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INVARIANT ENVELOPES OF HOLOMORPHY IN THE COMPLEXIFICATION OF A HERMITIAN SYMMETRIC SPACE

by Laura GEATTI & Andrea IANNUZZI (*)

Abstract. — In this paper we investigate invariant domains in Ξ^+, a distinguished G-invariant, Stein domain in the complexification of an irreducible Hermitian symmetric space G/K. The domain Ξ^+, recently introduced by Krötz and Opdam, contains the crown domain Ξ and it is maximal with respect to properness of the G-action. In the tube case, it also contains S^+, an invariant Stein domain arising from the compactly causal structure of a symmetric orbit in the boundary of Ξ. We prove that the envelope of holomorphy of an invariant domain in Ξ^+, which is contained neither in Ξ nor in S^+, is univalent and coincides with Ξ^+. This fact, together with known results concerning Ξ and S^+, proves the univalence of the envelope of holomorphy of an arbitrary invariant domain in Ξ^+ and completes the classification of invariant Stein domains therein.

Résumé. — Cet article est consacré à l'étude des domaines invariants dans Ξ^+, un domaine de Stein particulier dans la complexification d'un espace symétrique Hermitien irréductible G/K. Le domaine Ξ^+, introduit récemment par Krötz et Opdam, contient la couronne Ξ et il est maximal en ce qui concerne la propreté de l'action de G. Dans le cas tubulaire, Ξ^+ contient aussi S^+, un domaine de Stein invariant lié à la structure causale d'une orbite symétrique dans le bord de Ξ.

On demontre que l'enveloppe d'holomorphie d'un domaine invariant dans Ξ^+, non contenu ni dans Ξ ni dans S^+, est univalent et coincide avec Ξ^+. Ce fait, en combination avec des résultats connus pour Ξ et S^+, démontre l'univalence de l'enveloppe d'holomorphie d'un domaine arbitraire dans Ξ^+ et complète la classification des domains de Stein invariants dans Ξ^+.

1. Introduction

Let G/K be a non-compact, irreducible, Riemannian symmetric space. Its Lie group complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ is a Stein manifold and left translations by elements of G are holomorphic transformations of $G^{\mathbb{C}}/K^{\mathbb{C}}$. In

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this situation, G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and their envelopes of holomorphy are natural objects to study.

A first example is given by the crown Ξ, introduced by D. N. Akhiezer and S. G. Gindikin in [1]. This Stein invariant domain carries an invariant Kähler structure intrinsically associated with the Riemannian structure of the symmetric space G/K and, in many respects, can be regarded as its canonical complexification. In recent years, it has been extensively studied in connection with harmonic analysis on G/K (see, e.g [13], [14]).

If G/K is a Hermitian symmetric space of tube type, two additional distinguished invariant Stein domains $S^{\pm}$ arise from the compactly casual structure of a pseudo-Riemannian symmetric space G/H lying on the boundary of Ξ. The complex geometry of $S^{\pm}$ was studied by K. H. Neeb in [16]. Inside the crown Ξ, as well as inside $S^{\pm}$, an invariant domain can be described via a semisimple abelian slice, its envelope of holomorphy is univalent and Steiness is characterized by logarithmic convexity of such a slice.

One may ask how far the above results are from a complete description of envelopes of holomorphy of invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and a classification of invariant Stein domains therein. In [6], a univalence result for G-equivariant Riemann domains over $G^{\mathbb{C}}/K^{\mathbb{C}}$, and in particular for envelopes of holomorphy, was proven in the rank-one case. In addition, the complete classification of invariant Stein domains was obtained. From the latter result one sees that, up to finitely many exceptions, all invariant Stein domains are contained either in a copy of Ξ or, in the Hermitian case of tube type, in $S^{\pm}$. The study of the CR-structure of principal G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ (i.e. closed orbits of maximal dimension) carried out in [5], suggests that this fact holds true also in the higher rank case, the exceptions being finitely many invariant domains whose boundary entirely consists of non-principal G-orbits.

In this paper we focus on G/K irreducible non-compact of Hermitian type. In this case, B. Krötz and E. Opdam recently singled out two Stein invariant domains Ξ^+ and Ξ^- in $G^{\mathbb{C}}/K^{\mathbb{C}}$, which satisfy $\Xi^+ \cap \Xi^- = \Xi$ and are maximal with respect to properness of the G-action. The relevance of the crown Ξ and of the domains Ξ^+ and Ξ^- for the representation theory of G was underlined in Theorem 1.1 in [11]. Since Ξ^+ and Ξ^- are G-equivariantly anti-biholomorphic, in the sequel we simply refer to Ξ^+. If G/K is Hermitian of tube type, then Ξ^+ contains both the crown Ξ and the domain S^+ ([4], Prop. 7.5). Moreover, for $r := \mathrm{rank}(G/K) > 1$, the complement of $\Xi \cup S^+$ in Ξ^+ has non-empty interior. Our main result is as follows.

Theorem. — *Let G/K be an irreducible non-compact Hermitian symmetric space. Given a G-invariant domain D in Ξ^+, denote by $\widehat{D}$ its envelope of holomorphy.*

(i) Assume G/K is of tube type. If D is not contained in Ξ nor in S^+, then $\widehat{D}$ is univalent and coincides with Ξ^+.
(ii) Assume G/K is not of tube type. If D is not contained in Ξ, then $\widehat{D}$ is univalent and coincides with Ξ^+.

The envelopes of holomorphy of invariant domains in Ξ or in S^+ are known to be univalent and their Steiness is characterized in terms of the aformentioned semisimple abelian slices. Hence, the above theorem implies the univalence of the envelope of holomorphy of an arbitrary invariant domain in Ξ^+ and implies the following classification.

Corollary. — *Let G/K be an irreducible non-compact Hermitian symmetric space and let D be a Stein G-invariant proper subdomain of Ξ^+.*

(i) If G/K is of tube type, then either $D \subseteq \Xi$ or $D \subseteq S^+$.
(ii) If G/K is not of tube type, then $D \subseteq \Xi$.

The theorem is proved by showing that the natural G-equivariant holomorphic embedding $f : D \to \widehat{D}$ admits a G-equivariant holomorphic extension $\hat{f} \colon \Xi^+ \to \widehat{D}$ to the whole Ξ^+. For this purpose, we use the unipotent, abelian slice of Ξ^+ introduced by B. Krötz and E. Opdam in [12]. Namely, one has

$$\Xi^+ = G \cdot \Sigma ,$$

where $\Sigma := \exp i\Lambda_r^{\llcorner} \cdot x_0$ and $\Lambda_r^{\llcorner}$ is a closed hyperoctant in an r-dimensional, nilpotent, abelian subalgebra of $\mathfrak{g}$, the Lie algebra of G. This sets a one-to-one correspondence

$$D \to \Sigma_D := D \cap \Sigma$$

between G-invariant domains in Ξ^+ and domains in Σ which are invariant under the action of an appropriate Weyl group (see Sect. 3).

Then a key ingredient is Lemma 4.7, which implies that a continuous extension of $f|_{\Sigma_D}$ to a domain $\widetilde{\Sigma}$ in Σ induces a G-equivariant, holomorphic extension of f to $G \cdot \widetilde{\Sigma}$, provided that certain compatibility conditions are satisfied. In order to obtain $\hat{f}$, we therefore construct a continuous extension of $f|_{\Sigma_D}$ to Σ satisfying such compatibility conditions.

This is done in stages, where $f|_{\Sigma_D}$ is extended to larger domains $\widetilde{\Sigma} \subset \Sigma$ properly containing Σ_D. Such extensions are obtained by equivariantly embedding into $G^{\mathbb{C}}/K^{\mathbb{C}}$ various lower dimensional complex homogenous manifolds $L^{\mathbb{C}}/H^{\mathbb{C}}$, all of whose L-invariant domains have univalent and well understood envelope of holomorphy. The embedding of each space $L^{\mathbb{C}}/H^{\mathbb{C}}$ is carefully chosen, so that it intersects D in some L-invariant domain $T \subset L^{\mathbb{C}}/H^{\mathbb{C}}$. By the universality property of the envelope of holomorphy, the map $f|_T : T \to \widehat{D}$ extends L-equivariantly to a holomorphic map $\widehat{T} \to \widehat{D}$ producing, in particular, a real-analytic extension of $f|_{\Sigma_D}$ along the submanifold $\widehat{T} \cap \Sigma$. Generally, the intersection $\widehat{T} \cap \Sigma$ is not open in Σ. In that case, an extension of $f|_{\Sigma_D}$ to an open set $\widetilde{\Sigma} \subset \Sigma$ is obtained by embedding a continuous family of copies of T into D.

The real homogenous manifolds L/H which play a role in our situation are: real r-dimensional vector spaces acted on by $(\mathbb{R}^r, +)$, the Euclidean plane acted on by its isometry group, and irreducible rank-one Hermitian symmetric spaces, both of tube-type and non-tube type. In the latter case, the univalence results on equivariant Riemann domains obtained in [6] are crucial. The above strategy was inspired by the work of K. H. Neeb on bi-invariant domains in the complexification of a Hermitian semisimple Lie group ([15]).

The paper is organized as follows. In section 2, we set up the notation and recall some preliminary facts which are needed in the paper. In section 3, we recall the unipotent paramentrization of Ξ^+ and of its G-invariant subdomains. In section 4, we recall some basic facts about envelopes of holomorphy and develope the tools used in the proof of the main theorem. In section 5 we prove the main theorem.

2. Preliminaries

Let G/K be an irreducible Hermitian symmetric space of the non-compact type. We may assume G to be a connected, non-compact, real simple Lie group contained in its simple, simply connected universal complexification $G^{\mathbb{C}}$, and K to be a maximal compact subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Denote by θ both the Cartan involution of G with respect to K and the associated involution of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. The *rank* of G/K is by definition $r = \dim \mathfrak{a}$. The adjoint action of $\mathfrak{a}$ decomposes $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{a} \oplus Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g},\mathfrak{a})} \mathfrak{g}^{\alpha},$$

where $Z_{\mathfrak{k}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, the joint eigenspace $\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, \text{ for every } H \in \mathfrak{a}\}$ is the α-restricted root space and $\Delta(\mathfrak{g}, \mathfrak{a})$ consists of those $\alpha \in \mathfrak{a}^*$ for which $\mathfrak{g}^{\alpha} \neq \{0\}$. A set of simple roots $\Pi_{\mathfrak{a}}$ in $\Delta(\mathfrak{g}, \mathfrak{a})$ uniquely determines a set of positive restricted roots $\Delta^+(\mathfrak{g}, \mathfrak{a})$ and an Iwasawa decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \qquad \text{where} \quad \mathfrak{n} = \bigoplus_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}^{\alpha}\,.$$

The restricted root system of a Lie algebra $\mathfrak{g}$ of Hermitian type is either of type C_r (if G/K is of tube type) or of type BC_r (if G/K is not of tube type), i.e. there exists a basis $\{e_1, \ldots, e_r\}$ of $\mathfrak{a}^*$ for which

$$\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm 2e_j,\ 1 \leqslant j \leqslant r,\ \pm e_j \pm e_k,\ 1 \leqslant j \neq k \leqslant r\}, \qquad \text{for type } C_r,$$

$$\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm e_j,\ \pm 2e_j,\ 1 \leqslant j \leqslant r,\ \pm e_j \pm e_k,\ 1 \leqslant j \neq k \leqslant r\}, \quad \text{for type } BC_r.$$

Since $\mathfrak{g}$ admits a compact Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$, there exists a set of r positive long strongly orthogonal restricted roots $\{\lambda_1, \ldots, \lambda_r\}$ (i.e. such that $\lambda_j \pm \lambda_k \notin \Delta(\mathfrak{g}, \mathfrak{a})$, for $j \neq k$), which are restrictions of *real* roots with respect to a maximally split θ-stable Cartan subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ extending $\mathfrak{a}$.

Taking as simple roots $\Pi_{\mathfrak{a}} = \{e_1 - e_2, \ldots, e_{r-1} - e_r, 2e_r\}$, for type C_r, and $\Pi_{\mathfrak{a}} = \{e_1 - e_2, \ldots, e_{r-1} - e_r, e_r\}$, for type BC_r, one has

$$\lambda_1 = 2e_1, \ldots, \lambda_r = 2e_r\,,$$

and $\dim \mathfrak{g}^{\lambda_j} = 1$, for $j = 1, \ldots, r$. Let Z_0 be the element in $Z(\mathfrak{k})$ defining the complex structure $J_0 = \mathrm{ad}_{Z_0}$ on G/K. For $j = 1, \ldots, r$, choose $E_j \in \mathfrak{g}^{\lambda_j}$ such that the $\mathfrak{sl}(2)$-triple

$$\{E_j,\ \theta E_j,\ A_j := [\theta E_j, E_j]\} \tag{2.1}$$

is normalized as follows

$$[A_j, E_j] = 2E_j, \quad [Z_0, E_j - \theta E_j] = A_j, \quad [Z_0, A_j] = -(E_j - \theta E_j)\,. \tag{2.2}$$

Then the vectors $\{A_1, \ldots, A_r\}$ form an orthogonal basis of $\mathfrak{a}$ (with respect to the restriction of the Killing form) and

$$[E_j, E_k] = [E_j, \theta E_k] = 0, \quad [A_j, E_k] = \lambda_k(A_j)E_k = 0, \quad \text{for } j \neq k\,. \tag{2.3}$$

That is, the above $\mathfrak{sl}(2)$-triples commute with each other. Moreover, under the above choices, the element Z_0 is given by

$$Z_0 = S + \frac{1}{2}\sum T_j, \tag{2.4}$$

where $T_j = E_j + \theta E_j$ and $S \in Z_{\mathfrak{k}}(\mathfrak{a})$ (see Lemma 2.4 in [4]). If G/K is of tube type one has $S = 0$.

