5), which was trivial before. This time, on the other hand, due to (8.156), we have $\operatorname{diam} Y_l \leqslant \gamma r_l$ by choosing $s = \gamma/6$, so (5) is also satisfied. Thus we have

$$\lambda \|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(\{x : \operatorname{dist}(x, Y_l) \leqslant r_l\}) \geqslant \omega_n r_l^n \nu. \tag{8.160}$$

We may assume after choosing a subsequence that $r_l^{-1} a_l$ converges to $\bar{a} \in B_{\lambda-1} \cap T$. By (8.106),

$$\begin{aligned} \lambda^n \omega_n d &= \lim_{l\to\infty} \|(f_{(r_l)})_\sharp(\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l})\|(B_\lambda(\bar{a})) \\ &= \lim_{l\to\infty} r_l^{-n} \|\Phi_{\varepsilon_{j_l}} * \partial\mathcal{E}_{j_l}\|(B_{\lambda r_l}(r_l \bar{a})). \end{aligned} \tag{8.161}$$

For large j_l, by (8.156) taking $s = (\sqrt{\lambda} - 1)/6$, $\{x : \mathrm{dist}\,(x, Y_l) \leqslant r_l\} \subset B_{\sqrt{\lambda} r_l}(a_l) \subset B_{\lambda r_l}(r_l \bar{a})$. Hence (8.160) and (8.161) show $\lambda^{n+1} d \geqslant \nu$ which is a contradiction to (8.111). This proves (8.159). Finally, we note that

(8.162)
$$\lim_{l\to\infty} r_l^{-n} \|T_\sharp \partial \mathcal{E}_{j_l}\|(B_{(\lambda-1)r_l} \setminus G_l^{**}) \leqslant \lim_{l\to\infty} r_l^{-n} \|\partial \mathcal{E}_{j_l}\|(B_{(\lambda-1)r_l} \setminus G_l^{**}) = 0$$

due to (8.154) while

(8.163)
$$\begin{aligned} \|T_\sharp \partial \mathcal{E}_{j_l}\|(G_l^{**}) &= \int_{B_{(\lambda-1)r_l} \cap T} \sum\nolimits_{\{x \in G_l^{**} \,:\, T(x) = a\}} \theta^n(\|\partial \mathcal{E}_{j_l}\|, x)\, d\mathcal{H}^n(a) \\ &\leqslant \omega_n((\lambda - 1) r_l)^n (\nu - 1) \end{aligned}$$

by (8.159) for all large j_l. By (8.106),

$$\begin{aligned} \lim_{l\to\infty} r_l^{-n} \|T_\sharp \partial \mathcal{E}_{j_l}\|(B_{(\lambda-1)r_l}) &= \lim_{l\to\infty} \|T_\sharp V_{j_l}\|(B_{\lambda-1}) \\ &= \|T_\sharp d|T|\|(B_{\lambda-1}) \\ &= \omega_n(\lambda-1)^n d \end{aligned} \tag{8.164}$$

and (8.162)–(8.164) show $d \leqslant \nu - 1$. By (8.110), this proves $d = \nu - 1$. □

9. Proof of Brakke's inequality

Here, the main objective is to prove the inequality (3.4) usually referred to as Brakke's inequality. We are interested in proving integral form instead of differential form as in [8]. The proof is different from [8] and we adopt the proof of [43] which we believe is more transparent.

Lemma 9.1. — *Let $\{\partial \mathcal{E}_{j_l}(t)\}_{t \in \mathbb{R}^+}$ ($l \in \mathbb{N}$) and $\{\mu_t\}_{t\in\mathbb{R}^+}$ be as in Proposition 6.4 satisfying* (6.18), (6.19) *and* (6.20)*. Then we have the following.*

(a) *For a.e. $t \in \mathbb{R}^+$, μ_t is integral, i.e., there exists $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ such that $\mu_t = \|V_t\|$.*

(b) *For a.e. $t \in \mathbb{R}^+$, if a subsequence $\{j'_l\}_{l=1}^\infty \subset \{j_l\}_{l=1}^\infty$ satisfies*

$$\sup_{l\in\mathbb{N}} \int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_{j'_l}} * \delta(\partial \mathcal{E}_{j'_l}(t))|^2 \Omega}{\Phi_{\varepsilon_{j'_l}} * \|\partial \mathcal{E}_{j'_l}(t)\| + \varepsilon_{j'_l} \Omega^{-1}}\, dx < \infty, \tag{9.1}$$

then we have $\lim_{l\to\infty} \partial \mathcal{E}_{j'_l}(t) = V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ as varifolds and $\mu_t = \|V_t\|$.

(c) *Furthermore, for a.e.* $t \in \mathbb{R}^+$, V_t *has a generalized mean curvature* $h(\cdot, V_t)$ *which satisfies*

(9.2)
$$\int_{\mathbb{R}^{n+1}} |h(\cdot, V_t)|^2 \phi \, d\|V_t\| \leqslant \liminf_{l\to\infty} \int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_{j'_l}} * \delta(\partial\mathcal{E}_{j'_l}(t))|^2 \phi}{\Phi_{\varepsilon_{j'_l}} * \|\partial\mathcal{E}_{j'_l}(t)\| + \varepsilon_{j'_l}\Omega^{-1}} \, dx < \infty$$

for any $\phi \in \cup_{i\in\mathbb{N}} \mathcal{A}_i$.

Proof. — Due to (6.19) and Fatou's Lemma, we have

$$\liminf_{l\to\infty} \int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_{j_l}} * \delta(\partial\mathcal{E}_{j_l}(t))|^2 \Omega}{\Phi_{\varepsilon_{j_l}} * \|\partial\mathcal{E}_{j_l}(t)\| + \varepsilon_{j_l}\Omega^{-1}} \, dx < \infty \tag{9.3}$$

for a.e. $t \in \mathbb{R}^+$ and for any $T < \infty$, $\sup_{l\in\mathbb{N},\, t\in[0,T]} \|\partial\mathcal{E}_{j_l}(t)\|(\Omega) < \infty$ due to (6.3). Suppose we have (9.3) and (6.20) at t. We check that the assumptions of Theorem 8.6 are all satisfied for $\{\mathcal{E}_{j_l}(t)\}_{l=1}^{\infty}$: (1) from (5.8), (2) from above, (3) by (9.3) , and (4) from (6.20). Thus, there exists a further converging subsequence of $\{\partial\mathcal{E}_{j'_l}(t)\}_{l=1}^{\infty}$ and a limit $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$, where $\{j'_l\}_{l=1}^{\infty} \subset \{j_l\}_{l=1}^{\infty}$. This convergence is in the sense of varifold, so in particular, we have $\lim_{l\to\infty} \|\partial\mathcal{E}_{j'_l}(t)\| = \|V_t\|$. Note that the left-hand side is μ_t by (6.18), so $\mu_t = \|V_t\|$. This proves (a). Note that rectifiable (thus integral) varifolds are determined by the weight measure, thus V_t is uniquely determined by μ_t independent of the subsequence $\{j'_l\}_{l=1}^{\infty}$. Let $\{\partial\mathcal{E}_{j'_l}(t)\}_{l=1}^{\infty}$ be any subsequence with (9.1), then we have already seen that any converging further subsequence converges to V_t. Since it is unique, the full sequence $\{\partial\mathcal{E}_{j'_l}(t)\}_{l=1}^{\infty}$ converges to V_t. This proves (b). The claim (c) follows from Proposition 5.6. □

Remark 9.2. — Note that we are NOT claiming that $\lim_{l\to\infty} \partial\mathcal{E}_{j_l}(t) = V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ for a.e. $t \in \mathbb{R}^+$, but only the one with uniform bound of (9.1).

Up to this point, we defined $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ for a.e. $t \in \mathbb{R}^+$. On the complement of such set of time which is $\mathcal{L}^1$ measure 0, we still have μ_t. For such t, we define an arbitrary varifold with the weight measure μ_t. For example, let $T \in \mathbf{G}(n+1, n)$ be fixed, and define $V_t(\phi) := \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} \phi(x, T) \, d\mu_t$ for $\phi \in C_c(\mathbf{G}_n(\mathbb{R}^{n+1}))$. Then we have $\|V_t\| = \mu_t$. By doing this, we now have a family of varifolds $\{V_t\}_{t\in\mathbb{R}^+}$ such that $\|V_t\| = \mu_t$ for all $t \in \mathbb{R}^+$ and $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ for a.e. $t \in \mathbb{R}^+$.

