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Étale cohomology, cofinite generation, and p-adic L-functions

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ÉTALE COHOMOLOGY, COFINITE GENERATION, AND p-ADIC L-FUNCTIONS

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ABSTRACT. — Let p be a prime number. We study certain étale cohomology groups with coefficients associated to a p-adic Artin representation of the Galois group of a number field k. These coefficients are equipped with a modified Tate twist involving a p-adic index. The groups are cofinitely generated, and we determine the additive Euler characteristic. If k is totally real and the representation is even, we study the relation between the behaviour or the value of the p-adic L-function at the point e in its domain, and the cohomology groups with p-adic twist $1 - e$. In certain cases this gives short proofs of a conjecture by Coates and Lichtenbaum, and the equivariant Tamagawa number conjecture for classical L-functions. For $p = 2$ our results involving p-adic L-functions depend on a conjecture in Iwasawa theory.

RÉSUMÉ. — Soit p un nombre premier. Nous étudions certains groupes de cohomologie étale à coefficients associés à une représentation d'Artin p-adique de groupe de Galois d'un corps des nombres k. Ces coefficients sont munis d'un tordu à la Tate modifié avec un indice p-adique. Ces groupes sont de type cofini, et nous déterminons la caractéristique d'Euler additive. Si k est totalement réel et la représentation est paire, nous étudions la relation entre le comportement ou la valeur de la fonction L p-adique en le point e de ce domaine et les groupes de cohomologie avec torsion p-adique $1 - e$. Dans certains cas, ceci donne une preuve courte d'une conjecture de Coates et Lichtenbaum, et de la conjecture équivariante des nombres de Tamagawa pour les fonctions L classiques. Pour $p = 2$ nos résultats impliquant des fonctions L p-adiques dépendent d'une conjecture de la théorie d'Iwasawa.

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1. Introduction

Let k be a number field, p a prime number, E a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_E$, and $\eta : G_k \to E$ an Artin character with dual character $\eta^\vee$, that is, the character of an Artin representation of $G_k = \mathrm{Gal}(\overline{k}/k)$. If S is a finite set of finite primes of k, $m \leqslant 0$ an integer, and $\sigma : E \to \mathbb{C}$ an embedding, then the value $L_S(m, \sigma \circ \eta^\vee, k)$ of the classical truncated Artin L-function is in $\sigma(E)$, and if we let $L_S^*(m, \eta^\vee, k) = \sigma^{-1}(L_S(m, \sigma \circ \eta^\vee, k))$ in E then this is independent of σ (see Section 3).

We call η realizable over E if the corresponding representation can be defined over E. This representation can then be obtained as $M(E, \eta) \otimes_{\mathcal{O}_E} E \simeq M(E, \eta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for some finitely generated torsion-free $\mathcal{O}_E$-module $M(E, \eta)$ on which G_k acts (we shall call $M(E, \eta)$ an $\mathcal{O}_E$-lattice for η). If S includes all the finite primes of k at which η is ramified, and $\mathcal{O}_{k,S}$ is obtained from the ring of algebraic integers $\mathcal{O}_k$ of k by inverting all primes in S, then we may view $M(E, \eta)$ and $M(E, \eta) \otimes_{\mathcal{O}_E} E/\mathcal{O}_E \simeq M(E, \eta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ as sheaves for the étale topology on the open subscheme $\mathrm{Spec}\, \mathcal{O}_{k,S}$ of $\mathrm{Spec}\, \mathcal{O}_k$. We let $\alpha : \mathrm{Spec}\, \mathcal{O}_{k,S} \to \mathrm{Spec}\, \mathcal{O}_k$ be the inclusion, but in étale cohomology groups we shall write $\mathcal{O}_k$ instead of $\mathrm{Spec}\, \mathcal{O}_k$ and similarly for $\mathcal{O}_{k,S}$.

In the special case that p is odd, $E = \mathbb{Q}_p$, $m < 0$, and $L_S^*(m, \eta^\vee, k) \neq 0$, according to Conjecture 1 of [10] we should have that the étale cohomology groups $H^i(\mathcal{O}_k, \alpha_!(M(E, \eta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p(m)))$ are finite for all $i \geqslant 0$, trivial for $i > 3$, and that

$$(1.1) \quad |L_S^*(m, \eta^\vee, k)|_p = \prod_{i=0}^{3} \# H^i(\mathcal{O}_k, \alpha_!(M(E, \eta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p(m)))^{(-1)^i}.$$

(Note that on page 502 of loc. cit. the inverse of the arithmetic Frobenius is used in the definition of the L-function for η, resulting in the standard L-function for $\eta^\vee$.) We observe here that by [26, VII Theorem 12.6] and the definition of the completed L-series, the non-vanishing of the L-value is equivalent with k being totally real, and $\eta(c) = (-1)^{m-1}\eta(\mathrm{id}_{\overline{k}})$ for all complex conjugations c in G_k.

Báyer and Neukirch proved this conjecture for the trivial character [2, Theorem 6.1] assuming the main conjecture of Iwasawa theory for this character (later proved by Wiles in far greater generality; see [35, Theorems 1.2 and 1.3]). In this case the conjecture is, in fact, equivalent to an earlier conjecture of Lichtenbaum (see [23, Conjecture 9.1] and [10, Conjecture 3.1]) because $H^i(\mathcal{O}_k, \alpha_!(M(E, \eta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p(m)))$ is dual to $H^{2-i}(\mathcal{O}_{k,S}, M(E, \eta^\vee) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p(1 - m))$ for some lattice $M(E, \eta^\vee)$ (see Remark 1.5). Using this duality the proof of Báyer-Neukirch relates the

right-hand side of (1.1) to the p-adic absolute value of the value of a certain p-adic L-function at m, which equals the left-hand side of (1.1) by an interpolation formula (see (1.7)).

Again letting p be any prime number, one may therefore expect the p-adic absolute value of the value of a p-adic L-function at an integer m to be related to the multiplicative Euler characteristic of certain étale cohomology groups with an m-th Tate twist. In order to generalize this interpretation from integers m to (almost) every point in the much larger p-adic domain of definition of the p-adic L-function, we now introduce modified Tate twists indexed by suitable p-adic numbers. If k is any number field and A any $\mathbb{Z}_p[G_k]$-module, then $A(m)$ is obtained from A by multiplying the action of G_k by the m-th power of the p-cyclotomic character $\psi_p : G_k \to \mathbb{Z}_p^\times$. For g in G_k write

$$(1.2) \qquad \psi_p(g) = \omega_p(g)\psi_p^{\langle\rangle}(g)$$

with $\psi_p^{\langle\rangle}(g)$ in $1 + 2p\mathbb{Z}_p$ and $\omega_p = \omega_{p,k} : G_k \to \mu_{\phi(2p)} \subset \mathbb{Z}_p^\times$ the Teichmüller character of G_k for p. Let k_∞/k be the cyclotomic $\mathbb{Z}_p$-extension of k, and $\widetilde{\gamma}_0$ in G_k a lift of a topological generator γ_0 of $\mathrm{Gal}(k_\infty/k)$. Let $q_k = \psi_p(\widetilde{\gamma}_0) = \psi_p^{\langle\rangle}(\widetilde{\gamma}_0)$, and for a finite extension E of $\mathbb{Q}_p$ (always with p-adic absolute value $|\cdot|_p$ normalized by $|p|_p = 1/p$), put

$$(1.3) \qquad \mathfrak{B}_k(E) = \{e \text{ in } E \text{ with } |e|_p < |q_k - 1|_p^{-1}p^{-1/(p-1)}\},$$

i.e., those e in E where q_k^e and $\psi_p^{\langle\rangle}(g)^e$ for g in G_k converge. Then $\mathfrak{B}_k(E)$ contains $\mathcal{O}_E$ but it can be much larger. For any $\mathcal{O}_E[G_k]$-module A and e in $\mathfrak{B}_k(E)$, we let $A\langle e\rangle$ be A with the action of G_k multiplied by $(\psi_p^{\langle\rangle})^e$. Note that for an integer m, $A(m)$ can be obtained from $A\langle m\rangle$ by twisting the action of G_k with ω_p^m. Since $\psi_p^{\langle\rangle}(g)$ is often denoted as $\langle\psi_p(g)\rangle$, we think of this as a 'diamond' twist and use notation and terminology accordingly. Using this twist systematically leads to a more general result and simpler proofs: it removes the need to adjoin the $2p$-th roots of unity (thus avoiding many technical complications when $p = 2$).

With $W(E, \eta) = M(E, \eta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ as before, Theorem 1.4 concerns the structure of the étale cohomology groups $H^i(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$. (We state our results for étale cohomology groups with torsion coefficients. For some reasons why we prefer these over alternatives we refer to Remark 1.5 below.) For $i \geqslant 3$, it is well-known that this group is trivial if $p \neq 2$, and is finite and easily computed if $p = 2$ (see Remark 2.8), so we only consider $i = 0, 1, 2$. For its statement, let us call an $\mathcal{O}_E$-module A cofinitely generated if its Pontryagin dual $A^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(A, \mathbb{Q}_p/\mathbb{Z}_p)$, on which $\mathcal{O}_E$ acts

through its action on A, is a finitely generated $\mathcal{O}_E$-module. In this case we define $\mathrm{corank}_{\mathcal{O}_E} A = \mathrm{rank}_{\mathcal{O}_E} A^\vee$. We note that then also the natural map $A \to (A^\vee)^\vee$ is an isomorphism (see Remark 2.5). We write $\mathfrak{B}_\eta(E)$ for $\mathfrak{B}_k(E)$ if η does not contain the trivial character, and for $\mathfrak{B}_k(E) \setminus \{1\}$ if it does. Finally, if k is totally real then we call an Artin character η of G_k even if $\eta(c) = \eta(\mathrm{id}_{\overline{k}})$ for every complex conjugation c in G_k.

We first extend results on cofinite generation that already have a long history. For example, the equivalent statement for Galois cohomology of the first part of Theorem 1.4(1) below follows already by combining [32, Theorems 2.1 and 3.1] with the Corollary on page 260 of [33]; a succinct general overview is given in Appendix A.1 of [28]. (For the relation between those cohomology groups and the groups we consider, we refer to Remark 1.5 or Remark 2.6.) However, the uniform bound across all e and E in the second part of Theorem 1.4(1) is new.

THEOREM 1.4. — *Let k be a number field, p a prime number, E a finite extension of $\mathbb{Q}_p$, and $\eta : G_k \to E$ an Artin character realizable over E. Let $M(E, \eta)$ be an $\mathcal{O}_E$-lattice for η and let $W(E, \eta) = M(E, \eta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. Assume that S is a finite set of finite primes of k containing the primes above p as well as the finite primes at which η is ramified. Then for e in $\mathfrak{B}_k(E)$ the following hold.*

(1) *$H^i(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$ for $i \geqslant 0$ is cofinitely generated. There is a constant $D = D(S, \eta, k)$ independent of e, E and the choice of $M(E, \eta)$, such that each of these groups can be cogenerated by at most D elements.*

(2) *Let $r_i = r_{i,S}(1 - e, \eta) = \mathrm{corank}_{\mathcal{O}_E} H^i(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$ for $i = 0, 1, 2$. Then r_i is independent of the choice of $M(E, \eta)$, and the Euler characteristic $r_0 - r_1 + r_2$ equals*

$$-[k : \mathbb{Q}] \cdot \eta(\mathrm{id}_{\overline{k}}) + \sum_{v \in \Sigma_\infty} \mathrm{corank}_{\mathcal{O}_E} H^0(\mathrm{Gal}(\overline{k_v}/k_v), W(E, \eta)),$$

where Σ_∞ is the set of all infinite places of k and k_v the completion of k at v. This quantity is independent of e, non-positive, and is zero if and only if k is totally real and η is even. Moreover, r_i is independent of S for $i = 0$, and also for $i = 1, 2$ if $e \neq 0$.

(3) *$r_{0,S}(1-e, \eta) = 0$ if $e \neq 1$, and $r_{0,S}(0, \eta)$ equals the multiplicity of the trivial character in η. Moreover, the size of $H^0(\mathcal{O}_{k,S}, W(E, \eta)\langle 1-e\rangle)$ is a locally constant function for e in $\mathfrak{B}_\eta(E)$.*

(4) If $p \neq 2$ then $H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$ is p-divisible. If $p = 2$ then

$$H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle) \simeq$$

$$2H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle) \bigoplus \oplus_{v \in \Sigma_\infty} H^0(\mathrm{Gal}(\overline{k_v}/k_v), W(E, \eta))/2,$$

and $2H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$ is 2-divisible.

We mention that if $H^i(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$ is finite for $i = 0$ or 1, then Proposition 2.29, Remark 2.32 and Remarks 2.33 express its size and structure using cohomology groups with finite coefficients. By Theorem 1.4(2), this can happen for $i = 1$ only when k is totally real and η is even, in which case this group is finite for $i = 0$ and trivial for $i = 2$ by that theorem. Of independent interest is that for such k and η a short exact sequence of coefficients gives rise to a nine term exact sequence involving only $H^i(\mathcal{O}_{k,S}, \cdot)$ for $i = 0, 1, 2$ (see Lemma 2.25).

Remark 1.5. — We formulated Theorem 1.4 (and Theorem 1.8 below) for the groups $H^i(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$ because this gives a uniform point of view for all primes and all modified Tate twists, unlike the groups $H^i(\mathcal{O}_k, \alpha_! W(E, \eta)\langle e\rangle)$ and $H^i_{cts}(\mathcal{O}_{k,S}, M(E, \eta)\langle 1 - e\rangle)$ discussed below. On the other hand, the groups $\varprojlim_n H^i_c(\mathcal{O}_k, \alpha_!(M(E, \eta)\langle e\rangle/p^n))$ below also provide a uniform approach but they are technically more difficult to handle. For example, in the proof of Theorem 5.5 the only cohomology groups that we have to analyse in detail in our approach are of low degree, and those can be described fairly explicitly.

We refer to Remark 2.6 and the proof of Proposition 2.12 for details of the following discussion.

Let k be any number field and $\eta : G_k \to E$ any Artin character. Fix a finite set S of finite primes of k containing all primes at which η is ramified as well as all primes of k lying above p, and let $\alpha : \mathrm{Spec}\, \mathcal{O}_{k,S} \to \mathrm{Spec}\, \mathcal{O}_k$ be the natural inclusion. Then for an appropriate choice of lattices we have

$$\varprojlim_n H^i_c(\mathcal{O}_k, \alpha_!(M(E, \eta)\langle e\rangle/p^n)) \simeq \varprojlim_n H^i_c(\mathcal{O}_{k,S}, M(E, \eta)\langle e\rangle/p^n)$$

$$\simeq \left(H^{3-i}(\mathcal{O}_{k,S}, W(E, \eta^\vee \omega_p)\langle 1 - e\rangle)\right)^\vee$$

for e in $\mathfrak{B}_k(E)$ and $i = 0, \ldots, 3$, where H^i_c denotes cohomology with compact support. If $p \neq 2$ and all $H^i(\mathcal{O}_{k,S}, W(E, \eta^\vee \omega_p)\langle 1 - e\rangle)$ are finite then also

$$(1.6) \qquad H^i(\mathcal{O}_k, \alpha_! W(E, \eta)\langle e\rangle) \simeq H^{2-i}(\mathcal{O}_{k,S}, W(E, \eta^\vee \omega_p)\langle 1 - e\rangle)$$

for e in $\mathfrak{B}_k(E)$, $i = 0, 1, 2$, the group on the left being trivial for $i \geqslant 3$.

For p again an arbitrary prime, let Ω_S be the maximal extension of k that is unramified outside of S and the infinite primes of k, and let $G_S = \mathrm{Gal}(\Omega_S/k)$. Then $H^i(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle) \simeq H^i(G_S, W(E,\eta)\langle 1-e\rangle)$ where the right-hand side denotes continuous group cohomology.

Finally, for continuous étale cohomology groups H^i_{cts} with as coefficients $M(E,\eta)\langle 1-e\rangle \simeq \varprojlim_n M(E,\eta)\langle 1-e\rangle/p^n$, we have a long exact sequence

$$\cdots \to H^i_{cts}(\mathcal{O}_{k,S}, M(E,\eta)\langle 1-e\rangle) \to H^i_{cts}(\mathcal{O}_{k,S}, M(E,\eta)\langle 1-e\rangle) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$
$$\to H^i(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle) \to H^{i+1}_{cts}(\mathcal{O}_{k,S}, M(E,\eta)\langle 1-e\rangle) \to \cdots,$$

where $H^i_{cts}(\mathcal{O}_{k,S}, M(E,\eta)\langle 1-e\rangle) \simeq \varprojlim_n H^i(\mathcal{O}_{k,S}, M(E,\eta)\langle 1-e\rangle/p^n)$ is a finitely generated $\mathcal{O}_E$-module. Hence $\mathrm{rank}_{\mathcal{O}_E} H^i_{cts}(\mathcal{O}_{k,S}, M(E,\eta)\langle 1-e\rangle) = \mathrm{corank}_{\mathcal{O}_E} H^i(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$. If $H^j(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$ is finite for $j = i$ and $i+1$, then $H^i(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle) \simeq H^{i+1}_{cts}(\mathcal{O}_{k,S}, M(E,\eta)\langle 1-e\rangle)$ as well.

Our second main result Theorem 1.8 is a relation between the cohomology groups $H^i(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$ and a certain p-adic L-function $L_{p,S}(e,\eta,k)$. For such a p-adic L-function to be defined and non-trivial, we must have k totally real and η even. By Theorem 1.4(2), this is also the only case where those cohomology groups can simultaneously be finite for $i = 0, 1$ and 2. For the sake of clarity we write χ for an even Artin character when k is a totally real number field, and η for an arbitrary Artin character when k is an arbitrary number field.

The conjecture of Coates and Lichtenbaum will be deduced from the purely p-adic result Theorem 1.8 for $\chi = \eta^\vee \omega_p^{1-m}$ and $e = m$, by using (1.6) and two small miracles that occur only at integers. The first is the relation between $A\langle m\rangle$ and $A(m)$ for any integer m, so that we may assume $W(E,\chi)\langle 1-m\rangle = W(E,\chi\omega_p^{m-1})(1-m)$. The other is the interpolation formula (1.7) for negative integers (and possibly 0). It is surely no coincidence that the exponent of ω_p is the same in both miracles.

In order to state Theorem 1.8 we now discuss p-adic L-functions (see Section 3 for details). Let k be a totally real number field, p a prime number, E a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_E$, and $\chi : G_k \to E$ an even Artin character realizable over E. If S is a finite set of finite primes of k containing the primes above p, then on $\mathfrak{B}_k(E)$ there exists a meromorphic E-valued p-adic L-function $L_{p,S}(s,\chi,k)$, defined at all negative integers m, where it satisfies the interpolation formula

$$(1.7) \qquad\qquad L_{p,S}(m,\chi,k) = L_S^*(m, \chi\omega_p^{m-1}, k).$$

The same statement sometimes also holds when $m = 0$. The usual p-adic L-function $L_p(m, \chi, k)$ is obtained when S consists only of the finite primes of k lying above p.

A consequence of [17, Proposition 5] is that the main conjecture of Iwasawa theory for p (i.e., the statements in Theorem 4.2 and Remark 4.3 for p) implies that $L_{p,S}(s, \chi, k)$ is analytic on $\mathfrak{B}_k(E)$ if χ does not contain the trivial character, and meromorphic on $\mathfrak{B}_k(E)$ with at most a pole at $s = 1$ if it does. If $p \neq 2$ we could use this by [35, Theorems 1.2 and 1.3], but our results imply this consequence of Greenberg's stronger result when S also contains the finite primes of k at which χ is ramified, and we remove this restriction by proving a lower bound for the corank of $H^1(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 \rangle)$ in Proposition 2.18. In fact, working directly with Theorem 4.2 if $p \neq 2$ or Assumption 4.4 if $p = 2$, i.e., working with a weaker version of the main conjecture of Iwasawa theory, we get a uniform approach for all p.

A zero of $L_{p,S}(s, \chi, k)$ always lies in $\mathfrak{B}_k(E')$ for some finite extension E' of E (see Remark 3.9), so will be interpreted by Theorem 1.8. Note that $\mathfrak{B}_\chi(E)$ is the expected domain for $L_p(s, \chi, k)$ and $L_{p,S}(s, \chi, k)$, as the Leopoldt conjecture implies that $L_{p,S}(s, \chi, k)$ should not be defined at 1 if χ contains the trivial character. By Theorem 1.4(3) it is also the set for which $H^0(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)$ is finite, and where Theorem 1.8(3) gives a precise correspondence between non-vanishing of the p-adic L-function and the finiteness of the cohomology groups.

We can now state our second main result.

THEOREM 1.8. — *If in Theorem 1.4 k is totally real and $\eta = \chi$ is even, then $H^i(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)$ for $i \geqslant 0$ is finite for all but finitely many e in $\mathfrak{B}_k(E)$. Moreover, the following hold, where for $p = 2$ we make Assumption 4.4.*

(1) *Let $\sum_{i=0}^{2}(-1)^i i \cdot r_i = r_2 - r_0$ be the secondary Euler characteristic and let $\nu = \nu_S(1 - e, \chi) = \mathrm{ord}_{s=e} L_{p,S}(e, \chi, k)$. Then we have that $\min(1 - r_0, \nu) \leqslant r_2 - r_0 \leqslant \nu$.*

(2) *$L_{p,S}(s, \chi, k)$ and $L_p(s, \chi, k)$ are meromorphic on $\mathfrak{B}_k(E)$; the only possible pole is at $s = 1$, with order at most the multiplicity of the trivial character in χ.*

(3) *For e in $\mathfrak{B}_\chi(E)$ the following are equivalent:*
  (a) *$L_{p,S}(e, \chi, k) \neq 0$;*
  (b) *$H^1(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)$ is finite;*
  (c) *$H^2(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)$ is finite;*
  (d) *$H^2(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)$ is trivial.*

If this is the case, then

$$|L_{p,S}(e, \chi, k)|_p = \left(\frac{\#H^0(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e\rangle)}{\#H^1(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e\rangle)} \right)^{1/[E:\mathbb{Q}_p]}.$$

The assumption for $p = 2$ in Theorem 1.8 is fulfilled if $k = \mathbb{Q}$ and χ is a multiple of the trivial character (see Remark 4.5). We also remark that part of Theorem 1.8 (for p odd, $e \neq 1$ an odd integer, and $\chi = \omega_p^{1-e}$) is contained in [2, Theorem 6.1], under an assumption in Iwasawa theory since proved by Wiles. A similar partial result (for p odd, e a negative odd integer, and χ an even Artin character) is outlined in the proof of [9, Proposition 6.15]. The case $p = 2$ is not discussed in either paper.

Theorem 1.8 would be even more complete if the upper bounds in part (1) and part (2) were sharp. The equality $\nu_S(0, \chi) = -r_{0,S}(0, \chi)$ in part (2) for all k is equivalent with the Leopoldt conjecture for all k (see Remarks 5.20). Equivalently, we always have $\min(r_{0,S}(0, \chi), r_{2,S}(0, \chi)) = 0$. The other equality, in part (1), we formulate as a conjecture, which is itself implied by some folklore conjectures in Iwasawa theory (see Conjecture 5.19).