In the sequel, we denote by $\mathfrak{g}_j$ the $\mathfrak{sl}(2)$-triple satisfying (2.1) and (2.2), and by G_j the corresponding connected θ-stable subgroup of G. In the non-tube case, to each λ_j one can also associate a connected, simple, Hermitian subgroup $G_j^\bullet$ of G of real rank one. The group $G_j^\bullet$ is by definition the connected, θ-stable subgroup of G with Lie algebra

$$\mathfrak{g}_j^\bullet = \mathbb{R}A_j \oplus \mathfrak{g}^{\pm\lambda_j/2} \oplus \mathfrak{g}^{\pm\lambda_j} \tag{2.5}$$

isomorphic to $\mathfrak{su}(m,1)$, for some $m > 1$.

Lemma 2.1. — *Let G/K be an irreducible Hermitian symmetric space, which is not of tube type. Let $G_j^\bullet$ be the simple Hermitian subgroup of real rank one, associated to the root λ_j, for some $j \in \{1,\dots,r\}$. Then $G_j^\bullet$ commmutes with the subgroups G_k, for every $k \neq j$.*

Proof. — By relations (2.3), one has $[\mathfrak{g}_j, \mathfrak{g}_k] \equiv 0$, for $k \neq j$. Futhermore, since $\pm e_j \pm 2e_k$, for $k \neq j$, are not roots in $\Delta(\mathfrak{g},\mathfrak{a})$ and $e_j(A_k) = \delta_{jk}$, one also has $[\mathfrak{g}^{\pm\lambda_j/2}, \mathfrak{g}_k] \equiv 0$. Summarizing, there is commutativity on the Lie algebra level and likewise on the group level, by connectedness. $\square$

3. Invariant subdomains of Ξ^+

A description of the domain Ξ^+ was given in [11], p.286, and [12], Sect.8, via its unipotent parametrization. Fix vectors $E_j \in \mathfrak{g}^{\lambda_j}$ normalized as in (2.2). Then

$$\Xi^+ = G \exp i \bigoplus_{j=1}^{r} (-1,\infty) E_j \cdot x_0.$$

Define the nilpotent abelian subalgebras

$$\Lambda_r := \operatorname{span}_{\mathbb{R}}\{E_1,\dots,E_r\} \quad \text{and} \quad \Lambda_r^{\mathbb{C}} := \operatorname{span}_{\mathbb{C}}\{E_1,\dots,E_r\}$$

of $\mathfrak{n}$ and $\mathfrak{n}^{\mathbb{C}}$, respectively. The exponential map of $G^{\mathbb{C}}$ defines a biholomorphism between $\Lambda_r^{\mathbb{C}}$ and the unipotent abelian complex subgroup $L^{\mathbb{C}} := \exp \Lambda_r^{\mathbb{C}}$. In particular, it restricts to a diffeomorphism between Λ_r and the real unipotent subgroup $L := \exp \Lambda_r$. Since the map

$$\iota : \mathfrak{n}^{\mathbb{C}} \to N^{\mathbb{C}} \cdot x_0, \qquad Z \to \exp Z \cdot x_0, \tag{3.1}$$

is a biholomorphism onto its image (cf. Prop. 1.3 in [13]), so is its restriction $\iota : \Lambda_r^{\mathbb{C}} \to L^{\mathbb{C}} \cdot x_0$.

Lemma 3.1. — *The intersection $\Xi^+ \cap L^{\mathbb{C}} \cdot x_0$ is a closed, r-dimensional, complex submanifold of Ξ^+, which is biholomorphic, via the map ι, to the Stein tube domain $\Lambda_r \times i \bigoplus_{j=1}^r (-1, \infty) E_j$ of $\Lambda_r^{\mathbb{C}}$.*

Proof. — By a result of Rosenlicht ([17], Thm. 2), the orbits of the unipotent subgroup $L^{\mathbb{C}}$ in the affine space $G^{\mathbb{C}}/K^{\mathbb{C}}$ are closed. In particular $L^{\mathbb{C}} \cdot x_0 \cap \Xi^+$ is closed in Ξ^+. Now the statement follows from the injectivity of map ι and the fact that the set $\{\, X \in \Lambda_r \ : \ \exp iX \cdot x_0 \in \Xi^+ \,\}$ coincides with $\bigoplus_{j=1}^r (-1, \infty) E_j$ (see [11], p. 286). □

By Lemma 3.1 in [4], the group $W_K(\Lambda_r) := N_K(\Lambda_r)/Z_K(\Lambda_r)$ is a proper subgroup of the Weyl group $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ and acts on Λ_r by permutations of the basis elements $\{E_1, \dots, E_r\}$.

As it was observed in [4], Remark 6.6, the intersection of a G-orbit in Ξ^+ with the closed slice $\exp(i \bigoplus_{j=1}^r (-1, +\infty) E_j) \cdot x_0$ is not just a $W_K(\Lambda_r)$-orbit. So, we consider the smaller slice given by the nilpotent cone in $\mathfrak{g}$

$$\mathcal{N}^+ := \mathrm{Ad}_K(\Lambda_r^{\llcorner}),$$

where $\Lambda_r^{\llcorner}$ is the $W_K(\Lambda_r)$-invariant, closed hyperoctant in Λ_r defined by $\Lambda_r^{\llcorner} := \mathrm{span}_{\mathbb{R}_{\geqslant 0}}\{E_1, \dots, E_r\}$. The following fact holds true.

Proposition 3.2. — ([4], *Prop. 4.7*) *The G-equivariant map*

$$\psi \colon G \times_K \mathcal{N}^+ \to \Xi^+, \qquad [g, X] \to g \exp iX \cdot x_0$$

is a homeomorphism.

Given a G-invariant domain $D \subset \Xi^+$, define the corresponding open subset of $\bigoplus_{j=1}^r (-1, \infty) E_j$ by $\mathcal{D} := \{\, X \in \Lambda_r \ : \ \exp iX \cdot x_0 \in D \,\}$. By the definition of $\mathcal{D}$ and Proposition 3.2, the domain D can be written as

$$D = G \exp i\mathcal{D} \cdot x_0 = G \exp i\mathcal{D}^{\llcorner} \cdot x_0,$$

where $\mathcal{D}^{\llcorner} := \mathcal{D} \cap \Lambda_r^{\llcorner}$ is a $W_K(\Lambda_r)$-invariant open subset of $\Lambda_r^{\llcorner}$.

Lemma 3.3. — ([4], *Prop. 6.4*) *Let X be an element in $\Lambda_r^{\llcorner}$. Then the Ad_K-orbit of X intersects Λ_r in the $W_K(\Lambda_r)$-orbit of X in $\Lambda_r^{\llcorner}$.*

Note that the above result together with Proposition 3.2 implies that given X in $\Lambda_r^{\llcorner}$, one has

$$G \exp iX \cdot x_0 \bigcap \exp i\Lambda_r^{\llcorner} \cdot x_0 = \exp i(W_K(\Lambda_r) \cdot X) \cdot x_0,$$

i.e. every G-orbit (not just a K-orbit) in Ξ^+ intersects the closed slice $\exp i\Lambda_r^{\llcorner} \cdot x_0$ exactly in a $W_K(\Lambda_r)$-orbit.

Consider the open Weyl chamber $(\Lambda_r^{\llcorner})^+ := \big\{ \sum_{j=1}^r x_jE_j \ : \ x_1 > \cdots > x_r > 0 \big\}$. Since $W_K(\Lambda_r)$ acts on Λ_r by permutations of the basis elements $\{E_1, \ldots, E_r\}$, its topological closure

$$\overline{(\Lambda_r^{\llcorner})^+} = \Big\{ \sum_{j=1}^r x_jE_j, \ : \ x_1 \geqslant \cdots \geqslant x_r \geqslant 0 \Big\}$$

is a perfect slice for the $W_K(\Lambda_r)$-action on $\Lambda_r^{\llcorner}$, implying that $\exp i\overline{(\Lambda_r^{\llcorner})^+} \cdot x_0$ is a perfect slice for the G-action on Ξ^+. It follows that for a G-invariant domain D of Ξ^+ one also has

$$D = G \exp i(\mathcal{D}^{\llcorner})^+ \cdot x_0, \quad \text{where} \quad (\mathcal{D}^{\llcorner})^+ := \mathcal{D}^{\llcorner} \cap \overline{(\Lambda_r^{\llcorner})^+} \tag{3.2}$$

is an open subset of $\overline{(\Lambda_r^{\llcorner})^+}$. In particular, $(\mathcal{D}^{\llcorner})^+$ is connected if and only if D is connected. In the sequel we also need the following fact.

Lemma 3.4. — *Let X be an element in $\Lambda_r^{\llcorner}$. Then every connected component of $Z_K(X)$ meets $Z_K(\Lambda_r)$.*

Proof. — Let X be an arbitrary element in $\Lambda_r^{\llcorner}$. By Lemma 3.1 (i) and Lemma 4.6 in [4], one has

$$Z_K(\Lambda_r) \cong Z_K(\mathfrak{a}) \quad \text{and} \quad Z_K(X) \cong Z_K(\Psi(X)),$$

where $\Psi(X) = [Z_0, X - \theta X] \in \mathfrak{a}$. Thus in order to prove the lemma, it is sufficient to show that for an arbitrary element $H \in \mathfrak{a}$, every connected component of $Z_K(H)$ meets $Z_K(\mathfrak{a})$.

The centralizer $Z_G(H)$ is a θ-stable reductive subgroup of G (see [10], Prop. 7.25, p. 452) of the same rank and real rank as G, with maximal compact subgroup $Z_K(H)$. A maximal abelian subspace of $Z_{\mathfrak{p}}(H)$ is $\mathfrak{a}$ and, as $Z_K(\mathfrak{a})$ is contained in $Z_K(H)$, one has that $Z_{Z_K(H)}(\mathfrak{a}) = Z_K(\mathfrak{a})$. Now Proposition 7.33 in [10], p. 457, applied to the reductive group $Z_G(H)$, states that $Z_K(\mathfrak{a})$ meets every connected component of $Z_K(H)$, as desired. □

In [4] it was shown that if G/K is of tube type, then Ξ^+ contains another distinguished Stein invariant domain, besides the crown Ξ. Such domain S^+ arises from the compactly causal structure of a pseudo-Riemannian symmetric G-orbit in the boundary of Ξ. The domain S^+ and its invariant subdomains were investigated in [16]. In the unipotent parametrization of Ξ^+, the domain Ξ is given as follows (see [12], Sect. 8, [4], Prop. 7.5):

$$\Xi = G \exp i \bigoplus_{j=1}^r [0,1)E_j \cdot x_0 \,. \tag{3.3}$$

If G/K is of tube type, then one has

$$S^+ = G \exp i \bigoplus_{j=1}^{r} (1,\infty) E_j \cdot x_0. \tag{3.4}$$

4. Envelopes of holomorphy of invariant domains in Ξ^+

In this section we prove some preliminary results supporting the three basic ingredients of the proof of the main theorem. A key result is Lemma 4.7, used to produce G-equivariant, holomorphic extensions of the embedding $f\colon D \to \widehat{D}$ to invariant domains properly containing D.

We begin by recalling some general facts about envelopes of holomorphy. Let X be a Stein manifold and let D be a domain in X. By Rossi's results [18], D admits an envelope of holomorphy $\widehat{D}$. This means that there exist an open holomorphic embedding $f\colon D \to \widehat{D}$ into a Stein manifold $\widehat{D}$ to which all holomorphic functions on D simultaneously extend. As a consequence, there is a local biholomorphism $q\colon \widehat{D} \to X$ such that $q \circ f = Id_D$ and the following holds true.

Proposition 4.1. — *Let D_1 and D_2 be complex manifolds, with envelopes of holomorphy $f_1\colon D_1 \to \widehat{D}_1$ and $f_2\colon D_2 \to \widehat{D}_2$, respectively. Let $F\colon D_1 \to D_2$ be a holomorphic map. Then there exists a unique holomorphic map $\widehat{F}\colon \widehat{D}_1 \to \widehat{D}_2$ such that $q \circ f = Id_D$.*

Proposition 4.2. — *Let X be a Stein manifold and let $D \subset X$ be a domain with envelope of holomorphy $f\colon D \to \widehat{D}$ and projection $q\colon \widehat{D} \to X$.*

(i) Let Ω be the smallest Stein domain in X containing D. Then $q(\widehat{D})$ is contained in Ω and coincides with Ω provided that q is univalent.

(ii) Let Ω be a domain in X containing D. Assume there exists a holomorphic map $\hat{f}\colon \Omega \to \widehat{D}$ extending f. Then $\widehat{\Omega} = \widehat{D}$.

Proof. — Let's start with Statement (i). By Proposition 4.1, an arbitrary Stein domain containing D necessarily contains $q(\widehat{D})$. Define

$$\Omega := \operatorname{int}\left(\cap_{C \in \mathcal{F}} C\right),$$

where $\mathcal{F}$ denotes the family of all Stein domains in X containing D and $\operatorname{int}(\,\cdot\,)$ denotes the interior of a set. By definition, Ω contains D and it is open. It remains to show that it is Stein and connected. When $X = \mathbb{C}^n$, the Steinness of Ω follows from Corollary 2.5.7 in [9]. When X is an arbitrary Stein manifold, let B an open domain in X biholomorphic to the unit ball of $\mathbb{C}^n$. From the identity $B \cap \Omega = \operatorname{Int}\left(\cap_{C \in \mathcal{F}} (B \cap C)\right)$ and Corollary 2.5.7

in [9], it follows that $B \cap \Omega$ is Stein, implying that Ω is locally Stein in X. Now a classical result of Docquier-Grauert ([3], Satz 11, p.113) applies, showing that Ω is Stein. Finally, Ω is connected, since so is D.