Theorem 9.3. — *For all* $T > 0$, *we have*

$$\int_0^T \int_{\mathbb{R}^{n+1}} |h(\cdot, V_t)|^2 \Omega \, d\|V_t\| dt < \infty \tag{9.4}$$

and for any $\phi \in C^1_c(\mathbb{R}^{n+1} \times \mathbb{R}^+; \mathbb{R}^+)$ *and* $0 \leqslant t_1 < t_2 < \infty$*, we have*

$$\|V_t\|(\phi(\cdot,t))\Big|_{t=t_1}^{t_2} \leqslant \int_{t_1}^{t_2} \left(\delta(V_t, \phi(\cdot,t))(h(\cdot,V_t)) + \|V_t\|(\frac{\partial \phi}{\partial t}(\cdot,t))\right) dt. \tag{9.5}$$

Proof. — (9.4) follows from (9.2), Fatou's Lemma and (6.19). We prove (9.5) for time independent ϕ first and let $\phi \in C^\infty_c(\mathbb{R}^{n+1}; \mathbb{R}^+)$ be arbitrary. Since it has a compact support, there exists $c > 0$ such that $c\phi(x) < \Omega(x)$ for all $x \in \mathbb{R}^{n+1}$. Due to the linear dependence on ϕ of (9.5), it suffices to prove (9.5) for $c\phi$ for C^∞_c case, and by suitable density argument for C^1_c case. Re-writing $c\phi$ as ϕ, we may as well assume that $\phi < \Omega$. Then for all sufficiently large $i \in \mathbb{N}$, we have $\hat{\phi} := \phi + i^{-1}\Omega < \Omega$. After fixing i, there exists $m \in \mathbb{N}$ such that $\hat{\phi} \in \mathcal{A}_m$. Fix $0 \leqslant t_1 < t_2$ and suppose that l is large enough so that $j_l > m$ and $j_l > t_2$. We use (6.5) with $\hat{\phi}$. With the notation of (6.2), we obtain

$$\begin{aligned}\|\partial\mathcal{E}_{j_l}(t)\|(\hat{\phi}) - \|\partial\mathcal{E}_{j_l}(t - \Delta t_{j_l})\|(\hat{\phi}) \\ \leqslant \Delta t_{j_l} \left(\delta(\partial\mathcal{E}_{j_l}(t), \hat{\phi})(h_{\varepsilon_{j_l}}(\cdot, \partial\mathcal{E}_{j_l}(t)) + \varepsilon_{j_l}^{\frac{1}{8}}\right)\end{aligned} \tag{9.6}$$

for $t = \Delta t_{j_l}, 2\Delta t_{j_l}, \ldots, j_l 2^{p_{j_l}} \Delta t_{j_l}$. There exist $k_1, k_2 \in \mathbb{N}$ such that $t_2 \in ((k_2-1)\Delta t_{j_l}, k_2\Delta t_{j_l}]$ and $t_1 \in ((k_1-2)\Delta t_{j_l}, (k_1-1)\Delta t_{j_l}]$, where we are assuming that $\Delta t_{j_l} < t_2 - t_1$. Summing (9.6) over $t = k_1\Delta t_{j_l}, \ldots, k_2\Delta t_{j_l}$, we obtain

$$\begin{aligned}\|\partial\mathcal{E}_{j_l}(t)\|(\hat{\phi})\Big|_{t=(k_1-1)\Delta t_{j_l}}^{k_2\Delta t_{j_l}} \\ \leqslant \sum_{k=k_1}^{k_2} \Delta t_{j_l} \left(\delta(\partial\mathcal{E}_{j_l}(k\Delta t_{j_l}), \hat{\phi})(h_{\varepsilon_{j_l}}(\cdot, \partial\mathcal{E}_{j_l}(k\Delta t_{j_l}))) + \varepsilon_{j_l}^{\frac{1}{8}}\right).\end{aligned} \tag{9.7}$$

Due to the definition of $\hat{\phi} = \phi + i^{-1}\Omega$, we have

$$\begin{aligned}\|\partial\mathcal{E}_{j_l}(t)\|(\hat{\phi})\Big|_{t=(k_1-1)\Delta t_{j_l}}^{k_2\Delta t_{j_l}} \\ \geqslant \|\partial\mathcal{E}_{j_l}(t_2)\|(\phi) - \|\partial\mathcal{E}_{j_l}(t_1)\|(\phi) - i^{-1}\|\partial\mathcal{E}_{j_l}(t_1)\|(\Omega).\end{aligned} \tag{9.8}$$

As $l \to \infty$, with (6.3), we obtain

$$\begin{aligned}\limsup_{l\to\infty} \|\partial\mathcal{E}_{j_l}(t)\|(\hat{\phi})\Big|_{t=(k_1-1)\Delta t_{j_l}}^{k_2\Delta t_{j_l}} \\ \geqslant \|V_t\|(\phi)\Big|_{t=t_1}^{t_2} - i^{-1}\|\partial\mathcal{E}_0\|(\Omega)\exp(\frac{c_1^2 t_1}{2}).\end{aligned} \tag{9.9}$$

For the right-hand side of (9.7), by (2.5) and writing $h_{\varepsilon_{j_l}} = h_{\varepsilon_{j_l}}(\cdot, \partial\mathcal{E}_{j_l}(t))$ and $\partial\mathcal{E}_{j_l} = \partial\mathcal{E}_{j_l}(t)$,

$$\delta(\partial\mathcal{E}_{j_l}, \hat{\phi})(h_{\varepsilon_{j_l}}) = \delta(\partial\mathcal{E}_{j_l})(\hat{\phi}h_{\varepsilon_{j_l}}) + \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} S^{\perp}(\nabla\hat{\phi}) \cdot h_{\varepsilon_{j_l}}\, d(\partial\mathcal{E}_{j_l}). \tag{9.10}$$

By (5.23) for all sufficiently large l and all evaluated at $t = k\Delta t_{j_l}$ and if we write

$$b_{j_l} := \int_{\mathbb{R}^{n+1}} \frac{\hat{\phi}|\Phi_{\varepsilon_{j_l}} * \delta(\partial\mathcal{E}_{j_l})|^2}{\Phi_{\varepsilon_{j_l}} * \|\partial\mathcal{E}_{j_l}\| + \varepsilon_{j_l}\Omega^{-1}}\, dx \tag{9.11}$$

for simplicity,

$$|\delta(\partial\mathcal{E}_{j_l})(\hat{\phi}h_{\varepsilon_{j_l}}) + b_{j_l}| \leqslant \varepsilon_{j_l}^{\frac{1}{4}}(b_{j_l} + 1) \tag{9.12}$$

and by the Cauchy–Schwarz inequality and (5.24), we have

$$\begin{aligned} &\left|\int_{\mathbf{G}_n(\mathbb{R}^{n+1})} S^{\perp}(\nabla\hat{\phi}) \cdot h_{\varepsilon_{j_l}}\, d(\partial\mathcal{E}_{j_l})\right| \\ &\qquad \leqslant \left(\int_{\mathbb{R}^{n+1}} \hat{\phi}^{-1}|\nabla\hat{\phi}|^2\, d\|\partial\mathcal{E}_{j_l}\|\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{n+1}} \hat{\phi}|h_{\varepsilon_{j_l}}|^2\, d\|\partial\mathcal{E}_{j_l}\|\right)^{\frac{1}{2}} \\ &\qquad \leqslant c\, \|\partial\mathcal{E}_{j_l}\|(\Omega)^{\frac{1}{2}} \left((1 + \varepsilon_{j_l}^{\frac{1}{4}})b_{j_l} + \varepsilon_{j_l}^{\frac{1}{4}}\right)^{\frac{1}{2}}, \end{aligned} \tag{9.13}$$

where we estimated as in (6.27) and c depends only on $\|\phi\|_{C^2}$, $\min_{x \in \operatorname{spt}\phi} \Omega$ and c_1 and independent of i. Since $\sup_{t\in[0,t_2]} \|\partial\mathcal{E}_{j_l}(t)\|(\Omega)$ is bounded uniformly, (9.10)–(9.13) show that for all sufficiently large l, we have

$$\sup_{t\in[t_1,t_2]} \delta(\partial\mathcal{E}_{j_l}(t), \hat{\phi})(h_{\varepsilon_{j_l}}(\cdot, \partial\mathcal{E}_{j_l}(t))) \leqslant c \tag{9.14}$$

where c depends only on $\|\partial\mathcal{E}_0\|(\Omega)$, t_2, $\|\phi\|_{C^2}$, $\min_{x\in\operatorname{spt}\phi}\Omega$ and c_1. Thus we have

$$\begin{aligned} &\limsup_{l\to\infty} \sum_{k=k_1}^{k_2} \Delta t_{j_l} \delta(\partial\mathcal{E}_{j_l}(k\Delta t_{j_l}), \hat{\phi})(h_{\varepsilon_{j_l}}(\cdot, \partial\mathcal{E}_{j_l}(k\Delta t_{j_l}))) \\ &\quad = \limsup_{l\to\infty} \int_{t_1}^{t_2} \delta(\partial\mathcal{E}_{j_l}(t), \hat{\phi})(h_{\varepsilon_{j_l}}(\cdot, \partial\mathcal{E}_{j_l}(t)))\, dt \\ &\quad = -\liminf_{l\to\infty} \int_{t_1}^{t_2} \Big(c - \delta(\partial\mathcal{E}_{j_l}(t), \hat{\phi})(h_{\varepsilon_{j_l}}(\cdot, \partial\mathcal{E}_{j_l}(t)))\Big)\, dt + c(t_2 - t_1) \\ &\quad \leqslant -\int_{t_1}^{t_2} \liminf_{l\to\infty} \Big(c - \delta(\partial\mathcal{E}_{j_l}(t), \hat{\phi})(h_{\varepsilon_{j_l}}(\cdot, \partial\mathcal{E}_{j_l}(t)))\Big)\, dt + c(t_2 - t_1) \\ &\quad = \int_{t_1}^{t_2} \limsup_{l\to\infty} \delta(\partial\mathcal{E}_{j_l}(t), \hat{\phi})(h_{\varepsilon_{j_l}}(\cdot, \partial\mathcal{E}_{j_l}(t)))\, dt \end{aligned} \tag{9.15}$$

where we used (9.14) and Fatou's Lemma. We estimate the integrand of (9.15) from above. Fix t. Let $\{j'_l\}_{l=1}^\infty \subset \{j_l\}_{l=1}^\infty$ be a subsequence such that the lim sup is achieved, i.e.,

$$(9.16)\quad \limsup_{l\to\infty} \delta(\partial\mathcal{E}_{j_l}(t),\hat\phi)(h_{\varepsilon_{j_l}}(\cdot,\partial\mathcal{E}_{j_l}(t))) = \lim_{l\to\infty} \delta(\partial\mathcal{E}_{j'_l}(t),\hat\phi)(h_{\varepsilon_{j'_l}}(\cdot,\partial\mathcal{E}_{j'_l}(t))).$$