CONJECTURE 1.9. — *We have* $\nu_S(1-e, \chi) = r_{2,S}(1-e, \chi) - r_{0,S}(1-e, \chi)$ *in Theorem 1.8(1).*

In this paper we give two applications of the p-adic statements in Theorem 1.8 to classical L-functions. One of them is contained in Section 7. It consists of a short proof of the equivariant Tamagawa number conjecture at negative integers for Artin motives of the right parity over a totally real number field k with coefficients in a maximal order. (In this case the conjecture is equivalent with the Bloch-Kato conjecture [4, §5], and the corresponding result for a Dirichlet motive over $\mathbb{Q}$ with p odd and p^2 not dividing the conductor of the underlying Dirichlet character, was already stated in §10.1(b) of [15].) For Dirichlet motives over $\mathbb{Q}$ with p odd, Huber and Kings proved this conjecture at all integers [18, Theorem 1.3.1], whereas in the same situation Burns and Greither proved the stronger statement where the maximal order is replaced by the group ring [8, Corollary 8.1]. (See Remark 7.6(2) for those and other statements.)

The other application we give here, namely a proof of a generalization of the conjecture of Coates and Lichtenbaum discussed above. Let m be a negative integer, $\eta : G_k \to E$ an Artin character that is realizable over E with $L_S^*(m, \eta^\vee, k) \neq 0$, where E is a finite extension of $\mathbb{Q}_p$. Then k is totally real and $\eta\omega_p^{m-1}$ is even. As in the paragraph containing (1.1), let S contain the finite primes of k at which η is ramified. Since the conjecture is independent of the choice of S by (the proof of) [10, Proposition 3.4],

we may assume S also contains the finite primes of k lying above p, hence all finite primes of k at which the even Artin character $\chi = \eta^\vee \omega_p^{1-m}$ is ramified. We see from (1.7) that $|L_{p,S}(m,\chi,k)|_p = |L_S^*(m,\eta^\vee,k)|_p \neq 0$, so by Theorem 1.8(3) the groups $H^i(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-m\rangle)$ are all finite. By Remark 1.5, with η replaced with $\eta\omega_p^m$, if $p \neq 2$ then for $i = 0, 1$ and 2,

$$H^i(\mathcal{O}_k, \alpha_! W(E,\eta)(m)) = H^i(\mathcal{O}_k, \alpha_! W(E,\eta\omega_p^m)\langle m\rangle)$$
$$\simeq H^{2-i}(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-m\rangle)^\vee$$

with appropriate choice of lattices (see Remark 2.6), and for $i = 3$ the left-hand side is trivial. Also, the left-hand side is trivial for $i > 3$ by Theorem II.3.1 and Proposition II.2.3(d) of [25]. By Theorem 1.8(3) it then follows that

$$|L_S^*(m,\eta^\vee,k)|_p^{[E:\mathbb{Q}_p]} = |L_{p,S}(m,\chi,k)|_p^{[E:\mathbb{Q}_p]}$$
$$= \frac{\#H^0(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-m\rangle)}{\#H^1(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-m\rangle)}$$
$$= \frac{\#H^2(\mathcal{O}_k, \alpha_! W(E,\eta)(m))}{\#H^1(\mathcal{O}_k, \alpha_! W(E,\eta)(m))}.$$

Similarly, if k is totally real, $\eta^\vee \omega_p$ even, S contains the primes lying above p, $L_S^*(0,\eta^\vee,k) \neq 0$, and (1.7) is true with $m = 0$ and $\chi = \eta^\vee \omega_p$, then those equalities also hold for $m = 0$.

If $p = 2$ then the first two equalities hold for $m \leqslant 0$ under the same conditions, where for the second we make Assumption 4.4. The third equality holds for all primes p provided that we replace the last quotient with $\#\overline{H}_c^3(\mathcal{O}_{k,S}, M(E,\eta)(m))/\#\overline{H}_c^2(\mathcal{O}_{k,S}, M(E,\eta)(m))$ for certain cohomology groups $\overline{H}_c^j$ with compact support (see Section 7). In fact, for $m < 0$, the resulting equality

$$|L_S^*(m,\eta^\vee,k)|_p^{[E:\mathbb{Q}_p]} = \frac{\#\overline{H}_c^3(\mathcal{O}_{k,S}, M(E,\eta)(m))}{\#\overline{H}_c^2(\mathcal{O}_{k,S}, M(E,\eta)(m))}$$

is part of the equivariant Tamagawa number conjecture, a generalization of the conjecture of Coates and Lichtenbaum.

The paper is organized as follows. In Section 2 we prove Theorem 1.4 and discuss how the cohomology groups depend on the choice of the lattice, the field E, and on the finite set of primes S. If the cohomology groups are finite then we describe them in terms of cohomology with finite coefficients. In Section 3 we give the definition of the p-adic L-function and discuss the interpolation formula relating it to the classical L-function. In Section 4 we discuss the main conjecture of Iwasawa theory, including the case $p = 2$.

In Section 5 we restrict ourselves to k totally real and χ even and prove Theorem 1.8 in four steps. We first prove most of it for a 1-dimensional character of order not divisible by p, then for any 1-dimensional character, followed by the case of all characters. The fourth step then strengthens this to Theorem 1.8 by varying the twist. In this section we also discuss some conjectures. In Section 6 we discuss examples based on computations by X.-F. Roblot for certain Galois extensions of $\mathbb{Q}$ with dihedral group G of order 8 and cyclic subgroup H of order 4. They include the existence of two $(H \times (1+\mathbb{Z}_5)) \ltimes \mathbb{Z}_5$-extensions of $\mathbb{Q}(\sqrt{41})$ and two $(H \times (1+\mathbb{Z}_5)) \ltimes \mathbb{Z}_5[\sqrt{5}]$-extensions of $\mathbb{Q}(\sqrt{793})$, which are inside a $(G \times (1+\mathbb{Z}_5)) \ltimes \mathbb{Z}_5^2$-extension of $\mathbb{Q}$ and a $(G \times (1+\mathbb{Z}_5)) \ltimes \mathbb{Z}_5[\sqrt{5}]^2$-extension of $\mathbb{Q}$ respectively; see Remarks 6.3 and 6.5. Finally, in Section 7, we give the first application mentioned after Conjecture 1.9, i.e., we show how the results from the preceding sections imply certain cases of the equivariant Tamagawa number conjecture.

2. Cofinite generation and additive Euler characteristics

In this section we prove Theorem 1.4. In the process, we also discuss some of the properties of the cohomology groups that we shall use in later sections. We note that the cohomology groups in the later sections have coefficients with modified Tate twist indexed by $1 - e$ instead of e, hence use the twist $1 - e$ also in this section for the sake of consistency.

Notation 2.1. — Unless stated otherwise, throughout this section k denotes a number field, p a prime number, and E a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_E$. For $\eta : G_k \to E$ an Artin character realizable over E we let $V(E, \eta)$ be an Artin representation of G_k over E with character η. Then there exists an $\mathcal{O}_E$-lattice for η, i.e., a finitely generated torsion-free $\mathcal{O}_E$-module $M(E, \eta)$ with G_k-action such that $M(E, \eta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq V(E, \eta)$ as $E[G_k]$-modules. Let $W(E, \eta) = M(E, \eta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$.

For a number field F we write F_∞ for the cyclotomic $\mathbb{Z}_p$-extension. Fix a topological generator γ_0 of $\Gamma_0 = Gal(k_\infty/k)$. Let $\omega_p : G_k \to \mu_{2p}$ and $\psi_p^{\langle\rangle} : G_k \to 1 + 2p\mathbb{Z}_p$ be as in (1.2), and let $q_k = \psi_p^{\langle\rangle}(\widetilde{\gamma}_0)$ in $1 + 2p\mathbb{Z}_p$, where $\widetilde{\gamma}_0$ is a lift of γ_0 to G_k. As defined around (1.3), for any $\mathcal{O}_E$-module A on which G_k acts and e in $\mathfrak{B}_k(E) = \{s \text{ in } E \text{ with } |s|_p < |q_k - 1|_p^{-1} p^{-1/(p-1)}\}$, we let $A\langle e \rangle$ be A with the action of G_k multiplied by the character $(\psi_p^{\langle\rangle})^e$.

Let Σ_∞ be the set of infinite places of k, and S a finite set of finite primes of k containing the set P of primes above p, as well as all finite primes at which η is ramified. For any subfield k' of an algebraic closure $\overline{k}$

of k containing k, we denote by $\mathcal{O}_{k',S}$ the ring of integers in $\mathcal{O}_{k'}$ with all primes of $\mathcal{O}_{k'}$ lying above primes in S inverted. For e in $\mathfrak{B}_k(E)$ we view $W(E,\eta)\langle e\rangle$ as a sheaf for the étale topology of $\mathcal{O}_{k,S}$. As in Remark 1.5, we let Ω_S denote the maximal extension of k that is unramified outside of $S \cup \Sigma_\infty$ and let $G_S = \mathrm{Gal}(\Omega_S/k)$.

Finally, for any finite or infinite prime v of k, we let k_v denote the completion of k at v.

Remarks 2.2.

(1) If η is 1-dimensional then any $M(E,\eta)$ is isomorphic as $\mathcal{O}_E[G_k]$-module to $\mathcal{O}_E$ with g in G_k acting as multiplication by $\eta(g)$. Hence in this case $W(E,\eta)$ is always isomorphic to $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \simeq E/\mathcal{O}_E$ with this action of G_k.

(2) Let K/k be a Galois extension with $G = \mathrm{Gal}(K/k) \simeq \mathbb{Z}/p\mathbb{Z}$, $M = \mathcal{O}_E[G]$, $M_1 = \mathcal{O}_E \sum_{g \in G} g \subset M$, and $M_2 = M/M_1$, so that $M_1 \oplus M_2$ and M are lattices for the same Artin character of G_k. If $K \cap k_\infty = k$ then $W\langle 1-e\rangle^{G_k}$ is isomorphic with $\mathcal{O}_E/(q_k^{1-e}-1)$ if $e \neq 1$ and $E/\mathcal{O}_E$ if $e = 1$, $W_1\langle 1-e\rangle^{G_k} \simeq W\langle 1-e\rangle^{G_k}$ under the natural map, and $W_2\langle 1-e\rangle^{G_k} \simeq \mathcal{O}_E/(p, q_k^{1-e}-1)$. This shows the coefficients W are not necessarily unique up to isomorphism for Abelian characters that are not 1-dimensional.

(3) Assume $p \neq 2$, E contains a primitive p-th root of unity ξ_p, and fix e in $\mathfrak{B}_k(E)$. Let K/k be a Galois extension with $G = \mathrm{Gal}(K/k)$ isomorphic to the dihedral group of order $2p$. Fix r and s in G with orders p and 2 respectively. Consider the two actions of G on $M = \mathcal{O}_E^2$ by letting r act as either $R_1 = \left(\begin{smallmatrix} \xi_p & 0 \\ 0 & \xi_p^{-1} \end{smallmatrix}\right)$ or $R_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & \xi_p+\xi_p^{-1} \end{smallmatrix}\right)$, and s as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Noting that $K \cap k_\infty = k$, one easily checks that $W\langle 1-e\rangle^{G_k} \simeq \mathcal{O}_E/(\xi_p-1)$ for the first action but is trivial for the second. Therefore the coefficients W are not necessarily unique up to isomorphism for non-Abelian characters that are irreducible over $\overline{\mathbb{Q}}_p$.

Remark 2.3. — If $\chi : G_k \to E$ is a 1-dimensional even Artin character of order prime to p, then we shall see in the proof of Theorem 5.5 that $H^1(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e\rangle)$ is isomorphic to the Pontryagin dual of $\oplus_j \mathcal{O}_E/(g_j(q_k^{1-e}-1))$ for some distinguished polynomials $g_j(T)$ in $\mathcal{O}_E[T]$. We remark here that in general even $H^0(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e\rangle)$ cannot be isomorphic to the Pontryagin dual of $\oplus_j \mathcal{O}_E/(h_j(q_k^e))$ for non-zero polynomials $h_j(T)$ in $\mathcal{O}_E[T]$ because either $h_j(q_k^e)$ is zero for some e in $\mathfrak{B}_k(E)$ or $|h_j(q_k^e)|_p$ is a constant function on $\mathfrak{B}_k(E)$. This would contradict the example $W_2\langle 1-e\rangle^{G_k} \simeq \mathcal{O}_E/(p, q_k^{1-e}-1)$ in Remark 2.2(2).

Recall that in the introduction we made the following definition to describe the structure of the cohomology groups $H^i(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$ for $i = 0, 1, 2$, which, in general are not even finitely generated.

DEFINITION 2.4. — *Let $\mathcal{O}$ be the valuation ring in a finite extension of $\mathbb{Q}_p$. For an $\mathcal{O}$-module A we denote its Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ by $A^\vee$. Then $\mathcal{O}$ acts on $A^\vee$ via its action on A. We say that A is cofinitely generated if $A^\vee$ is a finitely generated $\mathcal{O}$-module, and in this case we denote by $\mathrm{corank}_{\mathcal{O}} A$ the $\mathcal{O}$-rank of $A^\vee$.*

Remark 2.5. — Note that $\mathrm{Hom}_{\mathbb{Z}_p}(\cdot, \mathbb{Q}_p/\mathbb{Z}_p)$ is an exact functor on $\mathbb{Z}_p$-modules, hence submodules and quotients of cofinitely generated modules are cofinitely generated. In particular, if A is cofinitely generated then so is $H^0(G_S, A)$. Moreover, if A is finitely generated or cofinitely generated, then by Pontryagin duality [27, Theorem 1.1.11] the natural inclusion of A into $(A^\vee)^\vee$ is an isomorphism because the $\mathbb{Q}_p/\mathbb{Z}_p$-dual of its cokernel is trivial, hence the cokernel is trivial.

Remark 2.6. — Let e in $\mathfrak{B}_k(E)$. We shall mostly use the étale cohomology groups $H^i(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$, but they are isomorphic to various other groups as we now discuss.

For m in $\mathbb{Z}$, we may assume $V(E,\eta) = V(E, \eta\omega_p^m)$ as E-vectorspaces but with different G_k-actions. Then choosing $M(E, \eta\omega_p^m) = M(E,\eta)$ we get $M(E,\eta)\langle e\rangle(m) = M(E, \eta\omega_p^m)\langle e + m\rangle$ and similarly for W instead of M. We let $M(E, \eta^\vee) = \mathrm{Hom}_{\mathbb{Z}_p}(M(E,\eta), \mathbb{Z}_p)$ on which G_k acts via the inverse of its action on $M(E,\eta)$, so that on $M(E, \eta^\vee) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ this gives an Artin representation of G_k with character $\eta^\vee$. We then have $W(E, \eta^\vee) = M(E, \eta^\vee) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \simeq \mathrm{Hom}_{\mathbb{Z}_p}(M(E,\eta), \mathbb{Q}_p/\mathbb{Z}_p)$.

By [25, Proposition II.2.9] we have an isomorphism between $H^i(\mathcal{O}_{k,S}, X)$ and the continuous group cohomology group $H^i(G_S, X)$ for any finite $\mathbb{Z}_p$-module X with continuous G_S-action. This isomorphism is natural, so taking direct limits gives $H^i(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle) \simeq H^i(G_S, W(E,\eta)\langle 1-e\rangle)$. By [27, Theorem 8.3.20(i)] the $H^i(G_S, X)$ are finite for any such X and $i \geqslant 0$, so by [33, Corollary, p.261] we have $H^i(G_S, M(E,\eta)\langle 1-e\rangle) \simeq \varprojlim_n H^i(G_S, M(E,\eta)\langle 1-e\rangle/p^n)$ for $i \geqslant 1$, and for $i = 0$ this is obvious. Similarly $\varprojlim_n^1 H^1(\mathcal{O}_{k,S}, M(E,\eta)\langle 1-e\rangle/p^n) \simeq \varprojlim_n^1 H^1(G_S, M(E,\eta)\langle 1-e\rangle/p^n)$ is trivial, so if H_{cts}^i denotes Jannsen's continuous étale cohomology [21], then it follows from (3.1) in loc. cit., that for all i,

$$H_{cts}^i(\mathcal{O}_{k,S}, M(E,\eta)\langle 1-e\rangle) \simeq \varprojlim_n H^i(\mathcal{O}_{k,S}, M(E,\eta)\langle 1-e\rangle/p^n)$$

$$\simeq H^i(G_S, M(E,\eta)\langle 1-e\rangle).$$

We shall see in Proposition 2.12 that $H^{3-i}(\mathcal{O}_{k,S}, W(E, \eta^\vee \omega_p)\langle 1 - e\rangle)$ is cofinitely generated. Let $H^i_c(\mathcal{O}_{k,S}, \cdot)$ denote the i-th cohomology with compact support as in [25, Section II § 2, p.203], so that $H^i_c(\mathcal{O}_{k,S}, \cdot) \simeq H^i_{cts}(\mathcal{O}_{k,S}, \cdot)$ when $p \neq 2$. Let $\alpha : \operatorname{Spec} \mathcal{O}_{k,S} \to \operatorname{Spec} \mathcal{O}_k$ be the natural inclusion. Then by [25, II, Proposition 2.3(d) and Corollary 3.3] and [33, Corollary, p.261], we have

(2.7)
$$
\varprojlim_n H^i_c(\mathcal{O}_k, \alpha_!(M(E, \eta)\langle e\rangle/p^n))
$$
$$
\simeq \varprojlim_n H^i_c(\mathcal{O}_{k,S}, M(E, \eta)\langle e\rangle/p^n)
$$
$$
\simeq \varprojlim_n H^{3-i}(\mathcal{O}_{k,S}, \operatorname{Hom}(M(E, \eta)\langle e\rangle/p^n, \mu_{p^\infty}))^\vee
$$
$$
\simeq H^{3-i}(\mathcal{O}_{k,S}, \varinjlim_n \operatorname{Hom}(M(E, \eta)\langle e\rangle/p^n, \mu_{p^\infty}))^\vee
$$
$$
\simeq H^{3-i}(\mathcal{O}_{k,S}, \operatorname{Hom}(M(E, \eta)\langle e\rangle, \mu_{p^\infty}))^\vee
$$
$$
\simeq H^{3-i}(\mathcal{O}_{k,S}, W(E, \eta^\vee \omega_p)\langle 1 - e\rangle)^\vee ,
$$

for e in $\mathfrak{B}_k(E)$, $i = 0, 1, 2$ and 3.

If $H^j(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$ is finite for $j = i, i + 1$, then it will follow from the proof of Proposition 2.12 that $H^i(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle) \simeq H^{i+1}_{cts}(\mathcal{O}_{k,S}, M(E, \eta)\langle 1 - e\rangle)$. So, if $p \neq 2$ and all $H^i(\mathcal{O}_{k,S}, W(E, \eta)\langle e\rangle)$ are finite, then $H^i(\mathcal{O}_k, \alpha_! W(E, \eta)\langle e\rangle) \simeq H^{2-i}(\mathcal{O}_{k,S}, W(E, \eta^\vee \omega_p)\langle 1 - e\rangle)^\vee$ for $i = 0, 1, 2$ and both sides are trivial for $i \geqslant 3$ (cf. [10, Theorem 3.2]).

Remark 2.8. — If $p \neq 2$ then by [27, Proposition 8.3.18] we have $H^i(\mathcal{O}_{k,S}, X) \simeq H^i(G_S, X) = 0$ for $i \geqslant 3$ and any finite G_S-module X. By taking filtered direct limits it follows that $H^i(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$ is trivial for $i \geqslant 3$. But for $p = 2$ and $i \geqslant 3$ we have, by [27, 8.6.10(ii)] and a direct limit argument, that

$$
H^i(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle) \simeq H^i(G_S, W(E, \eta)\langle 1 - e\rangle)
$$
$$
\simeq \bigoplus_{v \in \Sigma_\infty} H^i(k_v, W(E, \eta)\langle 1 - e\rangle) .
$$

We now start the proof of Theorem 1.4. In the proof of Proposition 2.12 below we shall use results by Tate [32], [33] and Jannsen [21], although we shall use [27] as a reference instead of [32].

LEMMA 2.9. — *Let $\mathcal{O}$ be the valuation ring of a finite extension of $\mathbb{Q}_p$. If A is a (co)finitely generated $\mathcal{O}$-module on which a finite group G acts, then each $H^i(G, A)$ is (co)finitely generated and the number of (co)generators needed can be bounded in terms of $\#G$, i and the number of (co)generators*

of A. In particular, if $i > 0$ then $\#H^i(G, A)$ is bounded in terms of the same quantities.

Proof. — Since A is (co)finitely generated and G is finite it follows that each $\operatorname{Hom}_{\mathcal{O}[G]}(\mathcal{O}[G^{i+1}], A)$ is (co)finitely generated with the number of (co)generators needed bounded by a constant depending only on $\#G$, i and the number of (co)generators of A. The lemma now follows immediately from the definition of $H^i(G, A)$ because $\#G$ annihilates $H^i(G, A)$ if $i > 0$. $\qquad\square$

DEFINITION 2.10. — *For any $a \neq 0$ in $\mathcal{O}_E$ we let*

$$W(E, \eta)[a] = \ker(W(E, \eta) \xrightarrow{a} W(E, \eta)),$$

so that for e in $\mathfrak{B}_k(E)$ we have a short exact sequence

$$(2.11) \quad 0 \to W(E, \eta)[a]\langle 1 - e\rangle \to W(E, \eta)\langle 1 - e\rangle \xrightarrow{a} W(E, \eta)\langle 1 - e\rangle \to 0.$$

PROPOSITION 2.12. — *Let k be a number field, p a prime number, E a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_E$, $\eta : G_k \to E$ an Artin character, and e in $\mathfrak{B}_k(E)$. Assume that S contains P as well as all the finite primes of k at which η is ramified. Then the following hold.*

(1) *For $i \geqslant 0$, $H^i(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$ is a cofinitely generated $\mathcal{O}_E$-module.*

(2) *There is some constant $D = D(S, \eta, k)$ depending only on S, η and k, but not on e, E, or the choice of the lattice $M(E, \eta)$, such that each $H^i(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$ can be cogenerated by at most D elements.*

Proof. — Write M and W for $M(E, \eta)$ and $W(E, \eta)$ respectively. From the long exact sequence associated to $0 \to M\langle 1 - e\rangle \xrightarrow{p} M\langle 1 - e\rangle \to M\langle 1 - e\rangle/pM\langle 1 - e\rangle \to 0$ we see that, for $i \geqslant 0$, $H^i(G_S, M\langle 1 - e\rangle)/p$ is finite since $H^i(G_S, M\langle 1-e\rangle/pM\langle 1-e\rangle)$ is finite by [27, Theorem 8.3.20(i)]. Hence $H^i_{cts}(\mathcal{O}_{k,S}, M\langle 1 - e\rangle) \simeq H^i(G_S, M\langle 1 - e\rangle)$ is finitely generated for $i \geqslant 0$ by [33, Corollary, p.260]. From [21, Theorem 5.14(a)] we have an exact sequence

$$\cdots \to H^i_{cts}(\mathcal{O}_{k,S}, M\langle 1 - e\rangle) \to H^i_{cts}(\mathcal{O}_{k,S}, M\langle 1 - e\rangle) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$
$$\to H^i(\mathcal{O}_{k,S}, W\langle 1 - e\rangle) \to H^{i+1}_{cts}(\mathcal{O}_{k,S}, M\langle 1 - e\rangle) \to \cdots,$$

so it follows that $H^i(\mathcal{O}_{k,S}, W\langle 1-e\rangle)$ is cofinitely generated for $i \geqslant 0$. This proves part (1).