Statement (ii) is a straightforward consequence of Proposition 4.1. $\square$

Coming back to our situation, let $D = G \exp i\mathcal{D} \cdot x_0 = G \exp i\mathcal{D}^{\llcorner} \cdot x_0$ be a G-invariant domain in Ξ^+. Since Ξ^+ is Stein, there is a commutative diagram

$$
\begin{tikzcd}
 & \widehat{D} \arrow[d, "q"] \\
D \arrow[ur, "f"] \arrow[r, "Id"'] & \Xi^+
\end{tikzcd}
\tag{4.1}
$$

Moreover the G-action on D lifts to an action on $\widehat{D}$ and all the maps in diagram (4.1) are G-equivariant. We prove that under the assumption that D is not entirely contained in Ξ nor in S^+ (in the tube case), the map $f\colon D \to \widehat{D}$ can be G-equivariantly extended to the whole Ξ^+. Then by Proposition 4.2(ii), one concludes $\widehat{D} = \Xi^+$.

The strategy is to gradually enlarge the domain of definition of f by iterating the following arguments. By reduction 1, we show that f can be G-equivariantly extended to a domain $G \exp i\widetilde{\mathcal{D}}^{\llcorner} \cdot x_0$ with all the connected components of $\widetilde{\mathcal{D}}^{\llcorner}$ convex (Prop. 4.10). By reduction 2, we show that f can be G-equivariantly extended to a domain with $\widetilde{\mathcal{D}}^{\llcorner}$ connected (Prop. 4.13). The third key ingredient is the rank-one reduction. It is based on the univalence and the precise description of the envelope of holomorphy of an arbitrary G-invariant domain in the complexification of a rank-one Hermitian symmetric space (cf. [6], Thm.6.1, Thm.7.6). The approach is similar to the one used by Neeb in [15].

4.1. The rank-one case

For the reader's convenience we outline a proof of the relevant facts in the rank-one case, in the formulation which is needed in this paper. For $n \geqslant 1$, let $G = SU(n,1)$ be the subgroup of $SL(n+1,\mathbb{C})$ leaving invariant the hermitian form $I_{n,1}$ in $\mathbb{C}^{n+1}$ and let σ be the conjugation of $G^{\mathbb{C}} = SL(n+1,\mathbb{C})$ relative to G, namely $\sigma(g) = I_{n,1}{}^t\bar{g}^{-1} I_{n,1}$. Denote by $\overline{\mathfrak{p}}^n$ the complex projective space endowed with the opposite complex structure. The group $G^{\mathbb{C}}$ acts holomorphically on $\mathfrak{p}^n \times \overline{\mathfrak{p}}^n$ by $g \cdot ([p],[q]) := ([g \cdot p], [\sigma(g) \cdot q])$, and $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be identified with the open orbit $G^{\mathbb{C}} \cdot x_0$, where $x_0 = ([0 : \ldots : 0 : 1], [0 : \ldots : 0 : 1])$. Fix the element

$$E = \frac{1}{2}\begin{pmatrix} O & O \\ O & \mathbf{e} \end{pmatrix}, \quad \text{with} \quad \mathbf{e} = \begin{pmatrix} i & -i \\ i & -i \end{pmatrix},$$

in $\mathfrak{g}$ so that the triple $\{E, \theta E, A = [\theta E, E]\}$ is normalized as in (2.2). The nilpotent slice $\ell : [0, +\infty) \to \Xi^+$ is given by

$$\ell(t) = \exp itE \cdot x_0 = \left([0 : \ldots : 0 : \frac{t}{2} : \frac{t+2}{2}], [0 : \ldots : 0 : \frac{t}{2} : \frac{t-2}{2}]\right).$$

As we are in the rank-one case, an invariant domain in Ξ^+ can be written as $D = G \exp iIE \cdot x_0$, where I is an open interval in $[0, \infty)$.

Lemma 4.3. — *Let $G = SU(n, 1)$, for $n \geqslant 1$, and let D be a proper Stein G-invariant subdomain of Ξ^+.*

(i) If $n > 1$, then $D = G \exp i[0, b)E \cdot x_0$, for some $b \leqslant 1$.
(ii) If $n = 1$, then either $D = G \exp i[0, b)E \cdot x_0$, for some $b \leqslant 1$, or $D = G \exp i(a, \infty)E \cdot x_0$, for some $1 \leqslant a < \infty$.

Proof. — We obtain the above classification by computing the Levi form of hypersurface G-orbits in Ξ^+. We do this by exploiting a smooth G-invariant function $f \colon \Xi^+ \to \mathbb{R}$, all of whose level sets, but $\{f = -1\} = G \cdot x_0$, consist of a single hypersurface orbit in Ξ^+ (cf. [6], Ex.6.3). For every $t > 0$, the element $\ell(t)$ belongs to the chart $\Psi \colon \mathbb{C}^n \times \mathbb{C}^n \setminus Z \to G^{\mathbb{C}}/K^{\mathbb{C}}$ defined by

$$\begin{aligned} &((z_1, \ldots, z_n), (w_1, \ldots, w_n)) \\ &\qquad\qquad \to ([z_1 : \ldots : z_{n-1} : 1 : z_n], [\bar{w}_1 : \ldots : \bar{w}_{n-1} : 1 : \bar{w}_n]), \end{aligned}$$

where $Z := \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n \ : < z, \bar{w} >_{n-1,1} +1 = 0 \}$, and corresponds to the point $((0, \ldots, 0, \frac{t+2}{t}), (0, \ldots, 0, \frac{t-2}{t}))$ therein. On the above holomorphic chart the function f reads as

$$f(z, w) = -\frac{(\langle z, z\rangle_{n-1,1} + 1)(\langle w, w\rangle_{n-1,1} + 1)}{|\langle z, \bar{w}\rangle_{n-1,1} + 1|^2}.$$

The complex tangent space $T^{CR}_{\ell(t)}(G \cdot \ell(t)) := T_{\ell(t)}(G \cdot \ell(t)) \cap J_{\ell(t)} T_{\ell(t)}(G \cdot \ell(t))$ to the orbit $G \cdot \ell(t)$ at $\ell(t)$, which is the kernel of the complex gradient of f at $\ell(t)$, is given by

$$T^{CR}_{\ell(t)}(G \cdot \ell(t)) = \left\{ \big((\boldsymbol{\zeta}, (1+t)\eta), (\boldsymbol{\omega}, (1-t)\eta)\big) \in \mathbb{C}^n \times \mathbb{C}^n \right\},$$

where $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_{n-1})$, $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{n-1}) \in \mathbb{C}^{n-1}$ and $\eta \in \mathbb{C}$. The quadratic Levi form of f at $\ell(t)$ is given by

$$\begin{aligned}\mathcal{L}_{\ell(t)} f\big((\boldsymbol{\zeta}, (1+t)\eta), (\boldsymbol{\omega}, (1-t)\eta)\big) \\ = \frac{t^2}{4}\left((1+t)\|\boldsymbol{\zeta}\|^2 + (1-t)\|\boldsymbol{\omega}\|^2 + \frac{t^2(1-t^2)}{2}|\eta|^2\right).\end{aligned}$$

Assume first $n > 1$. The above formula shows that, for all $t > 1$, the hypersurfaces $G \cdot \ell(t)$ have indefinite Levi form. By [8], Thm.4, p.194, only hypersurface orbits with semidefinite Levi form can lie on the boundary of a Stein domain. It follows that D Stein is necessarily of the form $G \exp iIE \cdot x_0$, for some open interval I in $[0, 1)$. We claim that $I = [0, b)$, for some $b \leqslant 1$.

Assume by contraddiction that $I = (a, b)$, for some $a > 0$. Since f is strictly increasing on the slice ℓ and the Levi form $\mathcal{L}_{\ell(a)} f$ is positive definite for $0 < a < 1$, the domain $\{x \in \Xi^+ \ : \ f(\ell(a)) < f(x)\}$ is not Stein ([8], Thm.4, p.194). Therefore $a = 0$. Further, since the orbit $G \cdot x_0$ is totally real in $G^{\mathbb{C}}/K^{\mathbb{C}}$, every holomorphic function defined on $G \exp i(0, b)E \cdot x_0$ extends to $G \exp i[0, b)E \cdot x_0$. Consequently, $I = [0, b)$, for $b \leqslant 1$. It remains to prove that $G \exp i[0, b)E \cdot x_0$ is indeed Stein. Since $\mathcal{L}_{\ell(b)} f$ is positive semidefinite, and the signature of the Levi form is a biholomorphic invariant, the intersection $G \exp i[0, b)E \cdot x_0 \cap \Psi(\mathbb{C}^n \times \mathbb{C}^n \setminus Z)$ is Levi-pseudoconvex. Hence it is Stein, by [8], Thm.4, p.194. By G-invariance, $G \exp i[0, b)E \cdot x_0 \cap g \cdot \Psi(\mathbb{C}^n \times \mathbb{C}^n \setminus Z)$ is Stein as well, for every $g \in G$, implying that the domain $G \exp i[0, b)E \cdot x_0$ is locally Stein in Ξ^+. Then it is Stein by [3], Satz 11, p.113.

If $n = 1$, the Levi form of the orbits $G \cdot \ell(t)$ is positive definite for $0 < t < 1$, negative definite for $t > 1$, and zero for $t = 1$. Consequently a proper Stein subdomain is either contained in $G \exp i[0, 1)E \cdot x_0$ or in $G \exp i(1, \infty)E \cdot x_0$. To exclude Stein domains other than those indicated in statement (ii), one argues as in the previous case. □

Proposition 4.4. — *Let $G = SU(n, 1)$ and let D be a G-invariant domain in Ξ^+. Then the envelope of holomorphy $\widehat{D}$ of D is univalent and given as follows.*

(i) *If $D = G \exp i(a, b)E \cdot x_0$ or $D = G \exp i[0, b)E \cdot x_0$, with $b \leqslant 1$, then*

$$\widehat{D} = G \exp i[0, b)E \cdot x_0;$$

(ii) *If* $D = G\exp i(a,b)E\cdot x_0$ *or* $D = G\exp i(a,\infty)E\cdot x_0$, *with* $a \geqslant 1$, *then*

$$\widehat{D} = G\exp i(a,\infty)E\cdot x_0 \quad \text{if} \quad n=1$$
$$\widehat{D} = \Xi^+ \quad \text{if} \quad n>1\,.$$

(iii) *If* D *contains the orbit* $G\cdot \ell(1)$, *then* $\widehat{D} = \Xi^+$.

Proof. — The projection $q : \widehat{D} \to \Xi^+$ is G-equivariant. Note that for all $n \geqslant 1$ the center Z of $SU(n,1)$ acts trivially on $D \subset G^{\mathbb{C}}/K^{\mathbb{C}}$ and, by the analytic continuation principle, on $\widehat{D}$. In particular, for $n = 1$ the projection $q : \widehat{D} \to \Xi^+$ is, in fact, $SU(1,1)/Z \cong PSL(2,\mathbb{R})$-equivariant and Theorem 7.6 in [6] applies for every $n \geqslant 1$. Hence q is injective and consequently the envelope of holomorphy $\widehat{D}$ coincides with the smallest Stein G-invariant domain in Ξ^+ containing D (cf. Prop. 4.2). The classification of all Stein G-invariant domains in Ξ^+ contained in Lemma 4.3 completes the proof of the proposition. □

4.2. The extension lemma

Let $\mathcal{C}$ be an open subset of the hyperoctant $\Lambda_r^{\llcorner}$. The goal of this subsection is to prove the "extension lemma", which provides sufficient conditions for a continuous lift $f\colon \exp i\mathcal{C}\cdot x_0 \to \widehat{D}$ to extend to a G-equivariant holomorphic map

$$\hat{f}\colon G\exp i\mathcal{C}\cdot x_0 \to \widehat{D}\,.$$

One of the conditions involves the isotropy subgroups of points $z \in D$ and $f(z) \in \widehat{D}$ in G.

Since the projection $q\colon \widehat{D} \to \Xi^+$ is a G-equivariant local biholomorphism, the isotropy subgroup G_z of $z \in \widehat{D}$ consists of connected components of the isotropy subgroup $G_{q(z)}$ of $q(z) \in \Xi^+$. On the other hand, since $f\colon D \to \widehat{D}$ is a G-equivariant biholomorphism onto its image and $q|_{f(D)} \circ f = Id_D$, one has $G_z = G_{q(z)}$, for all $z \in f(D)$. In the sequel it will be crucial to have such an identity of isotropy subgroups for points lying in suitable submanifolds, to which the map f extends holomorphically.

Lemma 4.5. — *Let* $\mathcal{C}$ *be an open subset of* $\Lambda_r^{\llcorner}$ *and let* $f : \exp i\mathcal{C}\cdot x_0 \to \widehat{D}$ *be a continuous map such that* $q\circ f = Id$. *Assume that there exists an open subset* $\mathcal{F}$ *of* $\mathcal{C}$ *such that*

(i) $G_{f(\exp iX'\cdot x_0)} = G_{\exp iX'\cdot x_0}$ *for all* X' *in* $\mathcal{F}$,

(ii) *for every $X \in \mathcal{C}$, there exist an element $X' \in \mathcal{F}$, such that the segment $\{ X' + t(X - X') \ : \ t \in [0,1]\}$ is contained in $\mathcal{C}$, and a holomorphic extension of f to the submanifold $\mathcal{S} = \{ \exp(i(X' + \lambda(X - X'))) \cdot x_0 \ : \mathrm{Re}\lambda \in [0,1] \}$.*

Then $G_{f(\exp iX \cdot x_0)} = G_{\exp iX \cdot x_0}$, for every X in $\mathcal{C}$.

Proof. — Since q is G-equivariant and $q \circ f = Id$ on $\exp i\mathcal{C} \cdot x_0$, it is clear that $G_{f(\exp iX \cdot x_0)} \subset G_{\exp iX \cdot x_0}$ for all $X \in \mathcal{C}$. In order to prove the opposite inclusion, we consider first generic elements in $\mathcal{C}$.