The right-hand side of (9.10) then have the same property for this subsequence and

$$(9.17)\quad \lim_{l\to\infty}\left(-\delta(\partial\mathcal{E}_{j'_l})(\hat\phi h_{\varepsilon_{j'_l}}) - \int S^\perp(\nabla\hat\phi)\cdot h_{\varepsilon_{j'_l}}\, d(\partial\mathcal{E}_{j'_l})\right) = \liminf_{l\to\infty}\left(-\delta(\partial\mathcal{E}_{j_l})(\hat\phi h_{\varepsilon_{j_l}}) - \int S^\perp(\nabla\hat\phi)\cdot h_{\varepsilon_{j_l}}\, d(\partial\mathcal{E}_{j_l})\right).$$

Using (9.12) and (9.13), the right-hand side of (9.17) may be bounded by $\liminf_{l\to\infty} 2b_{j_l} + c$ from above. The left-hand side of (9.17) is similarly estimated from below by $\limsup_{l\to\infty} \frac{1}{2}b_{j'_l} - c$. Thus, for any subsequence satisfying (9.16), we have (evaluation at t)

$$(9.18)\quad \limsup_{l\to\infty}\int_{\mathbb{R}^{n+1}} \frac{\hat\phi|\Phi_{\varepsilon_{j'_l}} * \delta(\partial\mathcal{E}_{j'_l})|^2}{\Phi_{\varepsilon_{j'_l}} * \|\partial\mathcal{E}_{j'_l}\| + \varepsilon_{j'_l}\Omega^{-1}}\,dx \leqslant 4\liminf_{l\to\infty}\int_{\mathbb{R}^{n+1}} \frac{\hat\phi|\Phi_{\varepsilon_{j_l}} * \delta(\partial\mathcal{E}_{j_l})|^2}{\Phi_{\varepsilon_{j_l}} * \|\partial\mathcal{E}_{j_l}\| + \varepsilon_{j_l}\Omega^{-1}}\,dx + c$$

where c is a constant estimated from above in terms of $\|\partial\mathcal{E}_0\|(\Omega)$, t_2, $\|\phi\|_{C^2}$, $\min_{x\in\operatorname{spt}\phi}\Omega$ and c_1. Define the right-hand side of (9.18) as $\tilde M(t)$ in the following.

For any t with $\tilde M(t) < \infty$, by Lemma 9.1(b) (note $\hat\phi \geqslant i^{-1}\Omega$), the full sequence $\{\partial\mathcal{E}_{j'_l}\}_{l=1}^\infty$ converges to $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ with $\mu_t = \|V_t\|$. From $\Omega \leqslant i\hat\phi$, we also have

$$(9.19)\quad \limsup_{l\to\infty}\int_{\mathbb{R}^{n+1}} \frac{\Omega|\Phi_{\varepsilon_{j'_l}} * \delta(\partial\mathcal{E}_{j'_l}(t))|^2}{\Phi_{\varepsilon_{j'_l}} * \|\partial\mathcal{E}_{j'_l}(t)\| + \varepsilon_{j'_l}\Omega^{-1}}\,dx \leqslant i\tilde M(t).$$

Set $M := \|\partial\mathcal{E}_0\|(\Omega)\exp(c_1^2 t_2/2)$ so that we have

$$(9.20)\quad \limsup_{l\to\infty}\sup_{t\in[0,t_2]} \|\partial\mathcal{E}_{j_l}(t)\|(\Omega) \leqslant M.$$

By (9.16), (9.10), (9.12) and Lemma 9.1 (c), we have

$$\begin{aligned}(9.21)\quad \limsup_{l\to\infty}\ &\delta(\partial\mathcal{E}_{j_l}(t),\hat\phi)(h_{\varepsilon_{j_l}}(\cdot,\partial\mathcal{E}_{j_l}(t)))\\ &= \lim_{l\to\infty}\ \delta(\partial\mathcal{E}_{j'_l}(t),\hat\phi)(h_{\varepsilon_{j'_l}}(\cdot,\partial\mathcal{E}_{j'_l}(t)))\\ &\leqslant -\int_{\mathbb{R}^{n+1}}|h(\cdot,V_t)|^2\hat\phi\,d\|V_t\|\\ &\quad+\limsup_{l\to\infty}\int_{\mathbf{G}_n(\mathbb{R}^{n+1})}S^\perp(\nabla\hat\phi)\cdot h_{\varepsilon_{j'_l}}(\cdot,\partial\mathcal{E}_{j'_l}(t))\,d(\partial\mathcal{E}_{j'_l}(t)).\end{aligned}$$

Let $\epsilon>0$ be arbitrary. Since $V_t\in\mathbf{IV}_n(\mathbb{R}^{n+1})$, there exists $\|V_t\|$ measurable, countably n-rectifiable set $C\subset\mathbb{R}^{n+1}$ such that

$$(9.22)\quad \int_{\mathbf{G}_n(\mathbb{R}^{n+1})}S^\perp(\nabla\phi(x))\,dV_t(x,S)=\int_{\mathbb{R}^{n+1}}(\mathrm{Tan}^n(C,x))^\perp(\nabla\phi(x))\,d\|V_t\|(x)$$

and $x\longmapsto(\mathrm{Tan}^n(C,x))^\perp(\nabla\phi(x))\Omega(x)^{-\frac12}$ is a $\|V_t\|$ measurable function on $\mathbb{R}^{n+1}$. Hence, corresponding to $\epsilon>0$, there exist $g\in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ and $m'\in\mathbb{N}$ such that $g\in\mathcal{B}_{m'}$ and

$$(9.23)\qquad \int_{\mathbb{R}^{n+1}}|(\mathrm{Tan}^n(C,x))^\perp(\nabla\phi(x))-g(x)|^2\Omega(x)^{-1}\,d\|V_t\|(x)<\epsilon^2.$$

Now we compute as (omitting t dependence for simplicity)

$$\begin{aligned}(9.24)\quad &\int_{\mathbf{G}_n(\mathbb{R}^{n+1})}S^\perp(\nabla\hat\phi)\cdot h_{\varepsilon_{j'_l}}\,d(\partial\mathcal{E}_{j'_l})\\ &=\int_{\mathbf{G}_n(\mathbb{R}^{n+1})}(S^\perp(\nabla\hat\phi)-g)\cdot h_{\varepsilon_{j'_l}}\,d(\partial\mathcal{E}_{j'_l})\\ &\quad+\left(\int_{\mathbb{R}^{n+1}}g\cdot h_{\varepsilon_{j'_l}}d(\partial\mathcal{E}_{j'_l})+\delta(\partial\mathcal{E}_{j'_l})(g)\right)-\delta(\partial\mathcal{E}_{j'_l})(g)+\delta V_t(g)\\ &\quad+\int_{\mathbb{R}^{n+1}}h(\cdot,V_t)\cdot(g-(\mathrm{Tan}^n(C,x))^\perp(\nabla\hat\phi))\,d\|V_t\|\\ &\quad+\int_{\mathbf{G}_n(\mathbb{R}^{n+1})}h(\cdot,V_t)\cdot S^\perp(\nabla\hat\phi)\,dV_t(\cdot,S).\end{aligned}$$

We estimate each term of (9.24). We have

$$
\begin{aligned}
(9.25)\quad & \left|\int_{\mathbf{G}_n(\mathbb{R}^{n+1})} (S^{\perp}(\nabla\hat{\phi}) - g)\cdot h_{\varepsilon_{j'_l}}\, d(\partial\mathcal{E}_{j'_l})\right| \\
& \leqslant i^{-1}\int_{\mathbb{R}^{n+1}} |\nabla\Omega||h_{\varepsilon_{j'_l}}|\, d\|\partial\mathcal{E}_{j'_l}\| \\
& \quad + \left(\int_{\mathbf{G}_n(\mathbb{R}^{n+1})} |S^{\perp}(\nabla\phi) - g|^2\Omega^{-1}\, d(\partial\mathcal{E}_{j'_l})\right)^{\frac12} \left(\int_{\mathbb{R}^{n+1}} |h_{\varepsilon_{j'_l}}|^2\Omega\, d\|\partial\mathcal{E}_{j'_l}\|\right)^{\frac12} \\
& \leqslant i^{-1}c_1(\|\partial\mathcal{E}_{j'_l}\|(\Omega))^{\frac12} \left(\int_{\mathbb{R}^{n+1}} |h_{\varepsilon_{j'_l}}|^2\Omega\, d\|\partial\mathcal{E}_{j'_l}\|\right)^{\frac12} \\
& \quad + \left(\int_{\mathbf{G}_n(\mathbb{R}^{n+1})} |S^{\perp}(\nabla\phi) - g|^2\Omega^{-1}\, d(\partial\mathcal{E}_{j'_l})\right)^{\frac12} \left(\int_{\mathbb{R}^{n+1}} |h_{\varepsilon_{j'_l}}|^2\Omega\, d\|\partial\mathcal{E}_{j'_l}\|\right)^{\frac12}.
\end{aligned}
$$