Consider the long exact sequence associated to (2.11) with $a = p$, so that $H^i(\mathcal{O}_{k,S}, W\langle 1-e\rangle)[p]$ is a quotient of $H^i(\mathcal{O}_{k,S}, W[p]\langle 1-e\rangle)$ for $i \geqslant 0$. Since

p kills $W[p]\langle 1-e\rangle$, there exists an m depending only on k such that, with $K = k_\eta(\mu_{p^m}) \subset \Omega_S$, $W[p]\langle 1-e\rangle \simeq (\mathcal{O}_E/(p))^{\eta(\mathrm{id}_{\overline{k}})}$ as $\mathrm{Gal}(\Omega_S/K)$-modules for all e in $\mathfrak{B}_k(E)$. Note that this isomorphism is independent of the choice of M. By [27, Theorem 8.3.20(i)], each $H^j(\mathrm{Gal}(\Omega_S/K), (\mathcal{O}_E/(p))^{\eta(\mathrm{id}_{\overline{k}})})$ is finite and so the number of its generators can be bounded by a constant that depends only on S, k and $\eta(\mathrm{id}_{\overline{k}})$ and not on e or E. By Lemma 2.9 the number of generators of $H^i(\mathrm{Gal}(K/k), H^j(\mathrm{Gal}(\Omega_S/K), W[p]\langle 1-e\rangle))$ can be bounded by a constant that depends only on S, η, i, j and k. By the Hochschild-Serre spectral sequence and Remark 2.8 it follows that the number of generators of $H^i(\mathcal{O}_{k,S}, W[p]\langle 1-e\rangle) \simeq H^i(G_S, W[p]\langle 1-e\rangle)$ can be bounded by a constant that depends only on S, η and k. This shows that there is a constant $D = D(S, \eta, k)$ depending only on S, η and k such that each $H^i(\mathcal{O}_{k,S}, W\langle 1-e\rangle)[p]$, and hence each $H^i(\mathcal{O}_{k,S}, W\langle 1-e\rangle)$ can be cogenerated by at most D elements. $\qquad\square$

This proves part (1) of Theorem 1.4. We now start the proof of the remaining parts, obtaining various results for later use along the way.

Remarks 2.13.

(1) If M and M' are $\mathcal{O}_E$-lattices for η, then we can find an injection from M to M' with finite cokernel. Tensoring with $\mathbb{Q}_p/\mathbb{Z}_p$, we get a short exact sequence

$$(2.14) \qquad 0 \to X_E \to M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to M' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

of sheaves for the étale topology on $\mathrm{Spec}(\mathcal{O}_{k,S})$ with X_E finite. Each $H^i(\mathcal{O}_{k,S}, X_E\langle 1-e\rangle)$ is finite by [27, Theorem 8.3.20(i)], so the groups $H^i(\mathcal{O}_{k,S}, M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p\langle 1-e\rangle)$ and $H^i(\mathcal{O}_{k,S}, M' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p\langle 1-e\rangle)$ have the same corank.

(2) Suppose that E'/E is a finite extension and let $\mathcal{O}_{E'}$ denote the valuation ring of E'. Then $M' = M \otimes_{\mathcal{O}_E} \mathcal{O}_{E'}$ is an $\mathcal{O}_{E'}$-module of finite $\mathcal{O}_{E'}$-rank on which G_k acts such that $M' \otimes_{\mathcal{O}_{E'}} E' \simeq V \otimes_E E'$. Because $M' \simeq M^{[E':E]}$ as $\mathcal{O}_E[G]$-modules, we have, with $W = M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ and $W' = M' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \simeq W^{[E':E]}$,

$$H^i(\mathcal{O}_{k,S}, W'\langle 1-e\rangle) \simeq H^i(\mathcal{O}_{k,S}, W\langle 1-e\rangle)^{[E':E]}$$

for all e in $\mathfrak{B}_k(E)$.

(3) Consider two Artin characters $\eta_1, \eta_2 : G_k \to E$ that are realizable over E, and let M_j for $j = 1, 2$ be corresponding torsion-free $\mathcal{O}_E$-lattices. Then $M_1 \oplus M_2$ corresponds to the character $\eta_1 + \eta_2$, and

with $W_j = M_j \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ for $j = 1, 2$, we have

$$H^i(\mathcal{O}_{k,S}, (W_1 \oplus W_2)\langle 1 - e\rangle)$$
$$\simeq H^i(\mathcal{O}_{k,S}, W_1\langle 1 - e\rangle) \times H^i(\mathcal{O}_{k,S}, W_2\langle 1 - e\rangle).$$

(4) Let k'/k be a finite extension, η' an Artin character of $G_{k'}$ realizable over E, and M' an $\mathcal{O}_E$-lattice for η'. Then $M = \mathcal{O}_E[G_k] \otimes_{\mathcal{O}_E[G_{k'}]} M'$ is an $\mathcal{O}_E$-lattice for the Artin character $\eta = \mathrm{Ind}_{k'}^k(\eta')$ of G_k. If S is such that η (and hence η') is unramified outside of S, then by [27, Proposition 1.6.4] we have that $H^i(\mathcal{O}_{k,S}, M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p\langle 1 - e\rangle) \simeq H^i(\mathcal{O}_{k',S}, M' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p\langle 1 - e\rangle)$.

(5) Suppose that S' is a finite set of primes of k containing S. Taking $M(E, \eta\omega_p^{-1}) = M(E, \eta)$ as $\mathcal{O}_E$-modules but with different G_k-action, localization with respect to $Z = \mathrm{Spec}(\mathcal{O}_{k,S}) \setminus \mathrm{Spec}(\mathcal{O}_{k,S'})$ gives an exact Gysin sequence (cf. the proof of VI Corollary 5.3, VI Remark 5.4(b) of [24], and [19, Exposé I, Theorem 5.1]), giving an isomorphism $H^0(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle) \simeq H^0(\mathcal{O}_{k,S'}, W(E, \eta)\langle 1 - e\rangle)$ and an exact sequence

$$0 \to H^1(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle) \to H^1(\mathcal{O}_{k,S'}, W(E, \eta)\langle 1 - e\rangle)$$
$$(2.15) \quad \to H^0(Z, W(E, \eta\omega_p^{-1})\langle -e\rangle) \to H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$$
$$\to H^2(\mathcal{O}_{k,S'}, W(E, \eta)\langle 1 - e\rangle) \to H^1(Z, W(E, \eta\omega_p^{-1})\langle -e\rangle) \to 0.$$

Note that $H^3(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle) \to H^3(\mathcal{O}_{k,S'}, W(E, \eta)\langle 1 - e\rangle)$ is an isomorphism by Remark 2.8.

We observe that

$$H^i(Z, W(E, \eta\omega_p^{-1})\langle -e\rangle) \simeq \bigoplus_{v \in S' \setminus S} H^i(\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v), W(E, \eta\omega_p^{-1})\langle -e\rangle),$$

where $\mathbb{F}_v$ is the residue field at v. Because $\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v)$ is topologically generated by the Frobenius Fr_v, we have

$$H^0(\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v), W(E, \eta\omega_p^{-1})\langle -e\rangle) \simeq \ker(1 - \mathrm{Fr}_v|W(E, \eta\omega_p^{-1})\langle -e\rangle)$$

and

$$H^1(\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v), W(E, \eta\omega_p^{-1})\langle -e\rangle) \simeq \mathrm{coker}(1 - \mathrm{Fr}_v|W(E, \eta\omega_p^{-1})\langle -e\rangle).$$

But $\ker(1 - \mathrm{Fr}_v|V(E, \eta\omega_p^{-1})\langle -e\rangle) \simeq \ker(1 - \langle\mathrm{Nm}(v)\rangle^{-e}\mathrm{Fr}_v|V(E, \eta\omega_p^{-1}))$ because $\eta\omega_p^{-1}$ is unramified at all v in Z and $\psi_p(\mathrm{Fr}_v) = \mathrm{Nm}(v)$ in $\mathbb{Z}_p^\times$, with ψ_p as in (1.2). This group is trivial if $e \neq 0$ because all the eigenvalues of Fr_v are roots of unity and $\langle\mathrm{Nm}(v)\rangle^e$ is a root of unity

only when $e = 0$. Applying the snake lemma to $1 - \mathrm{Fr}_v$ acting on the short exact sequence

$$0 \to M(E, \eta\omega_p^{-1})\langle -e\rangle \to V(E, \eta\omega_p^{-1})\langle -e\rangle \to W(E, \eta\omega_p^{-1})\langle -e\rangle \to 0 \,,$$

we find that $\mathrm{corank}_{\mathcal{O}_E} H^i(\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v), W(E, \eta\omega_p^{-1})\langle -e\rangle)$, for $i = 0, 1$, is trivial when $e \neq 0$, and equals $\dim_E V(E, \eta\omega_p^{-1})^{\mathrm{Fr}_v=1}$ otherwise. If this corank is trivial then $1 - \mathrm{Fr}_v$ is an isomorphism on $V(E, \eta\omega_p^{-1})\langle -e\rangle$, so $H^1(\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v), W(E, \eta\omega_p^{-1})\langle -e\rangle)$ is trivial and

$$
(2.16) \quad
\begin{aligned}
&\#H^0(\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v), W(E, \eta\omega_p^{-1})\langle -e\rangle) \\
&\qquad = |\det(1 - \langle \mathrm{Nm}(v)\rangle^{-e}\mathrm{Fr}_v | V(E, \eta\omega_p^{-1}))|_p^{-[E:\mathbb{Q}_p]} \,.
\end{aligned}
$$

In Section 5 we shall use the following proposition for Conjecture 5.19. Here we let $r_{i,S}(1 - e, \eta) = \mathrm{corank}_{\mathcal{O}_E} H^i(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)$, just as in Theorem 1.8.

PROPOSITION 2.17. — *Let k be a number field, η an Artin character, S and S' finite set of primes of k such that $S \subseteq S'$, and e in $\mathfrak{B}_k(E)$. Then*

(1) *the map $H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle) \to H^2(\mathcal{O}_{k,S'}, W(E, \eta)\langle 1 - e\rangle)$ in (2.15) has finite kernel;*

(2) *$r_{1,S'}(1 - e, \eta) - r_{1,S}(1 - e, \eta) = r_{2,S'}(1 - e, \eta) - r_{2,S}(1 - e, \eta)$, which is equal to $\sum_{v \in S' \setminus S} \dim_E V(E, \eta\omega_p^{-1})^{\mathrm{Fr}_v=1}$ if $e = 0$ and to 0 if $e \neq 0$.*

Proof. — From Remark 2.13(5) we see this holds for $e \neq 0$, and that for $e = 0$ it is sufficient to prove part (1). We may assume $W(E, \eta)\langle 1\rangle = W(E, \eta\omega_p^{-1})(1)$, therefore we only need to show that the kernel of the map $H^2(\mathcal{O}_{k,S}, W(E, \eta\omega_p^{-1})(1)) \to H^2(G_k, W(E, \eta\omega_p^{-1})(1))$ is finite. We let K be a Galois extension of k such that the restriction of $\eta\omega_p^{-1}$ to G_K is a multiple of the trivial character, and let $G = \mathrm{Gal}(K/k)$. Because the $H^i(\mathcal{O}_{K,S}, W(E, \eta\omega_p^{-1})(1))$ are cofinitely generated by Theorem 1.4(1), we see from Lemma 2.9 and the spectral sequence

$$H^p(G, H^q(\mathcal{O}_{K,S}, W(E, \eta\omega_p^{-1})(1))) \Longrightarrow H^{p+q}(\mathcal{O}_{k,S}, W(E, \eta\omega_p^{-1})(1))$$

that it suffices to show that the map

$$H^2(\mathcal{O}_{K,S}, W(E, \eta\omega_p^{-1})(1))^G \to H^2(G_K, W(E, \eta\omega_p^{-1})(1))^G$$

has finite kernel. By our choice of K and Remark 2.13(1) this follows if $H^2(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \to H^2(G_K, \mathbb{Q}_p/\mathbb{Z}_p(1))$ has finite kernel.

Consider the boundary map $H^1(G_K, \mathbb{Q}_p/\mathbb{Z}_p(1)) \to H^0(Z, \mathbb{Q}_p/\mathbb{Z}_p)$ where $Z = \mathrm{Spec}(\mathcal{O}_K) \setminus S$ (cf. (2.15)). This is obtained by taking the direct limit of the maps $H^1(G_K, \mathbb{Z}/p^n\mathbb{Z}(1)) \to H^0(Z, \mathbb{Z}/p^n\mathbb{Z})$. The size of the cokernel of this last map is bounded by the class number of K. Taking the direct

limit over n, it follows that $H^1(G_K, \mathbb{Q}_p/\mathbb{Z}_p(1)) \to H^0(Z, \mathbb{Q}_p/\mathbb{Z}_p)$ has finite cokernel, so $H^2(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \to H^2(G_K, \mathbb{Q}_p/\mathbb{Z}_p(1))$ has finite kernel. $\square$

We also prove the following result, which provides a lower bound for the corank of $H^1(\mathcal{O}_{k,S}, W(E, \eta)\langle 1\rangle)$.

PROPOSITION 2.18. — *Let k be a number field, p a prime number, E a finite extension of $\mathbb{Q}_p$, $\eta : G_k \to E$ an Artin character realizable over E, $M(E, \eta)$ an $\mathcal{O}_E$-lattice for η, and $W(E, \eta) = M(E, \eta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. Write $\alpha : \operatorname{Spec} k \to \operatorname{Spec} \mathcal{O}_{k,P}$ for the natural map. Then $H^i(\mathcal{O}_P, \alpha_* W(E, \eta)\langle 1\rangle)$ is cofinitely generated for $i \geqslant 0$, and its corank is independent of the choice of $M(E, \eta)$. Moreover,*

$$\operatorname{corank}_{\mathcal{O}_E} H^1(\mathcal{O}_{k,S}, W(E, \eta)\langle 1\rangle)$$
$$(2.19) \quad = \operatorname{corank}_{\mathcal{O}_E} H^1(\mathcal{O}_P, \alpha_* W(E, \eta)\langle 1\rangle) + \sum_{v \in S \setminus P} \dim_E V(E, \eta\omega_p^{-1})^{G_{w_v}} ,$$

where w_v is a prime of $\overline{k}$ above v with decomposition group G_{w_v}.

Proof. — Write W for $W(E, \eta)\langle 1\rangle$ and Z for $\operatorname{Spec} \mathcal{O}_{k,P} \setminus \operatorname{Spec} \mathcal{O}_{k,S}$. Let $\beta : \operatorname{Spec} \mathcal{O}_{k,S} \to \operatorname{Spec} \mathcal{O}_{k,P}$ be the natural map. For $i \geqslant 0$, by [25, II. Lemma 2.4] we have an exact sequence

$$(2.20) \quad \begin{aligned} \cdots &\to \oplus_{v \in Z} H^{i-1}(\operatorname{Gal}(\overline{k}_v/k_v), W) \to H^i(\mathcal{O}_{k,P}, \beta_! W) \\ &\to H^i(\mathcal{O}_{k,S}, W) \to \oplus_{v \in Z} H^i(\operatorname{Gal}(\overline{k}_v/k_v), W) \to \cdots . \end{aligned}$$

For any finite prime v of k and a finite Galois extension K of k_v such that $\operatorname{Gal}(\overline{k}_v/K)$ acts trivially on $W(E, \eta\omega_p^{-1})$, [27, Theorem 7.1.8(i) and Proposition 7.3.10] imply that $H^i(\operatorname{Gal}(\overline{k}_v/K), W)$ is cofinitely generated, and using Lemma 2.9 one sees that the same holds for $\oplus_{v \in Z} H^i(\operatorname{Gal}(\overline{k}_v/k_v), W)$. From Proposition 2.12(1) and (2.20) it then follows that $H^i(\mathcal{O}_{k,P}, \beta_! W)$ is cofinitely generated. Since $H^i(\operatorname{Gal}(\overline{k}_v/k_v), X)$ is finite for finite X by [27, Theorem 7.1.8(iii)], it also follows from the long exact sequence associated to (2.14) that the corank of $H^i(\operatorname{Gal}(\overline{k}_v/k_v), W)$ does not depend on the choice of $M(E, \eta)$. Hence the same holds for the corank of $H^i(\mathcal{O}_{k,P}, \beta_! W)$ by Remark 2.13(1) and (2.20).

On the other hand, with $\gamma : Z \to \operatorname{Spec} \mathcal{O}_{k,P}$ the natural map, by [24, p.76] we have a short exact sequence

$$(2.21) \qquad 0 \to \beta_! \beta^* \alpha_* W \to \alpha_* W \to \gamma_* \gamma^* \alpha_* W \to 0$$

of sheaves for the étale topology on $\mathcal{O}_{k,P}$, which gives a long exact sequence of cohomology groups. We note that

$$(2.22) \qquad H^i(\mathcal{O}_{k,P}, \gamma_* \gamma^* \alpha_* W) \simeq \oplus_{v \in Z} H^i(\operatorname{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v), W^{I_{w_v}}) ,$$

where I_{w_v} denotes the decomposition group of a prime w_v in $\overline{k}$ lying above v. These groups are cofinitely generated and using (2.14) their coranks are independent of the choice of $M(E, \eta)$ because $H^i(\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v), X)$ is finite if X is finite. Since $H^i(\mathcal{O}_{k,P}, \beta_!\beta^*\alpha_*W) = H^i(\mathcal{O}_{k,P}, \beta_!W)$ it follows from the long exact sequence associated to (2.21) that $H^i(\mathcal{O}_{k,P}, \alpha_*W)$ is cofinitely generated and that its corank is independent of the choice of $M(E, \eta)$. Moreover, $H^i(\mathcal{O}_{k,P}, \alpha_*W)$ behaves well under extending E, as well as under induction and addition of characters. Combining Brauer induction with parts (3) and (4) of Remarks 2.13, one sees that it is enough to prove (2.19) for 1-dimensional characters η. With this S, if η is unramified at a prime v of k, then $V(E, \eta\omega_p^{-1})^{G_{w_v}} = V(E, \eta\omega_p^{-1})^{\mathrm{Fr}_v=1}$, so by Proposition 2.17, it is enough to prove the result for $S = S_{min}$, the smallest set of primes containing the finite primes of k lying above p and the primes at which η is ramified. In this case, if v is in Z then $V(E, \eta\omega_p^{-1})^{G_{w_v}}$ is trivial since $\eta\omega_p^{-1}$ is ramified at v. Therefore, we need to show that $\mathrm{corank}_{\mathcal{O}_E} H^1(\mathcal{O}_{k,S_{min}}, W) = \mathrm{corank}_{\mathcal{O}_E} H^1(\mathcal{O}_{k,P}, \alpha_*W)$ for 1-dimensional η.

The first term of (2.20), with $i = 1$ and $S = S_{min}$, is finite since $\mathrm{Gal}(\overline{k}_v/k_v)$ acts non-trivially on W. On the other hand, for any finite Abelian extension K of k_v, the coranks of $H^1(\mathrm{Gal}(\overline{K}/K), E/\mathcal{O}_E(1))$ and $\oplus_\psi H^1(\mathrm{Gal}(\overline{k}_v/k_v), W(E, \psi)(1))$ are the same, where the sum runs over all characters $\psi : \mathrm{Gal}(\overline{k}_v/k_v) \to E$ that are irreducible over E and factor through $\mathrm{Gal}(K/k_v)$. Since

$$H^1(\mathrm{Gal}(\overline{K}/K), E/\mathcal{O}_E(1)) \simeq E/\mathcal{O}_E \simeq H^1(\mathrm{Gal}(\overline{k}_v/k), E/\mathcal{O}_E(1))$$

by [27, Theorem 7.3.10(ii)], $W \simeq W(E, \eta\omega_p^{-1})(1)$ by Remark 2.2(1) and $\eta\omega_p^{-1}$ is non-trivial on $\mathrm{Gal}(\overline{k}_v/k_v)$ for v in Z, we see that $H^1(\mathrm{Gal}(\overline{k}_v/k_v), W)$ is finite for all such v. By (2.20), with $i = 1$ and $S = S_{min}$, it follows that the coranks of $H^1(\mathcal{O}_{k,P}, \beta_!W)$ and $H^1(\mathcal{O}_{k,S_{min}}, W)$ are equal. Moreover, in this case, the groups $H^i(\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v), W^{I_{w_v}})$ in (2.22) are also finite since $W^{I_{w_v}}$ is finite. Therefore, from the long exact sequence associated to (2.21) it follows that the coranks of $H^1(\mathcal{O}_{k,P}, \alpha_*W)$ and $H^1(\mathcal{O}_{k,P}, \beta_!W)$ are the same. This completes the proof. □

We need the following lemmas in order to prove Theorem 1.4(2), and also to show that Definition 5.1 is independent of the choice of lattice. Here, for a G_k-module A and a place v in Σ_∞, we let $A_{+,v} = H^0(k_v, A)$.

LEMMA 2.23. — Let A be a finite Abelian group with continuous G_S-action and $\#A$ in $\mathcal{O}_{k,S}^\times$. Then

$$(2.24) \qquad \frac{\#H^0(G_S, A) \cdot \#H^2(G_S, A)}{\#H^1(G_S, A)} = (\#A)^{-[k:\mathbb{Q}]} \cdot \prod_{v \in \Sigma_\infty} (\#A_{+,v}).$$

For non-trivial A, this last quantity equals 1 if and only if k is totally real and every complex conjugation in G_k acts trivially on A.