By definition, generic elements $X \in \Lambda_r^{\llcorner}$ are those for which $Z_K(X) = Z_K(\Lambda_r)$, and by Lemma 6.3 in [4], they are dense in $\Lambda_r^{\llcorner}$. Let X be a generic element in $\mathcal{C}$ and let g be an element in $G_{\exp iX \cdot x_0} = Z_K(\Lambda_r)$ (see Section 3). The fixed point set of g in $\widehat{D}$

$$Fix(g, \widehat{D}) := \{z \in \widehat{D} \mid g \cdot z = z\}$$

is a complex analytic subset of $\widehat{D}$. Let $X' \in \mathcal{F}$ be an element satisfying condition (ii) of the lemma. Since both $\mathcal{C}$ and $\mathcal{F}$ are open, X' can be chosen generic as well. Consider the strip $S := \{\lambda \in \mathbb{C} \ : \ \mathrm{Re}\lambda \in [0,1]\}$ and define the function

$$\phi \colon S \to \widehat{D}, \qquad \phi(\lambda) := f(\exp i(X' + \lambda(X - X') \cdot x_0))\,.$$

We are going to show that the set

$$A := \{\lambda \in S \ : \ g \cdot \phi(\lambda) = \phi(\lambda)\}$$

contains the element 1: this implies that $f(\exp iX \cdot x_0) \in Fix(g, \widehat{D})$ and proves the statement for X generic.

Since both X and X' are generic in $\Lambda_r^{\llcorner}$, one has that $G_{\exp iX' \cdot x_0} = G_{\exp iX \cdot x_0} = Z_K(\Lambda_r)$. Therefore $g \in G_{\exp iX' \cdot x_0}$ and, by condition (i), it follows that $f(\exp iX' \cdot x_0) \in Fix(g, \widehat{D})$. Consequently $0 \in A$. Since $\mathcal{F}$ is open, there exists $\varepsilon > 0$ such that $[0, \varepsilon) \subset A$. Let $[0, b)$ be the maximal open interval in $A \cap \mathbb{R}$ containing 0 and assume by contradiction that $b < 1$. Since A is closed, it follows that $b \in A$ and, by the definition of A, that $\phi(b) \in Fix(g, \widehat{D})$. Locally, in a neighbourhood U of $\phi(b)$ in $\widehat{D}$, the analytic set $Fix(g, \widehat{D})$ is given as

$$Fix(g, \widehat{D}) \cap U = \{z \in U \mid \psi_1(z) = \ldots = \psi_k(z) = 0\},$$

for some $\psi_1, \ldots, \psi_k \in \mathcal{O}(U)$. Thus, for each $j = 1, \ldots r$, the holomorphic function

$$\psi_j \circ \phi : \phi^{-1}(U) \to \mathbb{C}, \quad \lambda \mapsto \psi_j(f(\exp i(X' + \lambda(X - X')) \cdot x_0))$$

vanishes identically on $[0, b]$. Since $\phi^{-1}(U)$ is open in S, there exists $\varepsilon' > 0$ such that the restriction $\psi_j \circ \phi_{(b-\varepsilon', b+\varepsilon')}$ is real analytic and identically zero on $(b-\varepsilon', b]$. Hence it is identically zero on the whole interval $(b-\varepsilon', b+\varepsilon')$, contradicting the maximality of b. Thus $b = 1$ and $1 \in A$, as claimed. This concludes the case of generic elements in $\mathcal{C}$.

Consider now a non-generic element $X \in \mathcal{C}$. Since generic elements form an open dense subset of $\mathcal{C}$, and all have isotropy subgroup $Z_K(\Lambda_r)$, one obtains that $g \cdot f(\exp iX \cdot x_0) = f(\exp iX \cdot x_0)$, for all $g \in Z_K(\Lambda_r)$. This fact together with Lemma 3.4 implies that $G_{\exp iX \cdot x_0} \subset G_{f(\exp iX \cdot x_0)}$ for all $X \in \mathcal{C}$, and concludes the proof of the lemma. $\square$

Lemma 4.6. — *Let $D = G \exp i\mathcal{D}^{\llcorner} \cdot x_0$ be a G-invariant domain in Ξ^+ and let X be a G-space. A G-equivariant map $f : D \to X$ is continuous if and only if its restriction to $\exp i\mathcal{D}^{\llcorner} \cdot x_0$ is continuous.*

Proof. — One implication is clear. For the converse, we first prove that f is continuous on $K \exp i\mathcal{D}^{\llcorner} \cdot x_0 = \exp i\mathrm{Ad}_K\mathcal{D}^{\llcorner} \cdot x_0$. Consider the homeomorphism $\mathrm{Ad}_K\mathcal{D}^{\llcorner} \to \exp i\mathrm{Ad}_K\mathcal{D}^{\llcorner} \cdot x_0$ defined by $X \to \exp iX \cdot x_0$ (see Prop. 3.2) and let $X_n \to X_0$ be a converging sequence in $\mathrm{Ad}_K\mathcal{D}^{\llcorner}$. Choose elements k_n in K such that $\mathrm{Ad}_{k_n}X_n \in \mathcal{D}^{\llcorner}$. Since K is compact, we can assume that the sequence $\{k_n\}_n$ converges to an element $k_0 \in K$ and that $\mathrm{Ad}_{k_n}X_n \to \mathrm{Ad}_{k_0}X_0$.

Now observe that $\mathcal{D}^{\llcorner} = \Lambda_r^{\llcorner} \cap \mathrm{Ad}_K\mathcal{D}^{\llcorner}$ (see Lemma 3.3). It follows that $\mathcal{D}^{\llcorner}$ is closed in $\mathrm{Ad}_K\mathcal{D}^{\llcorner}$, implying that $\mathrm{Ad}_{k_0}X_0$ is contained in $\mathcal{D}^{\llcorner}$ (and not just in $\mathrm{Ad}_K\mathcal{D}^{\llcorner}$). Then one has

$$\begin{aligned} f(\exp iX_n \cdot x_0) &= k_n^{-1} \cdot f(\exp i(\mathrm{Ad}_{k_n}X_n) \cdot x_0) \\ &\to k_0^{-1} \cdot f(\exp i(\mathrm{Ad}_{k_0}X_0) \cdot x_0) = f(\exp iX_0 \cdot x_0)\,, \end{aligned}$$

which says that f is continuous on $\exp i\mathrm{Ad}_K\mathcal{D}^{\llcorner} \cdot x_0$, as claimed.

Next, consider the following commutative diagram

$$\begin{tikzcd} G \times \mathrm{Ad}_K\mathcal{D}^{\llcorner} \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ D \arrow[r, "f"] & X\,, \end{tikzcd}$$

where π is the map given by $(g, X) \to g \exp iX \cdot x_0$ and $\tilde{f}$ is the lift of f to $G \times \mathrm{Ad}_K\mathcal{D}^{\llcorner}$. As a consequence of Proposition 3.2, the map f is continuous if and only if so is $\tilde{f}$. So let $(g_n, X_n) \to (g_0, X_0)$ be a converging sequence in $G \times \mathrm{Ad}_K\mathcal{D}^{\llcorner}$. Since f is continuous on $\exp i\mathrm{Ad}_K\mathcal{D}^{\llcorner} \cdot x_0$, one has

$$\tilde{f}(g_n, X_n) = f(g_n \exp iX_n \cdot x_0) =$$

$g_n \cdot f(\exp iX_n \cdot x_0) \to g_0 \cdot f(\exp iX_0 \cdot x_0) = f(g_0 \exp iX_0 \cdot x_0) = \tilde{f}(g_0, X_0)$. Thus $\tilde{f}$ is continuous, implying that f is continuous. □

LEMMA 4.7 (Extension lemma). — *Let $\mathcal{C}$ be an open subset of $\Lambda_r^{\llcorner}$ and let $f\colon \exp i\mathcal{C} \cdot x_0 \to \widehat{D}$ be a continuous map such that $q \circ f = Id$ and $G_{\exp iX \cdot x_0} = G_{f(\exp iX \cdot x_0)}$, for every $X \in \mathcal{C}$. Assume that for every pair $X, X' \in \mathcal{C}$ on the same $W_K(\Lambda_r)$-orbit there exists $n \in N_K(\Lambda_r)$ such that*

$$X' = \mathrm{Ad}_n X \quad \text{and} \quad f(\exp iX' \cdot x_0) = n \cdot f(\exp iX \cdot x_0).$$

Then there exists a unique holomorphic map $\hat{f}\colon G\exp i\mathcal{C} \cdot x_0 \to \widehat{D}$ which is G-equivariant and extends f.

We point out that the G-invariant domain $G \exp i\mathcal{C} \cdot x_0$ coincides with $G \exp i(W_K(\Lambda_r) \cdot \mathcal{C}) \cdot x_0$.

Proof. — If one such $\hat{f}$ exists, it is uniquely determined by the relation

$$\hat{f}(g \exp iX \cdot x_0) := g \cdot f(\exp iX \cdot x_0), \qquad \text{for } X \in \mathcal{C} \text{ and } g \in G.$$

By Proposition 3.2 and Lemma 3.3, the above map $\hat{f}$ is well defined.

Since $G \exp i\mathcal{C} \cdot x_0 = G \exp i(W_K(\Lambda_r) \cdot \mathcal{C}) \cdot x_0$, in order to show that $\hat{f}$ is continuous, by Lemma 4.6, it is sufficient to show that $\hat{f}$ is continuous on $\exp(iW_K(\Lambda_r) \cdot \mathcal{C}) \cdot x_0$, i.e. on each set $\exp(i\gamma \cdot \mathcal{C}) \cdot x_0$, for γ in $W_K(\Lambda_r)$. By assumption, $\hat{f}$ is continuous on $\exp i\mathcal{C} \cdot x_0$. This settles the case when γ is the neutral element in $W_K(\Lambda_r)$. Otherwise, write $\gamma = nZ_K(\Lambda_r)$, for some $n \in N_K(\Lambda_r)$. Then by the G-equivariance of $\hat{f}$ one has

$$\hat{f}(\exp(i\gamma \cdot X) \cdot x_0) = \hat{f}(\exp i\mathrm{Ad}_n X \cdot x_0) = n \cdot \hat{f}(\exp iX \cdot x_0)\,,$$

for every $X \in \mathcal{C}$, proving that $\hat{f}$ is continuous on $\exp(i\gamma \cdot \mathcal{C}) \cdot x_0$, as wished.

Finally we show that $\hat{f}$ is holomorphic. Note that $q \circ \hat{f} = Id$, since by assumption such equality holds true on $\exp i\mathcal{C} \cdot x_0$ and $\hat{f}$ is G-equivariant. Let x be an element of $G \exp i\mathcal{C} \cdot x_0$ and choose a connected open neighborhood U of $\hat{f}(x)$ such that the restriction $q|_U : U \to \hat{f}(U)$ is a biholomorphism. Then, given a neighborhood V of x such that $\hat{f}(V) \subset U$, one has $\hat{f}|_V = (q|_U)^{-1} \circ Id$, implying that $\hat{f}$ is holomorphic. □

4.3. Reduction 1

Let

$$D = G \exp i\mathcal{D} \cdot x_0 = G \exp i\mathcal{D}^{\llcorner} \cdot x_0$$

be a G-invariant domain in Ξ^+ (see Section 3). The first reduction consists of showing that the map f in diagram (4.1) has a G-equivariant holomorphic extension to a domain $G \exp i\widetilde{\mathcal{D}}^{\llcorner} \cdot x_0$, with $\widetilde{\mathcal{D}}^{\llcorner}$ a set containing $\mathcal{D}^{\llcorner}$, all of whose connected components are convex. Recall that the set $(D^{\llcorner})^+ = D^{\llcorner} \cap (\Lambda_r)^+$ is a perfect slice for D and that it is connected (cf. (3.2)).

Definition 4.8. — *Denote by $\mathcal{D}_\circ$ (resp. by $\mathcal{D}^{\llcorner}_\circ$) the connected component of $\mathcal{D}$ (resp. of $\mathcal{D}^{\llcorner}$) containing $(\mathcal{D}^{\llcorner})^+$.*

Note that the set $\mathcal{D}_\circ$ is open in Λ_r; the set $\mathcal{D}^{\llcorner}_\circ$ is open in $\Lambda^{\llcorner}_r$, while need not be open in Λ_r. Both $\mathcal{D}_\circ$ and $\mathcal{D}^{\llcorner}_\circ$ need not be $W_K(\Lambda_r)$-invariant.

For $k \in \{1, \ldots, r-1\}$, denote by γ_{kk+1} the reflection flipping the k^{th} and the $(k+1)^{th}$ coordinates in $\Lambda^{\llcorner}_r$. By Lemma 3.1(iii) in [4], such reflections generate the Weyl group $W_K(\Lambda_r)$. Denote by Γ^0 the set of those γ_{kk+1} for which there exists a non-zero element in $Fix(\gamma_{kk+1}) \cap (\mathcal{D}^{\llcorner})^+$, i.e. whose fixed point hyperplane intersects $(\mathcal{D}^{\llcorner})^+$ non-trivially. Consider the subgroup of $W_K(\Lambda_r)$

$$W^0 := \langle \{ \gamma_{kk+1} \in \Gamma^0 \} \rangle ,$$

generated by the elements of Γ^0.

Lemma 4.9. — $W^0 \cdot (\mathcal{D}^{\llcorner})^+ = \mathcal{D}^{\llcorner}_\circ$.

Proof. — Set $\mathcal{C} := W^0 \cdot (\mathcal{D}^{\llcorner})^+$. We first show that $\mathcal{C}$ is contained in $\mathcal{D}^{\llcorner}_\circ$. For this note that $(\mathcal{D}^{\llcorner})^+ \cap \gamma_{kk+1} \cdot (\mathcal{D}^{\llcorner})^+ \neq \emptyset$, for all $\gamma_{kk+1} \in \Gamma^0$. Thus $\gamma_{kk+1} \cdot (\mathcal{D}^{\llcorner})^+ \subset \mathcal{D}^{\llcorner}_\circ$ and γ_{kk+1} stabilizes $\mathcal{D}^{\llcorner}_\circ$. Then the whole group W^0 stabilizes $\mathcal{D}^{\llcorner}_\circ$, implying that $\mathcal{C} \subset \mathcal{D}^{\llcorner}_\circ$.