Since $\partial\mathcal{E}_{j'_l}$ converges to V_t as varifold,

$$
\begin{aligned}
(9.26)\quad \lim_{l\to\infty} & \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} |S^{\perp}(\nabla\phi) - g|^2\Omega^{-1}\, d(\partial\mathcal{E}_{j'_l}) \\
&= \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} |S^{\perp}(\nabla\phi) - g|^2\Omega^{-1}\, dV_t \\
&= \int_{\mathbb{R}^{n+1}} |(\mathrm{Tan}^n(C,x))^{\perp}(\nabla\phi) - g|^2\Omega^{-1}\, d\|V_t\| < \epsilon^2
\end{aligned}
$$

where we used (9.23). Using (5.24) and (9.19), (9.20), (9.25) and (9.26), we have

$$
\begin{aligned}
(9.27)\quad \limsup_{l\to\infty} & \left|\int_{\mathbf{G}_n(\mathbb{R}^{n+1})} (S^{\perp}(\nabla\hat{\phi}) - g)\cdot h_{\varepsilon_{j'_l}}\, d(\partial\mathcal{E}_{j'_l})\right| \\
& \leqslant c_1 M^{\frac12}(\tilde{M}(t))^{\frac12} i^{-\frac12} + (i\tilde{M}(t))^{\frac12}\epsilon.
\end{aligned}
$$

By Proposition 5.5 and (9.19), we have

$$
(9.28)\qquad \lim_{l\to\infty}\left|\int_{\mathbb{R}^{n+1}} g\cdot h_{\varepsilon_{j'_l}}\, d(\partial\mathcal{E}_{j'_l}) + \delta(\partial\mathcal{E}_{j'_l})(g)\right| = 0
$$

and the varifold convergence shows

$$
(9.29)\qquad \lim_{l\to\infty} |-\delta(\partial\mathcal{E}_{j'_l})(g) + \delta V_t(g)| = 0.
$$

For the second last term of (9.24),

$$(9.30)\quad \left|\int_{\mathbb{R}^{n+1}} h(\cdot,V_t)\cdot(g-(\mathrm{Tan}^n(C,x))^{\perp}(\nabla\hat\phi))\,d\|V_t\|\right| \\ \leqslant i^{-1}\int_{\mathbb{R}^{n+1}}|h(\cdot,V_t)||\nabla\Omega|\,d\|V_t\| \\ +\int_{\mathbb{R}^{n+1}}|h(\cdot,V_t)||g-(\mathrm{Tan}^n(C,x))^{\perp}(\nabla\phi)|\,d\|V_t\| \\ \leqslant i^{-\frac12}c_1M^{\frac12}(\tilde M(t))^{\frac12}+(i\tilde M(t))^{\frac12}\epsilon$$

where we used the Cauchy–Schwarz inequality, (9.19), (9.20) (which also hold for the limiting quantities) and (9.23). For the last term of (9.24), estimating as in (9.30),

$$(9.31)\quad \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} h(\cdot,V_t)\cdot S^{\perp}(\nabla\hat\phi)\,dV_t \\ \leqslant \int_{\mathbf{G}_n(\mathbb{R}^{n+1})} h(\cdot,V_t)\cdot S^{\perp}(\nabla\phi)\,dV_t + i^{-\frac12}c_1M^{\frac12}(\tilde M(t))^{\frac12} \\ =\int_{\mathbb{R}^{n+1}} h(\cdot,V_t)\cdot\nabla\phi\,d\|V_t\| + i^{-\frac12}c_1M^{\frac12}(\tilde M(t))^{\frac12}$$

where we used (2.3). Finally, combining (9.24), (9.27)–(9.31) and letting $\epsilon\to 0$, we obtain

$$(9.32)\quad \limsup_{l\to\infty}\int_{\mathbf{G}_n(\mathbb{R}^{n+1})} S^{\perp}(\nabla\hat\phi)\cdot h_{\varepsilon_{j'_l}}\,d(\partial\mathcal{E}_{j'_l}) \\ \leqslant 3c_1 i^{-\frac12}M^{\frac12}(\tilde M(t))^{\frac12}+\int_{\mathbb{R}^{n+1}} h(\cdot,V_t)\cdot\nabla\phi\,d\|V_t\|.$$

From (9.21) and (9.32), we obtain

$$(9.33)\quad \limsup_{l\to\infty}\ \delta(\partial\mathcal{E}_{j_l}(t),\hat\phi)(h_{\varepsilon_{j_l}}(\cdot,\partial\mathcal{E}_{j_l}(t))) \\ \leqslant \delta(V_t,\phi)(h(\cdot,V_t))+3c_1i^{-\frac12}(M+\tilde M(t)).$$

Since $\hat\phi\leqslant\Omega$, we have by Fatou's Lemma that

$$(9.34)\quad \int_{t_1}^{t_2}\tilde M(t)\,dt \\ \leqslant 4\liminf_{l\to\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^{n+1}}\frac{\Omega|\Phi_{\varepsilon_{j_l}}*\delta(\partial\mathcal{E}_{j_l}(t))|^2}{\Phi_{\varepsilon_{j_l}}*\|\partial\mathcal{E}_{j_l}(t)\|+\varepsilon_{j_l}\Omega^{-1}}\,dxdt+c<\infty$$

by (6.19). Thus, by (9.7), (9.9), (9.15), (9.33), (9.34) and letting $i\to\infty$, we obtain (9.5) for time-independent $\phi\in C_c^{\infty}(\mathbb{R}^{n+1};\mathbb{R}^+)$. For time dependent

$\phi \in C_c^\infty(\mathbb{R}^{n+1}\times\mathbb{R}^+;\mathbb{R}^+)$, we repeat the same argument. We similarly define $\hat\phi$ and use (6.5) with $\hat\phi(\cdot,t)$. Instead of (9.6), we obtain a formula with one extra term, namely,

$$\begin{aligned}\|\partial\mathcal{E}_{j_l}(s)\|(\hat\phi(\cdot,s))\Big|_{s=t-\Delta t_{j_l}}^{t} &\leqslant \Delta t_{j_l}\big\{\delta(\partial\mathcal{E}_{j_l}(t),\hat\phi(\cdot,t))(h_{\varepsilon_{j_l}}(\cdot,\partial\mathcal{E}_{j_l}(t)))+\varepsilon_{j_l}^{\frac{1}{8}}\big\}\\ &\quad+\|\partial\mathcal{E}_{j_l}(t-\Delta t_{j_l})\|(\phi(\cdot,t)-\phi(\cdot,t-\Delta t_{j_l})).\end{aligned} \tag{9.35}$$

Note that the last term has ϕ instead of $\hat\phi$. A similar inequality to (9.7) will have the summation of the last term of (9.35). It is not difficult to check using (6.18) and Lemma 9.1 (a) that we have

$$\begin{aligned}\lim_{l\to\infty}\sum_{k=k_1}^{k_2}&\|\partial\mathcal{E}_{j_l}((k-1)\Delta t_{j_l})\|(\phi(\cdot,k\Delta t_{j_l})-\phi(\cdot,(k-1)\Delta t_{j_l})\\ &=\lim_{l\to\infty}\sum_{k=k_1}^{k_2}\|\partial\mathcal{E}_{j_l}(k\Delta t_{j_l})\|\left(\frac{\partial\phi}{\partial t}(\cdot,k\Delta t_{j_l})\right)\Delta t_{j_l}\\ &=\lim_{l\to\infty}\int_{t_1}^{t_2}\|\partial\mathcal{E}_{j_l}(t)\|\left(\frac{\partial\phi}{\partial t}(\cdot,t)\right)dt\\ &=\int_{t_1}^{t_2}\|V_t\|\left(\frac{\partial\phi}{\partial t}(\cdot,t)\right)dt,\end{aligned} \tag{9.36}$$

where we also used the dominated convergence theorem in the last step. The rest proceeds by the same argument with error estimates coming from the time-dependency of $\hat\phi$. For example, in (9.15), we need to regard $\hat\phi(\cdot,t)$ as a piecewise constant function with respect to time variable on $[t_1,t_2]$, namely, in place of $\hat\phi$, we need to have

$$\hat\phi_{j_l}(\cdot,t):=\hat\phi(\cdot,k\Delta t_{j_l})\text{ if }t\in((k-1)\Delta t_{j_l},k\Delta t_{j_l}]. \tag{9.37}$$

For $\delta(\partial\mathcal{E}_{j_l}(t),\hat\phi_{j_l}(\cdot,t))(h_{\varepsilon_{j_l}}(\cdot,\partial\mathcal{E}_{j_l}(t)))$ in the last line of (9.15), if we replace $\hat\phi_{j_l}(\cdot,t)$ by $\hat\phi(\cdot,t)$, it only results in errors of order Δt_{j_l} times certain negative power of ε_{j_l} which remains small and goes to 0 uniformly as $l\to\infty$. Thus we may subsequently proceed just like the time independent case and we have (9.5) for C_c^∞ case, and by approximation for C_c^1 case. □

Now, the proof of Theorem 3.2 is complete: (1) is clear from the construction using $\mathcal{E}_0=\{E_{0,i}\}_{i=1}^N$, (2) is by Lemma 9.1(a) and (c), (3) and (4) follow from Theorem 9.3. We note that the claim of Theorem 3.6 is slightly different from [32, 45] in that it is stated for $(x,t)\in\mathbb{R}^{n+1}\setminus S_t$ here instead of $\operatorname{spt}\|V_t\|\setminus S_t$, allowing a possibility of $O_{(x,t)}\cap\operatorname{spt}\mu$ being empty. But exactly the same proof of [32] gives this slightly stronger claim of partial regularity and we write the result in this form.

10. Proof of Theorem 3.5

Let μ be a measure on $\mathbb{R}^{n+1}\times\mathbb{R}^+$ defined as in Definition 3.3.

LEMMA 10.1. — *We have the following properties for μ and $\{V_t\}_{t\in\mathbb{R}^+}$.*

(1) $\operatorname{spt}\|V_t\|\subset\{x\in\mathbb{R}^{n+1}:(x,t)\in\operatorname{spt}\mu\}$ *for all* $t>0$.