Proof. — By [27, Theorem 8.3.20(i)] the groups $H^i(G_S, A)$ for $i \geqslant 0$ are finite, and by the Global Euler-Poincaré Characteristic formula [27, 8.7.4] we have

$$\frac{\#H^0(G_S, A) \cdot \#H^2(G_S, A)}{\#H^1(G_S, A)} = \prod_{v \in \Sigma_\infty} \frac{\#H^0(k_v, A)}{||\#A||_v},$$

where $||n||_v = n$ if v is real and $||n||_v = n^2$ if v is complex. This proves the first claim, and the second claim follows immediately.   □

LEMMA 2.25. — *Let $\mathcal{O}$ be the valuation ring of a finite extension of $\mathbb{Q}_p$. Suppose that $0 \to A_1 \to A_2 \to A_3 \to 0$ is a short exact sequence of cofinitely generated $\mathcal{O}$-modules with continuous G_S-action, with A_1 finite. Then*

  (1) *the size of the image of the connecting map $\delta : H^2(G_S, A_3) \to H^3(G_S, A_1)$ is bounded by a constant depending only on the number of generators of A_1 and the number of real places of k;*
  (2) *δ is trivial if and only if $(A_2)_{+,v} \to (A_3)_{+,v}$ is surjective for all v in Σ_∞. In this case we have a nine-term exact sequence*

$$0 \to H^0(G_S, A_1) \to H^0(G_S, A_2) \to H^0(G_S, A_3) \to \cdots$$
$$\cdots \to H^2(G_S, A_1) \to H^2(G_S, A_2) \to H^2(G_S, A_3) \to 0.$$

Proof. — By [27, 8.6.10(ii)] we have $H^3(G_S, A_1) \simeq \prod_{v \in \Sigma_\infty} H^3(k_v, A_1) \simeq \prod_{v \in \Sigma_\infty} H^1(k_v, A_1)$, the size of which is, by Lemma 2.9, bounded by a constant depending only on the number of generators of A_1 and the number of real places of k. This proves part (1). For part (2), we see this way that δ is trivial if and only if $H^1(k_v, A_1) \to H^1(k_v, A_2)$ is injective for all v in Σ_∞, and this is true if and only if $H^0(k_v, A_2) \to H^0(k_v, A_3)$ is surjective for all v in Σ_∞.   □

We shall now finish the proof of Theorem 1.4 by proving Propositions 2.26, 2.27 and 2.28.

PROPOSITION 2.26. — *Fix e in $\mathfrak{B}_k(E)$. Then $H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e \rangle)$ is p-divisible for $p \neq 2$. If $p = 2$ then*

$$H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e \rangle)$$
$$\simeq 2H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e \rangle) \oplus \bigoplus_{v \in \Sigma_\infty} W(E, \eta)_{+,v}/2,$$

and $2H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e \rangle)$ is 2-divisible.

Proof. — If $p \neq 2$ then this follows from Lemma 2.25(2) applied to (2.11) with $a = p$, so let us take $p = 2$. By the structure of cofinitely generated $\mathcal{O}_E$-modules we have $H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1-e\rangle) \simeq A \oplus B$ with A 2-divisible and B finite. Take $n \geqslant 1$ such that $H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1-e\rangle)/2^n \simeq B$. Note that $H^3(\mathcal{O}_{k,S}, W(E, \eta)[2^n]\langle 1-e\rangle) \simeq \oplus_{v \in \Sigma_\infty} H^3(k_v, W(E, \eta)[2^n]\langle 1-e\rangle)$ is killed by multiplication by 2, so from the long exact sequence associated to (2.11) with $a = 2^n$ we see $2B = 0$, hence $A = 2H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1-e\rangle)$. On the other hand,

$$
\begin{aligned}
B &\simeq H^2(\mathcal{O}_{k,S}, W(E, \eta)\langle 1-e\rangle)/2 \\
&\simeq \ker\big(H^3(\mathcal{O}_{k,S}, W(E, \eta)[2]\langle 1-e\rangle) \to H^3(\mathcal{O}_{k,S}, W(E, \eta)\langle 1-e\rangle)\big) \\
&\simeq \oplus_{v \in \Sigma_\infty} \ker\big(H^3(k_v, W(E, \eta)[2]\langle 1-e\rangle) \to H^3(k_v, W(E, \eta)\langle 1-e\rangle)\big) \\
&\simeq \oplus_{v \in \Sigma_\infty} \ker\big(H^1(k_v, W(E, \eta)[2]\langle 1-e\rangle) \to H^1(k_v, W(E, \eta)\langle 1-e\rangle)\big) \\
&\simeq \oplus_{v \in \Sigma_\infty} W(E, \eta)\langle 1-e\rangle_{+,v}/2 \\
&\simeq \oplus_{v \in \Sigma_\infty} W(E, \eta)_{+,v}/2 \, ,
\end{aligned}
$$

where we used that $W(E, \eta)\langle 1-e\rangle_{+,v} = W(E, \eta)_{+,v}$ since $\psi_p^{\langle\rangle}$ is trivial on every complex conjugation. $\qquad\square$

PROPOSITION 2.27. — *For e in $\mathfrak{B}_k(E)$, with*

$$r_{i,S}(1-e, \eta) = \mathrm{corank}_{\mathcal{O}_E} H^i(G_S, W(E, \eta)\langle 1-e\rangle)$$

one has

$$
\sum_{i=0}^2 (-1)^i r_{i,S}(1-e, \eta) = -[k : \mathbb{Q}] \cdot \mathrm{corank}_{\mathcal{O}_E} W(E, \eta)
$$

$$
+ \sum_{v \in \Sigma_\infty} \mathrm{corank}_{\mathcal{O}_E} W(E, \eta)_{+,v} \, ,
$$

which is independent of e and non-positive. It is zero if and only of k is totally real and η is even. Moreover, $r_{i,S}(1-e, \eta)$ is independent of S for $i = 0$, and also for $i = 1, 2$ if $e \neq 0$.

Proof. — Note that an $\mathcal{O}_E$-module A is cofinitely generated as an $\mathcal{O}_E$-module if and only if the same holds as a $\mathbb{Z}_p$-module, and in this case $\mathrm{corank}_{\mathbb{Z}_p} A = [E : \mathbb{Q}_p] \cdot \mathrm{corank}_{\mathcal{O}_E} A$.

Write W for $W(E, \eta)\langle 1-e\rangle$, so that $W_{+,v} = W(E, \eta)_{+,v}$ as in the proof of Proposition 2.26. For $i = 0, 1, 2$, let r_i be an integer and X_i a finite $\mathbb{Z}_p$-module such that $H^i(G_S, W) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{r_i} \times X_i$ and let $d = [k : \mathbb{Q}] \cdot \mathrm{corank}_{\mathbb{Z}_p} W - \sum_{v \in \Sigma_\infty} \mathrm{corank}_{\mathbb{Z}_p} W_{+,v}$. We need to show $r_0 - r_1 + r_2 = -d$.

Let $n > 0$ be such that p^n annihilates X_i for $i = 0, 1, 2$, as well as the torsion in $(W_{+,v})^\vee$ for every v in Σ_∞. Since $(W[p^n])_{+,v} = W_{+,v}[p^n]$,

the right-hand side of (2.24) with $A = W[p^n]$ equals p^{-nd+c} for some c independent of n.

On the other hand, consider the long exact sequence associated to (2.11) with $a = p^n$. We get an isomorphism $H^0(G_S, W[p^n]) \simeq H^0(G_S, W)[p^n]$ and the two short exact sequences

$$0 \to H^0(G_S, W)/p^n \to H^1(G_S, W[p^n]) \to H^1(G_S, W)[p^n] \to 0$$

and

$$0 \to H^1(G_S, W)/p^n \to H^2(G_S, W[p^n]) \to H^2(G_S, W)[p^n] \to 0.$$

Therefore the left-hand side of (2.24) for $A = W[p^n]$ equals

$$\frac{\#H^0(G_S, W[p^n]) \cdot \#H^2(G_S, W[p^n])}{\#H^1(G_S, W[p^n])}$$
$$= \frac{\#H^0(G_S, W)[p^n] \cdot \#H^2(G_S, W)[p^n] \cdot \#X_1}{\#H^1(G_S, W)[p^n] \cdot \#X_0}$$
$$= p^{(r_0 - r_1 + r_2)n + c'}$$

for some c' independent of n. Hence $r_0 - r_1 + r_2 = -d$ by Lemma 2.23, proving the displayed equality. Independence of e is then obvious. Comparing the coranks of $W(E, \eta)$ and $W(E, \eta)_{+,v}$ as $\mathcal{O}_E$-modules it is easy to see when the right-hand side is non-positive and when it is zero. The last statement follows from Remark 2.13(5) and Proposition 2.17(1). □

We next prove a result on $H^0(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e \rangle)$. Recall from just before Theorem 1.4 that $\mathfrak{B}_\eta(E)$ equals $\mathfrak{B}_k(E)$ if η does not contain the trivial character, and equals $\mathfrak{B}_k(E) \setminus \{1\}$ if it does.

PROPOSITION 2.28. — *Let k be a number field, E a finite extension of $\mathbb{Q}_p$, and $\eta : G_k \to E$ an Artin character realizable over E. Assume that S contains P as well as all the primes at which η is ramified. Then the following hold.*

(1) *For e in $\mathfrak{B}_\eta(E)$, $H^0(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e \rangle)$ is finite and its size is a locally constant function of e.*

(2) *The corank of $H^0(\mathcal{O}_{k,S}, W(E, \eta))$ equals the multiplicity of the trivial character in η.*

Proof. — For part (1), let $\rho : G_k \to GL(V)$ realize η on a finite dimensional E-vector space V and let K be the fixed field of the kernel of ρ. Since $M = M(E, \eta)$ is torsion-free, we have a short exact sequence $0 \to M \to V \to W \to 0$ with $W = W(E, \eta)$. Let $g_1, \ldots, g_n$ in G_k be

lifts of the elements in $\operatorname{Gal}(K/k)$, and let g_0 in G_k be a lift of a topological generator of $\operatorname{Gal}(K_\infty/K) \subseteq \operatorname{Gal}(K_\infty/k)$. Then we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \Phi_{M,e}} & & \downarrow{\scriptstyle \Phi_{V,e}} & & \downarrow{\scriptstyle \Phi_{W,e}} & & \\
0 & \longrightarrow & M^{n+1} & \longrightarrow & V^{n+1} & \longrightarrow & W^{n+1} & \longrightarrow & 0
\end{array}
$$

of $\mathcal{O}_E[G_k]$-modules, where $\Phi_{*,e}$ is given by

$$
(\operatorname{id} - \psi_p^{\langle\rangle}(g_0)^{1-e}\operatorname{id}) \times (\operatorname{id} - \psi_p^{\langle\rangle}(g_1)^{1-e}\rho(g_1)) \times \cdots \times (\operatorname{id} - \psi_p^{\langle\rangle}(g_n)^{1-e}\rho(g_n))
$$

with $\psi_p^{\langle\rangle}$ as in (1.2), and $H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle) = \ker(\Phi_{W,e})$ because the action of G_k factorizes through its quotient $\operatorname{Gal}(K_\infty/k)$. Note that $\Phi_{V,e}$ is injective: if $e \neq 1$ then $\operatorname{id} - \psi_p^{\langle\rangle}(g_0)^{1-e}\operatorname{id}$ is an automorphism of $V(E,\eta)$, and if $e = 1$ and η does not contain the trivial character then it follows by considering the other components of $\Phi_{V,1}$. Hence $H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle) = \ker(\Phi_{W,e})$ is isomorphic as $\mathcal{O}_E$-module with the torsion in $\operatorname{coker}(\Phi_{M,e})$, hence is finite. If we fix an isomorphism $M(\eta,E) \simeq \mathcal{O}_E^d$ of $\mathcal{O}_E$-modules, where $d = \dim_E V(E,\eta)$, then $\Phi_{M,e}$ is identified with an $(nd+d) \times d$-matrix A_e with entries in $\mathcal{O}_E$. The ideal I_e of $\mathcal{O}_E$ generated by the determinants of the $d \times d$-minors of A_e is not the zero ideal because $\Phi_{V,e}$ is injective, hence $\#H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle) = \#\operatorname{coker}(\Phi_{M,e})_{\operatorname{tor}} = \#\mathcal{O}_E/I_e$. Clearly, $I_{e'} = I_e$ for all e' in $\mathfrak{B}_\eta(E)$ close enough to e. This proves part (1).

For part (2), write $\eta = s\eta_0 + \eta'$, where η' does not contain the trivial character η_0, and is realizable over E. Then by Remark 2.13(1) we may choose $M(E,\eta) = M(E,\eta') \oplus \mathcal{O}_E^s$. Since $H^0(\mathcal{O}_{k,S}, W(E,\eta'))$ is finite by part (1) we find the corank of $H^0(\mathcal{O}_{k,S}, W(E,\eta))$ equals that of $H^0(\mathcal{O}_{k,S}, (E/\mathcal{O}_E)^s) \simeq (E/\mathcal{O}_E)^s$, which is s. $\qquad\square$

We have now proved all the statements of Theorem 1.4: part (1) is Proposition 2.12, part (2) follows from Remark 2.13(1) and Proposition 2.27, part (3) is Proposition 2.28 and part (4) is Proposition 2.26.

To conclude this section, we describe $H^i(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$ for $i = 0, 1$ when it is finite by means of cohomology groups with finite coefficients, and discuss how its structure varies with e. By Theorem 1.4(2), for $i = 1$ this can only apply when k is totally real and η is even, in which case this group is also finite for $i = 0$, and, by Proposition 2.26, trivial for $i = 2$.

PROPOSITION 2.29. — *Let k be a number field, p a prime number, E a finite extension of $\mathbb{Q}_p$, and $\eta : G_k \to E$ an Artin character realizable over E.*

(1) Assume that $H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$ is annihilated by $a_0 \neq 0$ in $\mathcal{O}_E$. Then

    (a) $H^0(\mathcal{O}_{k,S}, W(E,\eta)[a_0]\langle 1-e\rangle) \simeq H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$ as $\mathcal{O}_E$-modules under the natural map;

    (b) if, for e' in $\mathfrak{B}_\eta(E)$, a_0 also annihilates $H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e'\rangle)$, and $q_k^e - q_k^{e'}$ is in $a_0\mathcal{O}_E$, then

$$H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e'\rangle) \simeq H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$$

    as $\mathcal{O}_E$-modules.

(2) Assume that $H^1(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$ is annihilated by $a_1 \neq 0$ in $\mathcal{O}_E$. Then

    (a) $\#H^1(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle) = \dfrac{\#H^1(\mathcal{O}_{k,S}, W(E,\eta)[a_1]\langle 1-e\rangle)}{\#H^0(\mathcal{O}_{k,S}, W(E,\eta)[a_1]\langle 1-e\rangle)}$;

    (b) if $H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$ is annihilated by $a_0 \neq 0$ in $\mathcal{O}_E$ then $H^1(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle) \simeq \operatorname{coker}(f_{a_0,a_1})$ as $\mathcal{O}_E$-modules, with

$$H^0(\mathcal{O}_{k,S}, W(E,\eta)[a_0]\langle 1-e\rangle) \overset{f_{a_0,a_1}}{\to} H^1(\mathcal{O}_{k,S}, W(E,\eta)[a_1]\langle 1-e\rangle)$$

    the boundary map in the long exact sequence corresponding to the short exact sequence

(2.30)
$$0 \to W(E,\eta)[a_1]\langle 1-e\rangle \to W(E,\eta)[a_0a_1]\langle 1-e\rangle$$
$$\overset{a_1}{\to} W(E,\eta)[a_0]\langle 1-e\rangle \to 0\,,$$

    and f_{a_0,a_1} is injective if and only if $H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$ is annihilated by a_1 as well.

    (c) If, for e' in $\mathfrak{B}_\eta(E)$, a_1 also annihilates $H^1(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e'\rangle)$, and $q_k^e - q_k^{e'}$ is in $a_0a_1\mathcal{O}_E$, then

(2.31)
$$H^1(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e'\rangle) \simeq H^1(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$$

    as $\mathcal{O}_E$-modules.

Proof. — Parts (1)(a) and (2)(a) are immediate from the long exact sequence associated to (2.11) with a replaced with a_i.

For (2)(b), we compare the long exact sequences associated to (2.30) and to (2.11) with a replaced with a_1 via the natural map between them. This gives the commutative diagram (with $H^i(\mathcal{O}_{k,S}, W(E,\eta)\langle 1-e\rangle)$ abbreviated to $H^i(W)$ and similarly for the other coefficients)

$$
\begin{array}{ccccccccc}
\cdots \to & H^0(W[a_0a_1]) & \overset{a_1}{\to} & H^0(W[a_0]) & \overset{f_{a_0,a_1}}{\to} & H^1(W[a_1]) & \to & H^1(W[a_0a_1]) & \overset{a_1}{\to} \cdots \\
 & \downarrow{\simeq} & & \downarrow{\simeq} & & \| & & \downarrow & \\
\cdots \longrightarrow & H^0(W) & \overset{a_1}{\longrightarrow} & H^0(W) & \longrightarrow & H^1(W[a_1]) & \longrightarrow & H^1(W) & \overset{a_1}{\to} \cdots
\end{array}
$$

where we used (1)(a) for some vertical maps, and the result is immediate as multiplication on $H^1(W)$ is the zero map by assumption.

Parts (1)(b) and (2)(c) follow from (1)(a) and (2)(b) by using that $W(E,\eta)[b]\langle 1 - e'\rangle \simeq W(E,\eta)[b]\langle 1 - e\rangle$ as $\mathcal{O}_E[G_k]$-modules if $b \neq 0$ lies in $\mathcal{O}_E$ and $q_k^e - q_k^{e'}$ lies in $b\mathcal{O}_E$. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.32. — Let notation and assumptions be as in part (2)(b) of Proposition 2.29. Comparing the long exact sequences associated to (2.11) for $a = a_1$ and $a = a_0 a_1$, one sees that $H^1(\mathcal{O}_{k,S}, W(E,\eta)[a_0 a_1]\langle 1 - e\rangle)$ is the direct sum of the image of the natural map

$$H^1(\mathcal{O}_{k,S}, W(E,\eta)[a_1]\langle 1 - e\rangle) \rightarrow H^1(\mathcal{O}_{k,S}, W(E,\eta)[a_0 a_1]\langle 1 - e\rangle)$$

and that of the (injective) connecting map $H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1 - e\rangle) \rightarrow H^1(\mathcal{O}_{k,S}, W(E,\eta)[a_0 a_1]\langle 1 - e\rangle)$. Therefore $H^1(\mathcal{O}_{k,S}, W(E,\eta)\langle 1 - e\rangle)$ can be obtained as a direct summand, i.e.,

$$H^1(\mathcal{O}_{k,S}, W(E,\eta)[a_0 a_1]\langle 1 - e\rangle)$$
$$\simeq H^1(\mathcal{O}_{k,S}, W(E,\eta)\langle 1 - e\rangle) \oplus H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1 - e\rangle).$$

For $b \neq 0$ in $\mathcal{O}_E$, the natural map

$$H^1(\mathcal{O}_{k,S}, W(E,\eta)[a_0 a_1]\langle 1 - e\rangle) \rightarrow H^1(\mathcal{O}_{k,S}, W(E,\eta)[b a_0 a_1]\langle 1 - e\rangle)$$

corresponds to multiplication by b on $H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1 - e\rangle)$ and the identity map on $H^1(\mathcal{O}_{k,S}, W(E,\eta)\langle 1 - e\rangle)$.

Remarks 2.33. — Let the notation and assumptions be as in Proposition 2.29.

(1) Let $a \neq 0$ be in $\mathcal{O}_E$ but not in $\mathcal{O}_E^\times$, so that $W(E,\eta)[a] \neq 0$. Using Theorem 1.4(3) and the long exact sequence associated to (2.11), one sees that

$$H^1(\mathcal{O}_{k,S}, W(E,\eta)[a]\langle 1 - e\rangle) \rightarrow H^1(\mathcal{O}_{k,S}, W(E,\eta)\langle 1 - e\rangle)$$

is injective if and only if $H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1 - e\rangle) = 0$.

(2) If $a \neq 0$ in $\mathcal{O}_E$ annihilates $H^i(\mathcal{O}_{k,S}, W(E,\eta)\langle 1 - e\rangle)$ for $i = 0$ and 1, then from parts (1)(a) and (2)(a) of Proposition 2.29 we obtain

$$\frac{\#H^0(\mathcal{O}_{k,S}, W(E,\eta)\langle 1 - e\rangle)}{\#H^1(\mathcal{O}_{k,S}, W(E,\eta)\langle 1 - e\rangle)} = \frac{\#H^0(\mathcal{O}_{k,S}, W(E,\eta)[a]\langle 1 - e\rangle)^2}{\#H^1(\mathcal{O}_{k,S}, W(E,\eta)[a]\langle 1 - e\rangle)}.$$

(3) The isomorphism in (2.31) is natural, but some isomorphism must already exist if $q_k^e - q_k^{e'}$ is only in the intersection $a_0\mathcal{O}_E \cap a_1\mathcal{O}_E$

(as opposed to the product $a_0a_1\mathcal{O}_E$). Namely, from the long exact sequence associated to (2.11) we find, for $a \neq 0$ in $\mathcal{O}_E$,

$$\#H^1(\mathcal{O}_{k,S}, W(E, \eta)\langle 1 - e\rangle)[a] = \frac{\#H^1(\mathcal{O}_{k,S}, W(E, \eta)[a]\langle 1 - e\rangle)}{\#H^0(\mathcal{O}_{k,S}, W(E, \eta)[a]\langle 1 - e\rangle)}$$

and similarly with e replaced with e'. Applying this with a the powers of a uniformizer of $\mathcal{O}_E$, the result follows from the classification of finite $\mathcal{O}_E$-modules as direct sums of cyclic modules.

3. p-adic L-functions

Notation 3.1. — Throughout this section, k is a number field, p a prime number, E a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_E$, and $\eta : G_k \to E$ the character of an Artin representation. We let P denote the set of primes of k lying above p.

First assume that η is 1-dimensional. Fixing an embedding $\sigma : E \to \mathbb{C}$, it follows from [26, VII Corollary 9.9] that $L(m, \sigma \circ \eta, k)$ for an integer $m \leqslant 0$ is in E and that the value

$$(3.2) \qquad\qquad L^*(m, \eta, k) := \sigma^{-1}(L(m, \sigma \circ \eta, k))$$

is independent of the choice of σ.

Now we want to show this for η of arbitrary dimension. By what we have just proved we may assume that η does not contain the trivial character. If $L(m, \sigma \circ \eta, k) = 0$ for $m \leqslant 0$ and some σ then the same holds for every σ because such a zero will be determined by the Γ-factors in the functional equation which in turn are determined by $\sigma \circ \eta(\mathrm{id}_{\bar{k}})$ and the multiplicities of the eigenvalues ± 1 of the complex conjugations. We may therefore assume that $L(m, \sigma \circ \eta, k) \neq 0$ for every σ. Because η does not contain the trivial character, we see as just after (1.1) that this is the case if and only if k is totally real and

$$(3.3) \qquad\qquad \sigma \circ \eta(c) = (-1)^{m-1}\sigma \circ \eta(\mathrm{id}_{\bar{k}})$$

for every complex conjugation in G_k, which is independent of σ. Then $\eta\omega_p^{m-1}$ is even, so it factorizes through some $\mathrm{Gal}(K/k)$ with K/k totally real and finite. Using Brauer's theorem (see [31, Theorems 16 and 19]) we can write $\eta\omega_p^{m-1} = \sum_i a_i \mathrm{Ind}_{k_i}^k(\chi_i)$ for 1-dimensional Artin characters $\chi_i : G_{k_i} \to E'$ and integers a_i, where $k \subseteq k_i \subseteq K$, so χ_i is even, and E' is a suitable finite Galois extension of E. Then $\eta = \sum_i a_i \mathrm{Ind}_{k_i}^k(\eta_i)$ with $\eta_i = \chi_i\omega_p^{1-m}$, and (3.3) applies with η replaced with η_i (cf. the more complicated

version of Brauer's theorem used in the proof of [10, Theorem 1.2]). As induction of characters is compatible with applying $\sigma : E' \to \mathbb{C}$, we find $\sigma^{-1}(L(m, \sigma \circ \mathrm{Ind}_{k_i}^{k}(\eta_i), k_i)) = \sigma^{-1}(L(m, \sigma \circ \eta_i, k_i))$ is in $E'^{\times}$. Since this is independent of σ, the same holds for the value in (3.2) for the current η, and this value lies in $E^{\times}$ because $\tau L^*(m, \eta, k) = L^*(m, \tau\eta, k) = L^*(m, \eta, k)$ for all τ in $\mathrm{Gal}(E'/E)$.