Next, we claim that for $\gamma \in W_K(\Lambda_r)$, one has that $\gamma \cdot (\mathcal{D}^{\llcorner})^+ \cap \mathcal{C} \neq \emptyset$ if and only if $\gamma \in W^0$. One implication is clear, since $\gamma \cdot (\mathcal{D}^{\llcorner})^+ \subset \mathcal{C}$ if $\gamma \in W^0$. Conversely, if $\gamma \cdot (\mathcal{D}^{\llcorner})^+ \cap \mathcal{C} \neq \emptyset$, then there exists γ_1 in W^0 such that

$$\gamma_1 \gamma \cdot (\mathcal{D}^{\llcorner})^+ \cap (\mathcal{D}^{\llcorner})^+ \neq \emptyset .$$

Since $(\mathcal{D}^{\llcorner})^+$ is a fundamental region for the action of $W_K(\Lambda_r)$ on $\mathcal{D}^{\llcorner}$, it follows that there exists X in the boundary of $(\mathcal{D}^{\llcorner})^+$ such that $\gamma_1 \gamma \cdot X = X$. In other words, $\gamma_1 \gamma$ lies in the stabilizer subgroup $W_K(\Lambda_r)_X$ of X in $W_K(\Lambda_r)$. Since $W_K(\Lambda_r)_X$ is generated by the elements γ_{kk+1} in $\Gamma^0 \cap W_K(\Lambda_r)_X$ (see [2], Thm.4.1, p. 202), one has that $\gamma_1 \gamma \in W^0$. Then $\gamma \in W^0$, as claimed.

It follows that $\mathcal{D}^{\llcorner}$ is the union of the two disjoint subsets

$$\mathcal{C} \qquad \text{and} \qquad \bigcup_{\gamma \in W_K(\Lambda_r) \setminus W^0} \gamma \cdot (\mathcal{D}^{\llcorner})^+ \ .$$

As $(\mathcal{D}^{\llcorner})^+$ is closed in $\mathcal{D}^{\llcorner}$, both subsets are closed in $\mathcal{D}^{\llcorner}$. Thus $\mathcal{C}$ must be the union of connected components of $\mathcal{D}^{\llcorner}$. Since we already showed that $\mathcal{C} \subset \mathcal{D}^{\llcorner}_{\circ}$, it follows that $\mathcal{C} = \mathcal{D}^{\llcorner}_{\circ}$, as stated. □

Proposition 4.10 (Reduction 1). — *The inclusion $f\colon D \hookrightarrow \widehat{D}$ extends holomorphically and G-equivariantly to the G-invariant domain*

$$G \exp i\mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ}) \cdot x_0 = G \exp i\widetilde{\mathcal{D}}^{\llcorner} \cdot x_0\,,$$

where $\widetilde{\mathcal{D}}^{\llcorner} = W_K(\Lambda^{\llcorner}_r) \cdot \mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ})$.

Proof. — Let $\mathcal{D}_{\circ}$ be as in Definition 4.8. By Lemma 3.1, the intersection $D \cap L^{\mathbb{C}} \cdot x_0$ is a closed r-dimensional L-invariant complex submanifold of D, biholomorphic, via the map ι, to the tube domain $\Lambda_r \times i\mathcal{D}$.

Consider the connected component $L \exp i\mathcal{D}_0 \cdot x_0 \cong L \times \mathcal{D}_0$ of $D \cap L^{\mathbb{C}} \cdot x_0$. By Bochner's tube theorem, its envelope of holomorphy is univalent and given by $L \exp i\mathrm{Conv}(\mathcal{D}_{\circ}) \cdot x_0 \subset \Xi^+$. Then, by Proposition 4.1, the map f admits a holomorphic extension to an L-equivariant map

$$L \exp i\mathrm{Conv}(\mathcal{D}_{\circ}) \cdot x_0 \to \widehat{D}.$$

Note that the convexification $\mathrm{Conv}(\mathcal{D}_{\circ})$ contains $\mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ})$, which is an open subset of $\Lambda^{\llcorner}_r$ and coincides with $\mathrm{Conv}(\mathcal{D}_{\circ}) \cap \Lambda^{\llcorner}_r$. Moreover, given $X \in \mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ})$ and $X' \in \mathcal{D}^{\llcorner}_{\circ}$, the one-dimensional complex manifold

$$\begin{aligned}\mathcal{S} &= \{\, \exp(i(X' + \lambda(X - X'))) \cdot x_0 \ : \mathrm{Re}\lambda \in [0,1]\,\} \\ &= \{\, \exp s(X - X') \exp(i(X' + t(X - X'))) \cdot x_0 \ : s \in \mathbb{R},\ t \in [0,1]\,\}\end{aligned}$$

is contained in $L \exp i\mathrm{Conv}(\mathcal{D}_{\circ}) \cdot x_0$. Then by applying Lemma 4.5, with $\mathcal{F} = \mathcal{D}^{\llcorner}_{\circ}$ and $\mathcal{C} = \mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ})$, we obtain that $G_{f(\exp iX \cdot x_0)} = G_{\exp iX \cdot x_0}$, for every X in $\mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ})$.

Next, we check that the extension of f to $\exp i\mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ}) \cdot x_0$ satisfies the compatibility condition of Lemma 4.7. As a consequence of Lemma 4.9, the convexification $\mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ})$ is W^0-invariant. Denote by N^0 the preimage of W^0 in $N_K(\Lambda_r)$ under the canonical projection $\pi\colon N_K(\Lambda_r) \to W_K(\Lambda_r)$. Since Λ_r and $\mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ})$ are Ad_{N^0}-invariant, the domain $L \exp i\mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ}) \cdot x_0$ is N^0-invariant. Moreover, the map $f\colon L \exp i\mathcal{D}^{\llcorner}_{\circ} \cdot x_0 \to \widehat{D}$ is N^0-equivariant and so is its extension to $L \exp i\mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ}) \cdot x_0$. Hence the extension of f to $\exp i\mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ}) \cdot x_0$ satisfies all the assumptions of Lemma 4.7 and f extends to a holomorphic, G-equivariant map $G \exp i\mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ}) \cdot x_0 \to \widehat{D}$, as claimed. □

4.4. Reduction 2

Given a domain $D = G \exp i\mathcal{D}^{\llcorner} \cdot x_0$, the second reduction consists of showing that the map $f\colon D \to \widehat{D}$ has a G-equivariant holomorphic extension to the domain $\widetilde{D} = G \exp i\widetilde{\mathcal{D}}^{\llcorner} \cdot x_0$, where the set $\widetilde{\mathcal{D}}^{\llcorner}$ is the convex envelope of $\mathcal{D}^{\llcorner}$.

We first recall some properties of the universal covering of the isometry group of the Euclidean plane, namely the semidirect product Lie group $\widetilde{S} := \mathbb{R} \ltimes \mathbb{R}^2$ with the multiplication defined by

$$\left(t, \begin{pmatrix} a \\ b \end{pmatrix}\right) \cdot \left(t', \begin{pmatrix} a' \\ b' \end{pmatrix}\right) := \left(t + t', \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}\right) .$$

Its Lie algebra $\mathfrak{s}$ is isomorphic to $\mathbb{R}^3$ endowed with the Lie algebra structure defined by

$$[\widetilde{L}, \widetilde{M}] = \widetilde{N}\,, \quad [\widetilde{L}, \widetilde{N}] = -\widetilde{M}\,, \quad [\widetilde{M}, \widetilde{N}] = 0\,,$$

where $\{\widetilde{L},\, \widetilde{M},\, \widetilde{N}\}$ denotes the canonical basis of $\mathbb{R}^3$. In particular, $\widetilde{S}$ is a solvable Lie group. The universal complexification of $\widetilde{S}$ is given by $\widetilde{S}^{\mathbb{C}} := \mathbb{C} \ltimes \mathbb{C}^2$, endowed with the extended multiplication law. Consider the quotient of $\widetilde{S}^{\mathbb{C}}$ by the connected subgroup $\widetilde{H}^{\mathbb{C}}$ with Lie algebra $\mathbb{C}\widetilde{L}$. The following facts can be easily verified.

Lemma 4.11.

(i) The map $\mathbb{C}^2 \to \widetilde{S}^{\mathbb{C}}/\widetilde{H}^{\mathbb{C}}$, *defined by* $(z, w) \to \left(0, \begin{pmatrix} z \\ w \end{pmatrix}\right) \widetilde{H}^{\mathbb{C}}$, *is a biholomorphism.*

(ii) The orbit of the base point $e\widetilde{H}^{\mathbb{C}}$ *under the one-parameter subgroup* $\exp i\mathbb{R}\widetilde{M}$ *is a slice for the left* $\widetilde{S}$*-action on* $\widetilde{S}^{\mathbb{C}}/\widetilde{H}^{\mathbb{C}}$. *There is a homeomorphism*

$$\widetilde{S} \setminus \widetilde{H}^{\mathbb{C}}/\widetilde{S}^{\mathbb{C}} \cong \mathbb{R}\widetilde{M}/\mathbb{Z}_2,$$

where the $\mathbb{Z}_2$*-action on* $\mathbb{R}\widetilde{M}$ *is generated by the restriction of* $\mathrm{Ad}_{\exp \pi \widetilde{L}}$ *to* $\mathbb{R}\widetilde{M}$, *namely the reflection* $\widetilde{M} \to -\widetilde{M}$.

(iii) The $\widetilde{S}$*-invariant domains in* $\widetilde{S}^{\mathbb{C}}/\widetilde{H}^{\mathbb{C}}$ *correspond to tube domains* $\mathbb{R}^2 + i\Omega$ *in* $\mathbb{C}^2$, *whose bases are annuli.*

(iv) The envelope of holomorphy of any such tube domain is univalent and coincides with the tube domain over the smallest disc containing Ω, *namely* $\mathrm{Conv}(\Omega)$.

The crucial step of reduction 2 deals with the case of two convex connected components of $\mathcal{D}^{\llcorner}$ symmetrically placed with respect to the fixed point set of a reflection $\gamma \in W_K(\Lambda_r) \setminus W^0$. The action of γ decomposes Λ_r into the direct sum

$$\Lambda_r = Fix(\gamma) \oplus Fix(\gamma)^{\perp}.$$

Denote by $Z_G(Fix(\gamma))$ the centralizer of $Fix(\gamma)$ in G, and by $Z_{\mathfrak{g}}(Fix(\gamma))$ its Lie algebra.

Lemma 4.12.

(i) The Lie algebra $Z_{\mathfrak{g}}(Fix(\gamma))$ contains a 3-dimensional solvable subalgebra isomorphic to the Lie algebra $\mathfrak{s}$ of $\widetilde{S}$.

(ii) There exists a Lie group morphism $\psi : \widetilde{S}^{\mathbb{C}} \to G^{\mathbb{C}}$ mapping $\widetilde{H}^{\mathbb{C}}$ to $K^{\mathbb{C}}$, which induces a closed embedding $\widetilde{S}^{\mathbb{C}}/\widetilde{H}^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$.

Proof.

(i) Recall that the restricted root system of $\mathfrak{g}$ is either of type C_r or of type BC_r (see Sect. 2). For simplicity of exposition we assume $\gamma := \gamma_{12}$, the reflection flipping the first and the second coordinates (the remaining cases can be dealt in the same way). Then $Fix(\gamma)^{\perp} = \mathbb{R}(E_1 - E_2)$ and $Fix(\gamma) = \mathrm{span}\{E_1 + E_2, E_3, \dots E_r\}$. Take an arbitrary element $Q \in \mathfrak{g}^{e_1 - e_2}$ and set

$$L := Q + \theta Q, \quad M := E_1 - E_2, \quad N := [L, M].$$

We first show that L, M, N lie in the centralizer $Z_{\mathfrak{g}}(Fix(\gamma))$. By construction, one has that

$$L \in \mathfrak{k}, \quad M \in \mathfrak{g}^{2e_1} \oplus \mathfrak{g}^{2e_2}, \quad N \in \mathfrak{g}^{e_1+e_2}.$$

In order to see that $[L, E_1 + E_2] = 0$, let $Z_0 = \frac{1}{2}\sum_j T_j + S$, with $T_j = E_j + \theta E_j$ and $S \in Z_{\mathfrak{k}}(\mathfrak{a})$, be the central element in $\mathfrak{k}$ given in (2.4). Since $[L, T_j] = 0$ for $j = 3, \dots, r$, and the terms $[L, T_1 + T_2]$ and $[L, S]$ are linearly independent, the relation $[L, Z_0] = 0$ implies $[L, T_1 + T_2] = [L, S] = 0$. From $[L, T_1 + T_2] = 0$ and the identity $\theta L = L$, it follows that $[L, E_1 + E_2] + \theta[L, E_1 + E_2] = 0$. This is equivalent to $[L, E_1 + E_2] \in \mathfrak{g}^{e_1+e_2} \cap \mathfrak{p}$ and implies $[L, E_1 + E_2] = 0$, as desired. The remaining bracket relations

$$[L, E_j] = [M, E_j] = [N, E_j] = 0, \text{ for } j \geqslant 3,$$
$$[M, E_1 + E_2] = [N, E_1 + E_2] = 0,$$

are straightforward.