(2) $\operatorname{clos}\{(x,t):x\in\operatorname{spt}\|V_t\|,V_t\in\mathbf{IV}_n(\mathbb{R}^{n+1})\}\cap\{(x,t):t>0\}=\operatorname{spt}\mu\cap\{(x,t):t>0\}$.

Proof. — Suppose $x\in\operatorname{spt}\|V_t\|$ and $t>0$. Then for any $r>0$, there exists some $\phi\in C^2_c(U_{2r}(x);\mathbb{R}^+)$ with $\|V_t\|(\phi)>0$. For any $t'\in[0,t)$, by (9.5) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}(10.1)\quad \|V_t\|(\phi)-\|V_{t'}\|(\phi)&\\ \leqslant\int_{t'}^t\int_{U_{2r}(x)}&-|h(\cdot,V_s)|^2\phi+\nabla\phi\cdot h(\cdot,V_s)\,d\|V_s\|ds\\ \leqslant\int_{t'}^t\int_{U_{2r}(x)}&\frac{|\nabla\phi|^2}{2\phi}\,d\|V_s\|ds\\ \leqslant(t-t')&\|\phi\|_{C^2}\sup_{s\in[t',t]}\|V_s\|(U_{2r}(x)).\end{aligned}$$

Choosing t' sufficiently close to t, (10.1) shows that there exists some $t'<t$ such that $\frac{1}{2}\|V_t\|(\phi)\leqslant\|V_s\|(\phi)$ for all $s\in[t',t)$. Thus, $\int_{U_{2r}(x)\times[t',t)}\phi\,d\mu\geqslant\frac{1}{2}(t-t')\|V_t\|(\phi)>0$. If $(x,t)\notin\operatorname{spt}\mu$, there must be some open set U in $\mathbb{R}^{n+1}\times\mathbb{R}^+$ with $\mu(U)=0$, but this is a contradiction to the preceding sentence. Thus we have (1).

Suppose $(x,t)\in\operatorname{clos}\{(x,t):x\in\operatorname{spt}\|V_t\|,V_t\in\mathbf{IV}_n(\mathbb{R}^{n+1})\}\cap\{(x,t):t>0\}$. Then there exists a sequence $\{(x_i,t_i)\}_{i=1}^\infty$ such that $x_i\in\operatorname{spt}\|V_{t_i}\|$, $t_i>0$ and $\lim_{i\to\infty}(x_i,t_i)=(x,t)$. By (1), $(x_i,t_i)\in\operatorname{spt}\mu$. Since $\operatorname{spt}\mu$ is a closed set by definition, we have $(x,t)\in\operatorname{spt}\mu$, proving $\subset$ of (2). Given $(x,t)\in\operatorname{spt}\mu$ with $t>0$ and $\epsilon>0$, we have $\mu(B_\epsilon(x)\times(t-\epsilon,t+\epsilon))>0$. Then, there must be some $t'\in(t-\epsilon,t+\epsilon)$ such that $\|V_{t'}\|(B_\epsilon(x))>0$ and $V_{t'}\in\mathbf{IV}_n(\mathbb{R}^{n+1})$. If $\operatorname{spt}\|V_{t'}\|\cap B_\epsilon(x)=\emptyset$, then we would have $\|V_{t'}\|(B_\epsilon(x))=0$, a contradiction. Thus we have some $x'\in\operatorname{spt}\|V_{t'}\|\cap B_\epsilon(x)$ with $V_{t'}\in\mathbf{IV}_n(\mathbb{R}^{n+1})$ and $|t'-t|<\epsilon$. Since $\epsilon>0$ is arbitrary, this proves $\supset$ of (2). □

Remark 10.2. — In (1), it may happen that the left-hand side is strictly smaller than the right-hand side. For example, consider a shrinking sphere.

At the moment of vanishing, we have $\|V_t\| = 0$ since it is a point and has zero measure, thus $\operatorname{spt}\|V_t\| = \emptyset$. On the other hand, the vanishing point is in $\operatorname{spt}\mu$, and the right-hand side is not the empty set. We may also encounter a situation where some portion of measure vanishes, thus the difference between the left- and right-hand sides of (1) may be of positive $\mathcal{H}^n$ measure. We also point out that, in general, (1) and (2) are not true if $t = 0$ is included. We may have some portion of measure $\|\partial\mathcal{E}_0\|$ vanishing instantly at $t = 0$. For example, consider on $\mathbb{R}^2$ a line segment with two end points which is surrounded by one of open partitions. For the first Lipschitz deformation step, such line segment may be eliminated as we indicated in 4.3.2. Thus, even though we have some positive measure at $t = 0$, $\operatorname{spt}\mu$ may be empty nearby.

Let $\eta \in C_c^\infty(U_2;\mathbb{R}^+)$ be a radially symmetric function such that $\eta = 1$ on B_1, $|\nabla\eta| \leqslant 2$ and $\|\nabla^2\eta\| \leqslant 4$. Then define for $x, y \in \mathbb{R}^{n+1}$, $s, t \in \mathbb{R}$ with $s > t$ and $R > 0$

$$\begin{aligned} \rho_{(y,s)}(x,t) &:= \frac{1}{(4\pi(s-t))^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4(s-t)}\right), \\ \hat\rho_{(y,s)}(x,t) &:= \eta(x-y)\rho_{(y,s)}(x,t), \\ \hat\rho^R_{(y,s)}(x,t) &:= \eta\left(\frac{x-y}{R}\right)\rho_{(y,s)}(x,t). \end{aligned} \tag{10.2}$$

We often write $\rho_{(y,s)}$ or ρ for $\rho_{(y,s)}(x,t)$ when the meaning is clear from the context and the same for $\hat\rho_{(y,s)}$ and $\hat\rho^R_{(y,s)}$. The following is a variant of well-known Huisken's monotonicity formula [26]. We include the outline of proof and the reader is advised to see [32, Lemma 6.1] for more details.

Lemma 10.3. — *There exists c_6 depending only on n with the following property. For $0 \leqslant t_1 < t_2 < s < \infty$, $y \in \mathbb{R}^{n+1}$ and $R > 0$, we have*

$$\|V_t\|(\hat\rho^R_{(y,s)}(\cdot,t))\Big|_{t=t_1}^{t_2} \leqslant c_6 R^{-2}(t_2-t_1) \sup_{t'\in[t_1,t_2]} R^{-n}\|V_{t'}\|(B_{2R}(y)). \tag{10.3}$$

Proof. — After change of variables by $\tilde x = (x-y)/R$ and $\tilde t = (t-s)/R^2$, we may regard $R = 1$ and $(y,s) = (0,0)$. A direct computation shows that for any $S \in \mathbf{G}(n+1,n)$, we have

$$\frac{\partial\rho}{\partial t} + S\cdot\nabla_x^2\rho + \frac{|S^\perp(\nabla_x\rho)|^2}{\rho} = 0$$

for all $t < 0$ and $x \in \mathbb{R}^{n+1}$. The same computation for $\hat\rho$ has some extra terms coming from differentiations of η, and such terms are bounded by

$c(n)(-t)^{-\frac{n}{2}}\exp(1/4t)$ since $\operatorname{spt}|\nabla\eta| \subset B_2 \setminus U_1$. Thus we have

$$\left|\frac{\partial\hat\rho}{\partial t} + S\cdot\nabla_x^2\hat\rho + \frac{|S^\perp(\nabla_x\hat\rho)|^2}{\hat\rho}\right| \leqslant c_6\chi_{B_2\setminus U_1}. \tag{10.4}$$

Use $\hat\rho$ in (9.5) as well as (10.4) to find that

$$\|V_t\|(\hat\rho_{(0,0)}(\cdot,t))\Big|_{t=t_1}^{t_2} \leqslant c_6\int_{t_1}^{t_2}\|V_{t'}\|(B_2\setminus U_1)\,dt'. \tag{10.5}$$

Then (10.5) gives (10.3). □

Lemma 10.4. — *For any $\lambda > 1$, there exists $c_7 \in (1,\infty)$ depending only on n, λ, Ω and $\|\partial\mathcal{E}_0\|(\Omega)$ such that*

$$\sup_{x\in B_\lambda, r\in(0,1], t\in[\lambda^{-1},\lambda]} r^{-n}\|V_t\|(B_r(x)) \leqslant c_7. \tag{10.6}$$

Proof. — We use (10.3) with $s = t + r^2$, $t_2 = t \in [\lambda^{-1},\lambda]$, $t_1 = 0$, $R = 1$ and $y \in B_\lambda$. Then we obtain also using $\eta\lfloor_{B_1(y)} = 1$ that

$$\begin{aligned}\frac{e^{-\frac14}}{(4\pi r^2)^{\frac n2}}&\|V_t\|(B_r(y)) \\ &\leqslant \frac{1}{(4\pi t)^{\frac n2}}\|V_0\|(B_2(y)) + c_6 t\sup_{t'\in[0,t]}\|V_{t'}\|(B_2(y)).\end{aligned} \tag{10.7}$$

The quantities on the right-hand side of (10.7) are all controlled by the stated quantities thus we obtain (10.6). □

Remark 10.5. — If $\|\partial\mathcal{E}_0\|$ satisfies the density ratio upper bound

$$\sup_{x\in\mathbb{R}^{n+1}, r\in(0,1]} r^{-n}\|\partial\mathcal{E}_0\|(B_r(x)) < \infty, \tag{10.8}$$

then we may obtain up to the initial time estimate for (10.6).

The following is essentially Brakke's clearing out lemma [8, §6.3] proved using Huisken's monotonicity formula.