We also define, for arbitrary η, v a finite prime of k, and m in $\mathbb{Z}$, the reciprocal Euler factor $\mathrm{Eul}_v^*(m, \eta, k)$. For this, let V be an Artin representation of G_k over a finite extension E' of E with character η, D_w the decomposition group in G_k of a prime w lying above v, with inertia subgroup I_w and Frobenius Fr_w. With $F_v(t, \eta)$ the determinant of $1 - \mathrm{Fr}_w t$ we let

$$(3.4) \qquad \mathrm{Eul}_v^*(m, \eta, k) = F_v(\mathrm{Nm}(v)^{-m}, \eta).$$

Clearly $F_v(t, \eta)$ has coefficients in $\mathcal{O}_E$, and is independent of the choice of V, w and Fr_w.

As in Notation 2.1, we let k_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of k, γ_0 a topological generator of $\mathrm{Gal}(k_\infty/k)$, and let $q_k = \psi_p^{\langle\rangle}(\widetilde{\gamma}_0)$ in $1 + 2p\mathbb{Z}_p$, where $\widetilde{\gamma}_0$ is a lift of γ_0 to G_k.

Now assume k is totally real and that χ is even and 1-dimensional. Then there exists a unique $\mathbb{C}_p$-valued function $L_p(s, \chi, k)$ on

$$\mathfrak{B}_k = \{s \text{ in } \mathbb{C}_p \text{ with } |s|_p < |q_k - 1|_p^{-1} p^{-1/(p-1)}\}$$

that is analytic if χ is non-trivial and meromorphic with at most a pole of order 1 at $s = 1$ if $\chi = 1$, such that, in $\mathbb{C}_p$,

$$(3.5) \qquad L_p(m, \chi, k) = L^*(m, \chi\omega_p^{m-1}, k) \prod_{v \in P} \mathrm{Eul}_v^*(m, \chi\omega_p^{m-1}, k)$$

for all integers $m \leqslant 0$. (For the existence and uniqueness of such functions we refer to the overview statement in [3, Theorem 2.9], and for the radius of convergence to [17, p.82].) In particular, $L_p(s, \chi, k)$ is not identically zero because χ is even, so that the right-hand side of (3.5) is non-zero for $m < 0$.

If χ factorizes through $\mathrm{Gal}(k_\infty/k)$, we let $h_\chi(T) = \chi(\widetilde{\gamma}_0)(1+T) - 1$ where $\widetilde{\gamma}_0$ is a lift of γ_0 to G_k, and we let $h_\chi(T) = 1$ otherwise. Then from [17, Eq. (3)] we have an identity

$$(3.6) \qquad L_p(s, \chi, k) = \frac{\pi^{m_\chi} \widetilde{g}_\chi(q_k^{1-s} - 1) u_\chi(q_k^{1-s} - 1)}{h_\chi(q_k^{1-s} - 1)}$$

for s in $\mathfrak{B}_k$ if χ is not the trivial character and in $\mathfrak{B}_k \setminus \{1\}$ if it is, where π is a uniformizer of $\mathcal{O}_E$, m_χ a non-negative integer, $\widetilde{g}_\chi(T)$ a distinguished polynomial in $\mathcal{O}_E[T]$ and $u_\chi(T)$ a unit of $\mathcal{O}_E[[T]]$.

If $\chi : G_k \to E$ is any even Artin character, with k still totally real, then we can write χ, as we did following (3.3), in terms of even 1-dimensional characters of totally real number fields (enlarging E to E' if necessary). This gives a meromorphic function $L_p(s, \chi, k)$ on $\mathfrak{B}_k$ such that, for all $m < 0$ (and possibly for $m = 0$), $L_p(m, \chi, k)$ is defined, non-zero, and (3.5) holds in E. This interpolation property shows that $L_p(s, \chi, k)$ is independent of the way we write χ in terms of induced characters, hence $L_p(s, \chi, k)$ is compatible with induction of characters. Using the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E)$ and the p-adic Weierstraß preparation theorem for $\mathcal{O}_{E'}[[T]]$ one sees easily that

$$(3.7) \qquad L_p(s, \chi, k) = \pi^{m_\chi} P_\chi(q_k^{1-s} - 1) u_\chi(q_k^{1-s} - 1)/Q_\chi(q_k^{1-s} - 1)$$

with π a uniformizer of $\mathcal{O}_E$, m_χ an integer, $P_\chi(T)$ and $Q_\chi(T)$ relatively prime distinguished polynomials in $\mathcal{O}_E[T]$, and $u_\chi(T)$ in $\mathcal{O}_E[[T]]^\times$. So if $L_p(s, \chi, k)$ is defined at some s in $\mathfrak{B}_k(E) = \mathfrak{B}_k \cap E$ (see (1.3)), then its value lies in E.

Remark 3.8. — Greenberg has shown [17, Proposition 5] that the main conjecture of Iwasawa theory for the prime p (i.e., the statements in Theorem 4.2 and Remark 4.3 for p) implies that in (3.7) one can take $Q_\chi[T]$ to be a product of factors $\zeta \cdot (1 + T) - 1$ with ζ a p-th power root of unity. Moreover, the number of factors with $\zeta = 1$ is at most the multiplicity m of the trivial character in χ, so that $(s - 1)^m L_p(s, \chi, k)$ is analytic on $\mathfrak{B}_k$. We could use this for $p \neq 2$ by Theorem 4.2 and Remark 4.3, but as mentioned in the introduction, we take a uniform approach for all primes based on Theorem 4.2 and Assumption 4.4.

Remark 3.9. — Note that if $G(t)$ is a polynomial with coefficients in a finite extension E of $\mathbb{Q}_p$, q_k is in $1 + 2p\mathbb{Z}_p$, and $G(q_k^s - 1) = 0$ for some s in $\mathfrak{B}_k$, then q_k^s lies in a finite extension E' of E inside $\overline{\mathbb{Q}}_p$, hence s lies in $\mathfrak{B}_k(E')$. Therefore any zero or pole of $L_p(s, \chi, k)$ lies in some $\mathfrak{B}_k(E')$ with $E'/\mathbb{Q}_p$ finite.

In order to extend (3.5) to truncated L-functions we now introduce some more notation. We shall denote the image of z in $\mathbb{Z}_p^\times$ under the projection $\mathbb{Z}_p^\times = (1 + 2p\mathbb{Z}_p) \cdot \mu_{\varphi(2p)} \to 1 + 2p\mathbb{Z}_p$ by $\langle z \rangle$. Then mapping s to $\langle z \rangle^s$ defines an analytic function on $\{s \in \mathbb{C}_p \text{ with } |s|_p < |\langle z \rangle - 1|_p^{-1} p^{-1/(p-1)}\}$, and $|\langle z \rangle^s - 1|_p < p^{-1/(p-1)}$. Now let k be any number field, and $\eta : G_k \to E$ an Artin character. For a finite place v of k not in P, we define a p-adic 'diamond reciprocal Euler factor' $\mathrm{Eul}_v^{\langle \rangle}(s, \eta, k) = F_v(\langle \mathrm{Nm}(v) \rangle^{-s}, \eta)$ (cf. (2.16)), i.e., we use $\langle \mathrm{Nm}(v) \rangle$ instead of $\mathrm{Nm}(v)$ in the definition of $\mathrm{Eul}_v^*(m, \eta, k)$

in (3.4), which allows us to replace the integer m with s. With notation as just before (3.4), we observe that twisting the representation of G_k on V with ω_p^{-m} for m in $\mathbb{Z}$ gives a G_k-action on V with character $\eta\omega_p^{-m}$, and V^{I_w} is the same for both actions because ω_p is unramified at v. Since $\psi_p(\mathrm{Fr}_w) = \mathrm{Nm}(v)$ we have $\omega_p(\mathrm{Fr}_w) = \mathrm{Nm}(v)/\langle\mathrm{Nm}(v)\rangle$, hence

$$(3.10) \qquad F_v(\langle\mathrm{Nm}(v)\rangle^{-m}, \eta\omega_p^{-m}) = F_v(\mathrm{Nm}(v)^{-m}, \eta)$$

in E. Therefore $\mathrm{Eul}_v^{\langle\rangle}(s, \eta\omega_p^{-1}, k)$ is a p-adic analytic function interpolating the values $\mathrm{Eul}_v^*(m, \eta\omega_p^{m-1}, k)$ for m in $\mathbb{Z}$. It converges on $\mathfrak{B}_k$ since $\langle\mathrm{Nm}(v)\rangle^{-s} = \psi_p^{\langle\rangle}(\mathrm{Fr}_w)^{-s}$ converges on this domain.

Remark 3.11. — Since the eigenvalues of Frobenius are roots of unity, and $\langle\mathrm{Nm}(v)\rangle^{-s}$ is only a root of unity for $s = 0$, $\mathrm{Eul}_v^{\langle\rangle}(s, \eta, k) \neq 0$ for $s \neq 0$. Moreover, the order of vanishing of $\mathrm{Eul}_v^{\langle\rangle}(s, \eta, k)$ at $s = 0$ equals the dimension of $(V^{I_w})^{\mathrm{Fr}_w=1}$ where V realizes η (cf. Proposition 2.17(2)).

We now return to the case where k is totally real and χ is even. If S is a finite set of primes of k containing P, then we define a meromorphic p-adic L-function $L_{p,S}$ on $\mathfrak{B}_k$ by putting

$$(3.12) \qquad L_{p,S}(s, \chi, k) = L_p(s, \chi, k) \prod_{v \in S \setminus P} \mathrm{Eul}_v^{\langle\rangle}(s, \chi\omega_p^{-1}, k)$$

(cf. [16, Section 3, p.147]). Note that $L_p(s, \chi, k) = L_{p,P}(s, \chi, k)$, the latter being a better notation. Finally, for an integer $m < 0$ (and possibly for $m = 0$), $L_{p,S}(s, \chi, k)$ is defined at m, and in E we have

$$(3.13) \qquad L_{p,S}(m, \chi, k) = L_S^*(m, \chi\omega_p^{m-1}, k)$$

where $L_S^*(m, \chi\omega_p^{m-1}, k) = L^*(m, \chi\omega_p^{m-1}, k)\prod_{v \in S}\mathrm{Eul}_v^*(m, \chi\omega_p^{m-1}, k)$.

Remark 3.14. — Suppose that k'/k is a finite extension of number fields and $\psi : G_{k'} \to E$ an Artin character. For a finite prime v of k not in P one has $\mathrm{Eul}_v^{\langle\rangle}(s, (\mathrm{Ind}_{k'}^k \psi)\omega_p^{-1}, k) = \prod_{w|v}\mathrm{Eul}_w^{\langle\rangle}(s, \psi\omega_p^{-1}, k')$, because $\mathrm{Eul}_v^{\langle\rangle}$ interpolates the values of Eul_v^* as just after (3.10), and for those an analogous result holds. Therefore, if k' is totally real and $\chi : G_{k'} \to E$ is an even Artin character, then $L_{p,S}(s, \mathrm{Ind}_{k'}^k \chi, k) = L_{p,S'}(s, \chi, k')$ where S' is the set of primes of k' consisting of all the primes lying above those in S.

Remark 3.15. — If $f(T) \neq 0$ is in $\mathcal{O}_E[[T]]$ then the power series $g(s) = f(q_k^{1-s} - 1) = \sum_{n \geqslant 0} a_n s^n$ converges on $\mathfrak{B}_k$ and $|a_n|_p \leqslant |q_k - 1|_p^n p^{n/(p-1)}$. With $E' = E(\delta)$ for some δ algebraic over E with $|\delta|_p = |q_k-1|_p^{-1}p^{-1/(p-1)}$, we find that $g(\delta s')$ is in $\mathcal{O}_{E'}[[s']]$, so can be written as $\pi_{E'}^l P(s')u(s')$ with $\pi_{E'}$ a uniformizer of E', l a non-negative integer, $P(s')$ a distinguished

polynomial in $\mathcal{O}_{E'}[s']$, and $u(s')$ in $\mathcal{O}_{E'}[[s']]^\times$. By the discussion following (3.10), the same argument can be applied to a factor $\mathrm{Eul}_v^{\langle\rangle}(s, \chi\omega_p^{-1}, k) = F_v(\langle \mathrm{Nm}(v)\rangle^{-s}, \chi\omega_p^{-1})$ in (3.12). From (3.7) and (3.12) we have, for s in $\mathfrak{B}_k$,

$$L_{p,S}(s, \chi, k) = \pi_{E'}^l \frac{P(s/\delta)}{Q(s/\delta)} u(s/\delta)$$

with l in $\mathbb{Z}$, $P(s')$ and $Q(s')$ distinguished polynomials in $\mathcal{O}_{E'}[s']$, and $u(s')$ in $\mathcal{O}_{E'}[[s']]^\times$, so that all the zeroes and poles here come from $P(s')/Q(s')$. Therefore $L_{p,S}(s, \chi, k)$, which equals $L_p(s, \chi, k)$ if $S = P$, is analytic (i.e., given by a power series that converges on $\mathfrak{B}_k$) if and only if it is bounded on $\mathfrak{B}_k(E)$ for all finite extensions E of $\mathbb{Q}_p$.

4. Iwasawa theory

In this section we discuss the main conjecture of Iwasawa theory and prove some lemmas for later use.

Let k be a totally real number field, p a prime number, E a finite extension of $\mathbb{Q}_p$, $\chi : G_k \to E$ an even Artin character, K the (totally real) fixed field of the kernel of χ, and $G = \mathrm{Gal}(K/k)$. As in Notation 2.1 we let k_∞ and K_∞ be the cyclotomic $\mathbb{Z}_p$-extensions of k and K respectively, γ_0 a topological generator of $\Gamma_0 = \mathrm{Gal}(k_\infty/k)$. Let L_∞ be the maximal Abelian pro-p-extension of K_∞ that is unramified outside of the primes above p and the infinite primes. Write $\mathcal{M} = \mathrm{Gal}(L_\infty/K_\infty)$ and let $\mathrm{Gal}(K_\infty/k)$ act on the right on $\mathcal{M}$ by conjugation.

Now assume that $K \cap k_\infty = k$, so that $\mathrm{Gal}(K_\infty/k) \simeq G \times \Gamma_0$. We let Γ_0 and G act on $\mathcal{M}$ via this isomorphism, and thus view $\mathcal{M}$ as a $\mathbb{Z}_p[G][[T]]$-module where T acts as $\gamma_0 - 1$.
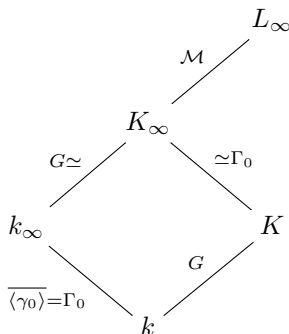
In the remainder of this section we assume χ is 1-dimensional. Then [34, Theorem 13.31] implies that $\mathcal{M} \otimes_{\mathbb{Z}_p[G]} \mathcal{O}_E$ is a finitely generated torsion $\mathcal{O}_E[[T]]$-module, where T acts on $\mathcal{M}$, and G acts on $\mathcal{O}_E$ via χ.

DEFINITION 4.1. — *Suppose $\mathcal{O}$ is the valuation ring in a finite extension of $\mathbb{Q}_p$. If Y is a finitely generated torsion $\mathcal{O}[[T]]$-module then by [5, Thm. 5 of §4.4] it is isogenous to $A \oplus B$ (i.e., there is a morphism $Y \to A \oplus B$ of $\mathcal{O}[[T]]$-modules with finite kernel and cokernel), with*

$$A = \bigoplus_{i=1}^{r} \frac{\mathcal{O}[[T]]}{(\pi^{\mu_i})}; \qquad B = \bigoplus_{j=1}^{s} \frac{\mathcal{O}[[T]]}{(g_j(T))},$$

where π is a uniformizer of $\mathcal{O}$, $r, s \geqslant 0$, the μ_i are positive integers, and the $g_j(T)$ are distinguished polynomials in $\mathcal{O}[T]$. We let $\mu_Y = \sum_{i=1}^{r} \mu_i$ and $g_Y = \prod_{j=1}^{s} g_j$.

We denote the g_Y and μ_Y in Definition 4.1 for $Y = \mathcal{M} \otimes_{\mathbb{Z}_p[G]} \mathcal{O}_E$ and $\mathcal{O} = \mathcal{O}_E$ by g_χ and μ_χ respectively. Then the following result is part of the main conjecture of Iwasawa theory as proved by Wiles for p odd in [35].

THEOREM 4.2. — *If p does not divide $[K : k]$ and $p \neq 2$ then, with notation as in (3.6), $g_\chi(T) = \tilde{g}_\chi(T)$ and $\mu_\chi = m_\chi$.*

Proof. — The first equality is proved in [35, Theorem 1.3], which is equivalent to [35, Theorem 1.2] via [16, Proposition 3]. Note that in [16] Greenberg uses the direct limit of class groups in the cyclotomic tower, but this is isogenous to the Galois group of the maximal Abelian unramified extension of K_∞ by [20, Theorem 11].

The equality of the μ-invariants follows from [35, Theorem 1.4]. Note that by [16, Proposition 1], combined with [20, Theorem 11], we have that μ_χ equals the μ-invariant considered in [35, Theorem 1.4].   □

Remark 4.3. —   If p divides $[K : k]$ but $K \cap k_\infty = k$, then

$$\mathcal{M} \otimes_{\mathbb{Z}_p[G]} E \; \simeq \; \bigoplus_{j=1}^{s} E[[T]]/(g_j(T)) \; \simeq \; \bigoplus_{j=1}^{s} E[T]/(g_j(T))$$

for some distinguished polynomials $g_j(T)$ in $\mathcal{O}_E[T]$, and, when $p \neq 2$, $\tilde{g}_\chi = \prod_{j=1}^{s} g_j$ by [35, Theorems 1.2 and 1.3].

Our proofs in Section 5 will also work for $p = 2$ if we assume the following weaker version of the main conjecture of Iwasawa theory.

ASSUMPTION 4.4. — *If $p = 2$ and χ is of odd order then $\tilde{g}_\chi(T) = g_\chi(T)$ and $m_\chi = \mu_\chi$.*

Remark 4.5. — Let χ be a 1-dimensional even Artin character of G_k such that $K \cap k_\infty = k$, and assume $p = 2$. If $k = \mathbb{Q}$ then by [35, Theorem 6.2] we have $\tilde{g}_\chi(T) = g_\chi(T)$ as well. In fact, this still holds for many more pairs (k, χ) by [35, Theorem 11.1]. We also note that $m_\chi \geqslant [k : \mathbb{Q}]$ by [29, (4.8)]. On the other hand, if k is Abelian over $\mathbb{Q}$ and χ the trivial character, then by combining the main theorem on [11, p.377] with [30, Section 6.4], we have $\mu_\chi = [k : \mathbb{Q}]$, hence $m_\chi \geqslant \mu_\chi$. Equality holds when $k = \mathbb{Q}$ because then $m_\chi = 1$ by [34, Lemma 7.12].

We shall later use the following lemma about the structure of finitely generated torsion $\mathcal{O}[[T]]$-modules.

LEMMA 4.6. — *Let $\mathcal{O}$ be the valuation ring in a finite extension of $\mathbb{Q}_p$, Y a finitely generated torsion $\mathcal{O}[[T]]$-module, so Y is isogenous to $A \oplus B$ with*

$$A = \bigoplus_{i=1}^{r} \frac{\mathcal{O}[[T]]}{(\pi^{\mu_i})} \qquad \text{and} \qquad B = \bigoplus_{j=1}^{s} \frac{\mathcal{O}[[T]]}{(g_j(T))},$$

where π is a uniformizer of $\mathcal{O}$, $r, s \geqslant 0$, the μ_i are positive integers, and the $g_j(T)$ are distinguished polynomials in $\mathcal{O}[T]$. Suppose that $\Gamma \simeq \mathbb{Z}_p$ acts on Y such that a topological generator acts as multiplication by $u \cdot (1 + T)$ for some u in $1 + \pi\mathcal{O}$. Then the following are equivalent.

(1) *Y^Γ is finite;*
(2) *Y_Γ is finite;*
(3) *$g_Y(u^{-1} - 1) \neq 0$.*

In particular, these are satisfied for all but finitely many u in $1 + \pi\mathcal{O}$. Moreover, if these hold, then $(\#Y_\Gamma)/(\#Y^\Gamma) = |\pi^{\mu_Y} g_Y(u^{-1} - 1)|_p^{-[\mathrm{Frac}(\mathcal{O}):\mathbb{Q}_p]}$.

Proof. — For any $\mu \geqslant 1$, we have a short exact sequence

$$0 \longrightarrow \frac{\mathcal{O}[[T]]}{(\pi^\mu)} \xrightarrow{u \cdot (1+T) - 1} \frac{\mathcal{O}[[T]]}{(\pi^\mu)} \longrightarrow \frac{\mathcal{O}}{(\pi^\mu)} \longrightarrow 0.$$

This shows that $A^\Gamma = 0$ and that $A_\Gamma \simeq \bigoplus_{i=1}^{r} \mathcal{O}/(\pi^{\mu_i})$. Similarly we get

$$(4.7) \qquad\qquad B_\Gamma \simeq \bigoplus_{j=1}^{s} \frac{\mathcal{O}}{(g_j(u^{-1} - 1))}.$$

Note that the $\mathcal{O}$-ranks of B_Γ and B^Γ both equal the number of g_j with $g_j(u^{-1} - 1) = 0$, and that B^Γ has no non-trivial finite subgroup. Therefore B_Γ is finite if and only if B^Γ is finite, hence trivial. From (4.7) it is clear that B_Γ is finite if and only if $g_Y(u^{-1} - 1) \neq 0$. Since g_Y is a polynomial, this last condition holds for all but finitely many u in $1 + \pi\mathcal{O}$.

Let Y' be a quotient of Y by a finite submodule such that Y' injects into $A \oplus B$. Note that finiteness of Y^{Γ} (resp. Y_{Γ}) is equivalent to that of $(Y')^{\Gamma}$ (resp. $(Y')_{\Gamma}$). We have an exact sequence of $\mathcal{O}[[T]]$-modules,

$$0 \to Y' \to A \oplus B \to C \to 0,$$

where C is a finite $\mathcal{O}[[T]]$-module. Since A_{Γ} is finite it follows that $(Y')_{\Gamma}$ is finite if and only if B_{Γ} is finite. Also, because A^{Γ} is trivial, $(Y')^{\Gamma}$ is finite if and only if B^{Γ} is finite, hence (1), (2) and (3) are equivalent. Moreover, in this case note that $(Y')^{\Gamma}$ is trivial, $\#C^{\Gamma} = \#C_{\Gamma}$, and therefore

$$\frac{\#Y_{\Gamma}}{\#Y^{\Gamma}} \;=\; \#(Y')_{\Gamma} \;=\; \#A_{\Gamma} \cdot \#B_{\Gamma} \;=\; |\pi^{\mu_Y} g_Y(u^{-1} - 1)|_p^{-[\mathrm{Frac}(\mathcal{O}):\mathbb{Q}_p]}.$$

$\square$

5. Multiplicative Euler characteristics and *p*-adic *L*-functions

In this section we prove Theorem 1.8, formulate Conjecture 5.19 (which would imply Conjecture 1.9) and briefly discuss the case of a 1-dimensional even Artin character in Example 5.21. Note that Theorem 1.8(3) implies Proposition 2.29 applies for almost all e.