Next we prove that the vectors $\{L, M, N\}$ generate a 3-dimensional solvable subalgebra of $\mathfrak{g}$ isomorphic to the algebra $\mathfrak{s}$ of $\widetilde{S}$, discussed above. Since $[M, N] = 0$, it remains to show that, by normalizing Q if necessary, one has $[L, N] = -M$. Endow the 3-dimensional subspace of $\mathfrak{g}$

$$V := \mathfrak{g}^{2e_1} \oplus \mathfrak{g}^{2e_2} \oplus \mathbb{R}N,$$

with the restriction of the Ad_K-invariant inner product of $\mathfrak{g}$, defined by $B_\theta(X,Y) := -B(X,\theta Y)$, for $X,Y\in\mathfrak{g}$. One can easily verify that the vectors $\{E_1+E_2, M=E_1-E_2, N=[L,M]\}$ form an orthogonal basis of V with respect to B_θ. Since ad_L is a skew-symmetric operator and $[L,E_1+E_2]=0$, the 2-dimensional subspace $Span\{M,N\}$ is ad_L-stable in V. Thus one can normalize Q so that $\mathrm{ad}_L(N)=-M$, as desired.

(ii) Under the identification of $\mathbb{C}^2$ with $\widetilde{S}^{\mathbb{C}}/\widetilde{H}^{\mathbb{C}}$ given in Lemma 4.11, the induced map is given by $(z,w)\to\exp(zM+wN)\cdot x_0$. Its image can be viewed as the orbit through the base point x_0 of the abelian subgroup with Lie algebra $\mathrm{span}_{\mathbb{C}}\{M,\ N\}$. Now the result follows from the injectivity of the map ι defined in (3.1) and Theorem 2 in [17], stating that the orbits of a unipotent subgroup in the affine space $G^{\mathbb{C}}/K^{\mathbb{C}}$ are closed. $\square$

Example. — As an example consider $G=Sp(r,\mathbb{R})$. Fix

$$Q=\begin{pmatrix}\check{Q} & O & O & O\\ O & O & O & O\\ O & O & -\check{Q}^t & O\\ O & O & O & O\end{pmatrix}\in\mathfrak{g}^{e_1-e_2},\qquad\text{with}\quad \check{Q}=\begin{pmatrix}0 & -1/2\\ 0 & 0\end{pmatrix}.$$

The corresponding 3-dimensional solvable Lie subalgebra of $\mathfrak{g}$ is generated by the matrices

$$L=\begin{pmatrix}\check{L} & O & O & O\\ O & O & O & O\\ O & O & \check{L} & O\\ O & O & O & O\end{pmatrix}\quad M=\begin{pmatrix}O & O & \check{M} & O\\ O & O & O & O\\ O & O & O & O\\ O & O & O & O\end{pmatrix}\quad N=\begin{pmatrix}O & O & \check{N} & O\\ O & O & O & O\\ O & O & O & O\\ O & O & O & O\end{pmatrix}$$

where

$$\check{L}=\begin{pmatrix}0 & -1/2\\ 1/2 & 0\end{pmatrix},\quad \check{M}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \check{N}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

and is isomorphic to $\mathfrak{s}$. The corresponding group is closed in $Sp(r,\mathbb{R})$ and given by

$$\begin{pmatrix}U & O & B & O\\ O & I_{r-2} & O & O\\ O & O & U & O\\ O & O & O & I_{r-2}\end{pmatrix},\qquad U\in SO(2),\ B={}^tB,\ tr(B)=0.$$

Proposition 4.13 (Reduction 2). — *Let $\mathcal{D}^{\llcorner}_{\circ}$ be a convex set in $\mathcal{D}^{\llcorner}$ and let γ be a reflection in $W_K(\Lambda_r)\setminus W^0$. The map*

$$f\colon G\exp i(\mathcal{D}^{\llcorner}_{\circ}\cup\gamma\cdot\mathcal{D}^{\llcorner}_{\circ})\cdot x_0\to\widehat{D}$$

has a G-equivariant, holomorphic extension to the domain

$$\widetilde{D} = G \exp i \ \mathrm{Conv}(\mathcal{D}_\circ^\llcorner \cup \gamma \cdot \mathcal{D}_\circ^\llcorner) \cdot x_0.$$

Proof. — For simplicity of exposition we assume $\gamma = \gamma_{12}$, which implies $Fix(\gamma) = \mathrm{span}\{E_1 + E_2, E_3, \ldots, E_r\}$ and $Fix(\gamma)^\perp = \mathbb{R}(E_1 - E_2)$. Set $M := E_1 - E_2$ and let N and L be as in the proof of Lemma 4.12. Denote by $\mathfrak{s}$ the Lie subalgebra of $Z_{\mathfrak{g}}(Fix(\gamma))$ generated by $\{L,\ M,\ N\}$ and by S the corresponding subgroup in $Z_G(Fix(\gamma))$. Denote by $\mathfrak{m}$ the abelian subalgebra of $\mathfrak{s}$ generated by $\{M,\ N\}$, and by H the (possibly non-closed) subgroup of $Z_G(Fix(\gamma)) \cap K$ with Lie algebra $\mathbb{R}L$.

An arbitrary element $X \in \mathcal{D}_\circ^\llcorner$ decomposes in a unique way as $X = Y + Z$, where $Y \in Fix(\gamma)$ and $Z \in Fix(\gamma)^\perp = \mathbb{R}M$ depend continuously on X. For a fixed $X \in \mathcal{D}_\circ^\llcorner$, define

$$\Sigma_Y := \mathbb{R}M \bigcap ((\mathcal{D}_\circ^\llcorner \cup \gamma \cdot \mathcal{D}_\circ^\llcorner) - Y)\,, \quad \text{and} \quad A_Y := \mathrm{Ad}_H \Sigma_Y.$$

Since the Adjoint action of H on $\mathfrak{m}$ is by rotations, the set A_Y is an annulus in $\mathfrak{m}$. Denote by

$$T_Y := \exp(iA_Y + \mathfrak{m}) \cdot x_0 = S \exp i\Sigma_Y \cdot x_0 \tag{4.2}$$

the image of the tube domain $iA_Y + \mathfrak{m}$ in $\mathfrak{m}^{\mathbb{C}} \cong \mathbb{C}^2$ under the embedding

$$\iota : \mathfrak{m}^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}, \quad W \to \exp W \cdot x_0 \tag{4.3}$$

(see Lemma 4.11(i) and Lemma 4.12(ii)). Note that $Y + \Sigma_Y$ is contained in $\mathcal{D}_\circ^\llcorner \cup \gamma \cdot \mathcal{D}_\circ^\llcorner$. Since $Y \in Fix(\gamma)$ and S centralizes $Fix(\gamma)$, the map

$$T_Y \to D, \qquad \exp W \cdot x_0 \mapsto \exp iY \ \exp W \cdot x_0, \qquad W \in iA_Y + \mathfrak{m}$$

is S-equivariant, and so is the holomorphic map

$$f_Y : T_Y \to \widehat{D}, \qquad \exp W \cdot x_0 \to f(\exp iY \exp W \cdot x_0)\,.$$

By Bochner's tube theorem, the envelope of holomorphy of T_Y is univalent and given by

$$\widehat{T}_Y = \exp(i\mathrm{Conv}(A_Y) + \mathfrak{m}) \cdot x_0 = S \exp i\mathrm{Conv}(\Sigma_Y) \cdot x_0$$

(note that $\mathrm{Conv}(\Sigma_Y) = \mathrm{Conv}(A_Y) \cap \mathbb{R}M$). In particular, it is contained in Ξ^+. By Proposition 4.1, the map f_Y extends holomorphically and S-equivariantly to $\hat{f}_Y : \widehat{T}_Y \to \widehat{D}$ and, as X varies in $\mathcal{D}_\circ^\llcorner$, one obtains a family of S-equivariant holomorphic maps $\hat{f}_Y$, parametrized by Y. Set

$$\widetilde{\mathcal{D}} := \bigcup_{X \in \mathcal{D}_\circ^\llcorner} Y + \mathrm{Conv}(\Sigma_Y) = \mathrm{Conv}(\mathcal{D}_\circ^\llcorner \cup \gamma \cdot \mathcal{D}_\circ^\llcorner)\,,$$

where the second equality follows from an argument similar to the one of Lemma 7.7 (iv) in [15]. We define a candidate for the desired extension $\hat{f}\colon \exp i\widetilde{\mathcal{D}} \cdot x_0 \to \widehat{D}$ as follows

$$\hat{f}(\exp iX \cdot x_0) := \hat{f}_Y(\exp iZ \cdot x_0)\,. \tag{4.4}$$

The map $\hat{f}$ coincides with f on $\exp i(\mathcal{D}^{\llcorner}_{\circ} \cup \gamma \cdot \mathcal{D}^{\llcorner}_{\circ}) \cdot x_0$, since for $X \in \mathcal{D}^{\llcorner}_{\circ} \cup \gamma \cdot \mathcal{D}^{\llcorner}_{\circ}$ one has that $Z \in \Sigma_Y$ and

$$\hat{f}(\exp iX \cdot x_0) = \hat{f}_Y(\exp iZ \cdot x_0) = f_Y(\exp iZ \cdot x_0) =$$

$$= f(\exp iY \exp iZ \cdot x_0) = f(\exp i(Y+Z) \cdot x_0) = f(\exp iX \cdot x_0)\,.$$

To complete the proof of the proposition, it remains to check that $\hat{f}$ satisfies all the assumptions of Lemma 4.7 and therefore extends to a G-equivariant holomorphic map $\hat{f}\colon G\exp i\widetilde{\mathcal{D}} \cdot x_0 \to \widehat{D}$.

- The map $\hat{f}$ is a lift of the natural inclusion $\exp i\widetilde{\mathcal{D}} \cdot x_0 \hookrightarrow \Xi^+$.
Since $\hat{f}$ extends f, one has $q \circ \hat{f}(\exp iX \cdot x_0) = \exp iX \cdot x_0$, for all $X \in \mathcal{D}^{\llcorner}_{\circ} \cup \gamma \cdot \mathcal{D}^{\llcorner}_{\circ}$. In particular, from (4.2), the S-equivariance of $q \circ f_Y$ and the fact that S centralizes Y, one has $q \circ f_Y(\exp iZ \cdot x_0) = \exp iY \exp iZ \cdot x_0$, for all $Z \in \Sigma_Y$. By applying the analytic continuation principle to each $q \circ \hat{f}_Y : \widehat{T}_Y \to G^{\mathbb{C}}/K^{\mathbb{C}}$, one obtains
$$q \circ \hat{f}(\exp iX \cdot x_0) = q \circ \hat{f}_Y(\exp iZ \cdot x_0) = \exp iY \exp iZ \cdot x_0 = \exp iX \cdot x_0$$
for all $X \in \widetilde{\mathcal{D}}$.
- The map $\hat{f}$ is continuous.
The Stein Riemann domain $\widehat{D}$ admits a holomorphic embedding into some $\mathbb{C}^N$. Then, in order to prove the continuity of $\hat{f}$, it is sufficient to show that the composition $F \circ \hat{f} : \exp i\widetilde{\mathcal{D}} \cdot x_0 \to \mathbb{C}$ is continuous, for every holomorphic function $F : \widehat{D} \to \mathbb{C}$. Since the map ι in (4.3) is an embedding, this is equivalent to checking that the map
$$F \circ \hat{f} \circ \iota|_{i\widetilde{\mathcal{D}}} : i\widetilde{\mathcal{D}} \to \mathbb{C}, \qquad iX \to F \circ \hat{f}(\exp iX \cdot x_0)$$
is continuous.
Choose an open set U in $Fix(\gamma)$ and an open γ-invariant subset Σ in $\mathbb{R}M = Fix(\gamma)^{\perp}$, such that $U + \Sigma \subset \mathcal{D}^{\llcorner}_{\circ} \cup \gamma \cdot \mathcal{D}^{\llcorner}_{\circ}$. By the definition of Σ, when Y varies in U, the functions f_Y are all defined on the tube domain $T_\Sigma = S\exp i\Sigma \cdot x_0$. Moreover, the map
$$U \to \mathcal{O}(T_\Sigma, \mathbb{C})\,, \quad Y \to F \circ f_Y|_{T_\Sigma}$$
is continuous with respect to the compact-open topology on the Fréchet algebra $\mathcal{O}(T_\Sigma, \mathbb{C})$ of holomorphic functions on T_Σ. Indeed, for $W \in$

$iA_Y + \mathfrak{m}$ and $Y \in U$, one has

$$F \circ f_Y(\exp W \cdot x_0) = F \circ f(\exp iY \exp W \cdot x_0) = F \circ f(\exp(iY + W) \cdot x_0)\,.$$

Thus, if $Y_n \to Y_0$, then $F \circ f_{Y_n} \to F \circ f_{Y_0}$ uniformly on compact subsets of T_Σ. Since the extension map $\mathcal{O}(T_\Sigma, \mathbb{C}) \to \mathcal{O}(\widehat{T}_\Sigma, \mathbb{C})$ is continuous (see cap. I in [8]), it follows that also the map

$$U \to \mathcal{O}(\widehat{T}_\Sigma, \mathbb{C})\,, \quad Y \to F \circ \hat{f}_Y|_{\widehat{T}_\Sigma}$$

is continuous with respect to the compact-open topology on $\mathcal{O}(\widehat{T}_\Sigma, \mathbb{C})$. As we already remarked, $\widehat{T}_\Sigma = S \exp i\mathrm{Conv}(\Sigma) \cdot x_0$. As a consequence, the map

$$F \circ \hat{f} \circ \iota|_{i(U + \mathrm{Conv}(\Sigma))} : i(U + \mathrm{Conv}(\Sigma)) \to \mathbb{C}\,,$$

defined by

$$iX \to F \circ \hat{f}(\exp iX \cdot x_0) = F \circ \hat{f}_Y(\exp iZ \cdot x_0)$$

is continuous. Since the domains of the form $i(U + \mathrm{Conv}(\Sigma))$ cover $i\widetilde{\mathcal{D}}$, the map $\hat{f}$ is continuous.

- For all $X \in \widetilde{\mathcal{D}}$, one has $G_{\hat{f}(\exp iX \cdot x_0)} = G_{\exp iX \cdot x_0}$.