Lemma 10.6. — *For any $\lambda > 1$, there exist positive constants $c_8, c_9 \in (0,1)$ depending only on n, λ, Ω and $\|\partial\mathcal{E}_0\|(\Omega)$ such that the following holds. For $(x,t) \in \operatorname{spt}\mu \cap (B_\lambda \times [\lambda^{-1},\lambda])$ and $r \in (0,\frac12]$ with $t - c_9 r^2 \geqslant (2\lambda)^{-1}$, we have*

$$\|V_{t-c_9r^2}\|(B_r(x)) \geqslant c_8 r^n. \tag{10.9}$$

Proof. — By Lemma 10.1(2), there exists a sequence $(x_i,t_i) \in \operatorname{spt}\|V_{t_i}\|$ with $\lim_{i\to\infty}(x_i,t_i) = (x,t)$. We may also have $V_{t_i} \in \mathbf{IV}_n(\mathbb{R}^{n+1})$, thus any neighborhood of x_i contains some point of integer density of $\|V_{t_i}\|$. Thus we may as well assume that $\theta^n(\|V_{t_i}\|, x_i) \geqslant 1$. One uses (10.3) with $R = r$, $t_1 = t - c_9 r^2$ (c_9 to be decided), $t_2 = t_i$, $y = x_i$ and $s = t_i + \epsilon$ to obtain

$$\text{(10.10)} \quad \|V_s\|(\hat{\rho}^r_{(x_i,t_i+\epsilon)}(\cdot,s))\Big|_{s=t-c_9r^2}^{t_i} \leqslant c_6 r^{-2}(t_i - t + c_9 r^2) \sup_{s\in[t-c_9r^2,t_i]} r^{-n}\|V_s\|(U_{2r}(x_i)).$$

By letting $\epsilon \to 0+$, $\theta^n(\|V_{t_i}\|, x_i) \geqslant 1$ and (10.10) give

$$\text{(10.11)} \quad 1 \leqslant \|V_{t-c_9r^2}\|(\hat{\rho}^r_{(x_i,t_i)}(\cdot, t - c_9 r^2)) + c_6 r^{-2}(t_i - t + c_9 r^2) \sup_{s\in[t-c_9r^2,t_i]} r^{-n}\|V_s\|(U_{2r}(x_i)).$$

Let $i \to \infty$ for (10.11) to obtain

$$\text{(10.12)} \quad 1 \leqslant \|V_{t-c_9r^2}\|(\hat{\rho}^r_{(x,t)}(\cdot, t-c_9r^2)) + c_6 c_9 \sup_{s\in[t-c_9r^2,t]} r^{-n}\|V_s\|(U_{2r}(x)).$$

We also have $\|V_{t-c_9r^2}\|(\hat{\rho}^r_{(x,t)}(\cdot, t-c_9r^2)) \leqslant (4\pi c_9)^{-\frac{n}{2}} r^{-n}\|V_{t-c_9r^2}\|(U_{2r}(x))$. Now, given λ, let c_7 be a constant obtained in Lemma 10.4 corresponding to λ there equals to 2λ. Suppose we choose $c_9 < (2\lambda)^{-1}$ and $t \geqslant \lambda^{-1}$ so that $t - c_9 r^2 \geqslant (2\lambda)^{-1}$. Then by (10.6) and (10.12), we have

$$\text{(10.13)} \quad 1 \leqslant (4\pi c_9)^{-\frac{n}{2}} r^{-n}\|V_{t-c_9r^2}\|(U_{2r}(x)) + c_6 c_9 2^n c_7.$$

Choose c_9 sufficiently small so that the last term is less than 1/2. Then we have a lower bound for $r^{-n}\|V_{t-c_9r^2}\|(B_{2r}(x))$. By adjusting constants again, we obtain (10.9). □

Remark 10.7. — If we have (10.8), then we may also obtain (10.9) up to $t = 0$, namely, we may replace $[\lambda^{-1}, \lambda]$ in the statement to $(0, \lambda]$ and for $r \in (0, \frac{1}{2}]$ with $t - c_9 r^2 \geqslant 0$.

COROLLARY 10.8. — *For any open set $U \subset B_\lambda$ and $t \in (\lambda^{-1}, \lambda]$, we have*

$$\text{(10.14)} \quad \mathcal{H}^n(\{x \in U : (x,t) \in \operatorname{spt}\mu\}) \leqslant \limsup_{s\to t-} \mathbf{B}_{n+1} c_8^{-1}\omega_n \|V_s\|(U).$$

Proof. — It is enough to prove the estimate for $K_t := \{x \in K : (x,t) \in \operatorname{spt}\mu\}$ where $K \subset U$ is compact and arbitrary. For each $x \in K_t$, for all sufficiently small r, $B_r(x) \subset U$ and by Lemma 10.6, $\|V_{t-c_9r^2}\|(B_r(x)) \geqslant c_8 r^n$. Applying the Besicovitch covering theorem to such family of balls, and recalling the definition of the Hausdorff measure, we have a disjoint family of balls $\{B_r(x_1),\dots,B_r(x_J)\}$ such that ($\mathcal{H}^n_{2r}$ is as defined in [18, Definition 2.1(i)])

$$\begin{aligned}\mathbf{B}_{n+1}^{-1}\mathcal{H}^n_{2r}(K_t) \leqslant J\omega_n r^n \leqslant c_8^{-1}\omega_n \sum_{i=1}^{J}\|V_{t-c_9r^2}\|(B_r(x_i)) \\ \leqslant c_8^{-1}\omega_n\|V_{t-c_9r^2}\|(U).\end{aligned} \tag{10.15}$$

By letting $r \to 0$ for (10.15), we obtain (10.14). □

Remark 10.9. — Lemma 10.1(1) and Corollary 10.8 prove (3.5) of Proposition 3.4.

Lemma 10.10. — *Let $\{\mathcal{E}_{j_l}(t)\}_{l=1}^{\infty}$ be a sequence obtained in Proposition 6.4 and denote the open partitions by $\{E_{j_l,k}(t)\}_{k=1}^N$ for each j_l and $t\in\mathbb{R}^+$, i.e., $\mathcal{E}_{j_l}(t)=\{E_{j_l,k}(t)\}_{k=1}^N$. For fixed $k\in\{1,\dots,N\}$, $0<r<\infty$, $x\in\mathbb{R}^{n+1}$ and $t>0$ with $t-r^2>0$, suppose*

$$\lim_{l\to\infty}\mathcal{L}^{n+1}(B_{2r}(x)\setminus E_{j_l,k}(t))=0 \tag{10.16}$$

and

$$\mu(B_{2r}(x)\times[t-r^2,t+r^2])=0. \tag{10.17}$$

Then for all $t'\in(t-r^2,t+r^2]$, we have

$$\lim_{l\to\infty}\mathcal{L}^{n+1}(B_r(x)\setminus E_{j_l,k}(t'))=0. \tag{10.18}$$

Proof. — For a contradiction, if (10.18) were not true for some $t'\in(t-r^2,t+r^2]$, by compactness of BV functions, there exists a subsequence $\{j_l'\}_{l=1}^{\infty}$ such that $\chi_{E_{j_l',k}(t')}$ converges to $\chi_{E_k(t')}$ in $L^1(B_{2r}(x))$ and $\mathcal{L}^{n+1}(B_r(x)\setminus E_k(t'))>0$. By the lower semicontinuity property, we have $\|\nabla\chi_{E_k(t')}\| \leqslant \|V_{t'}\|$. By Lemma 10.1(1) and (10.17), we have $\|\nabla\chi_{E_k(t')}\|(B_{2r}(x))=0$. Then, $\chi_{E_k(t')}$ is a constant function on $B_{2r}(x)$ and is identically 1 or 0. Since $\mathcal{L}^{n+1}(B_r(x)\setminus E_k(t'))>0$, $\chi_{E_k(t')}=0$ on $B_{2r}(x)$. Repeating the same argument, we may conclude that there exist some $k'\in\{1,\dots,N\}$, $k'\neq k$, and a subsequence (denoted again by $\{j_l'\}_{l=1}^{\infty}$) such that $\chi_{E_{j_l',k'}(t')}$ converges to $\chi_{E_{k'}(t')}$ and $\mathcal{L}^{n+1}(B_{2r}(x)\setminus E_{k'}(t'))=0$. Thus, we have a situation where, at time t, $E_{j_l',k}(t)$ occupies most of $B_{2r}(x)$ while at time t', $E_{j_l',k'}(t')$ occupies most of $B_{2r}(x)$ for all large l. In particular, for all sufficiently large l, we have $\mathcal{L}^{n+1}(B_{2r}(x)\setminus E_{j_l',k}(t))<\omega_{n+1}r^{n+1}/10$ and $\mathcal{L}^{n+1}(B_{2r}(x)\setminus E_{j_l',k'}(t'))<\omega_{n+1}r^{n+1}/10$. The maps f_1 and f_2 for the construction of $\{\mathcal{E}_{j,l}\}$ in Proposition 6.1 change volume of each open partitions very little at each step (note Definition 4.8(b) for f_1, and f_2 is diffeomorphism which is close to identity, see (5.59) and (5.60)), there exists some

$t_l \in (t, t')$ (or (t', t)) such that $\frac{1}{4}\omega_{n+1} r^{n+1} \leqslant \mathcal{L}^{n+1}(B_r(x) \cap E_{j'_l,k}(t_l)) \leqslant \frac{3}{4}\omega_{n+1} r^{n+1}$. By the relative isoperimetric inequality, there exists a positive constant c depending only on n such that

$$\|\partial \mathcal{E}_{j'_l}(t_l)\|(B_r(x)) \geqslant \|\nabla \chi_{E_{j'_l,k}(t_l)}\|(B_r(x)) \geqslant c r^n. \tag{10.19}$$