We use notation as in Notation 2.1. Also, for any Artin character η of G_k we let k_{η} denote the fixed field of the kernel of the corresponding representation.

As seen in Remarks 2.2(2) and (3), the size of $H^i(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)$ for $i = 0$ can depend on the choice of the lattice $M(E, \chi)$. We shall see that a (modified) multiplicative Euler characteristic can be defined independent of this choice when those cohomology groups are finite for $i = 0, 1$ and 2. By Proposition 2.27, this can only happen when k is totally real and the character even, so it will be denoted by χ. For $i = 2$ the finiteness of the cohomology group then implies it is trivial, so that the sizes for $i = 0$ and $i = 1$ depend on the choice of the lattice in the same way.

DEFINITION 5.1. — *Let E be a finite extension of $\mathbb{Q}_p$, k a totally real number field, $\chi : G_k \to E$ an even Artin character realizable over E, and e in $\mathfrak{B}_k(E)$. If $H^i(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)$ is finite for $i = 0, 1$ and 2 then it is*

trivial for $i = 2$ by Proposition 2.26, and we define

$$\widehat{EC}_{p,S}(e,\chi,k) = \left(\prod_{i=0}^{2} \left(\# H^i(\mathcal{O}_{k,S}, W(E,\chi)\langle 1 - e \rangle) \right)^{(-1)^i} \right)^{1/[E:\mathbb{Q}_p]}$$

$$= \left(\frac{\# H^0(G_S, W(E,\chi)\langle 1 - e \rangle)}{\# H^1(G_S, W(E,\chi)\langle 1 - e \rangle)} \right)^{1/[E:\mathbb{Q}_p]}.$$

Remark 5.2. — By Remark 2.8, if $p \neq 2$ then $\widehat{EC}_{p,S}(e,\chi,k)$ is a (modified) Euler characteristic, but if $p = 2$ then it is truncated. For $p = 2$, because every complex conjugation acts trivially, $H^i(k_v, W(E,\chi)\langle 1 - e \rangle)$ for $i \geqslant 3$ is trivial if i is even since $W(E,\chi)\langle 1 - e \rangle$ is 2-divisible, and isomorphic with $W(E,\chi)[2]$ for i odd.

Remarks 5.3.

(1) Definition 5.1 for a fixed E is independent of the lattice $M = M(E,\chi)$. Namely, if M' is another lattice for the same representation, then we obtain an exact sequence (2.14). The $H^i(\mathcal{O}_{k,S}, (M' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)\langle 1 - e \rangle)$ are finite for $i = 0, 1$ and 2 by Remark 2.13(1), and Lemma 2.23 applied with $A = X_E\langle 1 - e \rangle$ implies that $\widehat{EC}_{p,S}(e,\chi,k)$ is the same for $W(E,\chi) = M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ or $M' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$.

(2) Definition 5.1 is independent of the field E by Remark 2.13(2).

(3) If χ_j for $j = 1, 2$ are even Artin characters, then by Remark 2.13(3), the $\widehat{EC}_{p,S}(e,\chi_j,k)$ are defined if and only if $\widehat{EC}_{p,S}(e,\chi_1 + \chi_2,k)$ is defined, in which case $\widehat{EC}_{p,S}(e,\chi_1 + \chi_2,k) = \widehat{EC}_{p,S}(e,\chi_1,k) \cdot \widehat{EC}_{p,S}(e,\chi_2,k)$.

(4) Let k'/k be a finite extension of totally real fields, $\chi' : G_{k'} \to E$ an even Artin character realizable over E, and put $\chi = \mathrm{Ind}_{k'}^k \chi'$. If χ is unramified outside of S then by part (1) and Remark 2.13(4), $\widehat{EC}_{p,S}(e,\chi',k')$ is defined if and only if $\widehat{EC}_{p,S}(e,\chi,k)$ is defined, in which case the two are equal.

(5) Let S' be a finite set of primes of k with $S \subseteq S'$. Note that from the definition of $\mathrm{Eul}_v^{\langle\rangle}$ in Section 3, we have that

$$\mathrm{Eul}_v^{\langle\rangle}(e, \chi\omega_p^{-1}, k) = \det(1 - \langle \mathrm{Nm}(v) \rangle^{-e} \mathrm{Fr}_w | V(E, \chi\omega_p^{-1}))$$

for any v not in S and w above v. If this is non-zero then by Remark 2.13(5) we have $\# H^0(\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v), W(E,\chi\omega_p^{-1})\langle -e \rangle) = |\mathrm{Eul}_v^{\langle\rangle}(e, \chi\omega_p^{-1}, k)|_p^{-[E:\mathbb{Q}_p]}$. So if $\mathrm{Eul}_v^{\langle\rangle}(e, \chi\omega_p^{-1}, k) \neq 0$ for v in $S' \setminus S$ then from Remark 2.13(5) we see that $\widehat{EC}_{p,S}(e,\chi,k)$ is defined if and

only if $\widehat{EC}_{p,S'}(e,\chi,k)$ is defined, in which case

$$\widehat{EC}_{p,S'}(e,\chi,k) = \widehat{EC}_{p,S}(e,\chi,k) \cdot \prod_{v \in S' \setminus S} |\text{Eul}_v^{\langle\rangle}(e,\chi\omega_p^{-1},k)|_p\,.$$

We now prove the finiteness statement in Theorem 1.8.

LEMMA 5.4. — *Let k be a totally real number field, p a prime number, E a finite extension of $\mathbb{Q}_p$, and $\chi : G_k \to E$ an even Artin character realizable over E. Assume that S is a finite set of finite primes of k containing the primes above p as well as the finite primes at which χ is ramified. Then $H^i(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e\rangle)$ is finite for all but finitely many e in $\mathfrak{B}_k(E)$.*

Proof. — The result for $i = 0$ and $i \geqslant 3$ follows from Theorem 1.4(3) and Remark 2.8. Hence by Proposition 2.27, it is enough to consider $i = 1$.

Since χ occurs in $\text{Ind}_{k_\chi}^k(\chi|_{k_\chi})$ and $\mathfrak{B}_k(E) \subseteq \mathfrak{B}_{k_\chi}(E)$, parts (1), (3) and (4) of Remarks 2.13 imply that it is enough to prove the result for the trivial character. We therefore assume that χ is trivial and hence $W(E,\chi) = E/\mathcal{O}_E$. By the last statement in Proposition 2.27, we may also assume that $S = P$.

From the spectral sequence

$$H^j(\Gamma_0, H^{j'}(\text{Gal}(\Omega_P/k_\infty), E/\mathcal{O}_E\langle 1-e\rangle)) \Rightarrow H^{j+j'}(\mathcal{O}_{k,P}, E/\mathcal{O}_E\langle 1-e\rangle)$$

it is enough to show that $H^j(\Gamma_0, H^{1-j}(\text{Gal}(\Omega_P/k_\infty), E/\mathcal{O}_E\langle 1-e\rangle))$ is finite for $j = 0, 1$. Since $\text{Gal}(\Omega_P/k_\infty)$ acts trivially on $E/\mathcal{O}_E\langle 1-e\rangle$, for $j = 1$ this group equals the cokernel of multiplication by $q_k^{1-e} - 1$ on $E/\mathcal{O}_E$, which is finite for $e \neq 1$. For $j = 0$, this group is isomorphic to $\text{Hom}(\mathcal{M}, E/\mathcal{O}_E\langle 1-e\rangle)^{\Gamma_0} \simeq \text{Hom}((\mathcal{M} \otimes_{\mathbb{Z}_p} \mathcal{O}_E\langle e-1\rangle)_{\Gamma_0}, \mathbb{Q}_p/\mathbb{Z}_p)$, where $\mathcal{M}$ is the Galois group of the maximal Abelian extension of k_∞ that is unramified outside the primes above p and ∞, on which Γ_0 acts by conjugation, and Hom is the functor of continuous homomorphisms. Because $\mathcal{M} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ is a torsion $\mathcal{O}_E[[T]]$-module as mentioned at the beginning of Section 4, the result follows from Lemma 4.6 with $Y = \mathcal{M} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$, $\mathcal{O} = \mathcal{O}_E$, $\Gamma = \Gamma_0$ with topological generator corresponding to γ_0, and $u = q_k^{e-1}$. $\qquad\square$

In order to prove the rest of Theorem 1.8 we first prove two weaker results: Theorem 5.5 if χ is 1-dimensional, and Theorem 5.15 for general χ.

THEOREM 5.5. — *Let k be a totally real number field, p a prime number, E a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_E$, $\chi : G_k \to E$ a 1-dimensional even Artin character, and let be e in $\mathfrak{B}_k(E)$. Assume that S contains P as well as all the finite primes of k at which χ is ramified. Let $k \subseteq k' \subseteq k_\chi$ be such that $[k' : k] = p^l$ for some $l \geqslant 0$ and p does not divide $[k_\chi : k']$. Assume that $e \neq 1$ if $\chi_{|G_{k'}}$ is the trivial character, so that*

$L_{p,S}(s, \chi_{|G_{k'}}, k')$ is defined at $s = e$. If its value there is non-zero, then with Assumption 4.4 if $p = 2$, we have

   (1) $H^i(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)$ is finite for $i = 0, 1$ and trivial for $i = 2$;
   (2) $|L_{p,S}(e, \chi, k)|_p = \widehat{EC}_{p,S}(e, \chi, k)$.

   Proof. — We begin by fixing some notation. We write K for k_χ, G for $\mathrm{Gal}(K/k)$, and also view χ as a character of G. Also, for any 1-dimensional (even) Artin character χ with values in E we choose $M(E, \chi)$ to be $\mathcal{O}_E$ with action given through multiplication by the character, for which we write $M(\chi)$, and write $W(\chi) = W(E, \chi) = M(\chi) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$.

   We first prove the result for $l = 0$, so that $k' = k$, and we assume $L_{p,S}(e, \chi, k) \neq 0$. By (3.12) and Remark 5.3(5), it suffices to prove the theorem when S is the union of the set of primes of k where χ is ramified and P. Since $\chi\omega_p^{-1}$ is 1-dimensional and ramified at all v in $S \setminus P$ we then have $L_{p,S}(e, \chi, k) = L_p(e, \chi, k)$.

   Note that $\mathrm{Gal}(K_\infty/k) \simeq G \times \Gamma_0$ because p does not divide $[K : k]$. We shall consider γ_0 as a topological generator of $\mathrm{Gal}(K_\infty/K)$ as well via the isomorphism $\mathrm{Gal}(K_\infty/K) \simeq \mathrm{Gal}(k_\infty/k) = \Gamma_0$.

   We have

$$
(5.6) \quad
\begin{aligned}
H^0(\mathcal{O}_{k,S}, W(\chi)\langle 1 - e \rangle) &\simeq H^0(\mathrm{Gal}(K_\infty/k), W(\chi)\langle 1 - e \rangle) \\
&= \left(W(\chi)\langle 1 - e \rangle^G \right)^{\Gamma_0} .
\end{aligned}
$$

If χ is non-trivial then $\chi(g) - 1$ is in $\mathcal{O}_E^\times$ for any non-trivial g in G as $l = 0$, so $W(\chi)\langle 1 - e \rangle^G$ and (5.6) are trivial. If χ is trivial then (5.6) equals $(W(\chi)\langle 1 - e \rangle)^{\Gamma_0}$, which is isomorphic to the kernel of multiplication by $q_k^{1-e} - 1$ on $E/\mathcal{O}_E$. In this case $e \neq 1$ by assumption, so (5.6) is isomorphic to $\mathcal{O}_E/(q_k^{1-e} - 1)$. In either case, we see from the definition of h_χ in Section 3 that

$$
(5.7) \quad \#H^0(\mathcal{O}_{k,S}, W(\chi)\langle 1 - e \rangle)^{1/[E:\mathbb{Q}_p]} = |h_\chi(q_k^{1-e} - 1)|_p^{-1} .
$$

   We now consider $H^1(\mathcal{O}_{k,S}, W(\chi)\langle 1 - e \rangle)$. Note that $H^i(G, T) = 0$ for any $\mathbb{Z}_p$-module T if $i \geqslant 1$, because $\#G$ annihilates this group and $\#G$ is in $\mathbb{Z}_p^\times$. Hence taking G-invariants is exact and $H^i(\mathcal{O}_{k,S}, T) \simeq H^0(G, H^i(\mathcal{O}_{K,S}, T))$.

   With $Z = \mathrm{Spec}\,\mathcal{O}_{K,P} \setminus \mathrm{Spec}\,\mathcal{O}_{K,S}$ we have, for all j, that

$$
H^j(Z, W(\chi\omega_p^{-1})\langle -e \rangle) \simeq \prod_{w \in Z} H^j(\mathrm{Gal}(\overline{\mathbb{F}}_w/\mathbb{F}_w), W(\chi\omega_p^{-1})\langle -e \rangle) .
$$

Let G_w and I_w denote the decomposition and inertia subgroups for w inside G. Note that I_w is non-trivial for each w in Z since χ is ramified at w. Hence there exists g in I_w with $\chi(g)^{-1} - 1 \neq 0$, and this lies in $\mathcal{O}_E^\times$ as $l = 0$. The I_w-invariants, and hence G-invariants of $H^j(\mathrm{Gal}(\overline{\mathbb{F}}_w/\mathbb{F}_w), W(\chi\omega_p^{-1})\langle -e \rangle)$, are

therefore trivial because the action of I_w on this group is as on $W(\chi)$. From the Gysin sequence as in Remark 2.13(5), but with k, S and S' replaced with K, P and S, and with the current Z, we find by taking G-invariants that

$$H^i(\mathcal{O}_{k,S}, W(\chi)\langle 1-e\rangle)) \simeq H^i(\mathcal{O}_{K,S}, W(\chi)\langle 1-e\rangle)^G$$
$$\simeq H^i(\mathcal{O}_{K,P}, W(\chi)\langle 1-e\rangle)^G$$

for all $i \geqslant 0$.

For the remainder of the case $l = 0$ we can now follow the setup for the proof of [2, Theorem 6.1]. Let $\Omega_{K,P}$ denote the maximal extension of K that is unramified outside of the primes above p or ∞, so that by Remark 2.6 we have

$$H^i(\mathcal{O}_{K,P}, W(\chi)\langle 1-e\rangle)^G \simeq H^i(\mathrm{Gal}(\Omega_{K,P}/K), W(\chi)\langle 1-e\rangle)^G$$
$$\simeq H^i(\mathrm{Gal}(\Omega_{K,P}/k), W(\chi)\langle 1-e\rangle).$$

We obtain from the Hochschild-Serre spectral sequence for the normal subgroup $\mathrm{Gal}(\Omega_{K,P}/K_\infty)$ of $\mathrm{Gal}(\Omega_{K,P}/k)$ with quotient $\mathrm{Gal}(K_\infty/k) \simeq \Gamma_0 \times G$ a 5-term exact sequence. Using that $H^0(\mathrm{Gal}(\Omega_{K,P}/K_\infty), W(\chi)\langle 1-e\rangle) = W(\chi)\langle 1-e\rangle$, we find there is an exact sequence

(5.8)
$$0 \to H^1(\mathrm{Gal}(K_\infty/k), W(\chi)\langle 1-e\rangle)$$
$$\to H^1(\mathrm{Gal}(\Omega_{K,P}/k), W(\chi)\langle 1-e\rangle)$$
$$\to H^0(\mathrm{Gal}(K_\infty/k), H^1(\mathrm{Gal}(\Omega_{K,P}/K_\infty), W(\chi)\langle 1-e\rangle))$$
$$\to H^2(\mathrm{Gal}(K_\infty/k), W(\chi)\langle 1-e\rangle).$$

Note that $H^i(\mathrm{Gal}(K_\infty/k), \cdot) \simeq H^i(\mathrm{Gal}(K_\infty/K), \cdot)^G = 0$ for $i \geqslant 2$ as the cohomological dimension of $\mathrm{Gal}(K_\infty/K)$ is 1, so the last term in (5.8) is trivial. As $H^1(\mathrm{Gal}(K_\infty/K), W(\chi)\langle 1-e\rangle)$ is the cokernel of multiplication by $q_k^{1-e} - 1$ on $W(\chi)$, the first term in (5.8) is trivial when $e \neq 1$ and equals $W(\chi)^G$ when $e = 1$. In the last case χ is non-trivial by assumption, so $\chi(g) - 1$ is in $\mathcal{O}_E^\times$ for some g in G as $l = 0$. Hence $H^1(\mathcal{O}_{k,S}, W(\chi)\langle 1-e\rangle))$ is isomorphic with $H^0(\mathrm{Gal}(K_\infty/k), H^1(\mathrm{Gal}(\Omega_{K,P}/K_\infty), W(\chi)\langle 1-e\rangle))$. The action of $\mathrm{Gal}(\Omega_{K,P}/K_\infty)$ on $W(\chi)\langle 1-e\rangle$ is trivial, so with $\mathcal{M}$ the Galois group of the maximal Abelian pro-p-extension L_∞ of K_∞ that is unramified outside of the primes above p and the infinite primes (so $L_\infty \subseteq \Omega_{K,P}$), we

can rewrite this last group as

$$
\begin{aligned}
\text{Hom}&(\text{Gal}(\Omega_{K,P}/K_\infty), W(\chi)\langle 1-e\rangle)^{\text{Gal}(K_\infty/k)} \\
&\simeq \text{Hom}(\mathcal{M}, W(\chi)\langle 1-e\rangle)^{\text{Gal}(K_\infty/k)} \\
&\simeq \text{Hom}(\mathcal{M}\langle e-1\rangle \otimes_{\mathbb{Z}_p} M(\chi^\vee), \mathbb{Q}_p/\mathbb{Z}_p)^{\text{Gal}(K_\infty/k)} \\
&\simeq \text{Hom}\left(\left(\mathcal{M}\langle e-1\rangle \otimes_{\mathbb{Z}_p} M(\chi^\vee)\right)_{\text{Gal}(K_\infty/k)}, \mathbb{Q}_p/\mathbb{Z}_p\right),
\end{aligned}
$$
(5.9)

where Hom is again the functor of continuous homomorphisms. Note that the action of $\text{Gal}(K_\infty/k)$ on $\mathcal{M}\langle e-1\rangle\otimes_{\mathbb{Z}_p} M(\chi^\vee)$ is diagonal. We therefore have

$$
\begin{aligned}
\left(\mathcal{M}\langle e-1\rangle\right.&\left.\otimes_{\mathbb{Z}_p} M(\chi^\vee)\right)_{\text{Gal}(K_\infty/k)} \\
&\simeq \left(\left(\mathcal{M}\langle e-1\rangle \otimes_{\mathbb{Z}_p} M(\chi^\vee)\right)_G\right)_{\text{Gal}(K_\infty/K)} \\
&\simeq \left(\left(\mathcal{M}\langle e-1\rangle \otimes_{\mathbb{Z}_p[G]} M(\chi)\right)\right)_{\text{Gal}(K_\infty/K)}
\end{aligned}
$$
(5.10)

because for a right (resp. left) $\mathbb{Z}_p[G]$-module A (resp. B), with G acting on $A \otimes_{\mathbb{Z}_p} B$ as $g(a \otimes b) = a \cdot g \otimes g^{-1} \cdot b$, we have a natural isomorphism $(A \otimes B)_G \simeq A \otimes_{\mathbb{Z}_p[G]} B$ as $\mathbb{Z}_p$-modules.

Recall that, since k is totally real and χ is even, $\mathcal{M}$ is a torsion $\mathbb{Z}_p[G][[T]]$-module with the action of T given by that of $\gamma_0 - 1$. By [20, Theorem 18] and [30, Section 6.4], $\mathcal{M}$ has no non-trivial finite $\mathbb{Z}_p[[T]]$-submodule, hence the same holds for $\mathcal{M} \otimes_{\mathbb{Z}_p[G]} \mathcal{O}_E[G] \simeq \mathcal{M}^{[E:\mathbb{Q}_p]}$. Then the last group cannot contain a finite non-trivial $\mathcal{O}_E[[T]]$-submodule, and as we may view $M(\chi)$ as a direct summand of $\mathcal{O}_E[G]$ because p does not divide $\#G$ since $l=0$, we find that $\mathcal{M} \otimes_{\mathbb{Z}_p[G]} M(\chi)$ is a torsion $\mathcal{O}_E[[T]]$-module without non-trivial finite submodule.

Observe that we are now in the situation of Lemma 4.6 with $\mathcal{O} = \mathcal{O}_E$, $Y = \mathcal{M} \otimes_{\mathbb{Z}_p[G]} M(\chi)$, $\Gamma = \text{Gal}(K_\infty/K)$ with topological generator corresponding to γ_0, and $u = q_k^{e-1}$. As defined right after Definition 4.1, μ_χ and g_χ are the μ_Y and g_Y in Definition 4.1 but the action of γ_0 on Y in Lemma 4.6 for this u is the action on $\mathcal{M}\langle e-1\rangle \otimes_{\mathbb{Z}_p[G]} M(\chi)$.

We assume that $L_{p,S}(e, \chi, k) \neq 0$, so by Theorem 4.2 when $p \neq 2$ and Assumption 4.4 when $p = 2$, we get $g_\chi(q_k^{1-e} - 1) \neq 0$. Since $\mu_\chi = \mu_Y$ and $g_\chi = g_Y$ in Lemma 4.6, this lemma states that the group in (5.10) is finite and has cardinality $\left|\pi^{\mu_\chi} \cdot g_\chi(q_k^{1-e} - 1)\right|_p^{-[E:\mathbb{Q}_p]}$, with π a uniformizer of $\mathcal{O}_E$. The same holds for $H^1(\mathcal{O}_{k,S}, W(\chi)\langle 1-e\rangle)$ by (5.9). Then $H^2(\mathcal{O}_{k,S}, W(\chi)\langle 1-e\rangle)$ is also finite by Proposition 2.27, hence trivial by Proposition 2.26.

By Theorem 4.2 or Assumption 4.4, and (5.7), we then find, since $L_{p,S}(e, \chi, k) = L_p(e, \chi, k)$,

$$\begin{aligned}|L_{p,S}(e, \chi, k)|_p &= \frac{|\pi^{\mu_\chi} \cdot g_\chi(q_k^{1-e} - 1)|_p}{|h_\chi(q_k^{1-e} - 1)|_p}\\ &= \left(\frac{\#H^0(\mathcal{O}_{k,S}, W(\chi)\langle 1 - e\rangle)}{\#H^1(\mathcal{O}_{k,S}, W(\chi)\langle 1 - e\rangle)}\right)^{1/[E:\mathbb{Q}_p]}\\ &= \widehat{EC}_{p,S}(e, \chi, k).\end{aligned}$$

This completes the proof of the theorem when $l = 0$.