  We apply Lemma 4.5, with $\mathcal{C} = \widetilde{\mathcal{D}}$ and $\mathcal{F} = \mathcal{D}_\circ^\llcorner \cup \gamma \cdot \mathcal{D}_\circ^\llcorner$. In order to check condition (ii) of the lemma, let $X = Y + Z \in Y + \mathrm{Conv}(\Sigma_Y)$ be an arbitrary element of $\mathcal{C} \setminus \mathcal{F}$. Then there exists $Z' \in \Sigma_Y$ such that $X' = Y + Z' \in Y + \Sigma_Y \subset \mathcal{F}$ and the one dimensional complex submanifold in (ii) of Lemma 4.5 is given by

$$\begin{aligned}\mathcal{S} := \{\, \exp i(X' + \lambda(X - X')) \cdot x_0 \ &: \mathrm{Re}\lambda \in [0,1]\,\} \\ = \{\, \exp i(Y + Z' + \lambda(Z - Z')) \cdot x_0 \ &: \mathrm{Re}\lambda \in [0,1]\,\}\,.\end{aligned}$$

  Note that $Z - Z'$ belongs to $\mathbb{R}M$ and that the strip

$$\{i(Z' + \lambda(Z - Z'))) \cdot x_0 \ : \mathrm{Re}\lambda \in [0,1]\,\}$$

  is contained in $i\mathrm{Conv}(\Sigma_Y) + \mathfrak{m}$. Thus $\exp i(Z' + \lambda(Z - Z')) \cdot x_0 \in T_Y$ and one has a natural holomorphic extension of f to the one dimensional complex submanifold $\mathcal{S}$, namely

$$\hat{f}(\exp i(Y + Z' + \lambda(Z - Z')) \cdot x_0) = \hat{f}_Y(\exp i(Z' + \lambda(Z - Z')) \cdot x_0)\,.$$

  This shows that we can apply Lemma 4.5, as claimed.

- The map $\hat{f}$ satisfies the compatibility condition.

  Let $k_\gamma \in H$ be the element inducing the reflection with respect to the origin in $\mathbb{R}M$. Since H centralizes $Fix(\gamma)$, the element k_γ belongs to $N_K(\Lambda_r^\llcorner)$ and induces the reflection γ given in the statement. Hence, for every $X \in \widetilde{\mathcal{D}}$ one has $\gamma \cdot X = \mathrm{Ad}_{k_\gamma} X$. Moreover, by the H-equivariance of the maps $\hat{f}_Y$, one obtains the identity

$$\begin{aligned}\hat{f}(\exp(i\,\gamma \cdot X) \cdot x_0) &= \hat{f}(\exp i(Y + \gamma \cdot Z) \cdot x_0) = \hat{f}_Y(\exp i\mathrm{Ad}_{k_\gamma} Z \cdot x_0)\\ &= \hat{f}_Y(k_\gamma \exp iZ \cdot x_0) = k_\gamma \cdot \hat{f}_Y(\exp iZ \cdot x_0) = k_\gamma \cdot \hat{f}(\exp iX \cdot x_0)\,,\end{aligned}$$

which is the desired compatibility condition. □

Corollary 4.14. — *By iterating reductions 1 and 2 finitely many times, one obtains an extension of $f : D = G \exp i\mathcal{D}^{\llcorner} \cdot x_0 \to \widehat{D}$ to a G-equivariant holomorphic map*

$$\hat{f} \colon G \exp i\mathrm{Conv}(\mathcal{D}^{\llcorner}) \cdot x_0 \to \widehat{D}.$$

5. The main theorem

In this section we show that the envelope of holomorphy of a G-invariant domain in Ξ^+ is univalent (Cor. 5.4). Such a result is a consequence of Theorem 5.1. As a by-product we also obtain that every Stein G-invariant subdomain of Ξ^+ is either contained in Ξ or, in the tube case, in S^+ (Cor. 5.2). Togheter with the results in [7] and [16] and Remark 5.3 below, this completes the classification of all Stein G-invariant domains in Ξ^+.

Theorem 5.1. — *Let G/K be an irreducible non-compact Hermitian symmetric space. Given a G-invariant domain D in Ξ^+, denote by $\widehat{D}$ its envelope of holomorphy.*

(i) Assume G/K is of tube type. If D is not contained in Ξ nor in S^+, then $\widehat{D}$ is univalent and coincides with Ξ^+.

(ii) Assume G/K is not of tube type. If D is not contained in Ξ, then $\widehat{D}$ is univalent and coincides with Ξ^+.

Proof. — The proof of the theorem consists of a sequence of rank-one reductions and convexifications (reductions 1 and 2), until an extension $\hat{f}$ of the lift $f_{|\exp i\mathcal{D}^{\llcorner} \cdot x_0} \colon \exp i\mathcal{D}^{\llcorner} \cdot x_0 \to \widehat{D}$ in diagram (4.1) to the whole $\exp i\Lambda_r^{\llcorner} \cdot x_0$ is obtained. The map $\hat{f}$ is constructed so that it satisfies the assumptions of Lemma 4.7 and yields a G-equivariant holomorphic extension of the map $f \colon D \to \widehat{D}$ to the whole Ξ^+. Then the theorem follows from (ii) of Proposition 4.2. We need to distinguish several cases.

Case 1. — Let $D = G \exp i\mathcal{D}^{\llcorner} \cdot x_0$ in Ξ^+ be a domain satisfying the condition

$$\mathcal{D}^{\llcorner} \bigcap \Lambda_r^{\llcorner} \setminus \left(\bigoplus_{j=1}^{r} [0,1) E_j \bigcup \bigoplus_{j=1}^{r} (1,\infty) E_j \right) \neq \emptyset.$$

In the tube case, the above condition is equivalent to the assumptions in (i) (see (3.3) and (3.4)). By reductions 1 and 2, the set $\mathcal{D}^{\llcorner}$ may be assumed to be a $W_K(\Lambda_r)$-invariant, open convex subset of $\Lambda_r^{\llcorner}$. A simple argument shows that it contains a point X with exactly one coordinate equal to 1, and the other ones either all < 1 (Case 1.a) or all > 1 (Case 1.b).

For $j = 1, \ldots, r$, denote by G_j the rank-one subgroup of G with Lie algebra defined in (2.1). Then $K_j := G_j \cap K$ is a maximal compact subgroup of G_j and the quotient G_j/K_j is a rank-one Hermitian symmetric space of tube-type. The envelope of holomorphy of an invariant domain in $G_j^{\mathbb{C}}/K_j^{\mathbb{C}}$ is univalent and described by Theorem 4.4 (for $n = 1$).

Case 1.a. — In this case, in view of (3.2), the set $(\mathcal{D}^{\llcorner})^+$ contains a point

$$X = (1, x_2, \ldots, x_r), \qquad \text{with} \quad 1 > x_2 > \ldots x_r > 0. \tag{5.1}$$

Our first goal is to obtain an extension of f to $\exp i\widetilde{\mathcal{D}} \cdot x_0$, where $\widetilde{\mathcal{D}}$ is an open $W_K(\Lambda_r)$-invariant convex set in $\Lambda_r^{\llcorner}$ containing $\mathcal{D}^{\llcorner}$ and the point $(1, 0, \ldots, 0)$. This is done in stages, by gradually extending f to $W_K(\Lambda_r)$-invariant larger sets of the form $\exp i\mathcal{C} \cdot x_0$, where $\mathcal{C}$ contains $\mathcal{D}^{\llcorner}$ and, in order, the points

$$(1, x_2, \ldots, x_{r-2}, x_{r-1}, 0), \quad (1, x_2, \ldots, x_{r-2}, 0, 0), \ \ldots \ , \ (1, 0, \ldots, 0).$$

Denote by

$$\mathrm{int}(\overline{(\Lambda_r^{\llcorner})^+}) \qquad \text{and} \qquad \mathrm{int}((\mathcal{D}^{\llcorner})^+) = (\mathcal{D}^{\llcorner})^+ \cap \mathrm{int}(\overline{(\Lambda_r^{\llcorner})^+})$$

the interior of $\overline{(\Lambda_r^{\llcorner})^+}$ and of $(\mathcal{D}^{\llcorner})^+$ in $\Lambda_r^{\llcorner}$, respectively. Note that the former coincides with $\overline{(\Lambda_r^{\llcorner})^+} \setminus \mathcal{H}$, where $\mathcal{H} := \cup_{\gamma \in W_K(\Lambda_r)} \{Fix(\gamma)\}$ denotes the set of reflection hyperplanes in $\Lambda_r^{\llcorner}$. Under assumption (5.1), the interior of $(\mathcal{D}^{\llcorner})^+$ contains an open set of the form

$$U + V, \quad \text{with} \quad (1, x_2, \ldots x_{r-1}, 0) \in U \subset E_r^{\perp} \quad \text{and} \quad V = (a_r, b_r) E_r \quad (b_r < 1).$$

Decompose an element $W \in U + V$ as $W = Y + Z$, where $Y \in U$ and $Z \in V$ depend continuously on W, and define $D_r := G_r \exp iV \cdot x_0$. By (2.3), the group G_r commutes with G_j for $j \neq r$. Hence, for every $Y \in U$, the holomorphic map

$$f_Y \colon D_r \to \widehat{D}, \quad g \exp iZ \cdot x_0 \to f(\exp iY g \exp iZ \cdot x_0)$$

is G_r-equivariant and, by Prop. 4.4(i), extends to a G_r-equivariant holomorphic map

$$\hat{f}_Y \colon \widehat{D}_r \longrightarrow \widehat{D},$$

where $\widehat{D}_r = G_r \exp i\widehat{V} \cdot x_0$ and $\widehat{V} = [0, b_r)$. Define now a map

$$\hat{f}\colon\ \exp i(U + \widehat{V}) \cdot x_0 \to \widehat{D}, \quad \text{by} \quad \exp iW \cdot x_0 \to \hat{f}_Y(\exp iZ \cdot x_0). \tag{5.2}$$

The arguments used for the map (4.4) in the proof of Proposition 4.13, show that $\hat{f}$ defined above coincides with f on $\exp i(U + V) \cdot x_0$ and that it is a continuous lift of the natural inclusion of $\exp i(U + \widehat{V}) \cdot x_0$ into Ξ^+. In order to be able to apply Lemma 4.7, define

$$\mathcal{C} = (U + \widehat{V}) \bigcap \operatorname{int}(\overline{(\Lambda_r^{\llcorner})^+}) \tag{5.3}$$

and restrict the map (5.2) to the set $\exp i\mathcal{C} \cdot x_0$. We stress this point even if in this particular case $U + \widehat{V}$ is already entirely contained in the interior of $\overline{(\Lambda_r^{\llcorner})^+}$. However, this will not be the case in the next steps. Now $\hat{f}|_{\exp i\mathcal{C}\cdot x_0}$ satisfies all the assumptions of the extension Lemma 4.7 : the set $\mathcal{C}$ is open in $\Lambda_r^{\llcorner}$ and entirely contained in the perfect slice $\overline{(\Lambda_r^{\llcorner})^+}$. Hence $\hat{f}|_{\exp i\mathcal{C}\cdot x_0}$ satisfies the compatibility conditions. Finally, the identity $G_{\exp iX\cdot x_0} = G_{\hat{f}(\exp iX\cdot x_0)}$, for all $X \in \mathcal{C}$, follows from Lemma 4.5. For this set $\mathcal{F} = U + V$ and let $Y + Z$ be an arbitrary element in $\mathcal{C} \setminus \mathcal{F}$. Choose an element in $\mathcal{F}$ of the form $Y + Z'$. Then condition (ii) of Lemma 4.5 is satisfied, since $\hat{f}$ is holomorphic on the one dimensional complex submanifold

$$\{\exp i(Y + Z' + \lambda(Z - Z')) \cdot x_0 \ : \mathrm{Re}\lambda \in [0, 1]\}.$$

As a consequence of Lemma 4.7, the map (5.2) extends to a G-equivariant holomorphic map

$$\hat{f}\colon G \exp i\mathcal{C} \cdot x_0 \to \widehat{D}. \tag{5.4}$$

Note that the open G-invariant subset $G \exp i\mathcal{C} \cdot x_0$ of Ξ^+ coincides with $G \exp i\,(W_K(\Lambda_r) \cdot \mathcal{C}) \cdot x_0$ and has open intersection with D. By the analytic continuation principle, the map (5.4) coincides with f on the points of D, and determines a G-equivariant holomorphic extension $\hat{f}\colon G \exp i\widetilde{\mathcal{D}} \cdot x_0 \to \widehat{D}$ where

$$\widetilde{\mathcal{D}} = \mathcal{D}^{\llcorner} \bigcup W_K(\Lambda_r) \cdot \mathcal{C}.$$

The set $\widetilde{\mathcal{D}}$ contains the point $(1, x_2, \ldots x_{r-1}, 0)$, projection of the initial point X onto the hyperplane $x_r = 0$. By reductions 1 and 2, it may be assumed to be convex.

Iterating the above procedure for the coordinates $x_{r-1},\ x_{r-2},\dots,x_2$ produces G-equivariant holomorphic extensions of f to open $W_K(\Lambda_r)$-invariant convex sets containing $\mathcal{D}^{\llcorner}$ and, in order, the points

$$(1,x_2,\dots,x_{r-2},0,0),\ \dots,(1,0,\dots,0).$$

For the final step, set $\mathcal{D}^{\llcorner}=\widetilde{\mathcal{D}}$ and take an open subset of $\mathrm{int}((\mathcal{D}^{\llcorner})^+)$ of the form $U+V$, with $U\subset E_1^{\perp}$ and $V=(a_1,b_1)E_1$, for $a_1<1<b_1$. This time $D_1=G_1\exp iV\cdot x_0$ is a G_1-invariant complex submanifold of $G^{\mathbb{C}}/K^{\mathbb{C}}$ whose envelope of holomorphy is given by $\widehat{D}_1=G_1\exp i\widehat{V}\cdot x_0$, with $\widehat{V}:=[0,\infty)E_1$ (see Prop. 4.4(iii)). The usual procedure produces then a holomorphic G-equivariant extension $\hat{f}\colon \Xi^+\setminus G/K\to\widehat{D}$. The fact that the orbit G/K is a priori excluded from the domain of $\hat{f}$ is due to an intersection like in (5.3). On the other hand, since the orbit G/K is a totally real submanifold of Ξ^+, the map $\hat{f}$ extends to the whole Ξ^+, as desired.