We may assume without loss of generality that $t_l \in 2_{\mathbb{Q}}$. Fix an arbitrary $\hat{t} \in 2_{\mathbb{Q}} \cap (t - r^2, \min\{t, t'\})$. Choose $\phi \in C^2_c(U_{2r}(x); \mathbb{R}^+)$ such that $\phi = 1$ on $B_r(x)$ and $0 \leqslant \phi \leqslant 1$ on $U_{2r}(x)$. Now, we repeat the same argument leading to (6.25) with $t_2 = t_l$ and $t_1 = \hat{t}$ to obtain

$$\begin{aligned}
\liminf_{l\to\infty} \Big(\|\partial \mathcal{E}_{j'_l}(t_l)\|(\phi) - \|\partial \mathcal{E}_{j'_l}(\hat{t})\|(\phi + i^{-1}\Omega) \Big) \\
\leqslant \liminf_{l\to\infty} \frac{1}{2} \int_{\hat{t}}^{t_l} \int_{\mathbb{R}^{n+1}} \frac{|\nabla(\phi + i^{-1}\Omega)|^2}{\phi + i^{-1}\Omega} \, d\|\partial \mathcal{E}_{j'_l}(t)\| dt \\
\leqslant \liminf_{l\to\infty} \int_{\hat{t}}^{t_l} \int_{\mathbb{R}^{n+1}} \frac{|\nabla \phi|^2}{\phi} + i^{-1} c_1^2 \Omega \, d\|\partial \mathcal{E}_{j'_l}(t)\| dt \\
\leqslant i^{-1} c_1^2 \int_{\hat{t}}^{t+r^2} \|V_t\|(\Omega)\, dt,
\end{aligned} \tag{10.20}$$

where we used the dominated convergence theorem and $\|V_t\|(U_{2r}(x)) = 0$ which follows from (10.17). Since $\|\partial \mathcal{E}_{j'_l}(\hat{t})\|(\phi) \to \|V_{\hat{t}}\|(\phi) = 0$, (10.20) proves after letting $i \to \infty$ that $\liminf_{l\to\infty} \|\partial \mathcal{E}_{j'_l}(t_l)\|(\phi) = 0$. But this would be a contradiction to (10.19). □

Lemma 10.11. — *Let $\{\mathcal{E}_{j_l}(t)\}_{l=1}^{\infty}$ and $\{E_{j_l,k}(t)\}_{k=1}^{N}$ be the same as Lemma 10.10. For fixed $k \in \{1, \dots, N\}$, $0 < r < \infty$, $x \in \mathbb{R}^{n+1}$, suppose*

$$B_{2r}(x) \subset E_{j_l,k}(0) \tag{10.21}$$

for all $l \in \mathbb{N}$ and

$$\mu(B_{2r}(x) \times [0, r^2]) = 0. \tag{10.22}$$

Then, for all $t' \in (0, r^2]$, we have

$$\lim_{l\to\infty} \mathcal{L}^{n+1}(B_r(x) \setminus E_{j_l,k}(t')) = 0. \tag{10.23}$$

Proof. — By (10.21), we have $\|V_0\|(B_{2r}(x)) = 0$ and Proposition 10.1(1) and (10.22) show $\|V_t\|(U_{2r}(x)) = 0$ for $t \in (0, r^2]$. Then, we may argue just like the proof of Lemma 10.10, where we take $\hat{t}$ there by $\hat{t} = 0$. We omit the proof since it is similar. □

The following Lemma 10.12 is from [8, §3.7, "Sphere barrier to external varifolds"].

Lemma 10.12. — *For some $t \in \mathbb{R}^+$, $x \in \mathbb{R}^{n+1}$ and $r > 0$, suppose $\|V_t\|(U_r(x)) = 0$. Then for $t' \in [t, t+\frac{r^2}{2n}]$, we have $\|V_{t'}\|(U_{\sqrt{r^2-2n(t'-t)}}(x)) = 0$.*

Finally, we give a proof of Theorem 3.5.

Proof. — We may choose a subsequence so that for all $t \in 2_{\mathbb{Q}}$, each $\chi_{E_{j_l,k}(t)}$ converges in $L^1_{loc}(\mathbb{R}^{n+1})$ to $\chi_{E_k(t)}$ as $l \to \infty$. This is due to the mass bound and L^1 compactness of BV functions. Consider the complement of $\operatorname{spt}\mu \cup (\operatorname{spt}\|V_0\| \times \{0\})$ in $\mathbb{R}^{n+1} \times \mathbb{R}^+$ which is open in $\mathbb{R}^{n+1} \times \mathbb{R}^+$, and let S be a connected component. For any point $(x,t) \in S$, there exists $r > 0$ such that $B_{2r}(x) \times [t-r^2, t+r^2] \subset S$ if $t > 0$, and $B_{2r}(x) \times [0, r^2] \subset S$ if $t = 0$. First consider the case $t = 0$. Since $B_{2r}(x)$ is in the complement of $\operatorname{spt}\|V_0\| = \Gamma_0$, for some small enough $0 < t' \leqslant r^2$, Lemma 10.12 shows that $\operatorname{spt}\mu \cap (B_r(x) \times [0,t']) = \emptyset$. Since $B_{2r}(x) \subset \mathbb{R}^n \setminus \Gamma_0$, there exists some $i(x,0) \in \{1,\dots,N\}$ such that $B_{2r}(x) \subset E_{0,i(x,0)}$, thus $B_{2r}(x) \subset E_{j_l,i(x,0)}(0)$ for all l. Then, by Lemma 10.11, for some $r' \in (0, r/2)$, we have $\lim_{l\to\infty} \mathcal{L}^{n+1}(B_{r'}(x) \setminus E_{j_l,i(x,0)}(\tilde{t})) = 0$ for all $\tilde{t} \in (0, (r')^2)$. Similarly, for $t > 0$, using Lemma 10.10, there exist $i(x,t) \in \{1,\dots,N\}$ and $r' \in (0, r/2)$ such that $\lim_{l\to\infty} \mathcal{L}^{n+1}(B_{r'}(x) \setminus E_{j_l,i(x,t)}(\tilde{t})) = 0$ for all $\tilde{t} \in (t-(r')^2, t+(r')^2)$. By the connectedness of S, $i(x,t)$ has to be all equal to some $i \in \{1,\dots,N\}$ on S. This also shows that $\chi_{E_{j_l,i}}(t)$ converges to 1 in L^1 locally on $\{x : (x,t) \in S\}$ for all t. Now, for each $i \in \{1,\dots,N\}$, define $S(i)$ to be the union of all connected component with this property. Since $E_{0,i} = \{x : (x,0) \in S(i)\}$, each $S(i)$ is nonempty. They are open disjoint sets and $\cup_{i=1}^N S(i) = (\mathbb{R}^{n+1} \times \mathbb{R}^+) \setminus (\operatorname{spt}\mu \cup (\operatorname{spt}\|V_0\| \times \{0\}))$. Define $E_i(t) := \{x : (x,t) \in S(i)\}$. Then it is clear that $\chi_{E_{j_l,i}(t)}$ locally converges to $\chi_{E_i(t)}$ in L^1. Up to this point, the claims (1)–(5) of Theorem 3.5 are proved, in particular, (4) follows from the lower semicontinuity of BV norm.

To prove (6), let $i = \{1,\dots,N\}$ and $R > 0$ be fixed. Without loss of generality, we may assume $x = 0$. Consider $U_R \cap E_i(t)$ which is open. For $r > 0$, set $A_r := \{x \in U_{R-r} \cap E_i(t) : \operatorname{dist}(\partial(U_R \cap E_i(t)), x) < r\}$. Consider a family of closed balls $\{B_{2r}(x) : x \in A_r\}$. By Vitali's covering theorem, we may choose points $x_1,\dots,x_m \in A_r$ such that $\{B_{2r}(x_j)\}_{j=1}^m$ are mutually disjoint and $A_r \subset \cup_{j=1}^m B_{10r}(x_j)$. By the definition of A_r, there exist $\tilde{x}_j \in U_r(x_j) \cap \partial(E_i(t))$ for each $j = 1,\dots,m$. Since $(\partial(E_i(t)) \times \{t\}) \subset \operatorname{spt}\mu$, by Lemma 10.6, $\|V_{t-c_9r^2}\|(B_r(\tilde{x}_j)) \geqslant c_8 r^n$ for $0 < r < r_0$ (with a suitable λ chosen). Since $B_r(\tilde{x}_j) \subset B_{2r}(x_j)$, $\{B_r(\tilde{x}_j)\}_{j=1}^m$ are mutually disjoint. Thus we have

$$c_8 m r^n \leqslant \sum_{j=1}^{m} \|V_{t-c_9 r^2}\|(B_r(\tilde{x}_j)) = \|V_{t-c_9 r^2}\|(\cup_{j=1}^m B_r(\tilde{x}_j)) \leqslant \|V_{t-c_9 r^2}\|(U_{R+r}). \tag{10.24}$$

On the other hand,

$$\mathcal{L}^{n+1}(A_r) \leqslant m\omega_{n+1}(10r)^{n+1} \leqslant (c_8^{-1}\omega_{n+1}10^{n+1}\|V_{t-c_9 r^2}\|(U_{R+r}))r. \tag{10.25}$$