We now proceed by induction on l, so that we have $[K : k] = mp^l$ with $(p, m) = 1$ and $l \geqslant 1$. Recall that k' is the subfield of $K = k_\chi$ with $[k' : k] = p^l$. Since χ is even, K and k' are totally real. With $\chi' = \chi_{|G_{k'}}$ we have

(5.11) $$\mathrm{Ind}_{k'}^k(\chi') = \sum_{\varphi \in I} \varphi,$$

where I is the set of all 1-dimensional characters φ over $\overline{\mathbb{Q}}_p$ of $\mathrm{Gal}(\overline{k}/k)$ factoring through $\mathrm{Gal}(K/k)$ that restrict to χ' on $G_{k'}$ (hence the quotient of two such φ is a character of $\mathrm{Gal}(k'/k) \simeq \mathbb{Z}/p^l\mathbb{Z}$). Note that each φ is a power of χ, hence is even, takes values in E, and $\mathrm{Ind}_{k'}^k(\chi')$ is also unramified outside of S. Let $N_1 = \mathrm{Ind}_{k'}^k(M(\chi'))$ and $N_2 = \oplus_{\varphi \in I} M(\varphi)$. By (5.11) these two $\mathcal{O}_E[G_k]$-modules of finite $\mathcal{O}_E$-rank correspond to the same representation. From Remark 2.13(4) we know that

(5.12) $$H^i(\mathcal{O}_{k,S}, N_1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p\langle 1 - e\rangle) \simeq H^i(\mathcal{O}_{k',S}, W(\chi')\langle 1 - e\rangle)$$

for each $i \geqslant 0$.

By our assumptions, $L_{p,S}(e, \chi', k')$ is defined and non-zero, so by the case $l = 0$ the theorem is true for χ' and k'. Therefore the group in (5.12) is finite for $i = 0, 1$, trivial for $i = 2$, and

$$\widehat{EC}_{p,S}(e, \chi', k') = |L_{p,S}(e, \chi', k')|_p.$$

From Remark 5.3(1) we conclude that $H^i(\mathcal{O}_{k,S}, N_2 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p\langle 1 - e\rangle)$ is finite for $i = 0, 1, 2$, hence trivial for $i = 2$ by Proposition 2.26. The same statement therefore holds for each summand $H^i(\mathcal{O}_{k,S}, W(\varphi)\langle 1 - e\rangle)$. From

parts (3) and (4) of Remarks 5.3 we obtain

$$\prod_{\varphi \in I} \widehat{EC}_{p,S}(e, \varphi, k) = \widehat{EC}_{p,S}(e, \chi', k')$$

$$= |L_{p,S}(e, \chi', k')|_p$$

(5.13)

$$= \prod_{\varphi \in I} |L_{p,S}(e, \varphi, k)|_p \,,$$

with the last equality by (5.11) and the compatibility of L_p and $\mathrm{Eul}_v^{\langle\rangle}$ with induction (see Remark 3.14). Note that the p-adic L-functions involved here are of Abelian characters so cannot have a pole except possibly at 1, hence by our assumptions on e and χ', $L_{p,S}(e, \varphi, k)$ exists and is non-zero for every φ.

Let I_0 denote the subset of I consisting of all φ with order divisible by p^l. If $\varphi \in I \setminus I_0$ then $[k_\varphi : k]$ divides $p^{l-1}m$, hence by the induction hypothesis the theorem is true for φ. Thus from (5.13) we obtain

(5.14) $$\prod_{\varphi \in I_0} |L_{p,S}(e, \varphi, k)|_p = \prod_{\varphi \in I_0} \widehat{EC}_{p,S}(e, \varphi, k) \,.$$

Using Remark 5.3(2) we may replace E with its Galois closure over $\mathbb{Q}_p$ if necessary. All φ in I_0 are in the orbit of χ under the action of $\mathrm{Gal}(E/\mathbb{Q}_p)$ since this group acts transitively on the roots of unity of order p^l. But for σ in $\mathrm{Gal}(E/\mathbb{Q}_p)$ we have $L_{p,S}(e, \chi^\sigma, k) = L_{p,S}(e, \chi, k)^\sigma$ by [3, p.413], hence $|L_{p,S}(e, \chi^\sigma, k)|_p = |L_{p,S}(e, \chi, k)|_p$. On the other hand, σ induces an isomorphism $M(\chi) \simeq M(\chi^\sigma)$ of sheaves for the étale topology on $\mathcal{O}_{k,S}$, so that $\widehat{EC}_{p,S}(e, \chi^\sigma, k) = \widehat{EC}_{p,S}(e, \chi, k)$. Taking roots of both sides in (5.14) we find $|L_{p,S}(e, \chi, k)|_p = \widehat{EC}_{p,S}(e, \chi, k)$. This finishes the proof of the induction step, and of the theorem.   □

We now extend the statements of Theorem 5.5 to arbitrary Artin characters for all but finitely many e, which is the last main ingredient needed for the proof of Theorem 1.8.

THEOREM 5.15. — *Let k be a totally real number field, p a prime number, E a finite extension of $\mathbb{Q}_p$, and $\chi : G_k \to E$ an even Artin character realizable over E. Assume that S contains P as well as all the finite primes of k where χ is ramified. The following hold for all but finitely many e in $\mathfrak{B}_k(E)$ where for part (2) we make Assumption 4.4 if $p = 2$.*

(1) $H^i(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e\rangle)$ is finite for $i = 0, 1$ and trivial for $i = 2$.
(2) $|L_{p,S}(e, \chi, k)|_p = \widehat{EC}_{p,S}(e, \chi, k)$.

Proof. — Part (1) follows from Lemma 5.4 and Proposition 2.26. For part (2), note that we may enlarge E if we want because of Remark 5.3(2). Using Brauer's Theorem we can then write

$$\chi + \sum_i \operatorname{Ind}_{k_i}^k(\varphi_i) = \sum_j \operatorname{Ind}_{k'_j}^k(\varphi'_j)$$

for 1-dimensional E-valued Artin characters φ_i (resp. φ'_j) of G_{k_i} (resp. $G_{k'_j}$) with all k_i, k'_j intermediate fields of the extension k_χ/k, so that G_{k_χ} is in the kernel of all φ_i and φ'_j. Since k_χ is totally real, so are all k_i and k'_j, and all φ_i and φ'_j are even. Because $\mathfrak{B}_k(E)$ is contained in all $\mathfrak{B}_{k_i}(E)$ and $\mathfrak{B}_{k'_j}(E)$, we may apply Theorem 5.5 to each pair (φ_i, k_i) or (φ'_j, k'_j) for all but finitely many e in $\mathfrak{B}_k(E)$. For all such e, parts (3) and (4) of Remarks 5.3 give that

$$\widehat{EC}_{p,S}(e, \chi, k) \prod_i \widehat{EC}_{p,S}(e, \varphi_i, k_i) = \prod_j \widehat{EC}_{p,S}(e, \varphi'_j, k'_j),$$

whereas Remark 3.14 implies

$$L_{p,S}(e, \chi, k) \prod_i L_{p,S}(e, \varphi_i, k_i) = \prod_j L_{p,S}(e, \varphi'_j, k'_j),$$

so that $\widehat{EC}_{p,S}(e, \chi, k) = |L_{p,S}(e, \chi, k)|_p$. $\qquad\square$

We can now give the proof of Theorem 1.8. We make extensive use of a counting argument while approximating a fixed e with suitable e'. In particular, this method enables us to treat the finitely many e excluded in Theorem 5.15.

Proof of Theorem 1.8. — The first statement is Theorem 5.15(1). For the remaining statements we make Assumption 4.4 if $p = 2$.

Let $r_i = r_{i,S}(1 - e, \chi) = \operatorname{corank}_{\mathcal{O}_E} H^i(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e\rangle)$. By Theorem 1.4(3), $r_0(1 - e, \chi)$ is zero if $e \neq 1$ and equals the multiplicity of the trivial character in χ otherwise. Since $r_2 \geqslant 0$, the statements about analyticity and poles of $L_{p,S}(s, \chi, k)$ in part (2) follow from the inequality $r_2 - r_0 \leqslant \nu$ in part (1) and Remark 3.15. The claims in part (2) for $L_p(s, \chi, k)$ then follow from this in turn by using Remark 3.15, (3.12), Remark 3.11, and Proposition 2.18 (note that for $e = 0$ we have $r_0 = 0$ and $r_1 = r_2$ by parts (2) and (3) of Theorem 1.4). It thus suffices to prove parts (1) and (3).

For the proof of (1), let $r = r_2 - r_0$. We shall prove the inequalities $\min(1 - r_0, \nu) \leqslant r \leqslant \nu$ in (1) when $e = 1$ and χ contains the trivial character later. So let us fix e in $\mathfrak{B}_\chi(E)$, hence $r_0 = 0$ by Theorem 1.4(3)

and therefore $r = r_1 = r_2$ by Proposition 2.27. Let X be a finite $\mathcal{O}_E$-module of size c_1 such that $H^1(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e\rangle)^\vee \simeq \mathcal{O}_E^r \times X$ and hence $H^1(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e\rangle) \simeq (E/\mathcal{O}_E)^r \times X^\vee$ as $\mathcal{O}_E$-modules. Using Theorem 1.4(3) for condition (b) below, and Theorem 5.15 for condition (c), we now only consider $n > 0$ such that

(a) c_1 divides p^n;

(b) $\#H^0(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e'\rangle)$ equals a constant c_0 if $|e'-e|_p \leqslant p^{-n}$, and c_0 divides p^n;

(c) if $0 < |e'-e|_p \leqslant p^{-n}$ then e' is in $\mathfrak{B}_\chi(E)$, $H^i(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e'\rangle)$ is finite for $i = 0, 1, 2$, $L_{p,S}(e',\chi,k) \neq 0$, and $|L_{p,S}(e',\chi,k)|_p = \widehat{EC}_{p,S}(e',\chi,k)$.

Recall that, for $a \neq 0$ in $\mathcal{O}_E$, $W(E,\chi)[a] = \ker(W(E,\chi) \xrightarrow{a} W(E,\chi))$. For $|e'-e|_p \leqslant p^{-n}$ the long exact sequence associated to (2.11) gives

$$\#H^1(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e'\rangle)[p^n] = \frac{\#H^1(\mathcal{O}_{k,S}, W(E,\chi)[p^n]\langle 1-e'\rangle)}{\#H^0(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e'\rangle)}.$$

For $|e'-e|_p \leqslant p^{-n}$ we have $W(E,\chi)[p^n]\langle 1-e'\rangle \simeq W(E,\chi)[p^n]\langle 1-e\rangle$ as G_k-modules, hence

$$
\begin{aligned}
(5.16) \qquad \#H^1(\mathcal{O}_{k,S}, &W(E,\chi)\langle 1-e'\rangle)[p^n] \\
&= \#H^1(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e\rangle)[p^n] \\
&= c_1 p^{nr[E:\mathbb{Q}_p]},
\end{aligned}
$$

so if also $e' \neq e$ then by (c) and (b) we find

$$
\begin{aligned}
(5.17) \qquad |L_{p,S}(e',\chi,k)|_p^{[E:\mathbb{Q}_p]} &= \widehat{EC}_{p,S}(e',\chi,k)^{[E:\mathbb{Q}_p]} \\
&= \frac{\#H^0(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e'\rangle)}{\#H^1(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e'\rangle)} \\
&\leqslant c_0 c_1^{-1} p^{-nr[E:\mathbb{Q}_p]}.
\end{aligned}
$$

For $n \gg 0$ the left-hand side equals $c|e'-e|_p^{\nu[E:\mathbb{Q}_p]}$ for some $c \neq 0$ and so by choosing e' with $|e'-e|_p = p^{-n}$ we get $\nu \geqslant r$.

For proving that $\min(1,\nu) \leqslant r$ we may take $r = 0$. For $|e'-e|_p \leqslant p^{-n-1}$ we see from (5.16) that the part of $H^1(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e'\rangle)$ annihilated by p^n is the same as that annihilated by p^{n+1}. Hence this is the entire group, and $\#H^1(\mathcal{O}_{k,S}, W(E,\chi)\langle 1-e'\rangle) = c_1$ for such e'. From (5.17) we then see that $|L_{p,S}(e',\chi,k)|_p^{[E:\mathbb{Q}_p]} = c_0/c_1$ for $e' \neq e$ close enough to e, so by continuity we have $|L_{p,S}(e,\chi,k)|_p^{[E:\mathbb{Q}_p]} = c_0/c_1 \neq 0$ and $\nu = 0$.

We now assume $e = 1$ and χ contains the trivial character χ_0. Note that the two sides of the inequality $r(e,\chi) \leqslant \nu(e,\chi)$ are additive in χ,

therefore it is enough to prove this for $\chi = \chi_0$. In this case we may take $E = \mathbb{Q}_p$ and $M = \mathbb{Z}_p$. Moreover, we may take $S = P$ by Theorem 1.4(2) because $\mathrm{Eul}_v^{\langle\rangle}(1, \chi_0\omega_p^{-1}, k) \neq 0$. Clearly $H^0(\mathcal{O}_{k,P}, \mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{Q}_p/\mathbb{Z}_p$, hence $r_0(0, \chi_0) = 1$. On the other hand, by the discussion after (5.8) and (5.9) we have $r_1(0, \chi_0) = 1 + \mathrm{rank}_{\mathbb{Z}_p}\mathcal{M}$, where $\mathcal{M}$ is the Galois group of the maximal extension of k_∞ unramified outside of the primes above p and infinite primes. Since $\mathcal{M}$ is isogenous to $\bigoplus_{i=1}^r \mathbb{Z}_p/\pi^{m_i} \oplus \bigoplus_{j=1}^s \mathbb{Z}_p/(g_j(0))$, we find $\mathrm{rank}_{\mathbb{Z}_p}\mathcal{M} \leqslant \mathrm{ord}_{T=0}(\tilde{g}_{\chi_0}(T))$ by Theorem 4.2 if $p \neq 2$, and Assumption 4.4 if $p = 2$. As $h_{\chi_0}(T) = T$, $\mathrm{ord}_{T=0}(\tilde{g}_{\chi_0}(T)) = \nu(0, \chi_0) + 1$. Thus we have $r_1(0, \chi_0) \leqslant \nu(0, \chi_0) + 2$ and $r_2(0, \chi_0) \leqslant \nu(0, \chi_0) + 1$ by Theorem 1.4(2). Thus $r(0, \chi_0) \leqslant \nu(0, \chi_0)$. This proves the inequality $r \leqslant \nu$ in all the cases.

The inequality $\min(1 - r_0, \nu) \leqslant r = r_2 - r_0$ is trivial unless $r_2 = 0$. Write $\chi = \chi' + s\chi_0$ with χ' not containing the trivial character and $s \geqslant 1$, then $r_2(0, \chi) = r_2(0, \chi') + sr_2(0, \chi_0)$. This is zero only if $r_2(0, \chi') = r_2(0, \chi_0) = 0$. We have just proved Theorem 1.8(1) for χ', so using Theorem 1.4(3) for χ' we see $L_p(1, \chi', k) \neq 0$. Moreover, $r_2(0, \chi_0) = 0$ implies that $r_0(0, \chi_0) = r_1(0, \chi_0) = 1$ by parts (2) and (3) of Theorem 1.4. Hence $\mathrm{rank}_{\mathbb{Z}_p}\mathcal{M} = 0$, $\tilde{g}_{\chi_0}(0) \neq 0$ and therefore $\nu(0, \chi_0) = -1$. This proves $\nu(0, \chi) = -s$ and $\min(1 - r_0, \nu) = -s = r_2 - r_0$. This completes the proof of part (1) of the theorem.

For part (3), we fix e in $\mathfrak{B}_\chi(E)$, so $r_0 = 0$ by Theorem 1.4(3). Then the equivalences follow from the case $r_2 = 0$ in part (1) and Proposition 2.26. For the very last formula, we observe that this is covered by the case $r = 0$ in the paragraph following (5.17) since $\#H^i(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e\rangle) = c_i$ for $i = 0, 1$ and $r_1 = r_2$ by Theorem 1.4(2). This concludes the proof of Theorem 1.8.   □

Note that if the equality

$$(5.18) \qquad \nu_S(1 - e, \chi) = r_{2,S}(1 - e, \chi) - r_{0,S}(1 - e, \chi)$$

holds in Theorem 1.8(1) for some S then it holds for every choice of S by Proposition 2.17(2), Theorem 1.4(2) and Remark 3.11. If it holds for all totally real k and all 1-dimensional even Artin characters $\chi : G_k \to E$ with $[k_\chi : k]$ not divisible by p, then it holds for all even Artin characters of G_k for all totally real k. This is because such an equality is preserved under the induction argument in the proof of Theorem 5.5 and also under the arguments involving Brauer induction in the proof of Theorem 5.15. Since $r_{0,S}(1 - e, \chi)$ is independent of S as stated in Theorem 1.4(2), we see from Proposition 2.17(2) and Remark 3.11, the proof of the case $l = 0$ of Theorem 5.5, and the proof of Lemma 4.6, that the equality in (5.18) for such an Artin character χ is equivalent to all distinguished polynomials $g_j(T)$ in

$\mathcal{O}_E[T]$ having order at most 1 at $T = q_k^{e-1} - 1$, where $Y = \mathcal{M} \otimes_{\mathbb{Z}_p[G]} M(\chi)$ as in the application of Lemma 4.6 in the proof of Theorem 5.5 is isogenous to $\bigoplus_{i=1}^r \frac{\mathcal{O}[[T]]}{(\pi^{\mu_i})} \oplus \bigoplus_{j=1}^s \frac{\mathcal{O}[[T]]}{(g_j(T))}$. We therefore formulate the following (partly folklore) conjecture.

CONJECTURE 5.19. — *Let k be a totally real number field, p a prime number, E a finite extension of $\mathbb{Q}_p$, and $\chi : G_k \to E$ an even Artin character realizable over E. If χ is 1-dimensional then let $M(\chi) = \mathcal{O}_E$ on which G_k acts through χ. Assume that S contains the primes above p as well as the finite primes of k at which k_χ/k is ramified. Let $\mathcal{M}$ be the Galois group of the maximal Abelian extension of $(k_\chi)_\infty$ that is unramified outside of the primes above p and infinity. Then for e in $\mathfrak{B}_k(E)$ we have the following.*

(1) *If χ is 1-dimensional of order prime to p, then the g_j corresponding to $Y = \mathcal{M} \otimes_{\mathbb{Z}_p[G]} M(\chi)$ are square-free.*

(2) *If χ is 1-dimensional of order prime to p, then the g_j corresponding to $Y = \mathcal{M} \otimes_{\mathbb{Z}_p[G]} M(\chi)$ have no multiple roots z in E for which $|z - 1|_p < p^{-1/(p-1)}$.*

(3) *If χ is 1-dimensional of order prime to p, then equality holds in (5.18).*

(4) *Equality holds in (5.18).*

For the same k, e and χ, (1) implies (2), and, as discussed before the statement of the conjecture, (2) is equivalent to (3). As also implied by the discussion there, if we fix e, then (3) for all k, χ and E implies (4), and the converse is clear.

Remarks 5.20. — In this remarks we make Assumption 4.4 if $p = 2$.

(1) If $e = 1$ and χ is the trivial character then parts (2) and (3) of Theorem 1.4 and Theorem 1.8(1) imply the following are equivalent:

    (a) $\zeta_p(s, k)$ has a simple pole at $s = 1$;

    (b) $\operatorname{corank}_{\mathbb{Z}_p} H^1(\mathcal{O}_{k,S}, \mathbb{Q}_p/\mathbb{Z}_p) = 1$;

    (c) $H^2(\mathcal{O}_{k,S}, \mathbb{Q}_p/\mathbb{Z}_p) = 0$.

Those statements are all equivalent with the Leopoldt conjecture for k (see [27, Theorem 10.3.6]).

Clearly, an equality $\nu_S(0, \chi) = -r_{0,S}(0, \chi)$ with χ the trivial character implies (a) by Theorem 1.4(3), hence the Leopoldt conjecture for k. For other characters, recall that by Brauer's theorem $\chi = \sum_i a_i \operatorname{Ind}_{H_i}^G(\chi_i)$ for some non-zero integers a_i and 1-dimensional characters χ_i of certain solvable subgroups H_i. If H_i is nontrivial and $\chi_i = 1_H$, then $\operatorname{Ind}_{[H_i,H_i]}^G(1_{[H_i,H_i]}) = \operatorname{Ind}_{H_i}^G(1_{H_i}) + \sum_j \operatorname{Ind}_{H_i}^G(\chi_j)$ where the sum runs through the non-trivial 1-dimensional characters of H_i. Repeating this process if necessary, we may assume that χ_i is

trivial only if H_i is trivial. In that case, Frobenius reciprocity for 1_G implies that if χ does not contain the trivial character of G, then all χ_i are non-trivial. Therefore the Leopoldt conjecture for all totally real number fields implies $L_p(s, \chi, k)$, for an even character χ of G_k not containing the trivial character, is defined and non-zero at $s = 1$, so by Theorem 1.4(3) and Theorem 1.8(3), $\nu_S(0, \chi) = -r_{0,S}(0, \chi) = 0$. Since the Leopoldt conjecture for k implies $\nu_S(0, \chi) = -r_{0,S}(0, \chi)$ if χ is the trivial character, the Leopoldt conjecture for all totally real k implies equality for all k and all even Artin characters χ of G_k.

(2) For e in $B_\chi(E)$, $H^1(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)$ is finite if and only if $L_{p,S}(e, \chi, k)$ is defined and non-zero by Theorem 1.8(3). If χ contains the trivial character and $e = 1$, then $H^1(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)$ is infinite by parts (2) and (3) of Theorem 1.4. The Leopoldt conjecture (for all totally real k) would imply that then $L_{p,S}(e, \chi, k)$ should not be defined at $e = 1$, extending this equivalence to e in $B_k(E)$.

To conclude this section, we briefly discuss the case of 1-dimensional χ.

Example 5.21. — With notation as in Theorem 1.8, suppose that k is totally real, χ is a 1-dimensional even Artin character, and e is in $\mathfrak{B}_\chi(E)$ with $L_p(e, \chi, k) \neq 0$. Then $h_\chi(T) = 1$ unless $k_\chi \subseteq k_\infty$, in which case $h_\chi(T) = \chi(\gamma_0)(T + 1) - 1$. Note that in the latter case

$$H^0(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle) = \ker\left(W(E, \chi)\langle 1 - e \rangle \xrightarrow{1 - \gamma_0} W(E, \chi)\langle 1 - e \rangle\right).$$

Since γ_0 acts on $W(E, \chi)\langle 1 - e \rangle$ as multiplication by $q_k^{1-e}\chi(\gamma_0)$ in this case, we have $\#H^0(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle) = |h_\chi(q_k^{1-e} - 1)|_p^{-[E:\mathbb{Q}_p]}$. Hence for any 1-dimensional χ we always have

$$(5.22) \qquad \#H^0(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)) \geqslant |h_\chi(q_k^{1-e} - 1)|_p^{-[E:\mathbb{Q}_p]}.$$

So by (3.6) and Theorem 1.8(3) we also have

$$(5.23) \qquad \#H^1(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)) \geqslant |\pi^{\mu_\chi} \tilde{g}_\chi(q_k^{1-e} - 1)|_p^{-[E:\mathbb{Q}_p]}.$$

We distinguish three cases, with the third motivating why one needs to use $\widehat{EC}_{p,S}(e, \chi, k)$ rather than the individual cohomology groups (and to some extent necessitating the complexity of our proofs).