Case 1.b. — In this case in view of (3.2), the set $(\mathcal{D}^{\llcorner})^+$ contains a point

$$X=(x_1,x_2,\dots,1),\qquad \text{with}\quad x_1>x_2>\dots x_{r-1}>1. \tag{5.5}$$

Our goal is to reduce to the previous case (1.a) by contructing an extension of f to a set $\exp i\widetilde{\mathcal{D}}\cdot x_0$, where $\widetilde{\mathcal{D}}$ is an open $W_K(\Lambda_r)$-invariant convex set in $\Lambda_r^{\llcorner}$ containing $\mathcal{D}^{\llcorner}$ and the point $(1,0,\dots,0)$. By proceeding as in the first step of case (1.a), we obtain a G-equivariant holomorphic extension $\hat{f}\colon G\exp i\widetilde{\mathcal{D}}\cdot x_0\to\widehat{D}$, where $\widetilde{\mathcal{D}}$ is an open $W_K(\Lambda_r)$-invariant convex set in $\Lambda_r^{\llcorner}$, containing $\mathcal{D}^{\llcorner}$ and $(x_1,x_2,\dots x_{r-1},0)$, the projection of X onto the hyperplane $x_r=0$. Then $\widetilde{\mathcal{D}}$ also contains the point $(x_1,x_2,\dots,0,x_{r-1})$ and the segment

$$(x_1,x_2,\dots,tx_{r-1},(1-t)x_{r-1}),\quad \text{for } t\in[0,1].$$

In particular, it contains the point $(x_1,x_2,\dots,2/3\,x_{r-1},1/3\,x_{r-1})$, which lies in $(\mathcal{D}^{\llcorner})^+$ and has a smaller $(r-1)^{th}$ coordinate than $(x_1,x_2,\dots x_{r-1},0)$. By iterating this argument, we infer that $\widetilde{\mathcal{D}}$ contains $(x_1,x_2,\dots,x'_{r-1},0)$, for some $x'_{r-1}<1$. Then by the convexity of $\widetilde{\mathcal{D}}$ and the inequality $x_{r-1}>1$, it also contains the point $(x_1,\dots,x_{r-2},1,0)$.

By applying the above procedure to the coordinates $x_{r-1},x_{r-2},\dots,x_2$, we obtain G-equivariant holomorphic extensions $\hat{f}\colon G\exp i\widetilde{\mathcal{D}}\cdot x_0\to\widehat{D}$, where $\widetilde{\mathcal{D}}$ is an open $W_K(\Lambda_r)$-invariant convex set in $\Lambda_r^{\llcorner}$, containing $\mathcal{D}^{\llcorner}$ and, in order, the points

$$(x_1,\dots,x_{r-3},1,0,0),\dots,(1,0,\dots,0).$$

Case 2. — We finally consider the case where G/K is not of tube-type, the set $\mathcal{D}^{\llcorner}$ is contained in $\bigoplus_{j=1}^{r}(1,\infty)E_j$ and, in view of (3.2), contains a point

$$X=(x_1,x_2,\dots x_r),\qquad \text{with}\quad x_1>x_2>\dots>x_r>1.$$

Our goal is to reduce to the case (1.a) by showing that the map f extends to a set $\exp i\widetilde{\mathcal{D}}\cdot x_0$, where $\widetilde{\mathcal{D}}$ is an open $W_K(\Lambda_r)$-invariant convex set in $\Lambda_r^{\llcorner}$ containing $\mathcal{D}^{\llcorner}$ and the point $(1,0,\dots,0)$. As in case (1.a), this is done in stages, by gradually extending f to $W_K(\Lambda_r)$-invariant larger sets of the form $\exp i\mathcal{C}\cdot x_0$, where $\mathcal{C}$ contains $\mathcal{D}^{\llcorner}$ and, in order, the points

$$(1,x_2,\dots,x_{r-2},x_{r-1},0),\quad (1,x_2,\dots,x_{r-2},0,0),\ \dots,(1,0,\dots,0).$$

The procedure almost literally follows the one used in case (1.a). The difference is that when dealing with the j^{th}-coordinate, the rank-one reduction is done by using the subgroup $G_j^{\bullet}$ with Lie algebra (2.5). An important fact is that, by Lemma 2.1, the group $G_j^{\bullet}$ commutes with the rank-one subgroups G_k, for all $k\neq j$. One has that $K_j^{\bullet}:=K\cap G_j^{\bullet}$ is a maximal compact subgroup of $G_j^{\bullet}$ and $G_j^{\bullet}/K_j^{\bullet}$ is a rank-one Hermitian symmetric space, not of tube type. In particular, the envelopes of holomorphy of the $G_j^{\bullet}$-invariant domains in the complexification $(G_j^{\bullet})^{\mathbb{C}}/(K_j^{\bullet})^{\mathbb{C}}$ are described by Proposition 4.4, for $n>1$. □

Inside the crown Ξ and inside S^+, an invariant domain can be described via a semisimple abelian slice. Its Steiness is characterized by logarithmic convexity of such slice (cf. [7] and [16]). These results together with the above theorem conclude the classification of Stein G-invariant domains in Ξ^+.

Corollary 5.2. — *Let G/K be an irreducible non-compact Hermitian symmetric space and let D be a Stein G-invariant proper subdomain of Ξ^+.*

(i) If G/K is of tube type, then either $D\subseteq\Xi$ or $D\subseteq S^+$.

(ii) If G/K is not of tube type, then $D\subseteq\Xi$.

Remark 5.3. — Let G/K be an arbitrary irreducible, non-compact, Riemannian symmetric space and let $\Xi=G\exp i\Omega_{AG}\cdot x_0$ be the crown domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$, where $\Omega_{AG}:=\{H\in\mathfrak{a}\ :\ |\alpha(H)|<\frac{\pi}{2},\ \text{ for all } \alpha\in\Delta(\mathfrak{g},\mathfrak{a})\}$. An invariant domain in Ξ is given by $D=G\exp i\Omega\cdot x_0$, for some $W_K(\mathfrak{a})$-invariant open set $\Omega\subset\Omega_{AG}$, and it is Stein if and only if Ω is convex (cf. [7]). However we are not aware of a proof of the fact that $\widehat{D}=G\exp i\mathrm{Conv}(\Omega)\cdot x_0$. For the sake of completeness, we outline one here. Let $\overline{\mathfrak{a}^+}$ be the closure of a fixed Weyl chamber in $\mathfrak{a}$. Define $\Omega^+:=\Omega\cap\overline{\mathfrak{a}^+}$ and let $\Omega_\circ$ be the connected component of Ω containing Ω^+. Denote by Γ^0

the set of simple reflections in $\mathfrak{a}$ whose fixed point hyperplanes contain a non-zero element of Ω^+ and by W^0 the subgroup of $W_K(\mathfrak{a})$ generated by Γ^0. Arguing as in Lemma 4.9, one obtains $W^0 \cdot \Omega^+ = \Omega_\circ$. Set $A := \exp \mathfrak{a}$. The r-dimensional complex submanifold $A \exp i\Omega_\circ \cdot x_0$ of D is biholomorphic to a tube domain in $\mathbb{C}^r$ with base $\Omega_\circ$. Then, by the argument of Proposition 4.10, the inclusion $f\colon D \to \widehat{D}$ admits a G-equivariant holomorphic extension to the domain $G \exp i\mathrm{Conv}(\Omega_\circ) \cdot x_0$. In other words, all connected components of Ω may be assumed to be convex.

The second part of the proof consists of showing that the map f in diagram (4.1) admits a G-equivariant holomorphic extension to the domain $G \exp i\mathrm{Conv}(\Omega) \cdot x_0$. The relevant case is that of Ω consisting of two connected components $\Omega_\circ$ and $s_\alpha \cdot \Omega_\circ$, simmetrically placed with respect to the fixed point hyperplane $Fix(s_\alpha)$ of a reflection $s_\alpha \in W_K(\mathfrak{a}) \setminus W^0$. Fix a generator H_α of $Fix(s_\alpha)^\perp$ and $X_\alpha \in \mathfrak{g}^\alpha$ so that the vectors $\{X_\alpha, \theta X_\alpha, H_\alpha\}$ generate an $\mathfrak{sl}(2)$-subalgebra. Denote by G_α the corresponding rank-one subgroup of G. From now on the proof follows the one of Proposition 4.13, whereas the solvable group S is replaced by the rank-one subgroup G_α, and Lemma 4.11 is replaced by Proposition 4.4(i). Decompose an element $X \in \Omega_\circ$ as $X = Y + Z$, where $Y \in Fix(s_\alpha)$ and $Z \in \mathbb{R}H_\alpha$ depend continuosly on X, and define

$$\Sigma_Y := \mathbb{R}H_\alpha \cap ((\Omega_\circ \cup s_\alpha \cdot \Omega_\circ) - Y) \quad \text{and} \quad D_Y := G_\alpha \exp i\Sigma_Y \cdot x_0.$$

Then D_Y is biholomorphic to a G_α-invariant domain inside the crown $\Xi_\alpha \subset G_\alpha^{\mathbb{C}}/K_\alpha^{\mathbb{C}}$. The group G_α centralizes $Fix(s_\alpha)$ and, as X varies in $\Omega_\circ$, the family of G_α-equivariant holomorphic maps

$$f_Y\colon D_Y \to \widehat{D}, \qquad g \exp iZ \cdot x_0 \mapsto f(\exp iY g \exp iZ \cdot x_0),$$

determines a G-equivariant holomorphic extension of $f\colon D \to \widehat{D}$ to

$$\hat{f}\colon G \exp i\mathrm{Conv}(\Omega) \cdot x_0 \to \widehat{D}.$$

Since the domain $G \exp i\mathrm{Conv}(\Omega) \cdot x_0$ is Stein (see [7]), this shows that the envelope of holomorphy of $D = G \exp i\Omega \cdot x_0$ is univalent and coincides with $G \exp i\mathrm{Conv}(\Omega) \cdot x_0$. □

By [16], the envelope of holomorphy of a G-invariant domain in S^+ is univalent. Thus one has the following corollary.

Corollary 5.4. — *Let G/K be an irreducible non-compact Hermitian symmetric space. The envelope of holomorphy of a G-invariant domain in Ξ^+ is univalent.*

Remark 5.5. — In general a similar univalence result does not hold true for equivariant Stein Riemann domains over Ξ^+ other than envelopes of holomorphy. If G/K is a Hermitian symmetric space of tube type, one can construct a non-trivial G-equivariant covering of the Stein G-invariant subdomain S^+ of Ξ^+.

BIBLIOGRAPHY

[1] D. N. Akhiezer & S. G. Gindikin, "On Stein extensions of real symmetric spaces", *Math. Ann.* **286** (1990), no. 1-3, p. 1-12.

[2] T. Bröcker & T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1985, x+313 pages.

[3] F. Docquier & H. Grauert, "Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten", *Math. Ann.* **140** (1960), p. 94-123.

[4] L. Geatti & A. Iannuzzi, "Orbit structure of a distinguished Stein invariant domain in the complexification of a Hermitian symmetric space.", *Math. Z.* **278** (2014), no. 3-4, p. 769-793.

[5] L. Geatti, "Invariant domains in the complexification of a noncompact Riemannian symmetric space", *J. Algebra* **251** (2002), no. 2, p. 619-685.

[6] L. Geatti & A. Iannuzzi, "Univalence of equivariant Riemann domains over the complexifications of rank-one Riemannian symmetric spaces", *Pacific J. Math.* **238** (2008), no. 2, p. 275-330.

[7] S. Gindikin & B. Krötz, "Invariant Stein domains in Stein symmetric spaces and a nonlinear complex convexity theorem", *Int. Math. Res. Not.* (2002), no. 18, p. 959-971.

[8] R. C. Gunning, *Introduction to holomorphic functions of several variables. Vol. I*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1990, Function theory, xx+203 pages.

[9] L. Hörmander, *An introduction to complex analysis in several variables*, revised ed., North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973, North-Holland Mathematical Library, Vol. 7, x+213 pages.

[10] A. W. Knapp, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002, xviii+812 pages.

[11] B. Krötz, "Domains of holomorphy for irreducible unitary representations of simple Lie groups", *Invent. Math.* **172** (2008), no. 2, p. 277-288.

[12] B. Krötz & E. Opdam, "Analysis on the crown domain", *Geom. Funct. Anal.* **18** (2008), no. 4, p. 1326-1421.

[13] B. Krötz & R. J. Stanton, "Holomorphic extensions of representations. I. Automorphic functions", *Ann. of Math. (2)* **159** (2004), no. 2, p. 641-724.

[14] ———, "Holomorphic extensions of representations. II. Geometry and harmonic analysis", *Geom. Funct. Anal.* **15** (2005), no. 1, p. 190-245.

[15] K.-H. Neeb, "On the complex and convex geometry of Ol′shanskiĭ semigroups", *Ann. Inst. Fourier (Grenoble)* **48** (1998), no. 1, p. 149-203.

[16] ———, "On the complex geometry of invariant domains in complexified symmetric spaces", *Ann. Inst. Fourier (Grenoble)* **49** (1999), no. 1, p. vi, x, 177-225.

[17] M. Rosenlicht, "On quotient varieties and the affine embedding of certain homogeneous spaces", *Trans. Amer. Math. Soc.* **101** (1961), p. 211-223.

[18] H. Rossi, "On envelopes of holomorphy", *Comm. Pure Appl. Math.* **16** (1963), p. 9-17.

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