For any $x \in (U_{R-r} \cap E_i(t)) \setminus A_r$, $U_r(x) \subset E_i(t)$ and $\|V_t\|(U_r(x)) = 0$. Thus by Lemma 10.12, there exists $c_{10} > 0$ depending only on n such that $B_{r/2}(x) \subset E_i(\tilde{t})$ for all $\tilde{t} \in [t, t + c_{10}r^2]$. This means $(U_{R-r} \cap E_i(t)) \setminus A_r \subset E_i(\tilde{t})$ for all $\tilde{t} \in [t, t + c_{10}r^2]$. Thus, for such $\tilde{t}$,

$$\begin{aligned}\mathcal{L}^{n+1}&(U_R \cap E_i(t) \setminus E_i(\tilde{t})) \\ &\leqslant \mathcal{L}^{n+1}((U_R \setminus U_{R-r}) \cup A_r) \\ &\leqslant ((n+1)\omega_{n+1}R^n + c_8^{-1}\omega_{n+1}10^{n+1}\|V_{t-c_9 r^2}\|(U_{R+r}))r \\ &=: c_{11}(r)r,\end{aligned} \tag{10.26}$$

where c_{11} is uniformly bounded for small r. The estimate (10.26) holds for any i with the same c_{11}. $\{E_i(t) \cap U_R\}_{i=1}^N$ is mutually disjoint and the union has full $\mathcal{L}^{n+1}$ measure of U_R, and so is $\{E_i(\tilde{t}) \cap U_R\}_{i=1}^N$. Thus, except for a $\mathcal{L}^{n+1}$ zero measure set, we have $E_i(\tilde{t}) \cap U_R \setminus E_i(t) \subset U_R \cap \cup_{i' \neq i} E_{i'}(t) \setminus E_{i'}(\tilde{t})$. Thus

$$\begin{aligned}\mathcal{L}^{n+1}&(U_R \cap E_i(\tilde{t}) \setminus E_i(t)) \\ &\leqslant \sum_{i' \neq i} \mathcal{L}^{n+1}(U_R \cap E_{i'}(t) \setminus E_{i'}(\tilde{t})) \leqslant (N-1)c_{11}r.\end{aligned} \tag{10.27}$$

(10.26) and (10.27) prove that

$$\mathcal{L}^{n+1}(U_R \cap (E_i(t) \triangle E_i(\tilde{t}))) \leqslant N c_{11} r \tag{10.28}$$

for $\tilde{t} \in [t, t + c_{10}r^2]$ and $r < r_0$. We may exchange the role of t and $\tilde{t}$ to obtain the similar estimate for $\tilde{t} < t$. Once this is obtained, local $\frac{1}{2}$-Hölder continuity for g as defined in (6) follows for $t > 0$ using $(A \triangle B) \triangle (A \triangle C) = B \triangle C$ for any sets A, B, C. For $t = 0$, we cannot estimate as above, but we may still prove continuity using Lemma 10.12. If we assume an extra property on $\mathcal{E}_0 = \{E_{0,i}\}_{i=1}^N$, such as, for each $i = 1, \ldots, N$ and $R > 0$, $\mathcal{L}^{n+1}(\{x \in B_{R-r} \cap E_{0,i} : \operatorname{dist}(x, \partial E_{0,i}) < r) \leqslant c(R)r$ for all sufficiently small r, then we can proceed just like above and prove $\frac{1}{2}$-Hölder continuity of g up to $t = 0$. □

11. Additional comments

11.1. Tangent flow

For Brakke flow $\{V_t\}_{t\in\mathbb{R}^+}$, at each point (x,t) in space-time, $t>0$, there exists a tangent flow (see [30, 47] for the definition and proofs) which is again a Brakke flow and which tells the local behavior of the flow at that point. Just like tangent cones of minimal surfaces, tangent flows have a certain homogeneous property and one can stratify the singularity depending on the dimensions of the homogeneity. In this regard, due to the minimizing step in the construction of approximate solutions, one may wonder if some extra property of tangent flow may be derived. As far as the approximate solutions are concerned, as indicated in Section 4.3, unstable singularities are likely to break up into more stable ones by Lipschitz deformation. There should be some aspects on tangent flow which are affected by the choice of $f_1 \in \mathbf{E}(\mathcal{E}_{j,l}, j)$ in (6.9) as elaborated in Remark 6.5. It is a challenging problem to analyze this finer point of the Brakke flow obtained in this paper.

11.2. A short-time regularity

Suppose in addition that Γ_0 satisfies the following density ratio upper bound condition. There exist some $\nu \in (0,1)$ and $r_0 \in (0,\infty)$ such that $\mathcal{H}^n(\Gamma_0 \cap B_r(x)) \leqslant (2-\nu)\omega_n r^n$ for all $r \in (0,r_0)$ and $x \in \mathbb{R}^{n+1}$. Nontrivial examples with singularities satisfying such condition are suitably regular 1-dimensional networks with finite number of triple junctions, since such junctions have density $\frac{3}{2}$. Others are suitably regular 2-dimensional "soap bubble clusters" with singularities of three surfaces with boundaries meeting along a curve, or 6 surfaces with boundaries meeting at a point and 4 curves. They can have densities strictly less than 2. These are interesting classes of examples which are also physically relevant. Under this condition, by using Lemma 10.3, one can prove that there exists $T>0$ such that $\theta^n(\|V_t\|, x) = 1$ for $\|V_t\|$ almost all $x \in \mathbb{R}^{n+1}$ and for almost all $t \in (0,T)$. In other words, there cannot be any points of integer density greater than or equal to 2. Thus the solution of the present paper is guaranteed to remain unit density flow for $t \in (0,T)$. Then Theorem 3.6 applies and $\operatorname{spt}\mu$ is partially regular as described there for $(0,T)$. In the case of $n=1$, this implies further that any nontrivial static tangent flow within the time interval $(0,T)$ is either a line, or a regular triple junction, both of single-multiplicity. This is precisely the situation that we may apply [46, Theorem 2.2]. The result concludes that there exists a closed set $S \subset \mathbb{R}^2 \times [0,T)$ of parabolic Hausdorff dimension at most 1 such that, outside of S, spt $\|V_t\|$ is locally a smooth curve or a regular triple junction of 120 degree angle moving smoothly by the mean curvature. We mention that the short-time existence of one-dimensional network flow is recently obtained in [31]. We allow more general Γ_0 than [31] but our flow may have singularities of small dimension in general. Due to the minimizing step of the approximate solution, it is likely in the one-dimensional case that any static tangent flow constructed in this paper is either a line or a regular triple junction even for later time. This should require a finer look into the singularities and pose an interesting open question. In any case, away from space-time region with higher integer multiplicities ($\geqslant 2$), Brakke flow constructed in this paper is partially regular as in Theorem 3.6. Higher integer multiplicities pose outstanding regularity questions even for stationary integral varifolds.

We also mention that there is an initial time regularity property for regular points of Γ_0 for any n in the following sense. If Γ_0 is locally a C^1 hypersurface at a point x which is not an interior boundary point of some $E_{0,i}$ (i.e., there exist $i, i' \in \{1,\ldots,N\}$, $i \neq i'$, such that $x \in \partial E_{0,i} \cap \partial E_{0,i'}$), then there exists a space-time neighborhood of $(x,0)$ in which the constructed flow is C^1 in the parabolic sense up to $t = 0$ and C^∞ for $t > 0$. This can be proved by using a $C^{1,\alpha}$ regularity theorem in [32] as demonstrated in [43, Theorem 2.3(4)] for a phase field setting.

11.3. Other settings

If we replace $\mathbb{R}^{n+1}$ by the flat torus $\mathbb{T}^{n+1}$, we may simply change everything by setting quantities periodic on $\mathbb{R}^{n+1}$ with period 1. We would have finite open partitions defined on $\mathbb{T}^{n+1}$ and all convergence takes place accordingly. For general Riemannian manifolds, by adapting definitions and assumptions, similar results should follow with little change. All the key points of the paper such as the proofs of rectifiability and integrality are local estimates. On the other hand, if one is interested in the MCF with "Dirichlet condition" or "Neumann condition" in a suitable sense, the presence of such boundary condition may pose a nontrivial problem near the boundary and further studies are expected. From a geometric point of view in connection with the Plateau problem, such problem is natural and interesting. As a related matter, one aspect that may puzzle the reader is the finiteness of open partition, i.e., we always fix N of $\mathcal{OP}_\Omega^N$ even though we do not see any quantitative statement in the main results concerning N. One may naturally wonder if countably infinite open partition $\mathcal{OP}_\Omega^\infty$ can be allowed. In fact, $N=\infty$ can be dealt with all the way just before the last step of taking $j_l \to \infty$. For example, in Lemma 10.10, we want to conclude that a subsequence of $\chi_{E_{j_l,k}(t)}$ converges in $L^1_{loc}(\mathbb{R}^{n+1})$ to some $\chi_{E_k(t)}$ and $\sum_{k=1}^N \chi_{E_k(t)} \equiv 1$ a.e. on $\mathbb{R}^{n+1}$. However, if $N=\infty$, we need to exclude a possibility that $\sum_{k=1}^\infty \chi_{E_k(t)} < 1$ on a positive measure set. This is because, even though $\sum_{k=1}^\infty \chi_{E_{j_l,k}(t)} \equiv 1$ for all j_l, if there are infinite number of sets, the fear is that all of them become finer and finer as j_l increases and the limit may all vanish. This scenario seems unlikely to happen for a.e. t, but there has to be some extra argument to eliminate such possibility. Since the finite N case is interesting enough, we did not pursue $N=\infty$ for the technicality. It is also possible to first find Brakke flow for each N and take a limit $N \to \infty$. One can argue that there exists a converging subsequence whose limit is also a Brakke flow as described in the present paper and that the limit is nontrivial using the continuity property of the "grains".

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