(1) $[k_\chi : k]$ is divisible by a prime other than p: then $h_\chi(T) = 1$ and H^0 is trivial because it is contained in the kernel of multiplication by $1 - \xi_q$ where ξ_q is a root of unity of order a prime $q \neq p$. Therefore both entries in (5.22) are 1, and equality holds in both (5.22) and (5.23) because of Theorem 1.8(3).

(2) $[k_\chi : k]$ is a power of p and $k_\chi \subseteq k_\infty$. As already mentioned, equality holds in (5.22), hence it also holds in (5.23), but in (5.22) both sides are bigger than 1.

(3) $[k_\chi : k]$ is a power of p but $k_\chi \not\subseteq k_\infty$. In this case $h_\chi(T) = 1$. Note that any g in G acts on $W(E, \chi)\langle 1 - e \rangle \simeq E/\mathcal{O}_E\langle 1 - e \rangle$ as multiplication by an element of $\mathcal{O}_E^\times$ that reduces to 1 in the residue field of $\mathcal{O}_E$, hence $H^0(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)$ is non-trivial, and the inequalities in (5.22) and (5.23) are strict.

6. Numerical examples

The approximations $\widetilde{P}(T)$ of the distinguished polynomials $P(T)$ and the triviality of the μ-invariants in the following examples were kindly provided to us by X.-F. Roblot. In all cases we have a totally real field K such that $K/\mathbb{Q}$ is Galois with dihedral Galois group G of order 8. If k is the fixed field of the cyclic subgroup H of order 4 then we consider $L_5(s) := L_5(s, \chi, k) = L_5(s, \mathrm{Ind}_k^{\mathbb{Q}}(\chi), \mathbb{Q})$ where χ is a 1-dimensional character of order 4 of H with values in $\mu_4 \subset \mathbb{Q}_5$. Note that $\mathrm{Ind}_k^{\mathbb{Q}}(\chi)$ is the irreducible 2-dimensional character of $\mathrm{Gal}(K/\mathbb{Q})$ for either possibility for χ, so $L_5(s)$ is the same for either χ.

With notation as in (3.6) we have $h_\chi(T) = 1$, and used $q_k = 1 + 5$. In all cases below Roblot found that $m_\chi = 0$, so $L_5(s) = P(6^{1-s} - 1)u(6^{1-s} - 1)$ for s in $\mathfrak{B}_k = \{s \text{ in } \mathbb{C}_p \text{ with } |s|_5 < 5^{3/4}\}$, where $u(T)$ is in $\mathbb{Z}_5[[T]]^\times$ and $P(T) = \tilde{g}_\chi(T)$ is a distinguished polynomial in $\mathbb{Z}_5[T]$. So, in particular, $|L_5(s)|_5 = |P(6^{1-s} - 1)|_5$.

Note that if S is a finite set of primes of $\mathbb{Q}$ that includes 5 and the primes where $K/\mathbb{Q}$ is ramified, then by Remark 2.13(3) we have, for $E/\mathbb{Q}_5$ a finite extension and e in $\mathfrak{B}_\chi(E) = \mathfrak{B}_k(E)$, with the appropriate choice of lattices,

$$H^i(\mathcal{O}_{\mathbb{Q},S}, W(E, \mathrm{Ind}_k^{\mathbb{Q}}(\chi))\langle 1 - e \rangle) \simeq H^i(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle).$$

We recall from Remark 2.2(1) that the coefficients in the right-hand side are unique but that this does not necessarily hold in the left-hand side.

In the examples, K/k is unramified outside of the primes above 5. We discuss $H^i(\mathcal{O}_{k,S}, W(E, \chi)\langle 1 - e \rangle)$ only when S consists of those primes. The sizes of those groups for larger S can be calculated similarly by taking into account various $|\mathrm{Eul}_v^{\langle \rangle}(e, \chi\omega_5^{-1}, k)|_5$ as in (3.12). We leave this to the interested reader.

Example 6.1. — $k = \mathbb{Q}(\sqrt{145})$, K is the Hilbert class field of k, where $P(T) = 1$. Taking e in $\mathfrak{B}_k(\mathbb{Q}_5) = \mathbb{Z}_5$, one sees from Case (1) of Example 5.21 that $H^i(\mathcal{O}_{k,S}, W(E, \chi))\langle 1 - e\rangle)$ with $E = \mathbb{Q}_5$ is trivial for $i = 0$, hence the same holds for $i = 1$ and $i = 2$ by Theorem 1.8. In fact, the same statements are true if we use any finite extension E of $\mathbb{Q}_5$ and take e in $\mathfrak{B}_k(E)$.

Example 6.2. — $k = \mathbb{Q}(\sqrt{41})$, K is the ray class field of k modulo 5. Here $\widetilde{P}(T) = T - (5 + \varepsilon)$ with $|\varepsilon|_5 \leqslant 5^{-10}$. But $\mathrm{Eul}_P^*(s, \chi\omega_5^{-1}, k) = 1 - 5^{-s}$, so that $L_5(0, \chi, k) = 0$ by (3.5), and $P(T) = T - 5$. Fix a finite extension E of $\mathbb{Q}_5$ and e in $\mathfrak{B}_k(E)$. Again by Example 5.21 we have that $H^0(\mathcal{O}_{k,S}, W(E, \chi))\langle 1 - e\rangle) = 0$. Taking $e \neq 0$, so that $L_5(e) \neq 0$, by Theorem 5.5 we have $\#H^1(\mathcal{O}_{k,S}, W(E, \chi))\langle 1 - e\rangle) = \#\mathcal{O}_E/(6^{1-e} - 6) = \#\mathcal{O}_E/(6^{-e} - 1) = \#\mathcal{O}_E/(5e)$.

In fact, any $\mathbb{Z}_5[[T]]$-submodule of finite index in $\mathbb{Z}_5[[T]]/(T - 5) \simeq \mathbb{Z}_5$ is isomorphic with $\mathbb{Z}_5$ as $\mathbb{Z}_5[[T]]$-module, so that, in the proof of Theorem 5.5 for $l = 0$, the part of $\mathcal{M}$ on which H acts through χ is $\mathcal{M} \otimes_{\mathbb{Z}_5[H]} \mathbb{Z}_5(\chi) \simeq \mathbb{Z}_5$ with the action of γ_0 given by multiplication by 6. So by the proof of Lemma 4.6 (in particular (4.7)) we have $H^1(\mathcal{O}_{k,S}, W(E, \chi))\langle 1 - e\rangle) \simeq \mathcal{O}_E/(5e)$, also when $e = 0$.

Remark 6.3. — Note that the calculation in Example 6.2 for each of the two possibilities for χ implies the existence of L_χ/K_∞, such that L_χ/k is Galois with Galois group isomorphic to $(H \times (1 + 5\mathbb{Z}_5)) \ltimes \mathbb{Z}_5$, where the action of (h, u) on $\mathbb{Z}_5$ (the part of $\mathcal{M}$ on which H acts through χ) is given by multiplication by $\chi(h)u$.

Working over $\mathbb{Q}$ we find that the two L_χ together give rise to extensions $L_\infty/K_\infty/\mathbb{Q}$ with $\mathrm{Gal}(L_\infty/\mathbb{Q})$ isomorphic to $(G \times (1 + 5\mathbb{Z}_5)) \ltimes \mathbb{Z}_5^2$ where G acts on $\mathbb{Z}_5^2$ as on the irreducible 2-dimensional representation of G, and $1 + 5\mathbb{Z}_5$ by multiplication.

Example 6.4. — $k = \mathbb{Q}(\sqrt{793})$, K is the Hilbert class field of k, and

$$\widetilde{P}(T) = T^2 + ((0.341430342)_5 + \varepsilon_1) T + ((0.103034211)_5 + \varepsilon_2),$$

with $|\varepsilon_i|_5 \leqslant 5^{-10}$. Here $(0.a_1 \cdots a_m)_5 = \sum_{i=1}^m a_i 5^m$. Therefore $P(T)$ is Eisenstein, hence is irreducible over $\mathbb{Z}_5$ and $|L_5(s)|_5 = 1$ for s in $\mathbb{Z}_5$. But $\mathbb{Z}_5[[T]]/(P(T)) \simeq \mathbb{Z}_5[\sqrt{5}] = \mathcal{O}_{\mathbb{Q}_5(\sqrt{5})}$ so any $\mathbb{Z}_5[[T]]$-submodule of finite index is isomorphic to $\mathbb{Z}_5[[T]]/(P(T))$ with γ_0 acting as multiplication by $1 + T$. As in Example 6.2 we find that $H^1(\mathcal{O}_{k,S}, W(E, \chi))\langle 1 - e\rangle) \simeq \mathcal{O}_E/(P(6^{1-e} - 1))$ for each finite extension E of $\mathbb{Q}_5$ and e in $\mathfrak{B}_k(E)$ as $H^0(\mathcal{O}_{k,S}, W(E, \chi))\langle 1 - e\rangle)$ is trivial. Unless $|6^{1-e} - 1 - \alpha|_5 \leqslant 5^{-19/2}$ for a

root
$$\alpha = (0.104202323)_5 \pm (3.41423114)_5 \sqrt{5} + \varepsilon$$
of $P(T)$ (with $|\varepsilon|_5 \leqslant 5^{-19/2}$), we may replace $P(6^{1-e}-1)$ with $\widetilde{P}(6^{1-e}-1)$.

Remark 6.5. —   Just as in Remark 6.3, the calculation in Example 6.4 for each of the two possibilities implies the existence of L_χ/K_∞, such that L_χ/k is Galois with Galois group isomorphic to $(H \times (1+5\mathbb{Z}_5)) \ltimes \mathbb{Z}_5[\sqrt{5}]$, where the action of (h,u) on $\mathbb{Z}_5[\sqrt{5}]$ (the part of $\mathcal{M}$ on which H acts through χ) is given by multiplication by $\chi(h)(1+\alpha)^{\log(u)/\log(6)}$ for α in $\mathbb{Z}_5[\sqrt{5}]$ a root of $P(T)$.

Working over $\mathbb{Q}$ we find that the two L_χ together give rise to extensions $L_\infty/K_\infty/\mathbb{Q}$ with $\mathrm{Gal}(L_\infty/\mathbb{Q})$ isomorphic to $(G \times (1+5\mathbb{Z}_5)) \ltimes \mathbb{Z}_5[\sqrt{5}]^2$ where G acts as on $\mathbb{Z}_5[\sqrt{5}]^2$ as on the irreducible 2-dimensional representation of G, and u in $1+5\mathbb{Z}_5$ as multiplication by $(1+\alpha)^{\log(u)/\log(6)}$ for α a root of $P(T)$ in $\mathbb{Z}_5[\sqrt{5}]$.

7. The equivariant Tamagawa number conjecture

Let k be a number field, S a finite set of finite primes of k, Σ_∞ the set of infinite places of k and G_S the Galois group of the maximal extension of k that is unramified outside of $S \cup \Sigma_\infty$. For any place v of k let G_{w_v} be the decomposition group in G_k of a prime w_v of $\overline{k}$ lying above v. For a finitely generated $\mathbb{Z}_p$-module A with continuous G_S-action, define
$$R\Gamma_c(\mathcal{O}_{k,S}, A) := \mathrm{Cone}\left(C^\bullet(G_S, A) \to \oplus_{v \in S \cup \Sigma_\infty} C^\bullet(G_{w_v}, A)\right)[-1],$$
where $C^\bullet$ denotes the standard complex of (continuous) cochains, and the morphism is induced by the natural maps $G_{w_v} \to G_k \to G_S$. We denote the cohomology of $R\Gamma_c(\mathcal{O}_{k,S}, A)$ by $\overline{H}_c^*(\mathcal{O}_{k,S}, A)$. Applying inverse limits to the resulting long exact sequence of cohomology groups with coefficients $A/p^n A$ is exact because the $H^*(G_S, A/p^n A)$ and $H^*(G_{w_v}, A/p^n A)$ are finite. Therefore $\overline{H}_c^*(\mathcal{O}_{k,S}, A) \simeq \varprojlim_n \overline{H}_c^*(\mathcal{O}_{k,S}, A/p^n A)$ by the five lemma.

If R is a commutative ring and Λ a perfect complex of R-modules then $H^i(\Lambda)$ is trivial for all but finitely many i and $\mathrm{Det}_R \Lambda$ is defined (see [22] for the definition and properties of the determinant functor). Moreover, if $H^i(\Lambda)$ is projective for all i, then by [7, Proposition 2.1(e)] there is a canonical isomorphism $\mathrm{Det}_R \Lambda \to \otimes_i \mathrm{Det}_R^{(-1)^i} H^i(\Lambda)$. If R is the valuation ring $\mathcal{O}_E$ in a finite extension E of $\mathbb{Q}_p$, $H^i(\Lambda)$ is finite for all i, and trivial for all but finitely many i, then we have a canonical composition

$$(7.1) \quad \mathrm{Det}_{\mathcal{O}_E} \Lambda \simeq \otimes_i \mathrm{Det}_{\mathcal{O}_E}^{(-1)^i} H^i(\Lambda) \to \otimes_i \mathrm{Det}_E^{(-1)^i} (H^i(\Lambda) \otimes_{\mathcal{O}_E} E) \simeq E.$$

THEOREM 7.2. — *Let k be a totally real number field, p a prime number, E a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_E$, m a negative integer, η an Artin character of G_k realizable over E such that $\eta(c) = (-1)^{m-1}\eta(\mathrm{id}_{\overline{k}})$ for every complex conjugation c in G_k, $V(E, \eta^\vee)$ an Artin representation of G_k over E with character $\eta^\vee$, $M(E, \eta^\vee)$ an $\mathcal{O}_E$-lattice for $\eta^\vee$, and S a finite set of primes of k containing the primes above p as well as all the finite primes of k at which η is ramified. Then the following hold, where for $p = 2$ we make Assumption 4.4.*

(1) *$R\Gamma_c(\mathcal{O}_{k,S}, M(E, \eta^\vee)(m))$ and $R\Gamma_c(\mathcal{O}_{k,S}, V(E, \eta^\vee)(m))$ are perfect complexes.*

(2) *$\overline{H}_c^i(\mathcal{O}_{k,S}, M(E, \eta^\vee)(m))$ is finite for all i and trivial if $i \neq 2, 3$.*

(3) *In E we have*

$$(7.3) \qquad L_S^*(m, \eta, k) \cdot \mathcal{O}_E = \theta(\mathrm{Det}_{\mathcal{O}_E} R\Gamma_c(\mathcal{O}_{k,S}, M(E, \eta^\vee)(m))),$$

with $L_S^*(m, \eta, k)$ as in Section 3, and θ the composition (7.1) for $\Lambda = R\Gamma_c(\mathcal{O}_{k,S}, M(E, \eta^\vee)(m))$.

Proof. — Part (1) follows from [12, Theorem 5.1]. For parts (2) and (3), we abbreviate $M(E, \eta^\vee)(m)$ to M, and begin by computing $\overline{H}_c^*(\mathcal{O}_{k,S}, M)$. We have $\overline{H}_c^0(\mathcal{O}_{k,S}, M) = 0$ because $H^0(G_S, M) \to \oplus_{v \in S} H^0(G_{w_v}, M)$ is injective. For a real place v in S, note that the complex conjugation c_v in $\mathrm{Gal}(\overline{k_v}/k_v)$ acts on M as multiplication by -1, so that $H^0(G_{w_v}, M/p^n M) \simeq \widehat{H}^0(G_{w_v}, M/p^n M)$, where the right-hand side denotes Tate cohomology. Therefore, for $i \geqslant 1$, $\overline{H}_c^i(\mathcal{O}_{k,S}, M/p^n M) \simeq H_c^i(\mathcal{O}_{k,S}, M/p^n M)$ where the right-hand side denotes the cohomology with compact support as in [25, Section II § 2, p.203]. Taking inverse limits and using (2.7) we have

$$\overline{H}_c^i(\mathcal{O}_{k,S}, M) \simeq \left(H^{3-i}(\mathcal{O}_{k,S}, W(E, \eta\omega_p^{1-m})\langle 1 - m \rangle)\right)^\vee$$

for $i \geqslant 1$, where $W(E, \eta)$ and $W(E, \eta\omega_p^{1-m})$ are obtained from $M(E, \eta^\vee)$. We see as just after (1.1) that $L_S^*(m, \eta, k) \neq 0$ because $m < 0$, and hence $L_{p,S}(m, \eta\omega_p^{1-m}, k) \neq 0$ by (3.13). Then according to Theorem 1.8(3), $H^j(\mathcal{O}_{k,S}, W(E, \eta\omega_p^{1-m})\langle 1 - m \rangle)$ is finite for $j = 0, 1$ and trivial for $j = 2$, completing the proof of part (2).

For part (3), note that for any finite $\mathcal{O}_E$-module A, $I^{[E:\mathbb{Q}_p]} = (\#A) \cdot \mathcal{O}_E$ with I the image of the composition $\mathrm{Det}_{\mathcal{O}_E} A \to \mathrm{Det}_E(A \otimes_{\mathcal{O}_E} E) \simeq E$.

Therefore

$$
\begin{aligned}
\theta \left(\mathrm{Det}_{\mathcal{O}_E} \, R\Gamma_c(\mathcal{O}_{k,S}, M) \right)^{[E:\mathbb{Q}_p]} &= \frac{\# \overline{H}_c^2(\mathcal{O}_{k,S}, M)}{\# \overline{H}_c^3(\mathcal{O}_{k,S}, M)} \cdot \mathcal{O}_E \\
&= \frac{\# H^1(\mathcal{O}_{k,S}, W(E, \eta\omega_p^{1-m})\langle 1-m\rangle)}{\# H^0(\mathcal{O}_{k,S}, W(E, \eta\omega_p^{1-m})\langle 1-m\rangle)} \cdot \mathcal{O}_E \\
&= \widehat{EC}_{p,S}(m, \eta\omega_p^{1-m}, k)^{-[E:\mathbb{Q}_p]} \cdot \mathcal{O}_E \\
&= L_{p,S}(m, \eta\omega_p^{1-m}, k)^{[E:\mathbb{Q}_p]} \cdot \mathcal{O}_E \\
&= L_S^*(m, \eta, k)^{[E:\mathbb{Q}_p]} \cdot \mathcal{O}_E,
\end{aligned}
$$

where the last two equalities follow from Theorem 1.8(3) and the interpolation formula (3.13). $\square$

COROLLARY 7.4. — *Let k be a totally real number field, K/k a finite Galois extension with Galois group G and m a negative integer. Assume that K is totally real if m is odd and K is a CM field if m is even. Let $\mathfrak{M}$ denote a maximal $\mathbb{Z}[G]$-order inside $\mathbb{Q}[G]$. Let $\pi_m = 1$ if m is odd and $\pi_m = (1-c)/2$ if m is even, where c is the unique complex conjugation in G. Let p be a prime number and if $p = 2$ make Assumption 4.4. Then the p-part of the equivariant Tamagawa number conjecture ([7, Conjecture 6, p.535]) holds for the pair $(\pi_m h^0(\mathrm{Spec}\, K)(m), \pi_m \mathfrak{M})$.*

Proof. — Let F be a finite extension of $\mathbb{Q}$ such that all the irreducible $\overline{\mathbb{Q}}$-valued characters of G can be realized over F, and let $\mathfrak{M}_F$ be a maximal $\mathcal{O}_F[G]$-order in $F[G]$. By [7, Theorem 4.1], the equivariant Tamagawa number conjecture for the pair $(\pi_m h^0(\mathrm{Spec}\, K)(m), \pi_m \mathfrak{M})$ is then equivalent to that for the pair $(\pi_m h^0(\mathrm{Spec}\, K)(m), \pi_m \mathfrak{M}_F)$, which can be decomposed according to all irreducible characters ψ of G if K is totally real, and those ψ satisfying $\psi(c) = -\psi(\mathrm{id}_{\overline{k}})$ if K is a CM field. For such a character ψ, if $\pi_\psi = (\#G)^{-1} \sum_{g \in G} \psi(g)g^{-1}$ is the standard idempotent in $F[G]$ corresponding to ψ, then the special value of the motivic L-function associated to $\pi_\psi h^0(\mathrm{Spec}\, K)(m)$ is $L(m, \psi, k)$ in F. If we fix an embedding $\sigma : F \to E$, with E a finite extension of $\mathbb{Q}_p$, then $\sigma(L(m, \psi, k)) = L^*(m, \sigma \circ \psi, k)$, so the p-part of the equivariant Tamagawa number conjecture states that (7.3) holds with $\eta = \sigma \circ \psi$: in the formulation of the equivariant Tamagawa number conjecture in [7], the canonical composition θ in (7.1) is multiplied by $\prod_{v \in S \setminus P} \mathrm{Eul}_v^*(m, \eta, k)$ (see [7, Section 3] for details). $\square$

Remark 7.5. — Note that if the interpolation formula (3.13) holds for $m = 0$ and $L_S^*(0, \eta, k) \neq 0$ then the statements of Theorem 7.2 are also true for $m = 0$. Moreover, Corollary 7.4 holds in this case if we further assume

that the reciprocal Euler factors $\mathrm{Eul}_v^*(0, \eta, k)$ are non-trivial for all primes v in S.

Remarks 7.6.

(1) Assumption 4.4 holds if $p = 2$, $k = \mathbb{Q}$ and η is the trivial character (see Remark 4.5), so Corollary 7.4 holds without any assumptions when m is odd and $k = K = \mathbb{Q}$.

(2) A stronger version of Corollary 7.4, with the maximal order replaced by $\mathbb{Z}[G]$ but $p \neq 2$, is proven by Burns in [6, Corollary 2.10] under the assumption that certain Iwasawa-theoretic p-adic μ-invariants of K are trivial. (If G is Abelian then this was already proved under similar assumptions in [1, Theorem 3.3].) This is also proved without assumptions if $k = \mathbb{Q}$ and $K/\mathbb{Q}$ is Abelian by Burns and Greither in [8, Corollary 8.1] for $p \neq 2$, and by Flach in [13, Theorem 5.1] and [14, Theorem 1.2] for $p = 2$. Huber and Kings [18, Theorem 1.3.1] have also proved Corollary 7.4 in the case $k = \mathbb{Q}$, $K/\mathbb{Q}$ Abelian and p odd. In fact, in each case the results hold at every integer m, and in [8, 13, 14, 18] the results hold for the pair $(h^0(\mathrm{Spec}\, K)(m), \mathfrak{M})$ instead of $(\pi_m h^0(\mathrm{Spec}\, K)(m), \pi_m \mathfrak{M})$.

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